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MATHEMATICS

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Research article

Statistical properties of neutrosophic n -normed spaces in a non-Archimedean framework

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The notion of statistical convergence is a natural extension of ordinary convergence which has been studied by several authors in different spaces. Recently, several studies have shown a growing interest in convergence theory in neutrosophic spaces. In this research article, we investigate the statistical convergence and related notions of sequences in a non-Archimedean neutrosophic n -normed space. By combining natural density with the three neutrosophic components of truth-membership, indeterminacy-membership, and falsity-membership, we formulate statistical convergence with respect to the neutrosophic n -norm structure. Several equivalent characterizations of statistical convergence are established, including formulations in terms of exceptional index sets of natural density zero and componentwise statistical limits. We prove that the statistical limit, whenever it exists, is unique. Moreover, we show that ordinary convergence with respect to the neutrosophic n -norm implies statistical convergence, and we examine the corresponding behavior of algebraic operations, particularly the preservation of statistical convergence under addition. Statistical Cauchy sequences are also introduced and studied. In particular, every Cauchy sequence in the neutrosophic n -norm sense is shown to be statistically Cauchy. The notion of statistical completeness is then discussed through statistically Cauchy sequences and their statistical limits. Finally, illustrative examples in finite-dimensional real spaces are provided to clarify the abstract definitions and demonstrate sequences that are statistically convergent even when ordinary convergence fails. These results extend classical statistical convergence concepts to a neutrosophic non-Archimedean setting. The developed framework also provides a foundation for investigations of summability, approximation, and nonlinear analytical problems.

Keywords: natural density, statistical convergence, statistical Cauchy sequence, non-Archimedean field, neutrosophic norm, neutrosophic n -normed space, statistical completeness, statistical limit, non-Archimedean neutrosophic n -normed space (NA- Nn NS), neutrosophic Banach space.

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Introduction

The notion of statistical convergence was introduced by Steinhaus [1] and Fast [2], independently. Since then, statistical convergence has been studied in many generalized settings. Suja and Srinivasan considered statistically convergent and statistically Cauchy sequences in non-Archimedean fields [3], while Chu and Ku studied non-Archimedean n -normed spaces in connection with the Mazur–Ulam problem [4].

The development of generalized normed spaces is closely related to the theory of fuzzy sets introduced by Zadeh [5]. Fuzzy n -normed linear spaces were later introduced by Narayanan and Vijayabalaji [6]. Deferred

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statistical convergence in intuitionistic fuzzy normed spaces was studied by Melliani et al. [7], and generalized statistical convergence in non-Archimedean L -fuzzy normed spaces was investigated by Eghbali and Ganji [8]. Intuitionistic fuzzy n -normed linear spaces were developed by Vijayabalaji et al. [9], while statistical convergence in intuitionistic fuzzy n -normed linear spaces was studied by Sen and Debnath [10]. Konwar and Debnath further considered intuitionistic fuzzy n -normed algebra and continuous product structures [11]. Recently, Saranya and Suja examined statistical convergence on intuitionistic fuzzy n -normed spaces over non-Archimedean fields [12].

Further developments in convergence theory include multidimensional sliding window Pringsheim convergence for measurable functions studied by Savaş and Patterson [13], summability approximation theorems of double random variables [14], and multidimensional characterizations of asymptotically I_2^g -equivalent sequences [15]. In the neutrosophic setting, Shankar et al. investigated lacunary Δ^m -statistical convergence in neutrosophic normed spaces [16]. Debnath et al. studied deferred statistical convergence of sequences in neutrosophic normed spaces [17]. Kirişci and Şimşek introduced neutrosophic normed spaces and statistical convergence [18]. Mursaleen developed λ -statistical convergence [19], and Kumar et al. studied neutrosophic n -normed linear spaces [20]. Recently, Kişi studied ideal convergence in neutrosophic normed spaces [21] and also investigated I_θ -convergence in the same setting [22]. Kişi and Gürdal examined triple difference sequences in neutrosophic normed spaces [23]. Khan et al. studied neutrosophic normed spaces via Tribonacci convergent sequence spaces [24], while Khan et al. also considered neutrosophic λ -statistically convergent sequence spaces and their properties [25]. Gönül Bilgin investigated arithmetic statistical convergence [26] and Hilbert I -statistical convergence in neutrosophic normed spaces [27]. Khan et al. studied statistical convergence of sequences of functions in neutrosophic normed spaces [28], and rough lacunary statistical convergence was examined by Khan et al. [29]. Ersan considered ρ -statistical convergence in neutrosophic normed spaces [30]. Rough ideal statistical convergence was studied by Antal et al. [31], and λ - Δ^m -statistical convergence was investigated by Antal and Chawla [32]. Khan et al. introduced neutrosophic fuzzy Tribonacci $\mathcal{I}$ -lacunary statistical convergent sequence spaces [33]. Mursaleen et al. studied rough deferred statistical convergence in neutrosophic normed spaces [34]. Kaur et al. investigated neutrosophic Copson type rough I -statistically convergent sequences [35]. Jenifer et al. examined rough ideal statistical convergence in neutrosophic normed spaces [36]. Khan et al. studied Hilbert ideal convergent sequence spaces in neutrosophic normed spaces [37]. Khan and Arshad applied neutrosophic normed spaces to the convergence of sequences involving neutrosophic operators [38]. Most recently, Kişi et al. discussed recent advances in the convergence of function sequences in neutrosophic normed spaces via ideals [39].

Motivated by the above developments, this paper investigates statistical convergence in non-Archimedean neutrosophic n -normed spaces. We introduce statistical convergence with respect to the neutrosophic n -norm $(\Omega, \Upsilon, \Theta)^n$, establish equivalent characterizations, prove the uniqueness of the statistical limit, and study algebraic properties such as preservation under addition. Statistical Cauchy sequences are also introduced, and the relationship between ordinary Cauchy sequences and statistically Cauchy sequences is discussed. The notion of statistical completeness is then considered, and examples are provided to illustrate that statistical convergence may hold even when ordinary convergence fails.

In this study, we work with sequences defined over a non-Archimedean field $\mathcal{K}$. Unless stated otherwise, $\mathcal{K}$ refers to a complete non-Archimedean field equipped with a nontrivial valuation, and n represents a positive integer. The study begins with a brief overview of essential notations and definitions that will be used throughout.

1 Preliminary concepts

Definition 1. [12] Let $\mathcal{U}$ be a vector space of dimension $d \geq n$ over a valued field $\mathcal{K}$, equipped with a non-Archimedean valuation $|\cdot|$. A mapping

$$\|\cdot, \cdot, \dots, \cdot\| : \underbrace{\mathcal{U} \times \mathcal{U} \times \dots \times \mathcal{U}}_{n \text{ times}} \rightarrow [0, \infty)$$

is called a *non-Archimedean n -norm* if it satisfies the following conditions:

- (i) $\|\tilde{x}_1, \tilde{x}_2, \dots, \tilde{x}_n\| = 0 \Leftrightarrow \tilde{x}_1, \tilde{x}_2, \dots, \tilde{x}_n$ are linearly dependent;
- (ii) $\|\tilde{x}_1, \tilde{x}_2, \dots, \tilde{x}_n\| = \|\tilde{x}_{\tilde{r}_1}, \tilde{x}_{\tilde{r}_2}, \dots, \tilde{x}_{\tilde{r}_n}\|$ for every permutation $(\tilde{r}_1, \tilde{r}_2, \dots, \tilde{r}_n)$ of $(1, 2, \dots, n)$;
- (iii) $\|a\tilde{x}_1, \tilde{x}_2, \dots, \tilde{x}_n\| = |a| \|\tilde{x}_1, \tilde{x}_2, \dots, \tilde{x}_n\|$;
- (iv) $\|\tilde{x} + \tilde{y}, \tilde{x}_2, \tilde{x}_3, \dots, \tilde{x}_n\| \leq \max\{\|\tilde{x}, \tilde{x}_2, \dots, \tilde{x}_n\|, \|\tilde{y}, \tilde{x}_2, \dots, \tilde{x}_n\|\}$ for all $\alpha \in \mathcal{K}$ and $\tilde{x}, \tilde{y}, \tilde{x}_2, \dots, \tilde{x}_n \in \mathcal{U}$. Then $(\mathcal{U}, \|\cdot, \cdot, \dots, \cdot\|)$ is a non-Archimedean n -normed space.

Example 1. Let $\mathcal{U} = \mathbb{R}^n$ and define the n -norm $\|\cdot, \dots, \cdot\|_{\widetilde{M}}$ for vectors $\widetilde{x}_1, \widetilde{x}_2, \dots, \widetilde{x}_n \in \mathbb{R}^n$ by

$$\|\widetilde{x}_1, \widetilde{x}_2, \dots, \widetilde{x}_n\|_{\widetilde{M}} = \prod_{j=1}^n \max_{1 \leq i \leq n} |\widetilde{x}_{ij}|,$$

where $\widetilde{x}_i = (\widetilde{x}_{i1}, \widetilde{x}_{i2}, \dots, \widetilde{x}_{in})$ for $i = 1, 2, \dots, n$. This n -norm takes the maximum absolute value in each column across all vectors, then multiplies these maximums together. For example, if $n = 2$ and $\widetilde{x}_1 = (1, 3)$, $\widetilde{x}_2 = (4, -2)$, then

$$\|\widetilde{x}_1, \widetilde{x}_2\|_{\widetilde{M}} = \max(|1|, |4|) \cdot \max(|3|, |-2|) = 4 \cdot 3 = 12.$$

This function satisfies the properties of an n -norm and defines a normed structure on $\mathbb{R}^n$.

Definition 2. [17] A binary operation $\triangleleft: [0, 1] \times [0, 1] \rightarrow [0, 1]$ is called a continuous t -norm if it satisfies the following conditions:

- (a) $\triangleleft$ is associative and commutative;
- (b) $\triangleleft$ is continuous;
- (c) $\widetilde{a} \triangleleft 1 = \widetilde{a}$ for all $\widetilde{a} \in [0, 1]$;
- (d) $\widetilde{a} \triangleleft \widetilde{b} \leq \widetilde{c} \triangleleft \widetilde{d}$, whenever $\widetilde{a} \leq \widetilde{c}$ and $\widetilde{b} \leq \widetilde{d}$ for each $\widetilde{a}, \widetilde{b}, \widetilde{c}, \widetilde{d} \in [0, 1]$.

Definition 3. [12] A binary operation $\odot: [0, 1] \times [0, 1] \rightarrow [0, 1]$ is considered a continuous t -conorm if

- (a') $\odot$ is associative and commutative;
- (b') $\odot$ is continuous;
- (c') $\widetilde{a} \odot 0 = \widetilde{a}$ for all $\widetilde{a} \in [0, 1]$;
- (d') $\widetilde{a} \odot \widetilde{b} \leq \widetilde{c} \odot \widetilde{d}$ whenever $\widetilde{a} \leq \widetilde{c}$ and $\widetilde{b} \leq \widetilde{d}$ for each $\widetilde{a}, \widetilde{b}, \widetilde{c}, \widetilde{d} \in [0, 1]$.

Definition 4. The neutrosophic n -normed space over $\mathcal{K}$ is defined as follows. The six-duple $(\mathcal{U}, \Omega, \Upsilon, \Theta, \triangleleft, \odot)$ is said to be an NA-NnN space if $\mathcal{U}$ is a vector space over a field $\mathcal{K}$, $\odot$ is a continuous t -conorm, $\triangleleft$ is a continuous t -norm, and Ω, Υ, Θ are functions of $\mathcal{U} \times \mathbb{R}$ to $[0, 1]$ satisfying the following conditions. For every $(\widetilde{x}_1, \widetilde{x}_2, \dots, \widetilde{x}_n) \in \mathcal{U}^n$ and $u, v \in \mathcal{K}$

- (a) $\Omega(\widetilde{x}_1, \widetilde{x}_2, \dots, \widetilde{x}_n, u) + \Upsilon(\widetilde{x}_1, \widetilde{x}_2, \dots, \widetilde{x}_n, u) \leq 1$.
- (b) $\Omega(\widetilde{x}_1, \widetilde{x}_2, \dots, \widetilde{x}_n, u) > 0$.
- (c) $\Omega(\widetilde{x}_1, \widetilde{x}_2, \dots, \widetilde{x}_n, u) = 1 \iff \widetilde{x}_1, \widetilde{x}_2, \dots, \widetilde{x}_n$ are linearly dependent.
- (d) $\Omega(\widetilde{x}_1, \widetilde{x}_2, \dots, \widetilde{x}_n, u)$ is invariant under any permutation of $\widetilde{x}_1, \widetilde{x}_2, \dots, \widetilde{x}_n$.
- (e) $\Omega(\widetilde{x}_1, \widetilde{x}_2, \dots, c \cdot \widetilde{x}_n, u) = \Omega\left(\widetilde{x}_1, \widetilde{x}_2, \dots, \widetilde{x}_n, \frac{u}{|c|}\right)$ for $c \neq 0, c \in \mathcal{K}$.
- (f) $\Omega(\widetilde{x}_1, \widetilde{x}_2, \dots, \widetilde{x}_n, u) \triangleleft \Omega(\widetilde{x}_1, \widetilde{x}_2, \dots, \widetilde{x}'_n, v) \leq \Omega(\widetilde{x}_1, \widetilde{x}_2, \dots, \widetilde{x}_n + \widetilde{x}'_n, \max\{u, v\})$.
- (g) $\Omega(\widetilde{x}_1, \widetilde{x}_2, \dots, \widetilde{x}_n, \cdot): (0, \infty) \rightarrow [0, 1]$ is continuous.
- (h) $\lim_{u \rightarrow \infty} \Omega(\widetilde{x}_1, \widetilde{x}_2, \dots, \widetilde{x}_n, u) = 1$ and $\lim_{u \rightarrow 0} \Omega(\widetilde{x}_1, \widetilde{x}_2, \dots, \widetilde{x}_n, u) = 0$.
- (i) $\Upsilon(\widetilde{x}_1, \widetilde{x}_2, \dots, \widetilde{x}_n, u) < 1$.
- (j) $\Upsilon(\widetilde{x}_1, \widetilde{x}_2, \dots, \widetilde{x}_n, u) = 0 \iff \widetilde{x}_1, \widetilde{x}_2, \dots, c \cdot \widetilde{x}_n$ are linearly dependent.
- (k) $\Upsilon(\widetilde{x}_1, \widetilde{x}_2, \dots, \widetilde{x}_n, u)$ is invariant under any permutation of $\widetilde{x}_1, \widetilde{x}_2, \dots, \widetilde{x}_n$.
- (l) $\Upsilon(\widetilde{x}_1, \widetilde{x}_2, \dots, c \cdot \widetilde{x}_n, u) = \Upsilon\left(\widetilde{x}_1, \widetilde{x}_2, \dots, \widetilde{x}_n, \frac{u}{|c|}\right)$ for each $c \neq 0, c \in \mathcal{K}$.
- (m) $\Upsilon(\widetilde{x}_1, \widetilde{x}_2, \dots, \widetilde{x}_n, u) \odot \Upsilon(\widetilde{x}_1, \widetilde{x}_2, \dots, \widetilde{x}'_n, v) \geq \Upsilon(\widetilde{x}_1, \widetilde{x}_2, \dots, \widetilde{x}_n + \widetilde{x}'_n, \max\{u, v\})$.
- (n) $\Upsilon(\widetilde{x}_1, \widetilde{x}_2, \dots, \widetilde{x}_n, \cdot): (0, \infty) \rightarrow [0, 1]$ is continuous.
- (o) $\lim_{u \rightarrow \infty} \Upsilon(\widetilde{x}_1, \widetilde{x}_2, \dots, \widetilde{x}_n, u) = 0$ and $\lim_{u \rightarrow 0} \Upsilon(\widetilde{x}_1, \widetilde{x}_2, \dots, \widetilde{x}_n, u) = 1$.
- (p) $\Theta(\widetilde{x}_1, \widetilde{x}_2, \dots, \widetilde{x}_n, u) < 1$.
- (q) $\Theta(\widetilde{x}_1, \widetilde{x}_2, \dots, \widetilde{x}_n, u) = 0 \iff \widetilde{x}_1, \widetilde{x}_2, \dots, c \cdot \widetilde{x}_n$ are linearly dependent.
- (r) $\Theta(\widetilde{x}_1, \widetilde{x}_2, \dots, \widetilde{x}_n, u)$ is invariant under any permutation of $\widetilde{x}_1, \widetilde{x}_2, \dots, \widetilde{x}_n$.
- (s) $\Theta(\widetilde{x}_1, \widetilde{x}_2, \dots, c \cdot \widetilde{x}_n, u) = \Theta\left(\widetilde{x}_1, \widetilde{x}_2, \dots, \widetilde{x}_n, \frac{u}{|c|}\right)$ for each $c \neq 0, c \in \mathcal{K}$.
- (t) $\Theta(\widetilde{x}_1, \widetilde{x}_2, \dots, \widetilde{x}_n, u) \odot \Theta(\widetilde{x}_1, \widetilde{x}_2, \dots, \widetilde{x}'_n, v) \geq \Theta(\widetilde{x}_1, \widetilde{x}_2, \dots, \widetilde{x}_n + \widetilde{x}'_n, \max\{u, v\})$.
- (u) $\Theta(\widetilde{x}_1, \widetilde{x}_2, \dots, \widetilde{x}_n, \cdot): (0, \infty) \rightarrow [0, 1]$ is continuous.
- (v) $\lim_{u \rightarrow \infty} \Theta(\widetilde{x}_1, \widetilde{x}_2, \dots, \widetilde{x}_n, u) = 0$ and $\lim_{u \rightarrow 0} \Theta(\widetilde{x}_1, \widetilde{x}_2, \dots, \widetilde{x}_n, u) = 1$.

Here, $(\Omega, \Upsilon, \Theta)$ is called a non-Archimedean neutrosophic n -norm.

Example 2. Let $(\mathcal{U}, \|\cdot\|)$ be an n -normed space over $\mathbb{R}^n$ with a standard n -norm $\|\cdot\|$ (e.g., determinant-based).

Define the operations and mappings as follows for $\tilde{a}, \tilde{b} \in [0, 1]$: $\tilde{a} \triangleleft \tilde{b} = \min\{\tilde{a}, \tilde{b}\}$, $\tilde{a} \odot \tilde{b} = \max\{\tilde{a}, \tilde{b}\}$,

$$\Omega(\tilde{x}_1, \dots, \tilde{x}_n) = \frac{1}{1 + \|\tilde{x}_1, \dots, \tilde{x}_n\|}, \quad \Upsilon(\tilde{x}_1, \dots, \tilde{x}_n, \tilde{u}) = \frac{\|u, \tilde{x}_2, \dots, \tilde{x}_n\|}{1 + \|\tilde{x}_1, \dots, \tilde{x}_n\|},$$

$$\Theta(\tilde{x}_1, \dots, \tilde{x}_n, \tilde{u}) = 1 - \Omega(\tilde{x}_1, \dots, \tilde{x}_n).$$

Then, the structure $(\mathcal{U}, \Omega, \Upsilon, \Theta, \triangleleft, \odot)$ forms a Neutrosophic n -Normed Space (NnNS).

For instance, let $\tilde{x}_1 = (1, 0)$, $\tilde{x}_2 = (0, 1)$ in $\mathbb{R}^2$, with $n = 2$ and the 2-norm defined by $\|\tilde{x}_1, \tilde{x}_2\| = |\det(\tilde{x}_1, \tilde{x}_2)| = 1$. Then,

$$\Omega(\tilde{x}_1, \tilde{x}_2) = \frac{1}{1+1} = \frac{1}{2}, \quad \Theta(\tilde{x}_1, \tilde{x}_2, \tilde{x}_1) = 1 - \frac{1}{2} = \frac{1}{2}, \quad \Upsilon(\tilde{x}_1, \tilde{x}_2, \tilde{x}_1) = \frac{|\det(\tilde{x}_1, \tilde{x}_2)|}{1 + |\det(\tilde{x}_1, \tilde{x}_2)|} = \frac{1}{2}.$$

This example satisfies the neutrosophic space structure using bounded norm-based mappings.

Definition 5. [18] Let $\{\tilde{x}_Q\}$ be a sequence in a non-Archimedean neutrosophic n -normed space $(\mathcal{U}, \Omega, \Upsilon, \Theta, \triangleleft, \odot)$. The sequence is said to converge to $\ell \in \mathcal{U}$ with respect to $(\Omega, \Upsilon, \Theta)^n$ if, for every $\epsilon > 0$ and $u > 0$, and for all fixed elements $\tilde{y}_2, \tilde{y}_3, \dots, \tilde{y}_n \in \mathcal{U}$, there exists an integer $Q_0 \in \mathbb{N}$ such that for all $Q \geq Q_0$, the following inequalities hold:

$$\left\{ \begin{array}{l} \Omega(\tilde{x}_Q - \ell, \tilde{y}_2, \dots, \tilde{y}_n, \tilde{u}) > 1 - \epsilon, \\ \Upsilon(\tilde{x}_Q - \ell, \tilde{y}_2, \dots, \tilde{y}_n, \tilde{u}) < \epsilon, \\ \Theta(\tilde{x}_Q - \ell, \tilde{y}_2, \dots, \tilde{y}_n, \tilde{u}) < \epsilon \end{array} \right\}.$$

This type of convergence is denoted by $(\Omega, \Upsilon, \Theta)^n$ - $\lim \tilde{x}_Q = \ell$.

Definition 6. [19] A sequence $x = \{\tilde{x}_Q\}$ is said to be *statistically convergent* to a real number ℓ if, for each $\epsilon > 0$, the set $\{Q \in \mathbb{N} : |\tilde{x}_Q - \ell| \geq \epsilon\} = 0$. In other words, the relative frequency of terms that differ from ℓ by at least ϵ approaches zero as $Q \rightarrow \infty$. This is denoted by: $\text{Stat-lim } \tilde{x}_Q = \ell$.

Example 3. Consider the sequence $y = \{\tilde{y}_Q\}$ defined by $\tilde{y}_Q = \begin{cases} \frac{1}{2^Q}, & \text{if } Q \text{ is prime} \\ 0, & \text{otherwise.} \end{cases}$ Now, we consider this

sequence in the 2-adic number field $\mathbb{Q}_2$, using the 2-adic absolute value $|\cdot|_2$.

Observe that for each prime Q , we have $|\tilde{y}_Q|_2 = |\frac{1}{2^Q}|_2 = 2^Q$, which tends to ∞ as $Q \rightarrow \infty$, but recall that in $\mathbb{Q}_2$, higher exponents correspond to smaller values in the usual sense (i.e., $|2^{-Q}|_2 = 2^Q$ is large, but the number itself is small).

In particular, $\tilde{y}_Q \rightarrow 0$ in the 2-adic topology on a sparse set (the primes), and $\tilde{y}_Q = 0$ elsewhere.

The density of the primes among the natural numbers is zero:

$$\lim_{n \rightarrow \infty} \frac{|\{Q \leq n : Q \text{ prime}\}|}{n} = 0.$$

Thus, for any $\epsilon > 0$, the set $A_\epsilon = \{Q \in \mathbb{N} : |\tilde{y}_Q|_2 > \epsilon\}$ is contained in the primes and hence has density 0. Therefore, $\{\tilde{y}_Q\}$ is statistically convergent to 0 in $\mathbb{Q}_2$.

2 Statistical convergence in NA-NnNS

Definition 7. [18] Let $(\mathcal{U}, \Omega, \Upsilon, \Theta, \triangleleft, \odot)$ be a non-Archimedean neutrosophic n -normed space. A sequence $x = \{\tilde{x}_Q\} \subseteq \mathcal{U}$ is said to be *statistically convergent* to $\ell \in \mathcal{U}$ with respect to the neutrosophic n -norm $(\Omega, \Upsilon, \Theta)^n$ if, for every $\epsilon > 0$, $u > 0$, and for any $\tilde{y}_2, \tilde{y}_3, \dots, \tilde{y}_n \in \mathcal{U}$, the following condition holds:

$$\lim_{n \rightarrow \infty} \frac{1}{n} \left| \left\{ Q \leq n : \begin{array}{l} \Omega(\tilde{x}_Q - \ell, \tilde{y}_2, \tilde{y}_3, \dots, \tilde{y}_n, \tilde{u}) \leq 1 - \epsilon \text{ or} \\ \Upsilon(\tilde{x}_Q - \ell, \tilde{y}_2, \tilde{y}_3, \dots, \tilde{y}_n, \tilde{u}) \geq \epsilon \text{ or} \\ \Theta(\tilde{x}_Q - \ell, \tilde{y}_2, \tilde{y}_3, \dots, \tilde{y}_n, \tilde{u}) \geq \epsilon \end{array} \right\} \right| = 0.$$

In this case, we write $\text{Stat}_{(\Omega, \Upsilon, \Theta)^n}$ - $\lim x = \ell$.

Lemma 1. Let $(\mathcal{U}, \Omega, \Upsilon, \Theta, \triangleleft, \odot)$ be a neutrosophic n -normed linear space (NnNLS). Then, for any $\epsilon > 0$ and $u > 0$, and for all fixed elements $\tilde{y}_1, \tilde{y}_2, \dots, \tilde{y}_{n-1} \in \mathcal{U}$, the following statements are mutually equivalent:

- (a) $\text{Stat}_{(\Omega, \Upsilon, \Theta)^n} - \lim x = \ell;$
- (b)
$$\lim_{n \rightarrow \infty} \frac{1}{n} \left| \left\{ Q \leq n : \begin{array}{l} \Omega(\tilde{x}_Q - \ell, \tilde{y}_2, \tilde{y}_3, \dots, \tilde{y}_n, \tilde{u}) \leq 1 - \epsilon, \\ \Upsilon(\tilde{x}_Q - \ell, \tilde{y}_2, \tilde{y}_3, \dots, \tilde{y}_n, \tilde{u}) \geq \epsilon, \\ \Theta(\tilde{x}_Q - \ell, \tilde{y}_2, \tilde{y}_3, \dots, \tilde{y}_n, \tilde{u}) \geq \epsilon \end{array} \right\} \right| = 0;$$
- (c)
$$\lim_{n \rightarrow \infty} \frac{1}{n} \left| \left\{ Q \leq n : \begin{array}{l} \Omega(\tilde{x}_Q - \ell, \tilde{y}_2, \tilde{y}_3, \dots, \tilde{y}_n, \tilde{u}) > 1 - \epsilon, \text{ and} \\ \Upsilon(\tilde{x}_Q - \ell, \tilde{y}_2, \tilde{y}_3, \dots, \tilde{y}_n, \tilde{u}) < \epsilon, \text{ and} \\ \Theta(\tilde{x}_Q - \ell, \tilde{y}_2, \tilde{y}_3, \dots, \tilde{y}_n, \tilde{u}) < \epsilon \end{array} \right\} \right| = 1;$$
- (d)
$$\lim_{n \rightarrow \infty} \frac{1}{n} \left| \left\{ Q \leq n : \begin{array}{l} \Omega(\tilde{x}_Q - \ell, \tilde{y}_2, \tilde{y}_3, \dots, \tilde{y}_n, \tilde{u}) > 1 - \epsilon, \\ \Upsilon(\tilde{x}_Q - \ell, \tilde{y}_2, \tilde{y}_3, \dots, \tilde{y}_n, \tilde{u}) < \epsilon, \\ \Theta(\tilde{x}_Q - \ell, \tilde{y}_2, \tilde{y}_3, \dots, \tilde{y}_n, \tilde{u}) < \epsilon \end{array} \right\} \right| = 1;$$
- (e)

$\text{Stat-lim } \Omega(\tilde{x}_Q - \ell, \tilde{y}_2, \tilde{y}_3, \dots, \tilde{y}_n, \tilde{u}) = 1,$ and $\text{Stat-lim } \Upsilon(\tilde{x}_Q - \ell, \tilde{y}_2, \tilde{y}_3, \dots, \tilde{y}_n, \tilde{u}) = 0,$
and $\text{Stat-lim } \Theta(\tilde{x}_Q - \ell, \tilde{y}_2, \tilde{y}_3, \dots, \tilde{y}_n, \tilde{u}) = 0.$

Theorem 1. Let $(\mathcal{U}, \Omega, \Upsilon, \Theta, \triangleleft, \odot)$ be a non-Archimedean neutrosophic n -normed space (NA-NnNS). If a sequence $x = \{\tilde{x}_Q\} \subseteq \mathcal{U}$ is statistically convergent with respect to the neutrosophic n -norm $(\Omega, \Upsilon, \Theta)^n$, then the statistical limit $\text{Stat}_{(\Omega, \Upsilon, \Theta)^n} - \lim \tilde{x}_Q$ exists and is uniquely determined.

Proof. Assume that $\text{Stat}_{(\Omega, \Upsilon, \Theta)^n} - \lim x_1 = \ell_1$ and $\text{Stat}_{(\Omega, \Upsilon, \Theta)^n} - \lim \tilde{x} = \ell_2$. Let $\epsilon > 0$ be given. Choose $d \in (0, 1)$ such that $(1 - \tilde{d}) \triangleleft (1 - \tilde{d}) > 1 - \epsilon$ and $d \odot d < \epsilon$. For any $u > 0$ and for all $\tilde{y}_1, \tilde{y}_2, \dots, \tilde{y}_{n-1} \in \mathcal{U}$, we define the following sets:

$$\begin{aligned} Q_{\Omega,1}(d, \tilde{u}) &:= \left\{ Q \in \mathbb{N} : \Omega(\tilde{x}_Q - \ell_1, \tilde{y}_2, \tilde{y}_3, \dots, \tilde{y}_n, \tilde{u}) > 1 - \tilde{d} \right\}, \\ Q_{\Omega,2}(d, \tilde{u}) &:= \left\{ Q \in \mathbb{N} : \Omega(\tilde{x}_Q - \ell_2, \tilde{y}_2, \tilde{y}_3, \dots, \tilde{y}_n, \tilde{u}) > 1 - \tilde{d} \right\}, \\ Q_{\Upsilon,1}(d, \tilde{u}) &:= \left\{ Q \in \mathbb{N} : \Upsilon(\tilde{x}_Q - \ell_1, \tilde{y}_2, \tilde{y}_3, \dots, \tilde{y}_n, \tilde{u}) \geq \tilde{d} \right\}, \\ Q_{\Upsilon,2}(d, \tilde{u}) &:= \left\{ Q \in \mathbb{N} : \Upsilon(\tilde{x}_Q - \ell_2, \tilde{y}_2, \tilde{y}_3, \dots, \tilde{y}_n, \tilde{u}) \geq \tilde{d} \right\}, \\ Q_{\Theta,1}(d, \tilde{u}) &:= \left\{ Q \in \mathbb{N} : \Theta(\tilde{x}_Q - \ell_1, \tilde{y}_2, \tilde{y}_3, \dots, \tilde{y}_n, \tilde{u}) \geq \tilde{d} \right\}, \\ Q_{\Theta,2}(d, \tilde{u}) &:= \left\{ Q \in \mathbb{N} : \Theta(\tilde{x}_Q - \ell_2, \tilde{y}_2, \tilde{y}_3, \dots, \tilde{y}_n, \tilde{u}) \geq \tilde{d} \right\}. \end{aligned}$$

Assuming $\text{Stat}_{(\Omega, \Upsilon, \Theta)^n} - \lim \tilde{x} = \ell_1$, we deduce from Lemma 1 that

$$\lim_n \frac{1}{n} |\{Q_{\Omega,1}(\epsilon, \tilde{u})\}| = \lim_n \frac{1}{n} |\{Q_{\Upsilon,1}(\epsilon, \tilde{u})\}| = \lim_n \frac{1}{n} |\{Q_{\Theta,1}(\epsilon, \tilde{u})\}| = 1$$

for all $u > 0$.

Moreover, based on the assumption that $\text{Stat}_{(\Omega, \Upsilon, \Theta)^n} - \lim \tilde{x} = \ell_2$, we conclude

$$\lim_n \frac{1}{n} |\{Q_{\Omega,2}(\epsilon, \tilde{u})\}| = \lim_n \frac{1}{n} |\{Q_{\Upsilon,2}(\epsilon, \tilde{u})\}| = \lim_n \frac{1}{n} |\{Q_{\Theta,2}(\epsilon, \tilde{u})\}| = 1$$

for all $u > 0$.

Let us now consider,

$$Q_{\Omega, \Upsilon, \Theta}(\epsilon, \tilde{u}) = (Q_{\Omega,1}(\epsilon, \tilde{u}) \cap Q_{\Omega,2}(\epsilon, \tilde{u})) \cup (Q_{\Upsilon,1}(\epsilon, \tilde{u}) \cap Q_{\Upsilon,2}(\epsilon, \tilde{u})) \cup (Q_{\Theta,1}(\epsilon, \tilde{u}) \cap Q_{\Theta,2}(\epsilon, \tilde{u})).$$

In the next step, $\lim_n \frac{1}{n} |\{Q_{\Omega, \Upsilon, \Theta}(\epsilon, \tilde{u})\}| = 1$. Suppose $Q \in Q_{\Omega, \Upsilon, \Theta}(\epsilon, \tilde{u})$. Our initial focus is on $Q \in (Q_{\Omega, 1}(\epsilon, \tilde{u}) \cap Q_{\Omega, 2}(\epsilon, \tilde{u}))$. Then we have

$$\begin{aligned} & \Omega(\ell_1 - \ell_2, \tilde{y}_2, \tilde{y}_3, \dots, \tilde{y}_n, \tilde{u}) \\ & \geq \Omega(\tilde{x}_Q - \ell_1, \tilde{y}_2, \tilde{y}_3, \dots, \tilde{y}_n, \tilde{u}) \triangleleft \Omega(\tilde{x}_Q - \ell_2, \tilde{y}_2, \tilde{y}_3, \dots, \tilde{y}_n, \tilde{u}) \\ & > (1 - \tilde{d}) \triangleleft (1 - \tilde{d}). \end{aligned}$$

Given that $(1 - \tilde{d}) \triangleleft (1 - \tilde{d}) > 1 - \epsilon$, it follows that $\Omega(\ell_1 - \ell_2, \tilde{y}_2, \tilde{y}_3, \dots, \tilde{y}_n, \tilde{u}) > 1 - \epsilon$. Since $\epsilon > 0$ is arbitrary, we conclude

$$\Omega(\ell_1 - \ell_2, \tilde{y}_2, \tilde{y}_3, \dots, \tilde{y}_n, \tilde{u}) = 1 \quad \text{for all } u > 0 \text{ and } \tilde{y}_2, \tilde{y}_3, \dots, \tilde{y}_n \in \mathcal{U}.$$

Hence, it must be that $\ell_1 = \ell_2$.

On the other hand, assuming $Q \in Q_{\Upsilon, 1}(\epsilon, \tilde{u}) \cap Q_{\Upsilon, 2}(\epsilon, \tilde{u})$, we have

$$\begin{aligned} & \Upsilon(\ell_1 - \ell_2, \tilde{y}_2, \tilde{y}_3, \dots, \tilde{y}_n, \tilde{u}) \\ & \leq \Upsilon(\tilde{x}_Q - \ell_1, \tilde{y}_2, \tilde{y}_3, \dots, \tilde{y}_n, \tilde{u}) \odot \Upsilon(\tilde{x}_Q - \ell_2, \tilde{y}_2, \tilde{y}_3, \dots, \tilde{y}_n, \tilde{u}) \\ & < d \odot d < \epsilon. \end{aligned}$$

Since ϵ can be chosen freely, it follows that $\Upsilon(\ell_1 - \ell_2, \tilde{y}_2, \tilde{y}_3, \dots, \tilde{y}_n, \tilde{u}) = 0$ for every $u > 0$ and all $\tilde{y}_2, \tilde{y}_3, \dots, \tilde{y}_n \in \mathcal{U}$. This implies that ℓ_1 must be equal to ℓ_2 .

As a result, if $Q \in (Q_{\Upsilon, 1}(\epsilon, \tilde{u}) \cap Q_{\Upsilon, 2}(\epsilon, \tilde{u}))$, then

$$\begin{aligned} & \Theta(\ell_1 - \ell_2, \tilde{y}_2, \tilde{y}_3, \dots, \tilde{y}_n, \tilde{u}) \\ & \leq \Theta(\tilde{x}_Q - \ell_1, \tilde{y}_2, \tilde{y}_3, \dots, \tilde{y}_n, \tilde{u}) \odot \Theta(\tilde{x}_Q - \ell_2, \tilde{y}_2, \tilde{y}_3, \dots, \tilde{y}_n, \tilde{u}) \\ & < d \odot d < \epsilon. \end{aligned}$$

Since ϵ is arbitrary, it follows that $\Theta(\ell_1 - \ell_2, \tilde{y}_2, \tilde{y}_3, \dots, \tilde{y}_n, u) = 0$ for all $u > 0$ and $\tilde{y}_2, \tilde{y}_3, \dots, \tilde{y}_n \in \mathcal{U}$, which implies $\ell_1 = \ell_2$. Therefore, the $\text{Stat}_{(\Omega, \Upsilon, \Theta)^n}$ - $\lim \tilde{x}$ is unique. $\square$

Theorem 2. Let $(\mathcal{U}, \Omega, \Upsilon, \Theta, \triangleleft, \odot)$ be an NA-NnNS. If $(\Omega, \Upsilon, \Theta)^n$ - $\lim \tilde{x} = \ell$, then $\text{Stat}_{(\Omega, \Upsilon, \Theta)^n}$ - $\lim x = \ell$.

Proof. Assume that $(\Omega, \Upsilon, \Theta)^n$ - $\lim \tilde{x}_Q = \ell$. Then, for any $\epsilon > 0$, $d > 0$, and fixed elements $\tilde{y}_1, \tilde{y}_2, \dots, \tilde{y}_{n-1} \in \mathcal{U}$, there exists a positive integer $n_0 \in \mathbb{N}$ such that for all $Q \geq n_0$, the following conditions are satisfied:

$$\left\{ \begin{array}{l} \Omega(\tilde{x}_n - \tilde{x}_m, \tilde{y}_2, \dots, \tilde{y}_n, \tilde{u}) > 1 - \epsilon, \\ \Upsilon(\tilde{x}_n - \tilde{x}_m, \tilde{y}_2, \dots, \tilde{y}_n, \tilde{u}) < \epsilon, \\ \Theta(\tilde{x}_n - \tilde{x}_m, \tilde{y}_2, \dots, \tilde{y}_n, \tilde{u}) < \epsilon \end{array} \right\}.$$

Thus, the set of indices where these inequalities are violated,

$$\left\{ Q \in \mathbb{N} \left| \begin{array}{l} \Omega(\tilde{x}_Q - \ell, \tilde{y}_2, \dots, \tilde{y}_n, \tilde{u}) \leq 1 - \epsilon \\ \text{or } \Upsilon(\tilde{x}_Q - \ell, \tilde{y}_2, \dots, \tilde{y}_n, \tilde{u}) \geq \epsilon \\ \text{or } \Theta(\tilde{x}_Q - \ell, \tilde{y}_2, \dots, \tilde{y}_n, \tilde{u}) \geq \epsilon \end{array} \right. \right\}$$

is finite, which implies

$$\lim_{n \rightarrow \infty} \frac{1}{n} \left| \left\{ Q \in \mathbb{N} : \begin{array}{l} \Omega(\tilde{x}_Q - \ell, \tilde{y}_2, \tilde{y}_3, \dots, \tilde{y}_n, \tilde{u}) \leq 1 - \epsilon \text{ or } \\ \Upsilon(\tilde{x}_Q - \ell, \tilde{y}_2, \tilde{y}_3, \dots, \tilde{y}_n, \tilde{u}) \geq \epsilon \text{ or } \\ \Theta(\tilde{x}_Q - \ell, \tilde{y}_2, \tilde{y}_3, \dots, \tilde{y}_n, \tilde{u}) \geq \epsilon \end{array} \right\} \right| = 0.$$

Therefore, $\text{Stat}_{(\Omega, \Upsilon, \Theta)^n}$ - $\lim \tilde{x} = \ell$.

Note: Interestingly, the converse of the above theorem, which is not classically true, is true in NA-NnNS as shown below.

Let $x = \{\tilde{x}_Q\}$ be $\text{Stat}_{(\Omega, \Upsilon, \Theta)^n}$ that converges to $\ell \in \mathcal{U}$. Subsequently, for every $\epsilon > 0$ and $u > 0$,

$$\lim_{n \rightarrow \infty} \frac{1}{n} \left| \left\{ Q \in \mathbb{N} : \begin{array}{l} \Omega(\tilde{x}_Q - \ell, \tilde{y}_2, \tilde{y}_3, \dots, \tilde{y}_n, \tilde{u}) \leq 1 - \epsilon \text{ or } \\ \Upsilon(\tilde{x}_Q - \ell, \tilde{y}_2, \tilde{y}_3, \dots, \tilde{y}_n, \tilde{u}) \geq \epsilon \text{ or } \\ \Theta(\tilde{x}_Q - \ell, \tilde{y}_2, \tilde{y}_3, \dots, \tilde{y}_n, \tilde{u}) \geq \epsilon \end{array} \right\} \right| = 0.$$

We now prove that $\{\tilde{x}_Q\}$ is $(\Omega, \Upsilon, \Theta)^n$ - convergent to $\ell \in \mathcal{U}$. i.e., to prove

$$\left\{ \begin{array}{l} \Omega(\tilde{x}_Q - \ell, \tilde{y}_2, \tilde{y}_3, \dots, \tilde{y}_n, \tilde{u}) > 1 - \epsilon \text{ and} \\ Q \in \mathbb{N} : \Upsilon(\tilde{x}_Q - \ell, \tilde{y}_2, \tilde{y}_3, \dots, \tilde{y}_n, \tilde{u}) < \epsilon \text{ and} \\ \Theta(\tilde{x}_Q - \ell, \tilde{y}_2, \tilde{y}_3, \dots, \tilde{y}_n, \tilde{u}) < \epsilon \end{array} \right\}.$$

We assume that

$$\begin{aligned} & \{\Omega(\tilde{x}_Q - \ell, \tilde{y}_2, \tilde{y}_3, \dots, \tilde{y}_n, \tilde{u}) \leq 1 - \epsilon \text{ or } \Upsilon(\tilde{x}_Q - \ell, \tilde{y}_2, \tilde{y}_3, \dots, \tilde{y}_n, \tilde{u}) \geq \epsilon \text{ or} \\ & \Theta(\tilde{x}_Q - \ell, \tilde{y}_2, \tilde{y}_3, \dots, \tilde{y}_n, \tilde{u}) \geq \epsilon\} \\ \Rightarrow & \{Q \in \mathbb{N} : \Omega(\tilde{x}_Q - \ell, \tilde{y}_2, \tilde{y}_3, \dots, \tilde{y}_n, \tilde{u}) \leq 1 - \epsilon \text{ or } \Upsilon(\tilde{x}_Q - \ell, \tilde{y}_2, \tilde{y}_3, \dots, \tilde{y}_n, \tilde{u}) \geq \epsilon \text{ or} \\ & \Theta(\tilde{x}_Q - \ell, \tilde{y}_2, \tilde{y}_3, \dots, \tilde{y}_n, \tilde{u}) \geq \epsilon\} \end{aligned}$$

has infinitely many terms.

$$\lim_n \frac{1}{n} \left| \left\{ \begin{array}{l} \Omega(\tilde{x}_Q - \ell, \tilde{y}_2, \tilde{y}_3, \dots, \tilde{y}_n, \tilde{u}) \leq 1 - \epsilon \text{ or} \\ Q \in \mathbb{N} : \Upsilon(\tilde{x}_Q - \ell, \tilde{y}_2, \tilde{y}_3, \dots, \tilde{y}_n, \tilde{u}) \geq \epsilon \text{ or} \\ \Theta(\tilde{x}_Q - \ell, \tilde{y}_2, \tilde{y}_3, \dots, \tilde{y}_n, \tilde{u}) \geq \epsilon \end{array} \right\} \right| \neq 0.$$

However, this finding is contradictory. Thus, $\{\tilde{x}_Q\}$ is $(\Omega, \Upsilon, \Theta)^n$, which converges to $\ell \in \mathcal{U}$. $\square$

Theorem 3. Let $(\mathcal{U}, \Omega, \Upsilon, \Theta, \triangleleft, \oslash)$ be an NA-NnNS. If $\text{Stat}_{(\Omega, \Upsilon, \Theta)^n} - \lim x_Q = \tilde{x}$ and $\text{Stat}_{(\Omega, \Upsilon, \Theta)^n} - \lim y_Q = \tilde{y}$, then $\text{Stat}_{(\Omega, \Upsilon, \Theta)^n} - \lim (\tilde{x}_Q + \tilde{y}_Q) = \tilde{x} + \tilde{y}$.

Proof. Given any $\epsilon > 0$, choose $d > 0$ such that $(1 - d)^2 > 1 - \epsilon$. Let ℓ_1 and ℓ_2 denote the statistical limit points of the sequences $(\tilde{x}_Q)$ and $(\tilde{y}_Q)$, respectively. Then, for any $\tilde{u} > 0$ and elements $\tilde{y}_2, \tilde{y}_3, \dots, \tilde{y}_n \in \mathcal{U}$, we define the following sets:

$$Q_{\Omega,1}(d, \tilde{u}) := \{Q \in \mathbb{N} : \Omega(\tilde{x}_Q - \ell_1, \tilde{y}_2, \tilde{y}_3, \dots, \tilde{y}_n, \tilde{u}) > 1 - d\},$$

$$Q_{\Omega,2}(d, \tilde{u}) := \{Q \in \mathbb{N} : \Omega(\tilde{x}_Q - \ell_2, \tilde{y}_2, \tilde{y}_3, \dots, \tilde{y}_n, \tilde{u}) > 1 - d\},$$

$$Q_{\Upsilon,1}(d, \tilde{u}) := \{Q \in \mathbb{N} : \Upsilon(\tilde{x}_Q - \ell_1, \tilde{y}_2, \tilde{y}_3, \dots, \tilde{y}_n, \tilde{u}) < d\},$$

$$Q_{\Upsilon,2}(d, \tilde{u}) := \{Q \in \mathbb{N} : \Upsilon(\tilde{x}_Q - \ell_2, \tilde{y}_2, \tilde{y}_3, \dots, \tilde{y}_n, \tilde{u}) < d\},$$

$$Q_{\Theta,1}(d, \tilde{u}) := \{Q \in \mathbb{N} : \Theta(\tilde{x}_Q - \ell_1, \tilde{y}_2, \tilde{y}_3, \dots, \tilde{y}_n, \tilde{u}) < d\},$$

$$Q_{\Theta,2}(d, \tilde{u}) := \{Q \in \mathbb{N} : \Theta(\tilde{x}_Q - \ell_2, \tilde{y}_2, \tilde{y}_3, \dots, \tilde{y}_n, \tilde{u}) < d\}.$$

As

$$\text{Stat}_{(\Omega, \Upsilon, \Theta)^n} - \lim x_Q = \tilde{x} \text{ and } \text{Stat}_{(\Omega, \Upsilon, \Theta)^n} - \lim y_Q = \tilde{y},$$

$$\lim_n \frac{1}{n} \left| \{Q_{\Omega,1}(\epsilon, \tilde{d})\} \right| = \lim_n \frac{1}{n} \left| \{Q_{\Upsilon,1}(\epsilon, \tilde{d})\} \right| = \lim_n \frac{1}{n} \left| \{Q_{\Theta,1}(\epsilon, \tilde{d})\} \right| = 1,$$

and

$$\lim_n \frac{1}{n} \left| \{Q_{\Omega,2}(\epsilon, \tilde{t})\} \right| = \lim_n \frac{1}{n} \left| \{Q_{\Upsilon,2}(\epsilon, \tilde{d})\} \right| = \lim_n \frac{1}{n} \left| \{Q_{\Theta,2}(\epsilon, \tilde{d})\} \right| = 1.$$

Let

$$Q_{\Omega,u} = (Q_{\Omega,1}(\epsilon, \tilde{u}) \cap Q_{\Omega,2}(\epsilon, \tilde{u})) \cup (Q_{\Upsilon,1}(\epsilon, \tilde{u}) \cap Q_{\Upsilon,2}(\epsilon, \tilde{u})) \cup (Q_{\Theta,1}(\epsilon, \tilde{u}) \cap Q_{\Theta,2}(\epsilon, \tilde{u})).$$

Then,

$$\lim_n \frac{1}{n} \left| \{Q_{\Omega, \Upsilon, \Theta}(\epsilon, \tilde{u})\} \right| = 1.$$

If $Q \in Q_{\Omega, \Upsilon, \Theta}(\epsilon, \tilde{u})$ at the outset, consider $Q \in (Q_{\Omega,1}(\epsilon, \tilde{u}) \cap Q_{\Omega,2}(\epsilon, \tilde{u}))$. Following that, we arrive at

$$\begin{aligned} & \Omega(\tilde{x}_Q + \tilde{y}_Q - \tilde{x} - \tilde{y}, \tilde{y}_2, \tilde{y}_3, \dots, \tilde{y}_n, \tilde{u}) \\ & \geq \Omega(\tilde{x}_Q - \tilde{x}, \tilde{y}_2, \tilde{y}_3, \dots, \tilde{y}_n, \tilde{u}) \triangleleft \Omega(\tilde{y}_Q - \tilde{y}, \tilde{y}_2, \tilde{y}_3, \dots, \tilde{y}_n, \tilde{u}) \\ & > (1 - d) \triangleleft (1 - d) \\ & > 1 - \epsilon. \end{aligned}$$

Since, ϵ is arbitrary, $\Omega(\tilde{x}_Q + \tilde{y}_Q - (\tilde{x} + \tilde{y}), \tilde{y}_2, \tilde{y}_3, \dots, \tilde{y}_n, \tilde{u}) = 1$.

In a similar way,

$$\Upsilon(\tilde{x}_Q + \tilde{y}_Q - (\tilde{x} + \tilde{y}), \tilde{y}_2, \tilde{y}_3, \dots, \tilde{y}_n, \tilde{u}) = 0, \text{ and also } \Theta(\tilde{x}_Q + \tilde{y}_Q - (\tilde{x} + \tilde{y}), \tilde{y}_2, \tilde{y}_3, \dots, \tilde{y}_n, \tilde{u}) = 0.$$

Thus, $\text{Stat}_{(\Omega, \Upsilon, \Theta)^n} - \lim(\tilde{x}_Q + \tilde{y}_Q) = \tilde{x} + \tilde{y}$. □

Theorem 4. Let $(\mathcal{U}, \Omega, \Upsilon, \Theta, \triangleleft, \odot)$ be an NA-NnNS over $\mathcal{K}$. If $\left\{ \begin{array}{l} \lim_{Q \rightarrow \infty} \Omega(\tilde{x}_Q - \ell, \tilde{y}_2, \dots, \tilde{y}_n, \tilde{u}) = 1, \\ \lim_{Q \rightarrow \infty} \Upsilon(\tilde{x}_Q - \ell, \tilde{y}_2, \dots, \tilde{y}_n, \tilde{u}) = 0, \text{ and} \\ \lim_{Q \rightarrow \infty} \Theta(\tilde{x}_Q - \ell, \tilde{y}_2, \dots, \tilde{y}_n, \tilde{u}) = 0 \end{array} \right\}$.

Then, $\text{Stat}_{(\Omega, \Upsilon, \Theta)^n} - \lim_{Q \rightarrow \infty} \tilde{x}_Q = \ell$.

Proof. Assume that the following limits hold:

$$\left\{ \begin{array}{l} \lim_{Q \rightarrow \infty} \Omega(\tilde{x}_Q - \ell, \tilde{y}_2, \dots, \tilde{y}_n, \tilde{u}) = 1, \\ \lim_{Q \rightarrow \infty} \Upsilon(\tilde{x}_Q - \ell, \tilde{y}_2, \dots, \tilde{y}_n, \tilde{u}) = 0, \\ \lim_{Q \rightarrow \infty} \Theta(\tilde{x}_Q - \ell, \tilde{y}_2, \dots, \tilde{y}_n, \tilde{u}) = 0 \end{array} \right\}.$$

Then, for any $\epsilon > 0$ and $u > 0$, there exists a natural number $Q_0 \in \mathbb{N}$ such that for all $Q \geq Q_0$, we have

$$\left\{ \begin{array}{l} \Omega(\tilde{x}_Q - \ell, \tilde{y}_2, \dots, \tilde{y}_n, \tilde{u}) > 1 - \epsilon, \\ \Upsilon(\tilde{x}_Q - \ell, \tilde{y}_2, \dots, \tilde{y}_n, \tilde{u}) < \epsilon, \\ \Theta(\tilde{x}_Q - \ell, \tilde{y}_2, \dots, \tilde{y}_n, \tilde{u}) < \epsilon \end{array} \right\}.$$

Hence, the count of elements $Q \leq n$ for which these conditions are violated tends to zero as $n \rightarrow \infty$.

$$\left\{ \begin{array}{l} \Omega(\tilde{x}_Q - \ell, \tilde{y}_2, \tilde{y}_3, \dots, \tilde{y}_n, \tilde{u}) \leq 1 - \epsilon \text{ or} \\ Q \in \mathbb{N} : \Upsilon(\tilde{x}_Q - \ell, \tilde{y}_2, \tilde{y}_3, \dots, \tilde{y}_n, \tilde{u}) \geq \epsilon \text{ or} \\ \Theta(\tilde{x}_Q - \ell, \tilde{y}_2, \tilde{y}_3, \dots, \tilde{y}_n, \tilde{u}) \geq \epsilon \end{array} \right\} = 0$$

is finite. So,

$$\lim_{n \rightarrow \infty} \frac{1}{n} \left| \left\{ \begin{array}{l} \Omega(\tilde{x}_Q - \ell, \tilde{y}_2, \tilde{y}_3, \dots, \tilde{y}_n, \tilde{u}) \leq 1 - \epsilon \text{ or} \\ Q \in \mathbb{N} : \Upsilon(\tilde{x}_Q - \ell, \tilde{y}_2, \tilde{y}_3, \dots, \tilde{y}_n, \tilde{u}) \geq \epsilon \text{ or} \\ \Theta(\tilde{x}_Q - \ell, \tilde{y}_2, \tilde{y}_3, \dots, \tilde{y}_n, \tilde{u}) \geq \epsilon \end{array} \right\} \right| = 0.$$

Thus, $\text{Stat}_{(\Omega, \Upsilon, \Theta)^n} - \lim \tilde{x}_Q = \ell$. □

3 Statistical cauchy sequence in NA-NnNS

Definition 8. [18] Let $(\mathcal{U}, \Omega, \Upsilon, \Theta, \triangleleft, \odot)$ be a non-Archimedean neutrosophic n -normed space (NA-NnNS). A sequence $\{\tilde{x}_n\} \subset \mathcal{U}$ is said to be a *Cauchy sequence* with respect to the neutrosophic n -norm $(\Omega, \Upsilon, \Theta)^n$ if for every $\epsilon > 0$ and $u > 0$, there exists $n_0 \in \mathbb{N}$ such that for all $n, m \geq n_0$ and for all $\tilde{y}_2, \dots, \tilde{y}_n \in \mathcal{U}$, the following conditions hold:

$$\left\{ \begin{array}{l} \Omega(\tilde{x}_n - \tilde{x}_m, \tilde{y}_2, \dots, \tilde{y}_n, \tilde{u}) > 1 - \epsilon, \\ \Upsilon(\tilde{x}_n - \tilde{x}_m, \tilde{y}_2, \dots, \tilde{y}_n, \tilde{u}) < \epsilon, \\ \Theta(\tilde{x}_n - \tilde{x}_m, \tilde{y}_2, \dots, \tilde{y}_n, \tilde{u}) < \epsilon \end{array} \right\}.$$

Definition 9. [18] Let $(\mathcal{U}, \Omega, \Upsilon, \Theta, \triangleleft, \odot)$ be a non-Archimedean neutrosophic n -normed space (NA-NnNS). A sequence $x = \{\tilde{x}_Q\} \subset \mathcal{U}$ is called a *statistically Cauchy sequence* if for every $\epsilon > 0$ and $u > 0$, there exists $N \in \mathbb{N}$ such that for all $\ell \geq N$,

$$\lim_{n \rightarrow \infty} \frac{1}{n} \left| \left\{ \begin{array}{l} \Omega(\tilde{x}_Q - \tilde{x}_\ell, \tilde{y}_2, \dots, \tilde{y}_n, \tilde{u}) \leq 1 - \epsilon \text{ or} \\ Q \in \mathbb{N} : \Upsilon(\tilde{x}_Q - \tilde{x}_\ell, \tilde{y}_2, \dots, \tilde{y}_n, \tilde{u}) \geq \epsilon \text{ or} \\ \Theta(\tilde{x}_Q - \tilde{x}_\ell, \tilde{y}_2, \dots, \tilde{y}_n, \tilde{u}) \geq \epsilon \end{array} \right\} \right| = 0.$$

The NA-NnNS is said to be *complete* if every $(\Omega, \Upsilon, \Theta)^n$ -statistically Cauchy sequence converges statistically in the same norm. Such a space is called a non-Archimedean neutrosophic Banach space.

Theorem 5. In the NA-NnNS $(\mathcal{U}, \Omega, \Upsilon, \Theta, \triangleleft, \odot)$ over $\mathcal{K}$, every Cauchy sequence with respect to $(\Omega, \Upsilon, \Theta)^n$ is statistically Cauchy.

Proof. Assume that $\{\tilde{x}_Q\}$ is a Cauchy sequence with respect to the neutrosophic n -norm $(\Omega, \Upsilon, \Theta)^n$. Then for every $\epsilon > 0$ and $u > 0$, there exist $n_0 \in \mathbb{N}$ and a constant $p \in \mathbb{N}$ such that for all $Q \geq n_0$,

$$\left\{ \begin{array}{l} \Omega(\tilde{x}_{Q+p} - \tilde{x}_Q, \tilde{y}_2, \dots, \tilde{y}_n, \tilde{u}) > 1 - \epsilon, \\ \Upsilon(\tilde{x}_{Q+p} - \tilde{x}_Q, \tilde{y}_2, \dots, \tilde{y}_n, \tilde{u}) < \epsilon, \\ \Theta(\tilde{x}_{Q+p} - \tilde{x}_Q, \tilde{y}_2, \dots, \tilde{y}_n, \tilde{u}) < \epsilon. \end{array} \right\}.$$

Moreover, the set where these conditions fail has zero natural density:

$$\lim_{n \rightarrow \infty} \frac{1}{n} \left| \left\{ Q \in \mathbb{N} : \begin{array}{l} \Omega(\tilde{x}_{Q+p} - \tilde{x}_Q, \tilde{y}_2, \dots, \tilde{y}_n, \tilde{u}) \leq 1 - \epsilon, \\ \Upsilon(\tilde{x}_{Q+p} - \tilde{x}_Q, \tilde{y}_2, \dots, \tilde{y}_n, \tilde{u}) \geq \epsilon, \\ \Theta(\tilde{x}_{Q+p} - \tilde{x}_Q, \tilde{y}_2, \dots, \tilde{y}_n, \tilde{u}) \geq \epsilon \end{array} \right\} \right| = 0.$$

Thus, $\{\tilde{x}_Q\}$ is statistically Cauchy with respect to $(\Omega, \Upsilon, \Theta)^n$. $\square$

Example 4. Let $\mathcal{U} = \mathbb{R}^n$ over $\mathcal{K} = \mathbb{R}$ and define the usual n -norm by $\|z_1, z_2, \dots, z_n\|_n = |\det(z_1, z_2, \dots, z_n)|$. For $u > 0$, define as

$$\Omega(z_1, \dots, z_n, u) = \frac{u}{u + \|z_1, \dots, z_n\|_n}, \quad \Upsilon(z_1, \dots, z_n, u) = \frac{\|z_1, \dots, z_n\|_n}{u + \|z_1, \dots, z_n\|_n},$$

and

$$\Theta(z_1, \dots, z_n, u) = \frac{\|z_1, \dots, z_n\|_n}{2u + \|z_1, \dots, z_n\|_n}.$$

Take $\tilde{x}_Q = \frac{1}{Q}e_1$, $Q \in \mathbb{N}$, where $e_1, e_2, \dots, e_n$ are the standard basis vectors of $\mathbb{R}^n$. We show that $\{\tilde{x}_Q\}$ is statistically Cauchy with respect to $(\Omega, \Upsilon, \Theta)^n$.

Fix $p \in \mathbb{N}$, $u > 0$, and let $\tilde{y}_2 = e_2, \tilde{y}_3 = e_3, \dots, \tilde{y}_n = e_n$. Then,

$$\tilde{x}_{Q+p} - \tilde{x}_Q = \left(\frac{1}{Q+p} - \frac{1}{Q} \right) e_1 = -\frac{p}{Q(Q+p)} e_1.$$

Hence,

$$\|\tilde{x}_{Q+p} - \tilde{x}_Q, e_2, \dots, e_n\|_n = \frac{p}{Q(Q+p)}.$$

Therefore,

$$\Omega(\tilde{x}_{Q+p} - \tilde{x}_Q, e_2, \dots, e_n, u) = \frac{u}{u + \frac{p}{Q(Q+p)}} \rightarrow 1$$

as $Q \rightarrow \infty$, while

$$\Upsilon(\tilde{x}_{Q+p} - \tilde{x}_Q, e_2, \dots, e_n, u) = \frac{\frac{p}{Q(Q+p)}}{u + \frac{p}{Q(Q+p)}} \rightarrow 0, \text{ and } \Theta(\tilde{x}_{Q+p} - \tilde{x}_Q, e_2, \dots, e_n, u) = \frac{\frac{p}{Q(Q+p)}}{2u + \frac{p}{Q(Q+p)}} \rightarrow 0.$$

Thus, for every $\epsilon > 0$, there exists $Q_0 \in \mathbb{N}$ such that for all $Q \geq Q_0$,

$$\Omega(\tilde{x}_{Q+p} - \tilde{x}_Q, e_2, \dots, e_n, u) > 1 - \epsilon, \quad \Upsilon(\tilde{x}_{Q+p} - \tilde{x}_Q, e_2, \dots, e_n, u) < \epsilon, \text{ and } \Theta(\tilde{x}_{Q+p} - \tilde{x}_Q, e_2, \dots, e_n, u) < \epsilon.$$

Consequently, the set

$$\left\{ Q \in \mathbb{N} : \begin{array}{l} \Omega(\tilde{x}_{Q+p} - \tilde{x}_Q, e_2, \dots, e_n, u) \leq 1 - \epsilon, \\ \text{or } \Upsilon(\tilde{x}_{Q+p} - \tilde{x}_Q, e_2, \dots, e_n, u) \geq \epsilon, \\ \text{or } \Theta(\tilde{x}_{Q+p} - \tilde{x}_Q, e_2, \dots, e_n, u) \geq \epsilon \end{array} \right\}$$

is contained in the finite set $\{1, 2, \dots, Q_0 - 1\}$. Hence its natural density is zero. That is,

$$\lim_{m \rightarrow \infty} \frac{1}{m} \left| \left\{ Q \leq m : \begin{array}{l} \Omega(\tilde{x}_{Q+p} - \tilde{x}_Q, e_2, \dots, e_n, u) \leq 1 - \epsilon, \\ \text{or } \Upsilon(\tilde{x}_{Q+p} - \tilde{x}_Q, e_2, \dots, e_n, u) \geq \epsilon, \\ \text{or } \Theta(\tilde{x}_{Q+p} - \tilde{x}_Q, e_2, \dots, e_n, u) \geq \epsilon \end{array} \right\} \right| = 0.$$

Therefore, $\{\tilde{x}_Q\}$ is statistically Cauchy with respect to $(\Omega, \Upsilon, \Theta)^n$.

Theorem 6. Let $(\mathcal{U}, \Omega, \Upsilon, \Theta, \triangleleft, \odot)$ be a non-Archimedean neutrosophic n -normed space over the field $\mathcal{K}$. If a sequence $\{\tilde{x}_k\}_{k \in \mathbb{N}}$ is statistically convergent, then its statistical limit is unique.

Proof. Suppose the sequence $\{\tilde{x}_Q\}$ is statistically convergent to ℓ . Then for every $\epsilon > 0$, $\tilde{u} > 0$, and $\tilde{y}_2, \tilde{y}_3, \dots, \tilde{y}_n \in \mathcal{U}$, it holds that

$$\lim_{n \rightarrow \infty} \frac{1}{n} \left| \left\{ Q \leq n : \begin{array}{l} \Omega(\tilde{x}_n - \tilde{x}, \tilde{y}_2, \dots, \tilde{y}_n, \tilde{u}) \leq 1 - \epsilon, \\ \text{or } \Upsilon(\tilde{x}_n - \tilde{x}, \tilde{y}_2, \dots, \tilde{y}_n, \tilde{u}) \geq \epsilon, \\ \text{or } \Theta(\tilde{x}_n - \tilde{x}, \tilde{y}_2, \dots, \tilde{y}_n, \tilde{u}) \geq \epsilon \end{array} \right\} \right| = 0.$$

Next, consider the set

$$\lim_{n \rightarrow \infty} \frac{1}{n} \left\{ Q \leq n : \begin{array}{l} \Omega(\tilde{x}_n - \tilde{x}, \tilde{y}_2, \dots, \tilde{y}_n, \tilde{u}) \leq 1 - \epsilon, \\ \text{or } \Upsilon(\tilde{x}_n - \tilde{x}, \tilde{y}_2, \dots, \tilde{y}_n, \tilde{u}) \leq 1 - \epsilon, \\ \text{or } \Theta(\tilde{x}_n - \tilde{x}, \tilde{y}_2, \dots, \tilde{y}_n, \tilde{u}) \leq 1 - \epsilon \end{array} \right\} = 0.$$

By using the operations $\triangleleft$ and $\odot$, it follows that

$$\lim_{n \rightarrow \infty} \frac{1}{n} \left| \left\{ Q \leq n : \begin{array}{l} \Omega(\tilde{x}_Q - \ell, \tilde{y}_2, \tilde{y}_3, \dots, \tilde{y}_n, \tilde{u}) \\ \triangleleft \Omega(\tilde{x}_\ell - \ell, \tilde{y}_2, \tilde{y}_3, \dots, \tilde{y}_n, \tilde{u}) \leq 1 - \epsilon, \\ \text{or } \Upsilon(\tilde{x}_Q - \ell, \tilde{y}_2, \tilde{y}_3, \dots, \tilde{y}_n, \tilde{u}) \\ \odot \Upsilon(\tilde{x}_\ell - \ell, \tilde{y}_2, \tilde{y}_3, \dots, \tilde{y}_n, \tilde{u}) \geq \epsilon, \\ \text{or } \Theta(\tilde{x}_Q - \ell, \tilde{y}_2, \tilde{y}_3, \dots, \tilde{y}_n, \tilde{u}) \\ \odot \Theta(\tilde{x}_\ell - \ell, \tilde{y}_2, \tilde{y}_3, \dots, \tilde{y}_n, \tilde{u}) \geq \epsilon \end{array} \right\} \right| = 0.$$

□

4 Statistical completeness in NA-NnNS

Definition 10. [20] A non-Archimedean neutrosophic n -normed space $(\mathcal{U}, \Omega, \Upsilon, \Theta, \triangleleft, \odot)$ over the field $\mathcal{K}$ is called statistically complete if every sequence that is statistically Cauchy with respect to the neutrosophic n -norm $(\Omega, \Upsilon, \Theta)^n$ also statistically converges within the same norm.

Theorem 7. Every NA-NnNS $(\mathcal{U}, \Omega, \Upsilon, \Theta, \triangleleft, \odot)$ over $\mathcal{K}$ is statistically complete with respect to $(\Omega, \Upsilon, \Theta)^n$.

Proof. Assume that $\{\tilde{x}_Q\}$ is statistically Cauchy but not statistically convergent to $\ell \in \mathcal{U}$; then, we have

$$\begin{aligned} & \lim_{n \rightarrow \infty} \frac{1}{n} \left| \left\{ Q \leq n : \begin{array}{l} \text{at least one of the following holds:} \\ \Omega(\tilde{x}_Q - \tilde{x}_\ell, \tilde{y}_2, \dots, \tilde{y}_n, \tilde{u}) \leq 1 - \epsilon, \\ \Upsilon(\tilde{x}_Q - \tilde{x}_\ell, \tilde{y}_2, \dots, \tilde{y}_n, \tilde{u}) \geq \epsilon, \\ \Theta(\tilde{x}_Q - \tilde{x}_\ell, \tilde{y}_2, \dots, \tilde{y}_n, \tilde{u}) \geq \epsilon \end{array} \right\} \right| \\ &= \lim_{n \rightarrow \infty} \frac{1}{n} \left| \left\{ Q \leq n : \begin{array}{l} \text{at least one of the following holds:} \\ \Omega(\tilde{x}_Q - \ell, \tilde{y}_2, \dots, \tilde{y}_n, \tilde{u}) \triangleleft \Omega(\tilde{x}_\ell - \ell, \tilde{y}_2, \dots, \tilde{y}_n, \tilde{u}) \leq 1 - \epsilon, \\ \Upsilon(\tilde{x}_Q - \ell, \tilde{y}_2, \dots, \tilde{y}_n, \tilde{u}) \odot \Upsilon(\tilde{x}_\ell - \ell, \tilde{y}_2, \dots, \tilde{y}_n, \tilde{u}) \geq \epsilon, \\ \Theta(\tilde{x}_Q - \ell, \tilde{y}_2, \dots, \tilde{y}_n, \tilde{u}) \odot \Theta(\tilde{x}_\ell - \ell, \tilde{y}_2, \dots, \tilde{y}_n, \tilde{u}) \geq \epsilon \end{array} \right\} \right| = 0. \end{aligned}$$

However, this finding is contradictory. □

Example 5. Let $\mathcal{U} = \mathbb{R}^n$ over $\mathcal{K} = \mathbb{R}$ and define the usual n -norm by $\|z_1, z_2, \dots, z_n\|_n = |\det(z_1, z_2, \dots, z_n)|$. For $u > 0$, define

$$\Omega(z_1, \dots, z_n, u) = \frac{u}{u + \|z_1, \dots, z_n\|_n}, \quad \Upsilon(z_1, \dots, z_n, u) = \frac{\|z_1, \dots, z_n\|_n}{u + \|z_1, \dots, z_n\|_n},$$

and

$$\Theta(z_1, \dots, z_n, u) = \frac{\|z_1, \dots, z_n\|_n}{2u + \|z_1, \dots, z_n\|_n}.$$

Together with $a \triangleleft b = ab$, $a \odot b = \min\{a + b, 1\}$, this gives a standard example of an NA-NnNS.

Now, we define a sequence $\{\tilde{x}_Q\}$ in $\mathbb{R}^n$ by

$$\tilde{x}_Q = \begin{cases} e_1, & Q = k^2 \text{ for some } k \in \mathbb{N}, \\ 0, & \text{otherwise,} \end{cases} \quad \text{where } e_1 = (1, 0, \dots, 0).$$

We claim that $\{\tilde{x}_Q\}$ is statistically convergent to 0 with respect to $(\Omega, \Upsilon, \Theta)^n$. Let $\epsilon > 0$, $u > 0$, and $\tilde{y}_2, \dots, \tilde{y}_n \in \mathbb{R}^n$ be fixed. Consider the set

$$A_{\epsilon, u} = \left\{ Q \in \mathbb{N} : \begin{array}{l} \Omega(\tilde{x}_Q - 0, \tilde{y}_2, \dots, \tilde{y}_n, u) \leq 1 - \epsilon, \\ \text{or } \Upsilon(\tilde{x}_Q - 0, \tilde{y}_2, \dots, \tilde{y}_n, u) \geq \epsilon, \\ \text{or } \Theta(\tilde{x}_Q - 0, \tilde{y}_2, \dots, \tilde{y}_n, u) \geq \epsilon \end{array} \right\}.$$

If Q is not a perfect square, then $\tilde{x}_Q = 0$. Hence,

$$\Omega(0, \tilde{y}_2, \dots, \tilde{y}_n, u) = 1, \quad \Upsilon(0, \tilde{y}_2, \dots, \tilde{y}_n, u) = 0, \quad \text{and} \quad \Theta(0, \tilde{y}_2, \dots, \tilde{y}_n, u) = 0.$$

Therefore, the above inequalities can fail only when Q is a perfect square. Thus, $A_{\epsilon, u} \subseteq \{k^2 : k \in \mathbb{N}\}$. Since the set of perfect squares has natural density zero, we have

$$\lim_{m \rightarrow \infty} \frac{1}{m} |\{Q \leq m : Q = k^2 \text{ for some } k \in \mathbb{N}\}| = \lim_{m \rightarrow \infty} \frac{\lfloor \sqrt{m} \rfloor}{m} = 0.$$

Hence, $\lim_{m \rightarrow \infty} \frac{1}{m} |A_{\epsilon, u} \cap \{1, 2, \dots, m\}| = 0$. Therefore, $\text{Stat} - \lim_{Q \rightarrow \infty} \tilde{x}_Q = 0$ with respect to $(\Omega, \Upsilon, \Theta)^n$.

Consequently, $\{\tilde{x}_Q\}$ is statistically convergent, and hence it is statistically Cauchy. This example shows how a sequence may fail to be ordinarily convergent but still becomes statistically convergent in an NA-NnNS because its exceptional terms occur only on a set of natural density zero.

5 Statistically continuous in NA-NnNS

Definition 11. [2] Consider the space $(\mathcal{U}, \Omega, \Upsilon, \Theta, \triangleleft, \odot)$, which forms a non-Archimedean neutrosophic n -normed space (NA-NnNS) over the field $\mathcal{K}$. A function $f : \mathcal{U} \rightarrow \mathcal{U}$ is said to be continuous at a point $x \in \mathcal{U}$ with respect to the neutrosophic n -norm $(\Omega, \Upsilon, \Theta)^n$ if whenever a sequence $\{\tilde{x}_n\}$ converges to $\tilde{x}$ in this space, the image sequence $\{f(\tilde{x}_n)\}$ converges to $f(\tilde{x})$ in the same neutrosophic n -norm.

Definition 12. [2, 18, 20] Let $(\mathcal{U}, \Omega, \Upsilon, \Theta, \triangleleft, \odot)$ be a non-Archimedean neutrosophic n -normed space (NA-NnNS) over the field $\mathcal{K}$. A mapping $f : \mathcal{U} \rightarrow \mathcal{U}$ is said to be statistically continuous at a point $\tilde{x} \in \mathcal{U}$ if for every sequence $\{\tilde{x}_n\}$ in $\mathcal{U}$,

$$\text{Stat}_{(\Omega, \Upsilon, \Theta)^n} - \lim \tilde{x}_n = \tilde{x}$$

implies that

$$\text{Stat}_{(\Omega, \Upsilon, \Theta)^n} - \lim f(\tilde{x}_n) = f(\tilde{x}).$$

Theorem 8. Consider $(\mathcal{U}, \Omega, \Upsilon, \Theta, \triangleleft, \odot)$ as a non-Archimedean neutrosophic n -normed space over the field $\mathcal{K}$. A function $f : \mathcal{U} \rightarrow \mathcal{U}$ is said to be statistically continuous if it is continuous with respect to the neutrosophic n -norm $(\Omega, \Upsilon, \Theta)^n$.

Proof. Let $\{\tilde{x}_n\}$ be a sequence in $\mathcal{U}$ such that $\text{Stat}_{(\Omega, \Upsilon, \Theta)^n} - \lim \tilde{x}_n = \tilde{x}$. Then, for every $\epsilon > 0$ and $\tilde{u} > 0$, the following conditions hold:

$$\begin{cases} \Omega(\tilde{x}_n - \tilde{x}, \tilde{y}_2, \tilde{y}_3, \dots, \tilde{y}_n, \tilde{u}) > 1 - \epsilon, \\ \Upsilon(\tilde{x}_n - \tilde{x}, \tilde{y}_2, \tilde{y}_3, \dots, \tilde{y}_n, \tilde{u}) < \epsilon, \\ \Theta(\tilde{x}_n - \tilde{x}, \tilde{y}_2, \tilde{y}_3, \dots, \tilde{y}_n, \tilde{u}) < \epsilon. \end{cases}$$

These imply:

$$\begin{cases} \Omega(f(\tilde{x}_n) - f(\tilde{x}), f(\tilde{y}_2), f(\tilde{y}_3), \dots, f(\tilde{y}_n), \tilde{u}) > 1 - \epsilon, \\ \Upsilon(f(\tilde{x}_n) - f(\tilde{x}), f(\tilde{y}_2), f(\tilde{y}_3), \dots, f(\tilde{y}_n), \tilde{u}) < \epsilon, \\ \Theta(f(\tilde{x}_n) - f(\tilde{x}), f(\tilde{y}_2), f(\tilde{y}_3), \dots, f(\tilde{y}_n), \tilde{u}) < \epsilon. \end{cases}$$

Due to the continuity of $\mathfrak{f}$, the set

$$\left\{ n \in \mathbb{N} : \begin{array}{l} \Omega(\mathfrak{f}(\tilde{x}_n) - \mathfrak{f}(\tilde{x}), \mathfrak{f}(\tilde{y}_2), \dots, \mathfrak{f}(\tilde{y}_n), \tilde{u}) \leq 1 - \epsilon, \\ \text{or } \Upsilon(\mathfrak{f}(\tilde{x}_n) - \mathfrak{f}(\tilde{x}), \mathfrak{f}(\tilde{y}_2), \dots, \mathfrak{f}(\tilde{y}_n), \tilde{u}) \geq \epsilon, \\ \text{or } \Theta(\mathfrak{f}(\tilde{x}_n) - \mathfrak{f}(\tilde{x}), \mathfrak{f}(\tilde{y}_2), \dots, \mathfrak{f}(\tilde{y}_n), \tilde{u}) \geq \epsilon \end{array} \right\}$$

is a subset of

$$\left\{ n \in \mathbb{N} : \begin{array}{l} \Omega(\tilde{x}_n - \tilde{x}, \tilde{y}_2, \dots, \tilde{y}_n, \tilde{u}) \leq 1 - \epsilon, \\ \text{or } \Upsilon(\tilde{x}_n - \tilde{x}, \tilde{y}_2, \dots, \tilde{y}_n, \tilde{u}) \geq \epsilon, \\ \text{or } \Theta(\tilde{x}_n - \tilde{x}, \tilde{y}_2, \dots, \tilde{y}_n, \tilde{u}) \geq \epsilon \end{array} \right\}.$$

Since $\text{Stat}(\Omega, \Upsilon, \Theta)^n\text{-}\lim \tilde{x}_n = \tilde{x}$, we get

$$\lim_{n \rightarrow \infty} \frac{1}{n} \left| \left\{ n \in \mathbb{N} : \begin{array}{l} \Omega(\tilde{x}_n - \tilde{x}, \tilde{y}_2, \dots, \tilde{y}_n, \tilde{u}) \leq 1 - \epsilon, \\ \text{or } \Upsilon(\tilde{x}_n - \tilde{x}, \tilde{y}_2, \dots, \tilde{y}_n, \tilde{u}) \geq \epsilon, \\ \text{or } \Theta(\tilde{x}_n - \tilde{x}, \tilde{y}_2, \dots, \tilde{y}_n, \tilde{u}) \geq \epsilon \end{array} \right\} \right| = 0.$$

Therefore,

$$\lim_{n \rightarrow \infty} \frac{1}{n} \left| \left\{ n \in \mathbb{N} : \begin{array}{l} \Omega(\mathfrak{f}(\tilde{x}_n) - \mathfrak{f}(\tilde{x}), \mathfrak{f}(\tilde{y}_2), \dots, \mathfrak{f}(\tilde{y}_n), \tilde{u}) \leq 1 - \epsilon, \\ \text{or } \Upsilon(\mathfrak{f}(\tilde{x}_n) - \mathfrak{f}(\tilde{x}), \mathfrak{f}(\tilde{y}_2), \dots, \mathfrak{f}(\tilde{y}_n), \tilde{u}) \geq \epsilon, \\ \text{or } \Theta(\mathfrak{f}(\tilde{x}_n) - \mathfrak{f}(\tilde{x}), \mathfrak{f}(\tilde{y}_2), \dots, \mathfrak{f}(\tilde{y}_n), \tilde{u}) \geq \epsilon \end{array} \right\} \right| = 0,$$

which means $\text{Stat}(\Omega, \Upsilon, \Theta)^n\text{-}\lim \mathfrak{f}(\tilde{x}_n) = \mathfrak{f}(\tilde{x})$. Hence, $\mathfrak{f}$ is statistically continuous. $\square$

Conclusion

In this study, we investigated statistical convergence in non-Archimedean neutrosophic n -normed spaces by combining natural density with the truth-membership, indeterminacy-membership, and falsity-membership components. Several basic properties were established, including equivalent characterizations, the uniqueness of the statistical limit, preservation under addition, and the relation between ordinary convergence and statistical convergence. Statistical Cauchy sequences and statistical completeness were also discussed. The examples show that this approach is flexible enough to handle sequences that are not ordinarily convergent but are statistically convergent. These results extend the classical theory of statistical convergence to the neutrosophic non-Archimedean n -normed setting and may motivate further studies on ideal convergence, lacunary, and rough statistical convergence in this framework. These results open avenues for a deeper exploration of convergence concepts in more generalized and uncertain environments.

Building on the results of this study, future research may focus on extending the concepts of statistical convergence and completeness to other generalized neutrosophic structures, such as fuzzy or neutrosophic n -normed spaces. Additionally, exploring the relationships between statistical convergence and other types of convergence (e.g., ideal convergence or lacunary statistical convergence) in the non-Archimedean neutrosophic setting could yield deeper insights. Applications of these concepts to functional analysis, fixed point theory, and decision-making problems under uncertainty also present promising directions for further investigation.

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Author contributions

All authors contributed equally to writing this paper. Furthermore, this manuscript was read and approved by all authors.

Conflict of interest

The authors declare that there is no conflict of interest.

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Singular discrete Sturm–Liouville problems with transmission conditions

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This paper examines singular discrete Sturm–Liouville equations subject to transmission conditions, focusing on their fundamental spectral and geometric properties. In mathematical physics and operator theory, understanding the asymptotic behavior of solutions at singular boundaries is highly essential. While the classical spectral theory for continuous systems has been comprehensively analyzed, the structural nuances of their discrete counterparts containing interior discontinuities require a more detailed investigation. For the discrete equations under consideration, Weyl’s famous alternative classification is successfully established, dividing the underlying equations into limit-circle and limit-point cases based on the behavior of their solutions. By incorporating transmission conditions at an interior point, this study bridges a critical gap between discrete calculus and boundary value problems with transmission interactions. We construct the corresponding Hilbert space framework and thoroughly investigate the core properties of the corresponding Weyl function. Furthermore, the geometric characteristics of Weyl circles are systematically derived. Ultimately, the generalized spectral characteristics established herein offer comprehensive insights into discrete systems with transmission conditions, contributing significantly to the ongoing advancement of the spectral analysis of discontinuous difference operators.

Keywords: difference equations, discrete Sturm–Liouville problem, boundary conditions, Weyl theory and its generalizations, limit circle case, limit point case, transmission conditions, singular case.

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Introduction

In differential equation theory, the Sturm–Liouville problem is one of the most studied problems. The significance of the topic is increased by the appearance of such issues in the solution of partial differential equations using the technique of separating their variables [1–3]. Such problems have been studied under a variety of boundary condition types and situations [4–6]. Similar to impulsive boundary conditions, these conditions have been extensively studied in the literature (see [7–9] and its references).

For many mathematicians today, discrete equations are one of the most important topics to study. Discrete equations come from the approximate solution of differential equations through discretization. Nonetheless, they often manifest independently as mathematical encodings of various real-life events. Using algorithmic techniques, computers can solve discrete equations with ease. Spectral theory of difference equations has been addressed in [10–12]. In [13], the authors investigated the spectral properties of the following second-order difference equation

$$\begin{aligned} & a(\zeta - 1)z(\zeta - 1) + b(\zeta)z(\zeta) + a(\zeta)z(\zeta + 1) \\ & = (2 \cosh y)z(\zeta), \quad \zeta \in \mathbb{N} \setminus \{k - 1, k, k + 1\} \quad (\mathbb{N} := \{1, 2, 3, \dots\}), \end{aligned}$$

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$$z(0) = 0, \quad \begin{cases} z(k+1) = \theta_1 z(k-1), \\ \Delta z(k+1) = \theta_2 \nabla z(k-1), \end{cases} \quad \theta_1, \theta_2 \in \mathbb{R},$$

where $\{a(\zeta)\}_{\zeta \in \mathbb{N}}, \{b(\zeta)\}_{\zeta \in \mathbb{N}}$ are real sequences. Ergün and Guseinov [14] studied the following difference equation

$$\begin{aligned} & \Delta^2 z(\zeta - 1) + q(\zeta) z(\zeta) \\ & = \lambda \rho(\zeta) z(\zeta), \quad \zeta \in \mathbb{Z} \setminus \{0, 1\}, \quad (\mathbb{Z} := \{0, \pm 1, \pm 2, \pm 3, \dots\}), \\ & z(-1) = z(1), \quad \Delta z(-1) = e^{2i\delta} \Delta z(1), \end{aligned}$$

where $z = z(\zeta)$ with $\zeta \in \mathbb{Z}$ is a desired solution, $q(\zeta) \in \mathbb{R}$ for $\zeta \in \mathbb{Z} \setminus \{0, 1\}$, $\delta \in [0, \frac{\pi}{2})$, and

$$\rho(\zeta) = \begin{cases} e^{2i\delta} & \text{if } n \leq -1, \\ e^{-2i\delta} & \text{if } n \geq 2. \end{cases}$$

They gave simple conditions for the discreteness of its spectrum. In [15], Bairamov et al. considered a discrete Schrödinger operator defined as

$$\begin{aligned} & a(\zeta - 1) z(\zeta - 1) + b(\zeta) z(\zeta) + a(\zeta) z(\zeta + 1) \\ & = \lambda z(\zeta), \quad \zeta \in \mathbb{N} \setminus \{m_0 - 1, m_0, m_0 + 1\}, \end{aligned}$$

$$\begin{aligned} & z(m_0 + 1) = \gamma_1 z(m_0 - 1), \\ & z(m_0 + 2) - z(m_0 + 1) = \gamma_2 (z(m_0 - 1) - z(m_0 - 2)), \end{aligned}$$

where $\gamma_1 \gamma_2 \neq 0, \gamma_1, \gamma_2 \in \mathbb{R}$; λ is a spectral parameter, $\{a(\zeta)\}_{\zeta \in \mathbb{N} \cup \{0\}}$ and $\{b(\zeta)\}_{\zeta \in \mathbb{N} \cup \{0\}}$ are real sequences satisfying the condition

$$\sum_{\zeta \in \mathbb{N}} \zeta (|1 - a(\zeta)| + |b(\zeta)|) < \infty.$$

By using the scattering solutions, they investigated the properties of the scattering function of this problem.

