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MATHEMATICS

https://doi.org/10.31489/2026M2/4-16 *Research article*

Fractional differential equations with nonlocal boundary conditions involving initial and final segments of the given domain

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We explore the existence and uniqueness criteria for solutions of a Liouville–Caputo fractional differential equation with the nonlinearity containing the unknown function as well as its lower order fractional derivative, and supplemented with a set of nonlocal fractional boundary data with respect to initial and final segments of the given domain. Integral boundary conditions offer an effective approach to model the flow and drag phenomena in arbitrary shaped vessels, heat conduction, biomedical computational fluid dynamics, engineering problems, etc. The notion of segmental type nonlocal fractional integral boundary conditions introduced in this paper is novel and specializes to periodic/anti-periodic boundary data under a suitable choice of the parameters involved in these conditions (see the second last paragraph of Introduction). We apply Krasnosel'skiĭ's fixed point theorem and Leray-Schauder's nonlinear alternative to prove two existence results for the problem at hand, while the uniqueness of its solutions is established via Banach's contraction mapping principle. Examples are constructed for illustrating the obtained results. Our work is useful in the given configuration as it leads to a new direction for research on fractional boundary value problems. The paper concludes with some interesting observations.

Keywords: fractional differential equations, boundary value problems, Caputo fractional derivative operator, periodic/anti-periodic segmental boundary data, existence, uniqueness, nonlocal boundary conditions, fixed point.

2020 Mathematics Subject Classification: 34A08, 34A34, 34B15.

Introduction

We introduce a novel concept of nonlocal boundary conditions with respect to the segments $(0,\xi)$ and (η,T) of the domain $[0,T]$ and solve a Liouville–Caputo fractional differential equation with the nonlinearity depending upon the unknown function together with its lower order fractional derivative complemented with these conditions. When it is assumed that ξ is close to 0 and η is close to T, the

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periodic/anti-periodic phenomenon with respect to the segments $(0,\xi)$ and (η,T) can be observed. In precise terms, we consider the fractional boundary value problem given by

$$\begin{cases} {}^cD^{\alpha}x(t)=\varphi(t,x(t),{}^cD^{\kappa}x(t)),\ 1<\alpha\leq 2,\ \ 0<\kappa\leq 1,\ \ t\in[0,T],\\ \displaystyle\int_0^{\xi}x(s)ds=\delta_1\int_{\eta}^{T}x(s)ds, \int_0^{\xi}({}^cD^{p_1}x(s))ds=\delta_2\int_{\eta}^{T}({}^cD^{p_2}x(s))ds,\ 0<p_1,p_2<1, \end{cases} \tag{1}$$

where ${}^cD^{\varrho}$ denotes the Caputo fractional derivative operator of order ϱ, where $\varrho\in\{\alpha,\kappa,p_1,p_2\}$, δ_1 and δ_2 are real constants, $0<\xi<\eta<T$ and $\varphi:[0,T]\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ is a continuous function.

Let us now dwell on some recent work on nonlinear nonlocal fractional boundary value problems. Fractional differential equations arise in a variety of disciplines of applied sciences, for example, physics and engineering [1], financial economics [2], relaxation filtration processes [3], chaos synchronization [4], etc. For theoretical aspects of fractional calculus, for example, see [5].

Nonlocal boundary conditions can model the physical phenomena experiencing the changes happening at arbitrary positions (nonlocal points and segments) inside the domain. Nonlocal integral boundary conditions can describe non-uniformities on the curved structures. For application details, see fluid flow problems [6], biomedical sciences [7], etc. One can find the engineering applications of strip type integral boundary conditions in the article [8]. For a variety of recent results on nonlocal fractional boundary value problems, we refer the reader to the book [9] and articles [10, 11].

Periodic/anti-periodic boundary value problems constitute a special form of non-separated (Sturm-Liouville) type boundary value problems. Anti-periodic fractional boundary conditions appear in a variety of applications and have been extensively studied in the literature. For a detailed description of anti-periodic boundary value problems involving different types of fractional derivative operators, for instance, see the articles [12, 13]. A new concept of dual anti-periodic boundary conditions was introduced in [14]. The authors in [15] studied p-Laplacian systems with rotating periodic boundary conditions.

The present work is motivated by the fact that a periodic event or pattern within the distinct initial and final sections of a specified range helps to understand its characteristics, verify its periodicity, or examine its behavior at critical points. A tiny segment of the sound wave reveals the pattern of changing air pressure. A brief recording of an electrocardiogram (EKG) will display the electrical activity associated with each heartbeat [16]. The daily temperature fluctuations often follow a somewhat periodic pattern and analyzing temperature data for a brief interval can reveal a part of this daily cycle. Some other examples include temporary oscillations that can occur in response to a sudden change in a system, and occurrence of spectral edges for periodic operators inside the $\mathbb{Z}$-periodic media [17].

The aim of the present study is to develop the existence theory for a fractional differential equation complemented with newly introduced segmental type nonlocal fractional boundary conditions. When ξ is close to 0 and η is close to T, the segmental fractional boundary conditions in (1) can be regarded as periodic and anti-periodic ones for $\delta_1=\delta_2=1$ and $\delta_1=\delta_2=-1$, respectively. On the other hand, the mixed periodic and anti-periodic boundary conditions follow by taking $\delta_1=1,\delta_2=-1$ or $\delta_1=-1,\delta_2=1$ (or vice versa) in (1).

We organize the rest of the article as follows. Section 1 contains some basic definitions and a subsidiary lemma. The two existence results for the problem (1), based on Krasnosel'skiĭ fixed point theorem and Leray-Schauder's nonlinear alternative, are derived in Section 2. We also prove a uniqueness result for the given problem by applying Banach's contraction mapping principle in this section. Illustrative examples for the main results are constructed in Section 3. In the last section, we describe some interesting observations.

1 Preliminaries

We begin this section by recalling some basic definitions.

Definition 1. [5] We define the (left) Riemann–Liouville fractional integral of order $\sigma > 0$ for the function $\vartheta \in L_1[a,b]$, denoted by $I_{a^+}^{\sigma}\vartheta$, as

$$I_{a^+}^{\sigma}\vartheta(v) = \int_a^v \frac{(v-\widehat{v})^{\sigma-1}}{\Gamma(\sigma)}\vartheta(\widehat{v})\,d\widehat{v},$$

where Γ represents the Euler gamma function.

Definition 2. [5] Let $\vartheta, \vartheta^{(m)} \in L_1[a,b]$, $a,b \in \mathbb{R}$. The Riemann–Liouville fractional derivative of order $\sigma \in (m-1,m)$, $m \in \mathbb{N}$, denoted by $D_{a^+}^{\sigma}\vartheta$, is given by

$$D_{a^+}^{\sigma}\vartheta(v) = \frac{d^m}{dv^m} I_{a^+}^{m-\sigma}\vartheta(v) = \frac{1}{\Gamma(m-\sigma)}\frac{d^m}{dv^m}\int_a^v (v-\widehat{v})^{m-1-\sigma}\vartheta(\widehat{v})d\widehat{v},$$

while the Caputo fractional derivative ${}^cD_{a^+}^{\sigma}\vartheta$ of order σ is defined by

$${}^cD_{a^+}^{\sigma}\vartheta(v) = D_{a^+}^{\sigma}\left[\vartheta(v) - \sum_{p=0}^{m-1}\vartheta^{(p)}(a)\frac{(v-a)^p}{p!}\right].$$

Remark 1. The (left) Caputo fractional derivative for a function $\vartheta \in AC^m[a,b]$ of order σ can also be defined as

$${}^cD_{a^+}^{\sigma}\vartheta(v) = \int_a^v \frac{(v-\widehat{v})^{m-\sigma-1}}{\Gamma(m-\sigma)}\vartheta^{(m)}(\widehat{v})d\widehat{v}.$$

In our article, we write the Riemann–Liouville fractional integral operator I^{σ} and the Caputo fractional derivative operator ${}^cD^{\sigma}$ instead of $I_{0^+}^{\sigma}$ and ${}^cD_{0^+}^{\sigma}$, respectively.
The following lemma deals with the linear version of the problem (1).

Lemma 1. Let $g \in C[0,T]$ and

$$\omega_1 = \frac{\xi^{2-p_1}}{\Gamma(3-p_1)} - \frac{\delta_2(T^{2-p_2}-\eta^{2-p_2})}{\Gamma(3-p_2)} \neq 0, \quad \omega_3 = \xi - \delta_1(T-\eta) \neq 0, \tag{2}$$

then the unique solution of the linear fractional differential equation

$${}^cD^{\alpha}x(t) = g(t), \; 1 < \alpha \leq 2, \tag{3}$$

equipped with the boundary conditions in (1) is

$$\begin{aligned} x(t) \;=\; & \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}g(s)ds + \frac{1}{\omega_3}\int_\eta^T\int_0^s\left[\frac{\delta_1(s-u)^{\alpha-1}}{\Gamma(\alpha)} + \frac{\delta_2(t\omega_3-\omega_2)}{\omega_1\Gamma(\alpha-p_2)}(s-u)^{\alpha-p_2-1}\right]g(u)duds \\ & -\frac{1}{\omega_3}\int_0^{\xi}\left[\frac{(\xi-s)^{\alpha}}{\Gamma(\alpha+1)} + \frac{(t\omega_3-\omega_2)}{\omega_1\Gamma(\alpha-p_1+1)}(\xi-s)^{\alpha-p_1}\right]g(s)ds, \end{aligned} \tag{4}$$

where

$$\omega_2 = \frac{\xi^2 - \delta_1(T^2-\eta^2)}{2}. \tag{5}$$

Proof. Operating the Riemann–Liouville fractional integral operator I^α on both sides of (3) and using the formula (3.5.13) in [5], we find that

$$x(t) = I^\alpha g(t) + c_0 + c_1 t, \tag{6}$$

where $c_0, c_1 \in \mathbb{R}$ are unknown arbitrary constants, and

$${}^cD^{p_i}x(t) = \int_0^t \frac{(t-s)^{\alpha-p_i-1}}{\Gamma(\alpha-p_i)} g(s)ds + c_1 \frac{t^{1-p_i}}{\Gamma(2-p_i)}, \; i = 1, 2. \tag{7}$$

Using (6) in the first boundary condition of (1), it is found that

$$\begin{aligned} c_0 \;=\; & \frac{1}{\xi - \delta_1(T-\eta)} \left[\delta_1 \int_\eta^T \int_0^s \frac{(s-u)^{\alpha-1}}{\Gamma(\alpha)} g(u)duds - \int_0^\xi \frac{(\xi-s)^\alpha}{\Gamma(\alpha+1)} g(s)ds \right] \\ & - c_1 \frac{(\xi^2 - \delta_1(T^2-\eta^2))}{2(\xi - \delta_1(T-\eta))}, \end{aligned}$$

which, on using the notation (2) and (5), takes the form

$$c_0 \;=\; \frac{1}{\omega_3} \left(\delta_1 \int_\eta^T \int_0^s \frac{(s-u)^{\alpha-1}}{\Gamma(\alpha)} g(u)duds - \int_0^\xi \frac{(\xi-s)^\alpha}{\Gamma(\alpha+1)} g(s)ds \right) - c_1 \frac{\omega_2}{\omega_3}. \tag{8}$$

Substituting (7) in the second boundary condition of (1) together with the notation (2), we obtain

$$c_1 = \frac{1}{\omega_1} \left[\delta_2 \int_\eta^T \int_0^s \frac{(s-u)^{\alpha-p_2-1}}{\Gamma(\alpha-p_2)} g(u)duds - \int_0^\xi \frac{(\xi-s)^{\alpha-p_1}}{\Gamma(\alpha-p_1+1)} g(s)ds \right]. \tag{9}$$

Inserting the value of c_1 from (9) into (8), we find that

$$\begin{aligned} c_0 \;=\; & \frac{1}{\omega_3} \left(\delta_1 \int_\eta^T \int_0^s \frac{(s-u)^{\alpha-1}}{\Gamma(\alpha)} g(u)duds - \int_0^\xi \frac{(\xi-s)^\alpha}{\Gamma(\alpha+1)} g(s)ds \right) \\ & - \frac{\omega_2}{\omega_1\omega_3} \left[\delta_2 \int_\eta^T \int_0^s \frac{(s-u)^{\alpha-p_2-1}}{\Gamma(\alpha-p_2)} g(u)duds - \int_0^\xi \frac{(\xi-s)^{\alpha-p_1}}{\Gamma(\alpha-p_1+1)} g(s)ds \right], \end{aligned}$$

where ω_1 and ω_3 are defined in (2), while ω_2 is given in (5). Substituting the above values of c_0 and c_1 in (6) yields the solution (4). The converse of the lemma follows by direct computation. □

2 *Existence and uniqueness results*

Let $\mathcal{U} = \{x : x, {}^cD^\kappa x \in C([0,T], \mathbb{R})\}$ be a Banach space of all continuous functions defined on $[0,T]$ and equipped with the norm $\|x\|_\mathcal{U} = \max\{|x(t)| + |{}^cD^\kappa x(t)|, t \in [0,T], 0 < \kappa \leq 1\}$.