In this study, we will transfer the famous classification given by Weyl [16] in 1910 for classical Sturm–Liouville problems to discrete Sturm–Liouville problems with transmission conditions. It has been studied in [10, 12] for discrete Sturm–Liouville problems in the singular case without transmission conditions.

1 Set up

In this paper, we consider the discrete Sturm–Liouville equation with transmission conditions

$$\Gamma(z) := \frac{1}{w(\zeta)} \nabla [p(\zeta) \Delta z(\zeta)] + q(\zeta) z(\zeta) = \lambda z(\zeta), \quad \zeta \in J, \quad (1)$$

$$\begin{aligned} & p(\eta + 1) z(\eta + 1) - dp(\eta - 1) z(\eta - 1) = 0, \\ & z(\eta + 2) - \frac{1}{d} z(\eta) = 0, \end{aligned} \quad (2)$$

where Δ is the forward difference operator ($\Delta y(\zeta) = y(\zeta + 1) - y(\zeta)$), and ∇ is the backward difference operator ($\nabla y(\zeta) = y(\zeta) - y(\zeta - 1)$), and $-\infty \leq a < \eta - 1 < \eta < \eta + 1 < b \leq \infty$, $a, \eta, b \in \mathbb{Z}$, and $J = [a, b] \cap \mathbb{Z} \setminus \{\eta, \eta + 1\}$, λ is a parameter, $p(\zeta)$, $q(\zeta)$, and $w(\zeta)$ are real numbers such that $w(\zeta) > 0$ for $\zeta \in J$, and $p(\zeta) > 0$ for $\zeta \in J$, $q(\zeta)$ is a finite at $\eta, \eta + 1$, and $p(\zeta)$ is a positive at $\eta, \eta + 1$, and $0 \neq d \in \mathbb{R}$.

Recently, the spectral properties of the problem (1)–(2) were investigated in the references [17, 18] in the regular and singular cases.

Theorem 1. [18] Let ζ_0 be a fixed point in $[a, b] \cap \mathbb{Z}$. Then Problem (1), (2) has a unique solution $(z(\zeta))_{\zeta \in [a, b] \cap \mathbb{Z}}$, such that

$$z(\zeta_0) = k_1, \quad \Delta z(\zeta_0) = k_2,$$

where $k_1, k_2 \in \mathbb{C}$.

Theorem 2. [18] Let Ξ and Θ be solutions of the equation $\Gamma z = \lambda z$. Then the Wronskian defined as

$$W[\Xi, \Theta](\zeta) := p(\zeta - 1) [\Xi(\zeta) \Delta \Theta(\zeta - 1) - \Theta(\zeta) \Delta \Xi(\zeta - 1)]$$

is constant on each of the sets $[a, \eta - 1] \cap \mathbb{Z}$ and $[\eta + 2, b] \cap \mathbb{Z}$:

$$W[\Xi, \Theta](\zeta) = \begin{cases} W^-, & \zeta \in [a, \eta - 1] \cap \mathbb{Z}, \\ W^+, & \zeta \in [\eta + 2, b] \cap \mathbb{Z}. \end{cases}$$

Furthermore,

$$W^- = W^+,$$

and

$$W[\Xi, \Theta](\eta) = -\frac{1}{d} W^+.$$

Definition 1. Problem (1)–(2) is singular when the interval $[a, b]$ is infinite length.

Denote by $l_w^2(J)$ the Hilbert space with the inner product

$$(f, g) := \sum_{\zeta=a}^{\eta-1} f(\zeta) \overline{g(\zeta)} w(\zeta) + \sum_{\zeta=\eta+2}^b f(\zeta) \overline{g(\zeta)} w(\zeta),$$

where $f, g \in l_w^2(J)$.

For any Ξ and Θ in $l_w^2(J)$, we have

$$\begin{aligned} & (\Gamma(\Xi), \Theta) - (\Xi, \Gamma(\Theta)) \\ &= p(a-1) [\Xi(a-1) \overline{\Theta(a)} - \Xi(a) \overline{\Theta(a-1)}] \\ & \quad - p(\eta-1) [\Xi(\eta-1) \overline{\Theta(\eta)} - \Xi(\eta) \overline{\Theta(\eta-1)}] \\ & \quad + p(\eta+1) [\Xi(\eta+1) \overline{\Theta(\eta+2)} - \Xi(\eta+2) \overline{\Theta(\eta+1)}] \\ & \quad - p(b) [\Xi(b) \overline{\Theta(b+1)} - \Xi(b+1) \overline{\Theta(b)}] \\ &= [\Xi, \Theta](a) - [\Xi, \Theta](\eta) + [\Xi, \Theta](\eta+2) - [\Xi, \Theta](b+1), \end{aligned}$$

where

$$[\Xi, \Theta](\zeta) := W[\Xi, \overline{\Theta}](\zeta) = p(\zeta-1) [\Xi(\zeta-1) \overline{\Theta(\zeta)} - \Xi(\zeta) \overline{\Theta(\zeta-1)}].$$

2 The nested circles at b

Let us consider $J_1 = \{[c, b'] \cap \mathbb{Z}\} \setminus \{\eta, \eta + 1\}$, where $a < c < \eta - 1 < \eta < \eta + 1 < b' < b$ and let χ_1 and χ_2 be solutions of Problem (1)–(2) satisfying

$$\chi_1(c) = 0, \quad \chi_1(c-1) = -\frac{1}{p(c-1)},$$

$$\chi_2(c) = 1, \quad \chi_2(c-1) = 0.$$

It is clear that

$$\begin{aligned} & W[\chi_1, \chi_2](n) \\ &= -\frac{1}{d} W^-[\chi_1, \chi_2](c) p(c-1) [\chi_1(c-1) \chi_2(c) - \chi_1(c) \chi_2(c-1)] = \frac{1}{d}, \quad n \in J_1. \end{aligned}$$

Now consider the following regular problem:

$$\Gamma(z) = \lambda z(\zeta), \quad \zeta \in J_1, \quad (3)$$

$$z(c-1) = 0, \quad (4)$$

$$z(b'+1) + kz(b') = 0, \quad k \in \mathbb{R}, \quad (5)$$

$$\begin{aligned} p(\eta+1)z(\eta+1) - dp(\eta-1)z(\eta-1) &= 0, \\ z(\eta+2) - \frac{1}{d}z(\eta) &= 0. \end{aligned} \quad (6)$$

For $\text{Im } \lambda \neq 0$, let $\varphi_{b'} = \chi_1 + m_{b'}\chi_2$ be the solution of Problem (3)–(6). The function $m_{b'}$ is called the Titchmarsh–Weyl function of Problem (3)–(6). Here $\varphi_{b'}$ and χ_2 must be linearly independent.

Proposition 1. Let Ξ be a solution of $\Gamma u = \lambda u$ and Θ is a solution of $\Gamma u = \mu u$, both satisfying (4) and (6). Then, for $\zeta \in J_1$,

$$\begin{aligned} & (\lambda - \mu) \left(\sum_{\gamma=a}^{\eta-1} \Xi(\gamma) \Theta(\gamma) w(\gamma) + \sum_{\gamma=\eta+2}^{\zeta} \Xi(\gamma) \Theta(\gamma) w(\gamma) \right) \\ &= p(\zeta) [\Xi(\zeta+1) \Theta(\zeta) - \Xi(\zeta) \Theta(\zeta+1)]. \end{aligned}$$

Proof. From (3), we obtain

$$\lambda w(\gamma) \Xi(\gamma) - q(\gamma) \Xi(\gamma) = \nabla [p(\gamma) \Delta \Xi(\gamma)]$$

and

$$\mu w(\gamma) \Theta(\gamma) - q(\gamma) \Theta(\gamma) = \nabla [p(\gamma) \Delta \Theta(\gamma)].$$

Multiplying these equations, respectively, by $\Theta(\gamma)$, $-\Xi(\gamma)$, adding gives

$$(\lambda - \mu) \Xi(\gamma) \Theta(\gamma) = \nabla [p(\gamma) \Delta \Xi(\gamma)] \Theta(\gamma) - \nabla [p(\gamma) \Delta \Theta(\gamma)] \Xi(\gamma),$$

i.e.,

$$\begin{aligned} & (\lambda - \mu) \Xi(\gamma) \Theta(\gamma) = p(\gamma) [\Xi(\gamma+1) \Theta(\gamma) - \Xi(\gamma) \Theta(\gamma+1)] \\ & - p(\gamma-1) [\Xi(\gamma) \Theta(\gamma-1) - \Xi(\gamma-1) \Theta(\gamma)]. \end{aligned} \quad (7)$$

Thus, we find

$$\begin{aligned}
 & (\lambda - \mu) \left(\sum_{\gamma=a}^{\eta-1} \Xi(\gamma) \Theta(\gamma) w(\gamma) + \sum_{\gamma=\eta+2}^{\zeta} \Xi(\gamma) \Theta(\gamma) w(\gamma) \right) \\
 &= \sum_{\gamma=a}^{\eta-1} p(\gamma) [\Xi(\gamma+1) \Theta(\gamma) - \Xi(\gamma) \Theta(\gamma+1)] \\
 &\quad - \sum_{\gamma=a}^{\eta-1} p(\gamma-1) [\Xi(\gamma) \Theta(\gamma-1) - \Xi(\gamma-1) \Theta(\gamma)] \\
 &\quad + \sum_{\gamma=\eta+2}^{\zeta} p(\gamma) [\Xi(\gamma+1) \Theta(\gamma) - \Xi(\gamma) \Theta(\gamma+1)] \\
 &\quad - \sum_{\gamma=\eta+2}^{\zeta} p(\gamma-1) [\Xi(\gamma) \Theta(\gamma-1) - \Xi(\gamma-1) \Theta(\gamma)].
 \end{aligned}$$

By virtue of (4) and (6), we see that

$$\begin{aligned}
 & (\lambda - \mu) \left(\sum_{\gamma=a}^{\eta-1} \Xi(\gamma) \Theta(\gamma) w(\gamma) + \sum_{\gamma=\eta+2}^{\zeta} \Xi(\gamma) \Theta(\gamma) w(\gamma) \right) \\
 &= p(\zeta) [\Xi(\zeta+1) \Theta(\zeta) - \Xi(\zeta) \Theta(\zeta+1)].
 \end{aligned}$$

□

Theorem 3. Let $\text{Im } \lambda \neq 0$. $\varphi_{b'}$ satisfies conditions (4)–(6) if and only if $m_{b'}$ lies on a circle $C_{b'}$ in the complex plane. The equation of $C_{b'}$ is

$$W[\varphi_{b'}, \overline{\varphi_{b'}}] (b' + 1) = 0, \quad (8)$$

with radius

$$r_{b'} = |W[\chi_2, \overline{\chi_2}] (b' + 1)|^{-1},$$

and center

$$\widetilde{m}_{b'} = -\frac{W[\chi_1, \overline{\chi_2}] (b' + 1)}{W[\chi_2, \overline{\chi_2}] (b' + 1)}.$$

Proof. From (5), we obtain

$$k = -\frac{\varphi_{b'}(b' + 1)}{\varphi_{b'}(b')}.$$

Since $k \in \mathbb{R}$, we have

$$\begin{aligned}
 0 = k - \overline{k} &= -\frac{\varphi_{b'}(b' + 1)}{\varphi_{b'}(b')} + \frac{\overline{\varphi_{b'}(b' + 1)}}{\overline{\varphi_{b'}(b')}} \\
 &= \frac{\varphi_{b'}(b') \overline{\varphi_{b'}(b' + 1)} - \varphi_{b'}(b' + 1) \overline{\varphi_{b'}(b')}}{|\varphi_{b'}(b')|^2},
 \end{aligned}$$

i.e.,

$$\varphi_{b'}(b') \overline{\varphi_{b'}(b' + 1)} - \varphi_{b'}(b' + 1) \overline{\varphi_{b'}(b')} = 0.$$

Hence,

$$W[\varphi_{b'}, \overline{\varphi_{b'}}] (b' + 1) = 0.$$

$\varphi_{b'} (b' + 1) + k\varphi_{b'} (b') = 0$ is equivalent to

$$m_{b'} = -\frac{\chi_1 (b' + 1) + k\chi_1 (b')}{\chi_2 (b' + 1) + k\chi_2 (b')},$$

which describes a circle in the complex plane. By (8), we find

$$\begin{aligned} |m_{b'}|^2 W[\chi_2, \overline{\chi_2}] (b' + 1) + m_{b'} W[\chi_2, \overline{\chi_1}] (b' + 1) \\ \overline{m_{b'}} W[\chi_1, \overline{\chi_2}] (b' + 1) + W[\chi_1, \overline{\chi_1}] (b' + 1) = 0. \end{aligned} \quad (9)$$

Let

$$m_{b'} = u + iv, \quad W[\chi_2, \overline{\chi_2}] (b' + 1) = 2iA,$$

and

$$-W[\chi_2, \overline{\chi_1}] (b' + 1) = B + iC.$$

Then we obtain

$$W[\chi_1, \overline{\chi_2}] (b' + 1) = B - iC,$$

and

$$W[\chi_1, \overline{\chi_1}] (b' + 1) = 2iD.$$

Thus, by (9), we conclude that

$$(2iA) (u^2 + v^2) - (u + iv) (B + iC) + (u - iv) (B - iC) + 2iD = 0.$$

Hence,

$$\left(u - \frac{C}{2A}\right)^2 + \left(v - \frac{B}{2A}\right)^2 = \frac{B^2 + C^2 - 4AD}{4A^2},$$

which implies

$$\widetilde{m_{b'}} = \frac{C + iB}{2A} = -\frac{B - iC}{2iA} = -\frac{W[\chi_1, \overline{\chi_2}] (b' + 1)}{W[\chi_2, \overline{\chi_2}] (b' + 1)},$$

and

$$r_{b'} = \sqrt{\frac{B^2 + C^2 - 4AD}{4A^2}} = |W[\chi_2, \overline{\chi_2}] (b' + 1)|^{-1}.$$

□

Theorem 4. $m_{b'}$ is inside or on $C_{b'}$ if and only if

$$\sum_{\zeta=c}^{\eta-1} |\varphi_{b'} (\zeta)|^2 w (\zeta) + \sum_{\zeta=\eta+2}^{b'} |\varphi_{b'} (\zeta)|^2 w (\zeta) \leq -\frac{\operatorname{Im} m_{b'}}{\operatorname{Im} \lambda}.$$

Furthermore, we have

$$\widetilde{m_{b'}} = \frac{1 - 2i \operatorname{Im} \lambda \left[\sum_{\zeta=c}^{\eta-1} \chi_1 (\zeta) \overline{\chi_2} (\zeta) w (\zeta) + \sum_{\zeta=\eta+2}^{b'} \chi_1 (\zeta) \overline{\chi_2} (\zeta) w (\zeta) \right]}{2i \operatorname{Im} \lambda \left[\sum_{\zeta=c}^{\eta-1} |\chi_2 (\zeta)|^2 w (\zeta) + \sum_{\zeta=\eta+2}^{b'} |\chi_2 (\zeta)|^2 w (\zeta) \right]}, \quad (10)$$

and

$$r_{b'} = \left| 2i \operatorname{Im} \lambda \left[\sum_{\zeta=c}^{\eta-1} |\chi_2(\zeta)|^2 w(\zeta) + \sum_{\zeta=\eta+2}^{b'} |\chi_2(\zeta)|^2 w(\zeta) \right] \right|^{-1}. \quad (11)$$

Proof. From (8) and (9), we see that $m_{b'}$ is inside or on $C_{b'}$ if and only if

$$\frac{W[\varphi_{b'}, \overline{\varphi_{b'}}](b'+1)}{W[\chi_2, \overline{\chi_2}](b'+1)} \leq 0. \quad (12)$$

Let Ξ and Θ be solutions of (3), (6). From (7), we find

$$\begin{aligned} & W[\Xi, \overline{\Theta}](b'+1) - W[\Xi, \overline{\Theta}](c) \\ &= 2i \operatorname{Im} \lambda \left[\sum_{\zeta=c}^{\eta-1} \Xi(\zeta) \overline{\Theta}(\zeta) w(\zeta) + \sum_{\zeta=\eta+2}^{b'} \Xi(\zeta) \overline{\Theta}(\zeta) w(\zeta) \right]. \end{aligned}$$

Thus, we obtain

$$\begin{aligned} & W[\chi_2, \overline{\chi_2}](b'+1) \\ &= 2i \operatorname{Im} \lambda \left[\sum_{\zeta=c}^{\eta-1} |\chi_2(\zeta)|^2 w(\zeta) + \sum_{\zeta=\eta+2}^{b'} |\chi_2(\zeta)|^2 w(\zeta) \right] = 2iA, \end{aligned}$$

and

$$\begin{aligned} & W[\chi_1, \overline{\chi_2}](b'+1) \\ &= -1 + 2i \operatorname{Im} \lambda \left[\sum_{\zeta=c}^{\eta-1} \chi_1(\zeta) \overline{\chi_2}(\zeta) w(\zeta) + \sum_{\zeta=\eta+2}^{b'} \chi_1(\zeta) \overline{\chi_2}(\zeta) w(\zeta) \right] = B - iC. \end{aligned}$$

Therefore, we have

$$\begin{aligned} & W[\chi_2, \overline{\chi_1}](b'+1) \\ &= 1 + 2i \operatorname{Im} \lambda \left[\sum_{\zeta=c}^{\eta-1} \chi_2(\zeta) \overline{\chi_1}(\zeta) w(\zeta) + \sum_{\zeta=\eta+2}^{b'} \chi_2(\zeta) \overline{\chi_1}(\zeta) w(\zeta) \right] = -(B + iC), \end{aligned}$$

and

$$\begin{aligned} & W[\chi_1, \overline{\chi_1}](b'+1) \\ &= 2i \operatorname{Im} \lambda \left[\sum_{\zeta=c}^{\eta-1} |\chi_1(\zeta)|^2 w(\zeta) + \sum_{\zeta=\eta+2}^{b'} |\chi_1(\zeta)|^2 w(\zeta) \right] = 2iD. \end{aligned}$$

It follows from (9) that

$$\begin{aligned} & W[\varphi_{b'}, \overline{\varphi_{b'}}](b'+1) \\ &= m_{b'} - \overline{m_{b'}} + 2i \operatorname{Im} \lambda \left[\sum_{\zeta=c}^{\eta-1} |\varphi_{b'}(\zeta)|^2 w(\zeta) + \sum_{\zeta=\eta+2}^{b'} |\varphi_{b'}(\zeta)|^2 w(\zeta) \right]. \end{aligned}$$

By virtue of (12), we conclude that $m_{b'}$ is inside or on $C_{b'}$ if and only if

$$\frac{m_{b'} - \overline{m_{b'}} + 2i \operatorname{Im} \lambda \left[\sum_{\zeta=c}^{\eta-1} |\varphi_{b'}(\zeta)|^2 w(\zeta) + \sum_{\zeta=\eta+2}^{b'} |\varphi_{b'}(\zeta)|^2 w(\zeta) \right]}{2i \operatorname{Im} \lambda \left[\sum_{\zeta=c}^{\eta-1} |\chi_2(\zeta)|^2 w(\zeta) + \sum_{\zeta=\eta+2}^{b'} |\chi_2(\zeta)|^2 w(\zeta) \right]} \leq 0.$$

From the previous calculations, we obtain (10) and (11). □

Theorem 5. The $C_{b'}$ circles are nested.

Proof. Let $b'' < b' < b$. Then we obtain

$$\begin{aligned} & \sum_{\zeta=c}^{\eta-1} |\varphi_{b'}(\zeta)|^2 w(\zeta) + \sum_{\zeta=\eta+2}^{b''} |\varphi_{b'}(\zeta)|^2 w(\zeta) \\ & \leq \sum_{\zeta=c}^{\eta-1} |\varphi_{b'}(\zeta)|^2 w(\zeta) + \sum_{\zeta=\eta+2}^{b'} |\varphi_{b'}(\zeta)|^2 w(\zeta) \leq -\frac{\operatorname{Im} m_{b'}}{\operatorname{Im} \lambda}. \end{aligned}$$

From Theorem 4, we see that $m_{b'}$ is inside or on $C_{b''}$. Hence the $C_{b'}$ circles are nested. □

Corollary 1. As $b' \rightarrow b = +\infty$, the $C_{b'}$ circles contract to a limit-circle or a limit-point. Let $m_{b'} \rightarrow m_b$ and let $\varphi_b = \chi_1 + m_b \chi_2$. Then we obtain

$$\sum_{\zeta=c}^{\eta-1} |\chi_1(\zeta) + m_b \chi_2(\zeta)|^2 w(\zeta) + \sum_{\zeta=\eta+2}^b |\chi_1(\zeta) + m_b \chi_2(\zeta)|^2 w(\zeta) \leq -\frac{\operatorname{Im} m_b}{\operatorname{Im} \lambda},$$

i.e., $\varphi_b \in l^2(J_3)$, where $J_3 = \{[c, b] \cap \mathbb{Z}\} \setminus \{\eta, \eta+1\}$. Moreover, $m_{b'}$ approaches m_b independently of k . If we take $k = 0$, then we have

$$m_b = -\lim_{b' \rightarrow b} \frac{\chi_1(b'+1)}{\chi_2(b'+1)}.$$

i) The limit-circle (LC) case: Let the $C_{b'}$ circles converge to a circle. By (11), we get

$$\sum_{\zeta=c}^{\eta-1} |\chi_2(\zeta)|^2 w(\zeta) + \sum_{\zeta=\eta+2}^b |\chi_2(\zeta)|^2 w(\zeta) < \infty,$$

i.e., all solutions of (3) in $l^2(J_3)$.

ii) The limit-point (LP) case: Let the $C_{b'}$ circles converge to a point. From (11), we find

$$\sum_{\zeta=c}^{\eta-1} |\chi_2(\zeta)|^2 w(\zeta) + \sum_{\zeta=\eta+2}^b |\chi_2(\zeta)|^2 w(\zeta) = \infty,$$

i.e., φ_b is the only linearly independent solution in $l^2(J_3)$.

Theorem 6. For some value λ , the limit-circle case holds at $b = +\infty$. Then the same is true for all $\mu \in \mathbb{C}$.

Proof. The proof is similar to [12, Theorem 3.2.7]. □

3 The nested circles at a

Let us consider the set $J_4 = \{[a', c-1] \cap \mathbb{Z}\} \setminus \{\eta, \eta+1\}$, where $a < a' < \eta-1 < \eta < \eta+1 < c-1$, $c < b$. Then the problem of finding the solution of

$$\Gamma(z) = \lambda z \quad (13)$$

satisfying the following conditions:

$$z(a'-1) + hz(a') = 0, \quad (14)$$

$$z(c) = 0, \quad (15)$$

$$\begin{aligned} p(\eta+1)z(\eta+1) - dp(\eta-1)z(\eta-1) &= 0, \\ z(\eta+2) - \frac{1}{d}z(\eta) &= 0, \end{aligned} \quad (16)$$

is a regular self-adjoint problem on J_4 . For $\text{Im } \lambda \neq 0$, let $\varphi_{a'} = \chi_1 + m_{a'}\chi_2$ be the solution of Problem (13)–(16). Here $\varphi_{a'}$ and χ_2 must be linearly independent.

Theorem 7. Let $\text{Im } \lambda \neq 0$. $\varphi_{a'}$ satisfies conditions (13)–(16) if and only if $m_{a'}$ lies on a circle $C_{a'}$ in the complex plane. The equation of $C_{a'}$ is

$$W[\varphi_{a'}, \overline{\varphi_{a'}}](a') = 0,$$

with radius

$$r_{a'} = |W[\chi_2, \overline{\chi_2}](a')|^{-1},$$

and center

$$\widetilde{m}_{a'} = -\frac{W[\chi_1, \overline{\chi_2}](a')}{W[\chi_2, \overline{\chi_2}](a')}.$$

Proof. The proof is similar to Theorem 3. □

Theorem 8. $m_{a'}$ is inside or on $C_{a'}$ if and only if

$$\sum_{\zeta=a'}^{\eta-1} |\varphi_{a'}(\zeta)|^2 w(\zeta) + \sum_{\zeta=\eta+2}^{c-1} |\varphi_{a'}(\zeta)|^2 w(\zeta) \leq -\frac{\text{Im } m_{a'}}{\text{Im } \lambda}.$$

Proof. $m_{a'}$ is inside or on $C_{a'}$ if and only if

$$\frac{W[\varphi_{a'}, \overline{\varphi_{a'}}](a')}{W[\varphi_{a'}, \overline{\varphi_{a'}}](a')} \leq 0. \quad (17)$$

Let Ξ and Θ be solutions of (13), (16). From (7), we obtain

$$\begin{aligned} & W[\Xi, \overline{\Theta}](a') \\ &= W[\Xi, \overline{\Theta}](c) - 2i \text{Im } \lambda \left[\sum_{\zeta=a'}^{\eta-1} \Xi(\zeta) \overline{\Theta}(\zeta) w(\zeta) + \sum_{\zeta=\eta+2}^{c-1} \Xi(\zeta) \overline{\Theta}(\zeta) w(\zeta) \right]. \end{aligned}$$

Thus, we get

$$W[\chi_2, \overline{\chi_2}](a') = -2i \text{Im } \lambda \left[\sum_{\zeta=a'}^{\eta-1} |\chi_2(\zeta)|^2 w(\zeta) + \sum_{\zeta=\eta+2}^{c-1} |\chi_2(\zeta)|^2 w(\zeta) \right],$$

$$\begin{aligned}
& W[\chi_1, \bar{\chi}_2](a') \\
&= -1 - 2i \operatorname{Im} \lambda \left[\sum_{\zeta=a'}^{\eta-1} \chi_1(\zeta) \bar{\chi}_2(\zeta) w(\zeta) + \sum_{\zeta=\eta+2}^{c-1} \chi_1(\zeta) \bar{\chi}_2(\zeta) w(\zeta) \right], \\
& W[\chi_2, \bar{\chi}_1](a') \\
&= -1 - 2i \operatorname{Im} \lambda \left[\sum_{\zeta=a'}^{\eta-1} \bar{\chi}_1(\zeta) \chi_2(\zeta) w(\zeta) + \sum_{\zeta=\eta+2}^{c-1} \bar{\chi}_1(\zeta) \chi_2(\zeta) w(\zeta) \right],
\end{aligned}$$

and

$$W[\chi_1, \bar{\chi}_1](a') = -2i \operatorname{Im} \lambda \left[\sum_{\zeta=a'}^{\eta-1} |\chi_1(\zeta)|^2 w(\zeta) + \sum_{\zeta=\eta+2}^{c-1} |\chi_1(\zeta)|^2 w(\zeta) \right].$$

Hence

$$\begin{aligned}
& W[\varphi_{a'}, \overline{\varphi_{a'}}](a') = m_{a'} - \overline{m_{a'}} \\
& -2i \operatorname{Im} \lambda \sum_{\zeta=a'}^{\eta-1} \left[|m_{a'}|^2 |\chi_2(\zeta)|^2 + m_{a'} \bar{\chi}_1(\zeta) \chi_2(\zeta) \right. \\
& \quad \left. + \overline{m_{a'}} \chi_1(\zeta) \bar{\chi}_2(\zeta) + |\chi_1(\zeta)|^2 \right] w(\zeta) \\
& -2i \operatorname{Im} \lambda \sum_{\zeta=\eta+2}^{c-1} \left[|m_{a'}|^2 |\chi_2(\zeta)|^2 + m_{a'} \bar{\chi}_1(\zeta) \chi_2(\zeta) \right. \\
& \quad \left. + \overline{m_{a'}} \chi_1(\zeta) \bar{\chi}_2(\zeta) + |\chi_1(\zeta)|^2 \right] w(\zeta) \\
&= 2i \operatorname{Im}(m_{a'}) - 2i \operatorname{Im} \lambda \sum_{\zeta=a'}^{\eta-1} |\varphi_{a'}(\zeta)|^2 w(\zeta) + \sum_{\zeta=\eta+2}^{c-1} |\varphi_{a'}(\zeta)|^2 w(\zeta).
\end{aligned}$$

By (17), we conclude that $m_{a'}$ is inside or on $C_{a'}$ if and only if

$$\frac{\operatorname{Im}(m_{a'}) - \operatorname{Im} \lambda \sum_{\zeta=a'}^{\eta-1} |\varphi_{a'}(\zeta)|^2 w(\zeta) + \sum_{\zeta=\eta+2}^{c-1} |\varphi_{a'}(\zeta)|^2 w(\zeta)}{-\operatorname{Im} \lambda \left[\sum_{\zeta=a'}^{\eta-1} |\chi_2(\zeta)|^2 w(\zeta) + \sum_{\zeta=\eta+2}^{c-1} |\chi_2(\zeta)|^2 w(\zeta) \right]} \leq 0.$$

□

From the previous calculations, we get

$$\widetilde{m_{a'}} = \frac{1 + 2i \operatorname{Im} \lambda \left[\sum_{\zeta=a'}^{\eta-1} \chi_1(\zeta) \bar{\chi}_2(\zeta) w(\zeta) + \sum_{\zeta=\eta+2}^{c-1} \chi_1(\zeta) \bar{\chi}_2(\zeta) w(\zeta) \right]}{-2i \operatorname{Im} \lambda \left[\sum_{\zeta=a'}^{\eta-1} |\chi_2(\zeta)|^2 w(\zeta) + \sum_{\zeta=\eta+2}^{c-1} |\chi_2(\zeta)|^2 w(\zeta) \right]}$$

and

$$r_{a'} = \left| 2i \operatorname{Im} \lambda \left[\sum_{\zeta=a'}^{\eta-1} |\chi_2(\zeta)|^2 w(\zeta) + \sum_{\zeta=\eta+2}^{c-1} |\chi_2(\zeta)|^2 w(\zeta) \right] \right|^{-1}.$$

Theorem 9. The $C_{a'}$ circles are nested.

Proof. The proof is similar to Theorem 5. □

Corollary 2. As $a' \rightarrow a = -\infty$, the $C_{a'}$ circles contract to a limit-circle or a limit-point. Let $m_{a'} \rightarrow m_a$ and let $\varphi_a = \chi_1 + m_a \chi_2$. Then we obtain

$$\sum_{\zeta=a'}^{\eta-1} |\chi_1(\zeta) + m_a \chi_2(\zeta)|^2 w(\zeta) + \sum_{\zeta=\eta+2}^{c-1} |\chi_1(\zeta) + m_a \chi_2(\zeta)|^2 w(\zeta) \leq \frac{\operatorname{Im} m_a}{\operatorname{Im} \lambda},$$

i.e., $\varphi_a \in l^2(J_5)$, where $J_5 = \{[a, c-1] \cap \mathbb{Z}\} \setminus \{\eta, \eta+1\}$. Moreover, $m_{a'}$ approaches m_a independently of k . If we take $k=0$, then we have

$$m_a = - \lim_{a' \rightarrow a} \frac{\chi_1(a'-1)}{\chi_2(a'-1)}.$$

i) The limit-circle (LC) case: Let the $C_{a'}$ circles converge to a circle. Then all solutions of (13) in $l^2(J_5)$.

ii) The limit-point (LP) case: Let the $C_{a'}$ circles converge to a point. Then φ_a is the only linearly independent solution in $l^2(J_5)$.

Theorem 10. For some value λ , the limit-circle case holds at a . Then the same is true for all $\mu \in \mathbb{C}$.

4 Existence of boundary conditions

Let λ_0 be a complex number (possibly real in LC cases, but with nonzero imaginary part in LP cases) and χ_2 be a solution of

$$\Gamma(z) = \lambda_0 z. \tag{18}$$

Let φ_b be a solution of Eq. (18) in $l^2(J_3)$ such that $W[\varphi_b, \chi_2] = -1$.

Theorem 11. Let $\Gamma(z) = f$, where $z, f \in l^2(J_3)$. Then

i)

$$\lim_{b' \rightarrow b} W[z, \chi_2] (b' + 1)$$

exists when $b = +\infty$ is LC case, and

ii)

$$\lim_{b' \rightarrow b} W[z, \varphi_b] (b' + 1)$$

exists in all limit cases.

Proof. i) By assumption, we obtain

$$\nabla [p(\zeta) \Delta z(\zeta)] + q(\zeta) z(\zeta) = w(\zeta) f(\zeta)$$

and

$$\nabla [p(\zeta) \Delta \chi_2(\zeta)] + q(\zeta) \chi_2(\zeta) = \lambda_0 w(\zeta) \chi_2(\zeta).$$

Multiplying these equations, respectively, by $\chi_2(\zeta)$, $-z(\zeta)$, adding gives

$$\begin{aligned} & \nabla [p(\zeta) \Delta z(\zeta)] \chi_2(\zeta) - \nabla [p(\zeta) \Delta \chi_2(\zeta)] z(\zeta) \\ &= w(\zeta) f(\zeta) \chi_2(\zeta) - \lambda_0 w(\zeta) \chi_2(\zeta) z(\zeta), \end{aligned}$$

i.e.,

$$W[z, \chi_2](\zeta) - W[z, \chi_2](\zeta + 1)$$

$$= w(\zeta) f(\zeta) \chi_2(\zeta) - \lambda_0 w(\zeta) \chi_2(\zeta) z(\zeta). \quad (19)$$

From (19), we find

$$\begin{aligned} W[z, \chi_2](c) - W[z, \chi_2](\eta) + W[z, \chi_2](\eta + 2) - W[z, \chi_2](b' + 1) \\ = \sum_{\zeta=c}^{\eta-1} [w(\zeta) f(\zeta) \chi_2(\zeta) - \lambda_0 w(\zeta) \chi_2(\zeta) z(\zeta)] \\ + \sum_{\zeta=\eta+2}^{b'} [w(\zeta) f(\zeta) \chi_2(\zeta) - \lambda_0 w(\zeta) \chi_2(\zeta) z(\zeta)]. \end{aligned}$$

Hence,

$$\begin{aligned} [z, \chi_2](b' + 1) &= W[z, \chi_2](c) - W[z, \chi_2](\eta) + W[z, \chi_2](\eta + 2) \\ &\quad - \left[\sum_{\zeta=c}^{\eta-1} w(\zeta) f(\zeta) \chi_2(\zeta) + \sum_{\zeta=\eta+2}^{b'} w(\zeta) f(\zeta) \chi_2(\zeta) \right] \\ &\quad + \lambda_0 \left[\sum_{\zeta=c}^{\eta-1} w(\zeta) \chi_2(\zeta) z(\zeta) + \sum_{\zeta=\eta+2}^{b'} [w(\zeta) \chi_2(\zeta) z(\zeta)] \right]. \end{aligned}$$

In the LC case at $b = +\infty$, $\chi_2 \in l^2(J_3)$. Consequently, $\lim_{b' \rightarrow b} W[z, \chi_2](b' + 1)$ exists.

ii) Similarly, we have

$$\nabla[p(\zeta) \Delta z(\zeta)] + q(\zeta) z(\zeta) = w(\zeta) f(\zeta)$$

and

$$\nabla[p(\zeta) \Delta \varphi_b(\zeta)] + q(\zeta) \varphi_b(\zeta) = \lambda_0 w(\zeta) \varphi_b(\zeta).$$

Then we find

$$\begin{aligned} [z, \varphi_b](b' + 1) &= W[z, \varphi_b](c) - W[z, \varphi_b](\eta) + W[z, \varphi_b](\eta + 2) \\ &\quad - \left[\sum_{\zeta=c}^{\eta-1} w(\zeta) f(\zeta) \varphi_b(\zeta) + \sum_{\zeta=\eta+2}^{b'} w(\zeta) f(\zeta) \varphi_b(\zeta) \right] \\ &\quad + \lambda_0 \left[\sum_{\zeta=c}^{\eta-1} w(\zeta) \varphi_b(\zeta) z(\zeta) + \sum_{\zeta=\eta+2}^{b'} [w(\zeta) \varphi_b(\zeta) z(\zeta)] \right]. \end{aligned}$$

Since $\varphi_b \in l^2(J_3)$, we see that $\lim_{b' \rightarrow b} W[z, \varphi_b](b' + 1)$ exists in all limit cases. $\square$

Similarly, let φ_a be a solution of Eq. (18) in $l^2(J_5)$ such that $W[\varphi_a, \chi_2] = -1$.

Theorem 12. Let $\Gamma(z) = f$, where $z, f \in l^2(J_5)$. Then

i)

$$\lim_{a' \rightarrow a} W[z, \chi_2](a')$$

exists when $a = -\infty$ is LC case, and

ii)

$$\lim_{a' \rightarrow a} W[z, \varphi_a](a')$$

exists in all limit cases.

Proof. The proof is similar to Theorem 11. $\square$

Conclusion

In this paper, we have successfully investigated the spectral and geometric properties of singular discrete Sturm–Liouville equations subject to interior transmission conditions. The core achievement of this research is the rigorous establishment of Weyl’s alternative classification for this class of discontinuous difference equations. By classifying the underlying configurations into limit-circle and limit-point cases based on the behavior of their solutions, this work extends classical continuous operator theory into the discrete and discontinuous domains.

Furthermore, we have constructed a robust Hilbert space framework that accommodates the discrete dynamics alongside the jump conditions at the interior singular point. Through this structure, the analytical properties of the corresponding Weyl function were thoroughly explored, and the geometric behavior of the associated Weyl circles was systematically derived. These results provide a deeper understanding of how interior transmission conditions influence the overall spectral landscape of discrete systems.

The generalized framework and methodologies developed in this study open new avenues for further research. Future work may focus on extending these results to discrete non-selfadjoint operators. Ultimately, the insights gained herein contribute significantly to the ongoing advancement of the spectral analysis of discontinuous difference operators.

Author contributions

All authors contributed equally to this work.

Conflict of interest

The authors declare no conflict of interest.

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On limited and almost limited operators between Banach lattices

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This article investigates theoretical questions concerning the class of (almost) limited operators on Banach lattices, focusing on their precise relations to L -weakly compact, semi-compact, and Dunford–Pettis operators. The authors establish several conditions under which bounded operators are AM-compact and limited, and characterize when regular operators are almost limited. Furthermore, several new notions of collectively qualified sets of operators are introduced. Using results obtained for individual operators, the authors derive new structural results concerning these collectively qualified sets. The lattice approximation property for collectively M -weakly compact families of operators is also thoroughly investigated. Finally, the paper examines some related topics in functional analysis, focusing on the specific behavior of various kinds of operators acting between Banach lattices and Banach spaces. The presented theorems extend existing results, providing deeper insights into the topological and order properties of bounded operators, and establishing new connections between collective compactness and lattice structures.

Keywords: Banach lattice, Banach space, Schur property, limited set, L -weakly compact set, semi-compact operator, Dunford–Pettis operator, collectively qualified set of operators, enveloping norm.

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Introduction

Limited operators and their relations to $L(M)$ -weakly compact, semi-compact, Dunford–Pettis, and other classes of operators attract permanent attention of many researchers (see, for example, [1–3] and references therein). Collectively qualified families of operators were studied recently in [4–6]. The aim of this work is to find conditions to ensure inclusion relation among major classes of operators that are in use. In trying to do so, we also touch upon properties of Banach lattices and note a method of producing new classes of operators in them. Finally, the collective versions of the results obtained for a single operator are investigated.

Throughout the paper, all vector spaces are real, operators are linear; letters X , Y , and Z denote Banach spaces; E , F , and G denote Banach lattices. A subset C of X is called *bounded* if C is norm bounded. We denote by B_X the closed unit ball of X , by $L(X, Y)$ ($K(X, Y)$, $W(X, Y)$) the space of bounded (compact, weakly compact) operators from X to Y , by Id_X the identity operator on X , by E_+ the positive cone of E , by $\text{sol}(A) = \bigcup_{a \in A} [-|a|, |a|]$ the solid hull of $A \subseteq E$, by $x_\alpha \downarrow 0$ a decreasing net (x_α) with $\inf_\alpha x_\alpha = 0$, and by E^a the order continuous part of E .

The following well-known lemma will be permanently used below.

Lemma 1. ([7, Theorem 4.59] and [8, 2.4.14]) The following assertions are equivalent:

1. The norm on E^* is order continuous.
2. E^* is a KB-space.
3. Every disjoint bounded sequence in E is weakly null.

Definition 1. A bounded subset C

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- 1) of X is *limited* if each w^* -null sequence in X^* is uniformly null on C ;
- 2) of X is *Dunford-Pettis* or a *DP set* if each weakly null sequence in X^* is uniformly null on C ;
- 3) of E is *L-weakly compact* or an *Lwc set* if each disjoint sequence in $\text{sol}(C)$ is norm null;
- 4) of E is *almost order bounded* if, for each $\varepsilon > 0$ there exist $u_\varepsilon \in E_+$ satisfying $C \subseteq [-u_\varepsilon, u_\varepsilon] + \varepsilon B_X$.

It is well known that B_X is not limited unless $\dim(X) < \infty$ (however, B_{c_0} is a limited subset of ℓ^∞ by Phillips' Lemma [7, Theorem 4.67]). For every Lwc subset A of E , we have $A \subseteq E^a$ [8, p. 212]. The following technical lemma is an extension of the Grothendieck theorem.

Lemma 2. Let C be a bounded subset of X (or E). If, for each $\varepsilon > 0$, there exists

- 1) a relatively compact subset $C_\varepsilon \subseteq X$, $C \subseteq C_\varepsilon + \varepsilon B_X$, then C is relatively compact;
- 2) a relatively weakly compact $C_\varepsilon \subseteq X$, $C \subseteq C_\varepsilon + \varepsilon B_X$, then C is relatively weakly compact;
- 3) a limited $C_\varepsilon \subseteq E$, $C \subseteq C_\varepsilon + \varepsilon B_E$, then C is limited;
- 4) an almost limited $C_\varepsilon \subseteq E$, $C \subseteq C_\varepsilon + \varepsilon B_E$, then C is almost limited.

Item 1 of Lemma 2 is a standard fact. For item 2, we refer to [7, Theorem 3.44]. Items 3 and 4 are contained in [9, Lemma 2.2].

The following two definitions will be permanently used through the paper.

Definition 2. A Banach space X has:

- 1) the *Dunford-Pettis property* (or $X \in (\text{DPP})$) if $f_n(x_n) \rightarrow 0$ for each weakly null (x_n) in X and each weakly null (f_n) in X^* [7, p. 341];
- 2) the *Gelfand-Phillips property* (or $X \in (\text{GPP})$) if limited subsets of X are relatively compact [10].

Separable Banach spaces and reflexive Banach spaces are in (GPP) by [10]. The closed unit ball of ℓ^∞ is almost limited, but the closed unit ball of c_0 is not almost limited.

Definition 3. A Banach lattice E has:

- 1) the *dual disjoint w^* -property* (or $E \in (\text{DDw}^*\text{P})$) if every disjoint bounded sequence in E^* is w^* -null;
- 2) the *positive Schur property* (or $E \in (\text{PSP})$) if each w -null sequence in E_+ is norm null [11];
- 3) the *dual disjoint Schur property* (or $E \in (\text{DDSP})$) if each disjoint w^* -null sequence in E^* is norm null [9, Definition 3.2];
- 4) the *property (d)* (or $E \in (\text{d})$) if, for every disjoint w^* -null (f_n) in E^* , the sequence $(|f_n|)$ is also w^* -null [12, Definition 1];
- 5) the *positive Grothendieck property* (or $E \in (\text{PGP})$) if each w^* -null sequence in E_+^* is weakly null [11, p. 760];
- 6) the *disjoint Grothendieck property* (or $E \in (\text{DGP})$) if disjoint w^* -null sequences in E^* are w -null.

The disjoint Grothendieck property was studied in [13] under the name “weak Grothendieck property”. The following assertion illustrates relations of the dual disjoint w^* -property with another ones.

Assertion 1. For every Banach lattice E the following holds:

$$E \in (\text{DGP}) \cap (\text{DDw}^*\text{P}) \implies E^{**} \text{ is a KB-space} \implies E \in (\text{DDw}^*\text{P}).$$

Proof. Let $E \in (\text{DGP}) \cap (\text{DDw}^*\text{P})$ but E^{**} is not a KB-space. Then, by Lemma 1, there is a disjoint bounded (f_n) in E^* which is not weakly null. However $f_n \xrightarrow{w^*} 0$ because $E \in (\text{DDw}^*\text{P})$, and hence $f_n \xrightarrow{w} 0$ as $E \in (\text{DGP})$. The obtained contradiction proves the first implication.

For the second implication, let E^{**} be KB. By Lemma 1, every disjoint (f_n) in B_{E^*} is weakly null and hence w^* -null. It follows $E \in (\text{DDw}^*\text{P})$. $\square$

Recall that E has *sequentially w^* -continuous lattice operations* whenever $f_n \xrightarrow{w^*} 0$ in E^* implies $|f_n| \xrightarrow{w^*} 0$. Any AM-space with unit has sequentially w^* -continuous lattice operations. The property (d) is weaker than the sequential w^* -continuity of the lattice operations, e.g. in ℓ^∞ . It was proved in

[11, Proposition 1.4] that every Dedekind σ -complete E has the property (d) [9, Corollary 2.4], and it was observed in [11, Remark 1.5] that $\ell^\infty/c_0 \in (d)$ but ℓ^∞/c_0 is not Dedekind σ -complete. The lattice $L^p[0, 1]$, $1 \leq p \leq \infty$, has the property (d), whereas $C[0, 1] \notin (d)$ (Example 1).

The following assertion [9, Theorem 3.3] describes the impact of the property (d) on dual Schur properties.

Assertion 2. The following holds for Banach lattices:

1. $(\text{DDSP}) \subseteq (\text{DPSP})$.
2. $((\text{DDSP}) \cap (d)) = ((\text{DPSP}) \cap (d))$.

Clearly $((\text{PGP}) \cap (d)) \subseteq ((\text{DGP}) \cap (d))$. For this inclusion, the property (d) is essential, as $C[0, 1] \in (\text{PGP}) \setminus (\text{DGP})$ by Example 1 below. We have no example of $E \in (\text{DGP}) \setminus (\text{PGP})$. The next example shows that the property (d) is essential in Assertion 2(2).

Example 1. Consider the Banach lattice $E = C[0, 1]$. Let (f_n) be a positive w^* -null sequence in E^* . It follows from $\|f_n\| = f_n(\mathbf{1}_{[0,1]}) \rightarrow 0$ that $C[0, 1] \in (\text{DPSP})$, and hence $C[0, 1] \in (\text{PGP})$. Take the sequence (r_n) of Rademacher functions on $[0, 1]$ and define a disjoint w^* -null sequence in E^* as follows: $g_n = 2^{n+1} \cdot r_{2n} \cdot \mathbf{1}_{[\frac{1}{2^{n+1}}, \frac{1}{2^n}]}$. Then (g_n) is a disjoint w^* -null sequence in E^* , yet $\|g_n\| = 1$ for all $n \in \mathbb{N}$. Therefore $C[0, 1] \notin (\text{DDSP})$, and hence by Assertion 2, $C[0, 1] \notin (d)$. Note that the set $\{g_n : n \in \mathbb{N}\}$ is not even relatively weakly compact. Indeed, for $y := \sum_{n=1}^\infty r_{2n} \cdot \mathbf{1}_{[\frac{1}{2^{n+1}}, \frac{1}{2^n}]} \in L^\infty[0, 1] \subseteq E^*$, where $y(g_n) \equiv 1$ for all $n \in \mathbb{N}$. It also shows $C[0, 1] \notin (\text{DGP})$.

The following two definitions are permanently used below.

Definition 4. A bounded operator $T : X \rightarrow Y$ is

- 1) *Dunford–Pettis* (or $T \in \text{DP}(X, Y)$) if T takes weakly null sequences to norm null ones;
- 2) *limited* (or $T \in \text{Lim}(X, Y)$) if $T(B_X)$ is a limited subset of Y .
- 3) A bounded operator $T : E \rightarrow Y$ is *almost M-weakly compact* (or $T \in \text{aMwc}(E, Y)$) if $f_n(Tx_n) \rightarrow 0$ for each weakly convergent (f_n) in Y^* and disjoint bounded (x_n) in E [14, Definition 2.2].

For instance, $\text{Id}_{c_0} \in \text{aMwc}(c_0)$. The canonical embedding of c_0 into ℓ^∞ is limited, but not compact. Note also that order intervals of a Banach lattice are not necessarily limited (e.g., $[-\mathbf{1}, \mathbf{1}]$ in c is not limited, where $\mathbf{1}$ denotes the constantly one sequence).

Definition 5. A bounded operator

- 1) $T : E \rightarrow Y$ is *AM-compact* ($T \in \text{AMc}(E, Y)$) if $T[-x, x]$ is relatively compact for every $x \in E_+$.
- 2) $T : E \rightarrow Y$ is *M-weakly compact* ($T \in \text{Mwc}(E, Y)$) if $\|Tx_n\| \rightarrow 0$ for every disjoint bounded (x_n) in E [8, Def. 3.6.9 iv)].
- 3) $T : X \rightarrow F$ is *semi-compact* ($T \in \text{semi-K}(X, F)$) if it carries bounded subsets of X onto almost order bounded subsets of F (or, equivalently, for each $\varepsilon > 0$, there exists $u_\varepsilon \in F_+$ with $T(B_X) \subseteq [-u_\varepsilon, u_\varepsilon] + \varepsilon B_F$).
- 4) $T : X \rightarrow F$ is *L-weakly compact* ($T \in \text{Lwc}(X, F)$) if T carries bounded subsets of X onto Lwc subsets of F .
- 5) $T : X \rightarrow F$ is *almost L-weakly compact* ($T \in \text{aLwc}(X, F)$) if T carries relatively weakly compact subsets of X onto Lwc subsets of F [14, Definition 2.1] (or, equivalently, if $T(X) \subseteq F^a$ and $f_n(Tx_n) \rightarrow 0$ for each w -null (x_n) in X and disjoint (f_n) in F^*).
- 6) $T : X \rightarrow F$ is *limitedly L-weakly compact* ($T \in \text{l-Lwc}(X, F)$) if T maps limited subsets of X onto Lwc subsets of F (or, equivalently, $T^*f_n \xrightarrow{w} 0$ for each disjoint bounded (f_n) in F^*).

The limitedly L(M)wc-operators were defined in [1] and [15] (in the second paper they were named weak L(M)wc-operators). It is well known that $\text{semi-K}(X, F) = \text{Lwc}(X, F)$ for every X , whenever the norm in F is o -continuous. Note that [7, Theorem 5.63] yields a method of producing many classes of operators. For example, if we consider sets of the form $A = T(C)$ with any weakly compact subset C of

X and $B = B_{F^*}$, then we obtain almost L(M)wc operators [14], whereas the choice of sets of the form $A = T[0, x]$ with any order interval $[0, x]$ in E and $B = B_{F^*}$ produces order L(M)wc operators [16].

In Section 1 we prove Theorem 1 that gives a condition on bounded operators to be AM-compact and Theorem 2 that gives condition on bounded operators to be limited. In Section 2, Theorem 12 provides conditions under which every regular operator is almost limited. Theorem 16 of Section 3 gives a tool for lattice approximation for collectively Mwc families of operators.

For unexplained terminology and notation, we refer to [7, 8, 10].

1 The effect of w^* -continuity of lattice operations in the dual

In this section, we study how order continuity of norm in E^* and continuity of lattice operations in E^* (which imply order intervals being limited in E [9]) affect operators. The limited order intervals will play an important role in what follows. So let us recall that lattice operations are weakly continuous if, for every w -null (x_n) in E , $|x_n| \xrightarrow{w} 0$. We need the following lemma [9, Proposition 2.3(1)].

Lemma 3. The lattice operations of E^* are w^* -sequentially continuous iff every almost order bounded subset of E is limited.

Proposition 1. Let $T \in \text{semi-K}(X, F)$.

1. If F has limited order intervals, then $T \in \text{Lim}(X, F)$.
2. If F has compact order intervals, then $T \in \text{K}(X, F)$.
3. If F has o -continuous norm, then $T \in \text{W}(X, F)$.

Proof. 1. We need to show that $T(B_X)$ is limited. Since $T \in \text{semi-K}(X, F)$ then, for every $\varepsilon > 0$, there exists $u_\varepsilon \in F_+$ with $T(B_X) \subseteq [-u_\varepsilon, u_\varepsilon] + \varepsilon B_F$. As each order interval in F is limited, $T(B_X)$ is limited by Lemma 2(3).

2. Similar to 1.

3. Since $F^a = F$, the order intervals in F are w -compact. The rest of the proof is similar to 1. $\square$

Corollary 1. Let E^* be a KB-space and let F have limited order intervals. The following holds:

1. $\text{DP}_+(E, F) \subseteq \text{Lim}(E, F)$.
2. If $E \in (\text{DPP})$, then $\text{W}_+(E, F) \subseteq \text{Lim}(E, F)$.

Proof. 1. Let $T \in \text{DP}_+(E, F)$. By [17, Theorem 3.7.10], $T \in \text{Mwc}(E, F)$. Thus $T \in \text{semi-K}(E, F)$, and hence $T \in \text{Lim}(E, F)$ by Proposition 1(1).

2. If $T \in \text{W}_+(E, F)$ then $T \in \text{DP}(E, F)$ by [7, Theorem 5.82]. The rest of the proof is as in 1. $\square$

Since L-weakly compact, regular M-weakly compact, and positive operators dominated by semi-compact operators are semi-compact ([7, Theorem 5.71, Theorem 5.72]), they are (almost) limited independent of their ranges due to Proposition 1. A semi-compact operator need not be compact, weakly compact, L- or M-weakly compact. The identity operator on ℓ^∞ is semi-compact, but it has none of the compactness properties mentioned.

Corollary 2. Let F have o -continuous norm and limited order intervals. Then, for every E ,

1. each semi-compact operator $T : E \rightarrow F$ is compact;
2. each order bounded Mwc operator $T : E \rightarrow F$ is compact;
3. each Lwc operator $T : E \rightarrow F$ is compact;
4. if E^* has o -continuous norm, then each order bounded aDP operator $T : E \rightarrow F$ is compact.

Proof. 1. Let $T \in \text{semi-K}(E, F)$. By Proposition 1, $T \in \text{Lim}(E, F)$, and so $T(B_E)$ is limited. As $F^a = F$, then F is Dedekind σ -complete. Then $F \in (\text{GPP})$ by [18]. Thus $T(B_E)$ is relatively compact, so $T \in \text{K}(E, F)$.

2. This follows from 1 in view of [7, Theorem 5.72].

3. This follows from 1 in view of [7, Theorem 5.71].

4. If E^* is a KB-space then, by [19, Proposition 2.2], $T \in \text{aDP}(E, F)$ and is order bounded. Hence $T \in \text{Mwc}(E, F)$ and therefore $T \in \text{semi-K}(E, F)$. By 1, $T \in \text{K}(E, F)$. $\square$

By [19, Proposition 2.2], if E^* is a KB-space, $T \in \text{aDP}(E, F)$ and is order bounded, then $T \in \text{semi-K}(E, F)$. Thus it follows easily that if, additionally, $S \in \text{AMc}(F, G)$, then $ST \in \text{K}(E, G)$. Note that not every weakly compact operator is AM-compact, e.g., the identity operator $I_{L^2[0,1]}$ is weakly compact but it is not AM-compact.

Definition 6. An operator $T : E \rightarrow Y$ is called

- 1) *order limited* (or $T \in \text{o-Lim}(E, Y)$) if $T[0, x]$ is limited for each $x \in E_+$;
- 2) *order weakly compact* (or $T \in \text{owc}(E, Y)$) if $T[0, x]$ is relatively weakly compact for each $x \in E_+$.

Theorem 1. Let all order intervals in E be limited. The following holds:

1. $\text{W}(E, Y) \subseteq \text{AMc}(E, Y)$ for all Y .
2. $\text{owc}(E, Y) \subseteq \text{AMc}(E, Y)$ for all Y .
3. If E^* and F both have o-continuous norms, then $\text{L}_+(E, F) \subseteq \text{AMc}(E, F)$.
4. If E^* is a KB-space and Y does not contain c_0 , then $\text{L}(E, Y) \subseteq \text{AMc}(E, Y)$.

Proof. 1. Let $T \in \text{W}(E, Y)$, and let $[-x, x] \subseteq E$. By [7, Theorem 5.38], T factors over a reflexive Z as $T = SR$ with $R \in \text{L}(E, Z)$ and $S \in \text{L}(Z, Y)$. The set $R[-x, x]$ is limited in Z as $[-x, x]$ is limited in E . Since $Z \in (\text{GPP})$ by [10], $R[-x, x]$ is relatively compact in Z , and hence $SR[-x, x]$ is relatively compact in Y . Thus, $T \in \text{AMc}(E, Y)$.

2. Let $T \in \text{owc}(E, Y)$ and let $[-x, x] \subseteq E$. Then T admits a factorization over some $F = F^a$ as $T = SQ$, where $Q : E \rightarrow F$ is a lattice homomorphism and $S \in \text{L}(F, Y)$ by [7, Theorem 5.58]. The set $Q[-x, x]$ is limited as $[-x, x]$ is limited. Since F has o-continuous norm, it is Dedekind σ -complete, hence $F \in (\text{GPP})$ by [18]. Therefore $Q[-x, x]$ is relatively compact in F , and hence $SQ[-x, x]$ is relatively compact in Y . Thus $T \in \text{AMc}(E, Y)$.

3. By [7, Exercise 15, p. 314], T has a factorization $T = SR$ over a Banach lattice G with o-continuous norm. Let $[a, b] \subseteq E$. Then $R[a, b]$ is limited in G . Since $G \in (\text{GPP})$ (by [18]), $R[a, b]$ is relatively compact in G , and hence $SR[a, b]$ is relatively compact in F . Since the interval $[a, b]$ is arbitrary, $T \in \text{AMc}(E, F)$.

4. By [7, Exercise 19, p. 315], the operator T admits a factorization over a reflexive Banach lattice G as $T = SR$, where $R : E \rightarrow G$ and $S : G \rightarrow Y$ are bounded. The rest is similar to proof of 1. $\square$

Corollary 3. Let E have limited order intervals, and $F = F^a$. The following holds.

1. $\text{semi-K}_+(E, F) \subseteq \text{AMc}(E, F)$.
2. $\text{Lwc}(E, F)$ and $\text{Mwc}(E, F)$ are $\text{AMc}(E, F)$.
3. Let $T : E \rightarrow F$ have o-continuous operator norm. Then $T \in \text{AMc}(E, F)$.

Proof. 1. As $T \in \text{semi-K}_+(E, F)$, for each $\varepsilon > 0$, we can find $u_\varepsilon \in F_+$ such that $T(B_E) \subseteq [-u_\varepsilon, u_\varepsilon] + \varepsilon B_F$. Since F has o-continuous norm, the interval $[-u_\varepsilon, u_\varepsilon]$ is weakly compact. Consequently $T \in \text{W}(E, F)$ by Lemma 2(2), and $T \in \text{AMc}(E, F)$ due to Proposition 1.

2. It follows from 1, as L- and M-weakly compact operators are weakly compact.

3. By [7, Theorem 5.68], $T \in \text{Lwc}(E, F)$. The rest follows from 2. $\square$

Corollary 4. Suppose E^* is a KB-space, E has limited order intervals, and $F \in (\text{PSP})$. Then $\text{W}_+(E, F) \subseteq \text{K}(E, F)$.

Proof. Let $T \in \text{W}_+(E, F)$. Then $T \in \text{AMc}(E, F)$ by Proposition 1. On the other hand, $T \in \text{Mwc}(E, F)$ by [20, Theorem 3.3]. As T is positive, it is semi-compact by [7, Theorem 5.72]. Since $F \in (\text{PSP})$, it has o-continuous norm. As E^* and F both have o-continuous norms, $T \in \text{K}(E, F)$ by [17, Theorem 125.5] because of $T \in \text{semi-K}(E, F) \cap \text{AMc}(E, F)$. $\square$

The next result gives some conditions on an operator to be limited.

Definition 7. An operator $T : E \rightarrow Y$ is called *almost Dunford–Pettis* (or $T \in \text{aDP}(E, Y)$) if T maps disjoint weakly null sequences to norm null sequences.

Note that $\text{Id}_{L^1[0,1]}$ is an almost Dunford–Pettis operator but not Dunford–Pettis one.

Theorem 2. The following assertions hold.

1. If E^* is a KB-space and F^* has w^* -continuous lattice operations, then

$$\text{aDP}_+(E, F) \subseteq \text{Lim}(E, F).$$

2. If E^* has w^* -sequentially continuous lattice operations, then, for every F ,

$$\text{Mwc}(E, F) \subseteq \text{Lim}(E, F).$$

Proof. 1. Let $T \in \text{aDP}_+(E, F)$. By [19, Proposition 2.2], $T \in \text{semi-K}(E, F)$. Thus, for each $\varepsilon > 0$, there exists some $u_\varepsilon \in F_+$ such that $T(B_E) \subseteq [-u_\varepsilon, u_\varepsilon] + \varepsilon B_F$. Since F^* has w^* -continuous lattice operations, F has limited order intervals, hence $T(B_E)$ is limited by Lemma 2(3), which means $T \in \text{Lim}(E, F)$.

2. Let $T \in \text{Mwc}(E, F)$ and take $(f_n) \xrightarrow{w^*} 0$ in F^* . We need to show $\|T^*f_n\| \rightarrow 0$. It suffices to prove $|T^*f_n| \xrightarrow{w^*} 0$ and $T^*(f_n)(x_n) \rightarrow 0$ for each norm bounded disjoint sequence (x_n) in E_+ . Since T is bounded, $T^*f_n \xrightarrow{w} 0$ in E^* . w^* -continuity of lattice operations in E^* gives $|T^*f_n| \xrightarrow{w^*} 0$. Consider

$$|T^*(f_n)(x_n)| = |f_n(Tx_n)| \leq \|f_n\| \|Tx_n\|.$$

Since (f_n) is norm bounded and T is an Mwc operator, the claim follows. $\square$

The next example shows that a limited operator need not be Mwc.

Example 2. Suppose E^* is not a KB-space and $F \neq \{0\}$. Then there exists a positive compact (and hence limited) operator $T : E \rightarrow F$ that is not M -weakly compact. Indeed, pick $y \in F_+ \setminus \{0\}$. Since the norm in E^* is not o -continuous, E^* contains a sublattice isomorphic to ℓ^∞ and a sublattice isomorphic to ℓ^1 ; and E contains a sublattice isomorphic to ℓ^1 ([7, Theorem 4.51] and [8, Proposition 2.3.12]). Let $i : \ell^1 \rightarrow E$ be a lattice embedding. Then $i(\ell^1)$ is the range of a positive projection $P : E \rightarrow E$, that is $i(\ell^1) = P(E)$ [8, Proposition 2.3.11]. Define $S : P(E) \rightarrow F$ by

$$S \left(i \left(\sum_{k=1}^{\infty} \beta_k e_k \right) \right) := \left(\sum_{k=1}^{\infty} \beta_k \right) y,$$

and let $T = SP$. Then $T : E \rightarrow F$ is positive. Since S has rank one, T is compact (and hence limited). But $T \notin \text{Mwc}(E, F)$. This follows from $T(i(e_n)) = SP(i(e_n)) = S(i(e_n)) = y$ and so $\|T(i(e_n))\| \not\rightarrow 0$.

In the opposite direction, we have

Theorem 3. If one of the following conditions holds:

1. $\text{Lim}_+(E, F) \subseteq \text{Mwc}(E, F)$ for every F ;
2. $\text{Lim}(E, F) \subseteq \text{aMwc}(E, F)$ for every F ;

then E^* is a KB-space.

Proof. 1. Suppose $F = E$ and E^* is not a KB-space. Then, in view of Lemma 1 and [7, Theorem 4.14], there exists a disjoint sequence $(f_n) \subseteq [0, f] \subseteq E^*$ such that $\|f_n\| \not\rightarrow 0$. Pick $y \in E_+$ such that $\|y\| = 1$ and a functional $g \in E^*$ with $\|g\| = 1$ and $g(y) = 1$. Define a positive operator $T : E \rightarrow E$ by $T(x) = f(x)y$ for $x \in E$. As T is compact, $T \in \text{Lim}(E, E)$. We claim $T \notin \text{Mwc}(E, E)$. By [7, Theorem 5.64], it is enough to show $T^* \notin \text{Lwc}(E^*, E^*)$. Let us observe that $T^*(g) = g(y)f = f$.

Thus (f_n) is a disjoint sequence in $\text{sol}(T^*(B_E))$ with $\|f_n\| \not\rightarrow 0$. Therefore $T^* \notin \text{Lwc}(E^*, E^*)$ and hence $T \notin \text{Mwc}(E, E)$ by [7, Theorem 5.64], by the assumption $\text{Lim}_+(E, E) \subseteq \text{Mwc}(E, E)$. This is a contradiction.

2. Under the assumption, $K(E, F) \subseteq \text{aMwc}(E, F)$, hence E^* is a KB-space by [21, Theorem 2]. $\square$

The following theorem deals with the conditions on operators to be (a)Mwc (compare with [2, Proposition 4.5]).

Theorem 4. Let E^* be a KB-space. Then the following holds.

1. If $F \in (\text{GPP})$ then $\text{Lim}(E, F) \subseteq \text{Mwc}(E, F)$.
2. $\text{Lim}_+(E, F) \subseteq \text{aMwc}(E, F)$ for every F .

Proof. 1. Let $T \in \text{Lim}(E, F)$ and (x_n) be a disjoint bounded sequence in E . Then, by Lemma 1, $x_n \xrightarrow{w} 0$ and hence $Tx_n \xrightarrow{w} 0$. Since $F \in (\text{GPP})$ and the set $\{Tx_n\}_{n=1}^\infty$ is limited, it is relatively compact. Then $\|Tx_n\| \rightarrow 0$, and hence $T \in \text{Mwc}(E, F)$.

2. If $T \in \text{Lim}_+(E, F)$ then $T \in \text{aLim}_+(E, F)$, and hence $T^* \in \text{aDP}_+(F^*, E^*)$ by [12, Theorem 1]. Thus $T^* \in \text{aLwc}_+(F^*, E^*)$ by [14, Proposition 2.4], and so $T \in \text{aMwc}(E, F)$ by [14, Theorem 2.5]. $\square$

The condition that E^* is a KB-space is essential in Theorem 4(1) due to [7, the example before Theorem 5.62].

The operator Id_{ℓ_2} is w-compact but not limited. Theorem 5 in [22] shows that if F is σ -Dedekind complete, then every almost limited $T : E \rightarrow F$ is w-compact iff E is reflexive or $F = F^a$.

Theorem 5. If $W(E, Y) \subseteq \text{Lim}(E, Y)$, then one of the following holds:

1. E^* is a KB-space;
2. $Y \in (\text{DPP})$.

Proof. Let E^* be not a KB-space and $Y \notin (\text{DPP})$. We construct $T \in W(E, Y)$ but $T \notin \text{Lim}(E, Y)$.

Since E^* is not a KB-space, there exists a disjoint sequence (u_n) in $(B_E)_+$ which is not weakly null by Lemma 1. By passing to a subsequence, we may assume that, for some $\theta \in E_+^*$ and $\varepsilon > 0$, $\theta(u_n) > \varepsilon$ for all n . Then the components θ_n of θ in the carriers C_n of u_n form an order bounded disjoint sequence in E_+^* such that $\theta_n(u_n) = \theta(u_n)$ for all n and $\theta_n(u_m) = 0$ if $n \neq m$ by [23, Theorem 116.3]. We define an operator $P : E \rightarrow \ell^1$ by

$$P(x) = \left(\frac{\theta_k(x)}{\theta(u_k)} \right)_{k=1}^\infty \quad (x \in E).$$

The operator P is well defined as

$$\|Px\| = \sum_{k=1}^\infty \left| \frac{\theta_k(x)}{\theta(u_k)} \right| \leq \sum_{k=1}^\infty \frac{|\theta_k(x)|}{\varepsilon} \leq \frac{1}{\varepsilon} \sum_{k=1}^\infty \theta_k(|x|) \leq \frac{1}{\varepsilon} \theta(|x|)$$

holds for each $x \in E$.

As $Y \notin (\text{DPP})$, there is a weakly null sequence (y_n) in Y and a weakly null (hence w^* -null) sequence (f_n) in Y^* such that $(f_n(y_n)) \not\rightarrow 0$. Passing to a subsequence, we may assume that, for some $\epsilon > 0$,

$$|f_n(y_n)| > \epsilon \quad (\forall n \in \mathbb{N}). \quad (1)$$

We define an operator $S : \ell^1 \rightarrow Y$ by $S\alpha = \sum_{k=1}^\infty \alpha_k y_k$. By [7, Theorem 5.26], S is weakly compact.

Then $T = SP \in W(E, Y)$. Suppose T is limited. Then the w^* -null sequence (f_n) is uniformly null on $T(B_E)$. Since $Tu_n = SPu_n = Se_n = y_n$, it follows from (1) that $|f_n(Tu_n)| > \epsilon$ for all n , contradicting uniform convergence to zero of (f_n) on $T(B_E)$. Therefore $T \notin \text{Lim}(E, Y)$, as desired. $\square$

We discuss some composition properties of semi-compact operators.

Proposition 2. Let $F = F^a$, $T \in \text{semi-K}(X, F)$, and $S \in \text{DP}(F, G)$. Then $ST \in \text{K}(X, G)$.

Proof. Since $T \in \text{semi-K}(X, F)$, for each $\varepsilon > 0$, there exists $u_\varepsilon \in F_+$ such that $T(B_X) \subseteq [-u_\varepsilon, u_\varepsilon] + \varepsilon B_F$. Since F has o-continuous norm, $[-u_\varepsilon, u_\varepsilon]$ is relatively weakly compact and hence $T(B_X)$ is weakly compact by Lemma 2(2). In order to show that $ST(B_E)$ is relatively compact, take a sequence (STx_n) in $ST(B_X)$. As $T(B_X)$ is relatively weakly compact then, by [7, Theorem 3.40], there exists a weakly convergent subsequence (Tx_{n_k}) . Since $S \in \text{DP}(F, G)$, the sequence (STx_{n_k}) converges in norm. As the sequence (STx_n) in $ST(B_X)$ is arbitrary, $ST(B_X)$ is compact in G , and hence $ST \in \text{K}(X, G)$. $\square$

If E^* is a KB -space and $F = F^a$ then, for each regular aDP operator T (i.e. there exist $T_1, T_2 \in \text{aDP}_+(E, F)$ such that $T = T_1 - T_2$) and $S \in \text{DP}(F, G)$, we have $ST \in \text{K}(E, G)$. This follows from the fact that each positive aDP operator is semi-compact by [19, Proposition 2.2].

The square of a semi-compact operator is not compact in general (e.g., Id_c is semi-compact but $\text{Id}_c^2 = \text{Id}_c$ is not compact). This problem was studied in [24, Theorem 2.4]. We give another sufficient condition.

Theorem 6. Let E have limited order intervals.

1. If E has o-continuous norm and $T \in \text{semi-K}_+(E)$, then $T^2 \in \text{K}(E)$.
2. If $T \in \text{semi-K}(E) \cap \text{W}(E)$, then $T^2 \in \text{K}(E)$.

Proof. 1. Let $T \in \text{semi-K}_+(E)$. Then, for $\varepsilon > 0$ there exists $u_\varepsilon \in E_+$ with $T(B_E) \subseteq [-u_\varepsilon, u_\varepsilon] + \varepsilon B_E$, which yields $T^2(B_E) \subseteq [-Tu_\varepsilon, Tu_\varepsilon] + \varepsilon \|T\| B_E$. By the assumption, each $[-Tu_\varepsilon, Tu_\varepsilon]$ is limited, and hence $T^2(B_E)$ is limited by Lemma 2(3). By [18], a Dedekind σ -complete Banach lattice has the Gelfand–Phillips property iff it has o-continuous norm. Therefore $E \in (\text{GPP})$. Then $T^2(B_E)$ is relatively compact, and hence $T^2 \in \text{K}(E)$.

2. As $T \in \text{W}(E)$, it factors over a reflexive space G as $T = SR$, where $R : E \rightarrow G$ and $S : G \rightarrow E$ are bounded by [7, Theorem 5.38].

Let $\varepsilon > 0$. Since $T \in \text{semi-K}(E)$, there exists $u_\varepsilon \in E_+$ such that $T(B_E) \subseteq [-u_\varepsilon, u_\varepsilon] + \varepsilon B_E$. The interval $[-u_\varepsilon, u_\varepsilon]$ is limited in E . Applying R to $[-u_\varepsilon, u_\varepsilon]$, we see that $R[-u_\varepsilon, u_\varepsilon]$ is limited in G . Being reflexive, $G \in (\text{GPP})$, and hence $R[-u_\varepsilon, u_\varepsilon]$ is relatively compact in G . Therefore, $SR[-u_\varepsilon, u_\varepsilon]$ is relatively compact in E and $T^2(B_E) \subseteq SR[-u_\varepsilon, u_\varepsilon] + \varepsilon SR(B_E)$. Since $\varepsilon > 0$ is arbitrary, Lemma 2(1) implies that $T^2(B_E)$ is relatively compact in E , and hence $T^2 \in \text{K}(E)$. $\square$

Proposition 3. $\text{aLwc}(X, F) \subseteq \text{l-Lwc}(X, F)$.

Proof. Let $T \in \text{aLwc}(X, F)$. As one of defining properties of aLwc operators, we have $T(X) \subseteq F^a$. Since F^a has o-continuous norm, $F^a \in (\text{DDw}^*\text{P})$ so that each limited subset in F^a is Lwc by [25, Theorem 2.6], and hence $T \in \text{l-Lwc}(X, F)$. $\square$

Proposition 4. If F has the Schur property then $\text{l-Mwc}(E, F) \subseteq \text{owc}(E, F)$.

Proof. Let $T \in \text{l-Mwc}(E, F)$. By [7, Theorem 5.57], we need to show $\|Tx_n\| \rightarrow 0$ for each order bounded disjoint sequence (x_n) in E . Such a sequence is w-null as it is order bounded and disjoint. Since $T \in \text{l-Mwc}(E, F)$, we get $Tx_n \xrightarrow{w} 0$ and so $\|Tx_n\| \rightarrow 0$ as F has the Schur property. $\square$

A subset A of E is called bi-bounded if $i(A)$ is order bounded in E^{**} , where $i : E \rightarrow E^{**}$ is the canonical embedding. An operator $T : E \rightarrow F$ is b-weakly compact if $T(A)$ is relatively weakly compact for each bi-bounded A in E . A similar proof to the given one in Proposition 4 shows that $\text{l-Mwc}(E, F)$ is actually contained in the narrower set of b-weakly compact operators [26].

2 Almost limited operators

In this section, we focus on almost limited operators and show they contain many interesting classes of operators, and the converse problem of almost limited operators contained in other classes.

A bounded subset A of a Banach space X is called *almost limited* (we write aLim-set) if each disjoint w^* -null sequence in X^* converges uniformly to zero on A . Almost limited sets were introduced in [25]. Let us also note that every order interval in E is aLim iff $E \in (d)$. An operator $T : X \rightarrow F$ is called *almost limited* (we write $T \in \text{aLim}(X, F)$) if $T(B_X)$ is an aLim-set . Almost limited sets and operators were studied in [9, 22, 25].

It was shown in [9, Proposition 4.5] that each order bounded Mwc operator $T : E \rightarrow F$ is almost limited whenever lattice operations in E^* are w^* -sequentially continuous or $F \in (d)$. By [12, Corollary 4], if lattice operations in E^* are w^* -sequentially continuous then $\text{aLim}(X, E) \subseteq \text{Lim}(X, E)$.

Theorem 7. Let E^* be a KB-space. The following holds.

1. If $F \in (d)$ or the lattice operations in E^* are w^* -sequentially continuous then each order bounded aLwc operator $T : E \rightarrow F$ is aLim .
2. If E is reflexive then $\text{aLwc}(E, F) \subseteq \text{aLim}(E, F)$ for every F .

Proof. 1. Take an order bounded $T \in \text{aLwc}(E, F)$. For $T \in \text{aLim}(E, F)$, it suffices to show $f_n(Tx_n) \rightarrow 0$ for every disjoint w^* -null (f_n) in F^* and disjoint (x_n) in $(B_E)_+$ ([9, Theorem 2.7]). Let (f_n) be disjoint w^* -null in F^* and let (x_n) be disjoint in $(B_E)_+$. Since $f_n \xrightarrow{w^*} 0$, it is bounded. As E^* is a KB-space, $x_n \xrightarrow{w} 0$. Since $T \in \text{aLwc}(E, F)$ then $f_n(Tx_n) \rightarrow 0$ by [14, Theorem 2.2].

2. Let $T \in \text{aLwc}(E, F)$, and let (f_n) be disjoint w^* -null in B_{F^*} . We need to show that (f_n) is uniformly null on $T(B_E)$. Assume to the contrary that $\sup_{x \in B_E} |f_n(Tx)| \not\rightarrow 0$ as $n \rightarrow \infty$. Then, for some $\varepsilon > 0$, there exists a sequence (x_{n_k}) in B_E such that

$$|f_{n_k}(Tx_{n_k})| \geq \varepsilon \quad (\forall k \in \mathbb{N}). \quad (2)$$

B_E is weakly compact because E is reflexive. So, $x_{n_{k_j}} \xrightarrow{w} x \in E$ for some further subsequence $(x_{n_{k_j}})$ by the Eberlein–Smulian theorem. Since $T \in \text{aLwc}(E, F)$, then by [14, Theorem 2.2], $f_{n_{k_j}}(Tx_{n_{k_j}}) \rightarrow 0$ which contradicts (2). Therefore (f_n) converges uniformly to zero on $T(B_E)$, and $T \in \text{aLim}(E, F)$. $\square$

Theorem 8. Let $F \in (d)$. Then the following holds:

1. If $F \in (\text{PGP})$, then $\text{span}(\text{aMwc}_+(E, F)) \subseteq \text{aLim}(E, F)$.
2. $l\text{-Lwc}(E, F) \cap \text{Mwc}(E, F) \subseteq \text{aLim}(E, F)$ for every E .

Proof. 1. Let $T \in \text{span}(\text{aMwc}_+(E, F))$. We may suppose T is positive. For proving $T \in \text{aLim}(E, F)$, it suffices to show $f_n(Tx_n) \rightarrow 0$ for every disjoint w^* -null (f_n) in F_+^* and disjoint (x_n) in $(B_E)_+$ ([9, Theorem 2.7]). Let (f_n) be disjoint w^* -null in F_+^* and (x_n) be disjoint in $(B_E)_+$. As $F \in (\text{PGP})$, $f_n \xrightarrow{w} 0$. By Definition 5 (5), $f_n(Tx_n) \rightarrow 0$. Consequently $T \in \text{aLim}(E, F)$.

2. Let $T \in l\text{-Lwc}(E, F) \cap \text{Mwc}(E, F)$. We need to show $T(B_E)$ is almost limited. It suffices to show $f_n(Tx_n) \rightarrow 0$ for every disjoint (x_n) in $(B_E)_+$ and every disjoint w^* -null (f_n) in B_{F^*} [9, Theorem 2.7]. Let (x_n) be disjoint in $(B_E)_+$ and (f_n) be disjoint w^* -null in B_{F^*} . Since $T \in l\text{-Lwc}(E, F)$, then $(T^*f_n) \xrightarrow{w^*} 0$ in E^* by [1, Theorem 2.7]. As $f_n \xrightarrow{w^*} 0$, $\|f_n\| \leq M$ for all n . Since $T \in \text{Mwc}$, then $\|Tx_n\| \rightarrow 0$, and hence

$$|f_n(Tx_n)| \leq \|f_n\| \cdot \|Tx_n\| \leq M \cdot \|Tx_n\|$$

implies $f_n(Tx_n) \rightarrow 0$ as desired. $\square$

The following presents conditions for almost limited operators to be L-weakly compact.

Theorem 9. The following are equivalent.

1. F has o-continuous norm.
2. For every X , $\text{aLim}(X, F) \subseteq \text{Lwc}(X, F)$.
3. For every E , each positive rank one operator $T : E \rightarrow F$ is Lwc.

Proof. $1 \implies 2$: Let $T \in \text{aLim}(X, F)$. Then $T(B_X)$ is an aLim-set in F . By [25, Theorem 2.6], $T(B_X)$ is an Lwc-set, and hence $T \in \text{Lwc}(X, F)$.

$2 \implies 3$: It is trivial.

$3 \implies 1$: Since, for each $y \in F_+$, there exists a positive rank one operator $T : E \rightarrow F$ such that $T(E) = \text{span}(y)$, then $y \in F^a$ by [7, Theorem 5.66], and hence F has o-continuous norm. $\square$

Order continuity of the norm in F is essential above as the identity of ℓ^∞ is almost limited but it is neither M- or L-weakly compact.

Theorem 10. Let E be an AL-space and let F have o-continuous norm. Then each $T \in \text{aLim}(E, F)$ has a modulus T which is almost limited.

Proof. As $F^a = F$ and T is almost limited, then T is Lwc. Hence T has the modulus $|T|$ that is Lwc by [20, Theorem 2.4]. Therefore $|T|$ is almost limited by [25, Theorem 2.6]. $\square$

However [20, Theorem 2.2] shows that there is a regular compact (hence limited) operator from $E = L^2[0, 1]$ to $F = c(L^2[0, 1])$ which is both M- and L-weakly compact whose modulus does not exist. The identity of ℓ^∞ is almost limited but it is neither M- nor L-weakly compact.

Theorem 11. Suppose E is infinite dimensional and has o-continuous norm, and $F \in (d)$. If $\text{aLim}(E, F) \subseteq \text{Mwc}(E, F)$, then E^* and F both have order continuous norms.

Proof. If $\text{aLim}(E, F) \subseteq \text{Mwc}(E, F)$, then $K(E, F) \subseteq \text{Mwc}(E, F)$. The fact that the norm in E^* is o-continuous follows from [21, Theorem 2].

Suppose now the norm of F is not o-continuous. Then there exist some $u \in F_+$ and a disjoint sequence (u_n) in $[0, u]$ such that $\|u_n\| \not\rightarrow 0$. We may assume $\|u_n\| = 1$ for all n . Since $\dim(E) = \infty$, there exists a disjoint normalized sequence (x_n) in E_+ . By [23, Theorem 116.3 (iii)], there exists a disjoint sequence (g_n) in $(B_{E^*})_+$ such that $\frac{1}{2} \leq g_n(x_n) \leq 1$ for all n and $g_n(x_m) = 0$ for $n \neq m$. Since E has o-continuous norm, then $g_n \xrightarrow{w^*} 0$ by [23, Theorem 116.2].

Define positive operator $R : E \rightarrow c_0$ by $Rx = (g_k(x))_{k=1}^\infty$. Clearly $R(B_E) \subseteq B_{c_0}$. Define positive operator $S : c_0 \rightarrow F$ by $S\alpha = \sum_n \alpha_n u_n$. Then $S(B_{c_0}) \subseteq [-u, u]$, and hence $SR(B_E) \subseteq [-u, u]$. As order intervals in F are almost limited by the (d)-property of F , $SR(B_E)$ is almost limited, and $SR \in \text{aLim}(E, F)$. Since $SR(x_n) = S(g_n(x_n) \cdot e_n) = g_n(x_n) \cdot u_n$, then $\|SR(x_n)\| \geq |g_n(x_n)| \geq \frac{1}{2}$ for all n . Therefore $SR \notin \text{Mwc}(E, F)$. A contradiction. $\square$

An operator $T : E \rightarrow F$ is called *positively limited* if $\|T^*f_n\| \rightarrow 0$ for each w^* -null (f_n) in F_+^* . We denote the set of such operators by $\text{p-Lim}(E, F)$. We need the following lemma [27, Theorem 2.2].

Lemma 4. $T : E \rightarrow F$ is positively limited iff $T^*(y_n^*)(x_n) \rightarrow 0$ for each $y_n^* \xrightarrow{w^*} 0$ in F_+^* and every disjoint bounded (x_n) in E_+ .

Theorem 12. Suppose E^* and F both have o-continuous norms, F has limited order intervals, and each disjoint bounded sequence of E is order bounded. Then each regular operator $T : E \rightarrow F$ is almost limited.

Proof. Suppose $T : E \rightarrow F$ is positive. T admits a factorization over a Banach lattice G with G and G^* both have o-continuous norms, say $T = SR$, where R and S are both positive as in [7, Problem 15 on page 314]. We claim R is positively limited. Let (z_n^*) be a positive w^* -null sequence in G^* and (x_n) be a positive disjoint bounded sequence in E . By the assumption, there exists x with $0 \leq x_n \leq x$ for each n . As R is positive, $0 \leq Rx_n \leq Rx$. Thus $0 \leq R^*(z_n^*)(x_n) \leq z_n^*(Rx_n) \leq z_n^*(Rx)$ which shows $R^*(z_n^*)(x_n) \rightarrow 0$. By Lemma 4, we conclude that R is positively limited.

Now we show $T = SR \in \text{aLim}(E, F)$. Let (u_n^*) be disjoint w^* -null in F^* . Since F^* has w^* -continuous lattice operations, $|u_n^*| \xrightarrow{w^*} 0$ in F^* . As R is positively limited, $\|R^*S^*|u_n^*|\| \rightarrow 0$. Since $R^*|S^*u_n^*| \leq R^*S^*|u_n^*|$, we have $|R^*S^*u_n^*| \leq R^*|S^*u_n^*| \leq R^*S^*|u_n^*|$. Resulting that $\|R^*S^*u_n^*\| = \|T^*u_n\| \rightarrow 0$. It follows $T \in \text{aLim}(E, F)$. $\square$

It is worth noting that both conditions on E in Theorem 12 can be replaced by a single condition that E is an AM-space in which every disjoint bounded sequence is order bounded. Indeed, by the Abramovich's result [28], if E is not isomorphic to an AM-space then E contains a positive disjoint normalized sequence (x_n) such that $\|\bigvee_{n=1}^m x_n\| \rightarrow \infty$, and hence such a sequence (x_n) cannot be order bounded [28]. Thus, E must be (isomorphic to) an AM-space, and hence the norm in E^* is o -continuous. Also note that this condition on E is weaker than the condition that E is an AM-space with unit, as the space $C_\omega(\Gamma)$ of real-valued bounded countably supported functions on an uncountable set Γ still satisfies the condition that every disjoint bounded sequence is order bounded, yet $C_\omega(\Gamma)$ has not got even a weak unit.

Theorem 13. Let $T : E \rightarrow E$ be a positive operator in the ideal center of E . Suppose $E \in (\text{DPSP})$. Then T is positively limited.

Proof. For simplicity suppose $0 \leq T \leq I$. Let (f_n) be a positive w^* -null sequence in E^* . Then $0 \leq T^* \leq I$. Hence $0 \leq T^*f_n \leq f_n$ for all n . Since $E \in (\text{DPSP})$, $\|f_n\| \rightarrow 0$ hence $T^*f_n \rightarrow 0$. $\square$

It is easy to see that $\text{p-Lim}(E, F)$ is a vector space.

Proposition 5. Suppose $\|T_n - T\| \rightarrow 0$ and $T_n \in \text{p-Lim}(E, F)$ for all n . Then $T \in \text{p-Lim}(E, F)$.

We shall also use a known result (see [29, Theorem 2.4] and [3, Proposition 1.10]): if P is a norm closed subspace of $L(E, F)$, then $\text{span}(P_+)$ is complete under the enveloping norm $\|T\|_{r-P} = \inf\{\|S\| : \pm T \leq S \in P\}$. Together with Proposition 5, this immediately implies the following

Assertion 3. If P is a norm closed subspace of $L(E, F)$, then $\text{span}(P_+)$ is complete under the enveloping norm $\|T\|_{r-P} = \inf\{\|S\| : \pm T \leq S \in P\}$.

Corollary 5. For every E and F , $\text{span}(\text{p-Lim}_+(E, F))$ is a Banach space under the enveloping norm.

It was shown in [21, Theorem 1] that $K(X, F) \subseteq \text{aLwc}(X, F)$ for all X iff F has o -continuous norm. We extend this to almost limited operators.

Theorem 14. $\text{aLim}(X, F) \subseteq \text{aLwc}(X, F)$ iff F has o -continuous norm.

Proof. The necessity follows from [21, Theorem 1] as $K(X, F) \subseteq \text{aLim}(X, F)$.

For the sufficiency, suppose in contrary that $\text{aLim}(X, F) \subseteq \text{aLwc}(X, F)$ but the norm in F is not o -continuous. Then there exists $y_0 \in F \setminus F^a$. We choose $x_0 \in X$ and $h \in X^*$ such that $h(x_0) = \|x_0\| = 1$ and define $Tx = h(x)y_0$. The operator T is almost limited as it is compact. However $T \notin \text{aLwc}(X, F)$ as $\{Tx_0\} = \{y_0\}$ is not an Lwc-set. A contradiction, as an almost Lwc operator must map X into the order continuous part of F . $\square$

By [21, Theorem 2], $K(E, X) \subseteq \text{aMwc}(E, X)$ iff E^* is a KB-space. We extend this result as follows.

Proposition 6. If E^* is a KB-space then $\text{aLim}_+(E, F) \subseteq \text{aMwc}(E, F)$.

Proof. Let E^* be a KB-space and $T \in \text{aLim}_+(E, F)$. Then $T^* \in \text{aDP}_+(F^*, E^*)$ [12, Theorem 1]. Consequently $T^* \in \text{aLwc}(F^*, E^*)$ by [14, Proposition 2.4]. Then $T \in \text{aMwc}(E, F)$ by [14, Theorem 2.5]. $\square$

Theorem 15. If E^* and F both have o -continuous norms then $\text{span}(\text{aLim}_+(E, F)) \subseteq \text{Mwc}(E, F)$.

Proof. Let $T \in \text{span}(\text{aLim}_+(E, F))$. Suppose $T \geq 0$. Let (x_n) be a disjoint sequence in B_E , we show $\|Tx_n\| \rightarrow 0$. Since E^* is a KB-space, $x_n \xrightarrow{w} 0$ in E . Since (x_n) is disjoint, $|x_n| \xrightarrow{w} 0$ [11, Proposition 1.3], and hence $T|x_n| \xrightarrow{w} 0$ in F . As T is positive, $|Tx_n| \leq T|x_n|$ and $|Tx_n| \xrightarrow{w} 0$ in F . It remains to show $f_n(Tx_n) \rightarrow 0$ for each disjoint bounded sequence (f_n) in F^* . This follows from $|f_n(Tx_n)| \leq \|T^*f_n\| \|x_n\|$, since $\|T^*f_n\| \rightarrow 0$ as T is almost limited. $\square$

We continue with a necessary conditions for positive w-compact operators to be almost limited (see Theorem 5).

Proposition 7. Suppose $E = E^a$ and $W(X, E) \subseteq \text{aLim}(X, E)$. Then E is a KB-space or $X \in (\text{DPP})$.

Proof. Suppose E is not a KB-space. We show $X \in (\text{DPP})$. Since E is not a KB-space, it contains a lattice copy (say F) of c_0 . Then F is a closed linear span of some disjoint sequence (y_n) in E_+ such that $d = \inf_n \|y_n\| > 0$ and $\sup_m \|\sum_{i=1}^m y_i\| < \infty$ by [17, Proposition 0.5.1]. The operator

$$S\beta = \sum_{k=1}^{\infty} \beta_k y_k \quad (\beta = (\beta_k)_k \in c_0)$$

is a lattice homomorphism from c_0 onto F . There exists a positive disjoint (f_n) in B_{E^*} such that $f_n(y_n) \geq d$ for all n and $f_n(y_m) = 0$ if $n \neq m$. Take $B_X \ni x_n \xrightarrow{w} 0$ and $B_{X^*} \ni x_n^* \xrightarrow{w} 0$. Define $R : X \rightarrow c_0$ by $Rx = (x_k^*(x))_k$. Then $R \in W(X, c_0)$ by [7, Theorem 5.26]. Therefore $T = SR \in W(X, E)$ and hence $T \in \text{aLim}(X, E)$ by the hypothesis. Since E has o-continuous norm, the bounded disjoint sequence (f_n) is w^* -null and therefore $\|T^*f_n\| \rightarrow 0$ as $T \in \text{aLim}(X, E)$. Since (x_n) is bounded and $|f_n(Tx_n)| \leq \|x_n\| \cdot \|T^*f_n\|$, these yield $|f_n(Tx_n)| \rightarrow 0$. Thus, it follows from

$$\begin{aligned} |f_n(Tx_n)| &= |f_n(S(R(x_n)))| = \left| f_n \left(\sum_{k=1}^{\infty} [R(x_n)]_k y_k \right) \right| = |f_n([R(x_n)]_n y_n)| = |f_n([(x_k^*(x_n))_k]_n y_n)| = \\ &= |f_n(x_n^*(x_n) y_n)| = |x_n^*(x_n) f_n(y_n)| = |x_n^*(x_n)| \cdot |f_n(y_n)| \geq |x_n^*(x_n)| \cdot d \end{aligned}$$

that $x_n^*(x_n) \rightarrow 0$, and hence $X \in (\text{DPP})$. $\square$

Corollary 6. $[W(X, c_0) \subseteq \text{aLim}(X, c_0)] \implies X \in (\text{DPP})$.

3 Collectively limited and almost limited operator families

Some of the results given previously in the present paper for individual operators have collective versions as we show below following the recent study of collectively qualified families of operators in [4–6]. At the expense, we can give more general results extending results obtained in previous sections.

Definition 8. A norm bounded family $\mathcal{T}$ of operators is

- 1) *collectively compact* (or $\mathcal{T} \in \mathbf{K}(X, Y)$) if $\mathcal{T}B_X = \bigcup_{T \in \mathcal{T}} T(B_X)$ is relatively compact in Y ;
- 2) *collectively limited* (or $\mathcal{T} \in \mathbf{Lim}(X, Y)$) if $\mathcal{T}B_X$ is limited in Y ;
- 3) *collectively weakly compact* (or $\mathcal{T} \in \mathbf{W}(X, Y)$) if $\mathcal{T}B_X$ is relatively weakly compact in Y ;
- 4) *collectively Lwc* (or $\mathcal{T} \in \mathbf{Lwc}(X, F)$) if $\mathcal{T}B_X$ is an Lwc set in F ;
- 5) *collectively Mwc* (or $\mathcal{T} \in \mathbf{Mwc}(E, Y)$) if $\sup_{T \in \mathcal{T}} \|Tx_n\| \rightarrow 0$ for every disjoint (x_n) in B_E ;
- 6) *collectively semi-compact* (or $\mathcal{T} \in \mathbf{semi-K}(X, F)$) if, for each $\varepsilon > 0$ there exists $u \in F_+$ such that $\mathcal{T}B_X \subseteq [-u, u] + \varepsilon B_F$;
- 7) *collectively order bounded* (or $\mathcal{T} \in \mathbf{Lob}(E, F)$) if, for each $a \in E_+$, there exists $b \in F_+$ with $\bigcup_{T \in \mathcal{T}} T[-a, a] \subseteq [-b, b]$;

- 8) *collectively almost Dunford–Pettis* (or $\mathcal{T} \in \mathbf{aDP}(E, F)$) if, for each disjoint $x_n \xrightarrow{w} 0$ on E ,

$$\sup_{T \in \mathcal{T}} \|Tx_n\| \rightarrow 0.$$

We give an example of collectively limited set consisting of compact operators that is not collectively compact.

Example 3. Let $E = \ell^2$ and $F = \ell^\infty$. Define operators $T_n : E \rightarrow F$ by $T_n(x) = x_n \cdot \mathbf{e}_n$. Then $\mathcal{T} = \{T_n : n = 1, \dots, \infty\} \subseteq K(\ell^2, \ell^\infty)$. The family $\mathcal{T}$ is not collectively compact since $\bigcup_{n=1}^\infty T_n(B_{\ell^2})$ contains the sequence $\mathbf{e}_n$ that is not a convergent sequence in ℓ^∞ . However, $\mathcal{T}$ is collectively limited by the Phillips lemma since $\bigcup_{n=1}^\infty T_n(B_{\ell^2}) \subseteq B_{c_0}$.

Let $\mathcal{T}$ be a norm bounded family of operators. The following proposition gives a method of producing collectively qualified families. We omit its straightforward proof.

Proposition 8. Let $\mathcal{T} \subseteq L(X, F)$ be norm bounded and $S \in L(F, G)$. The following is true.

1. If $S \in K(F, G)$, then $\mathcal{T} \circ S \in \mathbf{K}(E, G)$.
2. If $S \in W(F, G)$, then $\mathcal{T} \circ S \in \mathbf{W}(E, G)$.
3. If $S \in \text{Lim}(F, G)$, then $\mathcal{T} \circ S \in \mathbf{Lim}(E, G)$.
4. If $S \in \mathbf{aLim}(F, G)$, then $\mathcal{T} \circ S \in \mathbf{aLim}(E, G)$.
5. If $S \in \text{Lwc}(F, G)$, then $\mathcal{T} \circ S \in \mathbf{Lwc}(E, G)$.
6. If $\mathcal{T} \in \mathbf{semi-K}(E, F)$ and $S \in \text{Lwc}(F, G)$, then $\mathcal{T} \circ S \in \mathbf{K}(E, G)$.
7. If $\mathcal{T} \in \mathbf{semi-K}(E, F)$ and $S \in \text{owc}(F, G)$, then $\mathcal{T} \circ S \in \mathbf{W}(E, G)$.

The following result is a consequence of Lemma 2.

Proposition 9. For every E the following hold:

1. If $F = F^a$, then $\mathbf{semi-K}(E, F) \subseteq \mathbf{Lwc}(E, F)$.
2. If $F = F^a$, then $\mathbf{semi-K}(E, F) \subseteq \mathbf{W}(E, F)$.
3. If F is discrete and $F = F^a$, then $\mathbf{semi-K}(E, F) \subseteq \mathbf{K}(E, F)$.
4. If F has limited order intervals, then $\mathbf{semi-K}(E, F) \subseteq \mathbf{Lim}(E, F)$.

Proof. 1. The statement follows from the well-known fact that in a Banach lattice with order continuous norm, every almost order bounded set is L-weakly compact.

2. It follows from 1, since every collectively L-weakly compact set is collectively weakly compact by [5, Corollary 2.11].

3. By the assumption, each interval $[-u, u]$ is norm compact. The rest follows from Lemma 2(1).

4. It follows from Lemma 2(3). $\square$

The following theorem is an important tool in lattice approximation for collectively Mwc families of operators.

Theorem 16. Let $\mathcal{T} \in \mathbf{Mwc}(E, F)$ be norm bounded. For each $\varepsilon > 0$, there exists $u \in E_+$ such that $\sup_{T \in \mathcal{T}} \|T[(|x| - u)^+]\| \leq \varepsilon$ holds for all $x \in B_E$.

Proof. Let $\mathcal{T} \in \mathbf{Mwc}(E, F)$. Take $\varepsilon > 0$, $A = B_E$, and consider $\rho(x) = \sup_{T \in \mathcal{T}} \|Tx\|$. The seminorm ρ is norm continuous as $\rho(x) \leq \|T\|\|x\| \leq K\|x\|$ for all $T \in \mathcal{T}$ and some $K > 0$. If (x_n) is a disjoint sequence (x_n) in B_E then $\rho(x_n) \rightarrow 0$ by Definition 8. By [7, Theorem 4.36], there exists $u \in E_+$ such that $\sup_{T \in \mathcal{T}} \|T[(|x| - u)^+]\| = \rho(\text{Id}_E(|x| - u)^+) \leq \varepsilon$ for $x \in B_E$. $\square$

Corollary 7. $\mathbf{Lob}(E, F) \cap \mathbf{Mwc}(E, F) \subseteq \mathbf{semi-K}(E, F)$.

Proof. Let $\mathcal{T} \in \mathbf{L}_{ob}(E, F) \cap \mathbf{Mwc}(E, F)$ and $\varepsilon > 0$. Since $\mathcal{T} \in \mathbf{L}_{ob}(E, F)$, [4, Theorem 2.1] implies that $\mathcal{T}$ is norm bounded. As $\mathcal{T} \in \mathbf{Mwc}(E, F)$, by Theorem 16 there exists $u \in E_+$ with

$$\sup_{T \in \mathcal{T}} \|T(|x| - u)^+\| \leq \varepsilon \quad \text{for all } x \in B_E. \quad (3)$$

If $y \in B_E^+$ and $T \in \mathcal{T}$, then $Ty = T(y \wedge u) + T((y - u)^+) \in T[0, u] + \varepsilon B_F$ by (3). Therefore, for every $x \in B_E$ and $T \in \mathcal{T}$, we have $Tx = T(x^+) - T(x^-) \in T[0, u] - T[0, u] + 2\varepsilon B_F \subseteq T[-u, u] + 2\varepsilon B_F$. Thus, $\bigcup_{T \in \mathcal{T}} T(B_E) \subseteq [-u, u] + 2\varepsilon B_F$ proving $\mathcal{T}$ is collective semicompact. $\square$

Example 4. Consider $T : E \rightarrow F$ be a positive almost Dunford–Pettis operator. Then $[0, T] \in \mathbf{aDP}(E, F)$. Indeed, let (x_n) be a disjoint w-null sequence in E . Then $(|x_n|)$ is also disjoint and w-null. Hence $\|T|x_n|\| \rightarrow 0$. As $|Sx_n| \leq S|x_n| \leq T|x_n|$ for each $S \in [0, T]$, we have $\|Sx_n\| \leq \|T|x_n|\|$ and so $\sup_{S \in [0, T]} \|Sx_n\| \leq \|T|x_n|\| \rightarrow 0$.

The following is a collective version of [19, Proposition 2.1].

Proposition 10. If E^* is a KB-space, then $\mathbf{L}_{ob}(E, F) \cap \mathbf{aDP}(E, F) \subseteq \mathbf{semi-K}(E, F)$.

Proof. Since $\mathcal{T} \in \mathbf{L}_{ob}(E, F)$, then $\mathcal{T}$ is norm bounded by [4, Theorem 2.1]. In view of Corollary 7, it suffices to show $\mathcal{T} \in \mathbf{Mwc}(E, F)$. Let (x_n) be a disjoint sequence in B_E . Since E^* has order continuous norm, $x_n \xrightarrow{w} 0$ in E . Therefore, $\sup_{T \in \mathcal{T}} \|Tx_n\| \rightarrow 0$ as $\mathcal{T} \in \mathbf{aDP}(E, F)$. Thus $\mathcal{T} \in \mathbf{Mwc}(E, F)$. $\square$

Conclusion

This research represents a novel approach to the investigation of the classes of (almost) limited, L-weakly compact, semi-compact, and Dunford–Pettis operators between Banach lattices and Banach spaces. The main scientific contribution lies in the comprehensive analysis of collectively qualified families of operators in Banach lattices. Future study might investigate extensions to operators in locally solid lattices and ordered Banach spaces.

Author contributions

Both authors contributed equally to this work.

Conflict of interest

The authors declare no conflict of interest.

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On quasivarieties generated by (3-3)-snake lattices

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This paper addresses two classical and interrelated questions in universal algebra and lattice theory concerning the finite axiomatizability of quasivarieties and the standardness of topological quasivarieties generated by finite lattices. These problems have long attracted significant attention due to their central role in understanding the boundary between algebraic structure and logical definability. Although a general characterization remains elusive, specific families of finite lattices provide valuable insight into the complexity of quasi-equational theories. In this work, we investigate the quasivariety generated by the so-called (3-3)-snake lattice, a finite structure known for its intricate congruence behavior. We demonstrate that the quasivariety determined by this lattice is neither finitely axiomatizable nor standard in the topological sense. This result contributes to the broader program of identifying finite lattices whose quasi-identities inherently resist finite axiomatizability and whose associated Boolean topological algebras fail to exhibit profiniteness. In addition to establishing the main theorems, we present a generalization that expands the scope of the construction, offering an illustrative example and a corollary that highlight the robustness of the method. The findings underscore the continued importance of studying finite lattices with atypical quasi-equational properties and enrich the existing landscape of nonstandard quasivarieties.

Keywords: lattice, modular lattice, quasivariety, splitting pair, semi-splitting pair, profiniteness, finite basis of quasi-identities, non-standard topological quasivariety.

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Introduction

The following two closely related and well-known problems arise in universal algebra and lattice theory.

The first problem is “Which finite lattices generate finitely based quasivarieties?”

The second one is “Which finite lattices generate standard topological quasivarieties?”

The first problem, often referred to as the finite axiomatizability problem, was first brought to light by V.A. Gorbunov and D.M. Smirnov [1] in 1979. The problem, which concerns the standardness of (quasi)varieties, was formally introduced in 2008 by D.M. Clark, B.A. Davey, M.G. Jackson, and J.G. Pitkethly [2].

Historically, R. McKenzie [3] proved in 1970 that a finite lattice generates a finitely based variety. However, in 1978, V.P. Belkin [4] provided a finite lattice that generates not finitely based quasivariety. Belkin’s example was later extended in 1984 by V.I. Tumanov [5], who identified a broad class of finite lattices that generate not finitely axiomatizable quasivarieties. In that same article he also proposed a conjecture asserting that a finite (modular) lattice has an infinite quasi-identity basis precisely when the quasivariety it generates is proper. This claim, however, does not hold universally. In 1989, W. Dziobiak [6] constructed a counterexample: a finite non-modular lattice whose generated

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quasivariety is proper yet admits a finite quasi-identity basis. The modular case, however, remains open.

This problem is also connected with the second problem on standardness of topological quasivarieties. In [2] was established, in particular, that the topological quasivariety generated by Belkin's lattice is non-standard. The fact that a finite lattice generates a standard topological variety was proved in [7] (with a shorter proof given in [8]). More recently, in [9] was shown that the Tumanov's lattices generate non-standard topological quasivarieties. Further results related to finite axiomatizability and standardness for quasivarieties of lattices were obtained in [10–12]. Sufficient conditions implying the existence of continuum many non-standard subquasivarieties, along with several applications of this result, were established in [13].

The primary focus of the present work is to establish that the quasivariety generated by (3-3)-snake lattice is neither standard nor finitely axiomatizable. We conclude the paper by presenting a generalization of this result, along with an illustrative example and a corresponding corollary.

1 Preliminaries and basic concepts

We start with a short review of the fundamental notions and facts about quasivarieties that will be required in the sequel. Further background on the concepts of universal algebra and lattice theory employed here can be found in [14, 15].

A class of algebras axiomatized by a set of quasi-identities is called a *quasivariety*. By *quasi-identity* we mean a sentence of the form

$$(\forall \bar{x})[p_1(\bar{x}) \approx q_1(\bar{x}) \wedge \dots \wedge p_n(\bar{x}) \approx q_n(\bar{x}) \rightarrow p(\bar{x}) \approx q(\bar{x})],$$

where $\bar{x} = (x_1, \dots, x_m)$, and $p, q, p_1, q_1, \dots, p_n, q_n$ are terms of the algebraic signature. By the Birkhoff-like theorem (see [15]), a class of algebras is a quasivariety if and only if it is closed under subalgebras, direct products, and ultraproducts.

A *variety* is a class of algebras axiomatized by a set of identities, where an *identity* is a sentence of the form $(\forall \bar{x})[s(\bar{x}) \approx t(\bar{x})]$ for some terms $s(\bar{x})$ and $t(\bar{x})$. By Birkhoff's theorem [15], a variety is exactly a quasivariety that is closed under homomorphisms.

Let $\mathbf{Q}(A)$ ($\mathbf{V}(A)$) be a quasivariety (variety) generated by an algebra A .

Let $\mathcal{K}$ be a class of algebras quasivariety. A congruence α on an algebra A is called a $\mathcal{K}$ -congruence if A/α belongs to $\mathcal{K}$. It is easy to see that $A \in \mathcal{K}$ if and only if the diagonal relation Δ_A on A is a $\mathcal{K}$ -congruence. The set $\text{Con}_{\mathcal{K}}A$ of all $\mathcal{K}$ -congruences of A forms an algebraic lattice with respect to inclusion $\subseteq$. By $\theta_{\mathcal{K}}(A)$ we denote the least congruence with property $A/\theta_{\mathcal{K}}(A) \in \mathcal{K}$. Let an algebra $A \in \mathcal{K}$ and $(a, b) \in A \times A$. The least $\mathcal{K}$ -congruence on algebra A containing the pair (a, b) is called a *principal $\mathcal{K}$ -congruence*, and denoted by $\theta_{\mathcal{K}}(a, b)$. If $\mathcal{K}$ is a variety, the congruence $\theta_{\mathcal{K}}(a, b)$ is the usual principal congruence that we denote $\theta(a, b)$.

For a quasivariety $\mathcal{K}$, an algebra $A \in \mathcal{K}$ is *subdirectly $\mathcal{K}$ -irreducible* if the least congruence Δ_A is completely meet-irreducible in $\text{Con}_{\mathcal{K}}A$. By the Birkhoff-like theorem for quasivarieties, every algebra of a quasivariety $\mathcal{K}$ is a subdirect product of subdirectly $\mathcal{K}$ -irreducible algebras [15]. It is known that for a finite algebra A , every subdirectly $\mathbf{Q}(A)$ -irreducible algebra is isomorphic to some subalgebra of A .

A one-element algebra is called *trivial*. A congruence θ on algebra A is trivial if it is the diagonal relation Δ_A on A or $A \times A$. A subalgebra B of algebra A is *proper* if it is not trivial and $B \neq A$.

A triple $\langle A, \sigma, \tau \rangle$ is called a *topological algebra* if $\langle A, \sigma \rangle$ is an algebra of the signature σ , and $\langle A, \tau \rangle$ is a topological space such that each operation belonging to σ is continuous under the topology τ .

A topological space is *Boolean* if it is compact, Hausdorff, and totally disconnected. A topological algebra $A_{\tau} = \langle A, \sigma, \tau \rangle$ is *Boolean* if its topological reduct $\langle A, \tau \rangle$ is Boolean.

For a finite algebra A equipped with the discrete topology, the class of topological algebras consisting of all topologically closed subalgebras of non-zero direct powers of A , endowed with the product topology is called a *topological quasivariety* $\mathbf{Q}_\tau(A)$. An algebra is called *profinite* if it is isomorphic to the inverse limit of finite algebras. An algebra is *profinite with respect to class of algebras* $\mathcal{K}$ if it is isomorphic to the inverse limit of finite algebras from $\mathcal{K}$. It is well-known that profinite algebras are equipped with Boolean topologies. A quasivariety $\mathcal{K}$ is *profinite* if every profinite algebra in $\mathcal{K}$ is profinite with respect to $\mathcal{K}$.

A quasivariety $\mathbf{Q}(A)$ is *standard* if every Boolean topological algebra with its algebraic reduct in $\mathbf{Q}(A)$ is profinite with respect to $\mathbf{Q}(A)$, that is, belongs to $\mathbf{Q}_\tau(A)$. In this case, we say that the algebra A generates a standard (topological) quasivariety.

2 (3-3)-snake lattices and their basic properties

The lattices M_3 , $M_{3,3}$ and M_{3-3} are illustrated in Figure 1.

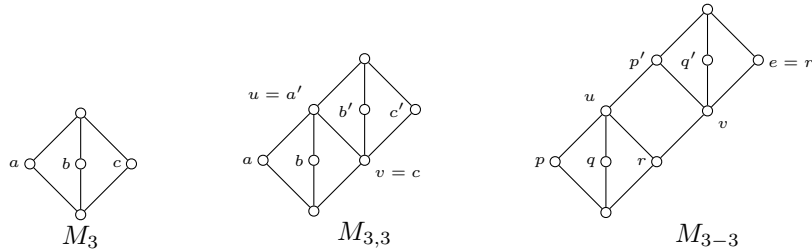


Figure 1. Lattices M_3 , $M_{3,3}$, M_{3-3}

Let L be a lattice and let $a \in L$. We write $a \downarrow$ ($a \uparrow$) for the principal ideal (filter) of a lattice L , that is, $a \downarrow = \{x \in L \mid x \leq a\}$ ($a \uparrow = \{x \in L \mid x \geq a\}$). A pair $(a, b) \in L \times L$ is said to be *splitting* (*semi-splitting*) whenever $L = a \downarrow \cup b \uparrow$ and $a \downarrow \cap b \uparrow = \emptyset$ ($L = a \downarrow \cup b \uparrow$ and $a \downarrow \cap b \uparrow \neq \emptyset$). For example, by notation of Figure 1, the pair (u, v) is a semi-splitting pair in $M_{3,3}$, and a splitting pair in M_{3-3} .

The following construction was suggested in [5].

Let L be a finite lattice with semi-splitting pair (u, v) . Let P be a finite modular lattice with coatom e . Define $L_{uv}(P, e)$ as the subset of $L \times P$ given by

$$L_{uv}(P, e) = \{(x, p), (y, e), (z, 1), (v, q) \mid x, y, z \in L, p, q \in P, \\ p < e, x \leq v, y \leq u, v \leq z, q \not\leq e, q < 1\}.$$

We show that $L_{uv}(P, e)$ is a sublattice of $L \times P$. One can see that $L_{uv}(P, e) = L_{xp} \cup L_{ye} \cup L_{z1} \cup L_{vq}$, where

$$\begin{aligned} L_{xp} &= \{(x, p) \mid p < e, x \leq v\}, \\ L_{ye} &= \{(y, e) \mid y \leq u\}, \\ L_{z1} &= \{(z, 1) \mid v \leq z\}, \\ L_{vq} &= \{(v, q) \mid q \not\leq e, q < 1\}. \end{aligned}$$

Suppose $(a, s), (b, t) \in L_{uv}(P, e)$. And let $(g, h) = (a, s) \wedge (b, t) = (a \wedge b, s \wedge t)$, and $(g', h') = (a, s) \vee (b, t) = (a \vee b, s \vee t)$. We show that $(g, h), (g', h') \in L_{uv}(P, e)$.

Let $(a, s) \in L_{xp}$. Since $a \leq v$, $s < p$, then $g = a \wedge b \leq v$ and $h = s \wedge t < p$. Hence $(g, h) \in L_{xp} \subseteq L_{uv}(P, e)$.

Let $(a, s) \in L_{ye}$. Since $a \leq u$, $s = e$, then $g = a \wedge b \leq u$ and $h = e \wedge t \leq e$. If $h = e$, then $(g, h) \in L_{ye}$. If $h < e$, then $(g, h) \in L_{vq}$. Hence $(g, h) \in L_{uv}(P, e)$.

Let $(a, s) \in L_{z1}$. Since $a \geq v$, $s = 1$, then $g = a \wedge b \geq v$ and $h = s \wedge t = t \leq 1$. If $h = 1$, then $(g, h) \in L_{z1}$. If $h < 1$, then $(b, t) \in L_{vq}$, whence $t \not\leq e$. Hence $(g, h) = (v, t) \in L_{uv}(P, e)$.

At last let $(a, s) \in L_{vq}$ and $(b, t) \in L_{vq}$. Thus $g = v$ and $h < 1$. If $h = e$, then $(g, h) \in L_{ye}$. If $h < e$, then $(g, h) \in L_{xp}$. If $h > e$, then $(g, h) \in L_{vq}$. Hence $(g, h) \in L_{uv}(P, e)$.

Therefore, $L_{uv}(P, e)$ is closed under intersection $\wedge$.

By the same arguments one can check that $L_{uv}(P, e)$ is closed under union $\vee$.

It means that $L_{uv}(P, e)$ forms a sublattice of $L \times P$. Moreover $L \leq_s L \times P$.

Let L and P be finite modular lattices. We say that a lattice K is obtained *by splitting* if $K \cong L_{uv}(P, e)$ for some semi-splitting pair (u, v) of L and a coatom e of P .

We provide an example of splitting.

Example 1. The lattice $L_{uv}(M_{3-3}, c')$ depicted in Figure 2 is obtained by splitting the lattice $M_{3,3}$ from Figure 1 by M_{3-3} . Note that the pair $((u, e), (v, 0))$ is a semi-splitting pair and the pair $((0, u), (v, v))$ is a splitting pair.

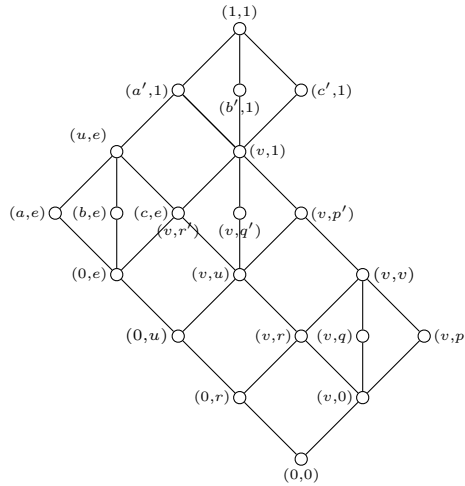


Figure 2. Lattices $L_{uv}(M_{3-3}, c')$

We note if P is a two-element lattice, then $L_{uv}(P, e)$ is a partial case of Day's doubling construction.

Let L be a modular lattice, and (u, v) is a splitting pair, and $\theta = \theta(u \wedge v, v)$. Since L is modular, then $(u/\theta, v/\theta)$ is a semi-splitting pair in L/θ . For a splitting pair (u, v) , we assume

$$L_{uv}(P, e) = L_{u/\theta v/\theta}(P, e).$$

Let L and P be finite modular lattices. We say that a lattice K is obtained *by gluing* the lattice P to L if $K \cong L_{uv}(P, e)$ for some splitting pair (u, v) of L and a coatom e of P .

Example 2. The lattice $L_{uv}(M_{3-3}, c')$ depicted in Figure 2 is obtained by gluing the lattice M_{3-3} to M_{3-3} .

We define the modular lattice S_n inductively by gluing M_{3-3} to S_{n-1} as follows:

1. $n = 0$. $S_0 \cong M_{3-3}$.
2. $n = 1$. $S_1 = L_{u_1 v_1}(M_{3-3}, e)$, where $L = S_0$ and $(u_1, v_1) = (a_1 \vee b_1, a_0 \wedge a'_1)$.
3. $n = 2$. $S_2 = L_{u_2 v_2}(M_{3-3}, e)$, where $L = S_1$ and $(u_2, v_2) = (a_2 \vee b_2, a_1 \wedge a'_2)$.
4. $n > 2$. $S_n = L_{u_n v_n}(M_{3-3}, e)$, where $L = S_{n-1}$ and $(u_n, v_n) = (a_n \vee b_n, a_{n-1} \wedge a'_n)$.

For $n = 0, 1, 2, 3$, the lattice S_n is depicted in Figures 3 and 4.

We refer to the lattice S_n as the $(3-3)$ -snake lattice. Based on its construction, it is clear that S_n is generated by its double irreducible elements, and the total number of these elements equals $s = 3n + 4$.

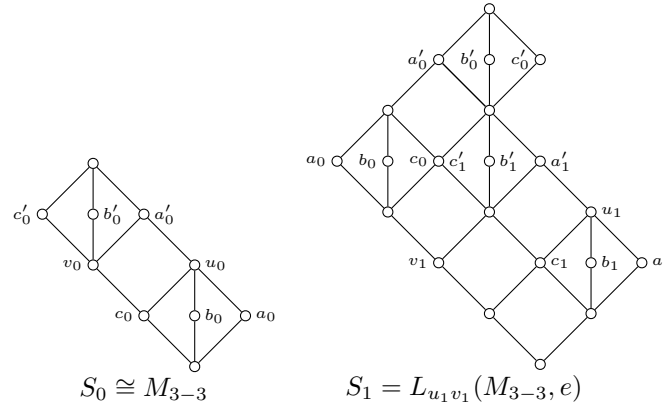


Figure 3. Lattices S_0, S_1

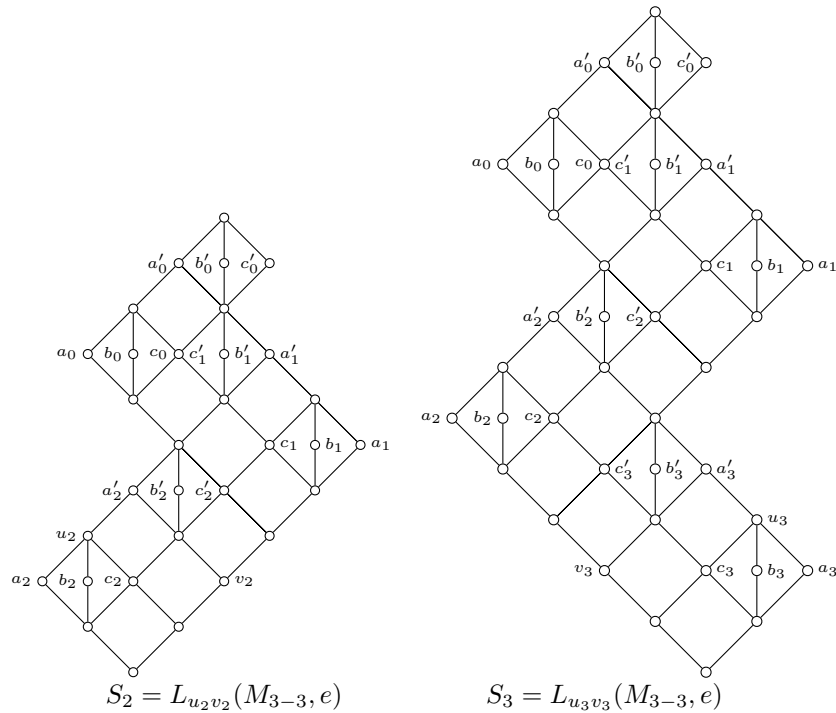


Figure 4. Lattices S_2, S_3

Lemma 1. Let $\mathcal{R}$ be a quasivariety not containing $M_{3,3}$. If $S_n \in \mathcal{R}$ for some $n \geq 0$, then $(a_0, b_0) \in \theta$ for every non-trivial $\mathcal{R}$ -congruence θ on S_n .

Proof. We proceed by induction on n .

For $n = 0$: $S_0 \cong M_{3-3}$. The lattice M_{3-3} has exactly two non-trivial congruences, let us call them α and β , whose quotients are isomorphic with $M_{3,3}$ and $\mathbf{2}$. Since $M_{3,3} \notin \mathcal{R}$, α cannot be an

$\mathcal{R}$ -congruence. Therefore, the lattice of $\mathcal{R}$ -congruences of M_{3-3} has exactly three congruences: the diagonal Δ , β , and the universal congruence ∇ . Any non-trivial $\mathcal{R}$ -congruence θ must therefore contain β , and since $(a_0, b_0) \in \beta$, the conclusion clearly holds.

For $n > 0$: Assume the statement holds for S_n . Let θ be a non-trivial $\mathcal{R}$ -congruence on S_{n+1} . Since θ is non-trivial, there exists a covering pair $x \prec y$ in S_{n+1} such that $(x, y) \in \theta$.

Recall that S_{n+1} is constructed by gluing a new M_{3-3} block, which we denote as $M^{(n+1)}$, to S_n . Based on the placement of x and y , we consider four exhaustive cases:

1. $x, y \in S_n$. In this case, both elements belong to the base lattice. The restriction $\theta|_{S_n}$ is a non-trivial $\mathcal{R}$ -congruence on S_n . By the induction hypothesis, we immediately obtain $(a_0, b_0) \in \theta|_{S_n} \subseteq \theta$.
2. $x, y \in M^{(n+1)}$. The restriction $\theta|_{M^{(n+1)}}$ is a non-trivial $\mathcal{R}$ -congruence on $M^{(n+1)} \cong M_{3-3}$. As shown in the base case, any non-trivial $\mathcal{R}$ -congruence on M_{3-3} must contain the congruence β , which collapses the elements of the gluing interval. Since the gluing interval belongs to S_n , θ inevitably identifies distinct elements within S_n . This makes the restriction $\theta|_{S_n}$ non-trivial. By the induction hypothesis, $(a_0, b_0) \in \theta$.
3. $y \in M^{(n+1)}$ and $x \notin M^{(n+1)}$. By taking the join with the zero element of S_n (denoted 0_{S_n}), the congruence yields $(x \vee 0_{S_n}, y \vee 0_{S_n}) \in \theta$. This operation projects the pair such that $y \vee 0_{S_n} = v_n \wedge u_n$ and $x \vee 0_{S_n} = 0_{S_n}$ where both resulting elements $0_{S_n}, v_n \wedge u_n$ lie strictly in S_n and are distinct. This implies that the restriction $\theta|_{S_n}$ is non-trivial. Thus, by the induction hypothesis, $(a_0, b_0) \in \theta$.
4. $x, y \notin M^{(n+1)}$. By taking the join with the zero element of M_{3-3} (denoted $0_{M_{3-3}}$), we get $(x \vee 0_{M_{3-3}}, y \vee 0_{M_{3-3}}) \in \theta$. These projected elements are distinct and both lie entirely within $M^{(n+1)}$. This means the congruence non-trivially restricts to $M^{(n+1)}$, which reduces our problem to Case 2. Hence, $(a_0, b_0) \in \theta$.

The case $y \notin S_n$ implies $x \notin S_n$. The case where $y \in S_n$ and $x \notin S_n$ implies $y = 0_{S_n}$ or $y \in M^{(n+1)}$. These both cases are covered by cases 2, 3 and 4. This strictly exhausts all possibilities for the covering pair $x \prec y$, completing the proof. $\square$

We construct a modular lattice K_m for S_n by induction on m .

1. $m = 0$. $K_0 = S_n$.
2. $m = 1$. K_1 is a modular lattice generated by $K_0 \cup \{a_{n+1}, b_{n+1}, c_{n+1}\}$ satisfying $\langle \{a_{n+1}, b_{n+1}, c_{n+1}\} \rangle \cong M_3$ and $c_{n+1} = (a_n \vee c_n) \wedge a_{n-1}$, $a_{n+1} \vee b_{n+1} = a'_n \wedge a_{n-1}$ and $a_{n+1} \wedge u, b_{n+1} \wedge u \notin K_0$ for any $u \in K_1$.
3. $m > 1$. K_m is a modular lattice generated by $K_{m-1} \cup \{a_{n+m}, b_{n+m}, c_{n+m}\}$ satisfying $\langle \{a_{n+m}, b_{n+m}, c_{n+m}\} \rangle \cong M_3$ and $c_{n+m} = (a_n \vee c_n) \wedge a_{n+m-1}$, $a_{n+m} \vee b_{n+m} = a_{n+m-1} \wedge c_{n+m-1}$ and $a_{n+m} \wedge u, b_{n+m} \wedge u \notin K_0$ for any $u \in K_{m-1}$.

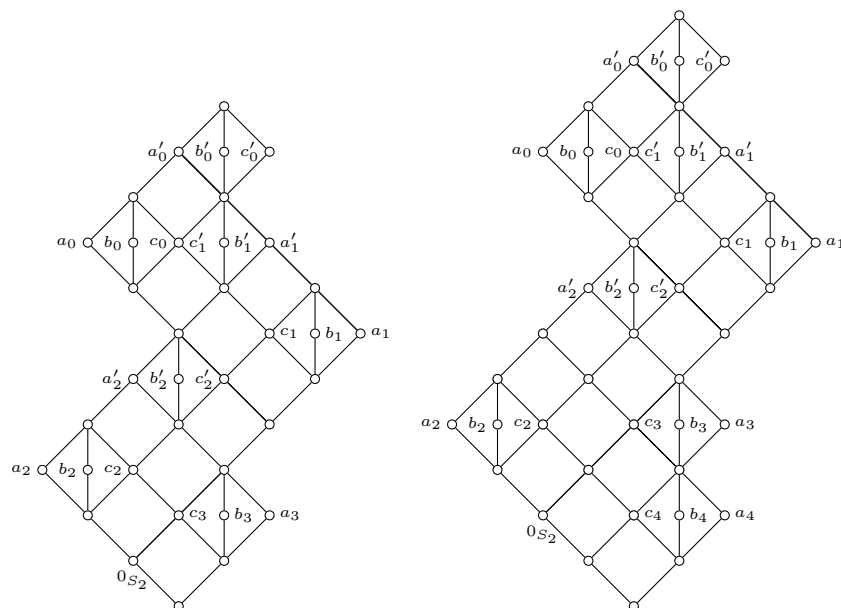
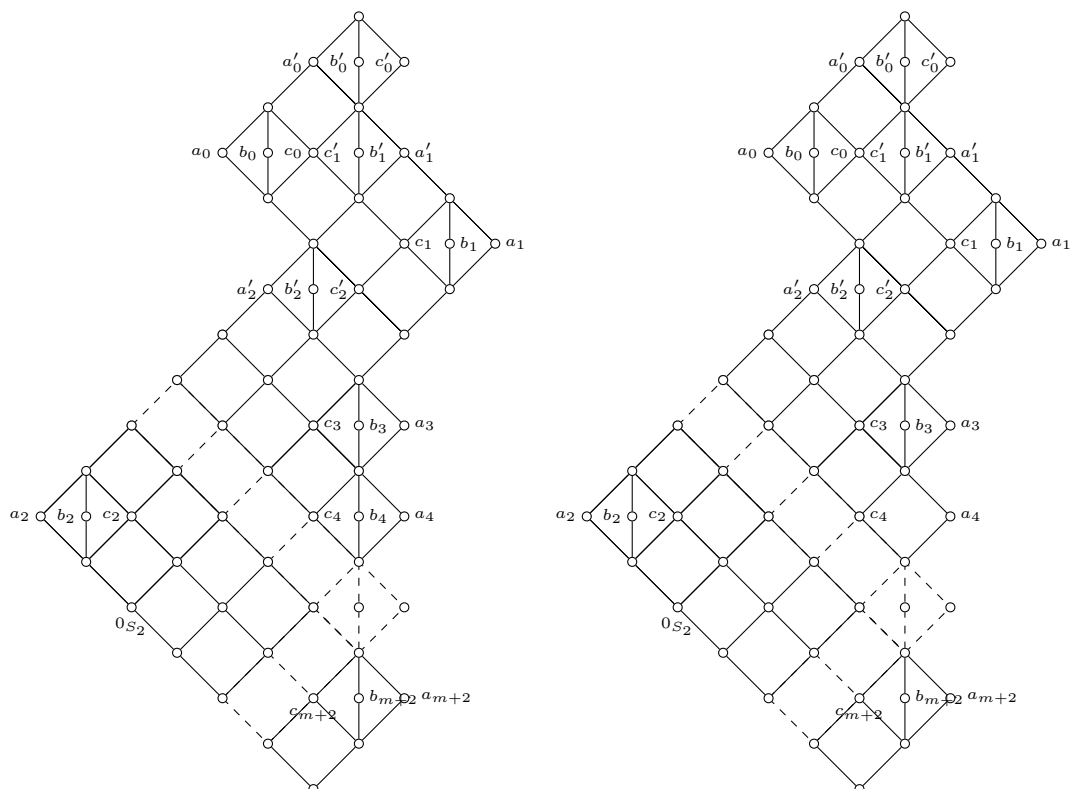
Equivalently, the lattice K_m may be described inductively by splitting the lattice K_{m-1} by M_3 as follows:

1. $m = 0$. $K_0 = S_n$.
2. $m = 1$. $K_1 = L_{u_1 v_1}(M_3, c)$ with $L = K_0$, where c is a coatom of M_3 and $(u_1, v_1) = (a_{n+1} \wedge c_{n+1}, a_n \vee c_n)$.
3. $m > 1$. $K_m = L_{u_m v_m}(M_3, c)$ with $L = K_{m-1}$ and $(u_m, v_m) = (a_{n+m} \wedge c_{n+m}, a_n \vee c_n)$.

For instance, the lattices K_1, K_2 for S_2 are depicted in Figure 5 and the lattice K_m for S_2 is depicted in Figure 6. We observe that (u_1, v_1) is a splitting pair in S_n , whereas (u_i, v_i) , $i > 1$ is a semi-splitting pair in K_{m-1} .

Lemma 2. Let $\mathcal{R}$ be the quasivariety generated by S_n . Then, for every $m \geq 1$, we have:

- i) $(a_0, b_0) \in \theta_{\mathcal{R}}(K_m)$ and $K_m \notin \mathcal{R}$;
- ii) Every subalgebra generated by m elements of K_m belongs to $\mathcal{R}$.


 Figure 5. Lattices K_1, K_2 for S_2

 Figure 6. Lattices $K_m, K_m \setminus \{b_4\}$ for S_2

Proof. i) We proceed by induction on m .

For $m = 1$: $K_1 = L_{u_1 v_1}(M_3, c)$ with $L = K_0$, where c is a coatom of M_3 and $(u_1, v_1) = (a_{n+1} \wedge$

$c_{n+1}, a_n \vee c_n$). Since K_1 is obtained by gluing M_3 , which we denote as $M^{(1)}$, to S_n , then $S_n \leq K_m$. Let θ be an $\mathcal{R}$ -congruence on K_1 and $(x, y) \in \theta$ with $x \prec y$.

Based on the placement of x and y , we consider two exhaustive cases:

1. If $x \in S_n$, then $y \in S_n$, and $(a_0, b_0) \in \theta$ by Lemma 1.
2. If $x \notin S_n$, then $x \in \{a_{n+1}, b_{n+1}, a_{n+1} \wedge b_{n+1}, 0_{K_1}\}$.

If $y \in M^{(1)}$, then θ collapses the elements of the gluing interval. If $y \notin M^{(1)}$, then θ collapses the elements 0_{S_n} and 0_{K_1} , hence two elements from $M^{(1)}$. It follows θ collapses the gluing interval. and we are in Case 1. Thus we get that $(a_0, b_0) \in \theta$ for any $\mathcal{R}$ -congruence θ .

Since $(a_0, b_0) \in \theta$ for every $\mathcal{R}$ -congruence on K_1 , then K_1 is subdirectly $\mathcal{R}$ -irreducible. Since every subdirectly $\mathcal{R}$ -irreducible lattice is a sublattice of S_n , it would follow $K_1 \leq S_n$. It's impossible because of the size of K_1 is strictly greater than the size of S_n . Therefore $K_1 \notin \mathcal{R}$.

For $m > 1$: We show that $(a_0, b_0) \in \theta$, where $\theta = \theta_{\mathcal{R}}(x, y)$ with $x \prec y$.

It is straightforward to verify that $\varphi : S_n \rightarrow K_m$ defined by $\varphi(x) = x$ for $x \in \{a_i, a'_i, b_i, b'_i, c_i, c'_i\}$, $i = 0, 1, \dots, n$, is an embedding.

Based on the placement of x and y relatively to S_n , we have the following four exhaustive cases:

1. Let $x, y \in S_n$. Then Lemma 1 yields $(a_0, b_0) \in \theta$.
2. Let $x \in S_n$ and $y \notin S_n$. Then $x = \{a_n \vee b_n, c_n, a_{n-1} \wedge c_n\}$ and $y = x \vee a_{n+m}$. It follows that $(a_{n+m}, a_{n+m} \wedge b_{n+m}) \in \theta$ as $x \wedge a_{n+m} = a_{n+m} \wedge b_{n+m}$ and $y \wedge a_{n+m} = a_{n+m}$. Hence, for $\theta' = \theta(x, y)$, we have $K_{m+1}/\theta' \cong K_m$ and $\theta_{\mathcal{R}}(x/\theta', y/\theta') = \theta_{\mathcal{R}}(\Delta) = \theta_{\mathcal{R}}(K_m)$, since $\theta' \subseteq \theta$. By induction $(a_0, b_0) \in \theta_{\mathcal{R}}(K_m)$. Therefore $(a_0, b_0) \in \theta$.
3. Let $x \notin S_n$ and $y \in S_n$. Then $y \in \{a'_n, a'_n \wedge b'_n, a'_n \wedge a_{n-1}, c_n \wedge a_{n-1}, a_n \wedge a_{n-1}\}$.
 - (a) If $y = a'_n$ or $y = a'_n \wedge b'_n$, then $x = a_n \vee c_{n+1}$ or $x = c_n \vee c_{n+1}$ respectively. It follows that $(a_{n+1}, a_{n+1} \wedge b_{n+1}) \in \theta$ as $x \wedge a_{n+1} = a_{n+1} \wedge b_{n+1}$ and $y \wedge a_{n+1} = a_{n+1}$.
 - (b) If $y = a'_n \wedge a_{n-1} = a_{n+1} \vee b_{n+1}$, then $x \in \{a_{n+1}, b_{n+1}, c_{n+1}\}$. It follows that $(a_{n+1}, a_{n+1} \wedge b_{n+1}) \in \theta$ as M_3 is a simple lattice.
 - (c) If $y \in \{c_n \wedge a_{n-1}, a_n \wedge a_{n-1}\}$, then $x = y \wedge a_{n+1}$. It follows that $(a_{n+1}, a_{n+1} \vee b_{n+1}) \in \theta$ as $x \vee a_{n+1} = a_{n+1}$ and $y \vee a_{n+1} = a_{n+1} \vee b_{n+1}$.
 Hence $K_{m+1}/\theta' \cong K_m$ and $\theta_{\mathcal{R}}(x/\theta', y/\theta') = \theta_{\mathcal{R}}(\Delta) = \theta_{\mathcal{R}}(K_m)$. By induction $(a_0, b_0) \in \theta_{\mathcal{R}}(K_m)$. Therefore $(a_0, b_0) \in \theta$.
4. Let $x, y \notin S_n$. Since K_{i+1} is obtained by splitting K_i by M_3 , we denote this lattice M_3 by $M^{(i)}$. There are two options. The first one – θ gluings two distinct elements in S_n . Then by Lemma 1 θ collapses (a_0, b_0) . The second one – θ gluings two distinct elements a_{n+i}, b_{n+i} in $M^{(i)}$ for some $i > 0$, then for $\theta' = \theta(a_{n+i}, b_{n+i})$ we have $K_{m+1}/\theta' \cong K_m$ and $\theta_{\mathcal{R}}(a_{n+i}/\theta', b_{n+i}/\theta') = \theta_{\mathcal{R}}(\Delta) = \theta_{\mathcal{R}}(K_m)$. By induction $(a_0, b_0) \in \theta_{\mathcal{R}}(K_m)$. Therefore $(a_0, b_0) \in \theta$.

Since $(a_0, b_0) \in \theta$ for every $\mathcal{R}$ -congruence on K_{m+1} , then K_{m+1} is subdirectly $\mathcal{R}$ -irreducible. Since each subdirectly $\mathcal{R}$ -irreducible lattice is a sublattice of S_n , this would imply $K_{m+1} \leq S_n$. However, it's impossible because of the size of K_{m+1} is strictly greater than the size of S_n . Therefore $K_{m+1} \notin \mathcal{R}$.

ii) The set of generators of the lattice K_m consists of

$$\{a_i, b_i, c_i, a'_i, b'_i, c'_i \mid 0 \leq i \leq n\} \cup \{a_i, b_i, c_i \mid n < i \leq n+m\}.$$

Here $\{a_i, b_i, c_i, a'_i, b'_i, c'_i \mid 0 \leq i \leq n\}$ generates S_n and $\{a_i, b_i, c_i\}$ generates M_3 for each $n < i \leq n+m$.

Let L_m be a sublattice of K_m generated by m elements. Because a_{n+i}, b_{n+i} are double irreducible elements, one necessarily has $L_m \leq K_m \setminus \{a_{n+i}\}$ or $L_m \leq K_m \setminus \{b_{n+i}\}$ for some index $0 < i \leq m$.

We show that $K_m \setminus \{a_{n+i}\}$ is a sublattice of $S_n \times M_3^{m-1}$.

Let α be a congruence on K_m generated by $\bigcup_{j \neq i} M^{(j)} \cup \{(0_{M^{(i)}}, c_{n+i})\}$.

Then $(K_m \setminus \{a_{n+i}\})/\alpha \cong S_n$.

Let β be a congruence on K_m generated by gluing all elements in S_n but asd

$$\beta = \theta\left(\bigcup_{i \leq n} \{(a_i, b_i) \mid i \leq n\}\right).$$

Then $(K_m \setminus \{a_i\})/\beta \leq \prod_{0 < j \leq m} M^{(j)}$.

We show that $\alpha \cap \beta = \Delta$. Let $x \prec y$ and $(x, y) \in \alpha \cap \beta$. By $(x, y) \in \alpha$, we get $x = a_n \vee z$ and $y = 0_{S_n} \vee z$ or $x = a_{n-1} \wedge z$ and $y = (a_{n-1} \vee b_{n-1}) \wedge z$ for some z on $K_m \setminus \{a_i\}$.

By $x \in \beta$, we get $x = 0_{k+n} \vee z'$, $y = a_{k+n} \vee z'$.

So $a_n \vee z = 0_{k+n} \vee z'$ and $0_{S_n} \vee z = a_{k+n} \vee z'$. One can see that it's impossible.

Thus $\alpha \cap \beta = \Delta$ and $K_m \setminus \{a_i\} \leq_s S_n \times M_3^{m-1}$. So $K_m \setminus \{a_i\} \in \mathcal{R}$. Since $L_m \leq K_m \setminus \{a_i\}$, then $L_m \in \mathcal{R}$. □

3 Main result

The following two theorems are the main results of this paper.

Theorem 1. A quasivariety generated by finite (3-3)-snake lattice is not finitely axiomatizable.

Proof. To establish non-finite axiomatizability, we use the following folklore lemma (see [15]):

A locally finite quasivariety $\mathcal{K}$ is not finitely axiomatizable whenever, for each positive integer n there exists a finite algebra X_n such that $X_n \notin \mathcal{K}$, and every subalgebra generated by n elements of X_n belongs to $\mathcal{K}$.

Let $\mathcal{R} = \mathbf{Q}(S_n)$. We show that for $\mathcal{R}$ the lattice $X_m = K_{m+n}$, $m > 0$, satisfies the condition of the above mentioned folklore lemma. Indeed by Lemma 2(i) $X_m \notin \mathcal{R}$ for all $m \geq 1$, by lemma 2(ii) every subalgebra generated by m elements of X_m belongs to $\mathcal{R}$. Consequently, the quasivariety $\mathcal{R}$ generated by finite (3-3)-snake lattice is not finitely axiomatizable. □

To demonstrate that the topological quasivariety generated by S_n is non-standard, we use the lemma below, derived from [2, Lemma 3.3].

Lemma 3. Let $\mathcal{K}$ be a quasivariety, and let $X = \varprojlim \{X_n \mid n \in \mathbb{N}\}$ be a surjective inverse limit of finite algebras. Assume that $X \in \mathcal{K}$ and there exist $a, b \in X$ with $a \neq b$ and $\varphi(a) = \varphi(b)$ for any homomorphism $\varphi : X \rightarrow M$, where $M \in \mathcal{K}$ and M is finite. Then $\mathcal{K}$ is not standard.

Theorem 2. The topological quasivariety generated by finite (3-3)-snake lattice is not standard.

Proof. Let $\mathcal{R} = \mathbf{Q}(S_n)$ for some (3-3)-snake lattice. Let $\varphi_{n,n-1}$ be a homomorphism from K_n to K_{n-1} such that $\ker \varphi_{n,n-1} = \theta(a_n, b_n)$, and let $\varphi_{n,n}$ be an identity map on K_n for all $n > 1$. For $m < n$, let

$$\varphi_{n,m} = \varphi_{m+1,m} \circ \cdots \circ \varphi_{n,n-1}.$$

Note that $\{K_n; \varphi_{n,m}, \mathbb{N}\}$ forms an inverse family where $\mathbb{N}$ is the linearly ordered set of positive integers. We denote $K = \varprojlim \{K_n \mid n \in \mathbb{N}\}$.

We first establish that $K \in \mathcal{R}$. Our reasoning follows the approach of [16] (cf. [11]). Consider a quasi-identity ξ of the form

$$\&_{i \leq r} p_i(x_0, \dots, x_{n-1}) \approx q_i(x_0, \dots, x_{n-1}) \rightarrow p(x_0, \dots, x_{n-1}) \approx q(x_0, \dots, x_{n-1}).$$

Suppose that ξ is valid on $\mathcal{R}$ and $K \models p_i(d_0, \dots, d_{n-1}) = q_i(d_0, \dots, d_{n-1})$ for all $i \leq r$ and some $d_0, \dots, d_{n-1} \in K$. By definition of the inverse limit, $K \leq_s \prod_{n \in \mathbb{N}} K_n$. Hence

$$K_s \models p_i(d_0(s), \dots, d_{n-1}(s)) = q_i(d_0(s), \dots, d_{n-1}(s)) \quad \text{for all } s > 0, i \leq r.$$

By Lemma 2(ii), every subalgebra generated by m elements of K_m belongs to $\mathcal{R}$ for all $s > n$. Therefore, ξ is true in K_s . Consequently,

$$K_s \models p(d_0(s), \dots, d_{n-1}(s)) = q(d_0(s), \dots, d_{n-1}(s)).$$

Since $d_i(m) = \varphi_{s,m}(d_i(s))$ for all $0 \leq i \leq n$ and $m < s$, it follows that

$$K_m \models p(d_0(m), \dots, d_{n-1}(m)) = q(d_0(m), \dots, d_{n-1}(m)) \quad \text{for all } m < s.$$

Hence, $K \models p(d_0, \dots, d_{n-1}) = q(d_0, \dots, d_{n-1})$. Thus $K \models \xi$, for every ξ that is valid on $\mathcal{R}$. Consequently, $K \in \mathcal{R}$.

From the definition of $\varphi_{n,n-1}$ it follows that $\varphi_{n,m}(a_0) = a_0$ and $\varphi_{n,m}(b_0) = b_0$. By the definition of inverse limit,

$$a = (a_0, \dots, a_0, \dots), \quad b = (b_0, \dots, b_0, \dots) \in K.$$

Let $\varphi : K \rightarrow M$ be a homomorphism, $M \in \mathcal{R}$ and M finite. By the universal property of inverse limit, there exists $n > 2$ and homomorphism $\psi_M : K_n \rightarrow M$ such that $\varphi = \varphi_n \circ \psi_M$ for some surjective homomorphism $\varphi_n : K \rightarrow K_n$.

Since $M \in \mathcal{R}$, we have $\ker \psi_M$ is an $\mathcal{R}$ -congruence on K_n .

By Lemma 2(i), we have $(a_0, b_0) \in \theta_{\mathcal{R}}(K_n)$. Since $\theta_{\mathcal{R}}(K_n) \in \ker \psi_M$, it follows that $\psi_M(a) = \psi_M(b)$, and hence $\varphi(a) = \varphi(b)$.

Applying Lemma 3, we conclude that $\mathcal{R}$ is not standard. $\square$

4 Generalization, examples, and corollary

By $h(L)$ we denote the height of a lattice L , that is, the maximum length among all chains in L .

Theorem 3. Let L be a finite lattice with $h(L) = k$. If there is a (3-3)-snake lattice $S_n \leq L$ and K_1 for S_n such that $h(K_1) \geq h(L)$ and $K_1 \notin \mathbf{Q}(L)$, then $\mathbf{Q}(L)$ is not finitely axiomatizable as well as not standard.

Sketch of the proof. It is straightforward to check that the lattices K_m from Lemma 2 satisfy the following conditions:

- i) $K_m \notin \mathbf{Q}(L)$ and $(a_0, b_0) \in \theta_{\mathbf{Q}(L)}(K_m)$;
- ii) Every subalgebra generated by m elements of K_m belongs to $\mathbf{Q}(S_n) \subseteq \mathbf{Q}(L)$.

Using arguments analogous to those in Theorems 1 and 2, the required statement follows.

Example 3. Let L be the lattice shown in Figure 7. It is clear that S_1 is a proper sublattice of L and $M_{3,3} \not\leq L$. Moreover, K_1 for S_1 does not belong to $\mathbf{Q}(L)$. Therefore, by Theorem 3, $\mathbf{Q}(L)$ is neither finitely axiomatizable nor standard. Note also that $\mathbf{Q}(S_1) \subset \mathbf{Q}(L)$, and, in addition, $\mathbf{V}(S_1) \subset \mathbf{V}(L)$.

The reader can construct many other examples that satisfy the conditions of Theorems 1 and 2.

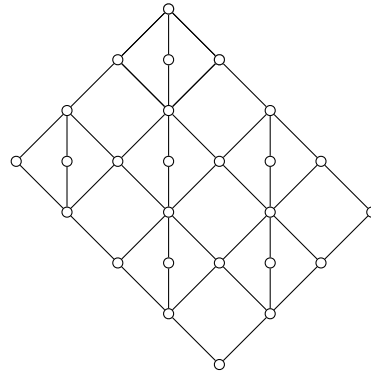
A direct corollary from Theorem 1 is V. P. Belkin's result in [4]:

Corollary 1. [4] A quasivariety generated by M_{3-3} is not finitely axiomatizable.

The reader can easily construct $(m-n)$ -snake lattices and adapt the proofs of the theorems to such lattices.

Conclusion

The results of this work contribute to the study of finite lattices whose quasi-identities resist finite axiomatizability and whose associated Boolean topological algebras do not exhibit profiniteness. We established the principal results, provided a generalization of the construction, and illustrated its applicability through an example and a corollary. These findings emphasize the ongoing significance

Figure 7. Lattice L

of investigating finite lattices with atypical quasi-equational behavior and show how such examples continue to shape and refine our understanding of nonstandard quasivarieties. The results presented here reinforce the view that even comparatively small finite lattices may encode complex logical properties, motivating further research into the connection between algebraic structure, axiomatizability, and topological properties.

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Author contributions

All authors contributed equally to this work.

Conflict of interest

The authors declare no conflict of interest.

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A note on the space-dependent source identification problem for the two-dimensional neutron transport equation

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The space-dependent source identification problem for the two-dimensional neutron transport equation with spatially variable coefficients and periodic boundary conditions is investigated. The neutron transport equation is a fundamental model for neutron propagation in a medium, accounting for both streaming and collision events, and is critical for nuclear reactor safety analysis. While the source identification problem has been extensively studied for parabolic, elliptic, and hyperbolic equations, recent research has extended these methods to transport equations as well. The investigation begins with the homogeneous problem with constant coefficients, where an explicit solution is derived using the method of characteristics. This solution serves as the basis for studying the variable-coefficient case. A function is introduced to quantify the deviation of variable coefficients from constant values. Under suitable assumptions, a stability estimate is established, showing that the solution operator is exponentially bounded in time. Building on these results, this study investigates the inverse problem of identifying space-dependent source terms. By analyzing the equation structure and applying the stability estimate for the homogeneous problem, the study derives conditions for stable recovery of source terms from measured data. We introduce a time interval parameter and a contraction condition to make sure the solution is unique and depends continuously on the data. The main result provides stability estimates, showing that small changes in measured data lead to small changes in the identified sources. These theoretical contributions enhance the understanding of inverse problems for transport equations and have potential applications in reactor physics and related fields.

Keywords: source identification problem, two-dimensional neutron transport equation, spatially variable coefficients, stability inequality, periodic boundary conditions, method of characteristics, inverse problem, space-dependent source term, homogeneous transport problem.

2020 Mathematics Subject Classification: 35R30, 82D75, 35Q49, 35B35, 65M25.

Introduction

The neutron transport equation (NTE) describes the distribution of neutrons as a function of spatial position, angular direction, energy, and time, accounting for both streaming and collision processes. Fundamental studies have provided comprehensive analyses of the NTE's computational methods, numerical techniques, and mathematical theory [1–3], while recent research has addressed the latest theoretical and computational developments [4]. The theory of linear differential equations in Banach spaces [5] and the method of positive operators [6] provide key tools for such analyses. The time-dependent NTE is fundamental to safety analysis for nuclear reactor systems.

Inverse problems, notably those involving the identification of unknown source terms, are central to numerous scientific disciplines. The literature presents extensive conceptual models and applications [7–9], including recent progress in fractional Schrödinger-type equations [10]. Parabolic equations

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have been investigated numerically for reconstructing distributed sources [11] and for using stability estimates in delay problems [12]. Further studies address well-posedness [13] and source identification in reverse problems [14]. For elliptic partial differential equations, research has focused on stable difference schemes for mixed boundary conditions [15] and source identification in elliptic-telegraph equations [16]. Investigations of hyperbolic equations include identification in hyperbolic-parabolic systems [17], source identification for hyperbolic differential equations [18], and combined parabolic-hyperbolic formulations [19]. Recent studies have expanded the scope of the neutron transport equation (NTE) by analyzing fractional spaces generated by neutron transport operators [20], examining their structure in two-dimensional contexts [21], and addressing source identification problems for neutron transport equations [22]. Related research explores nonlocal problems and operator positivity, with particular attention to the positivity of differential operators generated by hyperbolic systems [23]. Building on these foundations, the present study investigates the simultaneous determination of two space-dependent source terms in the NTE. This problem also arises in heat conduction, where similar simultaneous identification challenges have been addressed for two-dimensional heat conduction equations [24].

This paper presents stability estimates for the space-dependent source identification problem in the two-dimensional neutron transport equation with spatially variable coefficients and periodic boundary conditions

$$\left\{ \begin{array}{l} \frac{\partial u(t, x_1, x_2)}{\partial t} - a(x_1, x_2) \frac{\partial u(t, x_1, x_2)}{\partial x_1} - b(x_1, x_2) \frac{\partial u(t, x_1, x_2)}{\partial x_2} + \delta u(t, x_1, x_2) \\ = p(x_1, x_2) + f(t, x_1, x_2), \\ t \in (0, T), \quad x_1, x_2 \in (0, 1), \\ u(t, 0, x_2) = u(t, 1, x_2), \quad t \in [0, T], \quad x_2 \in [0, 1], \\ u(t, x_1, 0) = u(t, x_1, 1), \quad t \in [0, T], \quad x_1 \in [0, 1], \\ u(T_1, x_1, x_2) = u(0, x_1, x_2) + \psi_1(x_1, x_2), \quad T_1 \in (0, T), \\ u(T_2, x_1, x_2) = \psi_2(x_1, x_2), \quad x_1, x_2 \in [0, 1], \quad T_1 < T_2, \quad T_2 \in (0, T]. \end{array} \right. \quad (1)$$

Here, $u(t, x_1, x_2)$ and $p(x_1, x_2)$ denote unknown functions, whereas $f(t, x_1, x_2)$, $\psi_1(x_1, x_2)$, and $\psi_2(x_1, x_2)$ are sufficiently smooth functions. All compatibility conditions are assumed to be satisfied. $a(x_1, x_2)$ and $b(x_1, x_2)$ specify the directional components of the transport velocity. The parameter $\delta > 0$ is an absorption coefficient that models a sink or loss term. The problem is defined on the space-time domain $(t, x_1, x_2) \in (0, T) \times (0, 1)^2$, with periodic boundary conditions in both spatial directions.

1 Preliminaries

The Banach space $C([0, T], E) = C(E)$ is introduced to support the results. For all abstract continuous functions $\phi(t)$ with the values in E , which has the maximum norm, maps the interval $[0, T]$ into the set E ,

$$\|\phi\|_{C(E)} = \max_{0 \leq t \leq T} \|\phi(t)\|_E.$$

Let $\Omega = [0, 1] \times [0, 1]$ be the spatial domain, and define $E = C(\Omega)$ as the space of continuous functions $\psi(x_1, x_2)$ defined on Ω equipped with the norm

$$\|\psi\|_{C(\Omega)} = \max_{(x_1, x_2) \in \Omega} |\psi(x_1, x_2)|,$$

and let $C^{(1)}(\Omega)$ denote the space of continuously differentiable functions $\psi(x_1, x_2)$ on Ω with the norm

$$\|\psi\|_{C^{(1)}(\Omega)} = \|\psi\|_{C(\Omega)} + \max_{(x_1, x_2) \in \Omega} |\psi_{x_1}(x_1, x_2)| + \max_{(x_1, x_2) \in \Omega} |\psi_{x_2}(x_1, x_2)|.$$

Positive definite operator $A^{(x_1, x_2)}$ defined by the formula

$$A^{(x_1, x_2)}v = -\left(a(x_1, x_2)\frac{\partial v(x_1, x_2)}{\partial x_1} + b(x_1, x_2)\frac{\partial v(x_1, x_2)}{\partial x_2} - \delta v(x_1, x_2)\right) \quad (2)$$

which acts on the domain

$$D(A^{(x_1, x_2)}) = \{v \in C^{(1)}(\Omega) : v(0, x_2) = v(1, x_2), v(x_1, 0) = v(x_1, 1), \forall x_1, x_2 \in [0, 1]\}.$$

The following section presents a fundamental property of the homogeneous neutron transport equation, which is essential for stability analysis. It can be established that $v(t, x_1, x_2)$ is the solution of the following problem

$$\begin{cases} \frac{\partial v(t, x_1, x_2)}{\partial t} - a(x_1, x_2)\frac{\partial v(t, x_1, x_2)}{\partial x_1} - b(x_1, x_2)\frac{\partial v(t, x_1, x_2)}{\partial x_2} + \delta v(t, x_1, x_2) = 0, \\ t \in (0, T), x_1, x_2 \in [0, 1], \\ v(0, x_1, x_2) = \phi(x_1, x_2), x_1, x_2 \in [0, 1], \\ v(t, 0, x_2) = v(t, 1, x_2), t \in [0, T], x_2 \in [0, 1], \\ v(t, x_1, 0) = v(t, x_1, 1), t \in [0, T], x_1 \in [0, 1]. \end{cases} \quad (3)$$

Here, $v(t, x_1, x_2)$ is the unknown function, $\phi(x_1, x_2)$, $a(x_1, x_2)$, and $b(x_1, x_2)$ are periodic and smooth functions. $\delta > 0$ is a constant. By applying the method of characteristics to equation (3), one can demonstrate that for constant coefficients $a_0 = a(x_0, y_0)$ and $b_0 = b(x_0, y_0)$ with $\delta > 0$, equation (3) has the unique solution

$$v(t, x_1, x_2) = e^{-\delta t} \phi(x_1 + a_0 t, x_2 + b_0 t). \quad (4)$$

We can analyze this solution by examining the semigroup generated by the differential operator $A^{(x_0, y_0)}$,

$$e^{-tA^{(x_0, y_0)}} \phi(x_1, x_2) = e^{-\delta t} \phi(x_1 + a_0 t, x_2 + b_0 t). \quad (5)$$

Taking the maximum norm on $E = C(\Omega)$ with $\Omega = [0, 1]^2$ and from (4), we obtain the estimate

$$\|e^{-tA^{(x_0, y_0)}}\|_{E \rightarrow E} \leq e^{-\delta t}.$$

Consequently, the operator $I - e^{-TA^{(x_0, y_0)}}$ is invertible and its inverse obeys

$$\|(I - e^{-TA^{(x_0, y_0)}})^{-1}\|_{E \rightarrow E} \leq \frac{1}{1 - e^{-T\delta}}.$$

2 Stability analysis

Theorem 1. Let $\delta > 0$, and let $a(x_1, x_2)$ and $b(x_1, x_2)$ be smooth periodic functions on Ω . Set $a_0 = a(x_0, y_0) > 0$ and $b_0 = b(x_0, y_0) > 0$, and define

$$\alpha(x_1, x_2) = \frac{a(x_1, x_2) - a_0}{a_0} + \frac{b(x_1, x_2) - b_0}{b_0}. \quad (6)$$

Assume that

$$1 + \alpha(x_1, x_2) \neq 0, \quad (x_1, x_2) \in \Omega.$$

Introduce the constants

$$M_1 = \max_{(x_1, x_2) \in \Omega} \frac{1}{|1 + \alpha(x_1, x_2)|}, \quad M_2 = \max_{(x_1, x_2) \in \Omega} |1 + \alpha(x_1, x_2)|,$$

and

$$L = \max_{\substack{(x_1, x_2) \in \Omega \\ 0 \leq s < t \leq T}} \left| \delta \alpha(x_1 + a_0(t-s), x_2 + b_0(t-s)) + \frac{d}{ds} \alpha(x_1 + a_0(t-s), x_2 + b_0(t-s)) \right|.$$

Then, for $0 \leq t \leq T$, the semigroup $e^{-tA^{(x_1, x_2)}}$ satisfies the estimate

$$\left\| e^{-tA^{(x_1, x_2)}} \right\|_{C(\Omega) \rightarrow C(\Omega)} \leq M e^{-\delta t}, \quad (7)$$

where $M = M_1 M_2 e^{M_1 L T}$.

Proof. Now, we investigate $v(t, x_1, x_2)$, defined by equation (3) with $a_0, b_0, \delta > 0$, and periodic boundary conditions. We have that

$$\frac{\partial v(t, x_1, x_2)}{\partial t} = a_0 \frac{\partial v(t, x_1, x_2)}{\partial x_1} + b_0 \frac{\partial v(t, x_1, x_2)}{\partial x_2} - \delta v(t, x_1, x_2) + R(t, x_1, x_2), \quad (8)$$

where

$$R(t, x_1, x_2) = (a(x_1, x_2) - a_0) \frac{\partial v(t, x_1, x_2)}{\partial x_1} + (b(x_1, x_2) - b_0) \frac{\partial v(t, x_1, x_2)}{\partial x_2}. \quad (9)$$

We can rewrite (8) in abstract form

$$\frac{dv(t)}{dt} = -A^{(x_0, y_0)} v(t) + R(t), \quad v(0) = \phi, \quad (10)$$

where $R(t) = R(t, x_1, x_2)$. Therefore, we have the following formula (see [25])

$$v(t) = e^{-tA^{(x_0, y_0)}} \phi + \int_0^t e^{-(t-s)A^{(x_0, y_0)}} R(s) ds \quad (11)$$

for the solution of (10). Substituting (5) and (9) into (11), we obtain

$$\begin{aligned} v(t, x_1, x_2) &= e^{-\delta t} \phi(x_1 + a_0 t, x_2 + b_0 t) \\ &+ \int_0^t e^{-\delta(t-s)} R(s, x_1 + a_0(t-s), x_2 + b_0(t-s)) ds = G_0(t, x_1, x_2) + G_1(t, x_1, x_2), \end{aligned} \quad (12)$$

where

$$G_0(t, x_1, x_2) = e^{-\delta t} \phi(x_1 + a_0 t, x_2 + b_0 t),$$

$$\begin{aligned} G_1(t, x_1, x_2) &= \int_0^t e^{-\delta(t-s)} \left[(a(x_1 + a_0(t-s), x_2 + b_0(t-s)) - a_0) \frac{\partial v(s, x_1 + a_0(t-s), x_2 + b_0(t-s))}{\partial(x_1 + a_0(t-s))} \right. \\ &\quad \left. + (b(x_1 + a_0(t-s), x_2 + b_0(t-s)) - b_0) \frac{\partial v(s, x_1 + a_0(t-s), x_2 + b_0(t-s))}{\partial(x_2 + b_0(t-s))} \right] ds. \end{aligned}$$

Applying integration by parts to $G_1(t, x_1, x_2)$, we get

$$\begin{aligned}
 G_1(t, x_1, x_2) = & - \left\{ \frac{a(x_1, x_2) - a_0}{a_0} + \frac{b(x_1, x_2) - b_0}{b_0} \right\} v(t, x_1, x_2) \\
 & + \left\{ \frac{a(x_1 + a_0 t, x_2 + b_0 t) - a_0}{a_0} + \frac{b(x_1 + a_0 t, x_2 + b_0 t) - b_0}{b_0} \right\} e^{-\delta t} v(0, x_1 + a_0 t, x_2 + b_0 t) \\
 & + \delta \int_0^t \left\{ \frac{a(x_1 + a_0(t-s), x_2 + b_0(t-s)) - a_0}{a_0} + \frac{b(x_1 + a_0(t-s), x_2 + b_0(t-s)) - b_0}{b_0} \right\} \\
 & \quad \times e^{-\delta(t-s)} v(s, x_1 + a_0(t-s), x_2 + b_0(t-s)) ds \\
 & + \int_0^t \frac{d}{ds} \left\{ \frac{a(x_1 + a_0(t-s), x_2 + b_0(t-s)) - a_0}{a_0} + \frac{b(x_1 + a_0(t-s), x_2 + b_0(t-s)) - b_0}{b_0} \right\} \\
 & \quad \times e^{-\delta(t-s)} v(s, x_1 + a_0(t-s), x_2 + b_0(t-s)) ds.
 \end{aligned}$$

Multiplying (12) by the factor $e^{\delta t}$ gives

$$\begin{aligned}
 e^{\delta t} v(t, x_1, x_2) = & \phi(x_1 + a_0 t, x_2 + b_0 t) \\
 & - \left\{ \frac{a(x_1, x_2) - a_0}{a_0} + \frac{b(x_1, x_2) - b_0}{b_0} \right\} e^{\delta t} v(t, x_1, x_2) \\
 & + \left\{ \frac{a(x_1 + a_0 t, x_2 + b_0 t) - a_0}{a_0} + \frac{b(x_1 + a_0 t, x_2 + b_0 t) - b_0}{b_0} \right\} \phi(x_1 + a_0 t, x_2 + b_0 t) \\
 & + \delta \int_0^t e^{\delta s} \left\{ \frac{a(x_1 + a_0(t-s), x_2 + b_0(t-s)) - a_0}{a_0} + \frac{b(x_1 + a_0(t-s), x_2 + b_0(t-s)) - b_0}{b_0} \right\} \\
 & \quad \times v(s, x_1 + a_0(t-s), x_2 + b_0(t-s)) ds \\
 & + \int_0^t e^{\delta s} \frac{d}{ds} \left\{ \frac{a(x_1 + a_0(t-s), x_2 + b_0(t-s)) - a_0}{a_0} + \frac{b(x_1 + a_0(t-s), x_2 + b_0(t-s)) - b_0}{b_0} \right\} \\
 & \quad \times v(s, x_1 + a_0(t-s), x_2 + b_0(t-s)) ds.
 \end{aligned}$$

Applying the substitution

$$z(t, x_1, x_2) = e^{\delta t} v(t, x_1, x_2), \quad (13)$$

and using condition (6), we obtain

$$\begin{aligned}
 z(t, x_1, x_2) = & \phi(x_1 + a_0 t, x_2 + b_0 t) - \alpha(x_1, x_2) z(t, x_1, x_2) \\
 & + \alpha(x_1 + a_0 t, x_2 + b_0 t) \phi(x_1 + a_0 t, x_2 + b_0 t) \\
 & + \delta \int_0^t \alpha(x_1 + a_0(t-s), x_2 + b_0(t-s)) z(s, x_1 + a_0(t-s), x_2 + b_0(t-s)) ds \\
 & + \int_0^t \frac{d}{ds} \alpha(x_1 + a_0(t-s), x_2 + b_0(t-s)) z(s, x_1 + a_0(t-s), x_2 + b_0(t-s)) ds.
 \end{aligned}$$

Therefore,

$$\begin{aligned} (1 + \alpha(x_1, x_2))z(t, x_1, x_2) &= \phi(x_1 + a_0t, x_2 + b_0t)(1 + \alpha(x_1 + a_0t, x_2 + b_0t)) \\ &+ \delta \int_0^t \alpha(x_1 + a_0(t-s), x_2 + b_0(t-s))z(s, x_1 + a_0(t-s), x_2 + b_0(t-s)) ds \\ &+ \int_0^t \frac{d}{ds} \alpha(x_1 + a_0(t-s), x_2 + b_0(t-s))z(s, x_1 + a_0(t-s), x_2 + b_0(t-s)) ds. \end{aligned}$$

Equivalently, dividing by $1 + \alpha(x_1, x_2)$, we have

$$\begin{aligned} z(t, x_1, x_2) &= \frac{1}{1 + \alpha(x_1, x_2)} \left\{ \phi(x_1 + a_0t, x_2 + b_0t)(1 + \alpha(x_1 + a_0t, x_2 + b_0t)) \right. \\ &+ \delta \int_0^t \alpha(x_1 + a_0(t-s), x_2 + b_0(t-s))z(s, x_1 + a_0(t-s), x_2 + b_0(t-s)) ds \\ &\left. + \int_0^t \frac{d}{ds} \alpha(x_1 + a_0(t-s), x_2 + b_0(t-s))z(s, x_1 + a_0(t-s), x_2 + b_0(t-s)) ds \right\}. \end{aligned}$$

Combining the integrals, we get

$$\begin{aligned} z(t, x_1, x_2) &= \frac{1}{1 + \alpha(x_1, x_2)} \left\{ \phi(x_1 + a_0t, x_2 + b_0t)(1 + \alpha(x_1 + a_0t, x_2 + b_0t)) \right. \\ &+ \int_0^t \left(\delta \alpha(x_1 + a_0(t-s), x_2 + b_0(t-s)) + \frac{d}{ds} \alpha(x_1 + a_0(t-s), x_2 + b_0(t-s)) \right) \\ &\left. \times z(s, x_1 + a_0(t-s), x_2 + b_0(t-s)) ds \right\}. \end{aligned} \quad (14)$$

Using the triangle inequality and applying the maximum norm $\|\cdot\|_{C(\Omega)}$ to (14), we obtain

$$\|z(t)\|_{C(\Omega)} \leq M_1 M_2 \|\phi\|_{C(\Omega)} + M_1 L \int_0^t \|z(s)\|_{C(\Omega)} ds.$$

Applying Grönwall's inequality, we get

$$\|z(t)\|_{C(\Omega)} \leq M \|\phi\|_{C(\Omega)}, \quad 0 \leq t \leq T.$$

From this inequality and (13), it follows that

$$\|v(t)\|_{C(\Omega)} \leq M e^{-\delta t} \|\phi\|_{C(\Omega)}.$$

Therefore,

$$\left\| e^{-tA(x_1, x_2)} \right\|_{C(\Omega) \rightarrow C(\Omega)} \leq M e^{-\delta t}.$$

This establishes the result as stated in Theorem 1. □

Remark 1. The operator $I - e^{-TA(x_1, x_2)}$ is invertible for $\delta > \frac{1}{T} \ln M$, and the following estimate holds

$$\|(I - e^{-TA(x_1, x_2)})^{-1}\|_{C(\Omega) \rightarrow C(\Omega)} \leq \frac{1}{1 - M e^{-T\delta}}. \quad (15)$$

3 The main theorem

This section establishes stability estimates for the space-dependent source identification problem associated with the two-dimensional neutron transport equation with spatially variable coefficients and periodic boundary conditions, as given in (1).

Theorem 2. Assume that the conditions of Theorem 1 are satisfied. The following estimates hold

$$\|u\|_{C(C(\Omega))} + \left\| \left(A^{(x_1, x_2)} \right)^{-1} p \right\|_{C(\Omega)} \leq M_3 \left(\|\psi_1\|_{C(\Omega)} + \|\psi_2\|_{C(\Omega)} + \|f\|_{C(C(\Omega))} \right),$$

and

$$\begin{aligned} \|u_t\|_{C(C(\Omega))} + \|u\|_{C(C^{(1)}(\Omega))} + \|p\|_{C(\Omega)} &\leq M_3 \left(\|\psi_1\|_{C^{(1)}(\Omega)} + \|\psi_2\|_{C^{(1)}(\Omega)} \right. \\ &\quad \left. + \min \left\{ \|f\|_{C(C^{(1)}(\Omega))}, \|f(0)\|_{C(\Omega)} + \|f'\|_{C(C(\Omega))} \right\} \right). \end{aligned}$$

Here, M_3 depends only on T_1, T_2, δ, M .