By Lemma 1, we introduce a fixed point problem $x = \mathcal{H}x$, which is equivalent to the problem (1), where $\mathcal{H} : \mathcal{U} \to \mathcal{U}$ is given by

$$\begin{aligned} (\mathcal{H}x(t)) \;=\; & \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \varphi(s, x(s), {}^cD^\kappa x(s))ds \\ & + \frac{1}{\omega_3} \int_\eta^T \int_0^s \left[\frac{\delta_1(s-u)^{\alpha-1}}{\Gamma(\alpha)} + \frac{\delta_2(t\omega_3 - \omega_2)}{\omega_1\Gamma(\alpha-p_2)} (s-u)^{\alpha-p_2-1} \right] \varphi(u, x(u), {}^cD^\kappa x(u))duds \\ & - \frac{1}{\omega_3} \int_0^\xi \left[\frac{(\xi-s)^\alpha}{\Gamma(\alpha+1)} + \frac{(t\omega_3-\omega_2)}{\omega_1\Gamma(\alpha-p_1+1)} (\xi-s)^{\alpha-p_1} \right] \varphi(s, x(s), {}^cD^\kappa x(s))ds, \; t \in [0,T]. \end{aligned} \tag{10}$$

Next, we set

$$\tau_1 = \max_{t\in[0,T]}\left\{\frac{t^\alpha}{\Gamma(\alpha+1)}+\frac{|t\omega_3-\omega_2|}{|\omega_1\omega_3|}\left(\frac{|\delta_2(T^{\alpha-p_2+1}-\eta^{\alpha-p_2+1})|}{\Gamma(\alpha-p_2+2)}+\frac{\xi^{\alpha-p_1+1}}{\Gamma(\alpha-p_1+2)}\right)\right\}$$
$$+\frac{1}{|\omega_3|}\left(\frac{|\delta_1(T^{\alpha+1}-\eta^{\alpha+1})|+\xi^{\alpha+1}}{\Gamma(\alpha+2)}\right), \quad (11)$$

$$\tau_2 = \frac{T^{\alpha-\kappa}}{\Gamma(\alpha-\kappa+1)}+\frac{T^{1-\kappa}}{|\omega_1|\Gamma(2-\kappa)}\left[\frac{|\delta_2(T^{\alpha-p_2+1}-\eta^{\alpha-p_2+1})|}{\Gamma(\alpha-p_2+2)}+\frac{\xi^{\alpha-p_1+1}}{\Gamma(\alpha-p_1+2)}\right]. \quad (12)$$

In our first existence result for the problem (1), we make use of Krasnosel'skiĭ's fixed point theorem [18].

Theorem 1. Let $\varphi\in C([0,T]\times\mathbb{R}\times\mathbb{R},\mathbb{R})$ and the following assumptions hold:

(A_1) for all $t\in[0,T]$, x_i, $y_i\in\mathbb{R}$, $i=1,2$, there exists a positive constant $\mathcal{L}$ such that

$$|\varphi(t,x_1,x_2)-\varphi(t,y_1,y_2)|\le\mathcal{L}(|x_1-y_1|+|x_2-y_2|);$$

(A_2) there exists a function $\psi\in C([0,T],\mathbb{R}^+)$ such that

$$|\varphi(t,x(t),\,{}^cD^\kappa x(t))|\le\psi(t),\quad (t,x,\,{}^cD^\kappa x)\in[0,T]\times\mathbb{R}\times\mathbb{R}.$$

Then, at least one solution to the problem (1) exists on $[0,T]$ if $\mu\mathcal{L}<1$, where

$$\mu=(\tau_1+\tau_2)-\left(\frac{T^\alpha}{\Gamma(\alpha+1)}+\frac{T^{\alpha-\kappa}}{\Gamma(\alpha-\kappa+1)}\right), \quad (13)$$

τ_1 and τ_2 are given in (11) and (12), respectively.

Proof. We complete the proof by verifying the hypotheses of Krasnosel'skiĭ's fixed point theorem [18] in several steps. Let us first define the operators $\mathcal{H}_1,\mathcal{H}_2:B_\epsilon\to\mathcal{U}$ as follows:

$$(\mathcal{H}_1x)(t) = \int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\varphi(s,x(s),\,{}^cD^\kappa x(s))ds,$$
$$(\mathcal{H}_2x)(t) = \frac{1}{\omega_3}\int_\eta^T\int_0^s\left[\frac{\delta_1(s-u)^{\alpha-1}}{\Gamma(\alpha)}+\frac{\delta_2(t\omega_3-\omega_2)}{\omega_1\Gamma(\alpha-p_2)}(s-u)^{\alpha-p_2-1}\right]\varphi(u,x(u),\,{}^cD^\kappa x(u))duds$$
$$-\frac{1}{\omega_3}\int_0^\xi\left[\frac{(\xi-s)^\alpha}{\Gamma(\alpha+1)}+\frac{(t\omega_3-\omega_2)}{\omega_1\Gamma(\alpha-p_1+1)}(\xi-s)^{\alpha-p_1}\right]\varphi(s,x(s),\,{}^cD^\kappa x(s))ds,$$

where $B_\epsilon=\{x\in\mathcal{U}:\|x\|\le\epsilon\}$ with $\epsilon\ge\|\psi\|(\tau_1+\tau_2)$. Observe that $\mathcal{H}=\mathcal{H}_1+\mathcal{H}_2$.

Step I. Setting $\max_{t\in[0,T]}|\psi(t)|=\|\psi\|$, and taking any $x,y\in B_\epsilon$, we find that

$$\|(\mathcal{H}_1x)+(\mathcal{H}_2y)\| = \max_{t\in[0,T]}|(\mathcal{H}_1x)(t)+(\mathcal{H}_2y)(t)|$$
$$\le \max_{t\in[0,T]}\left\{\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}|\varphi(s,x(s),\,{}^cD^\kappa x(s))|ds\right.$$
$$+\frac{1}{|\omega_3|}\int_\eta^T\int_0^s\left[\frac{|\delta_1|(s-u)^{\alpha-1}}{\Gamma(\alpha)}+\frac{|\delta_2||t\omega_3-\omega_2|}{|\omega_1|\Gamma(\alpha-p_2)}(s-u)^{\alpha-p_2-1}\right]$$
$$\times|\varphi(u,y(u),\,{}^cD^\kappa y(u))|duds$$
$$+\frac{1}{|\omega_3|}\int_0^\xi\left[\frac{(\xi-s)^\alpha}{\Gamma(\alpha+1)}+\frac{|t\omega_3-\omega_2|}{|\omega_1|\Gamma(\alpha-p_1+1)}(\xi-s)^{\alpha-p_1}\right]$$
$$\left.\times|\varphi(s,y(s),\,{}^cD^\kappa y(s))|ds\right\}\le\|\psi\|\tau_1,$$

and

$$\begin{aligned}\|({}^cD^\kappa\mathcal{H}_1x)+({}^cD^\kappa\mathcal{H}_2y)\| &= \max_{t\in[0,T]}|({}^cD^\kappa\mathcal{H}_1x)(t)+({}^cD^\kappa\mathcal{H}_2y)(t)|\\ &\le \max_{t\in[0,T]}\Bigg\{\int_0^t\frac{(t-s)^{\alpha-\kappa-1}}{\Gamma(\alpha-\kappa)}|\varphi(s,x(s),\,{}^cD^\kappa x(s))|ds\\ &\quad+\frac{t^{1-\kappa}}{|\omega_1|\Gamma(2-\kappa)}\Bigg(|\delta_2|\int_\eta^T\int_0^s\frac{(s-u)^{\alpha-p_2-1}}{\Gamma(\alpha-p_2)}|\varphi(u,y(u),\,{}^cD^\kappa y(u))|duds\\ &\quad+\int_0^\xi\frac{(\xi-s)^{\alpha-p_1}}{\Gamma(\alpha-p_1+1)}|\varphi(s,y(s),\,{}^cD^\kappa y(s))|ds\Bigg)\Bigg\}\\ &\le \|\psi\|\tau_2.\end{aligned}$$

Thus, $\|\mathcal{H}_1x+\mathcal{H}_2y\|_\mathcal{U}\le\|\psi\|(\tau_1+\tau_2)\le\epsilon$. Therefore, $\mathcal{H}_1x+\mathcal{H}_2y\in B_\epsilon$.

Step II. We show that $\mathcal{H}_2$ is a contraction. By the assumption (A_1) and (13), for $x,y\in B_\epsilon$, we have

$$\begin{aligned}\|(\mathcal{H}_2x)-(\mathcal{H}_2y)\| &= \max_{t\in[0,T]}|(\mathcal{H}_2x)(t)-(\mathcal{H}_2y)(t)|\\ &\le \max_{t\in[0,T]}\Bigg\{\frac{1}{|\omega_3|}\int_\eta^T\int_0^s\Bigg[\frac{|\delta_1|(s-u)^{\alpha-1}}{\Gamma(\alpha)}+\frac{|\delta_2||t\omega_3-\omega_2|}{|\omega_1|\Gamma(\alpha-p_2)}(s-u)^{\alpha-p_2-1}\Bigg]\\ &\quad\times|\varphi(u,x(u),\,{}^cD^\kappa x(u))-\varphi(u,y(u),\,{}^cD^\kappa y(u))|duds\\ &\quad+\frac{1}{|\omega_3|}\int_0^\xi\Bigg[\frac{(\xi-s)^\alpha}{\Gamma(\alpha+1)}+\frac{|t\omega_3-\omega_2|}{|\omega_1|\Gamma(\alpha-p_1+1)}(\xi-s)^{\alpha-p_1}\Bigg]\\ &\quad\times|\varphi(s,x(s),\,{}^cD^\kappa x(s))-\varphi(s,y(s),\,{}^cD^\kappa y(s))|ds\Bigg\}\\ &\le \mathcal{L}\Bigg(\tau_1-\frac{T^\alpha}{\Gamma(\alpha+1)}\Bigg)\|x-y\|,\end{aligned}$$

and

$$\begin{aligned}&\|({}^cD^\kappa\mathcal{H}_2x)-({}^cD^\kappa\mathcal{H}_2y)\|\\ =\;&\max_{t\in[0,T]}|({}^cD^\kappa\mathcal{H}_2x)(t)-({}^cD^\kappa\mathcal{H}_2y)(t)|\\ \le\;&\max_{t\in[0,T]}\Bigg\{\frac{t^{1-\kappa}}{|\omega_1|\Gamma(2-\kappa)}\Bigg(|\delta_2|\int_\eta^T\int_0^s\frac{(s-u)^{\alpha-p_2-1}}{\Gamma(\alpha-p_2)}|\varphi(u,x(u),\,{}^cD^\kappa x(u))\\ &-\varphi(u,y(u),\,{}^cD^\kappa y(u))|duds+\int_0^\xi\frac{(\xi-s)^{\alpha-p_1}}{\Gamma(\alpha-p_1+1)}|\varphi(s,x(s),\,{}^cD^\kappa x(s))\\ &-\varphi(s,y(s),\,{}^cD^\kappa y(s))|ds\Bigg)\Bigg\}\le\mathcal{L}\Bigg(\tau_2-\frac{T^{\alpha-\kappa}}{\Gamma(\alpha-\kappa+1)}\Bigg)\|x-y\|.\end{aligned}$$

From the foregoing inequalities, we find that

$$\|(\mathcal{H}_2x)-(\mathcal{H}_2y)\|_\mathcal{U}=\|(\mathcal{H}_2x)-(\mathcal{H}_2y)\|+\|({}^cD^\kappa\mathcal{H}_2x)-({}^cD^\kappa\mathcal{H}_2y)\|\le\mathcal{L}\mu\|x-y\|.$$

Therefore, $\mathcal{H}_2$ is a contraction according to the assumption $\mu\mathcal{L}<1$.

Step III. Here, it will be shown that $\mathcal{H}_1$ is completely continuous.