Proof. We can rewrite problem (1) in the following abstract form:

$$\begin{cases} u'(t) + Au(t) = p + f(t), & t \in (0, T), \\ u(T_1) = u(0) + \psi_1, & T_1 \in (0, T), \\ u(T_2) = \psi_2, & T_2 \in (0, T], \end{cases} \quad (16)$$

in a Banach space $E = C(\Omega)$ with a positive operator $A = A^{(x_1, x_2)}$ defined by (2). The proof of Theorem 2 is based on the positivity of the operator A in $E = C(\Omega)$ and the following abstract theorem. $\square$

Theorem 3. Let $\psi_1, \psi_2 \in D(A)$ and $f \in C(D(A)) \cap C^{(1)}(E)$. The following estimates are valid for the solution of problem (1)

$$\|u(t)\|_E + \|A^{-1}p\|_E \leq M_4 (\|\psi_1\|_E + \|\psi_2\|_E + \|f\|_{C(E)}),$$

and

$$\begin{aligned} \|u_t\|_{C(E)} + \|u\|_{C(E)} + \|p\|_E &\leq M_4 \left(\|\psi_1\|_{D(A)} + \|\psi_2\|_{D(A)} \right. \\ &\quad \left. + \min \{ \|f\|_{C(D(A))}, \|f(0)\|_E + \|f'\|_{C(E)} \} \right). \end{aligned} \quad (17)$$

Here, M_4 depends only on T_1, T_2, δ, M .

Proof. Applying the formula (16), we obtain

$$e^{-(t-s)A} \frac{du(s)}{ds} + e^{-(t-s)A} Au(s) = e^{-(t-s)A} (p + f(s))$$

for $0 \leq s \leq t$. Then, taking the integral with respect to s from 0 to t , we get

$$u(t) = e^{-tA} u(0) + \int_0^t e^{-(t-s)A} (p + f(s)) ds. \quad (18)$$

Evaluating (18) at $t = T_1$ and using $u(T_1) = u(0) + \psi_1$ yields

$$u(0) + \psi_1 = e^{-T_1 A} u(0) + \int_0^{T_1} e^{-(T_1-s)A} (p + f(s)) ds,$$

which can be rearranged as

$$(I - e^{-T_1 A})u(0) = -\psi_1 + \int_0^{T_1} e^{-(T_1-s)A} f(s) ds + (I - e^{-T_1 A})A^{-1}p.$$

Since $I - e^{-T_1 A}$ is invertible, we obtain

$$u(0) = -(I - e^{-T_1 A})^{-1}\psi_1 + (I - e^{-T_1 A})^{-1} \int_0^{T_1} e^{-(T_1-s)A} f(s) ds + A^{-1}p. \quad (19)$$

Now evaluating (18) at $t = T_2$ and using $u(T_2) = \psi_2$ yields

$$\psi_2 = e^{-T_2 A}u(0) + \int_0^{T_2} e^{-(T_2-s)A} f(s) ds + (I - e^{-T_2 A})A^{-1}p. \quad (20)$$

Substituting (19) into (20) gives

$$\begin{aligned} \psi_2 = & -e^{-T_2 A}(I - e^{-T_1 A})^{-1}\psi_1 + e^{-T_2 A}(I - e^{-T_1 A})^{-1} \int_0^{T_1} e^{-(T_1-s)A} f(s) ds \\ & + \int_0^{T_2} e^{-(T_2-s)A} f(s) ds + A^{-1}p. \end{aligned}$$

Solving for $A^{-1}p$, we obtain

$$\begin{aligned} A^{-1}p = & \psi_2 + e^{-T_2 A}(I - e^{-T_1 A})^{-1}\psi_1 - e^{-T_2 A}(I - e^{-T_1 A})^{-1} \int_0^{T_1} e^{-(T_1-s)A} f(s) ds \\ & - \int_0^{T_2} e^{-(T_2-s)A} f(s) ds. \end{aligned} \quad (21)$$

After taking the norm in E and applying the triangle inequality, we obtain

$$\|A^{-1}p\|_E \leq M_5 \left(\|\psi_1\|_E + \|\psi_2\|_E + \|f\|_{C(E)} \right), \quad (22)$$

where M_5 is defined by

$$M_5 = \max \left\{ 1, \frac{Me^{-\delta T_2}}{1 - Me^{-\delta T_1}}, \frac{M(2 - e^{-\delta T_1})}{\delta} \right\}.$$

From (21), we can obtain

$$\begin{aligned} p = & A\psi_2 + e^{-T_2 A}(I - e^{-T_1 A})^{-1}A\psi_1 - e^{-T_2 A}(I - e^{-T_1 A})^{-1} \int_0^{T_1} e^{-(T_1-s)A} Af(s) ds \\ & - \int_0^{T_2} e^{-(T_2-s)A} Af(s) ds. \end{aligned}$$

Taking norms in E and using the triangle inequality, we obtain

$$\begin{aligned} \|p\|_E \leq & \|A\psi_2\|_E + \|e^{-T_2 A}(I - e^{-T_1 A})^{-1}A\psi_1\|_E \\ & + \left\| e^{-T_2 A}(I - e^{-T_1 A})^{-1} \int_0^{T_1} e^{-(T_1-s)A} Af(s) ds \right\|_E + \left\| \int_0^{T_2} e^{-(T_2-s)A} Af(s) ds \right\|_E. \end{aligned}$$

By using the estimates (7) and (15), we obtain

$$\|p\|_E \leq \|\psi_2\|_{D(A)} + \frac{Me^{-\delta T_2}}{1 - Me^{-\delta T_1}} \|\psi_1\|_{D(A)} + \left(\frac{M^2 e^{-\delta T_2} (1 - e^{-\delta T_1})}{\delta (1 - Me^{-\delta T_1})} + \frac{M(1 - e^{-\delta T_2})}{\delta} \right) \|f\|_{C(D(A))}.$$

Therefore, it follows that

$$\|p\|_E \leq M_6 \left(\|\psi_1\|_{D(A)} + \|\psi_2\|_{D(A)} + \|f\|_{C(D(A))} \right). \quad (23)$$

Here, M_6 depends only on T_1, T_2, δ, M .

Now using (20) and substituting (19), we obtain

$$\begin{aligned} (I - e^{-T_2 A})u(0) &= - (I - e^{-T_1 A})^{-1} \psi_1 + (I - e^{-T_1 A})^{-1} \int_0^{T_1} e^{-(T_1-s)A} f(s) ds - \psi_2 \\ &\quad + \int_0^{T_2} e^{-(T_2-s)A} f(s) ds + (2I - e^{-T_2 A})A^{-1}p. \end{aligned}$$

Since $I - e^{-T_2 A}$ is invertible, we obtain

$$\begin{aligned} u(0) &= (I - e^{-T_2 A})^{-1} \left[- (I - e^{-T_1 A})^{-1} \psi_1 + (I - e^{-T_1 A})^{-1} \int_0^{T_1} e^{-(T_1-s)A} f(s) ds - \psi_2 \right. \\ &\quad \left. + \int_0^{T_2} e^{-(T_2-s)A} f(s) ds + (2I - e^{-T_2 A})A^{-1}p \right]. \end{aligned}$$

By applying the norm in E and using the triangle inequality, we obtain

$$\begin{aligned} \|u(0)\|_E &\leq \frac{1}{1 - Me^{-\delta T_2}} \left(\frac{1}{1 - Me^{-\delta T_1}} \|\psi_1\|_E + \frac{M(1 - e^{-\delta T_1})}{\delta(1 - Me^{-\delta T_1})} \|f\|_{C(E)} \right. \\ &\quad \left. + \|\psi_2\|_E + \frac{M(1 - e^{-\delta T_2})}{\delta} \|f\|_{C(E)} + (2 + Me^{-\delta T_2}) \|A^{-1}p\|_E \right). \end{aligned}$$

Therefore, it follows that

$$\|u(0)\|_E \leq M_7 \left(\|\psi_1\|_E + \|\psi_2\|_E + \|f\|_{C(E)} + \|A^{-1}p\|_E \right). \quad (24)$$

Here, M_7 depends only on T_1, T_2, δ, M .

Substituting the estimate for (22) into the inequality for (24) gives

$$\|u(0)\|_E \leq M_8 (\|\psi_1\|_E + \|\psi_2\|_E + \|f\|_{C(E)}),$$

where the constant M_8 is depending only on T_1, T_2, δ, M .

Rewriting the term involving p in (18) leads to

$$u(t) = e^{-tA}u(0) + \int_0^t e^{-(t-s)A} f(s) ds + A^{-1}(I - e^{-tA})p. \quad (25)$$

By applying the norm in E and using the triangle inequality, we obtain

$$\|u(t)\|_E \leq \|e^{-tA}\|_{E \rightarrow E} \|u(0)\|_E + \int_0^t \|e^{-(t-s)A}\|_{E \rightarrow E} \|f(s)\|_E ds + \|A^{-1}(I - e^{-tA})p\|_E.$$

From the semigroup estimate (7), the inequality $\|I - e^{-tA}\|_E \leq 1 + Me^{-\delta t} \leq 1 + M$, and estimate

$$\int_0^t \|e^{-(t-s)A}\|_{E \rightarrow E} \|f(s)\|_E ds \leq \frac{M}{\delta} \|f\|_{C(E)}$$

it follows that

$$\|u(t)\|_E \leq M\|u(0)\|_E + \frac{M}{\delta} \|f\|_{C(E)} + (1 + M)\|A^{-1}p\|_E.$$

Substituting the estimate for (22) into the inequality for (24) yields

$$\|u(t)\|_E \leq M_9(\|\psi_1\|_E + \|\psi_2\|_E + \|f\|_{C(E)}). \quad (26)$$

Here, M_9 depends only on T_1, T_2, δ, M . Now we take derivative of formula (25)

$$u_t(t) = -e^{-tA}Au(0) + f(t) - \int_0^t e^{-(t-s)A}Af(s) ds + e^{-tA}p. \quad (27)$$

By applying the norm in E and using the triangle inequality, we obtain

$$\begin{aligned} \|u_t(t)\|_E &\leq \|e^{-tA}\|_{E \rightarrow E} \|Au(0)\|_E + \|A^{-1}\|_{E \rightarrow E} \|Af(t)\|_E + \|e^{-tA}\|_{E \rightarrow E} \|p\|_E \\ &\quad + \int_0^t \|e^{-(t-s)A}\|_{E \rightarrow E} \|Af(s)\|_E ds. \end{aligned} \quad (28)$$

It is easy to see that

$$\int_0^t \|e^{-(t-s)A}\|_{E \rightarrow E} \|Af(s)\|_E ds \leq \frac{M}{\delta} (1 - e^{-\delta t}) \|f\|_{C(D(A))}. \quad (29)$$

Substituting the semigroup estimate $\|e^{-tA}\|_{E \rightarrow E} \leq Me^{-\delta t}$ from (7) and (29) into (28) yields

$$\|u_t(t)\|_E \leq Me^{-\delta t} \|Au(0)\|_E + \|A^{-1}\|_{E \rightarrow E} \|Af(t)\|_E + Me^{-\delta t} \|p\|_E + \frac{M}{\delta} (1 - e^{-\delta t}) \|f\|_{C(D(A))}, \quad (30)$$

for all $t \in [0, T]$. Applying integration by parts to the integral term in (27), we obtain

$$\int_0^t e^{-(t-s)A}Af(s) ds = f(t) - e^{-tA}f(0) - \int_0^t e^{-(t-s)A}f'(s) ds,$$

and then equation (27) becomes

$$u_t(t) = e^{-tA}(p + f(0) - Au(0)) + \int_0^t e^{-(t-s)A}f'(s) ds.$$

By applying the norm in E and using the triangle inequality, we obtain

$$\|u_t(t)\|_E \leq Me^{-\delta t} \|Au(0)\|_E + Me^{-\delta t} \|f(0)\|_E + Me^{-\delta t} \|p\|_E + \frac{M}{\delta} (1 - e^{-\delta t}) \|f'\|_{C(E)}, \quad (31)$$

for all $t \in [0, T]$. Combining (30) and (31) with the estimates for $\|u\|_{C(E)}$ and $\|p\|_E$ in (23) and (26) and taking the maximum norm yields inequality (17).

This establishes the result as stated in Theorem 3. $\square$

Conclusion

This article investigated the space-dependent source identification problem for the 2D NTE with spatially variable coefficients and periodic boundary conditions. We first presented a fundamental formula for an explicit solution to the homogeneous problem with constant coefficients. Using this result, we then established a stability estimate for the homogeneous problem with variable coefficients. Finally, we employed these findings to derive stability estimates for the space-dependent source identification problem in the two-dimensional periodic transport equation with variable coefficients.

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Author contributions

All authors contributed equally to this work.

Conflict of interest

The authors declare no conflict of interest.

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Almost hyperbolic Ricci solitons on almost co-Kähler manifolds

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This research investigates almost hyperbolic Ricci solitons on almost co-Kähler manifolds, an important class of odd-dimensional manifolds closely related to Kähler geometry. Motivated by recent developments in hyperbolic geometric flows and their self-similar solutions, we study the existence and geometric properties of almost hyperbolic Ricci solitons within the framework of almost co-Kähler manifolds. First, we derive fundamental relations for almost hyperbolic Ricci solitons whose potential vector field is collinear with the Reeb vector field and obtain a differential equation governing the associated smooth function. We then examine almost hyperbolic Ricci solitons under several geometric conditions, including conformal Killing vector fields, torse-forming vector fields, Ricci bi-conformal vector fields, and $(V(\text{Ric}))$ -vector fields. In each case, necessary conditions are established, leading to characterization results and Einstein-type properties of the underlying manifolds. In particular, we obtain criteria under which an almost co-Kähler manifold is Einstein or Ricci-flat. Furthermore, for closed almost co-Kähler manifolds admitting a $(V(\text{Ric}))$ -vector field satisfying suitable trace conditions, we show that the manifold is Ricci-flat and the associated vector field is parallel. To illustrate the theoretical results, we construct two explicit three-dimensional examples of almost co-Kähler manifolds admitting hyperbolic Ricci solitons.

Keywords: almost co-Kähler manifolds, hyperbolic Ricci solitons, torse-forming vector fields, hyperbolic geometric flow, conformal Killing vector fields, Reeb vector field, Ricci-flat manifolds, contact metric geometry.

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Introduction

The geometry of (almost) co-Kähler manifolds, a specialized category of almost contact metric manifolds akin to Kähler manifolds, was initially introduced by Blair [1] and further explored by Goldberg and Yano [2], and Olszak [3]. These manifolds are essentially the almost cosymplectic manifolds referenced in the aforementioned studies. Following Li's [4] research, a number of authors have adopted this terminology in their works. Li's [5] findings indicate that co-Kähler manifolds serve as odd-dimensional counterparts to Kähler manifolds. A recent survey has compiled various new insights regarding (almost) co-Kähler manifolds from both geometric and topological perspectives. De, Majhi, and Suh [6] investigated semi-symmetric and φ -symmetric curvature properties, including Codazzi-type behaviour of the Ricci tensor and the role of transversal Killing fields. Wang [7] studied curvature homogeneity and ball-homogeneity, relating these notions to the algebraic structure of some operator; and the interplay between co-Kähler geometry and the Ricci-soliton equation has been clarified by Wang, who extended the analysis to Ricci solitons on general almost co-Kähler manifolds [8], where rigidity forces the underlying manifold to be Einstein or Ricci-flat under suitable hypotheses on the potential field. Notably, Perrone [9, 10] achieved a comprehensive classification of three-dimensional almost co-Kähler manifolds characterized by homogeneity or minimal Reeb vector fields, along with providing a local characterization of these manifolds. Co-Kähler manifolds represent a significant

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area of inquiry within contact geometry, serving as intriguing odd-dimensional counterparts to Kähler manifolds. A closed manifold qualifies as a co-Kähler manifold if and only if it can be characterized as a Kähler mapping torus [4].

In recent years, there has been a surge of classification results pertaining to three-dimensional almost co-Kähler (ACK) manifolds. For example, some authors have examined local ϕ -symmetry, curvature, and ball homogeneities within the context of three-dimensional almost co-Kähler manifolds. Further studies have investigated various symmetry properties related to the Ricci operators, including Codazzi-type conditions, and transversal Killing fields in three-dimensional ACK manifolds [11, 12].

The study of geometric solitons on Riemannian and semi-Riemannian manifolds represents a significant area of research within differential geometry and its applications in physics. In recent years, there has been a notable surge in interest surrounding geometric solitons and their extensions, as they offer innovative methods for exploring the geometry and topology of Riemannian manifolds. This growing fascination with geometric solitons and their variations across various geometric frameworks is largely attributed to their relevance to the principles of general relativity. Geometric flows and their solitons play a pivotal role in differential geometry, facilitating the discovery of canonical metrics on the corresponding Riemannian manifolds. Among these flows, the Ricci flow is particularly significant. Hamilton defined the Ricci soliton (RS) on a Riemannian manifold as a unique solution to the Ricci flow. A manifold denoted as (M, g) qualifies as a Ricci soliton if it adheres to the equation: $S + \frac{1}{2}\mathcal{L}_W g + \Lambda g = 0$, where Λ is a real constant, S represents the Ricci tensor of g , and $\mathcal{L}_W$ is the Lie derivative operator in the direction of W . This relationship is encapsulated in the triplet (g, W, Λ) . Furthermore, the metric g is identified as a gradient Ricci soliton if there exists a smooth function ψ such that $W = \text{grad } \psi$.

Investigations on existence and categorization of geometric solitons on (almost) co-Kähler manifolds presents a significant challenge. To date, the literature reveals only a limited number of findings pertaining to this subject. Current literature indicates that findings on this topic are relatively sparse. The initial study in this area was carried out by Cho [13], who illustrated that a co-Kähler 3-manifold qualifies as a Ricci soliton if the potential vector field aligns with the Reeb vector field ξ or is a unit vector field orthogonal to ξ . This leads to the conclusion that the manifold is locally flat, characterized by vanishing Riemann curvature tensor. Expanding on this work, Wang [14] determined that for a co-Kähler 3-manifold to be classified as a Ricci soliton, it must either be locally flat or the potential vector field must signify an infinitesimal contact transformation. Furthermore, Wang previously established that if a compact almost co-Kähler manifold operates as a Ricci soliton, with the potential vector field being pointwise collinear to the Reeb vector field, it is identified as a Ricci-flat co-Kähler manifold, resulting in a steady soliton [15]. Recently, new geometric criteria have been introduced to clarify the conditions under which a compact Einstein ACK manifold can be recognized as a co-Kähler manifold [16].

Another type of geometric flows is characterized by hyperbolic geometric flows, which bear a strong resemblance to the wave equation associated with metrics on the underlying manifold. This can be understood as a system of second-order nonlinear evolution partial differential equations, expressed as follows:

$$\frac{\partial^2}{\partial t^2} g = -2S, \quad g(0) = g_0, \quad \frac{\partial g}{\partial t}(0) = k_0. \quad (1)$$

Here, k_0 denotes a symmetric 2-tensor field defined on the manifold M . The investigation into the existence and uniqueness of the equation (1) has been conducted in [17], particularly in the context of closed Riemannian manifolds. The self-similar solution for this flow can be represented by the equation:

$$S(g) + \lambda \mathcal{L}_X g + (\mathcal{L}_X \circ \mathcal{L}_X)g = \mu g, \quad (2)$$

where λ and μ are constants, and X denotes a vector field. In this framework, g is identified as a hyperbolic Ricci soliton (HRS), which can be represented by the tuple (g, X, λ, μ) or simply as

(g, X, λ, μ) . In cases where X is zero, the HRS corresponds to an Einstein metric. Moreover, if λ is specified as $\frac{1}{2}$ and the vector field X is a 2-Killing vector field that satisfies the condition $(\mathcal{L}_X \circ \mathcal{L}_X)g = 0$ (see [18, 19]), the HRS simplifies to a Ricci soliton (RS). Furthermore, if X is expressed as ∇f for some smooth function f defined on M , we categorize this structure as a gradient HRS, denoting it as $(g, \nabla f, \lambda, \mu)$. If we consider μ to be a smooth function on M , then the soliton is called almost HRS (AHRS). Azami et al. studied HRS on some manifolds. Azami and Fasihi-Ramandi [20] initiated this line of investigation on warped product manifolds, deriving rigidity and classification results for HRSs whose potential vector field satisfies additional geometric constraints, and the same first author with Jafari, Jamal, and Haseeb [21] extended the analysis to perfect fluid spacetimes, where the hyperbolic soliton equation interacts with the physical restrictions imposed by the energy-momentum tensor. The three-dimensional case was studied in depth by Faraji, Azami, and Fasihi-Ramandi [22], who classified homogeneous hyperbolic Ricci solitons on 3-manifolds, and by Fasihi-Ramandi, Azami, and Choudhary [23], who established general existence and structural theorems for HRSs on Riemannian manifolds, including gradient and almost-HRS cases. Closely related results on sequential warped product manifolds were obtained by Kaya and Özgür [24], who clarified how the HRS equation interacts with the fibre-wise geometry of the warped decomposition, while Siddiqi and Mofarreh [25] further developed the physical interpretation by analysing hyperbolic Ricci solitons and their gradient analogue on relativistic perfect fluid spacetimes. In a parallel direction, Blaga and Özgür [26] investigated the neighbouring notion of hyperbolic Yamabe solitons, establishing properties that complement the HRS framework since the Yamabe equation reduces, under suitable conditions, to the Ricci-soliton equation.

Building upon the studies previously mentioned and the references cited in [27, 28], this paper investigates AHRS in the framework of ACK manifolds, providing two specific examples of HRS on these manifolds. The organization of the paper is as follows: Section 2 reviews key concepts and formulas pertinent to ACK manifolds, which will be crucial for our analysis. In Section 3, we provide the main results along with their proofs. Lastly, Section 4 is focused on presenting two examples of ACK manifolds that demonstrate hyperbolic Ricci solitons.

1 Preliminaries

In the upcoming sections, we will review essential principles related to contact geometry that will be significant for our later discussions. Additionally, we will consider the vector fields W_1, W_2 , and U on M as general unless stated otherwise.

Let M^{2n+1} represent an almost contact manifold characterized by a structure (ϕ, ξ, η) , where ξ denotes a unit vector field, ϕ denotes a $(1, 1)$ -tensor field, and η is a 1-form that satisfies the following conditions:

$$\phi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1. \quad (3)$$

If a metric g exists that fulfills the equation

$$g(\phi W_1, \phi W_2) = g(W_1, W_2) - \eta(W_1)\eta(W_2), \quad (4)$$

the manifold is referred to as an almost contact metric manifold. In this context, it is noted that $\eta \circ \phi = 0$, $\phi\xi = 0$, $g(\phi W_1, W_2) = -g(W_1, \phi W_2)$, and $\eta = g(\cdot, \xi)$. The fundamental 2-form Φ of M is determined by

$$\Phi(W_1, W_2) = g(W_1, \phi W_2).$$

An almost contact manifold M is classified as an almost co-Kähler (ACK) manifold if $d\Phi = 0$ and $d\eta = 0$. An ACK manifold qualifies as a co-Kähler manifold when it meets the condition of being normal, which is equivalent to the statement $\nabla\Phi = 0$, where ∇ represents the Levi-Civita connection.

Furthermore, an ACK structure is referred to as strictly ACK if it does not conform to the definition of a co-Kähler structure. Given that (almost) cosymplectic manifolds serve as the odd-dimensional counterpart to Kähler manifolds, Blair [29] proposed the term (almost) co-Kähler manifold as a new designation for (almost) cosymplectic manifolds.

On an ACK, let us define two operators, l and h , where l is expressed as $l = R(\cdot, \xi)\xi$ and h is given by $h = \frac{1}{2}\mathcal{L}_\xi\phi$, with R representing the curvature tensor. These operators adhere to specific conditions [30, 31]:

$$h\xi = 0, \quad \text{trace}(h) = 0, \quad \text{trace}(h\phi) = 0, \quad \phi h + h\phi = 0, \quad (5)$$

$$\begin{aligned} \nabla_W \xi &= h\phi W, \\ (\nabla_{W_1} \eta)(W_2) &= g(h\phi W_1, W_2), \\ \phi l \phi - l &= 2h^2. \end{aligned} \quad (6)$$

From equation (6), it can be inferred that

$$(\nabla_{W_1} \eta)W_2 = (\nabla_{W_2} \eta)W_1.$$

An ACK manifold is called integrable if

$$(\nabla_{W_1} \phi)W_2 = g(hW_1, W_2)\xi - \eta(W_2)hW_1.$$

An ACK manifold is called a K-ACK manifold if the Reeb vector field ξ is Killing. Any co-Kähler manifold is recognized as a K-almost co-Kähler manifold; however, the reverse is generally not the case. Nonetheless, this relationship holds true in the context of a three-dimensional manifold.

Lemma 1. [2] In three-dimensional space, an ACK manifold qualifies as a co-Kähler manifold if and only if the vector field ξ is Killing.

In a co-Kähler manifold of dimension 3, we have $QW = \frac{r}{2}(W - \eta(W)\xi)$ and consequently we have [32] $\xi r = 0$ where Q is the Ricci operator, and r is the scalar curvature.

2 AHS on ACK manifolds

In the subsequent sections, we will outline our results and provide the necessary proofs to support them. It is crucial to emphasize that all vector fields, namely W_1, W_2, Z, U, U_1, U_2 , and W , defined on the manifold M , are treated as arbitrary unless stated otherwise. A manifold is designated as η -Einstein if it satisfies the condition

$$S = ag + b\eta \otimes \eta,$$

where a and b are smooth functions defined on M . Let us consider a ACK manifold denoted as M that adheres to the HRS equation (2). We define V as $f\xi$, where f is a function on M . Utilizing equation (5), we can derive the following expression:

$$\begin{aligned} (\mathcal{L}_{f\xi}g)(W_1, W_2) &= (W_1f)\eta(W_2) + fg(h\phi(W_1), W_2) + (W_2f)\eta(W_1) \\ &\quad + fg(W_1, h\phi(W_2)) \\ &= (W_1f)\eta(W_2) + (W_2f)\eta(W_1) + 2fg(h\phi(W_1), W_2), \end{aligned}$$

hence

$$\begin{aligned} &(\mathcal{L}_{f\xi}(\mathcal{L}_{f\xi}g))(W_1, W_2) \\ &= f\xi((\mathcal{L}_{f\xi}g)(W_1, W_2)) - (\mathcal{L}_{f\xi}g)(\mathcal{L}_{f\xi}W_1, W_2) - (\mathcal{L}_{f\xi}g)(W_1, \mathcal{L}_{f\xi}W_2) \\ &= f\xi((W_1f)\eta(W_2) + (W_2f)\eta(W_1) + 2fg(h\phi(W_1), W_2)) \\ &\quad - (((\mathcal{L}_{f\xi}W_1)f)\eta(W_2) + (W_2f)\eta(\mathcal{L}_{f\xi}W_1) + 2fg(h\phi(\mathcal{L}_{f\xi}W_1), W_2)) \\ &\quad - ((W_1f)\eta(\mathcal{L}_{f\xi}W_2) + ((\mathcal{L}_{f\xi}W_2)f)\eta(W_1) + 2fg(h\phi(W_1), \mathcal{L}_{f\xi}W_2)). \end{aligned}$$

By utilizing the equation $V = f\xi$ along with the previously mentioned equation in (2), we deduce

$$\begin{aligned} & S(W_1, W_2) + \lambda((W_1 f)\eta(W_2) + (W_2 f)\eta(W_1) + 2fg(h\phi(W_1), W_2)) \\ & + f\xi((W_1 f)\eta(W_2) + (W_2 f)\eta(W_1) + 2fg(h\phi(W_1), W_2)) \\ & - (((\mathcal{L}_f \xi W_1)f)\eta(W_2) + (W_2 f)\eta(\mathcal{L}_f \xi W_1) + 2fg(h\phi(\mathcal{L}_f \xi W_1), W_2)) \\ & - ((W_1 f)\eta(\mathcal{L}_f \xi W_2) + ((\mathcal{L}_f \xi W_2)f)\eta(W_1) + 2fg(h\phi(W_1), \mathcal{L}_f \xi W_2)) \\ & - \mu g(W_1, W_2) = 0. \end{aligned}$$

We substitute $W_1 = W_2 = \xi$ into the final equation, employing equations (3), $\mathcal{L}_f \xi = -\xi f$, and $S(\xi, \xi) = -\text{trace}(h^2)$ to derive the result

$$-\text{trace}(h^2) + 2f\xi(\xi(f)) + 2\lambda\xi(f) + 4(\xi(f))^2 - \mu = 0.$$

This leads to the emergence of the following theorem.

Theorem 1. Consider an ACK manifold characterized by the tuple (M, g, ϕ, ξ, η) , which features an HRS denoted by $(g, f\xi, \lambda, \mu)$, with f being a smooth function on the manifold M . Consequently, the following equation holds:

$$-\text{trace}(h^2) + 2f\xi(\xi(f)) + 2\lambda\xi(f) + 4(\xi(f))^2 - \mu = 0.$$

Corollary 1. Consider an ACK manifold characterized by the tuple (M, g, ϕ, ξ, η) , which features a HRS denoted as (g, ξ, λ, μ) . Consequently, the following equation holds:

$$\mu = -\text{trace}(h^2).$$

Definition 1. If a vector field V satisfies the equation

$$\mathcal{L}_V g = 2\psi g, \tag{7}$$

then it is called a conformal Killing where ψ represents a function defined on the manifold M .

Let V represent a conformal Killing vector field that fulfills the condition stated in equation (7). Consequently,

$$\begin{aligned} ((\mathcal{L}_V \circ \mathcal{L}_V)g)(U, W) &= V((\mathcal{L}_V g)(U, W)) - \mathcal{L}_V g(\mathcal{L}_V U, W) - (\mathcal{L}_V g)(U, \mathcal{L}_V W) \\ &= V(2\psi g(U, W)) - 2\psi g(\mathcal{L}_V U, W) - 2\psi g(U, \mathcal{L}_V W) \\ &= 2V(\psi)g(U, W) + 2\psi \mathcal{L}_V g(U, W) \\ &= (2V(\psi) + 4\psi^2)g(U, W). \end{aligned} \tag{8}$$

By substituting equation (8) into equation (2), we arrive at the expression

$$S(U, W) + 2\psi \lambda g(U, W) + (2V(\psi) + 4\psi^2)g(U, W) - \mu g(U, W) = 0.$$

If we replace U and W with ξ in the preceding equation, we derive

$$-\text{trace}(h^2) + 2\psi \lambda + 2V(\psi) + 4\psi^2 - \mu = 0.$$

Consequently, we get:

Theorem 2. In the context of an ACK manifold defined by the metric g and characterized by the HRS parameters (g, V, λ, μ) , where V satisfies $\mathcal{L}_V g = 2\psi g$, it can be concluded that the manifold M exhibits Einstein properties. This yields

$$-\text{trace}(h^2) + 2\psi \lambda + 2V(\psi) + 4\psi^2 - \mu = 0.$$

Definition 2. A torse-forming vector field W on a pseudo-Riemannian manifold (M, g) is defined as a nonvanishing vector field that satisfies the equation

$$\nabla_U W = \psi U + \omega(U)W, \quad (9)$$

where ω denotes a 1-form and ψ is a smooth function. The vector field W can be further classified in various ways: it is called concircular, when $\omega = 0$; it is termed concurrent if $\omega = 0$ and ψ equals one; it is recognized as a parallel vector field when both ψ and ω are zero; and it is classified as torqued if the condition $\omega(W) = 0$ is satisfied.

Let (g, W, λ, μ) denote an HRS on an ACK manifold, with W being a torse-forming vector field and satisfies the condition outlined in (9). Then

$$(\mathcal{L}_W g)(U, Y) = 2\psi g(U, Y) + \omega(U)g(W, Y) + \omega(Y)g(W, U) \quad (10)$$

and

$$\begin{aligned} (\mathcal{L}_W(\mathcal{L}_W g))(U, Y) &= W(2\psi g(U, Y) + \omega(U)g(W, Y) + \omega(Y)g(W, U)) \\ &\quad - 2\psi g(\mathcal{L}_W U, Y) - \omega(\mathcal{L}_W U)g(W, Y) - \omega(Y)g(W, \mathcal{L}_W U) \\ &\quad - 2\psi g(U, \mathcal{L}_W Y) - \omega(U)g(W, \mathcal{L}_W Y) - \omega(\mathcal{L}_W Y)g(W, U). \end{aligned}$$

The subsequent equations can be represented as follows:

$$g(\mathcal{L}_W U, Y) = g(\nabla_W U, Y) - \psi g(U, Y) - \omega(Y)g(W, U),$$

similarly

$$g(U, \mathcal{L}_W Y) = g(U, \nabla_W Y) - \psi g(U, Y) - \omega(U)g(W, Y).$$

Then

$$g(\mathcal{L}_W U, Y) + g(U, \mathcal{L}_W Y) = W(g(U, Y)) - 2\psi g(U, Y) - \omega(Y)g(W, U) - \omega(U)g(W, Y).$$

Also,

$$\begin{aligned} \omega(\mathcal{L}_W U) &= \omega(\nabla_W U - \nabla_U W) = \omega(\nabla_W U - \psi U - \omega(U)W) \\ &= \omega(\nabla_W U) - \psi \omega(U) - \omega(U)\omega(W), \end{aligned}$$

similarly

$$\omega(\mathcal{L}_W Y) = \omega(\nabla_W Y) - \psi \omega(Y) - \omega(Y)\omega(W).$$

Therefore,

$$\begin{aligned} (\mathcal{L}_W(\mathcal{L}_W g))(U, Y) &= (2W(\psi) + 4\psi^2)g(U, Y) + W(\omega(U)g(W, Y) + \omega(Y)g(W, U)) \\ &\quad + 4h\omega(U)g(W, Y) + 4h\omega(Y)g(W, U) \\ &\quad - \omega(\nabla_W U)g(W, Y) + 2\omega(U)\omega(W)g(W, Y) - \omega(Y)g(\nabla_W U, W) \\ &\quad + 2\omega(Y)\omega(W)g(W, U) - \omega(\nabla_W Y)g(W, U) - \omega(U)g(\nabla_W Y, W). \end{aligned} \quad (11)$$

By substituting $U = Y = \xi$ into equations (10) and (11), it can be concluded that.

$$(\mathcal{L}_W g)(\xi, \xi) = -2\psi + 2\omega(\xi)\eta(W) \quad (12)$$

and

$$\begin{aligned}
(\mathcal{L}_W(\mathcal{L}_W g))(\xi, \xi) &= -(2W(\psi) + 4\psi^2) + W(\omega(\xi)\eta(W) + \omega(\xi)\eta(W)) \\
&\quad + 4\psi\omega(\xi)\eta(W) + 4h\omega(\xi)\eta(W) \\
&\quad - 2\omega(\nabla_W \xi)\eta(W) + 4\omega(\xi)\omega(W)\eta(W) - 2\omega(\xi)g(\nabla_W \xi, W) \\
&= -(2W(\psi) + 4\psi^2) + W(\omega(\xi)\eta(W) + \omega(\xi)\eta(W)) \\
&\quad + 4\psi\omega(\xi)\eta(W) + 4\psi\omega(\xi)\eta(W) + 4\omega(\xi)\omega(W)\eta(W) \\
&\quad + 2\omega(\xi)\eta(W) + 4(\eta(W))^2\omega(\xi) + 2\omega(\xi)|W|^2.
\end{aligned} \tag{13}$$

By replacing equations (12) and (13) in equation (2), we derive

$$\begin{aligned}
&-\text{trace}(h^2) - \mu + \lambda(-2\psi + 2\omega(\xi)\eta(W)) - (2W(\psi) + 4\psi^2) \\
&+ W(\omega(\xi)\eta(W) + \omega(\xi)\eta(W)) + 4\psi\omega(\xi)\eta(W) + 4\psi\omega(\xi)\eta(W) \\
&+ 4\omega(\xi)\omega(W)\eta(W) + 2\omega(\xi)\eta(W) + 4(\eta(W))^2\omega(\xi) + 2\omega(\xi)|W|^2 = 0.
\end{aligned} \tag{14}$$

Consequently, we arrive at the subsequent conclusions.

Theorem 3. If an ACK manifold meets the HRS condition (g, W, λ, μ) , where W admits (9), then equation (14) holds true.

Corollary 2. In the framework of an ACK manifold, when the metric g satisfies the HRS condition (g, W, λ, μ) , with W denoting a concircular vector field characterized by the equation $\nabla_U W = \psi U$ for all vector fields U , it can be concluded that the equation $\text{trace}(h^2) + \mu + 2\psi\lambda + 2W(\psi) + 4\psi^2 = 0$ is valid.

In the work presented in [33], the authors introduced the concept of bi-conformal vector fields utilizing two $(0, 2)$ tensor fields. Subsequently, De et al. in [34] expanded on this by defining Ricci bi-conformal vector fields, employing the two tensor fields denoted as g and S in their formulation.

Definition 3. On a Riemannian manifold (M, g) , a vector field X is referred to as a Ricci bi-conformal vector field if it satisfies the following equations

$$\mathcal{L}_X g = \alpha g + \gamma S, \tag{15}$$

and

$$\mathcal{L}_X S = \alpha S + \gamma g, \tag{16}$$

for certain non-zero smooth functions denoted as α and γ .

Consider an ACK manifold (M^n, g) that satisfies the HRS conditions (g, W, λ, μ) , where W denotes the Ricci bi-conformal vector field and meets the criteria specified in equations (15) and (16). Consequently, the expression can be expressed as follows:

$$\mathcal{L}_W(\mathcal{L}_W g) = (\alpha^2 + \gamma^2 + W(\alpha))g + (2\alpha\gamma + W(\gamma))S. \tag{17}$$

By incorporating equations (15) and (17) into equation (2), we obtain

$$(1 + \lambda\gamma + 2\alpha\gamma + W(\gamma))S(X, Y) + (\lambda\alpha - \mu + \alpha^2 + \gamma^2 + W(\alpha))g(X, Y) = 0. \tag{18}$$

Replacing $X = Y = \xi$ in (18) yields

$$-(1 + \lambda\gamma + 2\alpha\gamma + W(\gamma))(\text{trace}(h^2)) + (\lambda\alpha - \mu + \alpha^2 + \gamma^2 + W(\alpha)) = 0$$

and

$$(1 + \lambda\gamma + 2\alpha\gamma + W(\gamma))(S(X, Y) + \text{trace}(h^2)g(X, Y)) = 0.$$

Now, if $1 + \lambda\gamma + 2\alpha\gamma + W(\gamma) \neq 0$ then the above equation yields M is an Einstein manifold. Thus, we get:

Theorem 4. If an ACK manifold admits the HRS conditions (g, W, λ, μ) , where W adheres to the equations (15) and (16) such that $1 + \lambda\gamma + 2\alpha\gamma + W(\gamma) \neq 0$, then M is classified as an Einstein manifold.

Definition 4. A vector field V defined on a manifold M is referred to as a $V(Ric)$ -tensor field if there exists a constant a such that $\nabla V = aQ$.

Let V be a $V(Ric)$ -tensor field on ACK manifold M^{2n+1} such that $\nabla V = aQ$. Thus, we deduce

$$\operatorname{div}(V) = ar, \quad \mathcal{L}_V g = 2aS, \quad \mathcal{L}_V \mathcal{L}_V g = 2a\mathcal{L}_V S$$

and HRS equation becomes

$$(1 + 2a\lambda)S + 2a\mathcal{L}_V S - \mu g = 0. \quad (19)$$

Now, if we assume that $\operatorname{trace}(\mathcal{L}_V S) = 0$, then by taking the trace of (19), we get $(1 + 2a\lambda)r = \mu(2n + 1)$. Let $1 + 2a\lambda \neq 0$. In this case, we conclude that $r = \frac{(2n+1)\mu}{1+2a\lambda}$ is a constant. From [35, 36], we have

$$\operatorname{div}(\mathcal{L}_V g) = 2(d(\operatorname{div}(V)) + i_{QV}g) \quad (20)$$

and

$$\operatorname{trace}(\mathcal{L}_V \mathcal{L}_V g) = 2(\|\nabla V\|^2 + \operatorname{div}(\nabla_V V) - S(V, V)). \quad (21)$$

Since $\operatorname{trace}(\mathcal{L}_V \mathcal{L}_V g) = 0$, from (21) we obtain

$$\|\nabla V\|^2 + \operatorname{div}(\nabla_V V) - S(V, V) = 0.$$

Also, from (20) we find

$$S(V, V) = \frac{1}{2}\operatorname{div}(\mathcal{L}_V g)(V) - V(\operatorname{div}(V)) = 0.$$

Since $\|\nabla V\|^2 = a^2\|Q\|^2$ and $\operatorname{div}(\nabla_V V) = a\operatorname{div}(QV)$, we infer

$$a^2\|Q\|^2 + a\operatorname{div}(QV) = 0. \quad (22)$$

By integrating (22), we get $Q = 0$. Then $S = 0$ and $\nabla V = 0$. Therefore, the manifold M exhibits Ricci-flat, and the vector field V is parallel. This leads us to the subsequent theorem.

Theorem 5. Suppose that a closed ACK manifold M^{2n+1} satisfies the HRS (g, V, λ, μ) with a $V(Ric)$ -tensor field such that $\nabla V = aQ$ for some constant a , and $\operatorname{trace}(\mathcal{L}_V S) = 0$ with $1 + 2a\lambda \neq 0$. Then M becomes Ricci-flat, and V becomes a parallel vector field.

3 Examples

We give two examples of ACK manifolds admitting HRS.

Example 1. Let $M = \mathbb{R}^3$ be a manifold with the standard coordinates (y_1, y_2, y_3) . We proceed to establish the metric g on M accordingly

$$g = dy_1^2 + dy_2^2 + e^{2y_3}(y_1^2 e^{2y_3} + y_2^2 e^{2y_3} + 1)dy_3^2 - 2y_2 e^{2y_3} dy_1 dy_3 + 2y_1 e^{2y_3} dy_2 dy_3.$$

We examine the orthonormal vector fields defined as follows:

$$w_1 = \frac{\partial}{\partial y_1}, \quad w_2 = \frac{\partial}{\partial y_2}, \quad w_3 = y_2 e^{y_3} \frac{\partial}{\partial y_1} - y_1 e^{y_3} \frac{\partial}{\partial y_2} + e^{-y_3} \frac{\partial}{\partial y_3}.$$

We get $[w_1, w_2] = 0$, $[w_1, w_3] = -e^{y_3}w_2$, $[w_2, w_3] = e^{y_3}w_1$ and other Lie brackets are equal zero. We define the structure (ϕ, ξ, η) on M through the following means

$$\xi = w_3, \quad \eta = e^{y_3}dy_3, \quad \phi = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The 2-form Φ is determined by

$$\Phi = -2dy_1 \wedge dy_2 - 2y_2e^{2y_3}dy_2 \wedge dy_3 + 2y_1e^{2y_3}dy_3 \wedge dy_1.$$

It is important to observe that the relationships defined in equations (3) and (4) are satisfied. Forms η and Φ are closed. Hence, M is an ACK manifold. The Levi-Civita connection, denoted as ∇ , is defined by

$$\nabla_{w_i}w_j = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ e^{y_3}w_2 & -e^{y_3}w_1 & 0 \end{pmatrix}.$$

By direct calculation, it follows that the Ricci tensor vanishes in M . We have $\mathcal{L}_\xi g = 0$, and applying the Lie derivative twice yields $(\mathcal{L}_\xi \circ \mathcal{L}_\xi)g = 0$. Consequently, the tuple (g, ξ, λ, μ) constitutes a steady HRS on the manifold M .

Example 2. Let (y_1, y_2, y_3) represent the standard coordinates in $\mathbb{R}^3$, and define the set $M = \{(y_1, y_2, y_3) \in \mathbb{R}^3 \mid y_3 \neq 0\}$. The vector fields given by

$$w_1 = \frac{\partial}{\partial y_1}, \quad w_2 = \frac{\partial}{\partial y_2}, \quad w_3 = \alpha y_1 \frac{\partial}{\partial y_1} - \alpha y_2 \frac{\partial}{\partial y_2} + \frac{\partial}{\partial y_3}$$

are recognized as linearly independent vector fields for some constant α . We define the metric g as follows

$$g(w_j, w_j) = 1, \quad 1 \leq j \leq 3,$$

and otherwise is equal zero. We give $\xi = w_3$, $\eta(Y) = g(Y, w_3)$, and

$$\phi(w_1) = w_2, \quad \phi(w_2) = -w_1, \quad \phi(w_3) = 0.$$

The statements presented in equations (3) and (4) are indeed valid. Also, $[w_1, w_2] = 0$, $[w_1, w_3] = \alpha w_1$, $[w_2, w_3] = -\alpha w_2$ and other Lie brackets are equal zero. Also, we get

$$\nabla_{w_i}w_j = \begin{pmatrix} -\alpha w_3 & 0 & \alpha w_1 \\ 0 & -\alpha w_3 & -\alpha w_2 \\ 0 & 0 & 0 \end{pmatrix}.$$

Therefore, for any vector fields $X = \sum_{i=1}^3 X^i w_i$ and $Y = \sum_{i=1}^3 Y^i w_i$ we have $\nabla_X w_3 = \alpha(X^1 w_1 - X^2 w_2)$ and

$$g(Y, \nabla_X w_3) = \alpha(X^1 Y^1 - X^2 Y^2) = g(X, \nabla_Y w_3).$$

Now, for any vector fields X, Y we have

$$\begin{aligned} d\eta(X, Y) &= X(\eta(Y)) - Y(\eta(X)) - \eta([X, Y]) \\ &= X(g(Y, w_3)) - Y(g(X, w_3)) - g([X, Y], w_3) \\ &= g(\nabla_X Y, w_3) + g(Y, \nabla_X w_3) - g(\nabla_Y X, w_3) - g(X, \nabla_Y w_3) \\ &\quad - g(\nabla_X Y - \nabla_Y X, w_3) = 0. \end{aligned}$$

Hence, $d\eta = 0$. The 2-form Φ is determined by

$$\Phi = -2dy_1 \wedge dy_2 - 2\alpha y_2 dy_1 \wedge dy_3 + 2\alpha y_1 dy_3 \wedge dy_2.$$

Thus, $d\Phi = 0$ and (M, ϕ, ξ, η, g) constitutes an ACK manifold. The Ricci tensor is determined as $S = -2\alpha^2 dy_3^2$. Let $V = w_2$, then $\mathcal{L}_V g = 2\alpha dy_2 dy_3$ and $(\mathcal{L}_V \circ \mathcal{L}_V)g = 2\alpha^2 dy_3^2$. Therefore $(g, w_2, 0, 0)$ is an AHRs on the manifold M .

Conclusion

The primary focus of this paper is to derive the geometric conditions and characteristics of AHRs, with the intention of applying these findings to ACK manifolds. Initially, we consider the parameters $(g, f\xi, \lambda, \mu)$ that conform to an AHRs and present a differential equation that the function f must satisfy. Furthermore, we outline specific geometric conditions under which ACK manifolds can be classified as AHRs. Our investigation also extends to the case of a HRS (g, V, λ, μ) on ACK manifolds, particularly when the vector field V is characterized as a conformal Killing, a torse-forming, or a bi-conformal vector field. This study has prompted several questions that may serve as avenues for future research.

1. If we consider the hyperbolic Yamabe solitons instead of hyperbolic Ricci solitons, what results can be obtained?
2. If we examine the $*$ -Ricci tensor in place of the Ricci tensor, specifically in the context of hyperbolic $*$ -Ricci solitons, do the conclusions of this paper remain applicable?
3. Are the results of this paper still valid when we take into account an alternative connection, such as the Shouten-van Kampen connection?

Author contributions

All authors contributed equally in the preparation of this manuscript.

Conflict of interest

The authors declare no conflict of interest.

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Existence, uniqueness, and Ulam–Hyers stability of periodic solutions for a discrete Nicholson’s blowflies equation with proportional delay

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This paper investigates the existence, uniqueness, and stability of periodic solutions for a discrete Nicholson’s blowflies equation with proportional delay. Proportional delay, in which the delay is defined as a fraction of the current time, naturally arises in many discrete dynamical systems, including neural networks and biological population models. The original problem is transformed into an equivalent fixed-point equation by introducing a suitable operator on a Banach space of periodic functions. Sufficient conditions for the existence of N -periodic solutions are established using the Leray–Schauder nonlinear alternative, while uniqueness is obtained through Banach’s fixed-point theorem. In addition, the Ulam–Hyers stability of the periodic solution is investigated, showing that small perturbations in the governing equation lead to only small deviations from the exact solution. To validate the theoretical findings, numerical simulations are carried out using Python. Several representative examples are considered by varying the proportional-delay parameter, delay structure, birth coefficient, and nonlinear coefficient. The numerical results confirm the analytical predictions and clearly illustrate the influence of these parameters on the existence, stability, and qualitative behavior of the periodic solutions, thereby demonstrating the applicability and effectiveness of the proposed theoretical approach.

Keywords: difference equation, Nicholson’s blowflies model, proportional delay, periodic solution, fixed point theorem, Ulam–Hyers stability, existence of solution, uniqueness of solution.

2020 Mathematics Subject Classification: 39A23, 39A27, 39A30, 39A60.

Introduction

In this paper, we explore the existence, uniqueness, and Ulam–Hyers stability of solutions for a Nicholson’s blowflies difference equation with a proportional delay

$$\psi(k+1) - \psi(k) = -a(k)\psi(k) + b(k)\psi([q\tau(k)])e^{-c(k)\psi([q\tau(k)])}, \quad (1)$$

where $\psi(k)$ is the size of population at the discrete time k , $a(k)$ is the per capita daily adult death rate, $b(k)$ is the maximum per capita daily egg production rate, $[q\tau(k)]$ represents the delay term in the equation, $0 < q < 1$, the symbol $[\cdot]$ denotes the integer part. We assume that

- (i) ψ is N -periodic, $\psi(k) = \psi(k+N)$ for $k \in \mathbb{Z}$, $N \in \mathbb{N}$, $N \geq 1$.
- (ii) a, b, c are nonnegative, N -periodic functions and $0 < a(k) < 1$.
- (iii) $\tau: \mathbb{Z} \rightarrow \mathbb{R}_+$ is N -periodic function with $\tau(k) \leq k$.

Our objective is to analyze the delay difference equation (1) by applying fixed point theorems and conducting a numerical study for some specific cases. This model is a well-known example in population dynamics, particularly for studying insect populations. It was originally introduced by Nicholson in the 1950s to describe the population changes of the Australian sheep blowfly. In [1], Gurney et al. proposed a formal mathematical version of this model, known as the Nicholson blowflies equation, which was based on Nicholson’s experimental data, that is expressed as

$$\psi'(t) = -a\psi(t) + b\psi(t-\tau)e^{-c\psi(t-\tau)},$$

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where $\psi(t)$ is the population size at time t , $a > 0$ is the death rate of the adult flies, $b > 0$ is the maximum per capita daily egg production rate, τ is the time delay representing the developmental period from egg to adult, $1/c$ is the size at which the population reproduces at its maximum rate.

One interesting application of the Nicholson blowflies model is in modeling hematopoiesis, the process of forming blood cells (red blood cells, white blood cells, and platelets) in the bone marrow. In this context, Lasota [2] proposed the following discrete model for red blood cell production based on experimental results:

$$\psi(n+1) - \psi(n) = -a\psi(n) + (c\psi(n))^m e^{-c\psi(n)},$$

where $\psi(n)$ denotes the number of cells at time n , and $a\psi(n)$ the number of cells destroyed in the time interval $[n, n+1]$, $a \in (0, 1)$, c, m are positive constants. The term $(c\psi(n))^m e^{-c\psi(n)}$ represents the number of cells which are produced in the bone marrow during the same period.

This adaptation of the Nicholson blowflies model helps to understand the dynamics of red blood cell production and can provide a better understanding of the regulation and behavior of blood cell populations in the bone marrow.

Recently, the results of Lasota's paper were revisited in the paper [3], where the authors studied this model and applied it to population dynamics with specific seasons, adult survival rates, and density-dependent effects.

Nicholson's blowflies difference equations, as well as differential equations, have seen significant progress in the last decades. In the paper [4], the authors studied the following discrete Nicholson's blowflies type system:

$$\begin{aligned}\psi_1(k+1) &= a_{11}(k)\psi_1(k) + a_{12}(k)\psi_2(k) \\ &\quad + b(k)(\psi_1(k-\tau) + \psi_2(k-\tau))^m \\ &\quad e^{-c(k)(\psi_1(k-\tau) + \psi_2(k-\tau))}, \\ \psi_2(k+1) &= a_{21}(k)\psi_1(k) + a_{22}(k)\psi_2(k) \\ &\quad + b(k)(\psi_1(k-\tau) + \psi_2(k-\tau))^m \\ &\quad e^{-c(k)(\psi_1(k-\tau) + \psi_2(k-\tau))},\end{aligned}$$

where $m > 1$ is a constant, τ is a nonnegative integer, and a_{ij} , $i, j = 1, 2$, b and c are all N -periodic functions from $\mathbb{Z}$ to $\mathbb{R}$. They proved the existence of three nonnegative periodic solutions by using the Leggett–Williams fixed point theorem.

In the paper [5], the authors analyzed the global exponential convergence of Nicholson's blowflies model with proportional delays,

$$\psi'(t) = -a\psi(t) + \sum_{i=1}^m b_i(t)\psi(\tau_i t) e^{-c_i(t)\psi(\tau_i t)}, \quad t \geq 0.$$

Here $(\tau_i t)$ is the proportional delay term with $0 < \tau_i < 1$.

Proportional delays arise naturally in many mathematical and engineering models, particularly in neural networks and biological systems, where the delay depends on a proportion of time rather than a fixed constant. Since such delays are time-dependent and unbounded, their analysis is considerably more challenging. Foundational results on difference equations and delayed dynamical systems can be found in [6, 7]. Stability issues for delayed systems have been investigated in [8–10]. Related exponential stability results for neural networks with proportional delays were obtained in [11]. Nicholson-type models with proportional delays and convergence properties were studied in [12, 13]. The existence of periodic and almost periodic solutions for Nicholson-type systems was examined in [14–16]. Related

results for Nicholson-type models with nonlinear mortality terms and fractional dynamics can be found in [17, 18].

Most existing studies on Nicholson-type equations have focused on periodicity, stability, and extinction properties. Almost periodic solutions and related dynamics were investigated in [19–21]. Further results on periodic and almost periodic solutions were obtained in [22, 23]. Stability and extinction properties were analyzed in [24, 25]. However, the Ulam–Hyers stability of discrete Nicholson equations with proportional delays and integer-part arguments has not yet been studied.

In this paper, we investigate a discrete periodic Nicholson’s blowflies model with proportional delay of the form $\lfloor q\tau(k) \rfloor$, $0 < q < 1$.

The combination of the proportional parameter, the time-dependent delay function, and the integer-part operator produces a nonsmooth and piecewise constant delay structure, which increases the difficulty of the analysis.

By using the Leray–Schauder nonlinear alternative and Banach’s contraction principle, we establish the existence, uniqueness, and Ulam–Hyers stability of N -periodic solutions. Numerical simulations are also presented to illustrate the influence of the proportional-delay parameter and the delay functions on the dynamics of the system.

1 Analysis of the problem

Set $\theta(k) = 1 - a(k)$, and

$$\Lambda(k) = \prod_{s=0}^{k-1} \theta(s) = \theta(0) \theta(1) \dots \theta(k-1),$$

and

$$\Lambda^{-1}(k) = \prod_{s=0}^{k-1} (\theta(s))^{-1},$$

then equation (1) can be written as

$$\psi(k+1) = \theta(k) \psi(k) + b(k) \psi(\lfloor q\tau(k) \rfloor) e^{-c(k)\psi(\lfloor q\tau(k) \rfloor)}. \quad (2)$$

Since $0 < a(k) < 1$, then $0 < \theta(k) < 1$ and hence $0 < \Lambda(k) < 1$.

Lemma 1. A N -periodic function ψ is a solution of the difference equation (1) if and only if it satisfies

$$\psi(k) = \sum_{s=k}^{k+N-1} \mathfrak{G}(k, s) b(s) \psi(\lfloor q\tau(s) \rfloor) e^{-c(s)\psi(\lfloor q\tau(s) \rfloor)}, \quad k \in \mathbb{Z},$$

where

$$\mathfrak{G}(k, s) = (\Lambda^{-1}(N) - 1)^{-1} \Lambda(k) \Lambda^{-1}(s+1).$$

Proof. Taking into account equality (2) and that $0 < \Lambda(k) < 1$, then by computations it yields

$$\begin{aligned} & \Lambda^{-1}(s+1) \psi(s+1) - \Lambda^{-1}(s) \psi(s) \\ &= \Lambda^{-1}(s+1) \left[\theta(s) \psi(s) + b(s) \psi(\lfloor q\tau(s) \rfloor) e^{-c(s)\psi(\lfloor q\tau(s) \rfloor)} \right] - \Lambda^{-1}(s) \psi(s) \\ &= \Lambda^{-1}(s+1) b(s) \psi(\lfloor q\tau(s) \rfloor) e^{-c(s)\psi(\lfloor q\tau(s) \rfloor)}. \end{aligned}$$

On the other hand, we have for $l \geq k$:

$$\begin{aligned} & \Lambda^{-1}(l+1) \psi(l+1) - \Lambda^{-1}(k) \psi(k) \\ &= \sum_{s=k}^l [\Lambda^{-1}(s+1) \psi(s+1) - \Lambda^{-1}(s) \psi(s)] \\ &= \sum_{s=k}^l \Lambda^{-1}(s+1) b(s) \psi(\lfloor q\tau(s) \rfloor) e^{-c(s)\psi(\lfloor q\tau(s) \rfloor)}, \end{aligned}$$

hence

$$\begin{aligned}\Lambda^{-1}(l+1)\psi(l+1) &= \Lambda^{-1}(k)\psi(k) \\ &+ \sum_{s=k}^l \Lambda^{-1}(s+1)b(s)\psi([q\tau(s)])e^{-c(s)\psi([q\tau(s)])},\end{aligned}$$

for $l = k + N - 1$ yields

$$\begin{aligned}\Lambda^{-1}(k+N)\psi(k+N) &= \Lambda^{-1}(k)\psi(k) \\ &+ \sum_{s=k}^{k+N-1} \Lambda^{-1}(s+1)b(s)\psi([q\tau(s)])e^{-c(s)\psi([q\tau(s)])}.\end{aligned}$$

Taking into account that a and ψ are periodic and

$$\Lambda^{-1}(k+N) = \Lambda^{-1}(k)\Lambda^{-1}(N),$$

then

$$\psi(k) = (\Lambda^{-1}(N) - 1)^{-1} \Lambda(k) \sum_{s=k}^{k+N-1} \Lambda^{-1}(s+1)b(s)\psi([q\tau(s)])e^{-c(s)\psi([q\tau(s)])},$$

thus

$$\psi(k) = \sum_{s=k}^{k+N-1} \mathfrak{G}(k, s)b(s)\psi([q\tau(s)])e^{-c(s)\psi([q\tau(s)])}.$$

The properties of the function $\mathfrak{G}$ is given in the following lemma. □

Lemma 2. $\mathfrak{G}(k, s)$ is nonnegative and for $0 \leq k \leq N - 1$, $k \leq s \leq k + N - 1$, we have

$$A_0^N \leq \mathfrak{G}(k, s) \leq \frac{A_1^N}{A_0^N(1 - A_1^N)},$$

where $A_1 = \max_{0 \leq s \leq N-1} \theta(s)$ and $A_0 = \min_{0 \leq s \leq N-1} \theta(s)$.

Proof. Since $0 < \theta(k) < 1$, $k \in \mathbb{Z}$, then $0 < A_0 \leq \theta(k) \leq A_1 < 1$. From the definition of $\Lambda(N)$, we obtain

$$0 < \Lambda(N) = \Pi_{s=0}^{N-1} \theta(s) < 1, \quad A_0^N \leq \Lambda(N) \leq A_1^N.$$

Moreover, we have

$$\mathfrak{G}(k, s) = (1 - \Lambda(N))^{-1} \Lambda(N) \Lambda(k) \Lambda^{-1}(s+1).$$

Since

$$(\Lambda^{-1}(N) - 1)^{-1} = \frac{\Lambda(N)}{1 - \Lambda(N)}$$

we may write

$$\mathfrak{G}(k, s) = \frac{\Lambda(N)}{1 - \Lambda(N)} \prod_{r=k}^s \theta^{-1}(r).$$

Because $\theta(r) \geq A_0$, it follows that

$$\theta^{-1}(r) \leq A_0^{-1},$$

and therefore

$$\prod_{r=k}^s \theta^{-1}(r) \leq A_0^{-(s-k+1)}.$$

Using the inequalities

$$\Lambda(N) \leq A_1^N, \quad 1 - \Lambda(N) \geq 1 - A_1^N,$$

we deduce that

$$\mathfrak{G}(k, s) \leq \frac{A_1^N}{1 - A_1^N} A_0^{-(s-k+1)}.$$

Since $(s - k + 1) \leq N$, we obtain

$$\mathfrak{G}(k, s) \leq \frac{A_1^N}{A_0^N(1 - A_1^N)}.$$

For the lower bound, since $0 < \theta(r) < 1$, we have $\theta^{-1}(r) \geq 1$. Hence,

$$\prod_{r=k}^s \theta^{-1}(r) \geq 1.$$

Using also $\frac{\Lambda(N)}{1-\Lambda(N)} \geq \Lambda(N)$, we get

$$\begin{aligned} \mathfrak{G}(k, s) &= (1 - \Lambda(N))^{-1} \Lambda(N) \Lambda(k) \Lambda^{-1}(s+1) \geq \Lambda(N) \Lambda(k) \Lambda^{-1}(s+1) \\ &= \Lambda(N) \prod_{r=k}^s \theta^{-1}(r) \geq \Lambda(N) \geq A_0^N. \end{aligned}$$

Therefore,

$$A_0^N \leq \mathfrak{G}(k, s) \leq \frac{A_1^N}{A_0^N(1 - A_1^N)}.$$

The proof is complete. □

2 Existence and uniqueness results

We denote by $\mathbb{X}$ the Banach space of all N -periodic functions $\psi: \mathbb{Z} \rightarrow \mathbb{R}$ according to the norm $\|\psi\| = \max_{0 \leq k \leq N-1} |\psi(k)|$. Define the operator $\mathbb{A}$ on $\mathbb{X}$ by

$$\mathbb{A}\psi(k) = \sum_{s=k}^{k+N-1} \mathfrak{G}(k, s) b(s) \psi([q\tau(s)]) e^{-c(s)\psi([q\tau(s)])}.$$

If ψ is a fixed point for the operator $\mathbb{A}$, then it is a solution for the difference equation (1). We have the first existence result:

Theorem 1. Set $b_1 = \max_{0 \leq s \leq N-1} b(s)$, $c_0 = \min_{0 \leq s \leq N-1} c(s)$ and assume that

$$\frac{Nb_1 A_1^N}{A_0^N(1 - A_1^N)} < ec_0, \tag{3}$$

then the difference equation (1) has at least one solution in $\mathbb{X}$.

The proof of Theorem 1 is based on the following fixed point theorem.

Lemma 3 (Leray–Schauder nonlinear alternative). Let E be a Banach space and Ξ a bounded open subset of E , $0 \in \Xi$ and let $\mathcal{A}: \Xi \rightarrow E$ be a completely continuous operator. Then, either there exists $\psi \in \partial\Xi$, $\xi > 1$ such that $\mathcal{A}(\psi) = \xi\psi$, or there exists a fixed point $\psi^* \in \Xi$.

Proof of Theorem 1. Set $\Xi = \{\psi \in \mathbb{X} : \|\psi\| < r\}$, as an open bounded subset in $\mathbb{X}$. In view of the continuity of the functions a, b, c , the operator $\mathbb{A}$ is continuous. Moreover, the space of N -periodic sequences is finite dimensional, hence bounded subsets are relatively compact. Now, let $\psi \in \partial\Xi$, such that $A(\psi) = \xi\psi$, $\xi > 1$; hence, $\psi = \xi^{-1}\mathbb{A}\psi$, $0 < \xi^{-1} < 1$. Using the fact that the function $f(x) = xe^{-cx}$, $x \geq 0$, $c > 0$ satisfies $f(x) \leq \frac{1}{ce}$, and taking into account the estimate (3), we obtain

$$\begin{aligned} |\psi(k)| &= \xi^{-1}|\mathbb{A}\psi(k)| \leq |\mathbb{A}\psi(k)| = \sum_{s=k}^{k+N-1} \mathfrak{G}(k, s)b(s)\psi(\lfloor q\tau(s) \rfloor) e^{-c(s)\psi(\lfloor q\tau(s) \rfloor)} \\ &\leq \frac{Nb_1A_1^N}{A_0^N(1-A_1^N)c_0e} \|\psi\| = \frac{Nb_1A_1^N}{A_0^N(1-A_1^N)c_0e} r < r. \end{aligned}$$

Hence, $\|\psi\| < r$, this contradicts the fact that $\psi \in \partial\Xi$.

By Leray–Schauder nonlinear alternative, we conclude that $\mathbb{A}$ has a fixed point in $\bar{\Xi}$ and then the difference equation (1) has at least one solution in $\mathbb{X}$.

The following theorem provides a uniqueness result.

Theorem 2. Assume that

$$\frac{Nb_1A_1^N}{A_0^N(1-A_1^N)} < 1 \quad (4)$$

holds, then the difference equation (1) has a unique nonnegative solution in $\mathbb{X}_+ = \{\psi \in \mathbb{X}, \psi(k) \geq 0, \forall k \in \mathbb{Z}\}$, where $b_1 = \max_{0 \leq s \leq N-1} b(s)$.

Proof. First, let us prove that $\mathbb{A}(\mathbb{X}_+) \subset \mathbb{X}_+$. Let $\psi \in \mathbb{X}_+$. Then $\psi(k) \geq 0, \forall k \in \mathbb{Z}$. Since $b(s) \geq 0, c(s) \geq 0$, and from Lemma 2, $\mathfrak{G}(k, s) \geq 0$. We have $\psi(\lfloor q\tau(s) \rfloor) \geq 0$ and consequently $e^{-c(s)\psi(\lfloor q\tau(s) \rfloor)} > 0$. Therefore,

$$\mathfrak{G}(k, s)b(s)\psi(\lfloor q\tau(s) \rfloor) e^{-c(s)\psi(\lfloor q\tau(s) \rfloor)} \geq 0$$

for every $s \in \{k, k+1, \dots, k+N-1\}$. Summing these nonnegative terms, we obtain $(\mathbb{A}\psi)(k) \geq 0, \forall k \in \mathbb{Z}$. Hence, $\mathbb{A}\psi \in \mathbb{X}_+$, which proves that $\mathbb{A}(\mathbb{X}_+) \subset \mathbb{X}_+$.

Next, since the derivative of the function $f(x) = xe^{-cx}$, $x \geq 0, c > 0$, given by $f'(x) = e^{-cx}(1-cx)$ satisfies $|f'(x)| \leq 1$, then by the mean value theorem it yields $|xe^{-cx} - ye^{-cy}| \leq |x - y|, x, y \geq 0$.

Now, we prove that $\mathbb{A}$ is a contraction. Let $\psi, v \in \mathbb{X}$. Applying (4), the previous estimate and taking Lemma 2 into account, we get

$$\begin{aligned} |\mathbb{A}\psi(k) - \mathbb{A}v(k)| &\leq \sum_{s=k}^{k+N-1} \mathfrak{G}(k, s)b(s) \\ &\quad \times \left| \psi(\lfloor q\tau(s) \rfloor) e^{-c(s)\psi(\lfloor q\tau(s) \rfloor)} - v(\lfloor q\tau(s) \rfloor) e^{-c(s)v(\lfloor q\tau(s) \rfloor)} \right| \\ &\leq \frac{NA_1^Nb_1}{A_0^N(1-A_1^N)} \|\psi - v\|, \end{aligned}$$

thus

$$\|\mathbb{A}\psi - \mathbb{A}v\| \leq \frac{Nb_1A_1^N}{A_0^N(1-A_1^N)} \|\psi - v\|.$$

Thanks to inequality (4) we conclude that $\mathbb{A}$ is a contraction on $\mathbb{X}$. By Banach's fixed-point theorem, the operator $\mathbb{A}$ possesses a unique fixed point in $\mathbb{X}$. Consequently, the difference equation (1) admits a unique nonnegative N -periodic solution. $\square$

3 Ulam–Hyers stability

We recall that an equation is considered Ulam–Hyers stable if, for every approximate solution of the equation, there exists an exact solution that is close to it. More precisely:

Definition 1. The difference equation (1) is Ulam–Hyers stable if there exists a real number $\rho > 0$ such that for each $\epsilon > 0$ and for each solution φ of the inequality

$$|\varphi(k) - \sum_{s=k}^{k+N-1} \mathfrak{G}(k, s) b(s) \varphi([q\tau(s)]) e^{-c(s)\varphi([q\tau(s)])}| \leq \epsilon, \quad (5)$$

there exists a solution ψ of the difference equation (1) with

$$|\varphi(k) - \psi(k)| \leq \rho\epsilon, \quad k \in \mathbb{Z}.$$

Theorem 3. Assume that condition (4) holds, then the equation (1) is Ulam–Hyers stable.

Proof. Let ψ be a solution of the difference equation (1) and φ satisfying inequality (5), then

$$\begin{aligned} |\varphi(k) - \psi(k)| &= \left| \varphi(k) - \sum_{s=k}^{k+N-1} \mathfrak{G}(k, s) b(s) \psi([q\tau(s)]) e^{-c(s)\psi([q\tau(s)])} \right| \\ &\leq \left| \varphi(k) - \sum_{s=k}^{k+N-1} \mathfrak{G}(k, s) b(s) \varphi([q\tau(s)]) e^{-c(s)\varphi([q\tau(s)])} \right| \\ &\quad + \sum_{s=k}^{k+N-1} \mathfrak{G}(k, s) b(s) \left| \varphi([q\tau(s)]) e^{-c(s)\varphi([q\tau(s)])} - \psi([q\tau(s)]) e^{-c(s)\psi([q\tau(s)])} \right| \\ &\leq \epsilon + \frac{Nb_1 A_1^N}{A_0^N (1 - A_1^N)} \|\psi - \varphi\|. \end{aligned}$$

Set $\alpha = \frac{Nb_1 A_1^N}{A_0^N (1 - A_1^N)}$, hence

$$\|\psi - \varphi\| \leq \epsilon + \alpha \|\psi - \varphi\|,$$

that implies

$$\|\psi - \varphi\| \leq \frac{\epsilon}{1 - \alpha}.$$

Thus equation (1) is Ulam–Hyers stable, where $\rho = 1/(1 - \alpha)$. $\square$

4 Numerical simulation

To illustrate the theoretical results, we consider equation (1) with period $N = 4$. The coefficients are chosen as periodic functions satisfying assumptions (i)–(iii), namely

$$\begin{aligned} a(k) &= \frac{2}{3} - \frac{1}{6}(k \bmod 4), \quad b(k) = 0.001(k \bmod 4), \\ c(k) &= 1 + (k \bmod 4), \quad \tau(k) = \min(k, k \bmod 4). \end{aligned}$$

These choices ensure that $a(k) \in (0, 1)$, $b(k) \geq 0$, and $\tau(k) \leq k$, so that all assumptions of the theoretical analysis are satisfied. Moreover,

$$\theta(k) = \frac{1}{3} + \frac{1}{6}(k \bmod 4), \quad A_1 = \frac{5}{6}, \quad A_0 = \frac{1}{3}, \quad b_1 = 0.0015, \quad b_0 = 0,$$

and

$$\frac{Nb_1 A_1^N}{A_0^{N+1}(1 - A_1^N)} = 0.45268 < 1.$$

Hence, equation (1) admits a unique positive periodic solution in the space $\mathbb{X}$. In addition, the solution is Ulam–Hyers stable with $\rho = 1.8271$.

The periodic solution is computed iteratively through the operator scheme $\psi_{n+1} = \mathbb{A}\psi_n$, starting from a positive periodic initial function. To analyze the influence of the proportional delay parameter, we consider the values $q = 0.2$, $q = 0.4$, and $q = 0.8$, together with the two delay functions

$$\tau_1(k) = \begin{cases} 0, & k \equiv 0 \pmod{4}, \\ 0, & k \equiv 1 \pmod{4}, \\ 1, & k \equiv 2 \pmod{4}, \\ 2, & k \equiv 3 \pmod{4}, \end{cases} \quad \tau_2(k) = \min(k, k \pmod{4}).$$

For $q = 0.2$, the solutions associated with $\tau_1(k)$ and $\tau_2(k)$ are almost indistinguishable, as shown in Figure 1. Both solutions exhibit the same positive 4-periodic oscillatory behavior with nearly identical amplitudes and phases. This indicates that, for small proportional delays, the influence of the delay structure on the dynamics is weak.

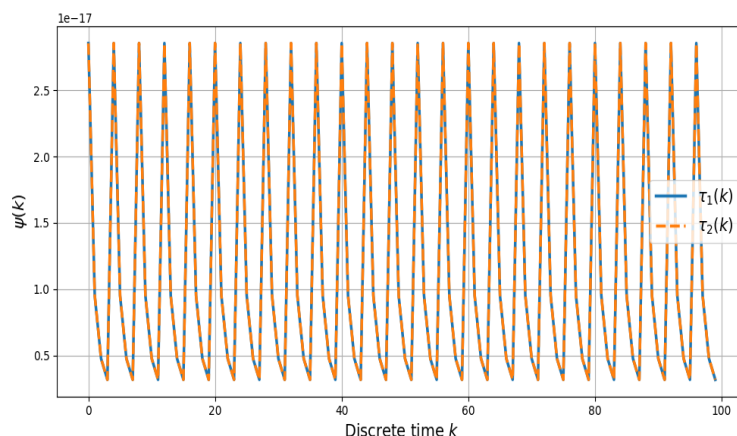


Figure 1. Positive 4-periodic solutions for $q = 0.2$

For $q = 0.4$, the difference between the two delay structures becomes more visible (Fig. 2). The solution corresponding to $\tau_1(k)$ attains larger oscillatory amplitudes, whereas the solution associated with $\tau_2(k)$ remains comparatively smaller. Although both solutions preserve their periodic character, increasing q enhances the contribution of the delayed states and therefore amplifies the sensitivity of the system to the delay structure.

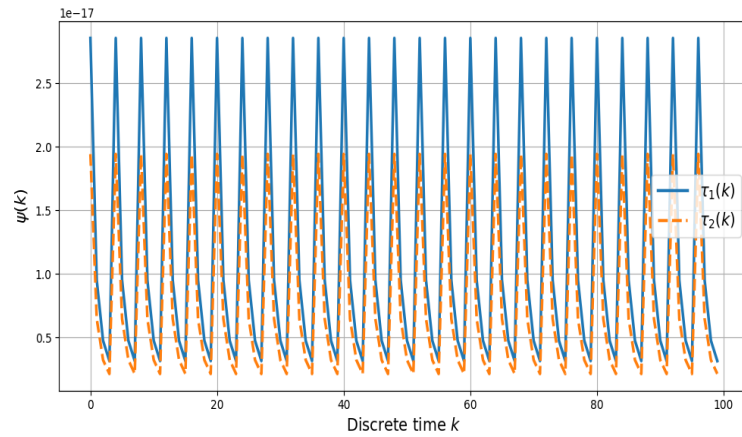


Figure 2. Positive 4-periodic solutions for $q = 0.4$

For $q = 0.8$, the effect of the delay becomes highly pronounced (Fig. 3). The solution associated with $\tau_2(k)$ still exhibits significant oscillations, while the solution generated by $\tau_1(k)$ remains close to zero. This demonstrates that large proportional delays substantially modify the amplitude of the periodic response. Nevertheless, all obtained solutions remain positive, bounded, and 4-periodic, confirming the theoretical predictions.

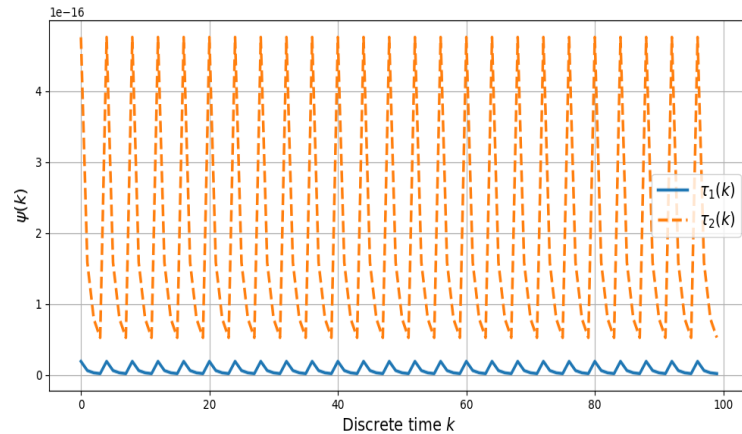


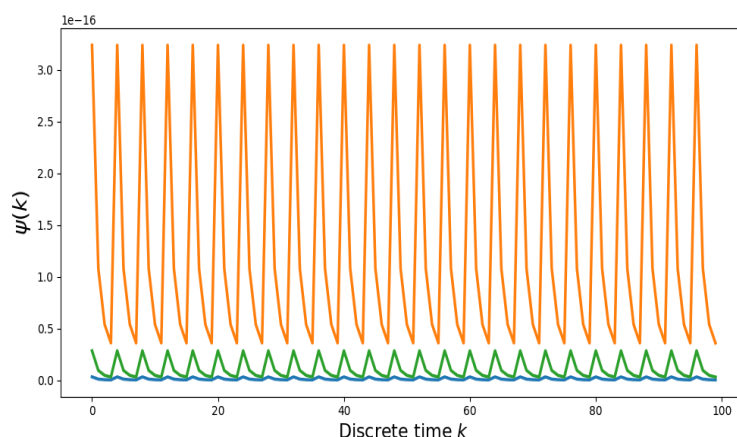
Figure 3. Positive 4-periodic solutions for $q = 0.8$

Next, we investigate the influence of the periodic birth coefficient. Let us choose

$$b(k) = \mu k \pmod{4},$$

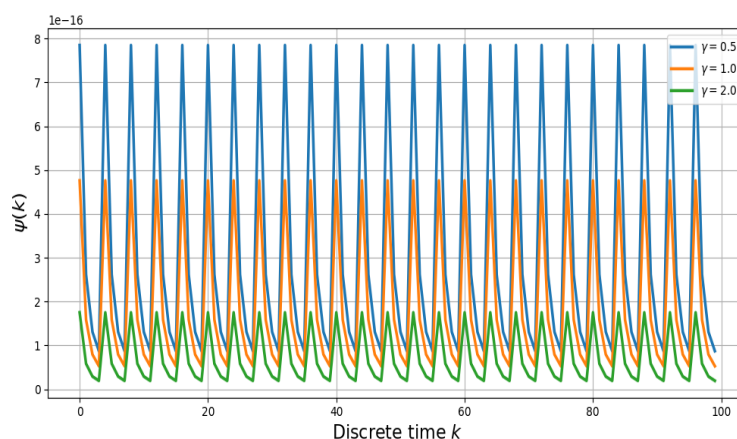
for $\mu = 0.0002$, 0.0005 , and 0.001 , while fixing $q = 0.6$. The results are displayed in Figure 4.

The parameter μ scales the delayed nonlinear birth term $b(k)\psi([q\tau(k)])e^{-c(k)\psi([q\tau(k)])}$, and therefore controls the strength of delayed feedback. For small values of μ , the oscillations remain close to zero, indicating weak reproduction effects. For the intermediate value $\mu = 0.0005$, the oscillatory amplitude becomes maximal, corresponding to the most effective amplification of the dynamics by delayed feedback. When μ increases further to 0.001 , the oscillations decrease again because the exponential damping term dominates the nonlinear response. Thus, the influence of μ is non-monotone due to the saturating behavior of the Nicholson-type nonlinearity xe^{-cx} .

Figure 4. Influence of the birth coefficient $b(k)$

Finally, Figure 5 illustrates the influence of the nonlinear coefficient

$$c(k) = \gamma + (k \bmod 4), \quad \gamma = 0.5, 1, 2.$$

Figure 5. Influence of the nonlinear coefficient $c(k)$

The numerical simulations show that the periodic structure of the solution is preserved for all tested values of γ , while the oscillatory amplitude decreases significantly as γ increases. In particular, $\gamma = 0.5$ produces the largest oscillations, whereas $\gamma = 2$ yields the smallest amplitudes. This behavior is explained by the exponential term $e^{-c(k)\psi([q\tau(k)])}$, which induces stronger damping for larger values of $c(k)$. Consequently, the coefficient $c(k)$ acts as a stabilizing factor that suppresses the magnitude of oscillations without destroying the periodic behavior of the solution.

Conclusion

In this paper, we investigated the existence, uniqueness, and Ulam–Hyers stability of positive periodic solutions for a Nicholson’s blowflies difference equation with proportional delay. The model involves a proportional-delay term combined with the integer-part operator, providing a more flexible framework for describing memory effects in discrete population dynamics. By reformulating the problem as an equivalent fixed-point equation, we established existence and uniqueness result via fixed point theorems. Furthermore, Ulam–Hyers stability of the periodic solution was obtained. Numerical simulations were presented to illustrate the theoretical results and to examine the effects of the

proportional-delay parameter, the delay structure, and the nonlinear coefficients on the qualitative behavior of the solutions. The simulations show that increasing the proportional delay parameter enhances the sensitivity of the dynamics to the delay structure, while larger nonlinear coefficients reduce the oscillation amplitudes due to stronger exponential damping. Future work may focus on asymptotic stability, bifurcation phenomena, and comparisons with classical fixed-delay Nicholson models in order to better understand the role of proportional delays in discrete dynamical systems.

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Author contributions

All authors contributed equally to this work.

Conflict of interest

The authors declare no conflict of interest.

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On a family of spectral problems for fourth-order ordinary differential operators

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Academician Olga Ladyzhenskaya highlighted the significance of developing a fundamental system in the space of solenoidal functions for canonical domains such as cubes, squares, and related geometries. In principle, such a construction could be obtained through the solution of the spectral problem associated with the Stokes operator in these standard domains. In the present paper, we investigate a class of fourth-order ordinary differential operators defined on a finite interval. In the corresponding spectral problems naturally arise in the construction of a fundamental system in the space of solenoidal functions for a cubic domain. It is established that each problem from the considered family possesses a discrete countable spectrum, while the associated eigenfunctions form an orthogonal basis in an appropriate Sobolev functional space. Moreover, the basis property remains valid for the triple direct product of the constructed system. The analysis relies on functional-analytic methods, compact operator theory, and Hilbert–Schmidt spectral theory, providing a rigorous framework for the proposed construction and further developments. By additionally exploiting the properties of the curl operator, one can achieve the original objective without explicitly solving the spectral problem for the Stokes operator.

Keywords: spectral problem, fourth-order ordinary differential operator, fundamental system, solenoidal functions, orthogonal basis, Sobolev space, compact operator, curl operator.

2020 Mathematics Subject Classification: 35P05, 35P10, 47A75, 53C65, 35A08.

Introduction

Throughout this work, let $\Omega_1 \subset \mathbb{R}^d$, with $d \geq 2$, be a domain with Lipschitz boundary $\partial\Omega_1$, and let $\Omega = (0, l)^d \subset \mathbb{R}^d$.

O.A. Ladyzhenskaya notes in [1, p. 4835]: “Our term Fundamental system $\{\varphi_k\}_{k=1}^\infty$ does not require the assumption of linear independence of elements of the system. Instead, the following property is required: for any $\varepsilon > 0$ and any $\varphi \in \overset{\circ}{W}_2^1(\Omega_1)$ there exists a sum $\varphi^{(\varepsilon)} = \sum_{k=1}^{N_\varepsilon} a_k \varphi_k$, $N_\varepsilon < \infty$, such that $\|\nabla(\varphi - \varphi^{(\varepsilon)})\|_{L^2(\Omega_1)} \leq \varepsilon$ ”.

We introduce the function spaces

$$L^2(\Omega_1) \equiv (L^2(\Omega_1))^d \equiv H(\Omega_1) \oplus H^\perp(\Omega_1), \quad (1)$$

$$H(\Omega_1) = \{\vec{w} \mid \vec{w} \in L^2(\Omega_1), \operatorname{div} \vec{w} = 0, \vec{w} \cdot \vec{n}|_{\partial\Omega_1} = 0\}, \quad (2)$$

$$H^\perp(\Omega_1) = \{\vec{w} \mid \vec{w} \in L^2(\Omega_1), \vec{w} = \operatorname{grad} q, q \in W_2^1(\Omega_1)\}, \quad (3)$$

$$V(\Omega_1) = \{\vec{w} \mid \vec{w} \in \overset{\circ}{W}_2^1(\Omega_1), \operatorname{div} \vec{w} = 0\}, \quad (4)$$

$$\overset{\circ}{W}_2^1(\Omega_1) \equiv (\overset{\circ}{W}_2^1(\Omega_1))^d. \quad (5)$$

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Note that here we have a dense and continuous embeddings

$$V(\Omega_1) \hookrightarrow H(\Omega_1) \equiv H'(\Omega_1) \hookrightarrow V'(\Omega_1).$$

Problem A: Construct a system of functions in $V(\Omega)$ (4) that is fundamental (in the sense of Ladyzhenskaya) for the space $H(\Omega)$ (2) (where $H(\Omega)$ is the orthogonal complement of $H^\perp(\Omega)$ (3)).

1 The spectral problem for the Stokes operator. The curl operator

Let $x = (x_1, \dots, x_d) \in \Omega_1 \subset \mathbb{R}^d$, where $d \geq 2$, and let Ω_1 be a bounded, simply connected open domain with boundary $\partial\Omega_1$. We seek nontrivial solutions $\{\vec{w}_k(x), p_k(x), \text{ with } x \in \Omega_1, k \in \mathbb{N}\}$, along with the corresponding eigenvalues $\{\lambda_k, k \in \mathbb{N}\}$ for the following boundary value problem:

$$-\Delta \vec{w}(x) + \nabla p(x) = \lambda \vec{w}(x), \quad x \in \Omega_1, \quad (6)$$

$$\operatorname{div}\{\vec{w}(x)\} = 0, \quad x \in \Omega_1, \quad (7)$$

$$\vec{w}(x) = 0, \quad x \in \partial\Omega_1. \quad (8)$$

First, let us consider the boundary value problem for the Stokes operator

$$-\Delta \vec{w}(x) + \nabla p(x) = \vec{f}(x), \quad x \in \Omega_1, \quad (9)$$

$$\operatorname{div}\{\vec{w}(x)\} = 0, \quad x \in \Omega_1, \quad (10)$$

$$\vec{w}(x) = 0, \quad x \in \partial\Omega_1. \quad (11)$$

As is well known, this problem admits the following variational formulation

$$\sum_{j=1}^d (\nabla w_j, \nabla v_j) \stackrel{\text{def}}{=} ((\vec{w}, \vec{v})) = (\vec{f}, \vec{v}) \quad \forall \vec{v} \in V(\Omega_1). \quad (12)$$

The following theorem holds (see [2, Chapter I, Section 2.2]).

Theorem 1. For an arbitrary open domain $\Omega_1 \subset \mathbb{R}^d$, bounded in at least one direction, and for every $\vec{f} \in L^2(\Omega_1)$, problem (9)–(11) admits a unique solution $w \in V(\Omega_1)$.

Let us return to the spectral problem (6)–(8). According to Theorem 1, the mapping $G : \vec{f} \mapsto w$ is linear and continuous from $L^2(\Omega_1)$ (1) into $V(\Omega_1)$ (4), and hence into $\overset{\circ}{W}_2^1(\Omega_1)$ (5). Since Ω_1 is bounded, the compactness of the embedding $\overset{\circ}{W}_2^1(\Omega_1) \hookrightarrow L^2(\Omega_1)$ implies that the linear operator $G : L^2(\Omega_1) \rightarrow L^2(\Omega_1)$ is compact. This operator is also self-adjoint because, in view of (12) and the symmetry of the operator $\Lambda = G^{-1}$, i.e.,

$$(\Lambda \vec{w}, \vec{v}) = (\vec{w}, \Lambda \vec{v}) \quad \forall w, v \in \mathcal{D}(\Lambda) \equiv V(\Omega_1) \cap \overset{\circ}{W}_2^2(\Omega_1), \quad \text{where} \quad \overset{\circ}{W}_2^2(\Omega_1) = (\overset{\circ}{W}_2^2(\Omega_1))^d,$$

the following equalities hold:

$$(G \vec{f}_1, \vec{f}_2) = (\vec{w}_1, \vec{f}_2) = (\vec{w}_1, \Lambda \vec{w}_2) = ((\vec{w}_1, \vec{w}_2)) = (\Lambda \vec{w}_1, \vec{w}_2) = (\vec{f}_1, \vec{w}_2) = (\vec{f}_1, G \vec{f}_2),$$

where $G f_j = w_j$, $j = 1, 2$.

As a result, we obtain the following theorem.

Theorem 2. [2, chapter I, Point 2.6] For a bounded open domain $\Omega_1 \subset \mathbb{R}^d$ with Lipschitz boundary $\partial\Omega_1$, the operator G possesses an orthonormal sequence of eigenfunctions $\{\vec{w}_k(x), k \in \mathbb{N}\}$ such that the following relations hold:

$$\Lambda \vec{w}_k = \lambda_k \vec{w}_k, \quad k \geq 1, \quad \lambda_k > 0, \quad \lambda_k \rightarrow +\infty, \quad k \rightarrow +\infty, \quad (13)$$

$$\vec{w}_k(x) \in V(\Omega_1) : ((\vec{w}_k, \vec{v})) = \lambda_k (\vec{w}_k, \vec{v}) \quad \forall \vec{v} \in V(\Omega_1), \quad (14)$$

$$(\vec{w}_k, \vec{w}_m) = \delta_{km}, \quad ((\vec{w}_k, \vec{w}_m)) = \lambda_k \delta_{km} \quad \forall k, m \in \mathbb{N}. \quad (15)$$

Thus, the spectral problem (6)–(8) admits a solution satisfying conditions (13)–(15). However, the explicit construction of the system of eigenfunctions $\{\vec{w}_k(x), k \in \mathbb{N}\}$ is an intractable problem, except for the case when the boundary conditions (8) are replaced by periodicity conditions. Similar spectral constructions and basis properties for higher-order differential operators have been extensively studied in recent years [3–5].

The main objective of the present work is to construct a system of functions that is fundamental (in the sense of O.A. Ladyzhenskaya) in the space $H(\Omega_1)$ without solving the spectral problem (6)–(8).

In this work, for simplicity, we will primarily consider the 3-D case. Assume that

$$d = 3, \quad x = (x_1, x_2, x_3) \in \Omega_1 \subset \mathbb{R}^3,$$

and let

$$\vec{U}(x) = (U_1(x), U_2(x), U_3(x)).$$

The components of the velocity vector $\vec{w}(x) = (w_1(x), w_2(x), w_3(x))$ are related to the vector-valued function $\vec{U}(x)$ via the curl operator as follows:

$$\vec{w}(x) = \text{curl}\{\vec{U}(x)\} \stackrel{\text{def}}{=} (\partial_{x_2} U_3 - \partial_{x_3} U_2, \partial_{x_3} U_1 - \partial_{x_1} U_3, \partial_{x_1} U_2 - \partial_{x_2} U_1). \quad (16)$$

In the 3-D case the stream (current) function was not defined.

The following propositions are true [2, Theorem I.1.2–I.1.4, Lemma I.1.3; Appendix I: Propositions 1.2, 2.1].

Proposition 1. Let $\Omega_1 = \Omega = (0, l)^3$. Then the mapping (16)

$$\text{curl} : (\overset{\circ}{W}_2^2(\Omega))^3 \cap (\ker\{\text{curl}\})^\perp \rightarrow \text{curl}\{(\overset{\circ}{W}_2^2(\Omega))^3\} \equiv V(\Omega)$$

is an isomorphism, where $\ker\{\text{curl}\} = (H(\Omega))^\perp$ in space $L^2(\Omega)$.

Proposition 2. Let $\Omega_1 = \Omega = (0, l)^3$. Then the mapping (16)

$$\text{curl} : (\overset{\circ}{W}_2^1(\Omega))^3 \cap (\ker\{\text{curl}\})^\perp \rightarrow \text{curl}\{(\overset{\circ}{W}_2^1(\Omega))^3\} \equiv H(\Omega)$$

is an isomorphism, where $\ker\{\text{curl}\} = (H(\Omega))^\perp$ in space $L^2(\Omega)$.

According to Propositions 1 and 2, it is sufficient to construct a fundamental system of functions in the space $(\overset{\circ}{W}_2^2(\Omega))^3$ that is orthogonal in the space $(\overset{\circ}{W}_2^1(\Omega))^3$. Then, by using the operator curl (16), we will be able to construct a fundamental system of functions (in the sense of O.A. Ladyzhenskaya) in the space $H(\Omega)$ (2).

At this point, there arises the necessity of solving a certain family of spectral problems for fourth-order ordinary differential operators. Related spectral problems for fourth-order differential operators and their self-adjoint extensions have also been investigated in various settings [6–8]. Compactness arguments, spectral asymptotics and basis properties for fourth-order operators continue to play an

important role in the modern theory of operator differential equations [9–11]. The remainder of the paper is devoted to this issue: the formulation of the main result (Theorem 3) is presented in Section 2, while its proof is given in Section 3.

As for the construction of a fundamental system of functions in the space $H(\Omega)$ (2), it can be carried out in a manner similar to that used in our previous works [12–14].

2 A family of spectral problems for fourth-order ordinary differential operators

Let $\Omega = (0, l)$. We introduce the following definitions.

Definition 1. Denote by $V_1(0, l)$ and $V_2(0, l)$ the Hilbert spaces endowed with the inner products and the norms

$$V_1(0, l) : \begin{cases} (u'(x), v'(x))_{L^2(0, l)} \quad \forall u, v \in \overset{\circ}{W}_2^1(0, l), \\ \|u\|_{V_1(0, l)} \equiv [(u'(x), u'(x))_{L^2(0, l)}]^{1/2}, \end{cases} \quad (17)$$

$$V_2(0, l) : \begin{cases} ((u, v)) = (u''(x), v''(x))_{L^2(0, l)} \quad \forall u, v \in \overset{\circ}{W}_2^2(0, l), \\ \|u\|_{V_2(0, l)} \equiv [((u(x), u(x)))]^{1/2}. \end{cases} \quad (18)$$

We consider the following family of spectral problems:

$$\begin{cases} X^{IV}(x) + \mu X(x) = -\lambda^2 X''(x), & x \in (0, l), \\ X(0) = X(l) = X'(0) = X'(l) = 0, & \mu \in \mathbb{R}^1, \end{cases} \quad (19)$$

for which the following theorem holds true.

Theorem 3 (Main result). For each fixed value of the parameter $\mu \in \mathbb{R}^1$, the spectral problem (19) has a countable spectrum $\{\lambda_n^2, n \in \mathbb{N}\}$ with a single limit point $\lambda^2 = +\infty$. The corresponding system of eigenfunctions $\{X_n(x), n \in \mathbb{N}\} \subset V_2(0, l)$ (18) forms an orthogonal basis in the space $V_1(0, l)$ (17).

3 Proof of Theorem 3

To solve $X(x) \in V_2(0, l)$ of the boundary value problem (19) we obtain the integral identity

$$((X, Y)) + \mu g(X, Y) = \lambda^2 (X', Y') \quad \text{for all } Y(x) \in V_2(0, l), \quad (20)$$

where

$$(X, Y)_{V_2(0, l)} \stackrel{\text{def}}{=} ((X, Y)) = \int_0^l X''(x) Y''(x) dx,$$

$$g(X, Y) = \int_0^l X(x) Y(x) dx \leq \tilde{c}_1 \|X\|_{V_1(0, l)} \|Y\|_{V_1(0, l)} \leq c_1 \|X\|_{V_2(0, l)} \|Y\|_{V_2(0, l)}, \quad (21)$$

$$(X', Y') = \int_0^l X'(x) Y'(x) dx \leq \tilde{c}_2 \|X\|_{V_1(0, l)} \|Y\|_{V_1(0, l)} \leq c_2 \|X\|_{V_2(0, l)} \|Y\|_{V_2(0, l)}. \quad (22)$$

By F. Riesz's theorem, the functional $g(X, Y)$ is uniquely representable as a scalar product

$$g(X, Y) = ((AX, Y)), \quad \forall Y \in V_2(0, l), \quad (23)$$

where A is a bounded linear operator in $V_2(0, l)$ with norm not exceeding c_1 from (21). Next, we have

$$((BX, Y)) = \int_0^l X'(x) Y'(x) dx, \quad \forall Y \in V_2(0, l), \quad (24)$$

where B is a bounded linear operator in $V_2(0, l)$ with norm not exceeding c_2 from (22). The operators A and B are linear and completely continuous in the space $V_2(0, l)$ (as will be shown below). Moreover, the operator A is symmetric and the operator B is symmetric and positive, i.e.

$$((BX, Y)) = ((X, BY)), \quad \forall X, Y \in V_2(0, l); \quad ((BX, X)) > 0 \quad \forall X \neq 0. \quad (25)$$

As a result, we arrive at the functional equation

$$X + \mu AX = \lambda^2 BX \quad (26)$$

in the space $V_2(0, l)$. We represent the equation (26) as

$$(I + \mu A + \lambda_0^2 B)X = (\lambda^2 + \lambda_0^2)BX, \quad (27)$$

where I is the identity operator. Obviously, for a sufficiently large real λ_0^2 , the operator $D = I + \mu A + \lambda_0^2 B$ has a bounded inverse, i.e.,

$$X = (\lambda^2 + \lambda_0^2)D^{-1}BX.$$

The operator $D^{-1}B$, as a product of a bounded operator and a completely continuous operator, is a completely continuous operator.

Let us show that for a sufficiently large real λ_0^2 , the operator $D = I + \mu A + \lambda_0^2 B$ has a bounded inverse. We denote $DX = Z$. According to (23)–(24), the equality (27) is equivalent to the integral identity (20) if we accept the assumption: $\lambda^2 = \lambda_0^2$. Then from the identity (20) and the inequality

$$((X, X)) + \mu g(X, X) \geq \|X\|_{V_2(0, l)}^2 - |\mu|c_3\|X\|_{V_1(0, l)}^2,$$

we obtain $((Z, X)) = ((X, X)) + \mu g(X, X) + \lambda_0^2\|X\|_{V_1(0, l)}^2$

$$\geq \|X\|_{V_2(0, l)}^2 + (\lambda_0^2 - |\mu|c_3)\|X\|_{V_1(0, l)}^2.$$

Hence, for $\lambda_0^2 \geq |\mu|c_3$ we obtain

$$\|X\|_{V_2(0, l)} \leq \|Z\|_{V_2(0, l)} = \|DX\|_{V_2(0, l)}, \quad \|D^{-1}Z\|_{V_2(0, l)} \leq \|Z\|_{V_2(0, l)},$$

i.e., the operator D for such λ_0^2 has a bounded inverse.

We use the following assertion: Corollary from the monograph [15, Chapter XII, §2, pp. 460–461]:

Corollary. A continuous inverse operator D^{-1} exists if and only if a continuous inverse operator $(D^*)^{-1}$ exists. Moreover,

$$(D^*)^{-1} = (D^{-1})^*. \quad (28)$$

Using the self-adjointness of the operator D , from (28) we obtain the self-adjointness of the inverse operator D^{-1} , i.e.,

$$D^{-1} = (D^*)^{-1} = (D^{-1})^*.$$

Along with the previous Corollary, one could also use Theorem 5.30 from [16, Chapter III, §5, p. 169]:

Theorem 5.30. Let $D \in \mathcal{C}(X, Y)$ be densely defined. If D^{-1} exists and belongs to $\mathcal{B}(Y, X)$, then $(D^*)^{-1}$ exists and belongs to $\mathcal{B}(X^*, Y^*)$, with

$$(D^*)^{-1} = (D^{-1})^*. \quad (29)$$

Conversely, if $(D^*)^{-1}$ exists and belongs to $\mathcal{B}(X^*, Y^*)$, then D^{-1} exists and belongs to $\mathcal{B}(Y, X)$ and (29) holds.

The set of all closed operators from X to Y is denoted by $\mathcal{C}(X, Y)$. The set of all bounded operators from X to Y is denoted by $\mathcal{B}(X, Y)$.

Furthermore, the following assertions are needed for the proof of our main result:

Theorem 2 from [17, chapter IV, § 6, p. 243]:

Theorem 2. If C is a compact operator and D is a bounded operator, then the operators CD and DC are compact.

Theorem 2 from [18, chapter IV, §18.2, p. 189]:

Theorem 2. Let operators C and D be self-adjoint. The operator CD is self-adjoint if and only if C and D are permutable.

Let us show that the self-adjoint operators D^{-1} and B from (27) are permutable. For this, we use Proposition g) from [19, chapter VIII, point 116, pp. 300–301]:

Proposition. Let A be a bounded linear operator and B is a (general) linear operator. Then the permutability relation satisfies the following rule:

(g) If the operator B^{-1} exists, then the permutability of A and B implies that A and B^{-1} are permutable.

Proof of point (g) of the above Proposition. For all $f \in \mathcal{D}(B^{-1})$, we have

$$Af = ABB^{-1}f = BAB^{-1}f,$$

therefore, $Af \in \mathcal{D}(B^{-1})$ and

$$B^{-1}Af = AB^{-1}f \quad \forall f \in \mathcal{D}(B^{-1}).$$

It remains to establish that the operators A and B are permutable. This will be shown below.

Thus, it remains to use the spectral properties of a completely continuous self-adjoint operator [17, chapter IV, §6, pp. 246–247, Hilbert–Schmidt Theorem], which completes the proof of Theorem 3.

With respect to relations (21) and (23) we have

$$\begin{aligned} -g(X, Y) &= -\int_0^l X(x) Y(x) dx = \int_0^l \int_0^x X'(\xi) d\xi \int_0^x Y'(\xi) d\xi dx \\ &= \int_0^l d \left(\int_0^x (x - \xi) X'(\xi) d\xi \right) \int_0^x Y'(\xi) d\xi = -\int_0^l \int_0^x (x - \xi) X'(\xi) d\xi Y'(x) dx \\ &= -\int_0^l d \left(\int_0^x \frac{(x - \xi)^2}{2} X'(\xi) d\xi \right) Y'(x) = \int_0^l \int_0^x \frac{(x - \xi)^3}{6} X''(\xi) d\xi Y''(x) dx \\ &= -((AX(x), Y(x))), \quad \text{where } A \in \mathcal{L}(V_2(0, l), V_2(0, l)). \end{aligned} \quad (30)$$

$$AX(x) = \int_0^x \frac{(x - \xi)^5}{120} X''(\xi) d\xi. \quad (31)$$

With respect to relations (22), (24)–(25) we have

$$\begin{aligned}
 & \int_0^l X'(x) Y'(x) dx = - \int_0^l X(x) Y''(x) dx \\
 &= - \int_0^l \int_0^x (x - \xi) X''(\xi) d\xi Y''(x) dx = \int_0^l [BX(x)]'' Y''(x) dx \\
 &= ((BX(x), Y(x))), \quad \text{where } B \in \mathcal{L}(V_2(0, l), V_2(0, l)),
 \end{aligned} \tag{32}$$

i.e.,

$$BX(x) = - \int_0^x \frac{(x - \xi)^3}{6} X''(\xi) d\xi. \tag{33}$$

Next, from (31) and (33) we obtain the equalities (where $X(x) \in V_2(0, l)$):

$$\begin{aligned}
 -720ABX &= \int_0^x (x - \xi_1)^5 \int_0^{\xi_1} (\xi_1 - \xi)^3 X''(\xi) d\xi d\xi_1 \\
 &= \frac{5}{4} \int_0^x \int_{\xi}^x (x - \xi_1)^4 (\xi_1 - \xi)^4 d\xi_1 X''(\xi) d\xi, \\
 -720BAX &= \int_0^x (x - \xi_1)^3 \int_0^{\xi_1} (\xi_1 - \xi)^5 X''(\xi) d\xi d\xi_1 \\
 &= \frac{5}{4} \int_0^x \int_{\xi}^x (x - \xi_1)^4 (\xi_1 - \xi)^4 d\xi_1 X''(\xi) d\xi, \\
 &= \int_{\xi}^x (x - \xi_1)^4 (\xi_1 - \xi)^4 d\xi_1 = \frac{(x - \xi)^9}{A_9^3}
 \end{aligned}$$

which implies that $AB = BA$. Therefore, A and B are permutable.

We show that A is a completely continuous operator in $V_2(0, l)$. For this purpose, we verify that any weakly convergent sequence $\{v_k\}_{k=1}^{\infty}$ in $V_2(0, l)$ is transformed by A into a strongly convergent sequence $\{Av_k\}_{k=1}^{\infty}$. Since A is bounded, $\{Av_k\}_{k=1}^{\infty}$ weakly converges to Au , where $u(x)$ is the weak limit of $\{v_k\}_{k=1}^{\infty}$.

Further, in view of the compactness of the embedding operator $V_2(0, l) \hookrightarrow V_1(0, l)$, the sequences $\{v_k\}_{k=1}^{\infty}$ and $\{Av_k\}_{k=1}^{\infty}$ converge strongly in $V_1(0, l)$ to u and Au , respectively. Using (23) and (30) of the operator A , we obtain the estimate

$$\begin{aligned}
 ((A(v_k - v_m), A(v_k - v_m))) &= g(v_k - v_m, A(v_k - v_m)) \\
 &\leq \tilde{c}_1 \|v_k - v_m\|_{V_1(0, l)} \|A(v_k - v_m)\|_{V_1(0, l)}.
 \end{aligned}$$

From this it is clear that as $k, m \rightarrow \infty$ the right-hand side tends to zero, and therefore $\{Av_k\}_{k=1}^{\infty}$ is indeed a sequence strongly convergent in $V_2(0, l)$. Note that the constant $\tilde{c}_1$ is taken from inequality (21).

Thus, the complete continuity of the operator A is proven.

We show that B is a completely continuous operator in $V_2(0, l)$. For this purpose, we verify that any weakly convergent sequence $\{v_k\}_{k=1}^\infty$ in $V_2(0, l)$ is transformed by B into a strongly convergent sequence $\{Bv_k\}_{k=1}^\infty$.

Since B is bounded, $\{Bv_k\}_{k=1}^\infty$ weakly converges to Bu , where $u(x)$ is the weak limit of $\{v_k\}_{k=1}^\infty$.

Further, in view of the compactness of the embedding operator $V_2(0, l) \hookrightarrow V_1(0, l)$, the sequences $\{v_k\}_{k=1}^\infty$ and $\{Bv_k\}_{k=1}^\infty$ converge strongly in $V_1(0, l)$ to u and Bu , respectively. Using (24) and (32) of the operator B , we obtain the estimate

$$\begin{aligned} ((B(v_k - v_m), B(v_k - v_m))) &= \int_0^l (v'_k - v'_m)[B(v_k - v_m)]' dx \\ &\leq \tilde{c}_2 \|v_k - v_m\|_{V_1(0, l)} \|B(v_k - v_m)\|_{V_1(0, l)}. \end{aligned}$$

From this it is clear that as $k, m \rightarrow \infty$ the right-hand side tends to zero, and therefore $\{Bv_k\}_{k=1}^\infty$ is indeed a sequence strongly convergent in $V_2(0, l)$. Note that the constant $\tilde{c}_2$ is taken from inequality (22).

Thus, the complete continuity of the operator B is proven.

Conclusion

In this work, we studied a family of spectral problems for fourth-order ordinary differential operators on a finite interval. These spectral problems arise in the construction of a fundamental system in the space of solenoidal functions for a cubic domain. We showed that each spectral problem in the considered family has a countable spectrum, while the corresponding system of eigenfunctions forms an orthogonal basis in a Sobolev functional space. These results can be used for the construction of a fundamental system in the space of solenoidal functions for a cubic domain. The proposed approach is also consistent with recent developments in spectral analysis of higher-order operators, biharmonic problems and generalized Helmholtz/curl decompositions [20–22].

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Author contributions

M.T. Jenaliyev served as the principal investigator of the research grant and supervised the research process. M.G. Yergaliyev and K.S. Sharipov collected and analyzed data, and led manuscript preparation. All authors participated in the revision of the manuscript and approved the final submission.

Conflict of interest

The authors declare no conflict of interest.

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Nonlocal problem for multi-parametric integral-differential equation

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This paper investigates a nonlocal boundary value problem for a multi-parametric integral-differential equation involving a Caputo–Prabhakar type operator in a bounded rectangular domain. This operator generalizes both the classical Caputo fractional derivative and the Prabhakar integral, incorporating memory effects through kernels expressed in terms of generalized Mittag-Leffler functions. The nonlocal conditions are prescribed by partial integral expressions involving the unknown function and given continuous kernels, supplemented by a boundary condition along a characteristic line. Using a known representation of the solution to the corresponding Goursat problem in terms of bivariate and trivariate Mittag-Leffler-type functions, we reduce the problem to a system of Volterra integral equations of the second kind for the unknown boundary traces. Two cases, depending on whether an auxiliary constant vanishes, are analysed separately. In the degenerate case, a first-kind Volterra equation arises and is transformed into a second-kind one by differentiation. Based on the boundedness properties of the special functions involved, sufficient conditions ensuring the existence and uniqueness of the solution are established, and an explicit representation of the solution is obtained through the derived integral system.

Keywords: integral-differential equation, nonlocal problem, integral equation, Goursat problem, multi-parametric integral-differential operator, Mittag-Leffler-type function, Caputo–Prabhakar operator, Volterra integral equation, existence and uniqueness.

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Introduction

Equations involving mixed partial derivatives of the form

$$\frac{\partial^2 u(t, x)}{\partial t \partial x} - \lambda u(t, x) = f(t, x)$$

play an important role in the theory of hyperbolic-type equations and arise naturally in the study of transport-type processes and related mathematical models (see [1–3]). Such equations are closely connected with Goursat-type problems, where boundary conditions are prescribed along characteristic curves.

A powerful approach to the analysis of such problems is based on the use of integral representations of solutions. These representations allow one to reduce boundary value problems to systems of Volterra integral equations of the second kind, which provide an effective framework for establishing existence and uniqueness results.

A significant direction in the development of this theory concerns boundary conditions of nonlocal type. In such problems, the values of the unknown function on the boundary are connected through integral relations with given kernels. Problems with integral boundary conditions have been widely studied; for instance, in [4–6], nonlocal conditions for related equations were considered, and the

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corresponding problems were reduced to Volterra integral equations. This approach made it possible to establish existence and uniqueness theorems and to obtain representations of solutions.

In recent years, significant attention has been devoted to the study of *integral-differential equations with multi-parametric operators*. Among them, Prabhakar-type operators [7] play an increasingly important role due to their flexibility and ability to incorporate memory effects through special kernels expressed in terms of Mittag-Leffler-type functions. These operators generalize classical fractional derivatives and allow the construction of more general analytical models. Such operators allow one to construct multi-parametric integral-differential models that extend classical equations while preserving their analytical structure.

The analysis of these generalized equations is closely connected with the theory of Mittag-Leffler-type functions. The main results and methodological contributions of the works under consideration are discussed in detail in the subsequent paragraphs. These functions arise naturally in the representation of solutions to integral-differential equations and play a role analogous to the exponential function in classical analysis.

Using representations involving Mittag-Leffler-type functions, boundary value problems for multi-parametric integral-differential equations can be reduced to systems of Volterra integral equations. This reduction provides an effective tool for studying solvability, since the theory of integral equations guarantees existence and uniqueness of solutions under appropriate conditions on the kernels and given functions.

Despite the significant progress achieved in the study of nonlocal problems and integral-differential generalizations, equations involving multi-parametric operators of Prabhakar type combined with mixed derivative structures remain insufficiently investigated. In particular, the extension of equation (2) to the case of multi-parametric integral-differential operators leads to new classes of problems that require further analysis. This motivates the present study.

In parallel, substantial progress has been made in the theory of special functions tailored to fractional-order equations. Generalized Mittag-Leffler-type functions – including multivariate and multiparameter extensions – have emerged as natural solution kernels for fractional differential equations. A significant contribution in this area is the introduction of the bivariate Mittag-Leffler-type functions $E_1(x, y)$ and $E_2(x, y)$ by [8]. Although the analytical properties of $E_1(x, y)$ were thoroughly investigated in that work, the corresponding properties of $E_2(x, y)$ remained unaddressed. This gap was subsequently filled by a series of studies [9–11], which systematically examined the mathematical characteristics of $E_2(x, y)$ and demonstrated its applicability to fractional-order partial differential equations.

In [12], a novel trivariate Mittag-Leffler function was introduced, and its complex integral representation, Laplace transform relations, and fractional derivative properties were systematically investigated, along with its applications to the solution of multi-term fractional differential equations. The work in [13] presents a new class of Mittag-Leffler functions defined through triple series, together with the development of associated fractional calculus operators, culminating in fully closed-form solutions. In [14], four new trivariate Mittag-Leffler-type functions $\overline{F}_A^{(3)}$, $\overline{F}_B^{(3)}$, $\overline{F}_C^{(3)}$, and $\overline{F}_D^{(3)}$ were introduced and thoroughly examined; these functions constitute natural generalizations of the classical Lauricella hypergeometric functions to three variables, and their fundamental properties, integral representations, and relevance to fractional differential equations were established in detail. Similarly, several further classes of trivariate Mittag-Leffler-type functions were introduced and investigated in [15–17].

Regarding boundary and nonlocal problems for fractional generalization of hyperbolic equations, a growing body of literature has emerged in recent years. R.A. Pshibikhova [4] studied a nonlocal boundary value problem for a generalized hyperbolic equation with Caputo fractional derivatives in a bounded rectangular domain of the plane with two independent variables. The equation under

consideration has the form

$$\partial_{0x}^\alpha \partial_{0y}^\beta u(x, y) + \lambda u(x, y) = f(x, y), \quad (1)$$

subject to nonlocal boundary conditions prescribed as partial integral expressions of the unknown solution with respect to each variable:

$$\begin{aligned} \int_0^b M(x, y) u(x, y) dy &= \tau(x), & 0 \leq x \leq a, \\ \int_0^a K(x, y) u(x, y) dx &= \eta(y), & 0 \leq y \leq b, \end{aligned}$$

where $M(x, y)$ and $K(x, y)$ are given continuous kernels. Employing a previously obtained representation of the solution to the Goursat problem for this equation in terms of Wright-type functions, the problem was reduced to a system of linear second-kind Volterra integral equations with respect to the traces of the unknown solution on a part of the domain boundary. As a result, the existence and uniqueness of the solution were established, and its representation was found in terms of the solutions to the obtained system of integral equations. In [5, 6], similar problems for equation (1) were also investigated under mixed boundary-integral conditions. Kh.N. Turdiev [18] investigated two nonlocal boundary value problems for a system of coupled hyperbolic equations in a mixed pentagonal domain, deriving associated second-kind Volterra integral equations and proving solution uniqueness via the general theory of integral equations. Further contributions to this direction appear in [19–21].

Despite this progress, nonlocal problems for hyperbolic-type equations governed by the Caputo–Prabhakar operator – a multi-parametric integral-differential operator generalizing both the classical Caputo derivative and the Prabhakar integral – have received virtually no attention in the literature. This gap motivates the present work.

Contribution of the paper.

In this work, we investigate a nonlocal boundary value problem for a multi-parametric integral-differential equation in a bounded domain. The main contributions are:

- reduction of the problem to a system of Volterra integral equations;
- proof of existence and uniqueness of the solution;
- derivation of an explicit representation of the solution in terms of Mittag-Leffler-type functions.

The obtained results contribute to the theory of nonlocal problems for integral-differential equations and extend existing methods based on integral equation techniques.

1 Formulation of problem

We consider the following multi-parametric integral-differential equation

$$\frac{\partial}{\partial x} {}^{PC}D_{0t}^{\alpha, \beta, \gamma, \delta} u(t, x) - a \frac{\partial}{\partial x} u(t, x) - b {}^{PC}D_{0t}^{\alpha, \beta, \gamma, \delta} u(t, x) = f(t, x) \quad (2)$$

in a domain $\Omega = \{(t, x) : 0 < t < q, 0 < x < p\}$. Here $f(t, x)$ is a given function and

$${}^{PC}D_{0t}^{\alpha, \beta, \gamma, \delta} y(t) = {}^PI_{0t}^{\alpha, m-\beta, -\gamma, \delta} \frac{d^m}{dt^m} y(t)$$

represents Caputo–Prabhakar multi-parametric integral-differential operator [22] and

$${}^PI_{0t}^{\alpha, \beta, \gamma, \delta} y(t) = \int_0^t (t-\xi)^{\beta-1} E_{\alpha, \beta}^\gamma [\delta (t-\xi)^\alpha] y(\xi) d\xi, \quad t > 0$$

represents Prabhakar fractional integral [7]. We note that above-given definitions are valid for $\alpha, \beta, \gamma, \delta, a, b, q, p \in \mathbb{R}$ such that $\alpha > 0$ and $m - 1 < \beta < m$.

Here $E_{\alpha, \beta}^{\gamma}(z)$ is the generalized Mittag-Leffler function [7]:

$$E_{\alpha, \beta}^{\gamma}(z) = \sum_{m=0}^{\infty} \frac{(\gamma)_m}{\Gamma(\alpha m + \beta)} \frac{z^m}{m!}.$$

Problem N. We are interested to find a regular solution of the equation (2) with $0 < \beta < 1$ in Ω , satisfying boundary condition

$$u(t, 0) = \varphi(t), \quad 0 \leq t \leq q \quad (3)$$

and nonlocal condition

$$u(0, x) - \int_0^q M(t)u(t, x)dt = \psi(x), \quad 0 \leq x \leq p, \quad (4)$$

where $\varphi(t)$, $\psi(x)$ and $M(t)$ are given functions ($M(t) \not\equiv 0$), such that $\varphi(0) - \int_0^q M(t)\varphi(t)dt = \psi(0)$.

Definition 1. We call a function $u(t, x)$ as a regular solution of problem (2), (3)-(4), if $u(t, x) \in C(\overline{\Omega})$, $u_x(t, x) \in C(\Omega)$, ${}^{PC}D_{0t}^{\alpha, \beta, \gamma, \delta} u(t, x) \in C(\Omega)$.

We note that at $\beta = 1$, $\delta = 0$, Eq. (2) becomes classical hyperbolic-type equation:

$$u_{tx}(t, x) - au_x(t, x) - bu_t(t, x) = f(t, x).$$

2 Main results

Let us introduce a notation

$$u(0, x) = \tau(x), \quad 0 \leq x \leq p. \quad (5)$$

Theorem 1. If $\varphi(0) = \tau(0)$, $\tau(x) \in C[0, 1] \cap C^1(0, 1)$ and $\varphi(t) \in C[0, q] \cap C^1(0, q)$, $f(t, x) = t^{-\varepsilon_1} x^{-\varepsilon_2} f_1(t, x)$, $f_1(t, x) \in C(\overline{\Omega})$ and $0 \leq \varepsilon_1 < \beta$, $0 \leq \varepsilon_2 < 1$, then the solution of Goursat problem multi-parametric integral-differential equation (2), that satisfies the initial condition (5) and boundary condition (3) will be represented as follows:

$$\begin{aligned} u(t, x) = & \tau(x) + (\varphi(t) - \varphi(0))e^{bx} + a\Gamma(\gamma)t^{\beta}\tau(x)E_2\left(\begin{matrix} \gamma, 1, \gamma; 0, 1; \\ \beta, \alpha, \beta + 1; \gamma, \gamma; 1, 1; \end{matrix} \middle| \frac{at^{\beta}}{\delta t^{\alpha}}\right) - \\ & a\varphi(0)t^{\beta}\overline{F}_E^{(3)}\left(\begin{matrix} \gamma, 1, \gamma; 1, 1, 2; \\ \beta, \alpha, \beta + 1; \gamma, \gamma; 1, 2; 1, 1; 1, 1; 1, 1; \end{matrix} \middle| \frac{at^{\beta}; bx; \delta t^{\alpha}}{\delta t^{\alpha}}\right) + \\ & abt^{\beta}\int_0^x \tau(\xi)\overline{F}_E^{(3)}\left(\begin{matrix} \gamma, 1, \gamma; 1, 1, 2; \\ \beta, \alpha, \beta + 1; \gamma, \gamma; 1, 1; 1, 1; 1, 1; 1, 1; 2; \end{matrix} \middle| \frac{at^{\beta}; b(x - \xi); \delta t^{\alpha}}{\delta t^{\alpha}}\right)d\xi + \\ & abx\int_0^t (t - \eta)^{\beta-1}\varphi(\eta)\overline{F}_E^{(3)}\left(\begin{matrix} \gamma, 1, \gamma; 1, 1, 2; \\ \beta, \alpha, \beta; \gamma, \gamma; 1, 2; 1, 1; 1, 1; 1, 1; 2; \end{matrix} \middle| \frac{a(t - \eta)^{\beta}; bx; \delta(t - \eta)^{\alpha}}{\delta(t - \eta)^{\alpha}}\right)d\eta + \\ & \int_0^t \int_0^x (t - \eta)^{\beta-1}f(\eta, \xi)\overline{F}_E^{(3)}\left(\begin{matrix} \gamma, 1, \gamma; 1, 1, 1; \\ \beta, \alpha, \beta; \gamma, \gamma; 1, 1; 1, 1; 1, 1; 1, 1; \end{matrix} \middle| \frac{a(t - \eta)^{\beta}; b(x - \xi); \delta(t - \eta)^{\alpha}}{\delta(t - \eta)^{\alpha}}\right)d\eta d\xi. \end{aligned} \quad (6)$$

The solution to the Goursat problem exists and is unique (see for details [15]).

Here, $E_2(\cdot)$ is the bivariate Mittag-Leffler type function [8]:

$$E_2 \left(\begin{matrix} \alpha_1, \beta_1, \gamma_1; \alpha_2, \gamma_2; \\ \alpha_3, \beta_2, \delta_1; \alpha_4, \delta_2; \beta_3, \delta_3; \end{matrix} \middle| \begin{matrix} x \\ y \end{matrix} \right) =$$

$$\sum_{m=0}^{\infty} \sum_{k=0}^{\infty} \frac{\Gamma(\alpha_1 m + \beta_1 k + \gamma_1) \Gamma(\alpha_2 m + \gamma_2)}{\Gamma(\gamma_1) \Gamma(\gamma_2) \Gamma(\alpha_3 m + \beta_2 k + \delta_1)} \frac{x^m}{\Gamma(\alpha_4 m + \delta_2)} \frac{y^k}{\Gamma(\beta_3 k + \delta_3)},$$

$$\gamma_1, \gamma_2, \delta_1, \delta_2, \delta_3, x, y \in \mathbb{R}, \quad \min \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \beta_1, \beta_2, \beta_3\} > 0,$$

in which the double series converges for $x, y \in \mathbb{R}$, if $\alpha_3 + \alpha_4 - \alpha_1 - \alpha_2 > 0$, and $\beta_2 + \beta_3 - \beta_1 > 0$.

$\overline{F}_E^{(3)}(\cdot)$ is the trivariate Mittag-Leffler type function [15]:

$$\overline{F}_E^{(3)} \left(\begin{matrix} \alpha_1, \beta_1, \delta_1; \alpha_2, \gamma_1, \delta_2; \\ \alpha_3, \beta_2, \delta_3; \alpha_4, \delta_4; \alpha_5, \delta_5; \beta_3, \delta_6; \gamma_2, \delta_7; \gamma_3, \delta_8; \end{matrix} \middle| \begin{matrix} x; y; z \end{matrix} \right) =$$

$$\sum_{m,j,k=0}^{+\infty} \frac{\Gamma(\alpha_1 m + \beta_1 k + \delta_1) \Gamma(\alpha_2 m + \gamma_1 j + \delta_2) x^m y^j z^k}{\Gamma(\alpha_3 m + \beta_2 k + \delta_3) \Gamma(\alpha_4 m + \delta_4) \Gamma(\alpha_5 m + \delta_5) \Gamma(\beta_3 k + \delta_6) \Gamma(\gamma_2 j + \delta_7) \Gamma(\gamma_3 j + \delta_8)},$$

$$(\alpha_l, \beta_i, \gamma_i, \delta_n, x, y, z \in \mathbb{R}; \quad \min \{\alpha_l, \beta_i, \gamma_i\} > 0, \quad (n = \overline{1, 8}, \quad l = \overline{1, 5}, \quad i = \overline{1, 3})),$$

in which the triple series converges for $x, y, z \in \mathbb{R}$, if $\Delta_1 > 0$, $\Delta_2 > 0$ and $\Delta_3 > 0$. Here

$$\Delta_1 = \alpha_3 + \alpha_4 + \alpha_5 - \alpha_1 - \alpha_2, \quad \Delta_2 = \gamma_2 + \gamma_3 - \gamma_1, \quad \Delta_3 = \beta_2 + \beta_3 - \beta_1.$$

To determine the function $\tau(x)$, we will use condition (4). Following the representation (6), and considering the nonlocal condition (4), we obtain

$$\tau(x) - \tau(x) \int_0^q M(t) dt - e^{bx} \int_0^q M(t) (\varphi(t) - \varphi(0)) dt -$$

$$a \Gamma(\gamma) \tau(x) \int_0^q M(t) t^\beta E_2 \left(\begin{matrix} \gamma, 1, \gamma; 0, 1; \\ \beta, \alpha, \beta + 1; \gamma, \gamma; 1, 1; \end{matrix} \middle| \begin{matrix} at^\beta \\ \delta t^\alpha \end{matrix} \right) dt +$$

$$a \varphi(0) \int_0^q M(t) t^\beta \overline{F}_E^{(3)} \left(\begin{matrix} \gamma, 1, \gamma; 1, 1, 2; \\ \beta, \alpha, \beta + 1; \gamma, \gamma; 1, 2; 1, 1; 1, 1; 1, 1; \end{matrix} \middle| \begin{matrix} at^\beta; bx; \delta t^\alpha \end{matrix} \right) dt -$$

$$ab \int_0^q M(t) \int_0^x \tau(\xi) t^\beta \overline{F}_E^{(3)} \left(\begin{matrix} \gamma, 1, \gamma; 1, 1, 2; \\ \beta, \alpha, \beta + 1; \gamma, \gamma; 1, 1; 1, 1; 1, 1; 1, 1; 1, 2; \end{matrix} \middle| \begin{matrix} at^\beta; b(x - \xi); \delta t^\alpha \end{matrix} \right) d\xi dt -$$

$$abx \int_0^q M(t) \int_0^t (t - \eta)^{\beta-1} \varphi(\eta) \times$$

$$\overline{F}_E^{(3)} \left(\begin{matrix} \gamma, 1, \gamma; 1, 1, 2; \\ \beta, \alpha, \beta; \gamma, \gamma; 1, 2; 1, 1; 1, 1; 1, 1; 1, 2; \end{matrix} \middle| \begin{matrix} a(t - \eta)^\beta; bx; \delta(t - \eta)^\alpha \end{matrix} \right) dt d\eta -$$

$$\int_0^q M(t) \int_0^t \int_0^x (t - \eta)^{\beta-1} f(\eta, \xi) \times$$

$$\overline{F}_E^{(3)} \left(\begin{matrix} \gamma, 1, \gamma; 1, 1, 1; \\ \beta, \alpha, \beta; \gamma, \gamma; 1, 1; 1, 1, 1, 1, 1, 1; \end{matrix} \middle| a(t-\eta)^\beta; b(x-\xi); \delta(t-\eta)^\alpha \right) d\eta d\xi dt = \psi(x). \quad (7)$$

By changing the order of integration, from equation (7), we deduce

$$\begin{aligned} & \tau(x) - \tau(x) \int_0^q M(t) dt - e^{bx} \int_0^q M(t) (\varphi(t) - \varphi(0)) dt - \\ & a\Gamma(\gamma)\tau(x) \int_0^q M(t)t^\beta E_2 \left(\begin{matrix} \gamma, 1, \gamma; 0, 1; \\ \beta, \alpha, \beta+1; \gamma, \gamma; 1, 1; \end{matrix} \middle| \frac{at^\beta}{\delta t^\alpha} \right) dt + \\ & a\varphi(0) \int_0^q M(t)t^\beta \overline{F}_E^{(3)} \left(\begin{matrix} \gamma, 1, \gamma; 1, 1, 2; \\ \beta, \alpha, \beta+1; \gamma, \gamma; 1, 2; 1, 1; 1, 1, 1, 1, 1; \end{matrix} \middle| at^\beta; bx; \delta t^\alpha \right) dt - \\ & ab \int_0^x \tau(\xi) \int_0^q M(t)t^\beta \overline{F}_E^{(3)} \left(\begin{matrix} \gamma, 1, \gamma; 1, 1, 2; \\ \beta, \alpha, \beta+1; \gamma, \gamma; 1, 1; 1, 1, 1, 1, 1, 1, 2; \end{matrix} \middle| at^\beta; b(x-\xi); \delta t^\alpha \right) dt d\xi - \\ & abx \int_0^q \varphi(\eta) \int_\eta^q M(t)(t-\eta)^{\beta-1} \times \\ & \overline{F}_E^{(3)} \left(\begin{matrix} \gamma, 1, \gamma; 1, 1, 2; \\ \beta, \alpha, \beta; \gamma, \gamma; 1, 2; 1, 1; 1, 1, 1, 1, 1, 2; \end{matrix} \middle| a(t-\eta)^\beta; bx; \delta(t-\eta)^\alpha \right) dt d\eta - \\ & \int_0^q M(t) \int_0^t \int_0^x (t-\eta)^{\beta-1} f(\eta, \xi) \times \\ & \overline{F}_E^{(3)} \left(\begin{matrix} \gamma, 1, \gamma; 1, 1, 1; \\ \beta, \alpha, \beta; \gamma, \gamma; 1, 1; 1, 1, 1, 1, 1, 1; \end{matrix} \middle| a(t-\eta)^\beta; b(x-\xi); \delta(t-\eta)^\alpha \right) d\eta d\xi dt = \psi(x). \quad (8) \end{aligned}$$

Let us introduce the following notation:

$$\begin{aligned} A &= 1 - \int_0^q M(t) \left(1 + a\Gamma(\gamma) t^\beta E_2 \left(\begin{matrix} \gamma, 1, \gamma; 0, 1; \\ \beta, \alpha, \beta+1; \gamma, \gamma; 1, 1; \end{matrix} \middle| \frac{at^\beta}{\delta t^\alpha} \right) \right) dt, \\ M_1(\xi, x) &= \int_0^q M(t)t^\beta \overline{F}_E^{(3)} \left(\begin{matrix} \gamma, 1, \gamma; 1, 1, 2; \\ \beta, \alpha, \beta+1; \gamma, \gamma; 1, 1; 1, 1, 1, 1, 1, 1, 2; \end{matrix} \middle| at^\beta; b(x-\xi); \delta t^\alpha \right) dt, \\ g(x) &= \psi(x) + e^{bx} \int_0^q M(t) (\varphi(t) - \varphi(0)) dt - \\ & a\varphi(0) \int_0^q M(t)t^\beta \overline{F}_E^{(3)} \left(\begin{matrix} \gamma, 1, \gamma; 1, 1, 2; \\ \beta, \alpha, \beta+1; \gamma, \gamma; 1, 2; 1, 1; 1, 1, 1, 1, 1; \end{matrix} \middle| at^\beta; bx; \delta t^\alpha \right) dt + \end{aligned}$$

$$\begin{aligned}
 & abx \int_0^q \varphi(\eta) \int_{\eta}^q M(t)(t-\eta)^{\beta-1} \times \\
 & \overline{F}_E^{(3)} \left(\begin{matrix} \gamma, 1, \gamma; 1, 1, 2; \\ \beta, \alpha, \beta; \gamma, \gamma; 1, 2; 1, 1, 1, 1, 1, 2; \end{matrix} \middle| a(t-\eta)^{\beta}; bx; \delta(t-\eta)^{\alpha} \right) dt d\eta + \\
 & \int_0^q M(t) \int_0^t \int_0^x (t-\eta)^{\beta-1} f(\eta, \xi) \times \\
 & \overline{F}_E^{(3)} \left(\begin{matrix} \gamma, 1, \gamma; 1, 1, 1; \\ \beta, \alpha, \beta; \gamma, \gamma; 1, 1, 1, 1, 1, 1, 1; \end{matrix} \middle| a(t-\eta)^{\beta}; b(x-\xi); \delta(t-\eta)^{\alpha} \right) d\eta d\xi dt.
 \end{aligned}$$

Considering the introduced notations, from equation (8) we deduce

$$A\tau(x) - ab \int_0^x \tau(\xi) M_1(\xi, x) d\xi = g(x). \quad (9)$$

Case 1. Let $A \neq 0$. Then, denoting

$$M_2(\xi, x) = \frac{M_1(\xi, x)}{A}, \quad G(x) = \frac{g(x)}{A},$$

from the last relation, we deduce

$$\tau(x) - ab \int_0^x \tau(\xi) M_2(\xi, x) d\xi = G(x). \quad (10)$$

Theorem 2. Let $A \neq 0$, $a < 0$, $b < 0$, $\delta < 0$. If $\varphi(0) = G(0)$, $\varphi(t) \in C[0, q] \cap C^1(0, q)$, $\psi(x) \in C[0, q] \cap C^1(0, q)$, $M(t) \in C[0, q]$, $f(t, x) = t^{-\varepsilon_1} x^{-\varepsilon_2} f_1(t, x)$, $f_1(t, x) \in C(\overline{\Omega})$ and $0 \leq \varepsilon_1 < \beta$, $0 \leq \varepsilon_2 < 1$, then the solution to the problem N exists and is unique.

Here, $G(0) = g(0)/A$,

$$g(0) = \varphi(0) - \varphi(0) \int_0^q M(t) dt - a\Gamma(\gamma)\varphi(0) \int_0^q M(t)t^{\beta} E_2 \left(\begin{matrix} \gamma, 1, \gamma; 0, 1; \\ \beta, \alpha, \beta+1; \gamma, \gamma; 1, 1; \end{matrix} \middle| \frac{at^{\beta}}{\delta t^{\alpha}} \right) dt.$$

Proof. According to the theory of integral equations, Volterra's second-kind integral equation exists and is unique, if the kernel $M_2(\xi, x)$ of the integral equation and the right-hand side function $G(x)$ are continuous.

For the integral equation (10) to have a continuous $M_2(\xi, x)$ kernel, the $M_1(\xi, x)$ kernel must be continuous. Let $M(t) \in C[0, q]$. Then, using the boundedness property of the function $\overline{F}_E^{(3)}$ (see Lemma 3.3, [15]) for negative arguments, we obtain

$$\begin{aligned}
 M_1(\xi, x) &= \int_0^q M(t)t^{\beta} \overline{F}_E^{(3)} \left(\begin{matrix} \gamma, 1, \gamma; 1, 1, 2; \\ \beta, \alpha, \beta+1; \gamma, \gamma; 1, 1, 1, 1, 1, 1, 2; \end{matrix} \middle| at^{\beta}; b(x-\xi); \delta t^{\alpha} \right) dt \leq \\
 & C_1 \int_0^q M(t)t^{\beta} dt \leq C_2,
 \end{aligned}$$

where $C_1, C_2 > 0$ are constants. Hence, the kernel $M_1(\xi, x)$ is continuous.

If $g(x)$ is continuous, then $G(x)$ will be also continuous. If $\psi(x) \in C[0, p]$, $\varphi(t) \in C[0, q]$, $M(t) \in C[0, q]$ and $f(t, x) = t^{-\varepsilon_1} x^{-\varepsilon_2} \tilde{f}_1(t, x)$, $\tilde{f}_1(t, x) \in C(\bar{\Omega})$ and $0 \leq \varepsilon_1 < \beta$, $0 \leq \varepsilon_2 < 1$, then we could prove the continuity of the function $g(x)$ using the boundedness property of the function $\bar{F}_E^{(3)}$ (see Lemma 3.3, [15]):

$$\begin{aligned} g(x) &= \psi(x) + e^{bx} \int_0^q M(t) (\varphi(t) - \varphi(0)) dt - \\ & a\varphi(0) \int_0^q M(t) t^\beta \bar{F}_E^{(3)} \left(\begin{matrix} \gamma, 1, \gamma; 1, 1, 2; \\ \beta, \alpha, \beta + 1; \gamma, \gamma; 1, 2; 1, 1; 1, 1; 1, 1; 1, 1; \end{matrix} \middle| at^\beta; bx; \delta t^\alpha \right) dt + \\ & abx \int_0^q \varphi(\eta) \int_\eta^q M(t) (t - \eta)^{\beta-1} \times \\ & \bar{F}_E^{(3)} \left(\begin{matrix} \gamma, 1, \gamma; 1, 1, 2; \\ \beta, \alpha, \beta; \gamma, \gamma; 1, 2; 1, 1; 1, 1; 1, 1; 1, 2; \end{matrix} \middle| a(t - \eta)^\beta; bx; \delta(t - \eta)^\alpha \right) dt d\eta + \\ & \int_0^q M(t) \int_0^t \int_0^x (t - \eta)^{\beta-1} f(\eta, \xi) \times \\ & \bar{F}_E^{(3)} \left(\begin{matrix} \gamma, 1, \gamma; 1, 1, 1; \\ \beta, \alpha, \beta; \gamma, \gamma; 1, 1; 1, 1; 1, 1; 1, 1; 1, 1; \end{matrix} \middle| a(t - \eta)^\beta; b(x - \xi); \delta(t - \eta)^\alpha \right) d\eta d\xi dt \leq C_3. \end{aligned}$$

□

Case 2. If $A = 0$, then the integral equation (9) will become the first-kind of Volterra equation

$$- ab \int_0^x \tau(\xi) M_1(\xi, x) d\xi = g(x). \quad (11)$$

By taking the once derivative of the integral equation (11), we then obtain the following integral equation:

$$\tau(x) + \int_0^x \tau(\xi) M^*(\xi, x) d\xi = g^*(x).$$

Here

$$\begin{aligned} M^*(\xi, x) &= \frac{[M_1(\xi, x)]_x}{B}, \quad g^*(x) = -\frac{1}{ab} \frac{g'(x)}{B}, \\ [M_1(\xi, x)]_x &= \int_0^q M(t) t^\beta \bar{F}_E^{(3)} \left(\begin{matrix} \gamma, 1, \gamma; 1, 1, 3; \\ \beta, \alpha, \beta + 1; \gamma, \gamma; 1, 1; 1, 1; 1, 1; 1, 1; 1, 3; \end{matrix} \middle| at^\beta; b(x - \xi); \delta t^\alpha \right) dt, \\ g'(x) &= \psi'(x) + be^{bx} \int_0^q M(t) (\varphi(t) - \varphi(0)) dt - \end{aligned}$$

$$\begin{aligned}
 & ab\varphi(0) \int_0^q M(t) t^\beta \overline{F}_E^{(3)} \left(\begin{matrix} \gamma, 1, \gamma; 1, 1, 3; \\ \beta, \alpha, \beta + 1; \gamma, \gamma; 1, 2; 1, 1; 1, 1, 1, 2; \end{matrix} \middle| at^\beta; bx; \delta t^\alpha \right) dt + \\
 & ab \int_0^q \varphi(\eta) \int_\eta^q M(t) (t - \eta)^{\beta-1} \times \\
 & \overline{F}_E^{(3)} \left(\begin{matrix} \gamma, 1, \gamma; 1, 1, 2; \\ \beta, \alpha, \beta; \gamma, \gamma; 1, 2; 1, 1; 1, 1, 1, 1; \end{matrix} \middle| a(t - \eta)^\beta; bx; \delta(t - \eta)^\alpha \right) dt d\eta + \\
 & \Gamma(\gamma) \int_0^q f(\eta, x) \int_\eta^q M(t) (t - \eta)^{\beta-1} E_2 \left(\begin{matrix} \gamma, 1, \gamma; 0, 1; \\ \beta, \alpha, \beta; \gamma, \gamma; 1, 1; \end{matrix} \middle| a(t - \eta)^\beta; \delta(t - \eta)^\alpha \right) d\eta dt + \\
 & b \int_0^q M(t) \int_0^t \int_0^x (t - \eta)^{\beta-1} f(\eta, \xi) \times \\
 & \overline{F}_E^{(3)} \left(\begin{matrix} \gamma, 1, \gamma; 1, 1, 2; \\ \beta, \alpha, \beta; \gamma, \gamma; 1, 1; 1, 1; 1, 1, 1, 1, 2; \end{matrix} \middle| a(t - \eta)^\beta; b(x - \xi); \delta(t - \eta)^\alpha \right) d\eta d\xi dt, \\
 & B = M_1(x, x) = \Gamma(\gamma) \int_0^q M(t) t^\beta E_2 \left(\begin{matrix} \gamma, 1, \gamma; 0, 1; \\ \beta, \alpha, \beta; \gamma, \gamma; 1, 1; \end{matrix} \middle| at^\beta; \delta t^\alpha \right) dt.
 \end{aligned}$$

Theorem 3. Let $A = 0$, $\alpha = 1$, $0 < \beta < 1$, $\gamma = \beta$, $a < 0$, $b < 0$, $\delta < 0$. If $\tau(0) = g^*(0)$, $\varphi(t) \in C[0, q] \cap C^1(0, q)$, $\psi(x) \in C^1[0, q] \cap C^2(0, q)$, $M(t) \in C[0, q]$, $f(t, x) = t^{-\varepsilon_1} x^{-\varepsilon_2} \tilde{f}_1(t, x)$, $\tilde{f}_1(t, x) \in C(\tilde{\Omega})$ and $0 \leq \varepsilon_1 < \beta$, $0 \leq \varepsilon_2 < 1$, then the solution to Problem N exists and is unique.

Here $g^*(0) = -[g'(0)]/[abB]$,

$$\begin{aligned}
 g'(0) &= \psi'(0) + b \int_0^q M(t) (\varphi(t) - \varphi(0)) dt - \\
 & ab\varphi(0) \int_0^q M(t) t^\beta E_{13} \left(\begin{matrix} \gamma, 1, \gamma; 1, 3; \\ \beta, \alpha, \beta + 1; \gamma, \gamma; 1, 2; 1, 1; \end{matrix} \middle| at^\beta; \delta t^\alpha \right) dt + \\
 & ab \int_0^q \varphi(\eta) \int_\eta^q M(t) (t - \eta)^{\beta-1} E_2 \left(\begin{matrix} \gamma, 1, \gamma; 0, 1; \\ \beta, \alpha, \beta; \gamma, \gamma; 1, 1; \end{matrix} \middle| a(t - \eta)^\beta; \delta(t - \eta)^\alpha \right) dt d\eta + \\
 & \Gamma(\gamma) \int_0^q f(\eta, x) \int_\eta^q M(t) (t - \eta)^{\beta-1} E_2 \left(\begin{matrix} \gamma, 1, \gamma; 0, 1; \\ \beta, \alpha, \beta; \gamma, \gamma; 1, 1; \end{matrix} \middle| a(t - \eta)^\beta; \delta(t - \eta)^\alpha \right) d\eta dt,
 \end{aligned}$$

where

$$\begin{aligned}
 & E_{13} \left(\begin{matrix} \alpha_1, \beta_1, \delta_1; \alpha_2, \delta_2; \\ \alpha_3, \beta_2, \delta_3; \alpha_4, \delta_4; \alpha_5, \delta_5; \beta_3, \delta_6; \end{matrix} \middle| \begin{matrix} x \\ y \end{matrix} \right) = \\
 & \sum_{m,k=0}^{+\infty} \frac{\Gamma(\alpha_1 m + \beta_1 k + \delta_1) \Gamma(\alpha_2 m + \delta_2) x^m y^k}{\Gamma(\alpha_3 m + \beta_2 k + \delta_3) \Gamma(\alpha_4 m + \delta_4) \Gamma(\alpha_5 m + \delta_5) \Gamma(\beta_3 k + \delta_6)},
 \end{aligned}$$

$$(\alpha_l, \beta_i, \delta_n, x, y \in \mathbb{R}; \min \{\alpha_l, \beta_i\} > 0; (n = \overline{1, 6}, l = \overline{1, 5}, i = \overline{1, 3})),$$

in which the double series converges for $x, y \in \mathbb{R}$, if $\alpha_3 + \alpha_4 + \alpha_5 - \alpha_1 - \alpha_2 > 0$, and $\beta_2 + \beta_3 - \beta_1 > 0$ ($E_{13}(\cdot)$ function was studied in [15]).

Proof. The proof of this theorem is similar to the proof of Theorem 2.

According to the integral representation of E_2 (see Lemma 2 in [11]), if $\alpha = 1$, $0 < \beta < 1$, $a < 0$, $\delta < 0$ and $\gamma = \beta$, then the following inequality holds for the bivariate Mittag-Leffler type function:

$$E_2 \left(\begin{matrix} \gamma, 1, \gamma; 0, 1; \\ \beta, \alpha, \beta + 1; \gamma, \gamma; 1, 1; \end{matrix} \middle| \begin{matrix} at^\beta \\ \delta t^\alpha \end{matrix} \right) > 0.$$

From this statement,

$$B = M_1(x, x) = \Gamma(\gamma) \int_0^q M(t) t^\beta E_2 \left(\begin{matrix} \gamma, 1, \gamma; 0, 1; \\ \beta, \alpha, \beta; \gamma, \gamma; 1, 1; \end{matrix} \middle| \begin{matrix} at^\beta \\ \delta t^\alpha \end{matrix} \right) dt \neq 0.$$

□

Conclusion

In this paper, a nonlocal boundary value problem was studied for a multi-parametric integral-differential equation involving the Caputo–Prabhakar operator in a bounded rectangular domain, subject to a boundary condition along a characteristic line and a nonlocal condition given as a partial integral expression of the unknown function. Relying on the representation of the solution to the corresponding Goursat problem in terms of the bivariate (E_2) and trivariate ($\overline{F}_E^{(3)}$) Mittag-Leffler-type functions, the problem was reduced to a Volterra integral equation with respect to the unknown trace $\tau(x)$.

Two cases were considered depending on the value of the constant A . When $A \neq 0$, a second-kind Volterra integral equation was obtained, and the existence and uniqueness of the solution were established under suitable conditions on the given data (Theorem 2). When $A = 0$, the resulting first-kind Volterra equation was transformed, by differentiation, into a second-kind equation, and the corresponding existence and uniqueness result was proved using the positivity of the bivariate Mittag-Leffler-type function (Theorem 3). In both cases, an explicit representation of the solution was derived through the solution of the obtained integral equation.

The results extend the integral-equation-based techniques to equations governed by multi-parametric Prabhakar-type operators and may serve as a basis for further investigation of nonlocal and inverse problems for fractional hyperbolic-type equations.

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Author contributions

All authors contributed equally to this work.

Conflict of interest

The authors declare no conflict of interest.

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Fundamental relation on hyper BCI -algebras

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This paper investigates several structural aspects of hyper BCI -algebras and their associated equivalence relations. In particular, we study a notion related to families of hyper BCI -algebras, provide illustrative examples, and analyze the relation β defined on these algebraic systems. Special attention is devoted to its transitive closure, denoted by β^* , and its role in the general theory of hyper BCI -algebras. We prove that β^* is the smallest equivalence relation for which the corresponding quotient structure is a classical BCI -algebra. Thus, β^* can naturally be regarded as the fundamental relation associated with a hyper BCI -algebra. We also establish a number of conditions that are equivalent to the transitivity of the relation β , thereby providing further characterizations of the algebraic structures under consideration and extending existing results in the theory of hyper BCI -algebras. Several examples are included to illustrate the introduced concepts and the obtained results.

Keywords: algebraic hyperstructure, hypergroupoid, hyperoperation, BCI -algebra, BCK -algebra, hyper BCI -algebra, hyper BCK -algebra, ideal, fundamental relation.

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1 Introduction and preliminaries

BCI -algebras are an important class of non-classical algebraic structures that arise in the study of abstract algebra and logic. These algebras generalize groups and are defined using a binary operation that satisfies specific conditions known as BCI -conditions. The BCI system is linked to substructural logics, which weaken some of the classical logical laws like associativity or distributivity. In a BCI -algebra, the operation is not necessarily associative or commutative, leading to a more flexible structure, useful for formalizing certain logical calculi. BCI -algebras find applications in areas such as automata theory, information systems, and fuzzy logic, where the relaxation of strict algebraic rules allows for new forms of reasoning and computation.

BCK -algebras and BCI -algebras, which are shortly called B -algebras, were introduced by two Japanese mathematicians, Imai and Iseki, in [1]. B -algebra structure comes from two different sources. The first origin is based on set theory. In set theory, there are three basic operations including union, intersection and set difference, so that Boolean algebras are derived from these three operations. The operations of union and intersection create distributable lattices; by examining only one of the union and intersection, we reach upper or lower semi-lattices. Until Iseki, the sets difference was not considered as a system. Among the logical operations, there are systems that include only one implicative operation such as positive implication system, weak positive implication system, BCK system and BCI system. There is a close relationship between the concept of sets difference in set theory and implicative operation in a logical device, for example see [2–5].

Let H be a non-empty set and $\mathcal{P}^*(H)$ be the family of all non-empty subsets of H . A mapping

$$\begin{aligned} \circ : H \times H &\rightarrow \mathcal{P}^*(H) \\ (x, y) &\mapsto x \circ y \subseteq H \end{aligned}$$

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is referred to a *binary hyperoperation* on H . The pair $(H, \circ)$ is called a *hypergroupoid*. Considering such definition, if A_1 and A_2 are two non-empty subsets of H as well as $x \in H$, we define

$$A_1 \circ A_2 = \bigcup_{\substack{a_1 \in A_1 \\ a_2 \in A_2}} a_1 \circ a_2, \quad x \circ A_1 = \{x\} \circ A_1 \text{ and } A_1 \circ x = A_1 \circ \{x\}.$$

A hypergroupoid $(H, \circ)$ is called as a hypergroup if

1. For all $a, a', a'' \in H$, $a \circ (a' \circ a'') = (a \circ a') \circ a''$;
2. For all $a \in H$, $a \circ H = H = H \circ a$.

For details about algebraic hyperstructures we refer to [6], and many other books.

There are various axiom systems for BCI -algebra, see [7]. In this paper, we will present the following axiom system related to BCI -algebra and give definition related to hyper BCI -algebra. The reader may refer to [8–11].

Definition 1. Consider A as a non-empty set equipped with a binary operation $\circ$ and a constant 0 . The structure $(A; \circ, 0)$ of type $(2, 0)$ is referred to a BCI -algebra if it meets the following conditions:

- (BCI-1) $((a_1 \circ a_2) \circ (a_1 \circ a_3)) \circ (a_3 \circ a_2) = 0$;
- (BCI-2) $(a_1 \circ (a_1 \circ a_2)) \circ a_2 = 0$;
- (BCI-3) $a_1 \circ a_1 = 0$;
- (BCI-4) $a_1 \circ a_2 = 0$ and $a_2 \circ a_1 = 0$ imply that $a_1 = a_2$.

Definition 2. A subset K of a BCI -algebra $(A; \circ, 0)$ is called a subalgebra of A if it meets the following criteria:

- (1) $0 \in K$,
- (2) $(K; \circ, 0)$ itself is a BCI -algebra.

A and $\{0\}$ are two subalgebras of A , which called the trivial subalgebra of A .

Definition 3. Consider a_1, a_2, a_3 as three element of a BCI -algebra $(A; \circ, 0)$. Then A is associative if the following identity holds:

$$(a_1 \circ a_2) \circ a_3 = a_1 \circ (a_2 \circ a_3).$$

Theorem 1. Suppose that $(A; \circ, 0)$ is a BCI -algebra. Then the following conditions are equivalent:

- (1) A is associative;
- (2) $0 \circ a = a$ for every $a \in A$;
- (3) $a_1 \circ a_2 = a_2 \circ a_1$ for every $a_1, a_2 \in A$.

It is well-established that while commutative BCK -algebra form a variety, but BCK -algebras in general do not. This observation prompts the search for specific classes of BCK -algebras such that each of them forms a variety. Consequently, in 1977, H. Yutani introduced the concept of quasi-commutative BCK -algebra [12]. Later, in 1980, K. Iseki also extended this concept to BCI -algebra [13]. For a BCI -algebra A , we inductively define a polynomial $Q_{m,n}(a, b)$ involving two variables a, b in A as follows:

$$\begin{aligned} Q_{0,0}(a, b) &= a * (a * b), \\ Q_{m+1,n}(a, b) &= Q_{m,n}(a, b) * (a * b), \\ Q_{m,n+1}(a, b) &= Q_{m,n}(a, b) * (b * a), \end{aligned}$$

where m, n are any natural numbers. In fact, $Q_{m,n}(a, b)$ can be achieved from right $*$ multiplying $Q_{0,0}(a, b)$ by $a * b$ m times and next by $b * a$ n times, that is,

$$Q_{m,n}(a, b) = (Q_{0,0}(a, b) * (a * b)^m) * (b * a)^n,$$

in other words, we have $Q_{m,n}(a, b) = (a * (a * b)^{m+1}) * (b * a)^n$.

Definition 4. A *BCI*-algebra A is referred to quasi-commutative of type $(i, j; m, n)$ if there exist two pairs of natural numbers i, j and m, n such that the following identity holds:

$$Q_{i,j}(a, b) = Q_{m,n}(b, a)$$

or equivalently,

$$(a * (a * b)^{i+1}) * (b * a)^j = (b * (b * a)^{m+1}) * (a * b)^n.$$

Correspondingly, we can explain a quasi-commutative *BCK*-algebra. If A is a quasi-commutative *BCI*-algebra of type $(i, j; m, n)$, the *BCK*-part of A is naturally a quasi-commutative *BCK*-algebra of the same type. Exchanging a and b in the identity $Q_{i,j}(a, b) = Q_{m,n}(b, a)$, we get $Q_{m,n}(a, b) = Q_{i,j}(b, a)$.

Definition 5. Suppose that $(A; \circ, 0)$ is a *BCI*-algebra and I is a subset of A and I is also non-empty. Then I is referred to an ideal of A , if for all $a, b \in A$, it meets the following conditions:

- (1) $0 \in I$;
- (2) $b \in I$ and $a \circ b \in I$ imply $a \in I$.

$\{0\}$ and A are obviously two ideals of A , which are called the trivial ideals of A . An ideal I is said to be proper if $I \subsetneq A$.

Definition 6. Let a_1, a_2, a_3 be any elements in a non-empty set H_B , $\circ$ be a binary hyperoperation on H_B as well as 0 be a constant. Then $(H_B, \circ, 0)$ is a hyper *BCK*-algebra if

- (HK-1) $(a_1 \circ a_3) \circ (a_2 \circ a_3) \ll a_1 \circ a_2$;
- (HK-2) $(a_1 \circ a_2) \circ a_3 = (a_1 \circ a_3) \circ a_2$;
- (HK-3) $a_1 \circ H_B \ll \{a_1\}$;
- (HK-4) $a_1 \ll a_2$ and $a_2 \ll a_1$ imply that $a_1 = a_2$;

where $a_1 \ll a_2$ is defined by $0 \in a_1 \circ a_2$. Let H_1, H_2 be two subsets of H_B , then we have $H_1 \ll H_2$ if and only if for all $h_1 \in H_1$ there exist $h_2 \in H_2$ such that $h_1 \ll h_2$.

Remark 1. In a hyper *BCK*-algebra $(H_B, \circ)$, the condition (HK-3) is equivalent to the following condition:

$$a_1 \circ a_2 \ll \{a_1\}, \text{ for all } a_1, a_2 \in H_B.$$

Definition 7. A hyper *BCI*-algebra is a structure $(H_B, \circ, 0)$ such that H_B is a non-empty set equipped with a hyperoperation $\circ : H_B \times H_B \rightarrow \mathcal{P}^*(H_B)$ and a constant 0 satisfying the following axioms:

- (HI-1) $(a_1 \circ a_3) \circ (a_2 \circ a_3) \ll a_1 \circ a_2$;
- (HI-2) $(a_1 \circ a_2) \circ a_3 = (a_1 \circ a_3) \circ a_2$;
- (HI-3) $a \ll a$;
- (HI-4) $a_1 \ll a_2$ and $a_2 \ll a_1$ imply $a_1 = a_2$;
- (HI-5) $0 \circ (0 \circ a_1) \ll a_1$

for all $a_1, a_2, a_3 \in H_B$.

Definition 8. Suppose that H_B is a hyper *BCI*-algebra. Then we say H_B is ordered if for any $a_1, a_2, a_3 \in H_B$ it satisfies the following condition:

$$a_1 \ll a_2 \text{ and } a_2 \ll a_3 \text{ imply } a_1 \ll a_3.$$

Definition 9. Suppose that $(H, \circ)$ is a hypergroupoid that includes 0 . We say H is a quasi-hyper *BCI*-algebra if it satisfies the following conditions:

- (QHCI-1) $(a_1 \circ a_3) \circ (a_2 \circ a_3) \ll a_1 \circ a_2$;
- (QHCI-2) $(a_1 \circ a_2) \circ a_3 = (a_1 \circ a_3) \circ a_2$;
- (QHCI-3) $a_1 \ll a_1$;
- (QHCI-4) $a_1 \ll a_2$ and $a_2 \ll a_1$ imply that $a_1 = a_2$;

- (QHCI-5) $a_1 \ll a_1 \circ 0$;
 (QHCI-6) $a_1 \ll 0 \Rightarrow a_1 = 0$.

Proposition 1. Every hyper BCK-algebra is a hyper BCI-algebra, but the reverse is not necessarily true.

Proposition 2. Assume that $(H_B, \circ)$ is a hyper BCI-algebra. Then the following are hold:

- (1) $a_1 \ll 0$ implies $a_1 = 0$,
- (2) $0 \in a_1 \circ (a_1 \circ 0)$,
- (3) $a_1 \ll a_1 \circ 0$,
- (4) $0 \circ (a_1 \circ a_2) \ll a_2 \circ a_1$,
- (5) $H_1 \ll H_1$,
- (6) $H_1 \subseteq H_2$ implies $H_1 \ll H_2$,
- (7) $H_1 \ll \{0\}$ implies $H_1 = \{0\}$,
- (8) $a_1 \circ 0 \ll \{a_2\}$ implies $a_1 \ll a_2$,
- (9) $a_2 \ll a_3$ implies $a_1 \circ a_3 \ll a_1 \circ a_2$,
- (10) $a_1 \circ a_2 = \{0\}$ implies $(a_1 \circ a_3) \circ (a_2 \circ a_3) = \{0\}$ and $a_1 \circ a_3 \ll a_2 \circ a_3$,
- (11) $H_1 \circ \{0\} = \{0\}$ implies $H_1 = \{0\}$,

for all $a_1, a_2, a_3 \in H_B$ and all non-empty subsets, H_1, H_2 of H_B .

Definition 10. Suppose that I is a non-empty subset of a hyper BCI-algebra H_B . Then I is said to be a hyper BCI-ideal of H_B if:

- (HI-1) $0 \in I$;
 (HI-2) $a_1 \circ a_2 \ll I$ and $a_2 \in I$ imply $a_1 \in I$ for all $a_1, a_2 \in H_B$.

Every hyper BCI-ideal of a hyper BCI-algebra H is a hyper subalgebra of H , but the reverse may not hold.

2 The fundamental relation on hyper BCI-algebras

Assume that $(H_B, \circ, 0)$ is a hyper BCI-algebra and R be an equivalence relation on H , $H_1, H_2 \subseteq H$ and $H_1, H_2 \neq \emptyset$ we define:

- $H_1 R H_2 \iff \exists h_1 \in H_1, h_2 \in H_2$ such that $h_1 R h_2$;
 $H_1 \bar{R} H_2 \iff \forall h_1 \in H_1, \exists h_2 \in H_2$ such that $h_1 R h_2$ and $\forall h_2 \in H_2, \exists h_1 \in H_1$ such that $h_1 R h_2$;
 $H_1 \bar{\bar{R}} H_2 \iff \forall h_1 \in H_1, \forall h_2 \in H_2$ we have $h_1 R h_2$.

Definition 11. Assume that $(H_B, \circ, 0)$ is a hyper BCI-algebra and R is an equivalence relation on H . Then R is referred to a regular relation if for all a of H , we have

$$x R y \implies (x \circ a) \bar{R} (y \circ a) \quad \text{and} \quad ((a \circ x) \bar{R} (a \circ y) \quad \text{respectively}).$$

Definition 12. Suppose that $(H_B, \circ, 0)$ is a hyper BCI-algebra and R is an equivalence relation on H_B . Then R is referred to a strongly regular relation if for all a of H , we have:

$$x R y \implies (x \circ a) \bar{\bar{R}} (y \circ a) \quad \text{and} \quad ((a \circ x) \bar{\bar{R}} (a \circ y) \quad \text{respectively}).$$

Let $(H_B; \circ, 0)$ be a BCI-algebra and U_H be the set of all finite combination of elements of H and define β on H as follows:

Definition 13. If $(H_B; \circ, 0)$ is a hyper BCI-algebra then we define $\beta_1 = \{(x, x) : x \in H_B\}$ as the diagonal relation on H_B .

For any integer $n > 1$, β_n is defined as

$$x \beta_n y \iff \exists (u_1, \dots, u_n) \in H_B^n, \quad \exists u \in U_{(u_1, \dots, u_n)} \quad \text{such that} \quad \{a, b\} \subseteq u.$$

It is evident that, for all $n \geq 1$, the relations β_n are symmetric and β is both reflexive and symmetric. Let β^* denote the transitive closure of β .

Theorem 2. Suppose that $(H_B; \circ, 0)$ is a hyper *BCI*-algebra. Then, β^* is strongly regular on H_B .

Proof. If $x\beta^*y$, then by definition of β^* , there exist $n \in \mathbb{N}$, $i = 1, 2, \dots, n$ $(z_0, \dots, z_n) \in H^{n+1}$ and $\exists k_i \in \mathbb{N}$ such that $z_0 = x$, $z_n = y$ and $x = z_0\beta_{k_1}z_1, z_1\beta_{k_2}z_2, \dots, z_{n-1}\beta_{k_n}z_n = y$ by definition of relation β , we have

$$z_0\beta_{k_1}z_1 \implies \exists(u_1^1, \dots, u_{k_1}^1) \in H_B^{k_1}, \exists u^1 \in U_{(u_1^1, \dots, u_{k_1}^1)} \text{ such that } \{z_0, z_1\} \subseteq u^1;$$

$$z_1\beta_{k_2}z_2 \implies \exists(u_1^2, \dots, u_{k_2}^2) \in H_B^{k_2}, \exists u^2 \in U_{(u_1^2, \dots, u_{k_2}^2)} \text{ such that } \{z_1, z_2\} \subseteq u^2;$$

$\vdots$

$$z_{n-1}\beta_{k_n}z_n \implies \exists(u_1^n, \dots, u_{k_n}^n) \in H_B^{k_n}, \exists u^n \in U_{(u_1^n, \dots, u_{k_n}^n)} \text{ such that } \{z_{n-1}, z_n\} \subseteq u^n.$$

Now let $a \in H_B$ be an arbitrary element of H_B . It follows that $\forall j \in \{0, \dots, n-1\}$ $z_j \circ a, z_{j+1} \circ a \subseteq u^j \circ a$. Hence if j is such that $0 \leq j \leq s$, $\forall u \in z_j \circ a, \forall v \in z_{j+1} \circ a$, we obtain $u\beta v$. Thus, $\forall u \in z_1 \circ a, \forall v \in z_n \circ a$ we have $u\beta^*v$. This means that β^* is strongly regular relation. $\square$

Definition 14. Suppose that $(H_B; \circ, 0)$ is a hyper *BCI*-algebra, R is a regular relation on H_B and $R(a), R(b)$ are two elements of H_B/R , where H_B/R is the set of all equivalence classes of R (i.e. $H_B/R = \{R(h) : h \in H\}$). Then we define hyperoperation $\bar{\circ}$ on H_B/R as follows:

$$R(a)\bar{\circ}R(b) := \{R(c) : c \in a \circ b\}.$$

Lemma 1. $\bar{\circ}$ is well-defined.

Proof. This proof is similar to the proof at [14]. $\square$

Definition 15. Let R be a regular relation on a hyper *BCI*-algebra $(H_B; \circ, 0)$ and $R(a), R(b)$ be two elements of H_B/R . Then we define

$$R(a) \ll R(b) \text{ if and only if } R(0) \in R(a)\bar{\circ}R(b).$$

Theorem 3. Suppose that $(H_B; \circ, 0)$ is a hyper *BCI*-algebra, R is an equivalence relation on H_B and hyperoperation $\bar{\circ}$ be one to one. Then R is regular relation if and only if $(H_B/R; \bar{\circ}, R(0))$ is a hyper *BCI*-algebra.

Proof. Assume that $R(a)$ is an arbitrary element of $(R(x)\bar{\circ}R(z))\bar{\circ}(R(y)\bar{\circ}R(z))$. By Definition 15, we have

$$\exists r \in x \circ z \text{ and } \exists s \in y \circ z : R(a) \in R(r)\bar{\circ}R(s).$$

Thus there exists $a' \in r \circ s$ such that $R(a) = R(a')$. Since $a' \in r \circ s$, we have

$$a' \in r \circ s \subseteq (x \circ z) \circ (y \circ z) \ll x \circ y.$$

So there exist $a'' \in x \circ y$ such that $a' \ll a''$. This means that $0 \in a' \circ a''$. So we have

$$R(0) \in R(a')\bar{\circ}R(a'') \implies R(a') \ll R(a'').$$

So $(R(x)\bar{\circ}R(z))\bar{\circ}(R(y)\bar{\circ}R(z)) \ll R(x)\bar{\circ}R(y)$.

Now, we investigate condition H_2 for H/R . In this case we must prove that $(R(x)\bar{\circ}R(y))\bar{\circ}R(z) = (R(x)\bar{\circ}R(z))\bar{\circ}R(y)$. For this purpose at first let $R(w) \in (R(x)\bar{\circ}R(y))\bar{\circ}R(z)$. Thus there is $u \in x \circ y$ such that $R(w) \in R(u)\bar{\circ}R(z)$. Then there exist $v \in u \circ z$ such that $R(w) = R(v)$. We have $v \in u \circ z \subseteq (x \circ y) \circ z$. As we know $(x \circ y) \circ z = (x \circ z) \circ y$. So $\exists r' \in x \circ z : v \in r' \circ y$, thus $R(w) = R(v) \in R(r')\bar{\circ}R(y) \subseteq (R(x)\bar{\circ}R(z))\bar{\circ}R(y)$. Consequently, it implies that $(R(x)\bar{\circ}R(y))\bar{\circ}R(z) \subseteq (R(x)\bar{\circ}R(z))\bar{\circ}R(y)$. In the same way, we earn the other side of equality.

We intend to prove condition H_3 for H_B/R . In other words, we need to prove that $R(x) \ll R(x)$ (i.e., $R(0) \in R(x) \bar{\circ} R(x)$). Since H_B is a hyper BCI -algebra, it follows that $0 \in x \circ x$. By using of the definition of hyperoperation $\bar{\circ}$ we have $R(0) \in R(x) \bar{\circ} R(x)$. Thus $R(x) \ll R(x)$.

For condition (H_4) , let $R(x) \ll R(y)$ and $R(y) \ll R(x)$, we want to prove that $R(x) = R(y)$. Thus we have, $R(0) \in R(x) \bar{\circ} R(y)$ and $R(0) \in R(y) \bar{\circ} R(x)$ so there exist $s \in x \circ y$ and $t \in y \circ x$ such that $R(0) = R(s)$ and $R(0) = R(t)$. Therefore we have $R(t) = R(s)$. This means that $R(x) \bar{\circ} R(y) = R(x) \bar{\circ} R(y)$, the hypothesis that $\bar{\circ}$ is one to one we conclude that $R(x) = R(y)$.

Finally we check condition H_5 . Let $a \in R(0) \bar{\circ} (R(0) \bar{\circ} R(x))$. So there is $s \in 0 \circ x$ and $h \in H_B$ such that $a = R(h) \in R(0) \bar{\circ} R(s)$. Therefore there exist $t \in 0 \circ s$ such that $R(h) = R(t)$. Since H_B is a hyper BCI -algebra, we have $0 \circ (0 \circ x) \ll x$. So $t \ll x$ and $R(t) \ll R(x)$. Consequently, $R(0) \bar{\circ} R(s) \ll R(x)$, which means that $R(0) \bar{\circ} (R(0) \bar{\circ} R(x)) \ll R(x)$.

Now, we want to show that R is a regular relation, first we have to show that

$$xRy \implies (x \circ a)R(y \circ a) \text{ and } (a \circ x)R(a \circ y).$$

For this purpose, let $x, y, a \in H_B$, xRy and $s \in x \circ a$, by the definition of hyperoperation $\bar{\circ}$ we have $R(s) \in R(x) \bar{\circ} R(a)$. Since $R(x) = R(y)$ so $R(s) \in R(x) \bar{\circ} R(a) = R(y) \bar{\circ} R(a)$ thus there exist $t \in y \circ a$ such that $R(s) = R(t)$. This means that $x \circ a \bar{R} y \circ a$. Therefore R is regular on the right. In the same way R is regular on the left. $\square$

Definition 16. Assume that $(H_B, \circ, 0)$ and $(H'_B, \circ', 0')$ are two hyper BCI -algebras. A function $f : H_B \rightarrow H'_B$ is referred to a homomorphism if the following conditions hold:

- (1) $f(x \circ y) = f(x) \circ' f(y)$;
- (2) $f(0) = 0'$ for every $(x, y) \in H_B^2$.

Theorem 4. Assume that $(H_B, \circ, 0)$ and $(H'_B, \circ', 0')$ are two hyper BCI -algebras and the function $f : H_B \rightarrow H'_B$ is a homomorphism. The equivalence R associated with f (i.e. xRy if and only if $f(x) = f(y)$) is regular as well as the function $g : f \rightarrow H_B/R$, $g(f(x)) = R(x)$ is an isomorphism.

Proof. It is similar to the proof of Theorem 3.9 in [14]. $\square$

Theorem 5. Suppose that $(H_B; \circ, 0)$ is a hyper BCI -algebra. Let R be a strongly regular relation on a hyper BCI -algebra $(H_B; \circ, 0)$. Then

- (1) H_B/R is a BCI -algebra (i.e. H_B/R is a fundamental BCI -algebra).
- (2) If H'_B is a BCI -algebra and $f : H_B \rightarrow H'_B$ is a homomorphism, then the equivalence γ associated with f is strongly regular.

Proof. (1) Let x, y two elements of H_B . By assumption R is strongly regular relation on H_B . So for we have $x \circ y \bar{R} y \circ x$. Hence $|R(x) \circ R(y)| = 1$. This means that H_B/R is a BCI -algebra.