Note that continuity of $\varphi(t,x(t),\,{}^cD^\kappa x(t))$ implies that of the operator $\mathcal{H}_1$. For $x\in B_\epsilon$, we obtain

$$\|(\mathcal{H}_1x)\|=\max_{t\in[0,T]}|(\mathcal{H}_1x)(t)|\;\le\;\max_{t\in[0,T]}\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}|\varphi(s,x(s),\,{}^cD^\kappa x(s))|ds\le\|\psi\|\frac{T^\alpha}{\Gamma(\alpha+1)},$$

and

$$\|({}^cD^\kappa\mathcal{H}_1x)\| = \max_{t\in[0,T]}|({}^cD^\kappa\mathcal{H}_1x)(t)| \leq \max_{t\in[0,T]}\int_0^t \frac{(t-s)^{\alpha-\kappa-1}}{\Gamma(\alpha-\kappa)}|\varphi(s,x(s),\,{}^cD^\kappa x(s))|ds$$
$$\leq \|\psi\|\frac{T^{\alpha-\kappa}}{\Gamma(\alpha-\kappa+1)}.$$

Hence, $\mathcal{H}_1$ is uniformly bounded.

Next, the operator $\mathcal{H}_1$ is shown to be equicontinuous. Let $\max\limits_{t\in[0,T]}|\varphi(t,x(t),\,{}^cD^\kappa x(t))| = \widehat{\varphi} < \infty$ for $(t,x,\,{}^cD^\kappa x) \in [0,T]\times B_\epsilon{}^2$. Then, for $0<\nu_1<\nu_2<T$, we have

$$\begin{aligned}|(\mathcal{H}_1x)(\nu_2)-(\mathcal{H}_1x)(\nu_1)| &\leq \int_0^{\nu_1}\frac{|(\nu_2-s)^{\alpha-1}-(\nu_1-s)^{\alpha-1}|}{\Gamma(\alpha)}|\varphi(s,x(s),\,{}^cD^\kappa x(s))|ds\\ &\quad+\int_{\nu_1}^{\nu_2}\frac{|(\nu_2-s)^{\alpha-1}|}{\Gamma(\alpha)}|\varphi(s,x(s),\,{}^cD^\kappa x(s))|ds\\ &\leq \widehat{\varphi}\left(\frac{2|(\nu_2-\nu_1)^\alpha|+|\nu_2^\alpha-\nu_1^\alpha|}{\Gamma(\alpha+1)}\right)\to 0 \text{ as } \nu_2\to\nu_1,\end{aligned}$$

independently of $x\in B_\epsilon$. Likewise, we have

$$\begin{aligned}|({}^cD^\kappa\mathcal{H}_1x)(\nu_2)-({}^cD^\kappa\mathcal{H}_1x)(\nu_1)| &\leq \int_0^{\nu_1}\frac{|(\nu_2-s)^{\alpha-\kappa-1}-(\nu_1-s)^{\alpha-\kappa-1}|}{\Gamma(\alpha-\kappa)}|\varphi(s,x(s),\,{}^cD^\kappa x(s))|ds\\ &\quad+\int_{\nu_1}^{\nu_2}\frac{|(\nu_2-s)^{\alpha-\kappa-1}|}{\Gamma(\alpha-\kappa)}|\varphi(s,x(s),\,{}^cD^\kappa x(s))|ds\\ &\leq \widehat{\varphi}\left(\frac{2|(\nu_2-\nu_1)^{\alpha-\kappa}|+|\nu_2^{\alpha-\kappa}-\nu_1^{\alpha-\kappa}|}{\Gamma(\alpha-\kappa+1)}\right)\to 0\end{aligned}$$

as $\nu_2\to\nu_1$ independently of $x\in B_\epsilon$. Therefore, the operator $\mathcal{H}_1$ is relatively compact on B_ϵ. In view of the foregoing steps, we deduce by the Arzelá-Ascoli theorem [18] that the operator $\mathcal{H}_1$ is compact on B_ϵ. As the hypotheses of Krasnosel'skiĭ's fixed point theorem [18] are verified, it follows by its conclusion that there exists at least one solution to the problem (1) on $[0,T]$. □

Our second existence result for the problem (1) is based on Leray-Schauder's nonlinear alternative [18].

Theorem 2. Let $\varphi\in C([0,T]\times\mathbb{R}\times\mathbb{R},\mathbb{R})$ and the following assumptions hold:

(A_3) there exist continuous nondecreasing functions $\Psi:[0,\infty)\to(0,\infty)$ and a function $\upsilon\in C([0,T],\mathbb{R}^+)$ satisfying $|\varphi(t,x,\,{}^cD^\kappa x)|\leq\upsilon(t)\Psi(\|x\|_{\mathcal{U}})$, for each $(t,x,\,{}^cD^\kappa x)\in[0,T]\times\mathbb{R}\times\mathbb{R}$;

(A_4) there exists a constant $\mathcal{K}>0$ such that

$$\frac{\mathcal{K}}{\Psi(\mathcal{K})\|\upsilon\|(\tau_1+\tau_2)}>1,$$

where τ_1 and τ_2 are respectively given in (11) and (12).

Then, at least one solution to the problem (1) exists on $[0,T]$.

Proof. We verify the assumptions of Leray-Schauder's nonlinear alternative [18] in different steps.
Step 1. The operator $\mathcal{H}$ maps bounded sets (balls) into bounded sets in $C([0,T],\mathbb{R})$.

Consider a bounded closed ball $B_R = \{x \in C([0,T],\mathbb{R}) : \|x\|_{\mathcal{U}} \leq R\}$ in $C([0,T],\mathbb{R})$. Then, for any $x \in B_R$, we obtain

$$\begin{aligned}
\|(\mathcal{H}x)\| &= \max_{t\in[0,T]} |\mathcal{H}x(t)| \\
&\leq \max_{t\in[0,T]} \Bigg\{ \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} |\varphi(s,x(s),\, {}^cD^\kappa x(s))| ds \\
&\quad + \frac{1}{|\omega_3|} \int_\eta^T \int_0^s \left[\frac{|\delta_1|(s-u)^{\alpha-1}}{\Gamma(\alpha)} + \frac{|\delta_2||t\omega_3-\omega_2|}{|\omega_1|\Gamma(\alpha-p_2)}(s-u)^{\alpha-p_2-1} \right] \\
&\quad \times |\varphi(u,x(u),\, {}^cD^\kappa x(u))| duds \\
&\quad + \frac{1}{|\omega_3|} \int_0^\xi \left[\frac{(\xi-s)^\alpha}{\Gamma(\alpha+1)} + \frac{|t\omega_3-\omega_2|}{|\omega_1|\Gamma(\alpha-p_1+1)}(\xi-s)^{\alpha-p_1} \right] |\varphi(s,x(s),\, {}^cD^\kappa x(s))| ds \Bigg\} \\
&\leq \|\upsilon\|\Psi(R) \Bigg[\max_{t\in[0,T]} \Bigg\{ \frac{t^\alpha}{\Gamma(\alpha+1)} + \frac{|t\omega_3-\omega_2|}{|\omega_1\omega_3|} \Bigg(\frac{|\delta_2(T^{\alpha-p_2+1}-\eta^{\alpha-p_2+1})|}{\Gamma(\alpha-p_2+2)} \\
&\quad + \frac{\xi^{\alpha-p_1+1}}{\Gamma(\alpha-p_1+2)} \Bigg) \Bigg\} + \frac{1}{|\omega_3|} \Bigg(\frac{|\delta_1(T^{\alpha+1}-\eta^{\alpha+1})| + \xi^{\alpha+1}}{\Gamma(\alpha+2)} \Bigg) \Bigg] \\
&= \|\upsilon\|\Psi(R)\tau_1.
\end{aligned}$$

Similarly, one can find that

$$\begin{aligned}
\|({}^cD^\kappa \mathcal{H}x)\| &= \max_{t\in[0,T]} |({}^cD^\kappa \mathcal{H}x)(t)| \\
&\leq \max_{t\in[0,T]} \Bigg\{ \int_0^t \frac{(t-s)^{\alpha-\kappa-1}}{\Gamma(\alpha-\kappa)} |\varphi(s,x(s),\, {}^cD^\kappa(s))| ds \\
&\quad + \frac{t^{1-\kappa}}{|\omega_1|\Gamma(2-\kappa)} \Bigg[|\delta_2| \int_\eta^T \int_0^s \frac{(s-u)^{\alpha-p_2-1}}{\Gamma(\alpha-p_2)} |\varphi(u,x(u),\, {}^cD^\kappa x(u))| duds \\
&\quad + \int_0^\xi \frac{(\xi-s)^{\alpha-p_1}}{\Gamma(\alpha-p_1+1)} |\varphi(s,x(s),\, {}^cD^\kappa x(s))| ds \Bigg] \Bigg\} \\
&\leq \|\upsilon\|\Psi(R) \Bigg[\frac{T^{\alpha-\kappa}}{\Gamma(\alpha-\kappa+1)} + \frac{T^{1-\kappa}}{|\omega_1|\Gamma(2-\kappa)} \Bigg(\frac{|\delta_2(T^{\alpha-p_2+1}-\eta^{\alpha-p_2+1})|}{\Gamma(\alpha-p_2+2)} \\
&\quad + \frac{\xi^{\alpha-p_1+1}}{\Gamma(\alpha-p_1+2)} \Bigg) \Bigg] \\
&= \|\upsilon\|\Psi(R)\tau_2.
\end{aligned}$$

From the last two inequalities, we have

$$\|\mathcal{H}x\|_{\mathcal{U}} = \|\mathcal{H}x\| + \|{}^cD^\kappa \mathcal{H}x\| \leq \|\upsilon\|\Psi(R)(\tau_1+\tau_2),$$

which shows that the operator $\mathcal{H}$ maps bounded sets (balls) into bounded sets in $C([0,T],\mathbb{R})$.

Step 2. The operator $\mathcal{H}$ maps bounded set into equicontinuous set of $C([0,T],\mathbb{R})$.

Letting $\gamma_1, \gamma_2 \in [0,T]$ with $\gamma_1 < \gamma_2$ and $x \in B_R$, we get

$$\begin{aligned}
|(\mathcal{H}x)(\gamma_2) - (\mathcal{H}x)(\gamma_1)| &\leq \|\upsilon\|\Psi(R) \Bigg\{ \frac{2|(\gamma_2-\gamma_1)^\alpha| + |\gamma_2^\alpha - \gamma_1^\alpha|}{\Gamma(\alpha+1)} \\
&\quad + \frac{|\gamma_2-\gamma_1|}{|\omega_1|} \Bigg(\frac{|\delta_2(T^{\alpha-p_2+1}-\eta^{\alpha-p_2+1})|}{\Gamma(\alpha-p_2+2)} + \frac{\xi^{\alpha-p_1+1}}{\Gamma(\alpha-p_1+2)} \Bigg) \Bigg\},
\end{aligned}$$

and

$$|({}^cD^\kappa\mathcal{H}x)(\gamma_2) - ({}^cD^\kappa\mathcal{H}x)(\gamma_1)| \le \|\upsilon\|\Psi(R)\bigg\{\frac{2|(\gamma_2-\gamma_1)^{\alpha-\kappa}| + |\gamma_2^{\alpha-\kappa} - \gamma_1^{\alpha-\kappa}|}{\Gamma(\alpha-\kappa+1)} + \frac{|\gamma_2^{1-\kappa} - \gamma_1^{1-\kappa}|}{|\omega_1|\Gamma(2-\kappa)}\bigg[\frac{|\delta_2(T^{\alpha-p_2+1} - \eta^{\alpha-p_2+1})|}{\Gamma(\alpha-p_2+2)} + \frac{\xi^{\alpha-p_1+1}}{\Gamma(\alpha-p_1+2)}\bigg]\bigg\}.$$

Since the right-hand sides of the last two inequalities tend to zero independently of $x \in B_R$ as $\gamma_2 \to \gamma_1$, so, the operator $\mathcal{H} : C([0,T],\mathbb{R}) \to C([0,T],\mathbb{R})$ is completely continuous by an application of the Arzelá–Ascoli theorem [18].