(2) To prove that γ is strongly regular relation, first we have to show that $x\gamma y \implies x \circ a \bar{\gamma} y \circ a$ and $a \circ x \bar{\gamma} a \circ y$. Let $a, x, y \in H_B$ and $x\gamma y$. It implies $\gamma(x) = \gamma(y)$ and $f(x) = f(y)$. Thus for every $s \in x \circ a$ and $t \in y \circ a$, we have $f(s), f(t) \in f(x \circ a) = f(x) \circ f(a) = f(y) \circ f(a) = f(y \circ a)$. As we know, H'_B is a BCI -algebra so cardinal of $f(x \circ a)$ and cardinal of $f(y \circ a)$ are equal to 1. It follows that $s \bar{R} t$. This means that $x \circ a \bar{R} y \circ a$. $\square$

Theorem 6. The relation β^* is the smallest equivalence relation such that the quotient H/β^* is a BCI -algebra.

Corollary 1. Relation β^* is the smallest strongly regular relation on a hyper BCI -algebra.

3 Transitivity condition for relation β

In this section, we want to identify the necessary and sufficient conditions for the transitivity of the relation β .

Definition 17. Consider X as a non-empty subset of H_B . If for every non-zero natural number n , for $i = 1, \dots, n$, $\forall k_i \in \mathbb{N}$, $\forall (u_{i1}, \dots, u_{ik_i}) \in H_B^{k_i}$ we have

$$U_{(u_{i1}, \dots, u_{ik_i})} \cap X \neq \emptyset \implies U_{(u_{i1}, \dots, u_{ik_i})} \subseteq X.$$

Then X is referred to a β -part of H_B .

Lemma 2. Assume that X is a non-empty subset of a hyper *BCI*-algebra of H_B . Then the following conditions are equivalent:

1. X is a β -part of H_B ;
2. if $x \in X$ and $x\beta y$, then $y \in X$;
3. if $x \in X$ and $x\beta^* y$, then $y \in X$.

Proof. (1 $\implies$ 2) If $(x, y) \in H_B$ is a pair such that $x \in X$ and $x\beta y$, then by the definition of relation β , there exist $n \in \mathbb{N}$, $(u_1, \dots, u_n) \in H^n$ such that $\{x, y\} \subseteq U_{(u_1, \dots, u_n)}$. By assumption X is a β -part and $U_{(u_1, \dots, u_n)} \cap X \neq \emptyset$, so we have $U_{(u_1, \dots, u_n)} \subseteq X$. This means that $y \in X$.

(2 $\implies$ 3) Assume that $(x, y) \in H^2$ such that $x \in X$ and $x\beta^* y$. By the definition of β^* , there exist $m \in \mathbb{N}$, $(u_0, \dots, u_m) \in H^{m+1}$ such that $u_0 = x$, $u_m = y$ and $x = u_0\beta u_1, u_1\beta u_2, \dots, u_{m-1}\beta u_m = y$. Since $x \in X$ and $x = u_0\beta u_1$ by applying (2), we obtain $u_1 \in X$. If we continue this process m times, we conclude that $y \in X$.

(3 $\implies$ 1) Now, we want to show that M is a β -part of H . So let $U_{(u_1, \dots, u_n)} \cap X \neq \emptyset$ and $x \in U_{(u_1, \dots, u_n)} \cap X$. For every $y \in U_{(u_1, \dots, u_n)}$ we have $x\beta y$. Thus, $x \in X$ and $x\beta^* y$. Eventually by applying (3) we conclude that $y \in X$. $\square$

At first we introduce several notations that they are applied to prove the next Theorem. Let H_B be a hyper *BCI*-algebra. For all $x \in H_B$, we set

1. $T_n(x) = \{(x_1, \dots, x_n) \in H^n | x \in U_{(x_1, \dots, x_n)}\}$;
2. $P_n(x) = \bigcup \{U_{(x_1, \dots, x_n)} | x \in U_{(x_1, \dots, x_n)}\}$;
3. $P_\sigma(x) = \bigcup_{n \geq 1} P_n(x)$.

It follows immediately the following from the previous notations and definitions:

Lemma 3. For every $x \in H_B$, $P_\sigma(x) = \{y \in H_B | x\beta y\}$.

Proof. For every $x, y \in H_B$, suppose that $x\beta y$ by definition of relation β , we have

$$\begin{aligned} x\beta y &\iff \exists n \in \mathbb{N}, \exists (u_1, \dots, u_n) \in H^n : \{x, y\} \subseteq U_{(u_1, \dots, u_n)} \\ &\iff \exists n \in \mathbb{N}, \exists (u_1, \dots, u_n) \in H^n : x \in U_{(u_1, \dots, u_n)} \\ &\iff \exists n \in \mathbb{N} : y \in P_n(x) \\ &\iff y \in P_\sigma(x). \end{aligned}$$

$\square$

Theorem 7. Let H be a hyper *BCI*-algebra. Then the following conditions are equivalent:

- (1) β is transitive;
- (2) $\beta^*(x) = P_\sigma(x)$ for all $x \in H$;
- (3) $P_\sigma(x)$ is a β -part of H .

Proof. (1 $\implies$ 2) Let $y \in \beta^*(x)$ by previous lemma, we have

$$\beta^*(x) = \beta^*(y) \iff x\beta^*y.$$

Since β is transitive, we have $x\beta y$. This means that $y \in P_\sigma(x)$.

(2 $\implies$ 3) By Lemma 2, if X is a non-empty subset of X , then M is a β -part of H_B if and only if it is a union of equivalence classes modulo β^* . As we know, every equivalence class modulo β^* is a β -part of H_B .

(3 $\implies$ 1) Now, we want to show that β is transitive. For this purpose, let $x\beta y$ and $y\beta z$, by the definition of β there exist $m, n \in \mathbb{N}$ and $(x_1, \dots, x_n) \in H^n$, $(y_1, \dots, y_m) \in H^m$ such that $\{x, y\} \subseteq U_{(x_1, \dots, x_n)}$ and $\{y, z\} \subseteq U_{(y_1, \dots, y_m)}$. Since $P_\sigma(x)$ is a β -part of H , we have

$$x \in U_{(x_1, \dots, x_n)} \cap P_\sigma(x) \quad \text{so} \quad y \in U_{(x_1, \dots, x_n)} \subseteq P_\sigma(x).$$

Thus $y \in P_\sigma(x)$ and $y \in U_{(y_1, \dots, y_m)}$ (i.e., $P_\sigma(x) \cap U_{(y_1, \dots, y_m)} \neq \emptyset$) since $P_\sigma(x)$ is a β -part of H , it implies that $z \in U_{(y_1, \dots, y_m)} \subseteq P_\sigma(x)$ so $z \in P_\sigma(x)$ and this means that there exist $n \in \mathbb{N}$ such that $z \in P_n(x)$. Therefore $x\beta z$ as well as β is transitive. $\square$

Conclusion

In this paper, we have studied structural properties of hyper *BCI*-algebras with respect to a naturally defined relation β . By considering families of hyper *BCI*-algebras, we provided several illustrative examples that clarify the behavior of the relation and its associated constructions. A central focus of this work was the investigation of the transitive closure β^* , which plays a crucial role in the structural analysis of these algebraic systems.

We showed that β^* is indeed the fundamental equivalence relation in the sense that it is the smallest equivalence relation on a hyper *BCI*-algebra whose corresponding quotient structure yields a classical *BCI*-algebra. This result highlights the importance of β^* in bridging hyper structures with their classical counterparts. Moreover, we demonstrated that β^* provides a natural framework for understanding the internal structure of hyper *BCI*-algebras.

Finally, we established several equivalent conditions for the transitivity of the relation β , offering additional characterizations of these algebraic structures. These results not only extend previous findings in the literature but also provide a foundation for further investigations in the theory of hyper *BCI*-algebras and related hyperstructures.

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Author contributions

All authors contributed equally to this work.

Conflict of interest

The authors declare no conflict of interest.

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Internal hybridization of Kaiser hulls and Morley-type criteria in K_T -normal Jonsson theories

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This article examines the internal hybridization of Kaiser hulls in Jonsson theory and the transfer of classical spectral properties to hybrid settings. The relevance of this study is determined by the need to distinguish internal hybrid constructions generated within a fixed Kaiser class from external hybrids of unrelated theories. The characteristic features of such internal constructions are analyzed within the framework of a fixed Kaiser-normal Jonsson theory. The necessity of stability conditions that preserve the Kaiser hull under admissible hybrid unification is identified and justified. Based on the conducted research, hybridization is considered as an internal operation on semantic fragments of a Jonsson theory. A technical criterion ensuring the preservation of the ambient Kaiser class of the semantic model under internal hybrid is formulated. We prove that, under Kaiser-normality, the semantic model of a hybrid remains within the original Kaiser class. It is shown that this fact makes it possible to transfer Morley-type categoricity criteria, Saracino-type perfection arguments, and Lindström-type model-completeness principles to Jonsson hybrid spectra. As a result, hybrid analogues of results on prime extensions, model perfection, and model completeness are obtained.

Keywords: Jonsson theory, Kaiser class, Kaiser hull, internal hybridization, hybrid model, hybrid spectrum, categoricity criterion, model perfection, model completeness, prime extension.

2020 Mathematics Subject Classification: 03C35, 03C48, 03C52, 03C65.

Introduction

One of the central directions in the theory of the Jonsson spectrum is the study of semantic models, existentially closed models, perfection, categoricity, and model completeness. The general model-theoretic background of these notions goes back to classical mathematical logic, while their development within Jonsson theory is based on the apparatus of semantic models, centers, fragments, and spectra of Jonsson theories [1, 2]. Traditionally, the apparatus of the Jonsson spectrum is applied to a Jonsson class $[T]$. The corresponding semantic model C_T determines the existentially closed model E_T and the center $T^* = Th(C_T)$ [2, 3].

Another direction of research concerns hybrids of theories generated by products of semantic models. Such hybrid constructions, as well as syntactic and semantic similarities between hybrids arising from the Jonsson spectrum and from S -acts, were studied in [4–6]. However, such hybrids have usually been treated as external constructions: one starts with arbitrary Jonsson theories T_1 and T_2 and then constructs a hybrid from them. The present paper proposes a fundamentally different approach. We fix a K_T -normal Jonsson theory T and work entirely within its Kaiser class K_T .

The Kaiser class of a Jonsson theory and the corresponding notion of K_T -equivalence provide a semantic invariant that refines the classical notion of cosemantiness [7]. Properties of cl -normal

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Jonsson theories, their Kaiser classes, and related criteria of perfection were further developed in [8]. Let $M_1, M_2 \in K_T$, and let their Kaiser hulls be denoted by $T_1 = M_1^0$ and $T_2 = M_2^0$. These Kaiser hulls generate a hybrid theory $H = H(T_1, T_2)$. Thus, the hybrid is generated internally from fragments of the same semantic Jonsson universe rather than externally from unrelated theories.

The main question is whether the semantic model C_H of the hybrid remains within the original Kaiser class K_T . Under K_T -normality, the answer is positive. This technical fact becomes the key mechanism that allows the transfer of classical spectral theorems to the hybrid properties. In particular, we obtain Jonsson-hybrid analogues of Morley-type criteria for ω_1 -categoricity. The relevance of Kaiser-class methods to concrete algebraic situations is also illustrated by the study of Kaiser classes of unars in expanded signatures [9].

Related model-theoretic investigations have considered semantic pairs and e.f.c.p. in the Jonsson spectrum, as well as J -non-multidimensional theories [10, 11]. The applicability of Kaiser-class methods to concrete algebraic structures was also demonstrated through the study of Kaiser classes of unars in expanded signatures [12].

It is important to emphasize that the Morley-type phenomenon considered in this paper is not merely the classical transfer principle stating that ω_1 -categoricity implies categoricity in all uncountable cardinalities. Rather, the crucial mechanism is a related Morley-type principle connecting ω_1 -categoricity with the existence of prime extensions over countable models. The classical model-theoretic background for categoricity and prime-model constructions is presented in [1]. The present paper transfers this principle to the Jonsson-hybrid context: if the hybrid center H^* is ω_1 -categorical, then every countable model in E_H admits a Jonsson-algebraically prime extension within E_H . This yields a genuine Jonsson-spectral generalization of Morley's criterion.

1 Preliminaries

We recall the basic definitions needed throughout the article, beginning with the definition of a Jonsson theory.

Definition 1. [1] A theory T in a countable language L is called Jonsson if:

- T has infinite models;
- T is inductive;
- T has the joint embedding property;
- T has the amalgamation property.

The semantic model and its center play a fundamental role in the structural analysis of Jonsson theories. In particular, they provide the framework for describing existentially closed models and studying spectral properties associated with the theory.

Definition 2. [2] A model C_T of a Jonsson theory T is called a semantic model of this theory if $|C_T| = 2^\omega$ and C_T is ω^+ -universal ω^+ -homogeneous model of T .

Definition 3. [2] Let T be a Jonsson theory. Then theory $T^* = Th(C_T)$ is called a center of T .

We give the definition of existentially closed models.

Definition 4. [2] A model $M \models T$ is existentially closed if every existential formula realizable in an extension of M inside $Mod(T)$ is already realizable in M . The class of existentially closed models is denoted by E_T .

We now recall the notion of the Kaiser hull, which is essential for the internal hybrid construction considered in this paper.

Let $M \models T$. The Kaiser hull of M is the $\forall\exists$ -theory generated by M . We denote it by M^0 :

$$M^0 = Th_{\forall\exists}(M).$$

We now recall the notion of the Kaiser class, which determines the internal domain from which the hybrid constructions of this paper are generated.

Definition 5. (A.R. Yeshkeyev) [7] Let T be a Jonsson theory. The Kaiser class K_T consists of those models $M \models T$ for which the Kaiser hull $M^0 = Th_{\forall\exists}(M)$ is a Jonsson theory. Equivalently,

$$M \in K_T \quad \text{if and only if} \quad M^0 \text{ is Jonsson.}$$

Thus, K_T is a model-theoretic class: its elements are models of T . This distinction is important, since the theories M^0 generated by such models may be used in the construction of hybrids, whereas the hybrid theories themselves are not elements of K_T .

Therefore hybrid theories themselves do not belong to K_T . The meaningful statement concerns semantic models of hybrids.

The following definition is the definition of K_T -normality.

Definition 6. (A.R. Yeshkeyev) A Jonsson theory T is called K_T -normal if the Kaiser class K_T is stable under admissible internal amalgamation constructions generated by models from K_T .

Informally, K_T -normality means that internal Jonsson constructions do not leave the Kaiser layer.

The main notion of the article is hybrid of Jonsson theories.

Definition 7. Let T_1 and T_2 be Jonsson theories with semantic models C_{T_1} and C_{T_2} . Their hybrid is $H(T_1, T_2) = Th_{\forall\exists}(C_{T_1} \times C_{T_2})$, provided this theory is Jonsson.

In the present work, the theories T_1 and T_2 are always internal Kaiser hulls: $T_1 = M_1^0$, $T_2 = M_2^0$, where $M_1, M_2 \in K_T$.

We now introduce the internal form of hybridization used throughout the paper. Let T be a fixed K_T -normal Jonsson theory. The aim is to construct hybrid theories not from arbitrary external Jonsson theories, but from Kaiser hulls of models belonging to the same Kaiser class K_T .

Choose models $M_1, M_2 \in K_T$ and put

$$T_1 = M_1^0, \quad T_2 = M_2^0,$$

where

$$M_i^0 = Th_{\forall\exists}(M_i), \quad i = 1, 2.$$

Since M_1 and M_2 belong to the Kaiser class K_T , their Kaiser hulls T_1 and T_2 are Jonsson theories. Therefore the corresponding semantic models C_{T_1} and C_{T_2} are well defined.

Definition 8. Let T be a K_T -normal Jonsson theory, and let $M_1, M_2 \in K_T$. Put $T_1 = M_1^0$ and $T_2 = M_2^0$. If the theory

$$H = H(T_1, T_2) = Th_{\forall\exists}(C_{T_1} \times C_{T_2})$$

is Jonsson, then H is called the internal hybrid generated by the models M_1 and M_2 inside the Kaiser class K_T .

The essential feature of this construction is its internal character. The theories T_1 and T_2 are not chosen independently of the original theory T ; they are obtained as Kaiser hulls of models from the fixed class K_T . Thus, the hybrid $H(T_1, T_2)$ is generated by internal Kaiser fragments of one ambient Jonsson universe.

This distinguishes the present construction from external hybridization, where one starts with two unrelated Jonsson theories and combines their semantic models. In the internal setting, both components of the hybrid arise from the same K_T -normal environment. Consequently, the main problem is not only to construct the hybrid theory H , but also to determine whether its semantic model C_H remains inside the original Kaiser class K_T .

Remark 1. The construction is internal in the sense that both component theories T_1 and T_2 are generated by models of the fixed Kaiser class K_T . Although the hybrid $H(T_1, T_2)$ is itself a theory, its components are generated by models from the same Kaiser class. Therefore, the natural semantic question is whether the semantic model C_H of the hybrid also belongs to K_T .

2 A technical lemma on the internality of the hybrid semantic model

We now prove the technical fact which justifies the transfer of classical Jonsson-spectral results to the hybrid properties. The point is that an internal hybrid generated by Kaiser hulls of models from K_T remains semantically controlled by the same Kaiser class.

Lemma 1. Let T be a K_T -normal Jonsson theory. Let $M_1, M_2 \in K_T$, and put

$$T_1 = M_1^0, \quad T_2 = M_2^0.$$

Suppose that the internal hybrid

$$H = H(T_1, T_2) = Th_{\forall\exists}(C_{T_1} \times C_{T_2})$$

is a Jonsson theory. Then the semantic model C_H of H belongs to the original Kaiser class K_T :

$$C_H \in K_T.$$

Proof. Since $M_1, M_2 \in K_T$, by the definition of the Kaiser class their Kaiser hulls

$$M_1^0 = T_1, \quad M_2^0 = T_2$$

are Jonsson theories. Hence the semantic models C_{T_1} and C_{T_2} are well defined.

The theory

$$H = Th_{\forall\exists}(C_{T_1} \times C_{T_2})$$

is generated by the semantic product of the Kaiser hulls of models belonging to the same class K_T . Therefore H is an admissible internal hybrid construction associated with the fixed theory T .

Now we use the K_T -normality of T . By this assumption, every admissible internal hybrid construction generated by models from K_T remains inside the Kaiser layer of the ambient theory. Consequently, the semantic model of the hybrid theory H is again a model belonging to the Kaiser class K_T . Thus

$$C_H \in K_T. \quad \square$$

Remark 2. Lemma 1 does not claim that the hybrid theory H itself is an element of K_T . This would be incorrect, since K_T is a class of models, not a class of theories. The assertion concerns the semantic model C_H of the hybrid. Thus, the hybrid theory is generated from internal Kaiser fragments, while its semantic model remains in the original Kaiser class.

Lemma 1 is the main technical reason why classical results on Jonsson spectrum can be transferred to the hybrid case. Indeed, after the lemma has been established, the hybrid theory H is not merely a formal combination of two Jonsson theories. Its semantic model is still located inside the same K_T -normal Kaiser class. This makes it possible to apply Jonsson-spectral arguments to H and to its center

$$H^* = Th(C_H).$$

2.1 The role of the internality lemma

Lemma 1 is the structural core of the paper. Without this lemma, the passage from classical Jonsson-spectral results to the hybrid setting would remain only a formal analogy. Indeed, a hybrid theory constructed from two components need not, a priori, belong to the original Kaiser class K_T .

The content of Lemma 1 is precisely that this obstruction does not occur in the internal case. Indeed, for an internal hybrid

$$H = H(T_1, T_2),$$

where

$$T_1 = M_1^0, \quad T_2 = M_2^0, \quad M_1, M_2 \in K_T,$$

we have the following facts:

- H is a hybrid Jonsson theory;
- C_H is the semantic model of H ;
- C_H belongs to the original Kaiser class K_T .

Thus, the semantic model of the hybrid remains within the original Kaiser class K_T from which the hybrid was generated. This is the point at which the hybrid construction can be regarded as an internal construction.

It is important to emphasize that Lemma 1 does not state that the theory H itself is an element of K_T . The class K_T is a class of models, not a class of theories. The assertion concerns the semantic model C_H of the hybrid theory. Therefore the hybrid theory is generated from internal Kaiser hulls, while its semantic model remains inside the original Kaiser class.

Consequently, the center

$$H^* = Th(C_H)$$

can be studied by the methods available in the ambient Jonsson-spectral setting. This provides the mathematical justification for transferring Morley-type categoricity criteria, perfection arguments, and model-completeness results to internal hybrid spectra. In this sense, Lemma 1 turns hybridization from an external operation on unrelated theories into an internal operation on Kaiser fragments of one semantic Jonsson universe.

3 Classical background results

In this section we recall the classical model-theoretic principles which will be used as prototypes for the hybrid results proved below. Three directions are especially relevant for the present paper: Morley-type categoricity criteria, Saracino-type perfection arguments, and Lindstrom-type model-completeness principles.

We emphasize that these classical results are not applied to the hybrid theory as a purely formal analogy. Their transfer to the hybrid setting becomes possible only after Lemma 1, which shows that the semantic model of an internal hybrid remains inside the original Kaiser class K_T . Thus, the hybrid theory remains semantically contained in the same K_T -normal Kaiser class.

3.1 Morley-type criteria

The classical Morley categoricity theorem states that, for a complete theory in a countable language, categoricity in one uncountable cardinality implies categoricity in all uncountable cardinalities. In particular, ω_1 -categoricity is a strong rigidity condition on the class of models of the theory.

For the purposes of the present paper, however, we use not only this categoricity-transfer aspect of Morley's theorem. The important point is a related Morley-type mechanism: uncountable categoricity imposes strong structural restrictions on countable models and on their prime or prime-like extensions.

In the classical setting, this principle is expressed through the existence of prime elementary extensions over countable models in the appropriate categorical environment.

The aim of the present paper is to transfer this mechanism to the internal Jonsson-hybrid spectrum. In the hybrid situation, ordinary elementary primeness is no longer the most natural notion. The relevant replacement is Jonsson-algebraic primeness inside the class of existentially closed models of the hybrid theory.

Let

$$H = H(T_1, T_2)$$

be an internal hybrid generated by Kaiser hulls

$$T_1 = M_1^0, \quad T_2 = M_2^0, \quad M_1, M_2 \in K_T.$$

By Lemma 1, the semantic model C_H of H belongs to the original Kaiser class K_T . Therefore the hybrid center

$$H^* = Th(C_H)$$

may be studied within the same Jonsson-spectral framework as the original theory T .

This allows us to formulate the hybrid Morley principle in the following form: if the center H^* is ω_1 -categorical, then countable models from the existentially closed layer E_H admit Jonsson-algebraically prime extensions inside E_H . Thus, the classical relation between categoricity and prime extensions is transformed into a Jonsson-spectral relation between ω_1 -categoricity of the hybrid center and Jonsson-algebraic primeness in the hybrid existentially closed class.

Consequently, the Morley-type result obtained below is not merely a repetition of the classical categoricity theorem. It is a hybrid version of the structural part of the Morley phenomenon: categoricity of the center produces internal prime-extension behavior in the corresponding Jonsson spectrum.

3.2 Saracino-type perfection criteria

The second classical direction used in the paper is related to Saracino-type criteria for perfection. In the theory of Jonsson spectrum, perfection is one of the main structural properties connecting the existentially closed part of a Jonsson class with the complete theory of its semantic model. In this sense, perfection allows one to pass from the incomplete Jonsson theory to its center and to work with a better controlled model-theoretic object.

Saracino-type arguments usually relate the following properties:

- ω -categoricity;
- existence of a model companion;
- model completeness of the companion or the center;
- perfection of the corresponding Jonsson class.

For the purposes of the present paper, these arguments serve as a prototype for proving perfection in the hybrid setting. Let

$$H = H(T_1, T_2)$$

be an internal hybrid generated by Kaiser hulls

$$T_1 = M_1^0, \quad T_2 = M_2^0, \quad M_1, M_2 \in K_T.$$

By Lemma 1, the semantic model C_H of the hybrid belongs to the original Kaiser class K_T . Consequently, the hybrid remains in the same semantic context as the original theory, allowing classical Jonsson-spectral results to be transferred to the hybrid case.

This observation is essential for the use of Saracino-type methods. Indeed, without the inclusion

$$C_H \in K_T,$$

one could only compare the hybrid theory with classical Jonsson spectrum formally. Lemma 1 shows that the comparison is semantic: the hybrid center

$$H^* = Th(C_H)$$

is generated by a semantic model still lying inside the original Kaiser environment. Consequently, the classical connection between categoricity, model companions, model completeness, and perfection can be transferred to internal hybrid spectrum.

Thus, in the hybrid context, Saracino-type arguments are used to prove that under suitable categoricity and completeness assumptions the hybrid theory H is perfect. In particular, the existentially closed layer E_H becomes controlled by the center H^* , and the hybrid spectrum acquires the same structural regularity as a classical perfect Jonsson spectrum.

3.3 Lindström-type model-completeness criterion

The third classical direction is provided by Lindström-type model-completeness criteria. In the form relevant to the present paper, the classical principle says that sufficiently strong $\forall\exists$ -categoricity assumptions on a complete theory imply model completeness. Thus, categoricity with respect to the $\forall\exists$ -fragment becomes a mechanism forcing elementary behavior of embeddings.

In the ordinary first-order setting, this principle can be expressed as a criterion passing from a categorical condition on the theory to model completeness of the theory itself. In the Jonsson-spectral setting, however, the natural complete object associated with a Jonsson theory is not the Jonsson theory itself, but its center. Therefore, for a hybrid Jonsson theory H , the corresponding complete object is

$$H^* = Th(C_H).$$

The internality lemma again plays a decisive role. Since

$$C_H \in K_T,$$

the center H^* is not an external complete theory unrelated to the original construction. It is the complete theory of a semantic model belonging to the same Kaiser class from which the hybrid was generated. This makes it possible to transfer Lindström-type model-completeness arguments to the hybrid setting.

Accordingly, the hybrid form of the Lindström-type principle can be stated as follows: if an internal hybrid H satisfies the appropriate $\forall\exists$ -categoricity condition, then its center

$$H^* = Th(C_H)$$

is model complete. Symbolically, the guiding implication is

$$\forall\exists\text{-categoricity of the internal hybrid } H \implies \text{model completeness of } H^*.$$

This formulation is important because model completeness is asserted not for the incomplete Jonsson theory H itself, but for the complete center H^* . Thus, the hybrid Lindström-type result obtained below is a Jonsson-spectral analogue of the classical model-completeness criterion: categoricity in the relevant fragment forces model completeness at the level of the hybrid center.

The three classical directions discussed above provide the background for the main results of the paper. The Morley-type criterion will be transferred to Jonsson-algebraically prime extensions inside E_H ; the Saracino-type criterion will be used to obtain perfection of internal hybrids; and the Lindström-type criterion will yield model completeness of the hybrid center H^* . In each case, Lemma 1 is the technical bridge that allows the classical argument to be applied in the internal hybrid setting.

4 A converse Morley-type criterion

We now formulate a converse form of the hybrid Morley criterion. The converse implication requires an additional uniqueness condition for Jonsson-algebraically prime extensions, which is needed to construct a back-and-forth system between models of cardinality ω_1 .

Theorem 1 (Converse Morley-type criterion). Let T be a K_T -normal Jonsson theory, and let

$$H = H(M_1^0, M_2^0)$$

be an internal hybrid generated by models $M_1, M_2 \in K_T$. Assume that H is perfect, so that

$$E_H = \text{Mod}(H^*).$$

Suppose that the following conditions hold:

- every countable model $A \in E_H$ has a Jonsson-algebraically prime extension in E_H ;
- such extensions are unique over their bases up to isomorphism inside E_H ;
- the class E_H admits the corresponding back-and-forth amalgamation over countable submodels.

Then the center

$$H^* = \text{Th}(C_H)$$

is ω_1 -categorical.

Proof. By Lemma 1, we have

$$C_H \in K_T.$$

Thus the semantic model of the hybrid remains inside the original K_T -normal Kaiser environment.

Let $A_0 \in E_H$ be a countable model. By assumption, A_0 admits a Jonsson-algebraically prime extension in E_H . Repeating this construction transfinitely, we obtain a continuous chain

$$A_0 \subseteq A_1 \subseteq \cdots \subseteq A_\alpha \subseteq \cdots \quad (\alpha < \omega_1),$$

where each successor step is chosen to be Jonsson-algebraically prime over the previous stage. Put

$$\overline{A} = \bigcup_{\alpha < \omega_1} A_\alpha.$$

Then $\overline{A}$ has cardinality ω_1 and belongs to E_H .

Let $B \models H^*$ be any model of cardinality ω_1 . Since H is perfect, we may regard B as an element of E_H . Write B as the union of a continuous chain of countable submodels:

$$B = \bigcup_{\alpha < \omega_1} B_\alpha.$$

Using the uniqueness of Jonsson-algebraically prime extensions over countable bases, together with the back-and-forth amalgamation property in E_H , we construct by induction a back-and-forth system between the chains $\{A_\alpha : \alpha < \omega_1\}$ and $\{B_\alpha : \alpha < \omega_1\}$.

This construction yields an isomorphism

$$\overline{A} \cong B.$$

Hence all models of H^* of cardinality ω_1 are isomorphic. Therefore H^* is ω_1 -categorical. $\square$

5 Hybrid perfection theorem

The next result gives sufficient conditions under which an internal hybrid Jonsson theory is perfect.

Theorem 2 (Hybrid perfection theorem). Let $H = H(M_1^0, M_2^0)$ be a $\forall\exists$ -complete internal hybrid generated inside a K_T -normal Jonsson theory T . Suppose that H satisfies one of the following categoricity assumptions:

- H is ω -categorical and the Saracino-type perfection criterion applies to the hybrid spectrum;
- H is κ -categorical for some uncountable cardinal κ and the Morley-type categoricity transfer applies to the hybrid center.

Then H is perfect.

Proof. By Lemma 1, the semantic model C_H of the hybrid belongs to the original Kaiser class:

$$C_H \in K_T.$$

Thus the hybrid spectrum is not external to the original Jonsson environment. This internality makes it legitimate to transfer classical Jonsson-spectral arguments to the hybrid setting.

If H is ω -categorical, the Saracino-type criterion applies to the hybrid spectrum. It yields that the existentially closed layer of H is controlled by the complete theory of the semantic model. Equivalently, the center

$$H^* = Th(C_H)$$

determines the existentially closed models of H .

If H is κ -categorical for an uncountable cardinal κ , the Morley-type categoricity argument gives the same conclusion at the level of the hybrid center.

In both cases, the class of existentially closed models of H is axiomatized by the center H^* . Therefore

$$E_H = \text{Mod}(H^*),$$

and hence H is perfect. □

6 Hybrid model companion

We now derive the existence of a model companion for ω -categorical internal hybrids. The result follows from the hybrid perfection theorem.

Theorem 3 (Hybrid model companion). Let $H = H(M_1^0, M_2^0)$ be a $\forall\exists$ -complete ω -categorical internal hybrid generated inside a K_T -normal Jonsson theory. Assume that the Saracino-type perfection criterion applies to H . Then H has an ω -categorical model companion. Moreover,

$$H^M = H^*,$$

where

$$H^* = Th(C_H).$$

Proof. By Theorem 2, the hybrid theory H is perfect. Therefore its existentially closed models are axiomatized by the center:

$$E_H = \text{Mod}(H^*).$$

Hence the center H^* is the model companion of H .

Thus

$$H^M = H^*.$$

Since H is ω -categorical and the center axiomatizes the existentially closed part of the hybrid spectrum, the model companion H^M inherits the corresponding ω -categoricity. Therefore H has an ω -categorical model companion, namely its center H^* . □

Theorems 1–3 show that the internality of the semantic model is the main mechanism behind the hybrid transfer of classical results. Lemma 1 places C_H inside the original Kaiser class K_T , Theorem 2 gives conditions for perfection, and Theorem 3 identifies the model companion of the hybrid with its center.

7 Hybrid Lindström-type theorem

We now formulate the hybrid analogue of the Lindström-type model-completeness criterion. The classical idea is that suitable $\forall\exists$ -categoricity assumptions imply model completeness. In the Jonsson setting, model completeness is naturally formulated not for the possibly incomplete Jonsson theory itself, but for its complete center.

Theorem 4 (Hybrid Lindström-type theorem). Let

$$H = H(M_1^0, M_2^0)$$

be a $\forall\exists$ -complete internal hybrid generated inside a K_T -normal Jonsson theory T . Suppose that H satisfies the categoricity assumptions of Theorem 2. Then the center

$$H^* = Th(C_H)$$

is model complete.

Proof. By Theorem 2, the internal hybrid H is perfect. Hence its existentially closed models are axiomatized by the complete theory of the semantic model:

$$E_H = \text{Mod}(H^*).$$

Therefore the center H^* is the model companion of H .

It is a standard property of model companions that they are model complete. Consequently, the center

$$H^* = Th(C_H)$$

is model complete. □

8 Examples of K_T -normal situations

We describe several natural situations in which the internal hybrid construction is expected to be stable. These examples illustrate the role of K_T -normality and show that the above results apply to a broad class of Jonsson-spectral environments.

Example 1 (Perfect Jonsson theories). Let T be a perfect Jonsson theory. In this case the existentially closed models of T are controlled by the center

$$T^* = Th(C_T).$$

If, in addition, the Kaiser class K_T is closed under the admissible internal hybrid-amalgamation constructions considered above, then T is K_T -normal. Hence internal hybrids generated by models from K_T remain semantically inside the original Kaiser environment.

Example 2 (The case $K_T = E_T$). Suppose that the Kaiser class coincides with the class of existentially closed models:

$$K_T = E_T.$$

Then internal hybridization is generated directly from existentially closed models of the original theory. Under the corresponding closure condition for Kaiser hulls and their semantic products, this gives a natural example of a K_T -normal situation.

Example 3 (Center- ω -stable Jonsson theories). Let T be a Jonsson theory whose center $T^* = Th(C_T)$ has a suitable ω -stability property. Stability of the center provides additional control over types and semantic extensions. Under admissibility and closure of the Kaiser class, such theories form another natural source of K_T -normal examples.

Example 4 (Cosemantic spectral classes). Spectral classes arising in the study of Jonsson spectrum and cosemanticness also provide natural environments for internal hybridization. In these classes, semantic models are compared through their complete theories and their existentially closed layers. Therefore, when the corresponding Kaiser classes are stable under admissible internal hybrid constructions, the hybridization procedure remains internal.

These examples show that K_T -normality is not an artificial assumption. It expresses the semantic stability of the Kaiser layer under internal constructions. In each of the situations above, once the required closure condition is verified, Lemma 1 applies and guarantees that the semantic model of the internal hybrid remains in the original class K_T .

Conclusion

The paper introduces internal hybridization of Kaiser hulls inside K_T -normal Jonsson theories.

The key technical fact is: $C_H \in K_T$.

This allows transfer of classical spectral results to hybrid Jonsson spectrum.

In particular, the paper constructs a Jonsson-hybrid analogue of Morley's criterion: ω_1 -categorical center iff Jonsson-algebraically prime extensions inside E_H .

The results open a new direction:

- hybrid Jonsson spectrum;
- internal hybridization of fragments;
- hybrid semantic geometry inside K_T ;
- future applications to diamonds, AMAL-configurations, and hybrid normality.

Author contributions

All authors contributed equally to this work.

Conflict of interest

The authors have no conflict of interest to declare.

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Estimates of solutions for a class of nonlinear equations in finite-dimensional space

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This paper investigates a class of nonlinear operator equations in a finite-dimensional Hilbert space. Specifically, equations of the form $u + L(u) = g$ are considered, where L is a nonlinear continuous operator acting on the space H , and g is a given element of this space. The main focus is on deriving a priori estimates for solutions and studying the conditions for their existence. For a sufficiently broad class of nonlinear operators, uniform estimates of solutions are established, expressed in terms of the parameters of the original problem and the properties of the operator under consideration. It is shown that these estimates do not depend on the index of the finite-dimensional approximation, which allows them to be used in the study of the corresponding infinite-dimensional problems. The results justify the method of finite-dimensional approximations and to carry out the limiting transition to the original problem. This makes it possible to obtain a priori estimates for solutions of infinite-dimensional nonlinear operator equations arising in problems of mathematical physics and functional analysis. The proposed approach can be applied to the study of a wide class of nonlinear problems and to the construction of approximate methods for their solution.

Keywords: nonlinear operator equation, finite-dimensional Hilbert space, a priori estimates of solutions, nonlinear operator, existence of solutions, approximation of solutions, operator equations, functional analysis.

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Introduction

In many problems of mathematical physics, energy estimates and identities play an important role, as they make it possible to obtain a priori estimates of solutions to the equations under consideration. The use of such estimates allows one to prove existence theorems for weak solutions of problems in mathematical physics. However, weak solutions do not always allow the application of perturbation theory methods or a more detailed analysis of the properties of the solution [1]. In this connection, there arises a need to obtain strong solutions to the corresponding problems [2–4].

One of the widely used methods of investigation is the use of finite-dimensional approximations of the original problem [5, 6]. An important condition in this approach is the derivation of a priori estimates that do not depend on the approximation index. The existence of such estimates makes it possible to pass to the limit and prove the existence of a solution to the original problem [7, 8].

In the present paper, a nonlinear equation in a finite-dimensional space is considered. For a sufficiently wide class of equations, a priori estimates of solutions are obtained, which explicitly depend on the constants appearing in the initial conditions of the problem. It is shown that in the case when finite-dimensional systems serve as approximations of the original problem, the obtained estimates do not depend on the approximation index. This makes it possible to pass to the limit and obtain a priori estimate for the solution of the original mathematical physics problem.

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1 Notation and basic assumptions

Throughout this section, H denotes a finite-dimensional real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$.

We consider equations of the form

$$u + L(u) = g \in H, \quad (1)$$

where $L(\cdot)$ is a nonlinear continuous mapping such that $L(0) = 0$, and g is an element of the space H . The solution u of equation (1) is sought in H .

We focus on finite-dimensional equations of type (1), which are finite-dimensional approximations of infinite-dimensional problems of the same form in an infinite-dimensional Hilbert space [9, 10]. It is of *particular importance to derive estimates that do not depend on the approximation index, allowing passage to the limit and obtaining, in the limit, an a priori estimate for the solution of the infinite-dimensional problem*. The infinite-dimensional problems of type (1), which are the ultimate goal of this study, typically arise in mathematical physics and are written in a weak (variational) form [11, 12].

In this section, we denote $f(u)$ as an operation of the form

$$f(u) := u + L(u).$$

If $\xi \in [0; +\infty)$ is a parameter and $u(\xi)$ is a vector-valued function continuously differentiable with respect to ξ , then we assume that $L(u(\xi))$, as well as the expressions involving $L(u)$ and $f(u)$ that appear later, are also continuously differentiable.

We introduce the notation L_u :

$$(L(u(\xi)))_\xi = L_{u(\xi)} u_\xi(\xi).$$

It is obvious that the operator L_u (for each $u \in H$) is linear

$$L_u v = (L(u(\xi))) \Big|_{u_\xi=v}.$$

Here, L_u denotes the Frechet derivative of the operator L at the point u :

$$L_u v = \frac{d}{d\xi} L(u - \xi v) \Big|_{\xi=0}.$$

We have

$$(f(u(\xi)))_\xi = u_\xi + L_u u_\xi = (E + L_u) u_\xi.$$

Further, if $u_0, v_0 \in H$, then the vector $L_{u_0} v_0$ is defined as follows: take a continuously differentiable vector-function $u(\xi)$ such that

$$u \Big|_{\xi=0} = u_0, \quad u_\xi(\xi) \Big|_{\xi=0} = v_0,$$

and define as $L_{u_0} v_0$ the vector

$$L_{u_0} v_0 = (L(u(\xi)))_\xi \Big|_{\xi=0}.$$

Here and throughout, E denotes the identity operator.

We denote

$$D_u = E + L_u, \quad D_u^* = E + L_u^*$$

$$D_u^* f(u) = (E + L_u^*) f(u).$$

The symbol $*$ stands for the adjoint operation.

$$M_u a = \left(D_{u(\xi)}^* f(u(\xi)) \right)_{\xi} \Big|_{\substack{u(\xi) = u \\ u_{\xi}(\xi) = a}} = M_u u_{\xi} \Big|_{u_{\xi} = a} = M_u a.$$

Condition U1. There exist linear invertible operators T and Q such that

$$\|T\| \leq C_T, \quad \|Q\| \leq C_T, \quad \|T^{-1}\| < \infty, \quad \|Q^{-1}\| < \infty,$$

and for any $u \in H$ the inequalities hold

$$\langle Tu, L(u) \rangle \geq 0, \quad \langle Tu, u \rangle \geq \|Qu\|^2 \geq C_{1T} \|Tu\|^2, \quad (2)$$

where C_T and C_{1T} are some fixed positive constants.

In what follows, by C or c (uppercase or lowercase, with or without indices, with or without tildes) we will denote constants (generally different in different places) that do not depend on the neighboring factors.

The following result holds

Theorem 1. Suppose that condition U1 is satisfied. Then for any $g \in H$ the problem

$$f(u) = g$$

has a solution $u \in H$, satisfying the estimate

$$C_{1T} \|Tu\|^2 \leq \|Qu\|^2 \leq C_T \|g\|^2, \quad (3)$$

where the operator Q , T and the constants C_T , C_{1T} are taken from condition U1.

The result of this theorem is well known and widely used. We present it in a form convenient for us.

2 Proof of Theorem 1

Proofs of theorems whose statements are similar to that of Theorem 1 are well known, and we could restrict ourselves to a reference [1, p. 550]. However, for completeness, we provide a proof of Theorem 1.

For the proof, we use a standard technique in a form convenient for us.

If $g \equiv 0$, the equation $u + L(u) = 0$ has the solution $u \equiv 0$. Therefore, in this case estimate (3) holds.

Let $0 \neq g \in H$. Denote by $M(g)$ the set of vectors

$$M(g) = \{u \in H : \langle Tu, u \rangle \leq 16 \langle Tg, g \rangle\},$$

where T is the operator from condition U1.

Assume that the equation $u + L(u) = g$ has no solution $u \in M(g)$. Define the mapping

$$F(u) = - \frac{u + L(u) - g}{\sqrt{\langle T(u + L(u) - g), u + L(u) - g \rangle}} 4\sqrt{\langle Tg, g \rangle}. \quad (4)$$

Since the equation $u + L(u) = g$ has no solution, it follows from inequality (2) in condition U1 that the mapping $F(\cdot)$ is continuous. It is easy to see that this mapping sends the set $M(g)$ into itself.

Since, according to condition U1, the inequality $\langle Tv, v \rangle \geq \|Qv\|^2$ holds for any $v \in H$ and the operator Q is invertible, the quantity $\langle Tu_0, u_0 \rangle$ can be taken as the square of the norm of the vector u_0 ,

and the quantity $\frac{1}{2}(\langle Tu_0, g \rangle + \langle u_0, T^*g \rangle)$ as the inner product (consistent with this norm) of the vectors u_0 and g in some real Hilbert space, which in this subsection will be denoted by $\tilde{H}$.*

In the space $\tilde{H}$ the set $M(g)$ is a ball. Therefore, by Brouwer's fixed point theorem and the finite dimensionality of H (see [2]), the mapping $F(\cdot)$ has a fixed point $u_0 \in M(g)$, that is,

$$F(u_0) = u_0. \quad (5)$$

Applying the operator T to (5) and then taking the inner product of the resulting equality with $u_0 + L(u_0) - g$ we obtain, using (4),

$$\begin{aligned} -A &\equiv -4\sqrt{\langle Tg, g \rangle} \sqrt{\langle T(u_0 + L(u_0) - g), u_0 + L(u_0) - g \rangle} = \\ &= \langle Tu_0, u_0 + L(u_0) - g \rangle \geq \langle Tu_0, u_0 \rangle - \langle Tu_0, g \rangle = \\ &= \langle Tu_0, u_0 \rangle - \frac{1}{2}(\langle Tu_0, g \rangle + \langle u_0, T^*g \rangle). \end{aligned} \quad (6)$$

In deriving (6) the inequality (2) from condition U1 was used in the penultimate step.

Therefore, applying the well-known Cauchy inequality, we obtain

$$-A \geq \langle Tu_0, u_0 \rangle - (\varepsilon^{-1} \langle Tu_0, u_0 \rangle + \varepsilon \langle Tg, g \rangle) = \langle Tu_0, u_0 \rangle (1 - \varepsilon^{-1}) - \varepsilon \langle Tg, g \rangle. \quad (7)$$

From (5) and (4) we have

$$\langle Tu_0, u_0 \rangle = \langle TF(u_0), F(u_0) \rangle = 16 \langle Tg, g \rangle.$$

Hence, from (7) we obtain

$$-A \geq [(1 - \varepsilon^{-1})16 - \varepsilon] \langle Tg, g \rangle.$$

Choosing $\varepsilon = 2$, we get $-A \geq 6 \langle Tg, g \rangle$.

The negativity of the left-hand side contradicts inequality (2) from condition U1. This proves that the equation $u + L(u) = g$ has a solution $u \in M(g)$.

For solving the equation $u + L(u) = g$, taking the inner product with Tu , we have

$$\langle Tu, u \rangle + \langle Tu, L(u) \rangle = \langle Tu, g \rangle = \frac{1}{2}(\langle Tu, g \rangle + \langle u, T^*g \rangle) = \langle u, g \rangle_{\tilde{H}}.$$

Here, on the right-hand side stands the inner product of the vectors Tu and g in the Hilbert space $\tilde{H}$. Applying the Cauchy inequality and taking into account the first inequality in (2) from condition U1, we obtain:

$$\langle Tu, u \rangle \leq \langle Tu, u \rangle + \langle Tu, L(u) \rangle = \langle u, g \rangle_{\tilde{H}} \leq \|u\|_{\tilde{H}} \|g\|_{\tilde{H}} = \sqrt{\langle Tu, u \rangle} \sqrt{\langle Tg, g \rangle}.$$

Hence,

$$\langle Tu, u \rangle \leq \langle Tg, g \rangle \leq \|Tg\| \|g\| \leq C_T \|g\|^2,$$

where C_T is the constant from condition U1.

Now from the second inequality (2) of the condition U1 we get

$$\|Qu\|^2 \leq C_T \|g\|^2.$$

Theorem 1 is proved.

*In a real Hilbert space, the equality $\langle a, b \rangle = \langle b, a \rangle$ holds for the inner product of vectors a and b . Then, for a vector $u \in H$ and a linear operator T , we have $\langle Tu, u \rangle = \langle u, Tu \rangle$. On the other hand, $\langle Tu, u \rangle = \langle u, T^*u \rangle$, where T^* denotes the operator adjoint to the operator T . Therefore, $\langle Tu, u \rangle = \langle \frac{1}{2}(T + T^*)u, u \rangle$. If the quantity $\langle Tu, u \rangle$ is strictly positive,

then it can be regarded as the square of a norm (which indeed it is): $\langle Tu, u \rangle = \left\| \sqrt{\frac{1}{2}(T + T^*)} u \right\|^2$. Thus, the inner product of vectors $a, b \in H$, consistent with this norm, is defined by the formula $\langle a, b \rangle_{\tilde{H}} = \frac{1}{2}(\langle Ta, b \rangle + \langle a, T^*b \rangle)$.

3 Formulation of the main finite-dimensional theorem

From Theorem 1, since the operators T and Q are invertible (under the given assumptions), we obtain the estimate

$$\|u\| \leq \tilde{C} \|f(u)\|. \quad (8)$$

In this inequality, the constant $\tilde{C}$ depends on the smallest eigenvalue of the operator T^*T (or Q^*Q).

In the case where $f(\cdot)$ is a finite-dimensional approximation of a nonlinear problem (written in weak form), the estimates (8) will depend on the approximation index and therefore cannot ensure passage to the limit.

To ensure the possibility of passing to the limit, it is sufficient, for finite-dimensional approximations, to obtain estimates that do not depend on the approximation index.

To obtain such estimates, we will use the conditions given below.

Condition U2. For the mapping $L(\cdot)$ and the operators L_u, M_u for any elements $u, v \in H$ the following inequalities hold:

$$\begin{aligned} \|M_u - M_v\|_{H \rightarrow H} + \|L_u - L_v\|_{H \rightarrow H} + \|L(u) - L(v)\| &\leq \\ &\leq (\psi(\|u\|) + \psi(\|v\|)) \|u - v\|; \end{aligned} \quad (9)$$

$$\|u\|^2 + \|L(u)\|^2 + \|L_u u\|^2 + \|M_u u\| \leq \psi(\|u\|). \quad (10)$$

In (9) and (10) $\psi(\cdot)$ is a function defined by

$$\psi(t) = C(t^2 + 1)^m, \quad (11)$$

where C and $m > 0$ are constants.

Condition U3. There exists an invertible linear operator T such that

$$\|T\| \leq C_0, \quad \|T^{-1}\| < \infty,$$

and for any $u \in H$ the following inequalities hold:

$$\langle f(u), Tu \rangle \geq 0, \quad \|Tu\| \leq d_0 \|f(u)\|, \quad (12)$$

where $d_0 > 0$ is a constant.

We introduce the notation

$$\tilde{L}(u) = L(u) + \|Tu\|^{2k+\varepsilon} Tu, \quad (13)$$

$$\tilde{f}(u) = u + \tilde{L}(u) = f(u) + \|Tu\|^{2k+\varepsilon} Tu, \quad (14)$$

where T is the operator from condition U3. The numbers ε, k from (13) and m from (11) are fixed constants such that $k > m + 3$ and $\varepsilon > 0$.

We denote by $\tilde{L}_u, \tilde{M}_u, \tilde{D}_u$ the operators analogous to L_u, M_u, D_u , in the definitions of which $f(u)$ and $L(u)$ are replaced by $\tilde{f}(u)$ and $\tilde{L}(u)$, respectively.

We note the following lemma that follows from condition U2.

Lemma 1. For the mapping $\tilde{L}(\cdot)$ and operators $\tilde{L}_u, \tilde{L}_u^*, \tilde{D}_u, \tilde{D}_u^*, \tilde{M}_u, \tilde{M}_u^*$ the following inequalities hold:

$$\begin{aligned} \|\tilde{M}_u - \tilde{M}_v\|_{H \rightarrow H} + \|\tilde{M}_u^* - \tilde{M}_v^*\|_{H \rightarrow H} + \|\tilde{L}_u - \tilde{L}_v\|_{H \rightarrow H} + \\ + \|\tilde{L}_u^* - \tilde{L}_v^*\|_{H \rightarrow H} + \|\tilde{D}_u - \tilde{D}_v\|_{H \rightarrow H} + \|\tilde{D}_u^* - \tilde{D}_v^*\|_{H \rightarrow H} \leq \\ \leq \tilde{\psi}(\|u\|) \tilde{\psi}(\|v\|) \|u - v\|; \end{aligned} \quad (15)$$

$$\begin{aligned} & \|\widetilde{L}(u)\| + \|\widetilde{L}_u\|_{H \rightarrow H} + \|\widetilde{M}_u\|_{H \rightarrow H} + \|\widetilde{D}_u\|_{H \rightarrow H} + \\ & + \|\widetilde{L}_u^*\|_{H \rightarrow H} + \|\widetilde{M}_u^*\|_{H \rightarrow H} + \|\widetilde{D}_u^*\|_{H \rightarrow H} \leq \widetilde{\psi}(\|u\|). \end{aligned} \quad (16)$$

In (15) and (16) $\widetilde{\psi}(\cdot)$ is strictly increasing function on $[0; \infty)$, $\widetilde{\psi}(0) \geq 1$.

Let us introduce one more condition used in what follows.

Condition U4. If the vector $u \in H$ satisfies condition

$$\|u\|^{4k+2} \geq \|Tu\|^{4k+2+2\varepsilon}, \quad \|u\|^2 \geq \widehat{C}, \quad \widehat{C} \geq 10,$$

then

$$\inf \frac{\langle \widetilde{M}_u P_u a, P_u a \rangle}{\|P_u a\|^2} < (k+1) \|u\|^{4k}, \quad (17)$$

where k and ε from (13)–(14), P_u is the projector defined by

$$P_u a = a - \frac{\langle a, u \rangle}{\|u\|^2} u.$$

In (17) *infimum* is taken over all $a \in H$ such that $P_u a \neq 0$.

The following result holds.

Theorem 2. If conditions U2, U3, and U4 are satisfied, then for any $u \in H$ the following estimate holds

$$\|u\|^2 \leq (\|f(u)\|^2 + 2)^\Theta,$$

where Θ is a constant independent of $u \in H$, depending only on the constant m , C_0 , d_0 , k , ε , appearing in conditions U2, U3, and U4.

Recall that by C (lowercase or uppercase, with or without indices) we denote constants (generally different in different places).

4 Main lemmas

In what follows, throughout this section we assume that conditions U2, U3, and U4 are satisfied.

We introduce the functionals $J(u)$, $\mathfrak{D}(u)$, $\gamma(u)$ and the vector $N(u) \in H$, depending on $u \in H$:

$$J(u) = \exp \left\{ \|u\|^{4k+2} - \|\widetilde{f}(u)\|^2 \right\}, \quad (18)$$

where $k > m + 3$, m is taken from (11) and $\widetilde{f}(\cdot)$ is defined in (14);

$$\mathfrak{D}(u) = \exp \left\{ \exp \{ \|u\|^2 \} - \|\widetilde{f}(u)\|^2 \right\}; \quad (19)$$

$$\gamma(u) = \frac{\langle \widetilde{D}_u^* \widetilde{f}(u), u \rangle}{\|u\|^2}; \quad (20)$$

$$N(u) = \widetilde{D}_u^* \widetilde{f}(u) - \gamma(u)u = P_u \widetilde{D}_u^* \widetilde{f}(u), \quad (21)$$

where P_u is the projector defined in (17).

The following holds.

Lemma 2. For any $c > 0$ the set

$$M^{(c)} = \{u \in H : J(u) \geq c\}$$

is bounded and compact.

Proof. Since H is finite-dimensional, it suffices to prove that the set $M^{(c)}$ is bounded.

Assume the contrary. Suppose that the set $M^{(c)}$ is unbounded. Then there exists a sequence $u_1, u_2, \dots, u_n, \dots$ such that

$$\lim_{n \rightarrow \infty} \|u_n\| = +\infty, \quad J(u_n) \geq c. \quad (22)$$

For $J(u_n)$ using (18) we have

$$\begin{aligned} 0 < c \leq J(u_n) &= \exp \left\{ \|u_n\|^{4k+2} - \|f(u_n) + \|Tu_n\|^{2k+\varepsilon} Tu_n\|^2 \right\} = \\ &= \exp \left\{ \|u_n\|^{4k+2} - \|f(u_n)\|^2 - 2\langle u_n, Tu_n \rangle \|Tu_n\|^{2k+\varepsilon} - \right. \\ &\quad \left. - 2\langle L(u_n), Tu_n \rangle \|Tu_n\|^{2k+\varepsilon} - \|Tu_n\|^{4k+2+2\varepsilon} \right\} \leq \\ &\leq \exp \left\{ \|u_n\|^{4k+2} + C \|u_n\|^{2k+2+\varepsilon} + C \|L(u_n)\| \|u_n\|^{2(k+1)+\varepsilon} - \|Tu_n\|^{4k+2+2\varepsilon} \right\} \leq \\ &\leq \exp \left\{ C_1 \|u_n\|^{4k+2} + C \|u_n\|^{2k+2+\varepsilon} + C_1 \|u_n\|^{3k-1+\varepsilon} - \|Tu_n\|^{4k+2+2\varepsilon} \right\}. \end{aligned} \quad (23)$$

In deriving (23) we used the boundedness of the operator T , condition U2 (see (11)) and the choice $k > m + 3$.

Now we use the invertibility of the operator T ($\|T^{-1}\|_{H \rightarrow H} < C_2 < \infty$). Then from (23) we obtain

$$0 < c \leq J(u_n) \leq \exp \left\{ C_3 \left[\|u_n\|^{4k+2+\varepsilon} - C_5 \|u_n\|^{4k+2+2\varepsilon} \right] \right\}.$$

Hence, as $n \rightarrow \infty$ using (22) and $\varepsilon > 0$ we arrive at a contradiction $0 < c \leq 0$.

From this contradiction it follows that the set $M^{(c)}$ is bounded.

The lemma is proved.

Let us note the following obvious result.

Lemma 3. Let $u \in H$. Then the equality holds

$$\langle N(u), u \rangle_H = 0.$$

Proof. The equality follows directly from the definitions of the functional $\gamma(\cdot)$ and the vector $N(\cdot)$ (see (20), (21)). The lemma is proved.

For $u_0 \in H$, suppose that

$$\tilde{\mathfrak{D}}(u_0) = \sup \mathfrak{D}(u), \quad (24)$$

where *supremum* is taken over all $u \in H$ such that

$$J(u) \geq J(u_0). \quad (25)$$

In formulas (24) and (25) the functionals $\mathfrak{D}(\cdot)$, $J(\cdot)$ are defined by (18) and (19).

The following holds.

Lemma 4. Let $0 \neq u_0 \in H$. Then there exists a vector $\tilde{u}_0$ that attains the *supremum* (24)–(25). For this vector $\tilde{u}_0$ the inequalities hold

$$\mathfrak{D}(\tilde{u}_0) \geq \mathfrak{D}(u_0), \quad (26)$$

$$J(\tilde{u}_0) \geq J(u_0). \quad (27)$$

Proof. The functionals $\mathfrak{D}(\cdot)$, $J(\cdot)$ are continuous. By Lemma 2 and inequality (25), *supremum* is taken over a compact set.

It is well known that *supremum* of a continuous functional over a compact set is attained at some point of that set [13, 14]. Thus, the existence of a vector $\tilde{u}_0$, realizing the *supremum* (24)–(25), is established [15, 16].

The inequalities (26)–(27) follow directly from the definitions.

The lemma is proved.

Define the vector-valued function $u(\xi)$ as the solution of the Cauchy problem

$$\begin{cases} u_\xi = x(\xi) u + P_u a \equiv x(\xi) u + a - \frac{\langle a, u \rangle}{\|u\|^2} u, \\ u|_{\xi=0} = \tilde{u}_0, \end{cases} \quad (28)$$

where $a \in H$ and $\tilde{u}_0$ is the vector realizing the *supremum* (24)–(25).

For $J(u(\xi))$ and $\mathfrak{D}(u(\xi))$ we have:

$$\begin{aligned} J_\xi(u(\xi)) &= 2J(u) \left[(2k+1) \|u\|^{4k} \langle u, u_\xi \rangle - \langle \tilde{f}(u), \tilde{D}_u u_\xi \rangle \right] = \\ &= 2J(u) \left\langle (2k+1) \|u\|^{4k} \cdot u - \gamma(u)u - N(u), u_\xi \right\rangle = \\ &= 2J(u) \left[((2k+1) \|u\|^{4k} - \gamma(u)) \|u\|^2 \cdot x - \langle N(u), P_u a \rangle \right]; \\ \mathfrak{D}_\xi(u(\xi)) &= 2\mathfrak{D}(u) \left[(\exp \{ \|u\|^2 \}) \langle u, u_\xi \rangle - \langle \tilde{f}(u), \tilde{D}_u u_\xi \rangle \right] = \\ &= 2\mathfrak{D}(u) \left[(\exp \{ \|u\|^2 \}) \langle u, u_\xi \rangle - \langle \tilde{D}_u^* \tilde{f}(u), u_\xi \rangle \right] = \\ &= 2\mathfrak{D}(u) \left[(\exp \{ \|u\|^2 \}) \langle u, u_\xi \rangle - \langle N(u) + \gamma(u)u, u_\xi \rangle \right] = \\ &= 2\mathfrak{D}(u) \left[(\exp \{ \|u\|^2 \} - \gamma(u)) \|u\|^2 \cdot x - \langle N(u), P_u a \rangle \right]. \end{aligned} \quad (29)$$

Lemma 5. Let $\tilde{u}_0$ be the vector realizing *supremum* (24)–(25). Then the following relations hold:

$$N(\tilde{u}_0) = 0; \quad (30)$$

$$(\exp \{ \|\tilde{u}_0\|^2 \} - \gamma(\tilde{u}_0)) \left((2k+1) \|\tilde{u}_0\|^{4k} - \gamma(\tilde{u}_0) \right) \leq 0. \quad (31)$$

Proof. We first prove (30).

Suppose that $N(\tilde{u}_0) \neq 0$. In the problem (28) we choose

$$\begin{cases} x = 0, \\ a = -N(u(\xi)). \end{cases}$$

Then for $J(u(\xi))$ and $\mathfrak{D}(u(\xi))$, using calculations (29) and equality

$$P_{\tilde{u}_0} a = N(\tilde{u}_0),$$

we have

$$\begin{aligned} J_\xi(u) &= \|N(u(\xi))\|^2 \cdot 2 J(u), \\ \mathfrak{D}_\xi(u) &= \|N(u(\xi))\|^2 \cdot 2 \mathfrak{D}(u). \end{aligned}$$

Next, if $N(\tilde{u}_0) \neq 0$, then we have the inequalities

$$J_\xi(u) > 0, \quad \mathfrak{D}_\xi(u) > 0,$$

which hold for $\xi \in [0; \xi_0]$, where $\xi_0 > 0$ is some number.

From this, we obtain the strict inequalities

$$J(u(\xi_0)) > J(\tilde{u}_0), \quad \mathfrak{D}(u(\xi_0)) > \mathfrak{D}(\tilde{u}_0), \quad (32)$$

which contradict the definition of the vector $\tilde{u}_0$.

Therefore, (30) holds.

Now let us prove (31). Suppose that inequality (31) does not hold.

In problem (28) we choose $a = 0$ and $x(\xi) = \exp \{\|\tilde{u}_0\|^2\} - \gamma(\tilde{u}_0)$. Then the vector-function $u = u(\xi)$ is defined as the solution of the Cauchy problem

$$\begin{cases} u_\xi = (\exp \{\|\tilde{u}_0\|^2\} - \gamma(\tilde{u}_0)) u(\xi), \\ u|_{\xi=0} = \tilde{u}_0. \end{cases} \quad (33)$$

From computations (29) using (33), we obtain:

$$\begin{aligned} J_\xi(u(\xi))|_{\xi=0} &= 2 J(\tilde{u}_0) \left[(2k+1) \|\tilde{u}_0\|^{4k} - \gamma(\tilde{u}_0) \right] \cdot \left[\exp \{\|\tilde{u}_0\|^2\} - \gamma(\tilde{u}_0) \right] \|\tilde{u}_0\|^2; \\ \mathfrak{D}_\xi(u(\xi))|_{\xi=0} &= 2 \mathfrak{D}(\tilde{u}_0) \left[\exp \{\|\tilde{u}_0\|^2\} - \gamma(\tilde{u}_0) \right]^2 \|\tilde{u}_0\|^2. \end{aligned}$$

Hence, from the assumption to the contrary, we obtain the strict inequalities

$$J_\xi(u(\xi))|_{\xi=0} > 0, \quad \mathfrak{D}_\xi(u(\xi))|_{\xi=0} > 0.$$

In turn, these inequalities imply that for some $\xi_0 > 0$ the strict inequalities (32) hold.

Since inequalities (32) contradict the definition of the vector $\tilde{u}_0$, we obtain a contradiction with the assumption. Therefore, (31) is established.

The lemma is proved.

Lemma 6. Suppose that $u \in H$ and the following condition holds

$$\exp \{\|u\|^2\} \leq |\gamma(u)|.$$

Then the following inequality holds

$$\|u\| \leq \tilde{C}_1, \quad (34)$$

where $\tilde{C}_1$ is a constant depending only on the constants $C, m, d_0, k, \varepsilon$, appearing in conditions U2, U3, and U4.

Proof. We may assume that $\|u\|^2 \geq 1$. From the assumption of the lemma ($\exp \{\|u\|^2\} \leq |\gamma(u)|$), using conditions U2 and U3 and the fact that $\|u\|^2 \geq 1$, we obtain

$$\exp \{\|u\|^2\} \leq |\gamma(u)| \leq |\gamma(u)| \|u\|^2 = \left| \langle \tilde{D}_u^* \tilde{f}(u), u \rangle \right| \leq \|\tilde{f}(u)\| \|\tilde{D}_u u\| \leq$$

$$\leq \left(\|u\| + \|\tilde{L}(u)\| \right) \left(\|u\| + \|\tilde{L}_u u\| \right) \leq C_0 (1 + \|u\|^2)^{2(2k+1+\varepsilon)},$$

where C_0 is a constant.

From this inequality, estimate (34) follows.

The lemma is proved.

The following result holds.

Lemma 7. Let $\tilde{u}_0 \in H$ be a vector realizing the *supremum* in (24)–(25). Then either a) or b) holds:

$$a) \|\tilde{u}_0\| \leq \tilde{C}_1, \quad \text{where } \tilde{C}_1 \text{ is from Lemma 6;}$$

$$b) N(\tilde{u}_0) = 0 \quad \text{and} \quad \gamma(\tilde{u}_0) \geq (2k+1) \|\tilde{u}_0\|^{4k}, \quad k \text{ is from (14).}$$

Proof. We may assume that $\|\tilde{u}_0\|^2 \geq 1$. From Lemma 5 we know that (30) and (31) hold. Therefore, if b) does not hold, then we obtain $\gamma(\tilde{u}_0) \leq (2k+1) \|\tilde{u}_0\|^{4k}$. Then from (31) it follows that

$$\exp \{ \|\tilde{u}_0\|^2 \} \leq \gamma(\tilde{u}_0) = |\gamma(\tilde{u}_0)|.$$

Hence, by Lemma 6, condition a) holds.

The lemma is proved.

5 Proof of Theorem 2

Let $u_0 \in H$ be a vector whose norm is to be estimated in terms of the norm of the vector $f(u_0) = u_0 + L(u_0)$. From the vector $f(u_0)$ construct the vector $f(u_0) + \|Tu_0\|^{2k+\varepsilon} Tu_0 = \tilde{f}(u_0)$. Next, for the vector u_0 construct the vector $\tilde{u}_0$, realizing the *supremum* in (24)–(25). According to Lemma 7, either a) or b) holds:

$$a) \|\tilde{u}_0\| \leq \tilde{C}_1; \tag{35}$$

$$b) N(\tilde{u}_0) = 0 \quad \text{and} \quad \gamma(\tilde{u}_0) \geq (2k+1) \|\tilde{u}_0\|^{4k},$$

where $\tilde{C}_1$ is a constant defined in Lemma 7.

If condition a), then from the definition of $J(u_0)$ and Lemma 4 we have

$$\begin{aligned} J(u_0) &\leq J(\tilde{u}_0) = \exp \left\{ \|\tilde{u}_0\|^{4k+2} - \|\tilde{f}(\tilde{u}_0)\|^2 \right\} \leq \\ &\leq \exp \left\{ \|\tilde{u}_0\|^{4k+2} \right\} \leq \exp \left\{ \tilde{C}_1^{4k+2} \right\}. \end{aligned} \tag{36}$$

From this and from the definition of $J(\cdot)$ we obtain

$$\begin{aligned} \|u_0\|^{2(2k+1)} &\leq \|\tilde{f}(u_0)\|^2 + \tilde{C}_1^{4k+2} = \|f(u_0) + \|Tu_0\|^{2k+\varepsilon} Tu_0\|^2 + \tilde{C}_1^{4k+2} \leq \\ &\leq [\|f(u_0)\| + \|Tu_0\|^{2k+1+\varepsilon}]^2 + \tilde{C}_1^{4k+2} \leq \\ &\leq [\|f(u_0)\| + (d_0 \|f(u_0)\|)^{2k+1+\varepsilon}]^2 + \tilde{C}_1^{4k+2}. \end{aligned} \tag{37}$$

In deriving (37) the last step uses condition U3.

From (37) we obtain the estimate

$$\|u_0\|^2 \leq (\|f(u_0)\|^2 + 2)^\Theta, \tag{38}$$

where Θ is a constant independent of $u_0 \in H$, depending only on the constants d_0 from condition U3, m from condition U2, and the choice $k > m$ and $\varepsilon \in \left(0; \frac{1}{10}\right)$ (see (14)).

Thus, in the case where condition $a)$ of (35) holds, Theorem 2 is proved.

In the case where condition $b)$ of (35) holds, we define the vector-function $u = u(\xi)$ as the solution of the Cauchy problem

$$\begin{cases} u_\xi = P_u a, \\ u|_{\xi=0} = \tilde{u}_0, \end{cases}$$

where P_u is the projector from condition U4, and the parameter $a \in H$ will be chosen later.

For $J(u(\xi))$ and $\mathfrak{D}(u(\xi))$, using (29), we obtain:

$$\begin{aligned} J_\xi(u(\xi)) &= 2 J(u) \left[\langle (2k+1) \|u\|^{4k} u - \tilde{D}_u^* \tilde{f}(u), u_\xi \rangle \right] = \\ &= 2 J(u) \langle -N(u), P_u a \rangle; \\ \mathfrak{D}_\xi(u(\xi)) &= 2 \mathfrak{D}(u) \left[\langle \exp \{ \|u\|^2 \} u - \tilde{D}_u^* \tilde{f}(u), u_\xi \rangle \right] = \\ &= 2 \mathfrak{D}(u) \langle -N(u), P_u a \rangle. \end{aligned} \tag{39}$$

From (39), taking into account that $N(\tilde{u}_0) = 0$, we obtain:

$$\begin{aligned} J_\xi(u(\xi)) &= 2 J(u) \left\langle - \int_0^\xi \left(\tilde{D}_u^* \tilde{f}(u) - \gamma(u) u \right)_\eta d\eta, P_{u(\xi)} a \right\rangle = \\ &= 2 J(u) \left\langle - \int_0^\xi \left(\tilde{D}_u^* \tilde{f}(u) - \gamma(u) u \right)_\eta d\eta, P_{\tilde{u}_0} a + \int_0^\xi (P_{u(\xi)} a)_\eta d\eta \right\rangle = \\ &= 2 J(u) \left[-\xi \left\langle \left(\tilde{D}_{u(\eta)}^* \tilde{f}(u(\eta)) - \gamma(u(\eta)) u_\eta \right) \Big|_{\eta=0}, P_{\tilde{u}_0} a \right\rangle + \xi^2 O(1) \right] = \\ &= 2 J(u) \left[-\xi \left\langle \tilde{M}_{\tilde{u}_0} P_u a, P_u a \right\rangle + \xi \gamma(\tilde{u}_0) \|P_{\tilde{u}_0} a\|^2 + \xi^2 O(1) \right] = \\ &= 2 J(u(\xi)) \left[\left(\gamma(\tilde{u}_0) - \frac{\langle \tilde{M}_{\tilde{u}_0} P_{\tilde{u}_0} a, P_{\tilde{u}_0} a \rangle}{\|P_{\tilde{u}_0} a\|^2} \right) \cdot \|P_{\tilde{u}_0} a\|^2 \xi + \xi^2 O(1) \right]; \\ \mathfrak{D}_\xi(u(\xi)) &= 2 \mathfrak{D}(u) \left[\left(\gamma(\tilde{u}_0) - \frac{\langle \tilde{M}_{\tilde{u}_0} P_{\tilde{u}_0} a, P_{\tilde{u}_0} a \rangle}{\|P_{\tilde{u}_0} a\|^2} \right) \cdot \|P_{\tilde{u}_0} a\|^2 \xi + \xi^2 O(1) \right]. \end{aligned} \tag{40}$$

Under condition $b)$ we have

$$\gamma(\tilde{u}_0) - \frac{\langle \tilde{M}_{\tilde{u}_0} P_{\tilde{u}_0} a, P_{\tilde{u}_0} a \rangle}{\|P_{\tilde{u}_0} a\|^2} \geq (2k+1) \|\tilde{u}_0\|^{4k} - \frac{\langle \tilde{M}_{\tilde{u}_0} P_{\tilde{u}_0} a, P_{\tilde{u}_0} a \rangle}{\|P_{\tilde{u}_0} a\|^2}.$$

If at least one of the inequalities $a')$, or $b')$ holds:

$$\begin{aligned} a') \quad & \|\tilde{u}_0\| \leq \tilde{C}, \text{ where } \tilde{C} \geq 10; \\ b') \quad & \|\tilde{u}_0\|^{4k+2} < \|T\tilde{u}_0\|^{4k+2+2\varepsilon}, \end{aligned} \tag{41}$$

then:

in the case where inequality $a')$ in (41) holds, it follows that Theorem 2 is proved;

in the case where inequality $b')$ in (41) holds, we have:

$$\begin{aligned} \|\tilde{u}_0\|^{4k+2} - \|\tilde{f}(\tilde{u}_0)\|^2 &= \|\tilde{u}_0\|^{4k+2} - \|f(\tilde{u}_0) + \|T\tilde{u}_0\|^{2k+\varepsilon} T\tilde{u}_0\|^2 = \\ &= \|\tilde{u}_0\|^{4k+2} - \|f(\tilde{u}_0)\|^2 - 2\|T\tilde{u}_0\|^{2k+\varepsilon} \langle f(\tilde{u}_0), T\tilde{u}_0 \rangle - \|T\tilde{u}_0\|^{4k+2+2\varepsilon} < 0. \end{aligned}$$

In deriving this inequality, condition U3 was used (see (12)).

From this, by the definition of $J(\cdot)$ we obtain $J(\tilde{u}_0) < 1$.

Therefore, by Lemma 4 (see (25)) we have

$$J(u_0) \leq J(\tilde{u}_0) < 1, \quad (42)$$

which holds if $b')$ in (41) is satisfied.

From (42), repeating the argument used to derive estimate (38) from (36), we again obtain inequality (38).

Thus, in case $b)$ of (35), if at least one of the inequalities $a')$, or $b')$ in (41) holds, then Theorem 2 is proved.

If neither $a')$ nor $b')$ in (41) holds, then the following inequalities are satisfied

$$\|\tilde{u}_0\|^{4k+2} \geq \|T\tilde{u}_0\|^{4k+2+2\varepsilon}, \quad \|\tilde{u}_0\|^2 > \tilde{C} \geq 10. \quad (43)$$

From (43) we can apply condition U4, and taking into account computations (40) we obtain

$$\begin{cases} J_\xi(u(\xi)) = 2J(\tilde{u}_0) \left(\gamma(\tilde{u}_0) - \theta(\tilde{u}_0) \right) \cdot \|P_{\tilde{u}_0} a\|^2 \xi + \xi^2 O(1), \\ \mathfrak{D}_\xi(u(\xi)) = 2\mathfrak{D}(\tilde{u}_0) \left(\gamma(\tilde{u}_0) - \theta(\tilde{u}_0) \right) \cdot \|P_{\tilde{u}_0} a\|^2 \xi + \xi^2 O(1), \end{cases} \quad (44)$$

where

$$\theta(\tilde{u}_0) = \inf_{\|P_{\tilde{u}_0} a\|=1} \langle \tilde{M}_{\tilde{u}_0} P_{\tilde{u}_0} a, P_{\tilde{u}_0} a \rangle.$$

Since, by condition $b)$ of (35)

$$\gamma(\tilde{u}_0) \geq (2k+1) \|\tilde{u}_0\|^{4k},$$

it follows from (44) taking into account condition U4, that the strict inequalities

$$J_\xi(u(\xi)) > 0, \quad \mathfrak{D}_\xi(u(\xi)) > 0 \quad (45)$$

hold for $\xi \in (0; \xi_0]$, ξ_0 is a sufficiently small number.

From (45) it follows that there exists $\xi_0 > 0$ such that for all $\xi \in (0; \xi_0]$ the strict inequalities hold

$$J(u(\xi)) > J(\tilde{u}_0), \quad \mathfrak{D}(u(\xi)) > \mathfrak{D}(\tilde{u}_0). \quad (46)$$

The inequalities (46) contradict the definition of the vector $\tilde{u}_0$, realizing the *supremum* in (24)–(25).

Therefore, in the case when condition $b)$ of (35) holds, either $a')$ or $b')$ in (41) must hold, and estimate (38) is valid, while case (43) does not occur.

Consequently, Theorem 2 is proved also in case $b)$.

Theorem 2 is proved.

Conclusion

The paper investigates a class of nonlinear operator equations in a finite-dimensional Hilbert space. The main focus is on obtaining a priori estimates for solutions of the considered equations and analyzing their properties. It is shown that for a sufficiently broad class of nonlinear operators, one can establish uniform estimates of solutions that depend only on the parameters of the original problem and do not depend on the approximation index.

The obtained estimates make it possible to apply the method of finite-dimensional approximations to the study of corresponding problems in infinite-dimensional spaces. The availability of a priori estimates ensures the possibility of passing to the limit and obtaining a priori estimates for solutions of the original problems of mathematical physics [17, 18]. This, in turn, is an important tool in proving the existence of solutions to nonlinear operator equations.

The proposed approach can be used to analyze a wider class of nonlinear problems arising in functional analysis and mathematical physics [3, 4]. The obtained results may serve as a basis for further research related to the study of the stability of solutions, their qualitative properties, and the construction of effective approximate methods for solving nonlinear operator equations.

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Author contributions

All authors contributed equally to this work. All authors participated in the conduct of the study and in the preparation of the final version of the manuscript.

Conflict of interest

The authors declare no conflict of interest.

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Asymptotics of solutions of a reaction-diffusion system with nonlinear boundary conditions

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This paper investigates the asymptotic properties of a system of parabolic equations with nonlinear boundary conditions. Systems of parabolic equations describe the behavior of various processes over time, such as heat propagation and material diffusion. Nonlinear boundary conditions involve complex properties that differ from the usual local conditions at the boundaries of the system, which influence the solutions of the system. In this article, a system of parabolic differential equations associated with nonlinear boundary conditions is reduced to a system of self-similar equations by means of certain transformations using the self-similar approach, and the asymptotic behavior of solutions with compact support is found. In addition, the reaction-diffusion system is approximated using implicit difference schemes and an iterative process for the numerical solution is constructed. It is known that the main problem for the iterative process is the choice of the best initial approximation. The article proposes to use the constructed asymptotic formula to solve this problem. Computational experiments were carried out with different values of numerical parameters and the results are presented graphically. The numerical results obtained in the case of slow diffusion $p > 2$ of a nonlinear reaction-diffusion system show that the reaction-diffusion process proceeds at a finite rate.

Keywords: system of nonlinear equations, reaction-diffusion, global solution, asymptotic, compact support, self-similar solution, sweep method, numerical solution.

2020 Mathematics Subject Classification: 35K55, 35K57, 35K65.

Introduction

In this paper, we investigate the asymptotics of self-similar solutions for the following system of perturbed parabolic-type equations in the domain $Q = \{(x, t) \mid x \in \mathbb{R}_+, 0 < t < T\}$

$$\partial_t u_i = \left(|\partial_x u_i|^{p_i-2} \partial_x u_i \right)_x, \quad i = 1, 2, \dots, k, \quad x > 0, \quad 0 < t < T, \quad (1)$$

with nonlinear boundary conditions

$$- |\partial_x u_i|^{p_i-2} \partial_x u_i(0, t) = u_{i+1}^{q_i}(0, t), \quad i = 1, 2, \dots, k, \quad u_{k+1} = u_1, \quad 0 < t < T, \quad (2)$$

and initial conditions

$$u_i(x, 0) = u_{i0}(x), \quad i = 1, 2, \dots, k, \quad x > 0. \quad (3)$$

The problem (1)–(3) is considered under the conditions $p_i > 2$, $q_i > 0$, where p_i is associated with the governing equation (1), and q_i arises in the nonlinear boundary condition (2). The initial functions $u_{i0}(x)$ ($i = 1, 2, \dots, k$) are nonnegative continuous functions with compact support in $\mathbb{R}_+$.

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The system of nonlinear parabolic equations (1) is widely used in the mathematical modeling of processes such as turbulent diffusion, population dynamics, chemical reactions, and heat transfer [1, 2]. Furthermore, system (1)–(3) describes many other physical processes [3–5].

Throughout the paper, we write u_i where the index i is assumed to range over $\{1, \dots, k\}$ unless otherwise specified. It is known that in the region where $u_i = 0$, the system of equations (1) is degenerate and does not have a solution in the classical sense. In such cases, we consider generalized solutions that satisfy the property $|\partial_x u_i|^{p_i-2} \partial_x u_i \in C(Q)$, $0 \leq u_i \in C(Q)$ and these solutions satisfy the system of equations (1) in the sense of an integral identity [1, 2, 6].

Unbounded (blow-up) and global solutions of nonlinear parabolic systems of equations, as well as the conditions for their existence, have been widely studied in [7–9].

In [10], the critical global existence curve was established for fast diffusive polytropic filtration equations with nonlinear boundary fluxes, and a Fujita-type critical curve was conjectured.

S.N. Zheng, X.F. Song, and Z.X. Jiang [11] considered the following problem with a nonlinear boundary conditions:

$$\begin{aligned} \partial_t u &= \partial_x (\partial_x u^m), \quad \partial_t \nu = \partial_x (\partial_x \nu^n), \quad x > 0, \quad 0 < t < T, \\ -\partial_x u^m(0, t) &= u^\alpha(0, t) \nu^\beta(0, t), \quad -\partial_x \nu^n(0, t) = u^\theta(0, t) \nu^\phi(0, t), \quad 0 < t < T, \\ u(x, 0) &= u_0(x), \quad \nu(x, 0) = \nu_0(x), \quad x > 0. \end{aligned} \quad (4)$$

It was proved that the solutions of problem (4) are global if the conditions $\alpha < \frac{1}{2}(n+1)$, $\beta < \frac{1}{2}(n+1)$, $pq \leq (\frac{1}{2}(m+1) - \alpha)(\frac{1}{2}(n+1) - \beta)$ are satisfied.

If, however $\alpha > \frac{1}{2}(m+1)$ or $\beta > \frac{1}{2}(n+1)$, then the solutions become unlimited (blow up) in finite time.

In the case when $\alpha > \frac{1}{2}(m+1)$, $\beta > \frac{1}{2}(n+1)$, $pq > (\frac{1}{2}(m+1) - \alpha)(\frac{1}{2}(n+1) - \beta)$ if at least one of the conditions $l_1 < k_1$, $l_2 < k_2$ or $l_1 = k_1$, $l_2 = k_2$ is being carried out, then any nonnegative nontrivial solution blows up in finite time.

On the other hand, $l_1 < k_1$, $l_1 > k_1$ it has been shown that for sufficiently large initial data the solutions blow up, whereas for small initial data the solutions remain global.

$$\begin{aligned} k_1 &= \frac{2p + n + 1 - 2\beta}{4pq - (n + 1 - 2\alpha)(m + 1 - 2\beta)}, \quad l_1 = \frac{1 - k_1(m - 1)}{2}, \\ k_2 &= \frac{2p + m + 1 - 2\beta}{4pq - (n + 1 - 2\alpha)(m + 1 - 2\beta)}, \quad l_2 = \frac{1 - k_2(n - 1)}{2}. \end{aligned}$$

Z.Y. Xiang, Q. Chen, and C.L. Mu [12] investigated the following system:

$$\begin{cases} \partial_t u_i = \partial_x (|\partial_x u_i|^{m_i-1} \partial_x u_i), & (i = 1, 2, \dots, k), \quad x > 0, \quad 0 < t < T, \\ -|\partial_x u_i|^{m_i-1} \partial_x u_i(0, t) = u_{i+1}^p(0, t), & (i = 1, 2, \dots, k), \quad u_{k+1} = u_1, \quad 0 < t < T, \\ u_i(x, 0) = u_{i0}(x), & (i = 1, 2, \dots, k), \quad x > 0. \end{cases} \quad (5)$$

Here the parameters are specified as follows: $k \geq 2$, $m > 1$, $p_i > 0$, u_{i0} ($i = 1, 2, \dots, k$) are continuous and nonnegative functions. The authors obtained the conditions for the global existence of solutions and the values of the Fujita-type critical exponent. In [13], the properties of solutions to problem (5) in the case of fast diffusion were studied.

Jiang Zheng [14] investigated the following equation:

$$\begin{cases} \partial_t u = \left(|\partial_x u|^\beta \partial_x u^m \right)_x, & x > 0, \quad 0 < t < T, \\ -|\partial_x u|^\beta \partial_x u^m(0, t) = u^p(0, t), & 0 < t < T, \\ u(x, 0) = u_0(x), & x > 0. \end{cases}$$

Here $m \geq 1$, $p > 0$, $\beta > 0$. They showed that $p_0 = \frac{2\beta+m+1}{\beta+2}$ is the critical exponent for global existence, and when $p_c = 2\beta + m + 1$ is the Fujita critical exponent. These results represent a generalization of the works of Galaktionov and Levine [15].

In [16–18], the main focus is on determining the critical conditions for the solutions of systems of degenerate parabolic equations, with particular emphasis on deriving theoretical results that characterize the threshold between global existence and blow-up phenomena, especially in the presence of nonlinear boundary fluxes.

In [19, 20], conditions for the existence of global-in-time and blow-up solutions were established for a model of heat and moisture transfer in a two-component medium, described by a cross-diffusion system with nonlinear boundary conditions.

In addition, in [21], the properties of solutions to the following diffusion problem with a source were investigated:

$$\frac{\partial u_i}{\partial t} = \frac{\partial}{\partial x} \left(\left| \frac{\partial u_i^k}{\partial x} \right|^{m-1} \frac{\partial u_i^k}{\partial x} \right) + u_i^{p_i}, \quad x \in \mathbb{R}_+, \quad t > 0, \quad i = 1, 2, \quad (6)$$

$$\begin{aligned} - \left| \frac{\partial u_i^k}{\partial x} \right|^{m-1} \frac{\partial u_i^k}{\partial x} \Big|_{x=0} &= u_{3-i}^{q_i}(0, t), \quad t > 0, \quad i = 1, 2, \\ u_i|_{t=0} &= u_{i0}(x), \quad i = 1, 2. \end{aligned} \quad (7)$$

Here $m > 1$, $k \geq 1$ and $q_i, p_i > 0$ are given numerical parameters, while $u_{i0}(x)$, $i = 1, 2$ are functions defined on $\mathbb{R}_+$, nonnegative and continuous.