Step 3. Suppose that there exists $x \in C([0,T],\mathbb{R})$ with $x = \zeta\mathcal{H}x$, $\zeta \in (0,1)$. As in the first part of the proof, we can obtain that

$$\frac{\|x\|_{\mathcal{U}}}{\|\upsilon\|\Psi(\|x\|_{\mathcal{U}})(\tau_1+\tau_2)} \le 1.$$

By the condition (A_4), there exists $\mathfrak{J} > 0$ satisfying $\|x\|_{\mathcal{U}} \ne \mathfrak{J}$. Define $\mathcal{M} = \{x \in \mathcal{U} : \|x\|_{\mathcal{U}} < \mathfrak{J}\}$ and observe that $\mathcal{H} : \overline{\mathcal{M}} \to C([0,T],\mathbb{R})$ is continuous and completely continuous. By the definition of the set $\mathcal{M}$, there does not exist any $x \in \partial\mathcal{M}$ satisfying $x = \zeta\mathcal{H}x$ for some $\zeta \in (0,1)$. In consequence, it follows by Leray–Schauder's nonlinear alternative [18] that the operator $\mathcal{H}$ has a fixed point $x \in \overline{\mathcal{M}}$. Hence, the problem (1) has at least one solution on $[0,T]$. □

Finally, we accomplish a uniqueness result for the problem (1) by applying Banach's fixed point theorem.

Theorem 3. Suppose that $\varphi \in C([0,T]\times\mathbb{R}\times\mathbb{R},\mathbb{R})$ satisfies the condition (A_1). If $\mathcal{L}(\tau_1+\tau_2) < 1$, where τ_1 and τ_2 are respectively given by (11) and (12), then there exists a unique solution to the problem (1) on $[0,T]$.

Proof. We first show that $\mathcal{H}B_\vartheta \subset B_\vartheta$, where $\mathcal{H} : \mathcal{U} \to \mathcal{U}$ is defined in (10) and $B_\vartheta = \{x \in \mathcal{U} : \|x\| \le \vartheta\}$ with $\vartheta \ge \dfrac{N(\tau_1+\tau_2)}{1-\mathcal{L}(\tau_1+\tau_2)}$ and $\max_{t\in[0,T]}|\varphi(t,0,0)| = N < \infty$. For $x \in B_\vartheta$, $t \in [0,T]$, by the assumption (A_1) and notation (11), we get

$$\begin{aligned}\|(\mathcal{H}x)\| &= \max_{t\in[0,T]}|(\mathcal{H}x)(t)| \\ &\le \max_{t\in[0,T]}\bigg\{\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}|\varphi(s,x(s),\,{}^cD^\kappa x(s))|ds \\ &\quad + \frac{1}{|\omega_3|}\int_\eta^T\int_0^s\bigg[\frac{|\delta_1|(s-u)^{\alpha-1}}{\Gamma(\alpha)} + \frac{|\delta_2||t\omega_3-\omega_2|}{|\omega_1|\Gamma(\alpha-p_2)}(s-u)^{\alpha-p_2-1}\bigg]|\varphi(u,x(u),\,{}^cD^\kappa x(u))|duds \\ &\quad + \frac{1}{|\omega_3|}\int_0^\xi\bigg[\frac{(\xi-s)^\alpha}{\Gamma(\alpha+1)} + \frac{|t\omega_3-\omega_2|}{|\omega_1|\Gamma(\alpha-p_1+1)}(\xi-s)^{\alpha-p_1}\bigg]|\varphi(s,x(s),\,{}^cD^\kappa x(s))|ds\bigg\} \\ &\le (\mathcal{L}\vartheta+N)\bigg[\max_{t\in[0,T]}\bigg\{\frac{t^\alpha}{\Gamma(\alpha+1)} + \frac{|t\omega_3-\omega_2|}{|\omega_1\omega_3|}\bigg(\frac{|\delta_2(T^{\alpha-p_2+1}-\eta^{\alpha-p_2+1})|}{\Gamma(\alpha-p_2+2)} \\ &\quad + \frac{\xi^{\alpha-p_1+1}}{\Gamma(\alpha-p_1+2)}\bigg)\bigg\} + \frac{1}{|\omega_3|}\bigg(\frac{|\delta_1(T^{\alpha+1}-\eta^{\alpha+1})| + \xi^{\alpha+1}}{\Gamma(\alpha+2)}\bigg)\bigg] = (\mathcal{L}\vartheta+N)\tau_1,\end{aligned}$$

and

$$\begin{aligned}\|({}^cD^\kappa\mathcal{H}x)\| &\le (\mathcal{L}\vartheta+N)\bigg[\frac{T^{\alpha-\kappa}}{\Gamma(\alpha-\kappa+1)} + \frac{T^{1-\kappa}}{|\omega_1|\Gamma(2-\kappa)}\bigg(\frac{|\delta_2(T^{\alpha-p_2+1}-\eta^{\alpha-p_2+1})|}{\Gamma(\alpha-p_2+2)} \\ &\quad + \frac{\xi^{\alpha-p_1+1}}{\Gamma(\alpha-p_1+2)}\bigg)\bigg] = (\mathcal{L}\vartheta+N)\tau_2.\end{aligned}$$

In consequence, we obtain

$$\|\mathcal{H}x\|_{\mathcal{U}} = \|\mathcal{H}x\| + \|{}^cD^{\kappa}\mathcal{H}x\| \le (\mathcal{L}\vartheta + N)(\tau_1 + \tau_2) \le \vartheta.$$

Therefore, $\mathcal{H}B_\vartheta \subset B_\vartheta$ as $x \in B_\vartheta$ is an arbitrary element. Next, we accomplish that $\mathcal{H}$ is a contraction. For $x, y \in B_\vartheta$ and $t \in [0,T]$, we get

$$\begin{aligned}
\|(\mathcal{H}x) - (\mathcal{H}y)\| &= \max_{t\in[0,T]} |(\mathcal{H}x)(t) - (\mathcal{H}y)(t)| \\
&\le \max_{t\in[0,T]} \Bigg\{ \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} |\varphi(s,x(s),\, {}^cD^{\kappa}x(s)) - \varphi(s,y(s),\, {}^cD^{\kappa}y(s))| ds \\
&\quad + \frac{1}{|\omega_3|} \int_\eta^T \int_0^s \left[\frac{|\delta_1|(s-u)^{\alpha-1}}{\Gamma(\alpha)} + \frac{|\delta_2||t\omega_3 - \omega_2|}{|\omega_1|\Gamma(\alpha - p_2)} (s-u)^{\alpha-p_2-1} \right] \\
&\quad \times |\varphi(u,x(u),\, {}^cD^{\kappa}x(u)) - \varphi(u,y(u),\, {}^cD^{\kappa}y(u))| duds \\
&\quad + \frac{1}{|\omega_3|} \int_0^\xi \left[\frac{(\xi-s)^{\alpha}}{\Gamma(\alpha+1)} + \frac{|t\omega_3 - \omega_2|}{|\omega_1|\Gamma(\alpha - p_1 + 1)} (\xi - s)^{\alpha-p_1} \right] \\
&\quad \times |\varphi(s,x(s),\, {}^cD^{\kappa}x(s)) - \varphi(s,y(s),\, {}^cD^{\kappa}y(s))| ds \Bigg\} \\
&\le \mathcal{L}\|x-y\| \Bigg[\max_{t\in[0,T]} \Bigg\{ \frac{t^{\alpha}}{\Gamma(\alpha+1)} + \frac{|t\omega_3 - \omega_2|}{|\omega_1\omega_3|} \Bigg(\frac{|\delta_2(T^{\alpha-p_2+1} - \eta^{\alpha-p_2+1})|}{\Gamma(\alpha-p_2+2)} \\
&\quad + \frac{\xi^{\alpha-p_1+1}}{\Gamma(\alpha-p_1+2)} \Bigg) \Bigg\} + \frac{1}{|\omega_3|} \Bigg(\frac{|\delta_1(T^{\alpha+1} - \eta^{\alpha+1})| + \xi^{\alpha+1}}{\Gamma(\alpha+2)} \Bigg) \Bigg] \\
&= \tau_1 \mathcal{L}\|x-y\|,
\end{aligned}$$

and

$$\begin{aligned}
\|({}^cD^{\kappa}\mathcal{H}x) - ({}^cD^{\kappa}\mathcal{H}y)\| &= \max_{t\in[0,T]} |({}^cD^{\kappa}\mathcal{H}x)(t) - ({}^cD^{\kappa}\mathcal{H}y)(t)| \\
&\le \mathcal{L}\|x-y\| \Bigg[\frac{T^{\alpha-\kappa}}{\Gamma(\alpha-\kappa+1)} + \frac{T^{1-\kappa}}{|\omega_1|\Gamma(2-\kappa)} \Bigg(\frac{|\delta_2(T^{\alpha-p_2+1} - \eta^{\alpha-p_2+1})|}{\Gamma(\alpha-p_2+2)} \\
&\quad + \frac{\xi^{\alpha-p_1+1}}{\Gamma(\alpha-p_1+2)} \Bigg) \Bigg] = \tau_2 \mathcal{L}\|x-y\|.
\end{aligned}$$

Combining the foregoing inequalities, we have $\|\mathcal{H}x - \mathcal{H}y\|_{\mathcal{U}} \le \mathcal{L}(\tau_1+\tau_2)\|x-y\|_{\mathcal{U}}$, which shows that $\mathcal{H}$ is a contraction since $\mathcal{L}(\tau_1+\tau_2) < 1$. Hence, by Banach's contraction mapping principle, there exists a unique fixed point for the operator $\mathcal{H}$. In consequence, a unique solution to the problem (1) exists on $[0,T]$. □

3 Examples

In this section, we construct examples to illustrate the results derived in the last two sections.

Example 1. Consider a segmental fractional boundary value problem given by

$$\begin{aligned}
&{}^cD^{\frac{3}{2}}x(t) = \varphi(t, x(t),\, {}^cD^{\frac{1}{3}}x(t)),\ t \in [0,1], \\
&\int_0^{\frac{1}{9}} x(s)ds = \frac{1}{6}\int_{\frac{9}{10}}^1 x(s)ds,\ \int_0^{\frac{1}{9}} ({}^cD^{\frac{1}{4}}x(s))ds = \frac{1}{5}\int_{\frac{9}{10}}^1 ({}^cD^{\frac{1}{2}}x(s))ds.
\end{aligned} \tag{14}$$

Here, $\alpha = \frac{3}{2}, T = 1,\ p_1 = \frac{1}{4},\ p_2 = \frac{1}{2},\ \delta_1 = \frac{1}{6},\ \delta_2 = \frac{1}{5},\ \xi = \frac{1}{9},\ \eta = \frac{9}{10},\ \kappa = \frac{1}{3}$. Using the given data, it is found that $\omega_1 \simeq -0.00869849$, $\omega_2 \simeq -0.00966049$, $\omega_3 \simeq 0.09444444$, $\tau_1 \simeq 3.65035558$, $\tau_2 \simeq 3.69959683$, where τ_1, τ_2 are given in (11) and (12), respectively.

(i) We illustrate Theorem 1 by considering

$$\varphi(t, x(t),\, {}^cD^{\frac{1}{3}}x(t)) = \frac{1}{t^2+36}\left(\frac{(1+x)^2}{1+(1+x)^2} + \frac{|{}^cD^{\frac{1}{3}}x(t)|}{1+|{}^cD^{\frac{1}{3}}x(t)|}\right) + \frac{e^{-t}}{\sqrt{t^2+4}}. \tag{15}$$

Observe that

$$|\varphi(t, x(t),\, {}^cD^{\frac{1}{3}}x(t))| \leq \frac{2}{t^2+36} + \frac{e^{-t}}{\sqrt{t^2+4}} = \psi(t),$$

and $\|\psi\| = \frac{5}{9}$, $\mathcal{L} = \frac{1}{36}$. Furthermore, $\mu \simeq 5.67377488$, and $\mu\mathcal{L} \simeq 0.15760486 < 1$, where μ is given in (13). Thus, by Theorem 1, there exists at least one solution to the problem (14) with $\varphi(t, x(t), {}^c D^{\frac{1}{3}}x(t))$ defined in (15) on $[0, 1]$.

(ii) For illustrating Theorem 2, we consider

$$\varphi(t, x(t),\, {}^cD^{\frac{1}{3}}x(t)) = \frac{1}{\sqrt{t^6+625}}\left(\frac{|x|^2}{1+|x|} + \sin({}^cD^{\frac{1}{3}}x(t)) + \frac{1}{2}\right), \tag{16}$$

and find that

$$|\varphi(t, x(t),\, {}^cD^{\frac{1}{3}}x(t))| \leq \frac{1}{\sqrt{t^6+625}}\left(\|x\|_{\mathcal{U}} + \frac{1}{2}\right).$$

Clearly, $\upsilon(t) = \frac{1}{\sqrt{t^6+625}}$ with $\|\upsilon\| = \frac{1}{25}$ and $\Psi(\|x\|_{\mathcal{U}}) = \|x\|_{\mathcal{U}} + \frac{1}{2}$. By the condition (A_4), we find that $\mathcal{K} > \mathcal{K}_1$, where $\mathcal{K}_1 \simeq 0.20821339$. Thus, the assumptions of Theorem 2 hold true and hence, its conclusion implies that the problem (14) with $\varphi(t, x(t),\, {}^cD^{\frac{1}{3}}x(t))$ given by (16) has at least one solution on $[0, 1]$.

(iii) For explaining Theorem 3, we take

$$\varphi(t, x(t),\, {}^cD^{\frac{1}{3}}x(t)) = \frac{1}{(t^2+25)}\left(\sqrt{([{}^cD^{\frac{1}{3}}x(t)]^2+4)} + \tan^{-1}x(t)\right) + \frac{3}{5+t^3}, \tag{17}$$

and note that $\varphi(t, x(t), {}^c D^{\frac{1}{3}}x(t))$ satisfies the condition (A_1) with $\mathcal{L} = \frac{1}{25}$ and $\mathcal{L}(\tau_1+\tau_2) \simeq 0.293998096 < 1$. As all the assumptions of Theorem 3 hold true, so its conclusion applies to the problem (14) with $\varphi(t, x(t),\, {}^cD^{\frac{1}{3}}x(t))$ given in (17).

Conclusion

We introduced a new notion of segmental type nonlocal fractional boundary conditions and obtained existence and uniqueness results for a more general type fractional differential equation complemented with these conditions. Though the standard tools of the fixed point theory are employed to study the problem (1), yet their imposition to the given problem produces new results for it. Moreover, the results for segmental type periodic, anti-periodic and mixed periodic-anti-periodic boundary conditions follow as special cases by fixing the parameters δ_1 and δ_2 in the results accomplished in this article as described in the second last paragraph of Introduction section. In case $p_1 = p_2 = p$, our results correspond to the ones with boundary conditions

$$\int_0^\xi x(s)ds = \delta_1 \int_\eta^T x(s)ds,\ \int_0^\xi ({}^cD^p x(s))ds = \delta_2 \int_\eta^T ({}^cD^p x(s))ds,\ 0 < p < 1.$$

Thus, our results are not only useful in the given configuration but also give rise a new avenue for research on fractional boundary value problems.

Author Contributions

All authors contributed equally to this work, participated in its revision and approved the final submission.

Conflict of Interest

The authors declare no conflict of interest.

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https://doi.org/10.31489/2026M2/17-32 *Research article*

Qualitative analysis of a system of non-homogeneous doubly nonlinear parabolic equations

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We consider the qualitative properties of solutions to a coupled system of nonhomogeneous doubly nonlinear parabolic equations on the whole line with an exponentially varying density. The characteristic features of degeneracy at vanishing values and gradients are analyzed, and the need for weak solutions and reliable comparison estimates is identified and justified. Using a nonlinear splitting method, we construct explicit comparison functions and, on this basis, apply a comparison principle to obtain global existence of nonnegative solutions for sufficiently small initial data in the slow-diffusion regime. In addition, a self-similar reduction is performed via a nonlinear change of variables, which converts the problem into an auxiliary system for similarity profiles. An asymptotic representation of these self-similar solutions is derived, and the dependence of the solution behaviour on the governing parameters is clarified. It is shown how the parameters affect spatial localization and finite-speed propagation, and a Fujita-type criterion is obtained that provides conditions for the existence and nonexistence of global solutions. To support the analytical results, numerical simulations implemented in Python produce solution profiles and graphical illustrations of the nonlinear diffusion dynamics. The computations agree with the qualitative predictions and help visualize the transition between parameter regimes.

Keywords: doubly nonlinear parabolic system, strong-coupling diffusion, exponential density, weak solutions, self-similar solution, comparison principle, Barenblatt profile, global solvability, asymptotic behaviour, numerical illustrations.

2020 Mathematics Subject Classification: 35B51, 35C06, 35D30, 35K45, 35K55.

Introduction

We investigate a system of parabolic partial differential equations expressed in divergence form, defined on the domain $Q = \{(t,x) \,|\, t > 0, x \in \mathbb{R}\}$:

$$\begin{cases} \rho(x)\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(u^{\sigma_1}\left|\frac{\partial u}{\partial x}\right|^{p-2}\frac{\partial u}{\partial x}\right) + \rho(x)v^{q_1}, \\ \rho(x)\frac{\partial v}{\partial t} = \frac{\partial}{\partial x}\left(v^{\sigma_2}\left|\frac{\partial v}{\partial x}\right|^{p-2}\frac{\partial v}{\partial x}\right) + \rho(x)u^{q_2}, \end{cases} \tag{1}$$

$$\begin{cases} u\big|_{t=0} = u_0(x) \geq 0, \\ v\big|_{t=0} = v_0(x) \geq 0, \end{cases} \qquad \forall x \in \mathbb{R}, \tag{2}$$

where $p \geq 2,\ \sigma_i \geq 0,\ q_i > 0\ (i = 1,2),\ \ q_1 q_2 \neq 1,\ \alpha \in \mathbb{R},\ \rho(x) = e^{\alpha x}$ are the numerical parameters and $u = u(t,x) \geq 0$, $v = v(t,x) \geq 0$.

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The system (1)–(2) arises in the mathematical modelling of several nonlinear processes observed in applied sciences, including heat propagation [1–3], nonlinear diffusion in a two-component medium [4–6], gas filtration through porous structures [7, 8], and fluid dynamics in heterogeneous domains. Furthermore, system (1)–(2) describes many physical processes [9–11].

H. Murakawa [12] investigates the connection between cross-diffusion and reaction-diffusion systems, examining their structure and the ways in which these two classes of systems are interconnected. The author offers new considerations on how to view cross-diffusion terms in mathematical models of species interactions or chemical processes as a special case of reaction-diffusion equations. This is beneficial to understand how complex dynamics behave in systems with multiple components.

X. Xu and T. Cheng studied a strongly coupled nonlinear filtration system with nonlocal source terms [13], arising in non-Newtonian fluid flow and a polytropic filtration system:

$$\begin{cases} u_t = \operatorname{div}(|\nabla u^m|^{p-2}\nabla u^m) + \int_\Omega u^\alpha\, dx \int_\Omega v^r\, dx, & (x,t) \in \Omega \times (0,\infty), \\ v_t = \operatorname{div}(|\nabla v^n|^{q-2}\nabla v^n) + \int_\Omega u^s\, dx \int_\Omega v^\beta\, dx, & (x,t) \in \Omega \times (0,\infty), \end{cases}$$

where $p, q > 1$, $m, n, r, s > 0$, $\alpha, \beta \geq 0$, $m(p-1), n(p-1) < 1$, $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary, and the non-negative non-trivial initial data (u_0, v_0) satisfy $(u_0^m, v_0^n) \in (L^\infty(\Omega) \cap W_0^{1,p}(\Omega), L^\infty(\Omega) \cap W_0^{1,q}(\Omega))$. The authors establish explicit conditions ensuring finite-time extinction of solutions. That is, if the parameters satisfy a certain balance condition between diffusion and reaction, the solution satisfies:

$$u(x,t) = v(x,t) = 0, \quad \forall\, t \geq T_{\text{ext}},$$

for some finite T_{ext}. If extinction does not occur, the authors derive the long-time decay rate for solutions:

$$\|u(\cdot,t)\|_{L^\infty} + \|v(\cdot,t)\|_{L^\infty} \leq Ct^{-\gamma},$$

where γ depends on the exponents $m, n, p, q, \alpha, r, s, \beta$.

X. Sun, B. Liu, and F. Li [14] considered the following system of parabolic equations with a time-dependent source

$$\begin{cases} u_t = \Delta u + t^{\sigma_1}\left(1+|x|^2\right)^{n/2} u^\alpha v^p, \\ v_t = \Delta v + t^{\sigma_2}\left(1+|x|^2\right)^{m/2} u^q v^\beta,\ x \in \Omega \subset \mathbb{R}^N,\ 0 < t < T. \\ u = v = 0,\ x \in \partial\Omega,\ 0 < t < T, \\ u\,(x,0) = u_0\,(x) \geq 0,\ v\,(x,0) = v_0\,(x) \geq 0,\ x \in \Omega, \end{cases}$$

where p, q, α, β, m, n, σ_1, σ_2 are non-negative numbers, and T is the limit of the existence time of classical solutions of the problem. X. Sun et al. have shown that the problem admits global solutions when the conditions $pq \leq (1-\alpha)(1-\beta)$, $\alpha \leq 1$, $\beta \leq 1$ are satisfied; They further proved that solutions blow up in finite time when the inequality $(1-\alpha)(1-\beta) < pq \leq (pq)_c$ or $1 < \alpha < \alpha_c$ or $1 < \beta \leq \beta_c$ is satisfied; $pq > (pq)_c$, $\alpha > \alpha_c$, $\beta > \beta_c$ and proved that both global and non-global solutions exist under the conditions where

$$(pq)_c = (1-\alpha)(1-\beta) + \frac{2}{N}\max\left\{\sigma\,(p, 1-\beta)\,, \sigma\,(1-\alpha, q)\,, 0\right\},$$
$$\alpha_c = 1 + \frac{n + 2\,(\sigma_1 + 1)}{N}, \beta_c = 1 + \frac{n + 2\,(\sigma_2 + 1)}{N},$$
$$\sigma\,(a,b) = a(1 + \frac{m}{2})(1+\sigma_2) + b(1 + \frac{n}{2})(1+\sigma_1).$$

R. Castillo, et al. [15] considered the following system in a time-dependent, heterogeneous environment

$$\begin{cases} u_t = \operatorname{div}\left(w(x)\nabla u\right) + t^r v^p, \\ v_t = \operatorname{div}\left(w(x)\nabla v\right) + t^s v^p,\ x \in \mathbb{R}^N,\ 0 < t < T, \\ u\left(x, 0\right) = u_0\left(x\right),\ v\left(x, 0\right) = v_0\left(x\right),\ x \in \mathbb{R}^N, \end{cases}$$

where $0 \leq u_0,\ v_0 \in L^\infty\left(\mathbb{R}^N\right)$,

$$p, q > 0,\ pq > 1,\ r, s > -1$$

and $w(x) = |x|^a$, $a > 0$. R. Castillo et al. showed that the Cauchy problem has local and global solutions, and that for the solution

$$||u(.,t)||_\infty \leq C(1+t)^{-\frac{N}{(2-\alpha)r_1}}, \quad ||v(.,t)||_\infty \leq C(1+t)^{-\frac{N}{(2-\alpha)r_2}}$$

proved that the bounds are valid, where $r_1 = \frac{N(pq-1)}{(2-\alpha)(r+1+p(s+1))}$, $r_2 = \frac{N(pq-1)}{(2-\alpha)(s+1+q(r+1))}$.

X. Tao and Z.B. Fang [16], L.E. Payne, and G.A. Philippin [17], and the following system with a time-dependent function was considered

$$\begin{cases} u_t = \Delta u + k_1(t) u^p v^q, \\ v_t = \Delta v + k_2(t) u^s v^r,\ x \in \Omega \subset \mathbb{R}^N,\ 0 < t < T, \\ u = v = 0,\ x \in \partial\Omega,\ 0 < t < T, \\ u\left(x, 0\right) = u_0\left(x\right),\ v\left(x, 0\right) = v_0\left(x\right),\ x \in \Omega, \end{cases}$$

where $p,\ q,\ r,\ s \geq 0,\ k_1(t), k_2(t) \in C^1$ are positive, and $u_0(x), v_0(x) \in C^1$ are non-negative functions. The authors showed that there exist global solutions to the Cauchy problem for sufficiently small initial functions under the conditions $p + q \leq 1,\ r + s \leq 1$ or $p > 1,\ r + s \geq 1$, that for sufficiently large initial functions under the conditions $p > 1,\ r + s \geq 1$ the solutions to the problem tend to infinity in a finite time, and that Sobolev estimates were obtained.