The authors proved that if the inequalities $q_1 q_2 > \left(\frac{m(k+1)}{m+1} \right)^2$, $\frac{1}{m} < p_i \leq 1$, ($i = 1, 2$) hold and if $\xi_i \rightarrow \left(\frac{d_i}{D_i} \right)^{\frac{m}{m+1}} = \xi_{i0}$, ($i = 1, 2$) then the solutions of problem (6)–(7) with compact support possess the following asymptotics:

$$u_i(x, t) = (T + t)^{\alpha_i} \bar{\varphi}_i(\xi_i) (1 + o(1)), \quad (i = 1, 2),$$

here

$$\alpha_i = \frac{1}{1 - p_i}, \quad \beta_i = \frac{p_i - km}{(p_i - 1)(m + 1)}, \quad \xi_i = x(T + t)^{-\beta_i},$$

$$\bar{\varphi}_i(\xi_i) = \left(d_i - D_i \xi_i^{\frac{m+1}{m}} \right)^{\frac{m}{mk-1}}, \quad d_i = \text{const} \geq 0, \quad D_i = \frac{\beta_i^{\frac{1}{m}} (mk - 1)}{k(m + 1)}.$$

Charkaoui A. and El Guermai G. [22] studied the following system and its particular cases:

$$\begin{cases} \partial_t u_i - d_i \Delta u_i = f_i(u_1, \dots, u_m) & \text{on } (0, T) \times \Omega, \\ \alpha_i \frac{\partial u_i}{\partial n} + (1 - \alpha_i) u_i = \beta_i & \text{on } (0, T) \times \partial\Omega, \\ u_i(0, \cdot) = u_{i0}, \end{cases} \quad (8)$$

where the $f_i : \mathbb{R}^m \rightarrow \mathbb{R}$ are C^1 functions of $u = (u_1, \dots, u_m)$, $d_i \in (0, \infty)$, $\alpha_i \in [0, 1]$, $0 \leq \beta_i \in C^2([0, T] \times \bar{\Omega})$, $T > 0$, $i = 1, \dots, m$, $\Omega \subset \mathbb{R}^N$.

The authors proved the existence of a global classical solution for the reaction-diffusion system (8) in the case $\alpha_i = 0$, under the assumptions of quasipositivity and at most polynomial growth for the function f . They utilized a lower triangular invertible matrix P with nonnegative entries that satisfies a specific linear growth constraint, along with L^∞ initial data.

They have extended upon the results of Pierre M. and Schmitt D. [23], showing that a blow-up phenomenon can occur under some conditions in the case $\alpha_i = 0$, $m = 2$. To support their findings, they used the following forms of solutions:

$$u(t, x) = \frac{a_1(T - t) + b_1|x|^2}{(T - t + |x|^2)^\gamma}, \quad v(t, x) = \frac{a_2(T - t) + b_2|x|^2}{(T - t + |x|^2)^\gamma}, \quad a_j, b_j > 0, \gamma > 1, j = 1, 2.$$

The authors further demonstrated the existence of global weak supersolutions for bounded L^1 -nonlinearities by reformulating the original problem (8) as an approximate problem. Additionally, they reviewed relevant literature and highlighted open problems related to specific cases of the system (8).

In [24], the authors reviewed the global existence of solutions to reaction-diffusion systems with positivity preservation and uniformly bounded mass. The paper summarizes existing theoretical results, highlights unresolved problems, and discusses conditions under which global solutions exist.

In [25], Charkaoui et al. propose a nonlinear parabolic system with variable growth conditions for image denoising and restoration. The authors establish the existence and uniqueness of weak solutions within the framework of Musielak–Orlicz spaces using the Faedo–Galerkin method. Numerical experiments on grayscale and MRI images demonstrate the efficiency and robustness of the proposed approach in noise removal compared with several existing methods.

The work in [26] investigates the asymptotic behavior of stochastic Allen–Cahn-type reaction-diffusion equations with time white noise. The authors show that the long-term dynamics are governed by curvature-driven interface motion under stochastic perturbations. Using the theory of stochastic solutions and generalized front propagation, they establish global-in-time results without imposing regularity assumptions on the moving interface.

In [27, 28], recent studies have focused on the qualitative properties of nonlinear parabolic and reaction-diffusion systems, including global existence, blow-up phenomena, periodic weak solutions, and problems with nonlinear boundary conditions. In particular, the existence and uniqueness of weak periodic solutions for quasilinear parabolic systems have been established using topological and variational approaches, while several works investigated global solvability together with mass control and dissipation effects in reaction-diffusion models. These studies provide important theoretical foundations for the analysis of nonlinear evolution equations and are methodologically related to the problem considered in the present work.

In the p -Laplacian model, the case $p_i > 2$ corresponds to the degenerate (slow) diffusion regime, whereas the range $1 < p_i < 2$ leads to singular (fast) diffusion. In this work, we consider the case $p_i > 2$, which is consistent with the physical and analytical framework of the problem under consideration.

The main contributions of this paper are summarized as follows. First, we investigate a system of p -Laplacian type equations coupled through nonlinear boundary conditions, generalizing existing models in the literature to a multi-component system of k equations. Second, we focus on the degenerate diffusion regime ($p_i > 2$) and establish novel asymptotic properties of the corresponding self-similar solutions. Finally, we develop a comprehensive analytical framework complemented by numerical experiments that characterize the qualitative behavior of the solutions.

1 Research methodology

In this work, we construct a self-similar solution to problem (1)–(3) and derive the asymptotics of the self-similar solution. We seek the self-similar solution in the following form:

$$\bar{u}_i(x, t) = (T + t)^{-\alpha_i} \varphi_i(\xi_i), \quad \xi_i = x(T + t)^{-\lambda_i}, \quad (9)$$

here $T > 0$, $\lambda_i = \frac{1-\alpha_i(p_i-2)}{p_i}$, $\alpha_{i+1} = \frac{(p_i-1)(2\alpha_i+1)}{p_i q_i}$, ($i = 1, 2, \dots, k$).

Substitution of (9) into (1)–(2) leads to the following system:

$$\begin{cases} \left(|\varphi'_i|^{p_i-2} \varphi'_i \right)' + \lambda_i \xi_i \varphi'_i + \alpha_i \varphi_i = 0, \\ -|\varphi'_i|^{p_i-2} \varphi'_i|_{\xi_i=0} = \varphi_{i+1}^{q_i}(0), \quad t > 0, \quad \varphi_{k+1} = \varphi_1, \quad i = 1, 2, \dots, k. \end{cases} \quad (10)$$

Let us consider the following function:

$$\overline{\varphi}_i = \left(a_i - b_i \xi_i^{\frac{p_i-1}{p_i-2}} \right)_+^{\frac{p_i-1}{p_i-2}},$$

here $a_i > 0$, $b_i = \frac{p_i-2}{p_i} \lambda_i^{\frac{1}{p_i-1}} > 0$, $(y)_+ = \max(0, y)$.

Theorem 1. For the solution of problem (10) with compact support, when $\xi_i \rightarrow \left(\frac{a_i}{b_i} \right)^{\frac{p_i-1}{p_i}}$ the following asymptotics holds:

$$\varphi_i(\xi_i) = \overline{\varphi}_i(1 + o(1)). \quad (11)$$

Proof. We seek the solution of system (10) in the following form:

$$\varphi_i(\xi_i) = \overline{\varphi}_i(\xi_i) w_i(\tau_i), \quad (12)$$

where $\tau_i = -\ln \left(a_i - b_i \xi_i^{\frac{p_i-1}{p_i-2}} \right)$ and the function $w_i(\tau_i)$ is nonnegative and bounded.

We first perform the basic calculations required for the transformation of arguments:

$$\begin{aligned} \frac{d\tau_i}{d\xi_i} &= \frac{p_i}{p_i-1} b_i^{\frac{p_i-1}{p_i}} e^{\tau_i} (a_i - e^{-\tau_i})^{1/p_i}, \\ \frac{d\varphi_i}{d\xi_i} &= \frac{d}{d\tau_i} \left(e^{-\frac{p_i-1}{p_i-2}\tau_i} w_i \right) \frac{d\tau_i}{d\xi_i} = \frac{p_i}{p_i-1} b_i^{\frac{p_i-1}{p_i}} e^{-\frac{1}{p_i-2}\tau_i} (a_i - e^{-\tau_i})^{1/p_i} \left(\frac{dw_i}{d\tau_i} - \frac{p_i-1}{p_i-2} w_i \right). \end{aligned} \quad (13)$$

Applying the relations (12)–(13) to (10), we obtain the following expression:

$$\frac{d}{d\tau_i} (L_i w_i) + a_{i1} L_i w_i + a_{i2} w_i^* + a_{3i} w_i = 0, \quad (14)$$

where $a_{i1} = \frac{p_i-1}{p_i} a_{i0} - \frac{p_i-1}{p_i-2}$, $a_{i0} = \frac{e^{-\tau_i}}{a_i - e^{-\tau_i}}$, $a_{i2} = \frac{\lambda_i p_i}{p_i-1} b_i^*$, $b_i^* = \left| \frac{p_i-1}{p_i} b_i^{\frac{1-p_i}{p_i}} \right|^{p_i}$, $a_{3i} = \alpha_i a_{i0} b_i^*$, $w_i^* = \frac{dw_i}{d\tau_i} - \frac{p_i-1}{p_i-2} w_i$, $L_i w_i = |w_i^*|^{p_i-2} w_i^*$.

The study of the solutions of the last equation is equivalent to the study of the solutions of equation (1), and each of them, on the interval $[\tau_{i0}, +\infty)$, satisfies the following inequalities [29]:

$$w_i > 0, \quad w_i^* \neq 0.$$

Let us verify that the solution $w_i(\tau_i)$ of equation (14) either has a finite limit w_{i0} or becomes unbounded as $\tau_i \rightarrow +\infty$.

Suppose that $\nu_i(\tau_i) = L_i w_i$. Then, for the derivative of the function $\nu_i(\tau_i)$ we get the following:

$$\nu'_i = -(a_{i1} \nu_i + a_{i2} w_i^* + a_{3i} w_i).$$

To analyze the solutions of the last equation, we introduce an auxiliary function

$$\theta_i(\tau_i, \mu_i) = -(a_{i1} \mu_i + a_{i2} w_i^* + a_{3i} w_i),$$

where μ_i is a real number. From this, it is easy to see that for each value of μ_i , the function $\theta_i(\tau_i, \mu_i)$ preserves its sign on some interval $[\tau_{i1}, +\infty) \subset [\tau_{i0}, +\infty)$, and for all $\tau_i \in [\tau_{i1}, +\infty)$, one of the inequalities holds:

$$\nu'_i(\tau_i) > 0, \quad \nu'_i(\tau_i) < 0.$$

Then, analyzing (14) with account of Bolzano's theorem, it follows that the function $\nu_i(\tau_i)$ has a limit as $\tau \rightarrow +\infty$. Now let us perform a limiting transition. Taking into account the last limit and the fact that $w'_i = 0$ from (14), for w_i we obtain the following results:

$$\lim_{\tau_i \rightarrow \infty} a_{i0} = \lim_{\tau_i \rightarrow \infty} a_{i3} = 0, \quad a_{i1}^0 = \lim_{\tau_i \rightarrow \infty} a_{i1} = -\frac{p_i - 1}{p_i - 2}, \quad a_{i2}^0 = \lim_{\tau_i \rightarrow \infty} a_{i2} = \frac{\lambda_i p_i}{p_i - 1} b_i^*,$$

$$w_i^* = \lim_{\tau_i \rightarrow \infty} w_i^* = -\frac{p_i - 1}{p_i - 2} w_i^0, \quad \mu_i^0 = \lim_{\tau_i \rightarrow \infty} L_i w_i = -\left| \frac{p_i - 1}{p_i - 2} \right|^{p_i - 1} (w_i^0)^{p_i - 1}, \quad w_i^0 = \lim_{\tau_i \rightarrow \infty} w_i.$$

Consequently, we obtain the following result:

$$a_{i1}^0 \mu_i^0 + a_{i2}^0 w_i^* = 0,$$

which is equivalent to this

$$w_i^0 \left[(w_i^0)^{p_i - 2} - \lambda_i \left| \frac{p_i - 2}{b_i p_i} \right|^{p_i - 1} \right] = 0,$$

whose solution is $w_i^0 = \left[\lambda_i \left| \frac{p_i - 2}{b_i p_i} \right|^{p_i - 1} \right]^{\frac{1}{p_i - 2}} = 1$, and by virtue of (12) $\varphi_i(\xi_i) = \bar{\varphi}(\xi_i)(1 + o(1))$.

□

2 Computational experiment and visualization results

In this section, we construct a numerical scheme to solve (1)–(3) for the case $k = 2$. In the domain $\Omega = \{(x, t) : 0 \leq x \leq b, 0 \leq t < T\}$, we discretize time and space into a grid as follows: $\tau = \frac{T}{n}$, $h = \frac{b}{n}$. For the numerical solution of problem (1)–(3), we use an implicit scheme:

$$V_\tau = \{t_j = j\tau, \tau > 0, j = 1, 2, \dots, m; m\tau = T, T > 0\}; S_h = \{x_\iota = \iota h; h > 0; \iota = 1, 2, \dots, n; nh = b\}$$

$$\begin{cases} \frac{y_\iota^{j+1} - y_\iota^j}{\tau} = \frac{1}{h^2} \left[D_{\iota+1}^j(y_{\iota+1}^j) (y_{\iota+1}^{j+1} - y_\iota^{j+1}) - D_\iota^j(y_\iota^j) (y_\iota^{j+1} - y_{\iota-1}^{j+1}) \right], \\ \frac{z_\iota^{j+1} - z_\iota^j}{\tau} = \frac{1}{h^2} \left[M_{\iota+1}^j(z_{\iota+1}^j) (z_{\iota+1}^{j+1} - z_\iota^{j+1}) - M_\iota^j(z_\iota^j) (z_\iota^{j+1} - z_{\iota-1}^{j+1}) \right], \\ \iota = 2, \dots, n-1; \quad j = 0, 1, \dots, m-1, \end{cases} \quad (15)$$

$$y_\iota^0 = y(x_\iota, 0), \quad z_\iota^0 = z(x_\iota, 0), \quad \iota = 0, 1, \dots, n,$$

$$\begin{cases} -D_1(y_1) \cdot \frac{y_1^{j+1} - y_0^{j+1}}{h} = (z_0^j)^{q_1}, \\ -M_1(z_1) \cdot \frac{z_1^{j+1} - z_0^{j+1}}{h} = (y_0^j)^{q_2}, \end{cases}$$

$$y_n^j = \varphi_1(t_j), \quad z_n^j = \varphi_2(t_j), \quad j = 2, 3, \dots, m.$$

a)

$$\begin{cases} D_t^j(y_t^j) = D_t^j\left(\frac{y_t^j + y_{t-1}^j}{2}\right), \\ M_t^j(z_t^j) = M_t^j\left(\frac{z_t^j + z_{t-1}^j}{2}\right), \end{cases} \quad (16)$$

b)

$$\begin{cases} D_t^j(y_t^j) = \frac{D_t^j(y_t^j) + D_{t-1}^j(y_{t-1}^j)}{2}, \\ M_t^j(z_t^j) = \frac{M_t^j(z_t^j) + M_{t-1}^j(z_{t-1}^j)}{2}. \end{cases} \quad (17)$$

Here, (16)–(17) are written using the Samarskii-Sobol scheme. If we take according to formula (16), then

$$\begin{cases} D_t^j(y_t^j) = \left| \frac{y_t^j - y_{t-1}^j}{h} \right|^{p_1-2}, \\ M_t^j(z_t^j) = \left| \frac{z_t^j - z_{t-1}^j}{h} \right|^{p_2-2}, \end{cases}$$

if we take according to formula (17), then

$$\begin{cases} D_t^j(y_t^j) = \frac{|y_t^j - y_{t-1}^j|^{p_1-2} + |y_{t-1}^j - y_{t-2}^j|^{p_1-2}}{2h^{p_1-2}}, \\ M_t^j(z_t^j) = \frac{|z_t^j - z_{t-1}^j|^{p_2-2} + |z_{t-1}^j - z_{t-2}^j|^{p_2-2}}{2h^{p_2-2}}, \end{cases}$$

system (15) is a system of nonlinear equations with respect to y^{j+1} and z^{j+1} . The system (18) is solved numerically by freezing the nonlinear coefficients at the s -th iteration to linearize the equations for the $(s+1)$ -th step.

$$\begin{cases} \frac{y_{t+1}^{s+1,j+1} - y_t^{s,j}}{\tau} = \frac{1}{h^2} \left[D_{t+1}^j(y_{t+1}^j) \left(y_{t+1}^{s+1,j+1} - y_t^{s+1,j+1} \right) - D_t^j(y_t^j) \left(y_t^{s+1,j+1} - y_{t-1}^{s+1,j+1} \right) \right], \\ \frac{z_{t+1}^{s+1,j+1} - z_t^{s,j}}{\tau} = \frac{1}{h^2} \left[M_{t+1}^j(z_{t+1}^j) \left(z_{t+1}^{s+1,j+1} - z_t^{s+1,j+1} \right) - M_t^j(z_t^j) \left(z_t^{s+1,j+1} - z_{t-1}^{s+1,j+1} \right) \right], \end{cases} \quad (18)$$

here, $s = 0, 1, 2, \dots$

The choice of the initial approximation determines the efficiency of the iterative method, since it ensures rapid convergence to the solution and preserves the physical meaning of the problem. As such an initial approximation, it is reasonable to use the asymptotic solutions of the problem [30]. At each time step, the initial values of the iterations $y_{t,0}^{s+1}$, $z_{t,0}^{s+1}$ are taken from the previous time step: $y_{t,0}^{j+1} = y_t^j$, $z_{t,0}^{j+1} = z_t^j$. In computations using the iterative scheme, the accuracy of the iterations is determined, and at the same time

$$\begin{cases} \max_{0 \leq \iota \leq n} \left| \bar{y}_{\iota}^{s+1} - \bar{y}_{\iota}^s \right| < \varepsilon, \\ \max_{0 \leq \iota \leq n} \left| \bar{z}_{\iota}^{s+1} - \bar{z}_{\iota}^s \right| < \varepsilon. \end{cases} \quad (19)$$

We introduce the following notation: $\bar{y}_{\iota} = y_{\iota}^{j+1}$, $\bar{z}_{\iota} = z_{\iota}^{j+1}$. Then the system (18) can be written in the following form:

$$\begin{cases} \bar{A}_{1\iota}^s \bar{y}_{\iota-1}^{s+1} - \bar{C}_{1\iota}^s \bar{y}_{\iota}^{s+1} + \bar{B}_{1\iota}^s \bar{y}_{\iota+1}^{s+1} = -\bar{F}_{1\iota}^s, \\ \bar{A}_{2\iota}^s \bar{z}_{\iota-1}^{s+1} - \bar{C}_{2\iota}^s \bar{z}_{\iota}^{s+1} + \bar{B}_{2\iota}^s \bar{z}_{\iota+1}^{s+1} = -\bar{F}_{2\iota}^s, \end{cases} \quad (20)$$

here

$$\bar{A}_{1\iota}^s = \frac{\tau}{h^2} D_{\iota}^j(y_{\iota}^j), \quad \bar{A}_{2\iota}^s = \frac{\tau}{h^2} M_{\iota}^j(z_{\iota}^j),$$

$$\bar{B}_{1\iota}^s = \frac{\tau}{h^2} D_{\iota+1}^j(y_{\iota+1}^j), \quad \bar{B}_{2\iota}^s = \frac{\tau}{h^2} M_{\iota+1}^j(z_{\iota+1}^j),$$

$$\bar{C}_{1\iota}^s = \frac{\tau}{h^2} \left(D_{\iota}^j(y_{\iota}^j) + D_{\iota+1}^j(y_{\iota+1}^j) \right) + 1, \quad \bar{C}_{2\iota}^s = \frac{\tau}{h^2} \left(M_{\iota}^j(z_{\iota}^j) + M_{\iota+1}^j(z_{\iota+1}^j) \right) + 1,$$

$$\bar{F}_{1\iota}^s = y_{\iota}^j, \quad \bar{F}_{2\iota}^s = z_{\iota}^j.$$

The values of the required quantities at the end of the interval $0 \leq x \leq b$ can be obtained using Milne's formula of second-order accuracy.

$$\begin{aligned} \left. \frac{\partial y}{\partial x} \right|_0 &\approx \frac{-y_2 + 4y_1 - 3y_0}{2h}, \quad \left. \frac{\partial y}{\partial x} \right|_n \approx \frac{3y_n - 4y_{n-1} + y_{n-2}}{2h}, \\ \left. \frac{\partial z}{\partial x} \right|_0 &\approx \frac{-z_2 + 4z_1 - 3z_0}{2h}, \quad \left. \frac{\partial z}{\partial x} \right|_n \approx \frac{3z_n - 4z_{n-1} + z_{n-2}}{2h}. \end{aligned} \quad (21)$$

To obtain numerical solutions of the system of linear algebraic equations (20), we use the sweep method (Thomas algorithm). According to the sweep method:

$$\begin{cases} \bar{y}_{\iota} = \alpha_{1\iota} \bar{y}_{\iota+1} + \beta_{1\iota}, \\ \bar{z}_{\iota} = \alpha_{2\iota} \bar{z}_{\iota+1} + \beta_{2\iota}. \end{cases} \quad (22)$$

Here, $\alpha_{1\iota}, \alpha_{2\iota}, \beta_{1\iota}, \beta_{2\iota}$ are coefficients that are computed using the following formulas:

$$\begin{cases} \alpha_{1\iota+1} = \frac{B_{1\iota}}{C_{1\iota} - \alpha_{1\iota} A_{1\iota}} \\ \beta_{1\iota+1} = \frac{A_{1\iota} \beta_{1\iota} + F_{1\iota}}{C_{1\iota} - \alpha_{1\iota} A_{1\iota}} \end{cases}, \quad \begin{cases} \alpha_{2\iota+1} = \frac{B_{2\iota}}{C_{2\iota} - \alpha_{2\iota} A_{2\iota}} \\ \beta_{2\iota+1} = \frac{A_{2\iota} \beta_{2\iota} + F_{2\iota}}{C_{2\iota} - \alpha_{2\iota} A_{2\iota}} \end{cases}. \quad (23)$$

Here $\iota = 1, 2, \dots, n$. The values $\alpha_{10}, \alpha_{20}, \beta_{10}, \beta_{20}$ are obtained from the boundary conditions (21).

It is known that inequality,

$$\left| C_{1l}^s \right| = \left| A_{1l}^s + B_{1l}^s + 1 \right| \geq \left| A_{1l}^s + B_{1l}^s \right|, \quad \left| C_{2l}^s \right| = \left| A_{2l}^s + B_{2l}^s + 1 \right| \geq \left| A_{2l}^s + B_{2l}^s \right|,$$

holds for system (20). Therefore, since the condition of diagonal dominance is satisfied for system (20), the applicability of the sweep method (22) and the stability of system (23) are ensured [31, Chapter 8], [32, Chapter 4], that is

$$\left| C_{1l}^s \right| \geq \left| A_{1l}^s + B_{1l}^s \right|, \quad \left| C_{2l}^s \right| \geq \left| A_{2l}^s + B_{2l}^s \right|.$$

The iterative process continues until condition (19) is satisfied. Using the numerical scheme constructed above, a computational experiment was carried out. We present some of its results. For the computations, the grid step was chosen as $h = 0.05$, the number of nodes $n = 300$ and the iteration accuracy $\varepsilon = 10^{-5}$. Formulas (9) and (11) were used as the initial approximation for the iterative process.

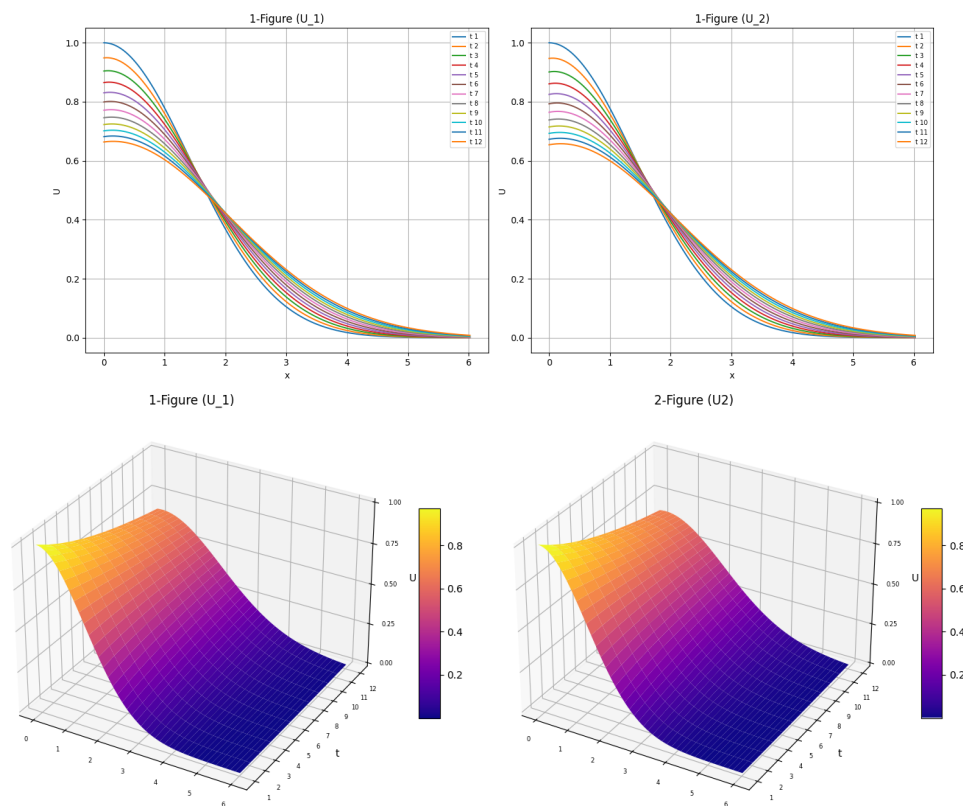


Figure 1. Numerical solution of problem (1)–(3).
 $p_1 = 2.0101$, $p_2 = 2.0105$, $q_1 = 1.85$, $q_2 = 1.95$

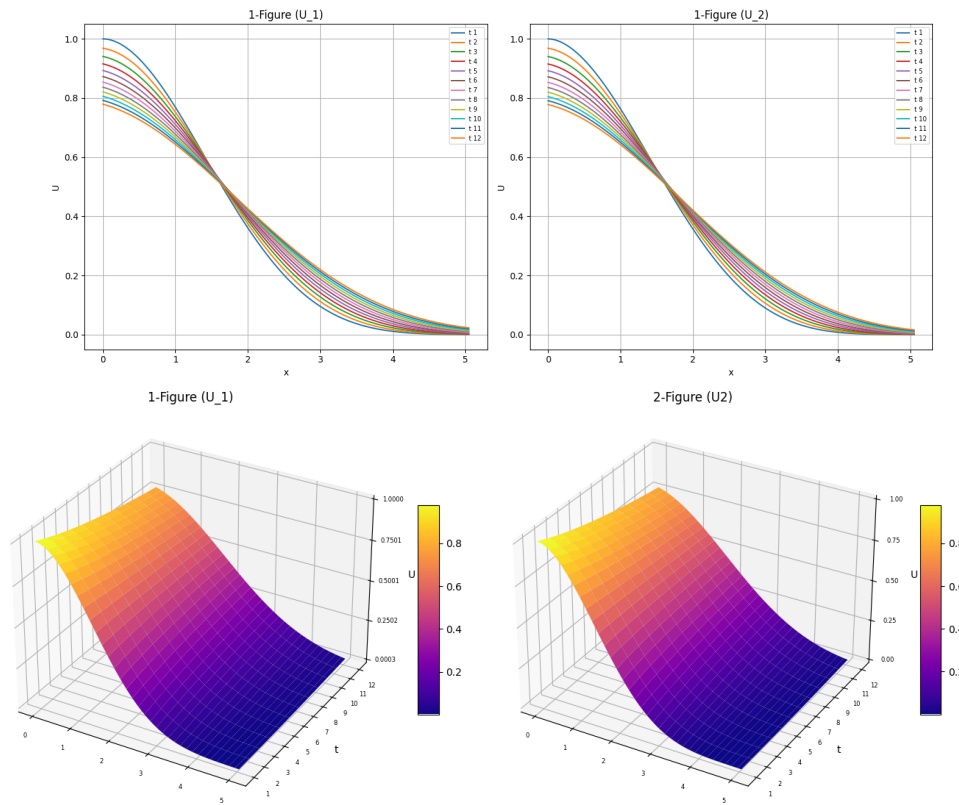


Figure 2. Numerical solution of problem (1)–(3). $p_1 = 2.11$, $p_2 = 2.18$, $q_1 = 2.6$, $q_2 = 2.7$

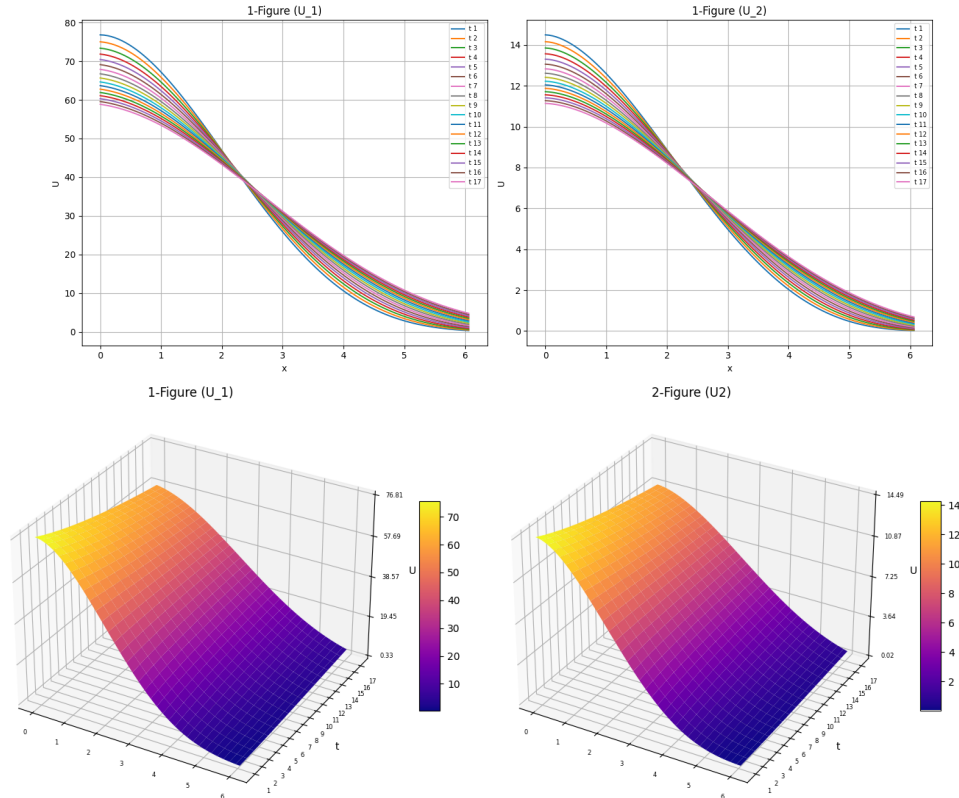


Figure 3. Numerical solution of problem (1)–(3). $p_1 = 2.19$, $p_2 = 2.35$, $q_1 = 2.6$, $q_2 = 2.7$

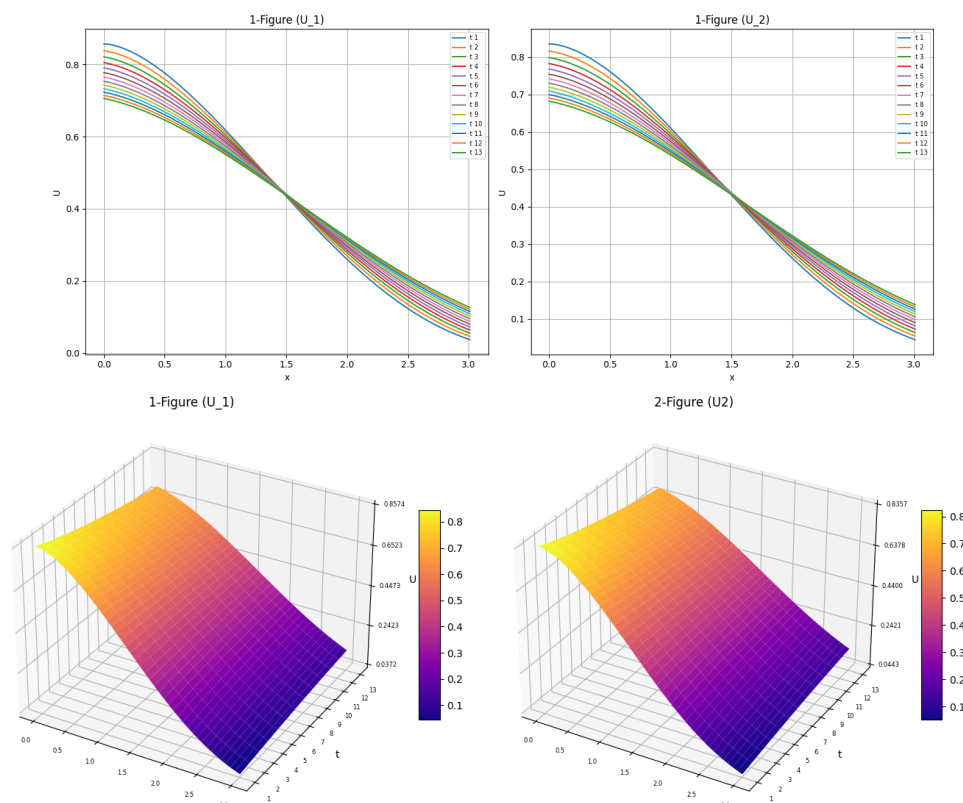


Figure 4. Numerical solution of problem (1)–(3).
 $p_1 = 2.5$, $p_2 = 2.4$, $q_1 = 3.201$, $q_2 = 3.302$

Figures 1–4 show the results of the numerical solution of the problem (1)–(3) for $p_i > 2$, corresponding to the case of slow diffusion. (9), (11) were taken as the initial approximation for the iterative process. For $p_i > 2$, as follows from the asymptotic formulas (11) of the graphs, the heat propagation occurs at a finite rate. The penetration depth of a thermal wave depends on the time and the wave front (the point at which $u_i(x, t)$ becomes zero) for each medium located at the final point:

$$x_{\phi_{u_i}} = \left(\frac{a_i}{b_i} \right)^{\frac{p_i-1}{p_i}} (T + t)^{\lambda_i} < \infty.$$

Numerical experiments have shown that the iterative processes converge within 10–12 steps. The asymptotics of self-similar solutions with compact support, taken as the initial approximation, played an important role in the convergence of the iterative process. The graphs presented above demonstrate that, for the given values of the numerical parameters, the reaction-diffusion processes occur at a finite speed.

Conclusion

It is well known that in the numerical solution of nonlinear problems, one of the main issues is the choice of the initial approximation, which ensures rapid convergence of the iterative process to the exact solution while preserving the qualitative properties of nonlinear processes. This problem was resolved by employing the asymptotic formulas constructed above, depending on the values of the numerical parameters, as the initial approximation.

The results of several computational experiments on the problem, the corresponding graphs, and

the analysis of the numerical solutions are presented. The outcomes of the computational experiments demonstrated the efficiency of the proposed methods, while the numerical solutions reflected the nonlinear features of the problem.

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Author contributions

All authors contributed equally to this work.

Conflict of interest

The authors declare no conflict of interest.

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Fokker–Planck equations on homogeneous time scales: well-posedness, stability, and applications to stochastic processes

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We study one-dimensional Fokker–Planck equations on homogeneous time scales with spatially dependent drift and diffusion coefficients. The model is formulated under reflective boundary conditions and is transformed, through the stationary density, into an abstract Cauchy problem governed by the backward Fokker–Planck operator in a weighted Hilbert space. Using semigroup methods on time scales, we prove the existence and uniqueness of mild solutions and characterize when such solutions are classical. The relation between the forward Fokker–Planck operator and the backward operator is clarified through a conjugacy with the stationary profile, which also preserves the spectral structure. Under the stated assumptions, the backward operator is positive, self-adjoint, and has a discrete spectrum. These properties yield well-posedness of the time-scale Fokker–Planck problem and asymptotic stability of solutions toward the normalized stationary density. The homogeneous time-scale setting covers both continuous and discrete dynamics in a unified framework and allows spectral representations of the resulting evolution. The applicability of the theory is illustrated by the Ornstein–Uhlenbeck process and the geometric Brownian motion, showing how classical stochastic models can be represented within the proposed time-scale formulation.

Keywords: Fokker–Planck equations, homogeneous time scales, reflective boundary conditions, abstract Cauchy problem, backward Fokker–Planck operator, asymptotic stability, Ornstein–Uhlenbeck process, geometric Brownian motion.

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Introduction

The Fokker–Planck equations (FPEs) are central in the study of stochastic dynamics, providing deterministic descriptions of probability density evolution for diffusion processes. Classically, FPEs arise from stochastic differential equations driven by Wiener processes and provide a fundamental framework for stochastic dynamics [1], with applications in control and spectral analysis [2,3], nonlinear dynamics [4], entropy production [5–7], plasma and kinetic models [8,9], biology [10], and economic and financial modeling [11,12]. For one-dimensional FPEs, Bluman [13] initiated the analysis of exact solutions. In general, the transition density $u(x, t)$ on a bounded interval with drift $f(x)$ and diffusion $D(x) > 0$ satisfies the forward form

$$\frac{\partial u}{\partial t} = -\frac{\partial}{\partial x}[f(x)u] + \frac{\partial^2}{\partial x^2}\left[\frac{D(x)}{2}u\right], \quad (1)$$

and the backward form

$$\frac{\partial u}{\partial t} = -f(x)u_x + \frac{D(x)}{2}u_{xx}$$

[14]. Both are parabolic PDEs, whose well-posedness follows classical results [15] and which relate closely to advection–diffusion equations [11,16]. This makes it natural to reformulate them as abstract Cauchy problems and apply C_0 -semigroup methods [17,18].

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In the literature, FPEs are treated either in continuous time, yielding PDEs on $\mathbb{R}^+$, or in discrete time, leading to recurrence equations on $\mathbb{Z}^+$. A general framework that unifies both cases, and potentially hybrid dynamics, remains underdeveloped. Time-scale calculus, introduced by Hilger [19], provides precisely such a framework by allowing dynamical equations on arbitrary closed subsets of $\mathbb{R}$ that may combine continuous and discrete intervals. Time-scale methods have been developed extensively for dynamic equations and differential or difference equations [20–22], variational problems [23], and controllability of dynamic equations [24, 25]. However, their application to stochastic processes, and to the FPE in particular, remains largely unexplored.

The extension of C_0 -semigroup theory from $\mathbb{R}^+$ to time scales has been investigated in recent works [26–28]. In particular, Hamza and Oraby [26, 27] proposed semigroup formulations under strong assumptions. In analogy with classical semigroup theory for infinite-dimensional systems [29], abstract dynamic equations on time scales have been developed through operator-semigroup and Cauchy-problem frameworks [27, 28, 30], with related models including Gompertz-type and heat equations on time scales [31, 32]; stability and controllability aspects are treated in [24, 26, 33].

This paper develops a unified time-scale theory of FPEs, covering continuous, discrete, and mixed temporal domains. Under suitable assumptions on the drift and diffusion coefficients, we prove existence and uniqueness of mild solutions, establish stability, and show consistency with the classical PDE formulation when $\mathbb{T} = \mathbb{R}$ and with difference-type FPEs when $\mathbb{T} = \mathbb{Z}$ or $h\mathbb{Z}$. The framework also reveals intermediate dynamics on hybrid grids. Its applicability is illustrated through the Ornstein–Uhlenbeck process (OUP) and the geometric Brownian motion (GBM), for which explicit or semi-explicit time-scale FPE solutions and numerical experiments show how nonuniform temporal domains modify distributional evolution, with implications for financial modeling, diffusion approximation, and systems subject to nonuniform interventions.

1 Preliminaries

The calculus on time scales unifies continuous and discrete analysis by allowing dynamic equations on closed subsets $\mathbb{T} \subset \mathbb{R}$. For $t \in \mathbb{T}$, the forward and backward jump operators are defined by

$$\sigma(t) := \inf\{s \in \mathbb{T} : s > t\}, \quad \rho(t) := \sup\{s \in \mathbb{T} : s < t\}.$$

The graininess function $\mu(t) := \sigma(t) - t$ measures the local step size. A time scale is called homogeneous when $\mu(t)$ is constant. Throughout this paper, we consider homogeneous time scales with constant graininess μ and assume $0 \in \mathbb{T}$. This setting provides a tractable framework for studying the associated Fokker–Planck operator and its C_0 -semigroup.

For $a, b \in \mathbb{T}$, let $[a, b]_{\mathbb{T}} := \{t \in \mathbb{T} : a \leq t \leq b\}$. If X is a Banach space, a function $f : \mathbb{T} \rightarrow X$ is called rd-continuous if it is continuous at right-dense points and has finite left-sided limits at left-dense points. The space $C_{\text{rd}}([a, b]_{\mathbb{T}}, X)$, equipped with the supremum norm, is a Banach space [20–22].

For $f : \mathbb{T} \rightarrow X$, write $f^\sigma = f \circ \sigma$. The Δ -derivative of f at $t \in \mathbb{T}^\kappa$ is the element $f^\Delta(t) \in X$, if it exists, such that

$$\|f^\sigma(t) - f(s) - f^\Delta(t)(\sigma(t) - s)\| \leq \varepsilon|\sigma(t) - s|$$

for all s sufficiently close to t . Equivalently [21, Theorem 1.16],

$$f^\Delta(t) = \begin{cases} \frac{f^\sigma(t) - f(t)}{\sigma(t) - t}, & \sigma(t) > t, \\ \lim_{s \rightarrow t} \frac{f(t) - f(s)}{t - s}, & \sigma(t) = t. \end{cases}$$

Basic rules mirror classical calculus:

$$(f + g)^\Delta = f^\Delta + g^\Delta, \quad (\alpha f)^\Delta = \alpha f^\Delta, \quad (fg)^\Delta = f^\Delta g + f^\sigma g^\Delta = f g^\Delta + f^\Delta g^\sigma.$$

The Laplace transform $\mathcal{L}$ on time scales [20, 21] extends the classical case. For $f : \mathbb{T} \rightarrow \mathbb{R}$ rd-continuous,

$$\mathcal{L}\{f\}(z) := \int_0^\infty f(t) e_{-z}(\sigma(t), 0) \Delta t,$$

with $e_z(t, s)$ is the generalized exponential. These standard properties remain valid on $\mathbb{T}$.

Similarly to the classical case, if an operator A generates a C_0 -semigroup T on the time scale $\mathbb{T}$, then T solves the abstract Cauchy problem

$$u^\Delta(t) = Au(t), \quad t \in \mathbb{T}, \quad u(0) = u_0,$$

with solution $u(t) = T(t)u_0$ [24]. The Laplace transform connects to the resolvent

$$\mathcal{L}\{T(\cdot)u_0\}(z) = (zI - A)^{-1}u_0,$$

valid for $\Re(z)$ large. Analogues of the Hille-Yosida theorem provide generation criteria for semigroups on $\mathbb{T}$ [34]. These tools form the foundation for analyzing well-posedness of Fokker-Planck equations on time scales.

2 Fokker-Planck equations on time scales

In the classical Fokker-Planck framework, boundary conditions may be absorbing, reflective, or periodic. We focus on reflective conditions, as they ensure conservation of probability and fit naturally with the time-scale setting. The dynamic FPEs (1) can be generalized with initial data and reflective boundaries, with $x \in [a, b]$ and $t \in \mathbb{T}_0^+$:

$$\begin{aligned} u^\Delta(x, t) &= -\frac{\partial}{\partial x}[f(x)u] + \frac{\partial^2}{\partial x^2}\left[\frac{D(x)}{2}u\right], \quad x \in (a, b), \quad t \in \mathbb{T}_0^+ \setminus \{0\}, \\ J(u)(a) &= J(u)(b) = 0, \\ u(0, x) &= u_0(x), \quad x \in [a, b], \end{aligned} \tag{2}$$

where

$$J(u)(x) := f(x)u(x) - \frac{1}{2}\frac{d}{dx}(D(x)u(x)).$$

The well-posedness and long-time behavior of (2) will be studied via semigroup methods.

Problem (2) can be written as a time-scale Cauchy problem

$$\begin{aligned} u^\Delta(t) &= \mathcal{A}u(t), \quad t \geq 0, \\ u(0) &= u_0, \end{aligned}$$

where $\mathcal{A}$ is the Fokker-Planck operator defined by

$$\mathcal{A}u := -\frac{d}{dx}[f(x)u] + \frac{d^2}{dx^2}\left[\frac{D(x)}{2}u\right]$$

on a domain $\mathcal{D}(\mathcal{A}) := \{u \in C^2(a, b) : J(u)(a) = J(u)(b) = 0\}$.

To simplify boundary conditions, we instead use the backward Kolmogorov operator A , after introducing the stationary solution u_s of (2), obtained from

$$-\frac{d}{dx}\left(f(x)u_s(x) - \frac{1}{2}\frac{d}{dx}(D(x)u_s(x))\right) = 0.$$

Integrating and imposing $J(u_s) = 0$ gives

$$u_s(x) = N^{-1} \frac{1}{D(x)} \exp \left(2 \int_0^x \frac{f(\tau)}{D(\tau)} d\tau \right), \quad (3)$$

with N the normalization constant given by

$$N = \int_a^b \frac{1}{D(x)} \exp \left(2 \int_0^x \frac{f(\tau)}{D(\tau)} d\tau \right) dx.$$

Writing $u(x, t) = u_s(x)v(x, t)$ leads to

$$v^\Delta(t) = Av(t), \quad v(0) = v_0 = u_s^{-1}u_0, \quad (4)$$

where

$$Av := f(x) \frac{dv}{dx} + \frac{D(x)}{2} \frac{d^2v}{dx^2} \quad (5)$$

on the domain $\mathcal{D}(A) = \{v \in C^2(a, b) : v'(a) = v'(b) = 0\}$.

Existence and uniqueness. Let $X = L^2(a, b)$ with inner product

$$\langle v, w \rangle_{u_s} = \int_a^b v(x)w(x)u_s(x) dx.$$

A mild solution of (4) is defined as

$$v(t) = v_0 + A \int_0^t v(s) \Delta s, \quad t \in \mathbb{T}_0^+, \quad (6)$$

where $\int_0^t v(s) \Delta s \in \mathcal{D}(A)$. Under sufficient regularity assumptions, classical and mild solutions coincide [28], as stated in the following theorem.

Theorem 1. A mild solution v of (4) is a classical solution if and only if $v \in C_{\text{rd}}^1(\mathbb{T}_0^+, X)$.

The following auxiliary results, adapted from [35], describe the relationship between the operators $\mathcal{A}$ and A , and summarize several basic properties of A in the Hilbert space X .

Lemma 1. If $u(x) = v(x)u_s(x)$ for $v \in \mathcal{D}(A)$, then $\mathcal{A}u = u_s Av$ and both have common eigenvalues.

Here, we write $u(x)$ and $v(x)$ in place of $u(x, \cdot)$ and $v(x, \cdot)$, respectively, since the operator A acts only on the spatial variable $x \in [a, b]$ and is independent of the time variable t .

Lemma 2. Operator A has properties:

- (a) $-A$ is a positive operator on X .
- (b) The null space of A consists of constants.
- (c) A is self-adjoint with a discrete spectrum in the Hilbert space X .

By Lemma 2, A is a closed self-adjoint operator on X with eigenvalues $0 = \lambda_0 < \lambda_1 < \lambda_2 < \dots$, whose associated eigenfunctions $\{\phi_n\}_{n \geq 0}$ form an orthonormal basis of X , with $\phi_0 = 1$. Hence,

$$Av = \sum_{n=0}^{\infty} -\lambda_n \langle v, \phi_n \rangle_{u_s} \phi_n, \quad x \in \mathcal{D}(A). \quad (7)$$

Moreover, operator A generates a C_0 -semigroup

$$T(t)v = \sum_{n=0}^{\infty} e_{-\lambda_n}^\sigma(t, 0) \langle v, \phi_n \rangle_{u_s} \phi_n, \quad v \in X. \quad (8)$$

On the other hand, if $\lambda_n \in \mathcal{R}$, by combining Theorem 3.3 in [28] with the remark following Theorem 2.3 in [24], we arrive at

$$|e_{-\lambda_n}^\sigma(t, 0)| \leq \frac{1}{1 + \lambda_n t}, \quad t \in \mathbb{T}_0^+. \quad (9)$$

Theorem 2. Let $T(t)$ be the C_0 -semigroup on X given in (8). Then:

- (a) $v(t) = T(t)v_0$ is a mild solution of (4) for all $v_0 \in X$;
- (b) $v(t) = T(t)v_0$ is a classical solution if and only if $v_0 \in \mathcal{D}(A)$.

Proof. (a) Let $w_n := \langle v_0, \phi_n \rangle_{u_s}$, $n \in \mathbb{N}$. We can easily verify that the scalar equation

$$v_n^\Delta(t) = -\lambda_n v_n(t), \quad v_n(0) = w_n, \quad t \in \mathbb{T}_0^+$$

has solution $v_n(t) = e_{-\lambda_n}^\sigma(t, 0)w_n$ satisfying, see (9),

$$|v_n(t)| \leq \frac{1}{1 + \lambda_n t} |w_n|. \quad (10)$$

Moreover, by virtue of the Fundamental Theorem of Calculus, we obtain

$$-\lambda_n \int_0^t v_n(\tau) \Delta\tau = v_n(t) - w_n, \quad t \in \mathbb{T}_0^+$$

which implies that

$$\left| \int_0^t v_n(\tau) \Delta\tau \right| \leq \frac{1}{\lambda_n} \left(1 + \frac{1}{1 + \lambda_n t} \right) |w_n|, \quad t \in \mathbb{T}_0^+. \quad (11)$$

We define

$$v(t) := \sum_{n=0}^{\infty} v_n(t) \phi_n, \quad t \in \mathbb{T}_0^+. \quad (12)$$

Equation (10), together with the convergence of series (8), ensures that this definition is well-defined. Moreover, the uniform convergence of the series (12) implies that

$$\int_0^t v(\tau) \Delta\tau = \sum_{n=0}^{\infty} \left(\int_0^t v_n(\tau) \Delta\tau \right) \phi_n, \quad t \in \mathbb{T}_0^+.$$

In addition, using (11), we obtain

$$\sum_{n=0}^{\infty} \lambda_n^2 \left| \int_0^t v_n(\tau) \Delta\tau \right|^2 \leq \sum_{n=0}^{\infty} \left(1 + \frac{1}{1 + \lambda_n t} \right)^2 |w_n|^2 \leq 4 \sum_{n=0}^{\infty} |w_n|^2 < \infty.$$

Therefore

$$\int_0^t v(\tau) \Delta\tau \in \mathcal{D}(A).$$

By (7) and the orthonormality of $\{\phi_n\}$, we have

$$A \int_0^t v(\tau) \Delta\tau = \sum_{n=0}^{\infty} -\lambda_n \left(\int_0^t v_n(\tau) \Delta\tau \right) \phi_n = \sum_{n=0}^{\infty} (v_n(t) - w_n) \phi_n = v(t) - v_0.$$

Thus, v is a mild solution of (4) and (12) yields $v(t) = T(t)v_0$ for $t \in \mathbb{T}_0^+$.

- (b) Let $v_0 \in \mathcal{D}(A)$. Since A is closed, the mild solution $v(t) = T(t)v_0$ in (6) can be expressed as

$$T(t)v_0 = v_0 + \int_0^t T(s)Av_0 \Delta s, \quad t \in \mathbb{T}_0^+.$$

Hence, it is clear that $T \in C_{\text{rd}}^1(\mathbb{T}_0^+, X)$, and the assertion follows from Theorem 1. Conversely, suppose that $v(t) = T(t)v_0$ is a classical solution. By definition, $v_0 = T(0)v_0 \in \mathcal{D}(A)$. $\square$

We see that if v_0 is the one referred to in (4), then w_n defined earlier in the proof to be $\langle u_0, \phi_n \rangle$, the usual inner product.

Theorem 3 (Stationary solution and stability). The Fokker–Planck problem (2) is well-posed. Moreover, for any u_0 , the solution satisfies

$$\lim_{t \rightarrow \infty} u(x, t) = w_0 u_s(x), \quad w_0 = \int_a^b u_0(x) dx. \quad (13)$$

Proof. For $u_0 \in X$ there is v_0 as in (4). For this v_0 , the function $v(t) = T(t)v_0$ is the solution of (4) giving

$$v(t) = \sum_{n=0}^{\infty} w_n e_{-\lambda_n}^{\sigma}(t, 0) \phi_n,$$

where $w_n = \langle u_0, \phi_n \rangle$. Lemma 1 concludes that the function

$$u(x, t) = w_0 u_s(x) + u_s(x) \sum_{n=1}^{\infty} w_n e_{-\lambda_n}^{\sigma}(t, 0) \phi_n(x), \quad t \in \mathbb{T}_0^+, \quad a \leq x \leq b,$$

is a unique solution of the Fokker–Planck problem (2) on the time scales $\mathbb{T}_0^+$. Lemma 2.18 of [28] gives $\lim_{t \rightarrow \infty} e_{-\lambda_n}^{\sigma}(t, 0) = 0$, and (13) follows. $\square$

Remark 1 (Mass conservation on time scales). Assume $u(\cdot, t) \in H^2(a, b)$ for each $t \in \mathbb{T}_0^+$. The regularity assumptions allows the interchange of the Δ -derivative in t with the spatial integral. Hence, integrating PDE (2) over $[a, b]$ and using the zero-flux boundary conditions yields

$$\frac{\Delta}{\Delta t} \int_a^b u(x, t) dx = \int_a^b u \Delta(x, t) dx = \int_a^b \mathcal{A}u(x, t) dx = J(u)(x, t) \Big|_{x=a}^{x=b} = 0.$$

Thus, the total mass is conserved on the time scale:

$$\int_a^b u(x, t) dx = \int_a^b u_0(x) dx =: w_0, \quad \forall t \in \mathbb{T}_0^+.$$

This identity shows that the total mass is conserved on the time scale, i.e.,

$$M(t) := \int_a^b u(x, t) dx, \quad t \in \mathbb{T}_0^+,$$

remains constant. In particular, the conservation holds both at right-dense and right-scattered points, ensuring that the total probability is preserved regardless of whether time evolves continuously, discretely, or in a hybrid manner.

Theorem 4 (Uniqueness of the stationary profile). Under the hypotheses of Theorem 3, any nontrivial stationary solution u_s is unique up to multiplication by a constant. In particular, if u_s is normalized so that $\int_a^b u_s(x) dx = 1$, then u_s is unique.

Proof. Stationary states solve the elliptic equation $\mathcal{A}u = 0$ with zero-flux boundary conditions. Under the hypotheses imposed on $f(x), D(x)$ (regularity and $D(x) > 0$ on $[a, b]$) the operator $-\mathcal{A}$ admits a discrete real spectrum

$$0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \cdots, \quad \lambda_n \rightarrow \infty,$$

see Lemma 1, and an orthonormal basis of eigenfunctions $\{\phi_n\}$ in the weighted Hilbert space (as used in Theorem 3). The eigenfunction corresponding to $\lambda_0 = 0$ is precisely the (nonnegative) steady profile u_s . Moreover, Lemma 2 provides that the kernel $\ker(\mathcal{A})$ is one-dimensional, hence any stationary solution is a scalar multiple of u_s . Normalizing the mass fixes the scalar and gives uniqueness. $\square$

Remark 2 (Rate of convergence). From the eigen-expansion used in Theorem 3,

$$u(\cdot, t) - w_0 u_s = u_s(\cdot) \sum_{n=1}^{\infty} w_n e_{-\lambda_n}^{\sigma}(t, 0) \phi_n(\cdot).$$

Moreover, since $\lambda_1 > 0$ and $|e_{-\lambda_n}^{\sigma}(t, 0)| \leq |e_{-\lambda_1}^{\sigma}(t, 0)|$ for all $n \in \mathbb{N}$, we can estimate the decay in terms of $e_{-\lambda_1}^{\sigma}(t, 0)$, so that there exists $C > 0$ with

$$\|u(\cdot, t) - w_0 u_s\|_X \leq C \sup_{n \geq 1} |w_n| e_{-\lambda_1}^{\sigma}(t, 0),$$

showing exponential-type decay governed by the spectral gap λ_1 .

3 Applications to stochastic processes

For Brownian motion in $\mathbb{R}$, there are several problems of the stochastic processes in one dimension corresponding to the FPEs that can be solved analytically. The Ornstein–Uhlenbeck process and geometric Brownian motion are two well-known examples, modeled as SDEs. In this section, we will discuss both processes when the Brownian motion is allowed to operate on the time scales.

3.1 The Ornstein–Uhlenbeck process

The Ornstein–Uhlenbeck process (OUP) is a fundamental stochastic model describing the velocity of a Brownian motion subject to friction. It is governed by the SDE:

$$dX_t = -\alpha X_t dt + \sqrt{2D} dW_t, \quad X(0) = x_0, \quad (14)$$

where $\alpha > 0$ denotes the drift (or relaxation) coefficient, and $D > 0$ represents the diffusion coefficient. The initial condition x_0 may be specified deterministically or randomly from a distribution independent of the Brownian motion W_t .

The OUP (14) can also be characterized through its associated probability density function $p(x, t)$, which describes the likelihood of the process occupying state x at time t . If t represents the time scale and $x \in \mathbb{R}$, OUP (14) is equivalently formulated in terms of the time-scale FPE

$$\begin{aligned} p^{\Delta}(x, t) &= \alpha \frac{\partial}{\partial x} [xp] + D \frac{\partial^2 p}{\partial x^2}, \quad x \in \mathbb{R}, \quad t \in \mathbb{T}_0^+ \setminus \{0\}, \\ \lim_{x \rightarrow -\infty} J(x, t) &= \lim_{x \rightarrow +\infty} J(x, t) = 0, \\ p(x, 0) &= p_0(x). \end{aligned} \quad (15)$$

In this case, we consider $f(x) = -\alpha x$ and $D(x) = 2D$, with the system constrained by reflective boundary conditions. By employing the eigenvalues and eigenfunctions of the generator of OUP, the time-dependent problem can be solved, leading to an explicit expression for the probability density function. It can be shown that the eigenfunctions of OUP are the Hermite polynomials. Referring to (3), we find the stationary solution of (15):

$$p_s(x) = \sqrt{\frac{\alpha}{2D\pi}} e^{-\frac{\alpha}{2D}x^2}.$$

The operator A in (5) has the form

$$Av = -\alpha x \frac{dv}{dx} + D \frac{d^2 v}{dx^2}$$

with $\mathcal{D}(A) = \{v \in C^2(\mathbb{R}) : \lim_{x \rightarrow \mp\infty} v'(x) = 0\}$. This operator is the generator of the OUP (14). Theorem 4.2 [1] implies that the operator $-A$ has the eigenvalues

$$\lambda_n = \alpha n, \quad n = 0, 1, 2, \dots$$

and the corresponding eigenfunctions are the normalized Hermite polynomials:

$$\phi_n(x) = \frac{1}{\sqrt{n!}} H_n \left(\sqrt{\frac{\alpha}{D}} x \right), \quad (16)$$

where

$$H_n(x) = (-1)^n e^{\frac{x^2}{2}} \frac{d^n}{dx^n} \left(e^{-\frac{x^2}{2}} \right).$$

We see that the orthonormality of the modified Hermite polynomials ϕ_n defined in (16) implies that

$$\int_{\mathbb{R}} \phi_n(x) \phi_m(x) p_s(x) dx = \delta_{nm}.$$

Therefore, the associated Cauchy problem (4) has the solution

$$v(x, t) = \sum_{n=0}^{\infty} w_n e^{\sigma_{-\lambda_n}(t, 0)} \phi_n(x), \quad w_n = \int_{\mathbb{R}} p_0(x) \phi_n(x) dx$$

for any initial distribution $p_0 \in L^2(\mathbb{R})_{p_s^{-1}}$. We note that w_n is equivalent to the one in the proof of Theorem 2. Finally, the solution of the problem (15) is given by

$$p(x, t) = p_s(x) \sum_{n=0}^{\infty} w_n e^{\sigma_{-\lambda_n}(t, 0)} \phi_n(x). \quad (17)$$

It is evident that, as the friction coefficient α goes to 0, the transition probability given in (17) converges to the transition probability of the Brownian motion. Moreover, we find that

$$\lim_{t \rightarrow \infty} p(x, t) = p_s(x), \quad t \in \mathbb{T}_0^+.$$

This corroborates the conclusion of Theorem 3. As anticipated, the OUP, being an ergodic Markov process with a Gaussian invariant distribution p_s , naturally leads to this outcome. The result closely mirrors the corresponding case for Brownian motion on $\mathbb{R}$.

The analysis reveals that the relaxation rate of the OUP is governed by the first nonzero eigenvalue $\lambda_1 = \alpha$, implying a universal convergence time scale $1/\alpha$, which represents the characteristic time required for the solution to approach equilibrium. In particular, the dominant decay mode behaves as $e^{-\alpha t}$, so that larger values of α produce faster relaxation. This convergence scale is independent of the underlying time domain. The effect of the time scale enters only through the decay factors $e^{\sigma_{-\lambda_n}(t, 0)}$, which are exponential in the continuous case, polynomial-like in the discrete case, and intermediate for hybrid domains. Consequently, finer graininess μ accelerates convergence while coarser μ slows it down. Nevertheless, the invariant distribution p_s remains Gaussian for all time scales, demonstrating that the ergodicity of the OUP is robust under continuous, discrete, and hybrid temporal structures.

We present simple illustrations of the dynamics of p on several representative time scales, namely $\mathbb{R}^+$, $\mathbb{Z}^+$ and $h\mathbb{Z}^+$. The corresponding C_0 -semigroups on these time scales are:

$$T(t)v = \sum_{n=0}^{\infty} c_n e^{-\lambda_n t} \phi_n, \quad t \in \mathbb{R}^+, \quad (18)$$

$$T(t)v = \sum_{n=0}^{\infty} c_n \frac{1}{(1 + \lambda_n)^{t+1}} \phi_n, \quad t \in \mathbb{Z}^+, \quad (19)$$

$$T(t)v = \sum_{n=0}^{\infty} c_n \frac{1}{(1 + \lambda_n h)^{t/h+1}} \phi_n, \quad t \in h\mathbb{Z}^+, \quad (20)$$

where $c_n := \langle v, \phi_n \rangle_{u_s}$ for $v \in L^2(\mathbb{R})$.

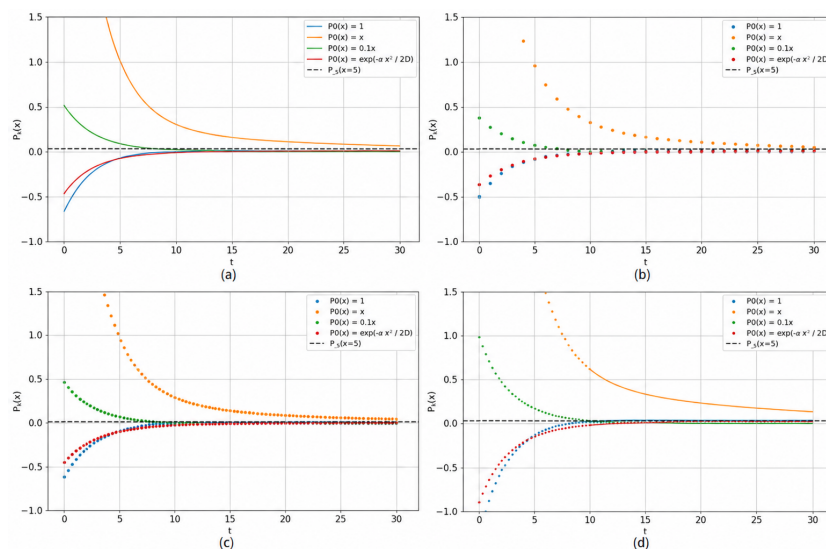


Figure 1. Dynamics of p for $\alpha = 0.05$, $D = 0.6$, $h = 0.2$ and $x = 5$ on: (a) $\mathbb{T} = \mathbb{R}$; (b) $\mathbb{T} = \mathbb{Z}^+$; (c) $\mathbb{T} = h\mathbb{Z}^+$; (d) $\mathbb{T} = h\mathbb{Z}^+ \cup \mathbb{R}^+$

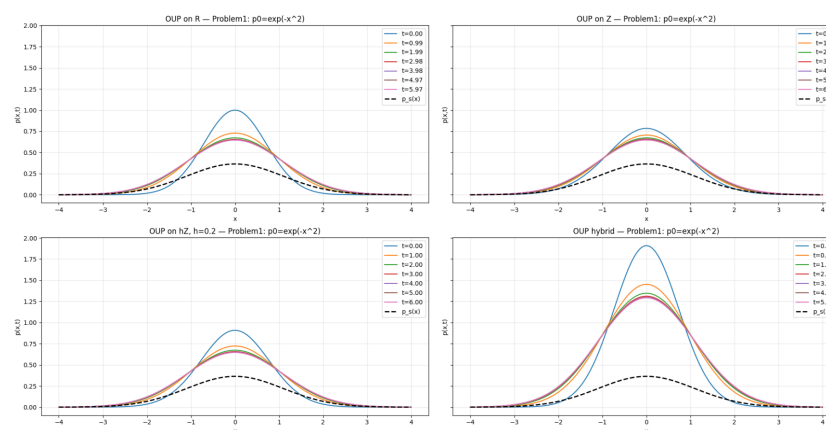


Figure 2. Dynamical behavior of p relative to p_s , under the initial distribution $p_0(x) = e^{-x^2}$

Figures 1–2 illustrate the implications of Theorem 3. Figure 1 shows the behavior on selected time scales $\mathbb{T}_0^+$ with respect to t for fixed x , while Figure 2 displays the spatial variation in x over its prescribed interval for fixed t . In particular, Figure 2 demonstrates that the OUP rapidly converges

to the Gaussian invariant distribution for large $|x|$, whereas the convergence is significantly slower in a small neighborhood of 0.

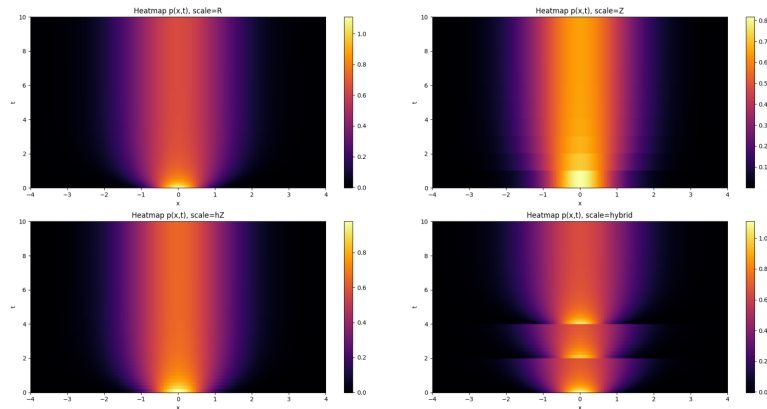


Figure 3. Heatmaps show how time scales shape convergence: smooth on $\mathbb{R}^+$, stepwise on $\mathbb{Z}^+$, nearly continuous on $h\mathbb{Z}^+$, and mixed freezing–stepwise behavior on the hybrid scale

Figure 3 shows that the bright region near $x = 0$ at $t = 0$ corresponds to the initial profile $p_0(x) = e^{-x^2}$. As time evolves, the density relaxes according to the underlying time scale toward the stationary Gaussian distribution $p_s(x)$. The symmetric contraction of the heatmap reflects the mean-reverting nature of the OUP, which progressively drives the probability mass toward equilibrium.

3.2 Geometric Brownian motion

Geometric Brownian Motion (GBM) is a well-known stochastic process, particularly in mathematical finance, for describing the dynamics of asset prices such as stocks. It is a continuous-time model in which the logarithm of the asset price evolves as a Brownian motion with a constant drift. This formulation captures both the inherent randomness of market movements and their tendency to fluctuate around an underlying trend. The model is governed by two key parameters: the drift α , representing the average rate of return, and the volatility β , representing the magnitude of fluctuations. Mathematically, GBM arises as the solution to a scalar linear SDE with multiplicative noise

$$dX_t = \alpha X_t dt + \beta X_t dW_t, \quad X(0) = x_0. \quad (21)$$

This represents a basic framework for describing population dynamics under environmental fluctuations.

In an alternative formulation, when we take $f(x) = \alpha x$ and $D(x) = \frac{\beta^2 x^2}{2}$ confined to the interval $[1, b]$ with decay at the boundaries, the process (21) simplifies to the FPE subject to absorbing boundary conditions. Furthermore, if the time is allowed to evolve over the time scale $\mathbb{T}_0^+$ rather than being purely continuous, this framework naturally extends to the FPE on the time scale $\mathbb{T}_0^+$ with absorbing boundary conditions

$$\begin{aligned} u^\Delta(x, t) &= -\frac{\partial}{\partial x} [\alpha x u] + \frac{\partial^2}{\partial x^2} \left[\frac{\beta^2 x^2}{2} u \right], \quad x \in [1, b], \quad t \in \mathbb{T}_0^+ \setminus \{0\}, \\ u(1, t) &= u(b, t) = 0, \quad t \in \mathbb{T}_0^+, \\ u(x, 0) &= u_0(x), \quad x \in [1, b]. \end{aligned} \quad (22)$$

The generator of this process is

$$\mathcal{A}u = -\frac{d}{dx} [\alpha x u] + \frac{d^2}{dx^2} \left[\frac{\beta^2 x^2}{2} u \right]$$

the linear operator defined on the domain

$$\mathcal{D}(\mathcal{A}) = \{u \in L^2(1, b) : u, u' \text{ are absolute continuous, } u'' \in L^2(1, b), u(1) = u(b) = 0\}.$$

We see that this operator is not uniformly elliptic due to the diffusion coefficient vanishes at $x = 0$. This is the reason why we consider the domain to be the interval $[1, b]$.

By applying the Dirichlet condition, we can easily find out the stationary solution of the forward FPE (22):

$$u_s(x) = 0, \quad 1 \leq x \leq b.$$

By reducing the eigenproblem $\mathcal{A}u = -\lambda u$ to a Cauchy-Euler equation, we obtain the eigenvalues and the corresponding normalized eigenfunctions of $\mathcal{A}$ as

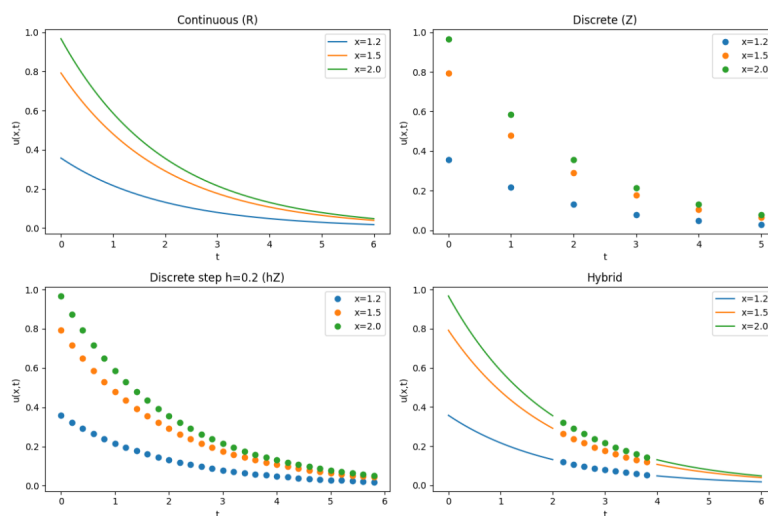


Figure 4. Dynamics of u with parameters $\alpha = 0.3$, $\beta = 0.6$, step size $h = 0.2$, for $u_0(x) = \sin\left(\frac{\pi(x-1)}{e-1}\right)$ on time scales: (a) $\mathbb{T} = \mathbb{R}^+$; (b) $\mathbb{T} = \mathbb{Z}^+$; (c) $\mathbb{T} = h\mathbb{Z}^+$; and (d) $\mathbb{T} = [0, 2] \cup (h\mathbb{Z}^+ \cap [2, 4]) \cup [4, 6]$

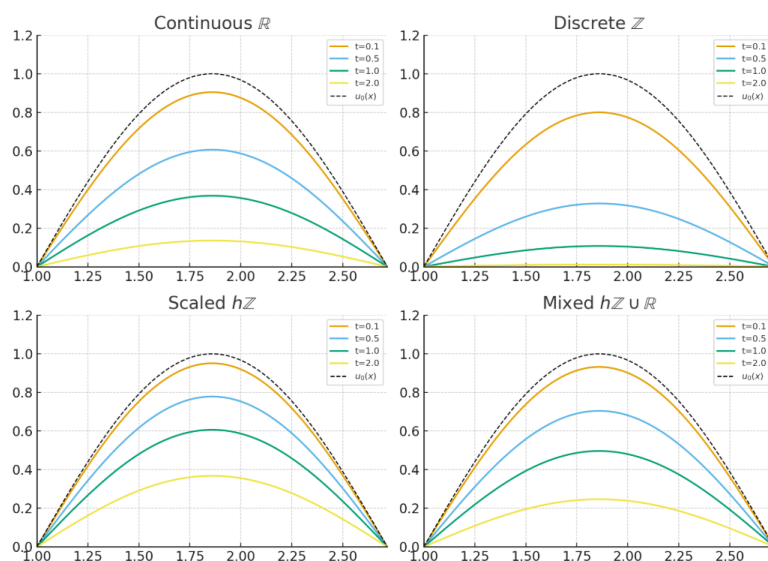


Figure 5. Dynamics of u for $\alpha = 0.3$ and $\beta = 0.6$, starting from $u_0(x) = \sin\left(\frac{\pi(x-1)}{e-1}\right)$, imposed with absorbing boundary conditions

$$\lambda_n = \frac{\beta^2}{2} \left[\left(\frac{n\pi}{\ln b} \right)^2 + \left(\frac{\alpha}{\beta^2} - \frac{1}{2} \right)^2 \right],$$

$$\phi_n(x) = \sqrt{\frac{2}{\ln b}} x^{r-1} \sin \left(\frac{n\pi}{\ln b} \ln x \right), \quad r = \frac{\alpha}{\beta^2} - \frac{1}{2}, \quad n \in \mathbb{N},$$

where the eigenfunctions are orthonormal with respect to the weight function $\rho(x) = x^{1-2r}$. Therefore, for an arbitrary initial condition $u_0 \in L^2(1, b)$, the mild solution of (22) is given by

$$u(x, t) = \sum_{n=1}^{\infty} w_n e^{-\lambda_n t} \phi_n(x), \quad (23)$$

where $w_n = \int_1^b u_0(x) \phi_n(x) \rho(x) dx$. It is evident that the presence of noise in equation (21) can fundamentally change the qualitative behavior of the solution. In particular, if $\alpha > 0$, the solution to the deterministic equation ($\beta = 0$) exhibits exponential growth as $t \rightarrow \infty$. In contrast, for the stochastic formulation, equation (23) demonstrates that the solution converges to 0 with probability one. This phenomenon is consistent with the fact that all Brownian particles are eventually absorbed at the boundary.

Figures 4–5 illustrate that under absorbing boundary conditions, the geometric Brownian motion $u(x, t)$ converges to the stationary state $u_s \equiv 0$ on all considered time scales. The convergence is smooth on the continuous time scale $\mathbb{R}^+$, stepwise on the discrete scales $\mathbb{Z}^+$ or $h\mathbb{Z}^+$, and exhibits intermediate behavior on the hybrid scale $\mathbb{T} = h\mathbb{Z}^+ \cup \mathbb{R}^+$. The eigenfunction expansion (23) shows that higher modes decay faster, so the long-time dynamics are dominated by the slowest-decaying mode, providing an estimate of the convergence rate. Moreover, the C_0 -semigroup simulations reveal that transient dynamics strongly depend on the time-scale structure, even though the ultimate absorption at the boundaries is time-scale independent. These findings highlight the influence of continuous, discrete, and hybrid temporal representations on the evolution of stochastic processes with multiplicative noise.

Referring to the semigroups in (18), (19), and (20), and replacing u_s with ρ , we perform simulations of the geometric Brownian motion on the time scales $\mathbb{R}^+$, $\mathbb{Z}^+$, and $h\mathbb{Z}^+$ for $x \in [1, e]$.

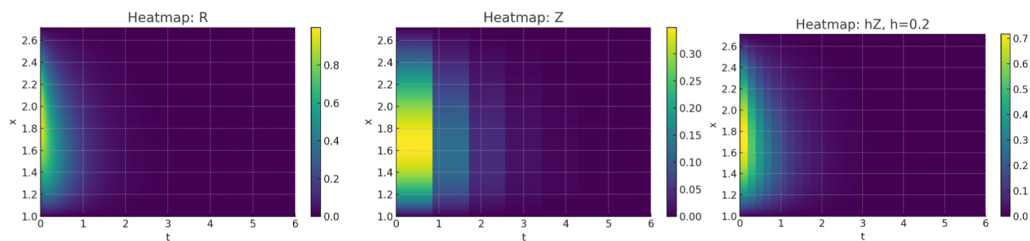


Figure 6. Heatmaps of solution decay to zero across the time scales $\mathbb{R}^+$, $\mathbb{Z}^+$, and $h\mathbb{Z}^+$, consistent with the asymptotic behavior of GBM with absorbing boundary conditions. The temporal evolution appears smooth in the continuous case and stepwise in the discrete cases

Figure 6 illustrates the temporal decay of the GBM solution on different time scales. In all cases, the solution converges toward zero as time increases, reflecting the dissipative nature of the system under absorbing boundary conditions. For the continuous time scale $\mathbb{R}^+$, the decay is smooth and exponential-like, corresponding to the classical continuous semigroup evolution. On the discrete time scales $\mathbb{Z}^+$ and $h\mathbb{Z}^+$, the evolution exhibits a stepwise structure induced by the discrete temporal dynamics. Moreover, the finer graininess in the $h\mathbb{Z}^+$ case produces a smoother transition and faster relaxation compared with the standard discrete case. These observations confirm that the qualitative asymptotic behavior of the GBM remains stable across different time-scale structures, while the temporal regularity and convergence profile are influenced by the underlying graininess.

Conclusion

We have shown the well-posedness of the one-dimensional Fokker–Planck equations on homogeneous time scales with reflective boundary conditions by employing the backward Fokker–Planck operator framework. The analysis ensures existence, uniqueness, and asymptotic stability of solutions, converging to a unique stationary distribution as $t \rightarrow \infty$. This unified time-scale formulation naturally incorporates continuous, discrete, and hybrid dynamics, as illustrated through the Ornstein–Uhlenbeck process and the geometric Brownian motion, and thus provides a versatile foundation for further studies of stochastic processes.

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Author contributions

S. Sutrima developed the theoretical framework and main proofs. S. Wibowo contributed to the stability analysis, asymptotic behavior, and OUP application. S. Sutanto worked on the GBM example, numerical simulations, and presentation of results. All authors discussed the results, revised the manuscript, and approved its submission.

Conflict of interest

The authors declare that there are no conflicts of interest related to this work. The funding sponsor had no role in the design of the study, analysis, interpretation of results, or in the decision to publish the manuscript.

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On an inverse boundary control of a system of integro-differential equations of fractional order in a locally convex space

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This paper investigates an inverse boundary optimal control problem for a nonlinear system of fractional-order integro-differential equations in a complete locally convex space. The state equation is supplemented with unknown boundary data, which are identified simultaneously with the optimal control. By introducing an adjoint final-value problem, a Pontryagin-type function is constructed, and a necessary optimality condition is derived in the form of a nonlinear integral equation for the control function. The original boundary value problem is transformed into an equivalent coupled system of nonlinear integral equations with respect to the state function, the boundary value, and the control. The solvability of this system is established by the Banach contraction principle in spaces whose topology is generated by a family of seminorms. Sufficient conditions guaranteeing the existence and uniqueness of the optimal control, the corresponding state trajectory, and the identified boundary value are obtained. In addition, the continuous dependence of the solution on the boundary data is proved. The proposed approach provides a constructive framework for studying inverse optimal control problems for fractional systems in locally convex spaces and can be extended to more general classes of nonlocal and infinite-dimensional dynamical systems.

Keywords: fractional order equation, locally convex algebra, equations with three unknown quantities, Pontryagin function, method of contraction mapping, existence, uniqueness, stability.

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Introduction

Optimal control of dynamic systems is a pressing area of modern mathematics. In classical optimal control theory, the state function is typically studied in finite-dimensional spaces or Banach spaces. However, many models arising in analysis, operator theory, integro-differential equations, memory problems, and distributed parameter problems are naturally formulated in a more general class of spaces, namely, locally convex spaces.

On the other hand, fractional calculus has become one of the effective tools for modeling processes with memory and heredity. Therefore, it is natural to transfer optimal control problems for fractional systems to locally convex spaces.

The study of differential equations in locally convex spaces is a current area of research in the theory of differential equations. A large number of papers have been published in this area.

Optimal control theory for differential and integro-differential equations has become one of the fundamental areas of modern applied mathematics owing to its numerous applications in engineering,

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mechanics, physics, biology, and economics. The foundations of optimal control theory for distributed parameter systems were established in the classical works of Butkovsky and Pustynnikov [1], while the general duality theorems and optimality conditions for nonconvex optimization problems were developed by Deumlich and Elster [2]. Methods of optimal control for distributed parameter systems were further investigated by Deineka in the context of thin compound plates [3]. Time-optimal control problems for heat transfer and evolutionary partial differential equations were studied in [4,5]. Optimal control of heat transfer processes with integral constraints was considered in [6]. Controllability of singularly perturbed discrete systems was analyzed in [7], whereas stability and null controllability of infinite systems of differential equations were investigated in [8]. Optimal control problems for impulsive functional-differential systems were addressed in [9]. Optimization problems involving multipoint and integral conditions were considered in [10,11]. Necessary optimality conditions for thermal control and tracking problems in nonlinear vector optimization were established in [12,13]. Convex analysis, infimal convolution, and duality methods for optimal control problems with second-order evolution differential inclusions were developed in [14].

During the last decades, considerable attention has been devoted to optimal control problems for systems governed by fractional-order differential equations because fractional derivatives provide an adequate mathematical description of memory and hereditary effects that cannot be captured by classical integer-order models. The mathematical foundations of fractional calculus are comprehensively presented in the monographs by Samko et al. [15], Podlubny [16], and Kilbas et al. [17]. Existence and uniqueness of solutions to nonlinear fractional differential equations were established in [18,19], while analytical properties and qualitative analysis of fractional differential equations were investigated in [20,21]. Applications of fractional calculus to viscoplasticity and damping models were presented in [22,23]. Fractional models in protein dynamics and continuum mechanics were developed in [24,25], whereas a broad collection of applications in physics is summarized in [26]. Boundary value problems for fractional differential equations with nonlocal conditions were investigated in [27]. A systematic treatment of partial differential equations of fractional order was presented by Pskhu [28], while inverse problems involving Hilfer-type fractional differential equations were studied in [29]. Recent investigations of loaded hyperbolic equations and heat equations involving fractional integro-differential operators are reported in [30,31].

Most existing results have been developed for finite-dimensional systems or for equations posed in Banach and Hilbert spaces. However, many models arising in functional analysis, operator theory, distributed parameter systems, and infinite-dimensional dynamics are naturally formulated in locally convex spaces (see, for example, [32–34]). Since such spaces are generated by families of seminorms rather than a single norm, many classical analytical techniques require substantial modification. Consequently, the development of constructive methods for inverse and optimal control problems in locally convex spaces remains an important and challenging topic.

Another significant feature of many practical applications is the presence of unknown boundary data that must be identified simultaneously with the optimal control. Such inverse boundary control problems arise in parameter identification, distributed systems, heat transfer, viscoelasticity, and other models involving hereditary effects. Nevertheless, the available literature contains relatively few results devoted to fractional integro-differential equations in locally convex spaces with simultaneous determination of the state, the control, and the unknown boundary value.

The purpose of this paper is to develop a constructive approach to solving an inverse boundary optimal control problem for a nonlinear system of fractional-order integro-differential equations in a complete locally convex space. By introducing an adjoint final-value problem, a Pontryagin-type function is constructed and a necessary optimality condition is obtained. The original boundary value problem is then reduced to a coupled system of nonlinear integral equations with respect to three unknown quantities: the state function, the control function, and the boundary value. The solvability

of the resulting system is established by applying the contraction mapping principle in locally convex spaces.

The main contributions of the paper can be summarized as follows. First, an equivalent integral formulation of the inverse boundary optimal control problem is obtained. Second, a nonlinear integral equation determining the optimal control is derived from the Pontryagin optimality condition. Third, sufficient conditions guaranteeing the existence and uniqueness of the control function, the corresponding state trajectory, and the unknown boundary value are established. Finally, the continuous dependence of the solution on the identified boundary data is proved. These results provide a rigorous theoretical basis for further investigations of inverse optimal control problems for fractional and nonlocal systems in locally convex spaces and locally convex algebras.

So, unlike the standard approach, in this paper we find the state function, boundary value, and control function simultaneously as the solution to a functional system consisting three nonlinear equations.

1 Preliminary information and statement of the problem

Let E be a linear space over the field F of real numbers. A seminorm on E is a non-negative function $E : p \rightarrow R_+$ satisfying the homogeneity condition $p(\lambda x) = |\lambda|p(x)$ for all $\lambda \in F$ and the triangle inequality $p(x + y) \leq p(x) + p(y)$ for all $x, y \in E$.