In [18], self-similar and approximate self-similar solutions to a nonlinear reaction-diffusion problem were studied by Sh.A. Sadullaeva:

$$\begin{cases} \frac{\partial(\rho(x)u)}{\partial t} = \operatorname{div}\left(|x|^n v^{m_1-1}|\nabla u|^{p-2}\nabla u\right) + \rho(x)\gamma(t)u^{\beta_1}, \\ \frac{\partial(\rho(x)v)}{\partial t} = \operatorname{div}\left(|x|^n u^{m_2-1}|\nabla v|^{p-2}\nabla v\right) + \rho(x)\gamma(t)v^{\beta_2}, \\ u(0,x) = u_0(x) \geq 0, \quad v(0,x) = v_0(x) \geq 0, \quad x \in \mathbb{R}^N, \end{cases}$$

where $m_i, n \in \mathbb{R}$, $\beta_i \geq 1,\ p \geq 2$ were given positive numbers, and $u_0(x),\ v_0(x) \geq 0$, $\rho(x) = |x|^{-l}$, $l > 0,\ 0 < \gamma(t) \in C(\mathbb{R}_+),\ i = 1, 2$.

In [19], D.B. Nigmanova examined the Fujita-type [20] global existence and blow-up conditions for a nonlinear parabolic system with initial conditions and variable density. Moreover, the author studied solution estimates and the asymptotic behaviour of self-similar solutions under slow, fast, and critical diffusion cases, highlighting the role of spatially varying density:

$$\begin{cases} |x|^{-l}\frac{\partial u}{\partial t} = \nabla\left(|x|^n|\nabla u^k|^{p-2}\nabla u^{l_1}\right) + \varepsilon|x|^{-l}u^{p_1}v^{q_1}, \\ |x|^{-l}\frac{\partial v}{\partial t} = \nabla\left(|x|^n|\nabla v^k|^{p-2}\nabla v^{l_2}\right) + \varepsilon|x|^{-l}u^{p_2}v^{q_2}, \\ u(0,x) = u_0(x) \geq 0, \quad v(0,x) = v_0(x) \geq 0, \quad x \in \mathbb{R}^N, \end{cases}$$

where $\varepsilon = \pm 1,\ k,\ m_i,\ q_i,\ l_i \geq 1,\ i = 1, 2,\ p \geq 2,\ n,\ l$ are given parameters.

In [21, 22], a nonlinear parabolic equation involving a source term and non-uniform density was investigated by the authors in the following form:

$$\rho(x)\frac{\partial u}{\partial t} = \operatorname{div}\left(u^{m-1}|\nabla u|^{p-2}\nabla u\right) + u^{\beta},$$

and

$$\rho(x)\frac{\partial u}{\partial t} = \operatorname{div}\left(u^{m-1}|\nabla u|^{p-2}\nabla u\right) + \rho(x)u^{\beta}, \quad x \in \mathbb{R}^N,$$

where $\rho(x) = |x|^{-n}$ or $\rho(x) = (1+|x|)^{-n}$, $n \geq 0$.

The authors established criteria for finite-time blow-up of solutions to the Cauchy problem.

D. Aronson and J. Graveleau focus on the porous medium equation and derive self-similar solutions to describe the hole-filling phenomenon [23]. The authors rewrite the radial dynamics using the logarithmic transformation $s = \log r$. This substitution converts the governing system into a weighted porous-medium equation

$$\rho(x)u_t = (u^m)_{xx} \tag{3}$$

defined on $Q = \mathbb{R}_+ \times \mathbb{R}^N$, characterized by the exponential density function $\rho(x) = e^{2x}$. This exponential structure is fundamental for determining support of the solution, $[-a, \infty)$, and analyzing the finite-time loss of the inner interface, where a is a free positive parameter describing the initial left endpoint of the support in the x variable or $r_0 = e^{-a} > 0$.

For the nonlinear equation, the density functions of the medium encountered in the $\rho(x) = \{|x|^{-\alpha}, e^{-x}, e^{-x^2}\}$ forms in the work of V. Galaktionov and J. King [24] relate to the asymptotic behaviour of blow-up solutions of the equation (3) with the Cauchy problem.

D. Andreucci and A. Tedeev take the density function as $\rho(x) = e^{g(|x|)}$ and proved for the solutions sup estimates or the decay rate at infinity, the property of finite speed of propagation and support estimates for the following equation [25]:

$$\rho(x)\frac{\partial u}{\partial t} = \operatorname{div}\left(\rho(x)u^{m-1}|\nabla u|^{p-2}\nabla u\right),$$

with the initial condition and under some assumptions for $g(|x|)$ function.

In [26], self-similar solutions to the Cauchy problem for a doubly nonlinear equation incorporating exponential effects were investigated. The equation is given by:

$$\rho(x)\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(u^{\sigma}\left|\frac{\partial u}{\partial x}\right|^{p-2}\frac{\partial u}{\partial x}\right), \quad (t,x) \in Q,$$

$$u\Big|_{t=0} = u_0(x) \geq 0, \quad x \in \mathbb{R},$$

where $Q = \{(t,x) : t > 0,\, x \in \mathbb{R}\}$, $p \geq 2$, $\sigma \in \mathbb{R}_+$, and $\rho(x) = e^{x}$.

They established the conditions for the existence of Fujita-type global solutions and identified the criteria under which a sub-solution exists for the equation.

The system (1)–(2) may be degenerate at the points where $u = 0$ or $\frac{\partial u}{\partial x} = 0$ and $v = 0$ or $\frac{\partial v}{\partial x} = 0$ [27–29]. Given that classical solutions may not exist in general, we focus on non-negative weak solutions defined by the following weak formulation.

Definition 1. A non-negative function $u(t,x)$ and $v(t,x)$ are a weak solution to the problem (1)–(2) in Q if for any compact subset $\Omega \subset \mathbb{R}$ and any sub-interval $[t_1, t_2] \subset (0,T)$:

$$0 \leq u, v \in C\left(0, T : L_2(\Omega)\right),\ u^{\frac{p+\sigma_1}{p}},\ v^{\frac{p+\sigma_2}{p}} \in L_p\left(0, T : W^1_{p,0}(\Omega)\right),$$

$$\int_0^T\int_\Omega \rho(x)v^{q_1+1}dxdt < \infty, \quad \int_0^T\int_\Omega \rho(x)u^{q_2+1}dxdt < \infty,$$

and for any test functions $\phi, \psi \in W_2^1\left(0, T : L_2(\Omega)\right) \cap L_p\left(0, T : W_{p,0}^1(\Omega)\right)$:

$$\int_\Omega u\phi dx\Big|_{t_1}^{t_2} + \int_{t_1}^{t_2}\int_\Omega \left(-u\phi_t + u^{\sigma_1}\left|u_x\right|^{p-2} u_x\phi_x\right) dxdt = \int_{t_1}^{t_2}\int_\Omega v^{q_1}\phi dxdt,$$
$$\int_\Omega v\psi dx\Big|_{t_1}^{t_2} + \int_{t_1}^{t_2}\int_\Omega \left(-v\psi_t + v^{\sigma_2}\left|v_x\right|^{p-2} v_x\psi_x\right) dxdt = \int_{t_1}^{t_2}\int_\Omega u^{q_2}\psi dxdt,$$

and take the initial data as follows

$$\lim_{t\to 0}\int_\Omega u(t,x)\varphi(x)dx = \int_\Omega u_0(x)\varphi(x)dx,$$
$$\lim_{t\to 0}\int_\Omega v(t,x)\chi(x)dx = \int_\Omega v_0(x)\chi(x)dx,$$

for any smooth compactly supported functions φ and χ.

A self-similar equation refers to a differential equation whose solution can be expressed as a function of a combination of independent variables, reflecting the scaling invariance of the process being considered. Self-similar solutions to differential equations are characterized by the fact that the solution depends on a particular combination of the independent variables, such as $\xi = xt^{-\alpha}$, rather than on each variable separately [30, 31]. This property allows the problem involving partial differential equations to be reduced to an ordinary differential equation, significantly simplifying its analysis.

1 *Formulas and theorems*

1.1 *Formulation of a self-similar system of equations*

By applying the following transformation, system (1)–(2) is reduced to the auxiliary system (4):

$$\begin{cases} u(t,x) = (t+T)^{-\alpha_1} f(\xi) \\ v(t,x) = (t+T)^{-\alpha_2}\varphi(\xi) \end{cases}, \quad \xi = (t+T)^{-\gamma}\cdot\left(\frac{p}{\alpha}e^{\frac{\alpha x}{p}}\right), \tag{4}$$

where $\alpha_i = \frac{q_i+1}{q_1 q_2 - 1}$, $\gamma p = 1 + \alpha_i(\sigma_i + p - 2),\ i = 1, 2,\ T > 0$, for $f(\xi)$ and $\varphi(\xi)$, we obtain the system of ordinary differential equations:

$$\begin{cases} \frac{1}{\xi^{p-1}}\frac{d}{d\xi}\left(\xi^{p-1} f^{\sigma_1}\left|\frac{df}{d\xi}\right|^{p-2}\frac{df}{d\xi}\right) + \alpha_1 f + \gamma\xi\frac{df}{d\xi} + \varphi^{q_1} = 0, \\ \frac{1}{\xi^{p-1}}\frac{d}{d\xi}\left(\xi^{p-1}\varphi^{\sigma_2}\left|\frac{d\varphi}{d\xi}\right|^{p-2}\frac{d\varphi}{d\xi}\right) + \alpha_2\varphi + \gamma\xi\frac{d\varphi}{d\xi} + f^{q_2} = 0. \end{cases} \tag{5}$$

Based on the initial formulation of the problem, our goal is to find a non-trivial, non-negative solution to equation (5) that satisfies the following condition:

$$\begin{cases} f'(0) = 0, & f(d_1) = 0, \quad 0 < d_1 < \infty, \\ \varphi'(0) = 0, & \varphi(d_2) = 0, \quad 0 < d_2 < \infty. \end{cases}$$

1.2 *Slow diffusion case: $\sigma_i + p - 2 > 0$ $(i = 1, 2)$. A global solution to the problem*

A comparison principle, as presented in [32, chapter I]; [33, p. 21] is employed to prove the global existence of weak solutions to system (1)–(2). Accordingly, a new system of equations is formulated based on the standard method outlined in [32, p. 19], [34,35] with the Barenblatt profile [7]:

$$\begin{cases} u_+(t,x) = (t+T)^{-\alpha_1}\,\overline{f}(\xi), \\ v_+(t,x) = (t+T)^{-\alpha_2}\,\overline{\varphi}(\xi), \end{cases} \tag{6}$$

$$\begin{cases} \overline{f}(\xi) = A_1 \left(a - \xi^{\frac{p}{p-1}}\right)_+^{\frac{p-1}{\sigma_1+p-2}}, \\ \overline{\varphi}(\xi) = A_2 \left(a - \xi^{\frac{p}{p-1}}\right)_+^{\frac{p-1}{\sigma_2+p-2}}, \end{cases} \tag{7}$$

where $a > 0,\ A_i = \left|\frac{\gamma(\sigma_i+p-2)^{p-1}}{p^{p-1}}\right|^{\frac{1}{\sigma_i+p-2}}, \quad i = 1,2, \quad (k)_+ = \max(0,k).$

For convenience, we introduce the following notation:

$$l_i = \frac{(p-1)q_i}{\sigma_{3-i}+p-2} - \frac{p-1}{\sigma_i+p-2}, \quad m_i = A_i^{-1}A_{3-i}^{q_i}, \quad i = 1,2.$$

Theorem 1. Let $\sigma_i + p - 2 > 0,\ q_i > \dfrac{\sigma_{3-i}+p-2}{\sigma_i+p-2}, \alpha_i(\sigma_i+p-2) + m_i a^{l_i} \leq 1,\ i = 1,2,$

$$u_+(0,x) \geq u_0(x), \quad v_+(0,x) \geq v_0(x), \quad x \in \mathbb{R}.$$

If the initial functions $u_0(x)$ and $v_0(x)$ are sufficiently small, then the following property holds:

$$u(t,x) \leq u_+(t,x), \quad v(t,x) \leq v_+(t,x) \quad \text{in } Q,$$

where $u_+(x)$ and $v_+(x)$ above-mentioned functions.