Definition 1. A sequence $\{x_n\} \in E$ is said to converge to an element $x \in E$ if for any $\varepsilon > 0$ and $p \in \mathcal{P}$ there exists a positive integer N such that $p(x_n - x) < \varepsilon$ for all $n \geq N$.

Definition 2. A sequence $\{x_n\} \in E$ is called a Cauchy sequence if for any $\varepsilon > 0$ and $p \in \mathcal{P}$ there exists a positive integer N such that $p(x_n - x_m) < \varepsilon$ for all $n, m \geq N$.

Definition 3. A locally convex space E is called complete if every Cauchy sequence is convergent.

So, let the space E be a complete locally convex space whose topology is defined by a family of seminorms

$$\mathcal{P} = \{p_\alpha : \alpha \in \mathcal{A}\}.$$

The operations of addition and multiplication by a number are continuous in any locally convex topology of the space E .

Let $\Omega = [0, T]$, $0 < T$, and let $C(\Omega, E)$ denote the space of all continuous functions $x : \Omega \rightarrow E$. For each seminorm $p_\alpha \in \mathcal{P}$ we define the induced seminorm on $C(\Omega, E)$:

$$\tilde{p}_\alpha(x) = \sup_{t \in \Omega} p_\alpha(x(t)).$$

Then $C(\Omega, E)$ is a locally convex space with respect to the family of seminorms $\{\tilde{p}_\alpha\}_{\alpha \in \mathcal{A}}$.

Let U be a control space. For definiteness, we assume that U is a locally convex space with the family of seminorms $\{q_\beta\}_{\beta \in \mathcal{B}}$, and by $C(\Omega, U)$ we denote the space of continuous control functions.

In addition, we suppose that the spaces E and U are topological algebra. For two elements $x, y \in E$ is defined factor $xy \in E$, where the mapping $xy : E \times E \rightarrow E$ is continuous. For each seminorm p_α there exist p_β, p_γ and $0 < C_\alpha$ that

$$p_\alpha(xy) \leq C_\alpha p_\beta(x) p_\gamma(y).$$

Let us consider the optimal control problem for the following nonlinear integro-differential equation of fractional order

$${}_CD_{0t}^\gamma x(t) = g(t, x(t)) \int_0^t f(s, x(s), u(s)) ds, \quad t \in \Omega \quad (1)$$

with boundary condition

$$a x(0) + b x(T) = x_r \in E, \quad (2)$$

where $x(t)$ is the state function, x_r is the redefinition value, $a, b \in R$, $a \neq 0$, $b \neq 0$, $a \pm b \neq 0$, $u(t)$ is the control function, ${}_C D_{0t}^\gamma$ denotes the Gerasimov–Caputo fractional operator of order $0 < \gamma < 1$:

$${}_C D_{0t}^\gamma x(t) = I_{0t}^{1-\gamma} x'(t) = \frac{1}{\Gamma(1-\gamma)} \int_0^t \frac{x'(s) ds}{(t-s)^\gamma}, \quad t \in \Omega,$$

and the Riemann–Liouville integral of order $0 < \gamma < 1$ is given by the expression

$$I_{0t}^\gamma x(t) = \frac{1}{\Gamma(\gamma)} \int_0^t \frac{x(s) ds}{(t-s)^{1-\gamma}}, \quad 0 < \gamma < 1,$$

$\Gamma(\gamma)$ is Gamma function: $\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt$, $\operatorname{Re} z > 0$, $g \in C(\Omega \times E, E)$, $f \in C(\Omega \times E \times U, E)$.

Let's consider the following quality functional

$$J[u] = (x_r - \eta)^2 + \mu \int_0^T \varepsilon(t) (u(t))^2 dt, \quad (3)$$

where $0 < \eta$, $\mu = \text{const}$, $0 < \varepsilon(t) \in C(\Omega)$, x_r will be defined from the following internal condition

$$x(t_1) = x_r, \quad 0 < t_1 < T. \quad (4)$$

In the present paper we use p_α is a fixed seminorm of the locally convex algebra E and q_β is a fixed seminorm of the locally convex algebra U .

Statement of the problem. It is required to find the control function $u^*(t) \in C(\Omega, U)$, the redefinition value $x_r^* \in E$, and the corresponding state function $x^*(t) \in C(\Omega, E)$ that minimize the functional (3) under the condition that $(x^*(t), x_r^*, u^*(t))$ satisfies the equation (1) and the boundary condition (2) and (4).

2 Reduction of the problem (1), (2) to an integral equation

Let $x(t) \in C(\Omega, E)$ be a solution to problem (1), (2). Apply the operator I_{0t}^γ to equation (1):

$$I_{0t}^\gamma {}_C D_{0t}^\gamma x(t) = I_{0t}^\gamma g(t, x(t)) \int_0^t f(s, x(s), u(s)) ds$$

and use the definition of the Gerasimov–Caputo fractional operator

$$I_{0t}^\gamma {}_C D_{0t}^\gamma (x(t)) = I_{0t}^\gamma I_{0t}^{1-\gamma} (x'(t)) = I_{0t}^1 (x'(t)) = \int_0^t x'(s) ds = x(t) - x(0),$$

$$I_{0t}^\gamma \eta_0(t, x(t)) = \frac{1}{\Gamma(\gamma)} \int_0^t (t-s)^{\gamma-1} \eta_0(s, x(s)) ds.$$

Then, applying these two formulas to the last fractional integro-differential equation, we obtain

$$x(t) = x(0) + \frac{1}{\Gamma(\gamma)} \int_0^t (t-s)^{\gamma-1} g(s, x(s)) \int_0^s f(\theta, x(\theta), u(\theta)) d\theta ds. \quad (5)$$

We consider

$$x(T) = x(0) + \frac{1}{\Gamma(\gamma)} \int_0^T (T-s)^{\gamma-1} g(s, x(s)) \int_0^s f(\theta, x(\theta), u(\theta)) d\theta ds.$$

Here we use the boundary condition (2) and the last equality, then we get that

$$x(0) = \frac{x_r}{a+b} - \frac{1}{\Gamma(\gamma)} \frac{b}{a+b} \int_0^T (T-s)^{\gamma-1} g(s, x(s)) \int_0^s f(\theta, x(\theta), u(\theta)) d\theta ds. \quad (6)$$

Now, we substitute the found value (6) of the state function at the initial point into the integral equation (5):

$$x(t) = \mathbf{K}_1(t; x, x_r, u) \equiv \frac{x_r}{a+b} + \int_0^T V(t, s) g(s, x(s)) \int_0^s f(\theta, x(\theta), u(\theta)) d\theta ds, \quad (7)$$

where

$$V(t, s) = \frac{1}{\Gamma(\gamma)} \begin{cases} (t-s)^{\gamma-1} - \frac{b}{a+b}(T-s)^{\gamma-1}, & 0 \leq s < t, \\ (T-s)^{\gamma-1}, & t \leq s \leq T. \end{cases}$$

Equation (7) is a nonlinear Fredholm integral equation of the second kind and satisfies problem (1), (2). We differentiate equation (7) with respect to t :

$$x'(t) = \int_0^t V'(t, s) g(s, x(s)) \int_0^s f(\theta, x(\theta), u(\theta)) d\theta ds, \quad (8)$$

where

$$\begin{aligned} V'(t, s) &= \left(\frac{1}{\Gamma(\gamma)} \begin{cases} (t-s)^{\gamma-1} - \frac{b}{a+b}(T-s)^{\gamma-1}, & 0 \leq s < t, \\ (T-s)^{\gamma-1}, & t \leq s \leq T, \end{cases} \right)'_t = \\ &= \frac{1}{\Gamma(\gamma)} \left((t-s)^{\gamma-1} \right)'_t = \frac{\gamma-1}{\Gamma(\gamma)} (t-s)^{\gamma-2}. \end{aligned}$$

3 Relation for redefinition value

We suppose that $a+b \neq 1$. Then we apply the intermediate condition (4) to the equation (7):

$$x(t_1) = \frac{x_r}{a+b} + \int_0^T V(t_1, s) g(s, x(s)) \int_0^s f(\theta, x(\theta), u(\theta)) d\theta ds = x_r.$$

From here, by a simple transformation, we find that

$$x_r = \mathbf{K}_2(x, x_r, u) \equiv \frac{a+b}{a+b-1} \int_0^T V(t_1, s) g(s, x(s)) \int_0^s f(\theta, x(\theta), u(\theta)) d\theta ds. \quad (9)$$

The formula (9), like (7), is a relation between unknown functions $(x(t), x_r, u(t))$.

4 Integral equations for the control function

Let $u^*(t) \in U$ be the optimal control function, where $u^*(t) + \Delta u^*(t) \in C(\Omega, U)$. Consider the problem of finding the control function $u^*(t) \in U$ and the corresponding state function $x^*(t)$ that ensure the minimum of the functional (3). To obtain the optimality conditions using the functional (3) and the equation (8), we introduce the Pontryagin function

$$H(u(t), x(t)) = \psi(t) \int_0^t V'(t, s) g(s, x(s)) \int_0^s f(\theta, x(\theta), u(\theta)) d\theta ds - \mu \varepsilon(t) u^2(t), \quad (10)$$

where the function $\psi(t)$ belongs to the dual space E' and is determined from the following equation:

$${}_CD_{0t}^\gamma \psi(t) = -\psi(t) \quad (11)$$

with final point condition

$$\psi(T) = -2[x_r - \eta]. \quad (12)$$

As a result of solving the fractional differential equation (11), we obtain the function

$$\psi(t) = \psi(0) - \frac{1}{\Gamma(\gamma)} \int_0^t (t-s)^{\gamma-1} \psi(s) ds.$$

We consider

$$\psi(T) = \psi(0) - \frac{1}{\Gamma(\gamma)} \int_0^T (T-s)^{\gamma-1} \psi(s) ds.$$

To determine the constant of integration $\psi(0)$, we use the condition specified at the end point (12):

$$\psi(0) - \frac{1}{\Gamma(\gamma)} \int_0^T (T-s)^{\gamma-1} \psi(s) ds = -2[x_r - \eta].$$

Hence, we obtain that

$$\psi(0) = -2x_r + 2\eta + \frac{1}{\Gamma(\gamma)} \int_0^T (T-s)^{\gamma-1} \psi(s) ds. \quad (13)$$

Therefore, by virtue of (13), we have

$$\psi(t) = -2x_r + 2\eta + \frac{1}{\Gamma(\gamma)} \int_0^T V_0(t, s) \psi(s) ds, \quad (14)$$

where

$$V_0(t, s) = \frac{1}{\Gamma(\gamma)} \begin{cases} (T-s)^{\gamma-1} - (t-s)^{\gamma-1}, & 0 \leq s < t, \\ -(T-s)^{\gamma-1}, & t \leq s \leq T. \end{cases}$$

Equation (14) is an integral equation of the second kind and has a unique solution $\psi(t, x_r) \in E'$, depending on x_r .

By differentiating the expression (10) with respect to the control function, we obtain an integral equation with respect to the control function

$$H_u(u(t), x(t)) = -2\mu \varepsilon(t) u(t) +$$

$$+ \psi(t, x_r) \int_0^t V'(t, s) g(s, x(s)) \int_0^s f_u(\theta, x(\theta), u(\theta)) d\theta ds. \quad (15)$$

We rewrite the equation (15) as an operator equation with respect to the control function

$$\begin{aligned} u(t) &= \mathbf{K}_3(t; x, x_r, u) \equiv \\ &\equiv a_0(t, x_r) \int_0^t V'(t, s) g(s, x(s)) \int_0^s f_u(\theta, x(\theta), u(\theta)) d\theta ds, \end{aligned} \quad (16)$$

where

$$a_0(t, x_r) = \frac{\psi(t, x_r)}{\mu \varepsilon(t)}.$$

The equation (16) is the third relation between the unknown functions $(x(t), x_r, u(t))$.

5 Key results

Let $C(\Omega, E) \times E \times C(\Omega, U)$ is a complete locally convex space with topological algebra. For each seminorm $p_{\alpha, 0, \beta} \in \mathcal{P}$ we define the family of seminorm on $C(\Omega, E) \times E \times C(\Omega, U)$:

$$\tilde{p}_{\alpha, 0, \beta}(x) = \sup_{t \in \Omega} p_{\alpha}(x(t)) + p_0(x(0)) + \sup_{t \in \Omega} q_{\beta}(u(t)).$$

Consider the right-hand side of three operators (7), (9) and (16): $\mathbf{K}_j(t; x, x_r, u)$, $j = 1, 2, 3$, as a system of equations.

Theorem 1. Assume that the following conditions are satisfied:

- 1) $a + b \neq 1$;
- 2) the functions g, f, f_u satisfy the inclusions:

$$g \in C(\Omega \times E, E), f \in C(\Omega \times E \times U, E), f_u \in C(\Omega \times E \times U, E)$$

and $\tilde{p}_{\alpha}(g) < \infty$, $\tilde{p}_{\alpha}(f) < \infty$, $\tilde{p}_{\alpha}(f_u) < \infty$;

- 3) for each seminorm $p_{\alpha} \in \mathcal{P}$ there exist constants $0 < L_1, L_{\kappa i}$, $\kappa = 2, 3$, $i = 1, 2$ such that the following inequalities hold:

$$p_{\alpha}(g(t, x_1) - g(t, x_2)) \leq L_1 p_{\alpha}(x_1 - x_2),$$

$$p_{\alpha, \beta}(f(t, x_1, u_1) - f(t, x_2, u_2)) \leq L_{21} p_{\alpha}(x_1 - x_2) + L_{22} q_{\beta}(u_1 - u_2),$$

$$p_{\alpha, \beta}(f_u(t, x_1, u_1) - f_u(t, x_2, u_2)) \leq L_{31} p_{\alpha}(x_1 - x_2) + L_{32} q_{\beta}(u_1 - u_2),$$

where q_{β} is a fixed seminorm in U ;

- 4) for each seminorm $p_{\alpha, 0, \beta}$, the following condition is satisfied

$$\tilde{\rho}_{\alpha, 0, \beta}(x(t), x_r, u(t)) = \tilde{\rho}_{\alpha}(x(t)) + \tilde{\rho}_0(x_r) + \tilde{\rho}_{\beta}(u(t)) = \chi_1 + \chi_2 + \chi_3 < 1,$$

where χ_1, χ_2, χ_3 are determined from (33), (21), (22), (24), (25) and (27)–(29) below.

Then the system of three equations (7), (9) and (16) has a unique set of solutions

$$\left\{ x^*(t) \in C(\Omega, E), \quad x_r^* \in E, \quad u^*(t) \in C(\Omega, U) \right\}.$$

Proof. We fix $\alpha \in \mathcal{A}$, $\beta \in \mathcal{B}$. Then, taking into account the conditions of the theorem, there the following estimate holds:

$$\begin{aligned} \tilde{p}_\alpha(\mathbf{K}_1 x(t)) &\leq C_1 p_0 \left(\frac{x_r}{a+b} \right) \sup_{t \in \Omega} \int_0^T p_\alpha(V(t, s)) p_\alpha(g(s, x(s))) \times \\ &\quad \times \int_0^s p_\alpha(f(\theta, x(\theta), u(\theta))) d\theta ds \leq \\ &\leq C_1 p_0 \left(\frac{x_r}{a+b} \right) \sup_{t \in \Omega} \int_0^T p_\alpha(V(t, s)) s ds p_\alpha(g(t, x(t))) p_\alpha(f(t, x(t), u(t))) = \delta_1 < \infty, \end{aligned} \quad (17)$$

where C_1, δ_1 are positive constants.

Similarly, we obtain the following estimate:

$$\begin{aligned} p_0(\mathbf{K}_2 x_r) &\leq C_2 p_0 \left(\frac{a+b}{a+b-1} \right) \int_0^T p_\alpha(V(t_1, s)) p_\alpha(g(s, x(s))) \times \\ &\quad \times \int_0^s p_\alpha(f(\theta, x(\theta), u(\theta))) d\theta ds \leq \\ &\leq C_2 b_0 \int_0^T p_\alpha(V(t_1, s)) s ds p_\alpha(g(t, x(t))) p_\alpha(f(t, x(t), u(t))) = \delta_2 < \infty, \end{aligned} \quad (18)$$

where C_2, δ_2 are positive constants, $b_0 = p_0 \left(\frac{a+b}{a+b-1} \right)$.

We also have that the estimate is valid

$$\begin{aligned} \tilde{q}_\beta(\mathbf{K}_3 u(t)) &\leq C_3 \sup_{t \in \Omega} p_0(a_0(t, x_r)) \int_0^t p_\alpha(V'(t, s)) p_\alpha(g(s, x(s))) \times \\ &\quad \times \int_0^s p_\alpha(f(\theta, x(\theta), u(\theta))) d\theta ds \leq \\ &\leq C_3 \sup_{t \in \Omega} p_\alpha(a_0(t, x_r)) \int_0^t p_\alpha(V'(t, s)) s ds p_\alpha(g(t, x(t))) p_\alpha(f(t, x(t), u(t))) = \delta_3 < \infty, \end{aligned} \quad (19)$$

where C_3, δ_3 are positive constants.

Now for two triples $(x_1(t), x_{r,1}, u_1(t))$ and $(x_2(t), x_{r,2}, u_2(t))$, using the Lipschitz conditions, we obtain

$$\tilde{p}_\alpha(\mathbf{K}_1 x_1(t) - \mathbf{K}_1 x_2(t)) \leq C_4 \left\{ p_0 \left(\frac{x_{r,1} - x_{r,2}}{a+b} \right) + \sup_{t \in \Omega} \int_0^T p_\alpha(V(t, s)) p_\alpha(g(s, x_1(s))) \times \right.$$

$$\begin{aligned}
 & \times \int_0^s [L_{21}p_\alpha(x_1(\theta) - x_2(\theta)) + L_{22}q_\beta(u_1(\theta) - u_2(\theta))] d\theta ds + \\
 & + \sup_{t \in \Omega} \int_0^T p_\alpha(V(t, s)) L_1 p_\alpha(x_1(s) - x_2(s)) \int_0^s p_\alpha(f(\theta, x_2(\theta), u_2(\theta))) d\theta ds \Big\} \leq \\
 & \leq C_4 \left\{ p_0 \left(\frac{x_{r,1} - x_{r,2}}{a + b} \right) + \sup_{t \in \Omega} \int_0^T p_\alpha(V(t, s)) s ds p_\alpha(g(t, x_1(t))) \times \right. \\
 & \quad \times [L_{21}p_\alpha(x_1(t) - x_2(t)) + L_{22}q_\beta(u_1(t) - u_2(t))] + \\
 & \quad \left. + \sup_{t \in \Omega} \int_0^T p_\alpha(V(t, s)) s ds p_\alpha(f(t, x_2(t), u_2(t))) L_1 p_\alpha(x_1(t) - x_2(t)) \right\} \leq \\
 & \leq C_4 p_0 \left(\frac{x_{r,1} - x_{r,2}}{a + b} \right) + \chi_{11} \tilde{p}_\alpha(x_1(t) - x_2(t)) + \chi_{12} \tilde{q}_\beta(u_1(t) - u_2(t)), \tag{20}
 \end{aligned}$$

where C_4 is a positive constant,

$$\chi_{11} = C_4 \sup_{t \in \Omega} \int_0^T p_\alpha(V(t, s)) s ds [L_{21}p_\alpha(g(t, x_1(t))) + L_1 p_\alpha(f(t, x_2(t), u_2(t)))], \tag{21}$$

$$\chi_{12} = C_4 L_{22} \sup_{t \in \Omega} \int_0^T p_\alpha(V(t, s)) s ds p_\alpha(g(t, x_1(t))). \tag{22}$$

Similarly, we obtain the estimate

$$\begin{aligned}
 p_0(x_{r,1} - x_{r,2}) & \leq C_5 \left\{ p_0 \left(\frac{a + b}{a + b - 1} \right) \int_0^T p_\alpha(V(t_1, s)) p_\alpha(g(s, x_1(s))) \times \right. \\
 & \quad \times \int_0^s [L_{21}p_\alpha(x_1(\theta) - x_2(\theta)) + L_{22}q_\beta(u_1(\theta) - u_2(\theta))] d\theta ds + \\
 & \quad \left. + p_0 \left(\frac{a + b}{a + b - 1} \right) \int_0^T p_\alpha(V(t_1, s)) L_1 p_\alpha(x_1(s) - x_2(s)) \int_0^s p_\alpha(f(\theta, x_2(\theta), u_2(\theta))) d\theta ds \right\} \leq \\
 & \leq C_5 \left\{ p_0 \left(\frac{a + b}{a + b - 1} \right) \int_0^T p_\alpha(V(t_1, s)) s ds p_\alpha(g(t, x_1(t))) \times \right. \\
 & \quad \times [L_{21}p_\alpha(x_1(t) - x_2(t)) + L_{22}q_\beta(u_1(t) - u_2(t))] + \\
 & \quad \left. + p_0 \left(\frac{a + b}{a + b - 1} \right) \int_0^T p_\alpha(V(t_1, s)) s ds p_\alpha(f(t, x_2(t), u_2(t))) L_1 p_\alpha(x_1(t) - x_2(t)) \right\} \leq \\
 & \leq \chi_{21} \tilde{p}_\alpha(x_1(t) - x_2(t)) + \chi_{22} \tilde{q}_\beta(u_1(t) - u_2(t)), \tag{23}
 \end{aligned}$$

where

$$\chi_{21} = C_5 b_0 \int_0^T p_\alpha(V(t_1, s)) s ds [L_{21} p_\alpha(g(t, x_1(t))) + L_1 p_\alpha(f(t, x_2(t), u_2(t)))] , \quad (24)$$

$$\chi_{22} = C_5 L_{22} b_0 \int_0^T p_\alpha(V(t_1, s)) s ds p_\alpha(g(t, x_1(t))) , \quad (25)$$

C_5 is a positive constant.

We continue the estimation process. Note that for the function $a_0(t, x_r)$ is true the estimate

$$p_\alpha(a_0(t, x_{r,1}) - a_0(t, x_{r,2})) \leq L_0 p_0(x_{r,1} - x_{r,2}), \quad 0 < L_0 = \text{const.}$$

Now for the third unknown function we obtain the estimate

$$\begin{aligned} \tilde{q}_\beta(u_1(t) - u_2(t)) &\leq C_6 \left\{ 3p_\alpha(a_0(t, x_{r,1})) \int_0^t p_\alpha(V'(t, s)) p_\alpha(g(s, x_1(s))) \times \right. \\ &\quad \times \int_0^s [L_{31} p_\alpha(x_1(\theta) - x_2(\theta)) + L_{32} q_\beta(u_1(\theta) - u_2(\theta))] d\theta ds + \\ &\quad + p_\alpha(a_0(t, x_{r,1})) \sup_{t \in \Omega} \int_0^t p_\alpha(V'(t, s)) L_1 p_\alpha(x_1(s) - x_2(s)) \int_0^s p_\alpha(f_u(\theta, x_2(\theta), u_2(\theta))) d\theta ds + \\ &\quad \left. + L_0 p_0(x_{r,1} - x_{r,2}) \sup_{t \in \Omega} \int_0^t p_\alpha(V'(t, s)) p_\alpha(g(s, x_2(s))) \int_0^s p_\alpha(f_u(\theta, x_2(\theta), u_2(\theta))) d\theta ds \right\} \leq \\ &\leq C_6 \left\{ 3p_\alpha(a_0(t, x_{r,1})) \sup_{t \in \Omega} \int_0^t p_\alpha(V'(t, s)) s ds p_\alpha(g(t, x_1(t))) \times \right. \\ &\quad \times [L_{31} p_\alpha(x_1(t) - x_2(t)) + L_{32} q_\beta(u_1(t) - u_2(t))] + \\ &\quad + L_1 p_\alpha(a_0(t, x_{r,1})) \sup_{t \in \Omega} \int_0^t p_\alpha(V'(t, s)) s ds p_\alpha(f_u(t, x_2(t), u_2(t))) p_\alpha(x_1(t) - x_2(t)) + \\ &\quad \left. + L_0 p_0(x_{r,1} - x_{r,2}) \sup_{t \in \Omega} \int_0^t p_\alpha(V'(t, s)) s ds p_\alpha(g(t, x_2(t))) p_\alpha(f_u(t, x_2(t), u_2(t))) \right\} \leq \\ &\leq \chi_{31} \tilde{p}_\alpha(x_1(t) - x_2(t)) + \chi_{32} p_0(x_{r,1} - x_{r,2}) + \chi_{33} \tilde{q}_\beta(u_1(t) - u_2(t)) , \end{aligned} \quad (26)$$

where $\tilde{a}_0 = p_\alpha(a_0(t))$,

$$\chi_{31} = 3C_6 p_\alpha(a_0(t, x_{r,1})) \sup_{t \in \Omega} \int_0^t p_\alpha(V'(t, s)) s ds p_\alpha(g(t, x_1(t))) , \quad (27)$$

$$\chi_{32} = C_6 L_0 \sup_{t \in \Omega} \int_0^t p_\alpha(V'(t, s)) s ds p_\alpha(g(t, x_1(t))) p_\alpha(f_u(t, x_2(t), u_2(t))) , \quad (28)$$

$$\chi_{33} = C_6 L_1 p_\alpha (a_0(t, x_{r,1})) \sup_{t \in \Omega} \int_0^t p_\alpha (V'(t, s)) s ds p_\alpha (f_u(t, x_2(t), u_2(t))), \quad (29)$$

C_6 is a positive constant.

We rewrite the estimates (20), (23) and (26) as follows:

$$\begin{aligned} \tilde{p}_\alpha (\mathbf{K}_1 x_1(t) - \mathbf{K}_1 x_2(t)) &\leq \\ &\leq \chi_1 [\tilde{p}_\alpha (x_1(t) - x_2(t)) + p_0 (x_{r,1} - x_{r,2}) + \tilde{q}_\beta (u_1(t) - u_2(t))], \end{aligned} \quad (30)$$

$$p_0 (\mathbf{K}_2 x_{r,1} - \mathbf{K}_2 x_{r,2}) \leq \chi_2 [\tilde{p}_\alpha (x_1(t) - x_2(t)) + p_0 (x_{r,1} - x_{r,2}) + \tilde{q}_\beta (u_1(t) - u_2(t))], \quad (31)$$

$$\begin{aligned} \tilde{q}_\beta (\mathbf{K}_3 u_1(t) - \mathbf{K}_3 u_2(t)) &\leq \\ &\leq \chi_3 [\tilde{p}_\alpha (x_1(t) - x_2(t)) + p_0 (x_{r,1} - x_{r,2}) + \tilde{q}_\beta (u_1(t) - u_2(t))], \end{aligned} \quad (32)$$

where

$$\chi_1 = \max \{p_0 (a + b), \chi_{11}, \chi_{12}\}, \quad \chi_2 = \max \{\chi_{21}, \chi_{22}\}, \quad \chi_3 = \max \{\chi_{31}, \chi_{32}, \chi_{33}\}. \quad (33)$$

Adding (30), (31) and (32) term by term, we obtain that

$$\begin{aligned} &\tilde{p}_\alpha (\mathbf{K}_1 x_1(t) - \mathbf{K}_1 x_2(t), \mathbf{K}_2 x_{r,1} - \mathbf{K}_2 x_{r,2}, \mathbf{K}_3 u_1(t) - \mathbf{K}_3 u_2(t)) = \\ &= \tilde{p}_\alpha (\mathbf{K}_1 x_1(t) - \mathbf{K}_1 x_2(t)) + p_0 (\mathbf{K}_2 x_{r,1} - \mathbf{K}_2 x_{r,2}) + \tilde{q}_\beta (\mathbf{K}_3 u_1(t) - \mathbf{K}_3 u_2(t)) \leq \\ &\leq \tilde{\rho}_\alpha \tilde{p}_\alpha (x_1(t) - x_2(t)) + \tilde{\rho}_0 p_0 (x_{r,1} - x_{r,2}) + \tilde{\rho}_\beta \tilde{q}_\beta (u_1(t) - u_2(t)) \leq \\ &\leq \tilde{\rho}_{\alpha,0,\beta} \tilde{p}_{\alpha,0,\beta} (x_1(t) - x_2(t), x_{r,1} - x_{r,2}, u_1(t) - u_2(t)), \end{aligned} \quad (34)$$

where

$$\tilde{\rho}_\alpha = \chi_1, \quad \tilde{\rho}_0 = \chi_2, \quad \tilde{\rho}_\beta = \chi_3, \quad \tilde{\rho}_{\alpha,0,\beta} = \chi_1 + \chi_2 + \chi_3.$$

Since $\tilde{\rho}_{\alpha,0,\beta} < 1$, it follows from the estimates (17)–(19), (34) that the operators given by the right-hand sides of equations (7), (9), and (16) are contracting with respect to seminorm of $\tilde{\rho}_{\alpha,0,\beta}$. Since the space $C(\Omega, E) \times E \times C(\Omega, U)$ is complete, from the estimates (17)–(19), (20), (23) and (26) imply the existence and uniqueness of the control function $u^*(t) \in C(\Omega, U)$, redefinition value $x_r^* \in E$ and the corresponding state function $x^*(t) \in C(\Omega, E)$. Denote the solution of the integral equation (16) by $u^*(t) = h(t)$. Then it is easy to establish the solvability of the differential equation (8). Theorem 1 is proved. $\square$

Theorem 2. Let the conditions of Theorem 1 be satisfied. Then the state function $x^*(t) \in E$ continuously depends on the boundary condition x_r^* .

Proof. In this case the equation (7) we rewrite as:

$$x(t) = \mathbf{K}_1(t; x, x_r, h) \equiv \frac{x_r}{a+b} + \int_0^T V(t, s) g(s, x(s)) \int_0^s f(\theta, x(\theta), h(\theta)) d\theta ds, \quad (35)$$

where $h(t)$ is known function.

Now we show the continuous dependence of the solution to problem (1)–(2) on the right-hand side of the boundary condition (2). Let $x_{r,1}, x_{r,2} \in E$ be two different numbers and the functions $x_1(t), x_2(t) \in C(\Omega, E)$ be the corresponding solutions of problem (1)–(2). Then, similarly to the estimates (20) and (23) from equation (35) we obtain that

$$p_\alpha (x_1(t) - x_2(t)) \leq p_0 (x_{r,1} - x_{r,2}) + (\chi_1 + \chi_2 + \chi_3) p_\alpha (x_1(t) - x_2(t)). \quad (36)$$

Since $\chi_1 + \chi_2 + \chi_3 < 1$, it follows from the inequality (36) that

$$p_\alpha(x_1(t) - x_2(t)) \leq \frac{p_0(x_{r,1} - x_{r,2})}{1 - (\chi_1 + \chi_2 + \chi_3)}.$$

The right-hand side tends to zero as $x_{r,1} \rightarrow x_{r,2}$, which proves the continuous dependence of the solution on the boundary value. Theorem 2 is proved. $\square$

Conclusion

In this paper, an inverse boundary optimal control problem for a nonlinear system of fractional-order integro-differential equations in a complete locally convex space has been investigated. The proposed approach combines the theory of fractional differential equations with methods of optimal control and functional analysis in locally convex spaces.

The original boundary value problem has been transformed into an equivalent coupled system of nonlinear integral equations involving the state function, the unknown boundary value, and the control function. By introducing an adjoint final-value problem, a Pontryagin-type function has been constructed, and a necessary optimality condition has been derived in the form of an integral equation for the control. Under suitable boundedness, continuity, and Lipschitz assumptions on the nonlinear operators, the existence and uniqueness of the solution to the resulting system have been established by means of the contraction mapping principle. Furthermore, the continuous dependence of the state function on the identified boundary value has been proved.

The obtained results considerably extend the applicability of optimal control techniques to fractional dynamical systems formulated in locally convex spaces, where the absence of a global norm requires the use of families of seminorms and appropriate topological arguments. The proposed framework is constructive and can be applied to a broad class of inverse problems involving nonlocal conditions and fractional operators.

Future research will be directed toward nonlinear locally convex algebras, optimal control problems with general nonlocal boundary conditions, impulsive and delay fractional systems, and distributed parameter models in infinite-dimensional locally convex spaces. Another promising direction is the development of numerical algorithms based on the proposed theoretical framework and the extension of the results to more general classes of fractional evolution equations.

Author contributions

All authors contributed equally to this work.

Conflict of interest

The authors declare no conflict of interest.

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Normality and superperfection of Jonsson spectra

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This paper investigates the structural properties of Jonsson spectra with respect to the notions of K_T -normality, cl -normality, and superperfection. Particular attention is paid to the role of the Kaiser class K_T as an internal structural layer of a Jonsson spectrum. In contrast to the classical approach based exclusively on the class of existentially closed models, a more general normalization framework for Jonsson spectra is developed through the consistency of Jonsson pregeometry with the Kaiser class. The study is carried out within the framework of Jonsson model theory and relies on methods of pregeometry and spectral analysis. The characteristic features of Jonsson spectra are examined from the standpoint of their interaction with structural closure operators and the Kaiser class. The notions of a K_T -normal Jonsson spectrum, a cl -normal Jonsson spectrum, an ideal Jonsson spectrum, and a K_T -beautiful pair are introduced and formally characterized. Criteria for the stability of Jonsson spectra in the superperfect case are obtained. Connections between spectrum normality, Jonsson pregeometry, central types, the heredity of Jonsson theories, and P -stability are established. The results contribute to the further development of the structural theory of Jonsson spectra and provide new tools for their investigation.

Keywords: Jonsson spectrum, Jonsson theory, Kaiser class, superperfection, K_T -normality, cl -normality, Jonsson pregeometry, central types, P -stability, beautiful pairs.

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Introduction

Jonsson theory is one of the most natural generalizations of the theory of complete first-order theories in modern model theory. It provides a framework for studying broad classes of incomplete theories while preserving many important semantic and geometric properties. During the last decades, methods of classical model theory have been successfully adapted to Jonsson theories, leading to the development of a rich theory of existentially closed models, semantic invariants, and geometric structures.

The study of Jonsson theories is based both on classical results in model theory obtained by B. Poizat [1], E. Bouscaren [2], E. Palyutin [3], and T.G. Mustafin [4], in which model-theoretic aspects of Jonsson theories were developed. Their investigations demonstrated that many classical notions of stability theory can be transferred to the context of Jonsson theories and provided the basis for studying Jonsson spectra, semantic models, and related geometric constructions.

A substantial contribution to the development of modern Jonsson theory has been made by A.R. Yeshkeyev. A systematic exposition of the theory, including semantic models, Jonsson spectra, central types, and positive Jonsson theories, was presented in the monograph [5]. This work summarizes the principal directions of the Karaganda school of model theory and serves as a foundation for further investigations.

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The study of companions of Jonsson sets was further developed in [6], where model-theoretic properties of the $\sharp$ -companion of a Jonsson set were investigated. These results provided new methods for analyzing semantic relations between Jonsson structures and their associated theories.

Further development of spectral methods was obtained in [7], where several classes of algebras of Jonsson spectra were introduced and studied. These constructions established new connections between algebraic and geometric properties of Jonsson spectra.

The structure of perfect classes associated with Jonsson spectra was investigated in [8], where the authors determined the number of fragments of a perfect class and described their fundamental properties. This direction was continued in [9], where the notion of 1-perfectness for fragments of normal Jonsson theories was studied.

An important role in modern Jonsson theory is played by positive Jonsson theories. Their amalgamation and joint embedding properties were analyzed in [10], providing additional tools for the study of existentially closed models and hereditary subclasses.

The interaction between positive logic and Jonsson theories was investigated by B. Poizat and A.R. Yeshkeyev in [11]. Their results established new back-and-forth techniques in positive logic and contributed to the understanding of semantic equivalence in positive Jonsson frameworks.

Recent investigations have focused on the Kaiser class and its role as a semantic invariant of Jonsson theories. In [12, 13], the Kaiser class K_T was studied as a fundamental component of the semantic structure of a Jonsson theory. The obtained results demonstrated that the Kaiser class provides a natural tool for analyzing geometric properties of Jonsson spectra and their associated semantic models.

Several related directions have also been developed within the Karaganda model-theoretic school. Semantic pairs and existentially fragmentally complete predicates in Jonsson spectra were investigated in [14]. Jonsson existentially closed unars of expanded signatures were studied in [15]. The properties of 1-model companions of Jonsson sets in normal Jonsson theories were analyzed in [16], while companions of hybrids of fragments of theoretical subsets were considered in [17]. More recently, syntactic similarity of hybrids of classes of normal Jonsson theories was investigated in [18].

These results collectively demonstrate the formation of a coherent geometric approach to the study of Jonsson theories, their spectra, semantic models, central types, and Kaiser classes. The present paper continues this line of research and is devoted to the investigation of normality and superperfection of Jonsson spectra.

Our main objective is to study the interaction between the Kaiser class, Jonsson pregeometry, and central types, and to describe conditions under which these structures become compatible. We introduce the notions of K_T -normality, cl -normality, ideal Jonsson spectra, and K_T -beautiful pairs. Particular attention is paid to superperfect Jonsson spectra characterized by the equality $K_T = E_T$, which establishes a strong connection between the Kaiser class and the internal geometry of the spectrum.

The obtained results contribute to the further development of geometric methods in Jonsson model theory and provide new tools for investigating hereditary structures, central types, semantic pairs, and spectral invariants associated with Kaiser classes.

1 Basic definitions and preliminary results

In this section we introduce the main notions and recall the basic structural framework used throughout the paper.

Let T be a Jonsson theory in a language L .

Recall that a Jonsson theory is defined as follows.

Definition 1. [19, p. 80] A theory T is called a *Jonsson theory* if the following conditions are satisfied:

1. The theory T has at least one infinite model;
2. T is *inductive*, that is, it is equivalent to a set of sentences of the form $\forall\exists$;
3. T has the *joint embedding property (JEP)*, meaning that for any two models $A, B \models T$, there exists a model $C \models T$ into which both A and B can be embedded isomorphically;
4. T has the *amalgamation property (AP)*: for any models $A, B_1, B_2 \models T$ and isomorphic embeddings $f_1: A \rightarrow B_1$ and $f_2: A \rightarrow B_2$, there exists a model $C \models T$ together with isomorphic embeddings $g_1: B_1 \rightarrow C$ and $g_2: B_2 \rightarrow C$ such that $g_1 f_1 = g_2 f_2$.

The semantic model plays a fundamental role in the study of Jonsson theories and serves as their principal semantic invariant.

Definition 2 (Ye.T. Mustafin [20]). A model C_T of a Jonsson theory T is called a semantic model of T if

$$|C_T| = 2^\omega$$

and C_T is an ω^+ -universal and ω^+ -homogeneous model of the theory T .

The semantic model determines the complete theory that serves as the main syntactic invariant of a Jonsson theory.

Definition 3 (T.G. Mustafin [5]). The theory

$$T^* = Th(C_T)$$

is called the center of the Jonsson theory T .

Since the center is a complete theory, many methods of classical model theory can be transferred to the study of Jonsson theories through their centers. An important subclass of Jonsson theories is formed by perfect theories.

Definition 4. [5] A Jonsson theory T is called perfect if its semantic model C_T is ω^+ -saturated.

We now define the Jonsson spectrum of a model, which describes the collection of Jonsson theories realized by a given structure. Previously, a number of results concerning Jonsson spectra, which generalize the concept of Jonsson theories, had already been obtained. The notion of a Jonsson spectrum was first introduced by A.R. Yeshkeyev [5].

Throughout the paper, let L be a first-order language, let σ be its signature, and let K be a class of σ -structures. We use the notation $A \models T$ to indicate that a structure A is a model of the theory T .

Definition 5 (A.R. Yeshkeyev [5]). The following set of Jonsson theories is called the *Jonsson spectrum* of the class K :

$$JSp(K) = \{T \mid T \text{ is a Jonsson theory and } A \models T \text{ for every } A \in K\}.$$

The notion of a Jonsson spectrum has been extensively studied in connection with fragments, companions, and algebraic structures associated with classes of Jonsson theories [6–8].

According to the semantic approach introduced by T.G. Mustafin, the properties of a Jonsson theory are determined by its semantic model. Therefore, Jonsson theories having the same semantic model are naturally considered as belonging to the same cosemantic class.

The relation of cosemanticity is an equivalence relation on $JSp(K)$. Hence, the quotient set

$$JSp(K)/\simeq$$

consists of the cosemantic classes of Jonsson theories.

Let $[T]$ denote the cosemantic class containing the theory T , where $T \in JSp(K)$. Since all theories in the class $[T]$ share the same semantic model, they also have the same center. We denote the common semantic model of the class by $C_{[T]}$, and its complete theory by

$$[T]^* = Th(C_{[T]}),$$

which is called the center of the Jonsson class.

Next, we recall the standard notion of existentially closed models associated with a Jonsson theory. By E_T we denote the class of all existentially closed models of the theory T .

We now introduce the structural notion of the Kaiser class, which plays a central role in the present work.

Definition 6 (A.R. Yeshkeyev [13]). Let T be a Jonsson theory. A class K_T of L -structures is called a *Kaiser class* of T if

$$K_T = \{M \mid M \models T \text{ and } Th_{\forall\exists}(M) \text{ is a Jonsson theory}\}.$$

We next introduce the operator of Jonsson pregeometry, which provides the geometric structure on the semantic model.

Let $cl : \mathcal{P}(C_T) \rightarrow \mathcal{P}(C_T)$ be the operator of Jonsson pregeometry.

Definition 7. [5] Let T be a Jonsson theory, and let $X \subseteq C_T$. The pair (X, cl) is called a Jonsson pregeometry τ , if there exists a closure operator $cl : \mathcal{P}(C_T) \rightarrow \mathcal{P}(C_T)$ such that the following conditions hold:

- (1) If $A \subseteq X$, then $A \subseteq cl(A)$ and $cl(cl(A)) = cl(A)$;
- (2) If $A \subseteq B \subseteq X$, then $cl(A) \subseteq cl(B)$;
- (3) (Exchange) If $A \subseteq X$, $a, b \in X$, and $a \in cl(A \cup \{b\}) \setminus cl(A)$, then $b \in cl(A \cup \{a\})$;
- (4) (Finite character) If $A \subseteq X$ and $a \in cl(A)$, then there exists a finite subset $A_0 \subseteq A$ such that $a \in cl(A_0)$.

Using this operator, we define several notions of normality for classes of Jonsson theories.

Definition 8 (A.R. Yeshkeyev). A class of Jonsson theories $[T]$ is called K_T -normal if for any models $M_1, M_2 \in K_T$ there exists a model $N \in K_T$, into which M_1 and M_2 embed isomorphically, and the Jonsson- cl -closures are preserved under these embeddings.

Definition 9 (A.R. Yeshkeyev). A class of Jonsson theories $[T]$ is called cl -normal if for any Jonsson subsets $A, B \subseteq C_T$ the following condition holds: $cl(A \cap B) = cl(A) \cap cl(B)$.

cl -normality is an internal consistency condition of a Jonsson pregeometry. It guarantees the consistent behavior of the Jonsson closure operator under intersections of Jonsson sets.

We now introduce the notion of an ideal class of Jonsson theories, which captures the compatibility between the two types of normality.

Definition 10 (A.R. Yeshkeyev). A class of Jonsson theories $[T]$ is called ideal if K_T -normality $\iff cl$ -normality.

Thus, ideality means the coincidence of two distinct geometries of the Jonsson spectrum:

- * the geometry induced by the Kaiser class;
- * the geometry induced by the Jonsson pregeometry.

Finally, we define the important special case of superperfect Jonsson theories.

Definition 11 (A.R. Yeshkeyev). A Jonsson theory T is called superperfect if $K_T = E_T$.

Note that superperfection does not imply the trivialization of the Jonsson spectrum.

On the contrary, the coincidence of K_T and E_T stabilizes the Jonsson spectrum and makes the Jonsson pregeometry consistent with the Kaiser class.

In the imperfect case, the inclusion

$$K_T \subseteq E_T$$

may be proper, and then the Jonsson spectrum loses its internal geometric consistency.

2 Superperfect Jonsson spectra

The notion of superperfection occupies a special place in the study of Jonsson spectra. The importance of this concept is closely related to the Kaiser class, which serves as a refined semantic invariant of a Jonsson theory. In contrast to the class of existentially closed models, the Kaiser class reflects a broader range of structural properties and allows one to investigate the interaction between semantic and geometric aspects of Jonsson spectra.

In the superperfect case, the distinction between the Kaiser class and the class of existentially closed models disappears. This fact significantly simplifies the internal structure of the spectrum and creates a natural connection between the center of a Jonsson theory, its semantic model, and the corresponding closure structures. As a consequence, several normality conditions admit a more transparent description and can be studied within a common geometric framework.

Throughout this section, we investigate the relationship between superperfection, normality conditions, and Jonsson pregeometry. The results obtained below demonstrate that superperfect spectra possess a particularly stable structure and provide a natural setting for the study of hereditary properties and stability-theoretic phenomena.

Recall that a Jonsson theory is called perfect if its semantic model is saturated.

Theorem 1. Let $[T]$ be a superperfect class of Jonsson theories. Then $\text{Mod}(T^*) = K_T = E_T$.

Proof. Since $[T]$ is superperfect, by definition of superperfection, $K_T = E_T$.

On the other hand, for a perfect Jonsson theory it is known that the class of existentially closed models coincides with the class of models of the center. Therefore, $E_T = \text{Mod}(T^*)$.

Combining these equalities, we obtain $\text{Mod}(T^*) = K_T = E_T$.

Thus the center, the Kaiser class, and the class of existentially closed models determine the same semantic layer of the spectrum. $\square$

Recall that a central type is a complete 1-type obtained from an admissible enrichment of a Jonsson theory. Central types were introduced by A.R. Yeshkeyev and play an important role in the study of hereditary Jonsson theories, semantic pairs, and admissible enrichments [5]. Since central types are closely related to the center T^* of a Jonsson theory, their realizability reflects essential structural properties of the corresponding Jonsson spectrum.

Corollary 1. In a superperfect class of Jonsson theories, every central type is realized inside K_T .

Proof. Since $K_T = E_T$, every existentially closed model belongs to the Kaiser class.

Central types are realized in existentially closed models. Therefore all realizations of central types belong to K_T . $\square$

The obtained result shows that, in the superperfect case, the Kaiser class becomes the natural carrier of central types.

In particular, central types cease to depend on the choice of a particular existentially closed model and become stabilized with respect to the entire K_T -structure.

Theorem 2. Let $[T]$ be a superperfect class of Jonsson theories. Then the following conditions are equivalent:

1. $[T]$ is ideal;
2. $[T]$ is both K_T -normal and cl -normal;
3. Jonsson pregeometry is consistent with K_T .

Proof. $(1 \Rightarrow 2)$. Assume that $[T]$ is ideal. By the definition of an ideal class of Jonsson theories, K_T -normality $\iff cl$ -normality.

Hence, whenever one of these properties holds, the other also holds. Therefore the spectrum is simultaneously K_T -normal and cl -normal.

$(2 \Rightarrow 3)$. Assume that $[T]$ is both K_T -normal and cl -normal.

It follows from K_T -normality that any Jonsson embeddings within K_T preserve the Jonsson- cl structure.

Indeed, if

$$M_1, M_2 \in K_T,$$

then there exists a model

$$N \in K_T,$$

into which M_1 and M_2 Jonsson-embed with preservation of Jonsson- cl closures.

Furthermore, cl -normality guarantees stability of the closure operator under intersections of Jonsson sets:

$$cl(A \cap B) = cl(A) \cap cl(B).$$

Therefore the closure structure behaves coherently with respect to the basic operations of Jonsson pregeometry. Hence Jonsson pregeometry is consistent with the Kaiser class.

$(3 \Rightarrow 1)$. Suppose that Jonsson pregeometry is consistent with K_T .

Consistency implies that embeddings inside the Kaiser class preserve the closure structure induced by the operator cl . As a consequence, K_T -normality yields cl -normality.

Conversely, preservation of the closure structure guarantees that the geometric behavior of the spectrum determines the corresponding closure-preserving embeddings in K_T . Therefore cl -normality implies K_T -normality.

Hence, K_T -normality $\iff cl$ -normality.

Thus $[T]$ is an ideal class of Jonsson theories. □

3 Structural properties of normal Jonsson spectra

The notion of a beautiful pair plays an important role in the study of stability-theoretic properties of model-theoretic structures. In the context of Jonsson theories, the Kaiser class provides a natural environment for the construction of beautiful pairs compatible with Jonsson pregeometry and central types. The following definition adapts the classical concept to the framework of superperfect Jonsson spectra.

Definition 12 (A.R. Yeshkeyev). A pair (C_T, M) is called a K_T -beautiful pair if:

1. $M \in K_T$;
2. $cl(M) = M$;
3. every central type over M is realized in C_T .

Theorem 3. Let $[T]$ be a superperfect ideal Jonsson class. Then every K_T -beautiful pair is hereditary.

Proof. Since $[T]$ is superperfect, by Theorem 1 we have $K_T = E_T$. Hence every model belonging to the Kaiser class is existentially closed.

Furthermore, since $[T]$ is ideal, K_T -normality $\iff cl$ -normality.

Therefore the closure structure induced by Jonsson pregeometry is preserved inside the Kaiser class.

Let (C_T, M) be a K_T -beautiful pair. By definition, M belongs to K_T , is cl -closed, and every central type over M is realized in the semantic model C_T .

Consider an admissible enrichment of the language. Since admissible enrichments preserve the Jonsson nature of the theory, the corresponding enriched theory remains within the same semantic framework. Moreover, the closure operator associated with Jonsson pregeometry remains compatible with the Kaiser class.

Since $cl(M) = M$, the model M remains cl -closed after enrichment. Realizability of central types is also preserved because all realizations remain inside the semantic model.

Consequently:

- M remains in K_T ;
- the Jonsson- cl structure is preserved;
- realizability of central types is preserved.

Therefore all defining properties of a K_T -beautiful pair are preserved under admissible enrichments. Hence the enriched theory remains a Jonsson theory and the pair remains hereditary. $\square$

The notion of P -stability extends classical stability-theoretic ideas to the setting of Jonsson spectra. In the superperfect case, the coincidence of the Kaiser class, the class of existentially closed models, and the class of models of the center allows one to describe P -stability in terms of the center of the corresponding Jonsson theory.

The definition of P -stability is based on an admissible enrichment of the language. Let

$$\sigma_\Gamma = \sigma \cup \{P\} \cup \{c\},$$

where P is a unary predicate symbol and c is a constant symbol. The interpretation of P is assumed to be an existentially closed submodel of the semantic model C_T . We denote by $S_P(A)$ the set of complete P -types over a Jonsson set A .

Definition 13. A class of Jonsson theories $[T]$ is called P -stable if for any Jonsson set $A \subseteq C_T$ the following holds:

$$|S_P(A)| \leq |A| + \omega.$$

Theorem 4. Let $[T]$ be a superperfect ideal class of Jonsson theories. Then the P -stability of the spectrum is determined by the center T^* .

Proof. Since $[T]$ is superperfect, we have

$$K_T = E_T = \text{Mod}(T^*).$$

Hence, all central types are realized within a single K_T -layer. Since $[T]$ is ideal, the Jonsson pregeometry is consistent with the Kaiser class. Therefore, any P -types are determined solely by the structure of the center T^* .

Now assume that the center T^* is stable. Then the number of types over any set is bounded by the cardinality of that set.

Since all Jonsson- P -types are induced by types of the center, we obtain:

$$|S_P(A)| \leq |A| + \omega.$$

Hence, $[T]$ is P -stable. Conversely, if $[T]$ is P -stable, then the number of central types is also bounded. Therefore, the center T^* is stable.

Thus,

$$[T] \text{ is } P\text{-stable} \iff T^* \text{ is stable.}$$

$\square$

Recall that a central type arises from an admissible enrichment of a Jonsson theory. It is precisely central types that allow one to control the behavior of the Jonsson spectrum with respect to admissible enrichments.

Definition 14 (A.R. Yeshkeyev). A central type p_c is called K_T -normal if all its realizations belong to models of the Kaiser class.

Definition 15 (A.R. Yeshkeyev). A central type p_c is called cl -normal if for any Jonsson set A the following holds:

$$cl(A \cup p_c) = cl(A) \cup cl(p_c).$$

Theorem 5. Let $[T]$ be an ideal class of Jonsson theories. Then every K_T -normal central type is cl -normal.

Proof. Since $[T]$ is ideal, K_T -normality $\iff cl$ -normality.

Let p_c be a K_T -normal central type. Then all of its realizations belong to models of the Kaiser class.

Since the Jonsson pregeometry is consistent with K_T , the Jonsson- cl closure is preserved with respect to all realizations of p_c .

Therefore,

$$cl(A \cup p_c) = cl(A) \cup cl(p_c).$$

Hence, p_c is cl -normal. □

Conclusion

In this paper, we investigated normality conditions for Jonsson spectra in the framework of the Kaiser class and Jonsson pregeometry.

The main objective of the study was to analyze the interaction between the Kaiser class, closure operators, central types, and stability-theoretic properties of Jonsson spectra. Particular attention was devoted to superperfect spectra, where the coincidence of the Kaiser class and the class of existentially closed models provides a natural semantic stabilization.

We showed that the Kaiser class constitutes a distinguished structural layer of a Jonsson spectrum and serves as a convenient environment for the study of normality conditions. The relationship between K_T -normality and cl -normality was investigated, and it was established that ideal Jonsson spectra are characterized by the compatibility of these two notions.

Furthermore, we demonstrated that superperfection leads to a strong connection between the center T^* , the Kaiser class, and existentially closed models. This connection allows one to describe stability-theoretic properties of the spectrum through the center of the corresponding Jonsson theory.

The notions of K_T -beautiful pairs and normal central types were studied within this framework. It was shown that hereditary properties of K_T -beautiful pairs are preserved under admissible enrichments, while normal central types remain compatible with the closure structure induced by Jonsson pregeometry.

The obtained results indicate that Jonsson spectra possess a rich internal geometry determined by the interaction between the Kaiser class, central types, and closure operators. These methods may be further applied to the study of hereditary Jonsson theories, admissible enrichments, and stability phenomena in classes of cosemantic Jonsson theories.

Author contributions

All authors contributed equally to the development of the study, including the formulation of main results, analysis, and writing of the manuscript. All authors have read and approved the final version.

Conflict of interest

The authors declare no conflict of interest.

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Linear maps on Banach algebras which are Lie centralizers at a point

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In this paper, we study continuous linear maps that behave as Lie centralizers at specific points, including identity products, zero products, and idempotent products. Our aim is to determine when local Lie centralizer conditions force a linear map to have a global centralizer-like structure. We first show that every continuous linear map satisfying the Lie centralizer condition at identity products is necessarily a commuting map. We then obtain explicit descriptions of such map on C^* -algebras, von Neumann algebras, standard operator algebras, and matrix algebras. For maps acting at zero products, we prove that under suitable assumptions they are Lie centralizers, and we derive corresponding structural characterizations. We also investigate Lie centralizers at idempotent products and establish several characterization theorems for von Neumann algebras and standard operator algebras. These results extend a number of earlier theorems on commuting maps, centralizers, and Lie centralizers, and provide new criteria under which local conditions imply global behavior. Moreover, our approach highlights the strong relationship between local algebraic conditions and the global structure of linear maps. Consequently, this paper contributes to the theory of linear preserver problems and centralizer-type mappings on Banach and operator algebras.

Keywords: centralizer, Lie centralizer, Lie triple centralizer, commuting map, C^* -algebra, von Neumann algebras, standard operator algebras, matrix algebras.

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Introduction

Let A be an associative algebra over $\mathbb{C}$ with center $Z(A)$. A linear map θ from algebra A into itself is called a *left centralizer* (*right centralizer*) if for all $a, b \in A$,

$$\theta(ab) = \theta(a)b, \quad (\theta(ab) = a\theta(b)),$$

and θ is called a *centralizer* if it is both left and right centralizer. In case $\theta : A \rightarrow A$ is a centralizer, then by [1, Theorem 2.3.2] there exists an element $\lambda \in C$ such that $\theta(a) = \lambda a$ for all $a \in A$, where C is the extended centroid of A .

By considering a new product $[a, b] = ab - ba$, called the *Lie product*, A takes on a structure of a Lie algebra. A linear map $\theta : A \rightarrow A$ is called a *Lie centralizer* if

$$\theta([a, b]) = [\theta(a), b], \quad a, b \in A. \quad (1)$$

Note that condition (1) is equivalent to $\theta([a, b]) = [a, \theta(b)]$ for all $a, b \in A$. A simple example of a Lie centralizer $\theta : A \rightarrow A$ is a map of the form

$$\theta(a) = \phi(a) + \xi(a), \quad a \in A,$$

where ϕ is a centralizer, and ξ is a linear map of A into $Z(A)$ vanishing at commutators $[a, b]$ for all $a, b \in A$. In particular, if $\phi(a) = ca$ for some $c \in Z(A)$, then $\theta(a) = ca + \xi(a)$ for all $a \in A$. The goal is to show that in certain cases these are in fact the only examples.

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A linear map θ from algebra A into itself is called *centralizing* if $[\theta(a), a] \in Z(A)$ for every $a \in A$. In the special case where $[\theta(a), a] = 0$ for all $a \in A$, the map θ is said to be a *commuting map*.

It is shown in [2, Theorem 1] that every centralizing map on a prime algebra A is a commuting map. Recall that A is called *prime* if $aAb = 0$ implies that $a = 0$ or $b = 0$.

Obviously, each Lie centralizer is a commuting map, but the converse fails in general; see Example 1 below. Moreover, every centralizer is a special case of a Lie centralizer, however, there exist Lie centralizers which are not centralizers, [3, Example 2.1].

Determining the structure of commuting maps in terms of centralizers or Lie centralizers is a matter of interest. In [4], Brešar proved that every commuting additive map θ on a prime ring A is of the form $\theta(a) = \lambda a + \xi(a)$ for all $a \in A$, where $\lambda \in C$ and ξ is an additive map from A into C . We refer the reader to the survey paper [5] for the background and history of commuting maps.

We say that a map θ is a Lie centralizer at a given point $z \in A$ if

$$a, b \in A, \quad ab = z \quad \implies \quad \theta([a, b]) = [\theta(a), b]. \quad (2)$$

This kind of maps were discussed by several authors. For example, it is shown [6] that if A is a prime ring containing a non-trivial idempotent and θ is an additive map satisfying $\theta(ab) = a\theta(b)$ for all $a, b \in A$ with $ab = 0$, then θ is a right centralizer. The same result was obtained [7] for unital C^* -algebra A and the continuous linear map $T : A \rightarrow X$ where X is a unital Banach A -bimodule. Article [8] investigated this problem for Banach algebras at the identity products.

Lie centralizers at zero products on a class of operator algebras are discussed in [3]. In [9], the authors have studied Lie centralizers on triangular rings. In [10], Ghahramani considered the problem of characterizing linear maps behaving like Lie centralizers at idempotent products. We refer the reader to [8, 11, 12] for a full account of the topic and a list of references.

Taking the ideas from the above, the aim of this paper is to characterize linear maps that satisfy in condition (2) at the identity products, zero products and idempotent products.

1 Lie centralizer at identity

We mention that every Lie centralizer is a commuting map, but the converse fails, in general. The following example illustrates this fact.

Example 1. Let

$$A = \left\{ \begin{bmatrix} 0 & z_1 & z_2 \\ 0 & 0 & z_3 \\ 0 & 0 & 0 \end{bmatrix} : z_1, z_2, z_3 \in \mathbb{C} \right\}.$$

Define $\theta : A \rightarrow A$ via

$$\theta \left(\begin{bmatrix} 0 & z_1 & z_2 \\ 0 & 0 & z_3 \\ 0 & 0 & 0 \end{bmatrix} \right) = \begin{bmatrix} 0 & z_1 & -z_2 \\ 0 & 0 & z_3 \\ 0 & 0 & 0 \end{bmatrix}.$$

Then $[\theta(a), a] = 0$ for all $a \in A$, and hence θ is a commuting map, however, θ is not a Lie centralizer.

The following result is the main theorem of [4].

Theorem 1. Let R be a prime ring and $\theta : R \rightarrow R$ be an additive centralizing map. If either $\text{char} R \neq 2$ or θ is a commuting map, then

$$\theta(a) = \lambda a + \xi(a), \quad a \in A,$$

where $\lambda \in C$, the extended centroid of R , and ξ is an additive map from A into C .

Combining Theorem 1 with the fact that the extended centroid of a prime C^* -algebra is isomorphic to the complex numbers, we see that if θ is an additive centralizing map on a prime C^* -algebra A , then there exist $\lambda \in Z(A)$ and an additive map $\xi : A \rightarrow Z(A)$ such that $\theta(a) = \lambda a + \xi(a)$, for all $a \in A$.

The next result states that this conclusion holds in general (not necessarily prime) von Neumann algebras.

Theorem 2 ([13, Theorem 2.1]). Let A be a von Neumann algebra. If $\theta : A \rightarrow A$ is an additive centralizing map, then $\theta(a) = ca + \xi(a)$ for all $a \in A$, where $c \in Z(A)$ and ξ is an additive map from A into $Z(A)$.

The second dual space A'' of a Banach algebra A admits two Banach algebra products known as the first and second Arens products, each extending the product on A . These products, which we denote by $\square$ and $\diamond$, respectively, can be defined as follows:

$$\Phi \square \Psi = \lim_i \lim_j a_i \cdot b_j, \quad \Psi \diamond \Phi = \lim_j \lim_i b_j \cdot a_i, \quad \Phi, \Psi \in A'',$$

where $\{a_i\}_{i \in I}$ and $\{b_j\}_{j \in I}$ are nets in A that converge in the w^* -topology to Φ and Ψ , respectively. If these products coincide on A'' , then A is said to be Arens regular. It is shown in [14] that every C^* -algebra A is Arens regular.

We regard A as a closed subalgebra of both $(A'', \square)$ and $(A'', \diamond)$ by identifying A with $k(A)$, where $k : A \rightarrow A''$ is the canonical embedding, that is

$$k(a)(f) = \widehat{a}(f) = f(a), \quad a \in A, f \in A',$$

and A is w^* -dense in A'' by Goldstine's theorem. In this regard, we have

$$\widehat{a} \square \Phi = \widehat{a} \diamond \Phi = \widehat{a} \cdot \Phi, \quad \text{and} \quad \Phi \square \widehat{a} = \Phi \diamond \widehat{a} = \Phi \cdot \widehat{a},$$

for all $a \in A$ and $\Phi \in A''$. One may refer to the monograph of Dales [14] for a full account of Arens product.

Next we generalize Theorem 2 for all C^* -algebras.

Theorem 3. Suppose that A is a C^* -algebra. If $\theta : A \rightarrow A$ is a continuous centralizing map, then $\theta(a) = ca + \xi(a)$ for all $a \in A$, where $c \in Z(A'')$ and ξ is an additive map from A into $Z(A'')$. In particular, θ is a commuting map.

Proof. Since θ is a centralizing map, we have $[\theta(a), a] \in Z(A)$ for all $a \in A$. Replacing a by $a + b$, we obtain

$$[\theta(a), b] + [\theta(b), a] \in Z(A), \quad a, b \in A.$$

Let $\Phi, \Psi \in A''$ and $\widehat{a}_i \rightarrow \Phi, \widehat{b}_j \rightarrow \Psi$ in the w^* -topology. Then for each $f \in A'$,

$$\begin{aligned} (\theta^{**}(\widehat{a}_i))f &= (\widehat{a}_i \circ \theta^*)f \\ &= \widehat{a}_i(\theta^*(f)) \rightarrow \Phi(\theta^*(f)) = (\theta^{**}(\Phi))f. \end{aligned}$$

Therefore, $\theta^{**}(\widehat{a}_i) \rightarrow \theta^{**}(\Phi)$ in the w^* -topology. Consequently,

$$\begin{aligned} [\theta^{**}(\Phi), \Psi] + [\theta^{**}(\Psi), \Phi] &= \lim_i \lim_j [\theta^{**}(\widehat{a}_i), \widehat{b}_j] + [\theta^{**}(\widehat{b}_j), \widehat{a}_i] \\ &= \lim_i \lim_j [\widehat{\theta(a_i)}, \widehat{b_j}] + [\widehat{\theta(b_j)}, \widehat{a_i}] \in Z(A''). \end{aligned}$$

In particular, $[\theta^{**}(\Phi), \Phi] \in Z(A'')$ for all $\Phi \in A''$. This means that θ^{**} is a centralizing map. It is known that the second dual of each C^* -algebra is a von Neumann algebra [14], hence by Theorem 2, we have

$$\theta^{**}(\Phi) = c \square \Phi + \rho(\Phi), \quad \Phi \in A'', \quad (3)$$

where $c \in Z(A'')$ and ρ is an additive map from A'' into $Z(A'')$.

Let ξ be the restriction of ρ on A , so $\xi(a) = \rho(a)$ for all $a \in A$. Therefore, by (3) we obtain

$$\theta(a) = ca + \xi(a),$$

for all $a \in A$, where $c \in Z(A'')$ and ξ is an additive map from A into $Z(A'')$.

Since $c, \xi(a) \in Z(A'')$, by the above equality we conclude that

$$\begin{aligned} [\theta(a), a] &= [ca + \xi(a), a] \\ &= [ca, a] + [\xi(a), a] = 0, \end{aligned}$$

for all $a \in A$. Therefore, θ is a commuting map. This completes the proof. $\square$

As a consequence of Theorem 3, we obtain the following result.

Corollary 1. Let A be a C^* -algebra and $\theta : A \rightarrow A$ be a continuous linear map. Then θ is a Lie centralizer if and only if $\theta(a) = ca + \xi(a)$ for all $a \in A$, where $c \in Z(A'')$ and ξ is a linear map from A into $Z(A'')$ such that $\xi([a, b]) = 0$ for all $a, b \in A$.

Next we prove that every Lie centralizer at the identity product is a commuting map.

Theorem 4. Let A be a unital Banach algebra and $\theta : A \rightarrow A$ be a continuous linear map satisfying

$$a, b \in A, \quad ab = e_A \quad \implies \quad \theta([a, b]) = [\theta(a), b]. \quad (4)$$

Then θ is a commuting map.

Proof. Let $a \in A$ be arbitrary. For $\lambda \in \mathbb{C}$ with $|\lambda| < 1/\|a\|$, $e_A - \lambda a$ is invertible and $(e_A - \lambda a)^{-1} = \sum_{n=0}^{\infty} \lambda^n a^n$. Noticing that $(e_A - \lambda a)(e_A - \lambda a)^{-1} = e_A$, thus by condition (4) we have

$$\theta([e_A - \lambda a, \sum_{n=0}^{\infty} \lambda^n a^n]) = [\theta(e_A - \lambda a), \sum_{n=0}^{\infty} \lambda^n a^n].$$

Obviously,

$$\theta([e_A - \lambda a, \sum_{n=0}^{\infty} \lambda^n a^n]) = 0, \quad a \in A.$$

Therefore,

$$\begin{aligned} 0 &= [\theta(e_A) - \lambda\theta(a), \sum_{n=0}^{\infty} \lambda^n a^n] \\ &= \sum_{n=0}^{\infty} \lambda^n [\theta(e_A), a^n] - \sum_{n=0}^{\infty} \lambda^{n+1} [\theta(a), a^n] \\ &= \sum_{n=0}^{\infty} \lambda^{n+1} [\theta(e_A), a^{n+1}] - \sum_{n=0}^{\infty} \lambda^{n+1} [\theta(a), a^n]. \end{aligned}$$

Thus,

$$\sum_{n=0}^{\infty} \lambda^{n+1} ([\theta(e_A), a^{n+1}] - [\theta(a), a^n]) = 0,$$

for all $\lambda \in \mathbb{C}$ with $|\lambda| < 1/\|a\|$. Hence

$$[\theta(e_A), a^{n+1}] = [\theta(a), a^n], \quad n = 0, 1, 2, \dots \quad (5)$$

Taking $n = 0$, we obtain $[\theta(e_A), a] = 0$ for all $a \in A$. Now it follows from (5) that

$$[\theta(a), a] = 0, \quad a \in A.$$

Consequently, θ is a commuting map. □

Let $B(X)$ be the operator algebra of all bounded linear operators on Banach space X . A *standard operator algebra* is any subalgebra of $B(X)$ which contains the identity, and the ideal $F(X)$ of all finite rank operators.

We mention that standard operator algebra and $M_n(\mathbb{C})$ are prime algebras, but C^* -algebras are not prime, in general.

Combining Theorem 4 and Theorem 1, we obtain the next result.

Corollary 2. Let A be either standard operator algebra or $M_n(\mathbb{C})$ and let $\theta : A \rightarrow A$ be a continuous linear map satisfying (4). Then θ is a commuting map, and

$$\theta(a) = \lambda a + \xi(a),$$

for all $a \in A$, where $\lambda \in \mathbb{C}$ and ξ is a linear map of A into $\mathbb{C}e_A$.

Corollary 3. Let A be a unital C^* -algebra and let $\theta : A \rightarrow A$ be a continuous linear map such that the condition (4) holds. Then

$$\theta(a) = ca + \xi(a),$$

for all $a \in A$, where $c \in Z(A'')$ and ξ is a linear map of A into $Z(A'')$ such that $\xi([a, b]) = 0$ for all $a, b \in A$ with $ab = e_A$.

Proof. By Theorem 4, θ is a commuting map, so it follows from Theorem 3 that θ is of the form $\theta(a) = ca + \xi(a)$ for all $a \in A$, where $c \in Z(A'')$ and $\xi : A \rightarrow Z(A'')$ is a linear map.

Now let $a, b \in A$ with $ab = e_A$. Then by (4), we have

$$\begin{aligned} \xi([a, b]) &= \theta([a, b]) - c[a, b] \\ &= [\theta(a), b] - [ca, b] = [\xi(a), b] = 0. \end{aligned}$$

This completes the proof. □

A linear map θ on algebra A is called a *Lie triple centralizer* if

$$\theta([a, [b, c]]) = [a, [b, \theta(c)]], \quad a, b, c \in A.$$

We mention that the above identity is equivalent to assuming that

$$\theta([a, [b, c]]) = [a, [\theta(b), c]], \quad a, b, c \in A.$$

This notion introduced in [15] and the general form of Lie triple centralizers on a generalized matrix algebra are characterized.

Clearly, every Lie centralizer is a Lie triple centralizer, but the converse fails, in general. For example, let A and θ be as in Example 1. Then θ is a Lie triple centralizer, but it is not a Lie centralizer.

The following result is an analogous version of Theorem 4 for Lie triple centralizers.

Theorem 5. Let A be a unital Banach algebra and $\theta : A \rightarrow A$ be a continuous linear map satisfying

$$a, b \in A, \quad ab = e_A \implies \theta([a, [b, a]]) = [a, [\theta(b), a]]. \quad (6)$$

Then θ is a centralizing map. Moreover, if A is prime, then θ is a commuting map.

Proof. Let $a \in A$ be arbitrary. Similar to the proof of Theorem 4, we have

$$\theta([e_A - \lambda a, [\sum_{n=0}^{\infty} \lambda^n a^n, e_A - \lambda a]]) = [e_A - \lambda a, [\sum_{n=0}^{\infty} \lambda^n \theta(a^n), e_A - \lambda a]].$$

Obviously, the left hand side of the above equality is zero, thus by a simple calculation we get

$$\sum_{n=0}^{\infty} \lambda^{n+1} ([a, [\theta(a^n), a]]) = 0,$$

for all $\lambda \in \mathbb{C}$ with $|\lambda| < 1/\|a\|$. Hence

$$[a, [\theta(a^n), a]] = 0, \quad n = 0, 1, 2, \dots$$

Taking $n = 1$, we obtain $[\theta(a), a] \in Z(A)$ for all $a \in A$, and so θ is a centralizing map. If A is prime, then by [2, Theorem 1], θ is a commuting map. $\square$

By Theorem 5 and Theorem 1, we conclude the next result.

Corollary 4. Let A be either standard operator algebra or $M_n(\mathbb{C})$ and let $\theta : A \longrightarrow A$ be a continuous linear map satisfying (6). Then θ is a commuting map, and

$$\theta(a) = \lambda a + \xi(a),$$

for all $a \in A$, where $\lambda \in \mathbb{C}$ and ξ is a linear map of A into $\mathbb{C}e_A$.

The corollary below follows from Theorem 5 and Theorem 3.

Corollary 5. Let A be a unital C^* -algebra and $\theta : A \longrightarrow A$ be a continuous linear map such that the condition (6) holds. Then

$$\theta(a) = ca + \xi(a), \quad a \in A,$$

where $c \in Z(A'')$ and $\xi : A \longrightarrow Z(A'')$ is a linear map such that $\xi([a, [b, a]]) = 0$ for every $a, b \in A$ with $ab = e_A$.

2 Lie centralizers at zero point

This section is devoted to characterize linear maps behaving like Lie centralizers at zero point. We begin with the following lemma.

Lemma 1. Let A be a unital algebra and $\theta : A \longrightarrow A$ be a linear map satisfying

$$a, b \in A, \quad ab = 0 \quad \implies \quad \theta([a, b]) = [\theta(a), b]. \quad (7)$$

Then for every idempotent $p, q \in A$,

$$\theta([p, q]) = [\theta(p), q].$$

Proof. From $(p - pq)q = 0$, we have

$$\theta([p - pq, q]) = [\theta(p - pq), q],$$

and hence

$$\theta([p, q]) - \theta([pq, q]) = [\theta(p), q] - [\theta(pq), q]. \quad (8)$$

Similarly, as $pq(q - e_A) = 0$, we get

$$\theta([pq, q - e_A]) = [\theta(pq), q - e_A],$$

and thus

$$\theta([pq, q]) = [\theta(pq), q]. \quad (9)$$

It follows from (8) and (9) that

$$\theta([p, q]) = [\theta(p), q],$$

for every idempotent $p, q \in A$. □

Theorem 6. Let A be a von Neumann algebra and $\theta : A \longrightarrow A$ be a continuous linear map satisfying (7). Then θ is a Lie centralizer.

Proof. By Lemma 1,

$$\theta([p, q]) = [\theta(p), q], \quad (10)$$

for every idempotent $p, q \in A$. Let A_{sa} denote the set of all self-adjoint elements of A and let $x \in A_{sa}$. Then x is the limit of a sequence of linear combinations of projections in A , i.e., self-adjoint idempotents, therefore

$$x = \lim_n \sum_{k=1}^n \lambda_k p_k,$$

and thus from (10), we get $\theta([x, y]) = [\theta(x), y]$ for all self-adjoint elements $x, y \in A$. Now let $a \in A$ be arbitrary. Then $a = x + iy$ for $x, y \in A_{sa}$ and hence

$$\theta([a, b]) = [\theta(a), b], \quad a, b \in A.$$

Consequently, θ is a Lie centralizer. □

Recall that a *bounded approximate identity* for A is a bounded net $\{e_\lambda\}_{\lambda \in I}$ in A such that $e_\lambda a \longrightarrow a$ and $ae_\lambda \longrightarrow a$ for every $a \in A$. It seems appropriate to mention that the group algebra $L^1(G)$ for a locally compact group G and every C^* -algebra has a bounded approximate identity bounded by one, see [14].

The Banach algebra A is called *zero product determined* if each continuous bilinear map $\phi : A \times A \longrightarrow X$, where X is an arbitrary Banach space, satisfying

$$a, b \in A, \quad ab = 0 \quad \implies \quad \phi(a, b) = 0,$$

can be written as

$$\phi(a, b) = T(ab), \quad a, b \in A,$$

for some continuous linear map $T : A \longrightarrow X$.

The concept of zero product determined appeared as a byproduct of the so-called property $(\mathbb{B})$ introduced in [16].

We now generalize Theorem 6 to zero product determined Banach algebras.

Theorem 7. Let A be a zero product determined Banach algebra equipped a bounded approximate identity. If $\theta : A \longrightarrow A$ is a continuous linear map satisfying (7), then θ is a Lie centralizer.

Proof. Define a continuous bilinear map $\phi : A \times A \longrightarrow A$ by

$$\phi(a, b) = \theta([a, b]) - [\theta(a), b], \quad a, b \in A.$$

Then $ab = 0$ implies that $\phi(a, b) = 0$. Since A is zero product determined, there exists a continuous linear map $T : A \longrightarrow A$ such that $\phi(a, b) = T(ab)$ for all $a, b \in A$. Hence,

$$\begin{aligned} \theta([a, bc]) - [\theta(a), bc] &= \phi(a, bc) \\ &= T(a(bc)) \\ &= T((ab)c) \\ &= \phi(ab, c) = \theta([ab, c]) - [\theta(ab), c]. \end{aligned}$$

Letting $c = e_\lambda$ in the above equality, we get $\theta([a, b]) = [\theta(a), b]$ for all $a, b \in A$. Therefore, θ is a Lie centralizer. $\square$

It should be pointed out that every C^* -algebra, as well as every group algebra $L^1(G)$ of locally compact group G , and the matrix algebra $M_n(\mathbb{C})$ are zero product determined Banach algebra [16, 17]. However, standard operator algebras are not zero product determined, in general.

Combining Theorem 7 and Corollary 1, we obtain the following result.

Corollary 6. Let A be a C^* -algebra and $\theta : A \longrightarrow A$ be a continuous linear map such that the condition (7) holds. Then

$$\theta(a) = ca + \xi(a), \quad a \in A,$$

where $c \in Z(A'')$ and $\xi : A \longrightarrow Z(A'')$ is a linear map such that $\xi([a, b]) = 0$ for all $a, b \in A$.

As a consequence of Theorem 7 and Theorem 1, we get the next corollary.

Corollary 7. Let $A = M_n(\mathbb{C})$ and $\theta : A \longrightarrow A$ be a continuous linear map such that condition (7) holds. Then

$$\theta(a) = \lambda a + \xi(a),$$

for all $a \in A$, where $\lambda \in \mathbb{C}$ and ξ is a linear map of A into $\mathbb{C}e_A$.

For a standard operator algebra we have the following result. Note that in the theorem below θ is not continuous, in general.

Theorem 8. Let A be a standard operator algebra, and let $\theta : A \longrightarrow A$ be a linear map such that condition (7) holds. Then $\theta(a) = \lambda a + \xi(a)$ for all $a \in A$, where $\lambda \in \mathbb{C}$ and ξ is a linear map of A into $\mathbb{C}e_A$.

Proof. By [3, Theorem 3.1], θ is a commuting map. Now Theorem 1 applies. $\square$

3 Lie centralizers at idempotents

In this section, we consider the question of characterizing Lie centralizers by acting at idempotents products on von Neumann and standard operator algebras.

The following result is another tool for characterizing linear maps at an idempotent. For its proof, see [18, Theorem 2.1].

Theorem 9. Let A be a unital Banach algebra and X a Banach space. Suppose that $\phi : A \times A \longrightarrow X$ is a continuous bilinear map such that

$$a, b \in A, \quad ab = p \implies \phi(a, b) = 0,$$

where p is an idempotent in A . Then, for all $a \in A$,

$$\phi(a, p) = \phi(e_A, ap), \quad \text{and} \quad \phi(p, a) = \phi(pa, e_A).$$

The following result for $p = 0$ is Theorem 6.

Theorem 10. Let A be a von Neumann algebra and $\theta : A \longrightarrow A$ be a continuous linear map satisfying

$$a, b \in A, \quad ab = p \implies \theta([a, b]) = [\theta(a), b], \quad (11)$$

for every non-trivial idempotent $p \in A$. Then θ is a Lie centralizer.

Proof. Let p be a non-trivial idempotent in A . Define a bilinear map $\phi : A \times A \longrightarrow A$ via

$$\phi(a, b) = \theta([a, b]) - [\theta(a), b], \quad a, b \in A.$$

Then $ab = p$ implies that $\phi(a, b) = 0$. Applying Theorem 9, we obtain

$$\begin{aligned} \theta([p, b]) - [\theta(p), b] &= \phi(p, b) \\ &= \phi(pb, e_A) \\ &= \theta([pb, e_A]) - [\theta(pb), e_A] \\ &= 0. \end{aligned}$$

Therefore,

$$\theta([p, b]) = [\theta(p), b], \quad (12)$$

for all $a \in A$ and every idempotent $p \in A$.

Since the linear span of projections is norm dense in every von Neumann algebra, thus from (12) we arrive at $\theta([a, b]) = [\theta(a), b]$ for all $a, b \in A$. Hence θ is a Lie centralizer. $\square$

From Theorem 10 and Theorem 2 we get the following result.

Corollary 8. Let A be a von Neumann algebra and $\theta : A \longrightarrow A$ be a continuous linear map such that the condition (11) holds. Then

$$\theta(a) = ca + \xi(a), \quad a \in A,$$

where $c \in Z(A)$ and $\xi : A \longrightarrow Z(A)$ is a linear map such that $\xi([a, b]) = 0$ for all $a, b \in A$.

Theorem 11. Let A be a standard operator algebra and let $\theta : A \longrightarrow A$ be a continuous linear map such that condition (11) holds. Then θ is a commuting map.

Proof. Let $p \in A$ be a non-trivial idempotent of rank-one. Define a bilinear map $\phi : A \times A \longrightarrow A$ via

$$\phi(a, b) = \theta([a, b]) - [\theta(a), b], \quad a, b \in A.$$

Then $ab = p$ implies that $\phi(a, b) = 0$. Applying Theorem 9, we get

$$\begin{aligned} \theta([a, p]) - [\theta(a), p] &= \phi(a, p) \\ &= \phi(e_A, ap) \\ &= \theta([e_A, ap]) - [\theta(e_A), ap] \\ &= [ap, \theta(e_A)]. \end{aligned} \quad (13)$$

Replacing a by p in (13), we obtain

$$[p, \theta(p)] = [p, \theta(e_A)].$$

On the other hand, since $pp = p$ the condition (11) implies that

$$[\theta(p), p] = \theta[p, p] = 0.$$

From the above two equalities, we get $[p, \theta(e_A)] = 0$. By [11, Lemma 1.1] each element $u \in F(X)$ is a linear combination of rank-one idempotents. Thus,

$$[u, \theta(e_A)] = 0, \quad u \in F(X),$$

and hence $[au, \theta(e_A)] = 0$ for all $a \in A$. Therefore, from (13) we conclude that

$$\theta(ap) = \theta(pa) + \theta(a)p - p\theta(a),$$

for all $a \in A$ and every idempotent $p \in A$. In particular,

$$\theta(au) = \theta(ua) + \theta(a)u - u\theta(a), \quad a \in A, u \in F(X). \quad (14)$$

Let $a, b \in A$. By applying (14), we obtain

$$\theta(abu) = \theta(uab) + \theta(ab)u - u\theta(ab), \quad u \in F(X).$$

Replacing u by bu in (14), we have

$$\begin{aligned} \theta(abu) &= \theta(bua) + \theta(a)bu - bu\theta(a) \\ &= \theta(uab) + \theta(b)ua - ua\theta(b) + \theta(a)bu - bu\theta(a). \end{aligned}$$

By comparing the two expressions for $\theta(abu)$, we arrive at

$$\theta(ab)u - u\theta(ab) = \theta(b)ua - ua\theta(b) + \theta(a)bu - bu\theta(a),$$

for all $a, b \in A$ and $u \in F(X)$.

Since $F(X)$ is dense in $B(X)$ with respect to the strong operator topology, there is a net u_i in $F(X)$ such that $u_i \rightarrow e_A$. Therefore,

$$\theta(ab)u_i - u_i\theta(ab) = \theta(b)u_i a - u_i a \theta(b) + \theta(a)bu_i - bu_i\theta(a).$$

From the separate continuity of product in $B(X)$, we get

$$\theta(b)a - a\theta(b) + \theta(a)b - b\theta(a) = 0.$$

By letting $a = b$ we get $[\theta(a), a] = 0$ for every $a \in A$. Thus, θ is a commuting map. $\square$

The corollary below follows from Theorem 11 and Theorem 1.

Corollary 9. Let A be a standard operator algebra and let $\theta : A \rightarrow A$ be a continuous linear map such that condition (11) holds. Then

$$\theta(a) = \lambda a + \xi(a), \quad a \in A,$$

where $\lambda \in \mathbb{C}$ and ξ is a linear map of A into $\mathbb{C}e_A$.

The set of idempotents of a given algebra A is denoted by $\mathcal{I}(A)$. A Banach algebra A is generated by idempotents, whenever $A = \text{alg}\mathcal{I}(A)$.

For example, von Neumann algebras, $B(\mathcal{H})$ for infinite-dimensional complex Hilbert space $\mathcal{H}$, topologically simple Banach algebras containing a non-trivial idempotent, the group algebra $L^1(G)$ for a compact group G , and the matrix algebra $M_n(A)$ over a unital Banach algebra A are generated by idempotents. Other classes of Banach algebras with this property can be found in [16].

Now, similar to Theorem 10, we can prove the next result.

Theorem 12. Let A be a unital Banach algebra with the property that $A = \overline{\text{alg}\mathcal{I}(A)}$. If $\theta : A \rightarrow A$ is a continuous linear map satisfying (11), then

$$\theta(a) = ca + \xi(a), \quad a \in A,$$

where $c \in Z(A)$ and $\xi : A \rightarrow Z(A)$ is a linear map such that $\xi([a, b]) = 0$ for all $a, b \in A$.

Conclusion

In this paper, we investigated linear maps behaving like Lie centralizers at identity products, zero products, and idempotent products on several classes of Banach algebras. We proved that under suitable assumptions, such mappings are closely related to commuting maps and Lie centralizers. In particular, we obtained structural characterizations for these maps on C^* -algebras, von Neumann algebras, standard operator algebras, and matrix algebras. These results extend several known characterizations of Lie centralizers and provide new perspectives on local conditions that force a linear map to exhibit global centralizer-like behavior.

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Conflict of interest

The author declares no conflict of interest.

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