Proof. Theorem 1 is established using the solution comparison method with $u_+(x)$ and $v_+(x)$ taken as auxiliary comparison functions. By substituting expression (6) into system (1)–(2), we derive the following inequality:

$$\begin{cases} \frac{1}{\xi^{p-1}}\frac{d}{d\xi}\left(\xi^{p-1}f^{\sigma_1}\left|\frac{df}{d\xi}\right|^{p-2}\frac{df}{d\xi}\right) + \alpha_1 f + \gamma\xi\frac{df}{d\xi} + \varphi^{q_1} & \leq 0, \\ \frac{1}{\xi^{p-1}}\frac{d}{d\xi}\left(\xi^{p-1}\varphi^{\sigma_2}\left|\frac{d\varphi}{d\xi}\right|^{p-2}\frac{d\varphi}{d\xi}\right) + \alpha_2\varphi + \gamma\xi\frac{d\varphi}{d\xi} + f^{q_2} & \leq 0, \end{cases} \tag{8}$$

With the specific expressions for $\overline{f}(\xi)$ and $\overline{\varphi}(\xi)$ given in (7), inequality (8) simplifies to the following form:

$$\begin{cases} -1 + \frac{1+q_1}{q_1q_2-1}(\sigma_1+p-2) + m_1\left(a - \xi^{\frac{p}{p-1}}\right)_+^{l_1} \leq 0, \\ -1 + \frac{1+q_2}{q_1q_2-1}(\sigma_2+p-2) + m_2\left(a - \xi^{\frac{p}{p-1}}\right)_+^{l_2} \leq 0. \end{cases}$$

It can be readily verified that

$$m_1\left(a - \xi^{\frac{p}{p-1}}\right)_+^{l_1} \leq m_1 a^{l_1}, \quad m_2\left(a - \xi^{\frac{p}{p-1}}\right)_+^{l_2} \leq m_2 a^{l_2}.$$

Then, taking into account the assumptions of Theorem 1, applying the comparison principle, and the initial functions satisfied the following inequality:

$$u_+(0,x) \geq u_0(x), \quad v_+(0,x) \geq v_0(x), \quad x \in \mathbb{R}.$$

We obtain the following result:

$$u(t,x) \leq u_+(t,x), \quad v(t,x) \leq v_+(t,x).$$

This completes the proof of the theorem. □

1.3 Asymptotic behaviour of the self-similar solutions

We now investigate the asymptotic behaviour of self-similar solutions corresponding to system (5). A self-similar solution is considered in the form:

$$f(\xi) = \overline{f}(\xi)\, y(\eta), \quad \varphi(\xi) = \overline{\varphi}(\xi)\, z(\eta), \tag{9}$$

where

$$\eta = -\ln\left(a - \xi^{\frac{p}{p-1}}\right), \quad \overline{f}(\xi) = \left(a - \xi^{\frac{p}{p-1}}\right)_+^{\frac{p-1}{\sigma_1+p-2}}, \quad \overline{\varphi}(\xi) = \left(a - \xi^{\frac{p}{p-1}}\right)_+^{\frac{p-1}{\sigma_2+p-2}}, \quad a > 0.$$

Under the assumption that $y(\eta) > 0$ and $z(\eta) > 0$, substitution of expression (9) into system (5) yields the following nonlinear system of equations:

$$\begin{cases} \frac{d}{d\eta}(L_1 y) + b_{11}(\eta) L_1 y + b_{12}(\eta)\left(\frac{dy}{d\eta} + b_{10}(\eta) y\right) + b_{13}(\eta) z^{q_1} + b_{14}(\eta) y = 0, \\ \frac{d}{d\eta}(L_2 z) + b_{21}(\eta) L_2 z + b_{22}(\eta)\left(\frac{dz}{d\eta} + b_{20}(\eta) z\right) + b_{23}(\eta) y^{q_2} + b_{24}(\eta) z = 0, \end{cases} \tag{10}$$

here

$$b_{i0}(\eta) = -\frac{p-1}{\sigma_i + p - 2}, \; b_{i1}(\eta) = (p-1)\left(\frac{e^{-\eta}}{a - e^{-\eta}} - \frac{p-1}{\sigma_i + p - 2}\right), \; b_{i2} = \gamma\left(\frac{p-1}{p}\right)^p,$$

$$b_{i3} = \left(\frac{p-1}{p}\right)^p \frac{e^{-s_i \eta}}{a - e^{-\eta}}, \; b_{i4} = \alpha_i \left(\frac{p-1}{p}\right)^p \cdot \frac{e^{-\eta}}{a - e^{-\eta}}, \; s_i = 1 + \frac{p-1}{\sigma_{3-i} + p - 2} q_i - \frac{p-1}{\sigma_i + p - 2} \quad (i = 1, 2),$$

$$L_1 y = y^{\sigma_1}\left|\frac{dy}{d\eta} + b_{10}(\eta) y\right|^{p-2}\left(\frac{dy}{d\eta} + b_{10}(\eta) y\right), \; L_2 z = z^{\sigma_2}\left|\frac{dz}{d\eta} + b_{20}(\eta) z\right|^{p-2}\left(\frac{dz}{d\eta} + b_{20}(\eta) z\right).$$

There was supposed to be a domain $\xi \in [\xi_0, \xi_1)$, where $0 < \xi_0 < \xi_1$, and $\xi_1 = a^{\frac{p-1}{p}}$.
Therefore, the function $\eta(\xi)$ satisfies the following properties [36, 37]:

$$\eta'(\xi) > 0 \quad \text{at } \xi \in [\xi_0, \xi_1), \quad \eta_0 = \eta(\xi_0) > 0, \quad \lim_{\xi \to \xi_1^-} \eta(\xi) = +\infty.$$

The auxiliary system of equations (10) is considered below under the following conditions:

$$\lim_{\eta \to +\infty} b_{ij}(\eta) = b_{ij}^0 \quad (i = 1, 2; \; j = 0, 1, 2, 3, 4),$$

are assumed to exist, be finite and non-negative, that is:

$$0 \le \left|b_{ij}^0\right| < +\infty.$$

To explore the behaviour of system solutions (1) as $\eta \to +\infty$, we first apply the transformations given in equations (4) and (9). This reformulates the original problem into the study of system (10), under the assumption that its solutions satisfy certain conditions in the vicinity of $+\infty$ [38, 39]. Specifically, the functions must remain positive and obey the inequalities:

$$y(\eta) > 0, \quad y' + b_{10}(\eta) y \ne 0,$$

$$z(\eta) > 0, \quad z' + b_{20}(\eta) z \ne 0.$$

Our focus now turns to examining the asymptotic properties of such positive solutions to system (10), particularly those that converge to a finite, nonzero value as $\eta \to +\infty$.

2 The main theoretical results

To simplify the presentation, we define the following notation:

$$c_{i1} = \Big(\frac{p-1}{\sigma_i + p - 2}\Big)^p, \quad c_{i2} = -\gamma \cdot \frac{p-1}{\sigma_i + p - 2} \cdot \Big(\frac{p-1}{p}\Big)^p, \quad c_{i3} = \Big(\frac{p-1}{p}\Big)^p \cdot \frac{1}{a} \quad (i = 1, 2).$$

Let $y(\eta) = y^0 + o(1),\ z(\eta) = z^0 + o(1),\ 0 < y^0, z^0 < +\infty$, as $\eta \to +\infty$, and suppose the following equality holds:

$$(1 + q_1)(\sigma_1 + p - 2) = (1 + q_2)(\sigma_2 + p - 2).$$

Then the following theorems are valid.

Theorem 2. Let $s_1 = s_2 = 0$, then the self-similar solution of equations (1) has the following asymptotic form:

$$\begin{cases} u_A(t,x) \simeq y^0 (T+t)^{\frac{1+q_1}{1-q_1 q_2}} \left(a - \left(\left(\frac{p}{\alpha} \cdot \frac{e^{\frac{\alpha x}{p}}}{(t+T)^\gamma} \right)^{\frac{p}{p-1}} \right) \right)^{\frac{p-1}{\sigma_1 + p - 2}} (1 + o(1)), \\ v_A(t,x) \simeq z^0 (T+t)^{\frac{1+q_2}{1-q_1 q_2}} \left(a - \left(\left(\frac{p}{\alpha} \cdot \frac{e^{\frac{\alpha x}{p}}}{(t+T)^\gamma} \right)^{\frac{p}{p-1}} \right) \right)^{\frac{p-1}{\sigma_2 + p - 2}} (1 + o(1)), \end{cases} \tag{11}$$

as $x \to \frac{p}{\alpha} \ln \left(\frac{\alpha}{p} a^{\frac{p-1}{p}} (T+t)^\gamma \right)$, where w_1, w_2 solutions of the following system:

$$c_{i1} w_i^{\sigma_i + p - 1} + c_{i2} w_i + c_{i3} w_{3-i}^{q_i} = 0 \quad (i = 1, 2).$$

Theorem 3. Let $s_1 = 0$ and $s_2 > 0$, then the self-similar solution of system (1)–(2) has the asymptotic expansion of the form of (11) as $x \to \frac{p}{\alpha} \ln \left(\frac{\alpha}{p} a^{\frac{p-1}{p}} (T+t)^\gamma \right)$, where w_1, w_2 solutions of the following system:

$$\begin{aligned} &c_{11} w_1^{\sigma_1 + p - 1} + c_{12} w_1 + c_{13} w_2^{q_1} = 0, \\ &c_{21} w_2^{\sigma_2 + p - 1} + c_{22} w_2 = 0. \end{aligned}$$

Theorem 4. Let $s_1 > 0$ and $s_2 = 0$, then the self-similar solution of system (1)–(2) has the asymptotic expansion of the form of (11) as

$$x \to \frac{p}{\alpha} \ln \left(\frac{\alpha}{p} a^{\frac{p-1}{p}} (T+t)^\gamma \right),$$

where w_1, w_2 solutions of the following system:

$$\begin{aligned} &c_{11} w_1^{\sigma_1 + p - 1} + c_{12} w_1 = 0, \\ &c_{21} w_2^{\sigma_2 + p - 1} + c_{22} w_2 + c_{23} w_1^{q_2} = 0. \end{aligned}$$

Theorem 5. Let $s_1 > 0$ and $s_2 > 0$, then the self-similar solution of system (1)–(2) has the asymptotic expansion of the form of (11) as

$$x \to \frac{p}{\alpha} \ln \left(\frac{\alpha}{p} a^{\frac{p-1}{p}} (T+t)^\gamma \right),$$

where w_1, w_2 solutions of the following system:

$$c_{i1} w_i^{\sigma_i + p - 1} + c_{i2} w_i = 0, \quad i = 1, 2.$$

Proof. Assuming that the system (10) takes the form

$$\mathfrak{g}_1(\eta) = L_1 y, \qquad \mathfrak{g}_2(\eta) = L_2 z, \tag{12}$$

we obtain the following identities:

$$\begin{cases} \mathfrak{g}_1'(\eta) & \equiv -b_{11}(\eta)\mathfrak{g}_1(\eta) - b_{12}(\eta)\mathfrak{g}_1(\eta)\eta^{\frac{1}{p-1}} y^{-\frac{\sigma_1}{p-1}} - b_{13}(\eta) z^{q_1} - b_{14}(\eta) y, \\ \mathfrak{g}_2'(\eta) & \equiv -b_{21}(\eta)\mathfrak{g}_2(\eta) - b_{22}(\eta)\mathfrak{g}_2(\eta)\eta^{\frac{1}{p-1}} z^{-\frac{\sigma_2}{p-1}} - b_{23}(\eta) y^{q_2} - b_{24}(\eta) z. \end{cases} \tag{13}$$

Now, consider the auxiliary functions

$$\begin{cases} h_1(\chi_1, \eta) & \equiv -b_{11}(\eta)\chi_1 - b_{12}(\eta)\chi_1\eta^{\frac{1}{p-1}} y^{-\frac{\sigma_1}{p-1}} - b_{13}(\eta) z^{q_1} - b_{14}(\eta) y, \\ h_2(\chi_2, \eta) & \equiv -b_{21}(\eta)\chi_2 - b_{22}(\eta)\chi_2\eta^{\frac{1}{p-1}} z^{-\frac{\sigma_2}{p-1}} - b_{23}(\eta) y^{q_2} - b_{24}(\eta) z, \end{cases}$$

where $\chi_i \in \mathbb{R}$ $(i = 1, 2)$.

Assume initially that $s_i = 0$ $(i = 1, 2)$. In this case, the functions $h_i(\chi_i, \eta)$ $(i = 1, 2)$ retain a constant sign on the interval $[\eta_1, +\infty) \subseteq [\eta_0, +\infty)$ for each fixed value of χ_i $(i = 1, 2)$, provided it differs from the value that satisfies the system

$$\begin{cases} -b_{11}^0\chi_1 - b_{12}^0\chi_1^{\frac{1}{p-1}} \left(y^0\right)^{\frac{-\sigma_1}{p-1}} - b_{13}^0 \left(z^0\right)^{q_1} - b_{14}^0 y^0 & = 0, \\ -b_{21}^0\chi_2 - b_{22}^0\chi_2^{\frac{1}{p-1}} \left(z^0\right)^{\frac{-\sigma_2}{p-1}} - b_{23}^0 \left(y^0\right)^{q_2} - b_{24}^0 z^0 & = 0. \end{cases}$$

Suppose $s_i > 0$ for $i = 1, 2$. Then, for every fixed $\chi_i \neq \chi_i^*$ not satisfying the corresponding system, the behaviour of the functions $h_i(\chi_i, \eta)$ $(i = 1, 2)$ differs from the special case $\chi_i = \chi_i^*$, where χ_i satisfies the following system of equations:

$$\begin{cases} -b_{11}^0\chi_1 - b_{12}^0\chi_1^{\frac{1}{p-1}} \left(y^0\right)^{\frac{-\sigma_1}{p-1}} - b_{14}^0 y^0 & = 0, \\ -b_{21}^0\chi_2 - b_{22}^0\chi_2^{\frac{1}{p-1}} \left(z^0\right)^{\frac{-\sigma_2}{p-1}} - b_{24}^0 z^0 & = 0. \end{cases}$$

□

The functions $h_i(\chi_i, \eta)$ $(i = 1, 2)$ preserve their sign throughout the interval $[\eta_2, +\infty) \subseteq [\eta_0, +\infty)$. When they can be expressed in an alternative form as follows:

$$\begin{cases} h_1(\chi_1, \eta) & \equiv -b_{11}(\eta)\chi_1 - b_{12}(\eta)\chi_1\, \eta^{\frac{1}{p-1}}\, y^{-\frac{\sigma_1}{p-1}} - b_{13}(\eta) y \left(y^{-1} z^{q_1} + b_{14}(\eta) b_{13}^{-1}(\eta)\right), \\ h_2(\chi_2, \eta) & \equiv -b_{21}(\eta)\chi_2 - b_{22}(\eta)\chi_2\, \eta^{\frac{1}{p-1}}\, z^{-\frac{\sigma_2}{p-1}} - b_{23}(\eta) z \left(z^{-1} y^{q_2} + b_{24}(\eta) b_{23}^{-1}(\eta)\right). \end{cases}$$

From here, we find

$$\lim_{\eta\to+\infty} b_{i1}(\eta) = -\frac{p-1}{\sigma_i + p - 2}, \quad \lim_{\eta\to+\infty} b_{i2}(\eta) = \gamma\Big(\frac{p-1}{p}\Big)^p,$$

$$\lim_{\eta\to+\infty} b_{i3}(\eta) = \begin{cases} (1 - 1/p)^p \frac{1}{a}, & s_i = 0, \\ 0, & s_i > 0, \end{cases} \quad \lim_{\eta\to+\infty} b_{i4}(\eta) = 0 \quad (i = 1, 2),$$

implies that the functions $h_i(\chi_i, \eta)$ $(i = 1, 2)$ maintain a constant sign on the interval $[\eta_2, +\infty) \subseteq [\eta_0, +\infty)$, where $\chi_i \neq 0$ $(i = 1, 2)$. That means the functions $h_i(\chi_i, \eta)$ $(i = 1, 2)$ for all $\eta \in [\eta_i, +\infty)$ $(i = 1, 2)$ satisfy one of the inequalities:

$$h_i(\chi_i, \eta) > 0 \quad \text{or} \quad h_i(\chi_i, \eta) < 0 \quad (i = 1, 2). \tag{14}$$

Assume now that the functions $\mathfrak{g}_i(\eta)$ $(i = 1, 2)$ do not have a limit as $\eta \to +\infty$. Let us consider the case where at least one of the inequalities in (14) holds.

Given the oscillatory nature of the functions $\mathfrak{g}_i(\eta)$ $(i = 1, 2)$, their graphs must intersect the straight line $\bar{\mathfrak{g}}_i(\eta)$ $(i = 1, 2)$ infinitely many times within the interval $[\eta_i, +\infty)$ $(i = 1, 2)$.

Then

$$\begin{cases} \mathfrak{g}_1(\eta) & = y^{\sigma_1} \left| \frac{dy}{d\eta} + b_{10}(\eta) y \right|^{p-2} \left(\frac{dy}{d\eta} + b_{10}(\eta) y \right) = \left(y^0\right)^{\sigma_1} |b_{10}^0 y^0|^{p-2} b_{10}^0 y^0 + o(1), \\ \mathfrak{g}_2(\eta) & = z^{\sigma_1} \left| \frac{dz}{d\eta} + b_{20}(\eta) z \right|^{p-2} \left(\frac{dz}{d\eta} + b_{20}(\eta) z \right) = \left(z^0\right)^{\sigma_1} |b_{20}^0 z^0|^{p-2} b_{20}^0 z^0 + o(1), \end{cases}$$

as $\eta \to +\infty$. And according to relation (13), the derivative of the functions $\mathfrak{g}_i(\eta)$ $(i = 1, 2)$ tends to a finite limit as $\eta \to +\infty$, and this limit is zero.

As a result, it is important to

$$\begin{cases} \lim_{\eta\to+\infty} \left(b_{11}(\eta)\mathfrak{g}_1(\eta) + b_{12}(\eta) y^{-\frac{\sigma_1}{p-1}} \eta^{\frac{1}{p-1}} \mathfrak{g}_1(\eta) \right) + \lim_{\eta\to+\infty} \left(b_{13}(\eta) z^{q_1} + b_{14}(\eta) y \right) & = 0, \\ \lim_{\eta\to+\infty} \left(b_{21}(\eta)\mathfrak{g}_2(\eta) + b_{22}(\eta) z^{-\frac{\sigma_2}{p-1}} \eta^{\frac{1}{p-1}} \mathfrak{g}_2(\eta) \right) + \lim_{\eta\to+\infty} \left(b_{23}(\eta) y^{q_2} + b_{24}(\eta) z \right) & = 0. \end{cases}$$

It readily follows from this that at $s_i < 0$ $(i = 1, 2)$, system (12) cannot have solutions $(y(\eta), z(\eta))$ with a finite non-zero limit as $\eta \to +\infty$, and at $s_i \geq 0$ $(i = 1, 2)$. For such solutions to exist, the conditions stated in Theorems 2, 3, 4, and 5 must be fulfilled.

Consequently, due to the transformations introduced in (4) and (9), the self-similar solution of system (1)–(2) exhibits the following asymptotic behaviour as $x \to \frac{p}{\alpha} \ln \left(\frac{\alpha}{p} a^{\frac{p-1}{p}} (T + t)^{\gamma} \right)$:

$$\begin{aligned} u_A(t, x) &\simeq y^0 (T + t)^{\frac{1+q_1}{1-q_1 q_2}} \left(a - \left(\left(\frac{p}{\alpha} \cdot \frac{e^{\frac{\alpha x}{p}}}{(t + T)^{\gamma}} \right)^{\frac{p}{p-1}} \right) \right)^{\frac{p-1}{\sigma_1 + p - 2}}, \\ v_A(t, x) &\simeq z^0 (T + t)^{\frac{1+q_2}{1-q_1 q_2}} \left(a - \left(\left(\frac{p}{\alpha} \cdot \frac{e^{\frac{\alpha x}{p}}}{(t + T)^{\gamma}} \right)^{\frac{p}{p-1}} \right) \right)^{\frac{p-1}{\sigma_2 + p - 2}}. \end{aligned}$$

The theorems are proved.

3 *Numerical experiments*

To complement the qualitative analysis, we computed representative solution profiles for system (1)–(2) with the exponential density $\rho(x) = e^{\alpha x}$. The computations follow the scheme and implementation principles described in [40] and are written in Python. The diffusion terms are discretized in conservative form, which is natural for divergence operators and helps control the support propagation [32, pp. 258–264]. The degeneracy at $u = 0$ and $u_x = 0$ is handled by a small regularization of the nonlinear mobility, which is a standard practice for degenerate parabolic equations and helps avoid spurious oscillations near interfaces [32, pp. 525–542], [41, pp. 332–356]. Non-negativity is enforced by the update choice and by using non-negative initial data. Figures 1–4 show typical solution profiles for several parameter sets. They illustrate finite-speed propagation, localization driven by the exponential density, and the change of growth rate, in agreement with the comparison estimates and the Fujita-type threshold discussed in the theoretical part [7, 41]. Furthermore, we listed the graphics of the solution in some particular cases:

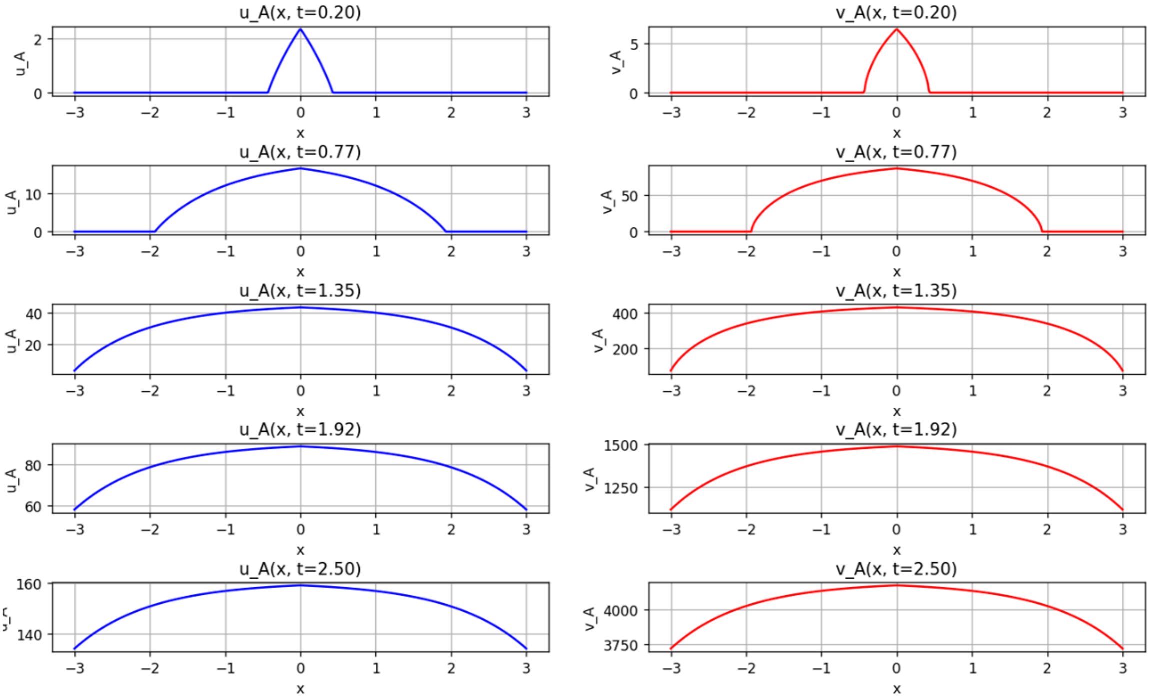

Figure 1. $y_0 = 1.5$, $z_0 = 1.0$, $a = 1.0$, $p = 2.2$, $T = 1.0$, $\alpha = 1.2$, $q_1 = 0.4$, $q_2 = 1.5$, $\sigma_1 = 1.1$, $\sigma_2 = 1.7$

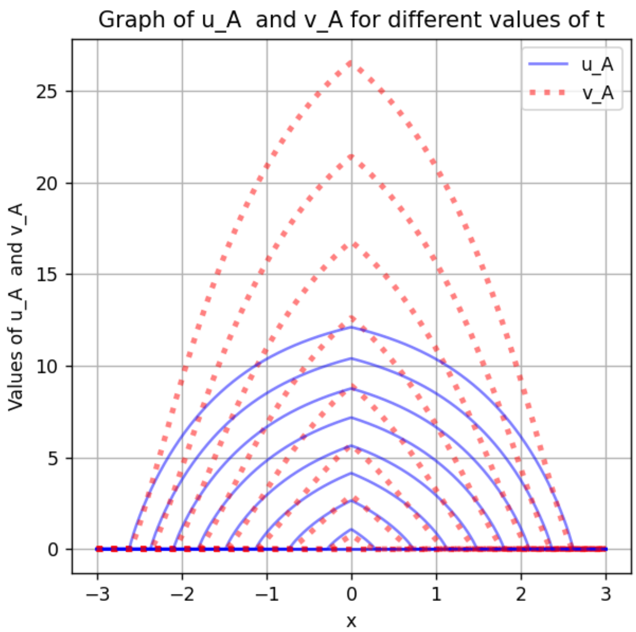

Figure 2. $y_0 = 1.5$, $z_0 = 1.5$, $a = 1.0$, $p = 2.2$, $T = 1.1$, $\alpha = 1.0$, $q_1 = 0.3$, $q_2 = 0.8$, $\sigma_1 = 1.2$, $\sigma_2 = 0.7$

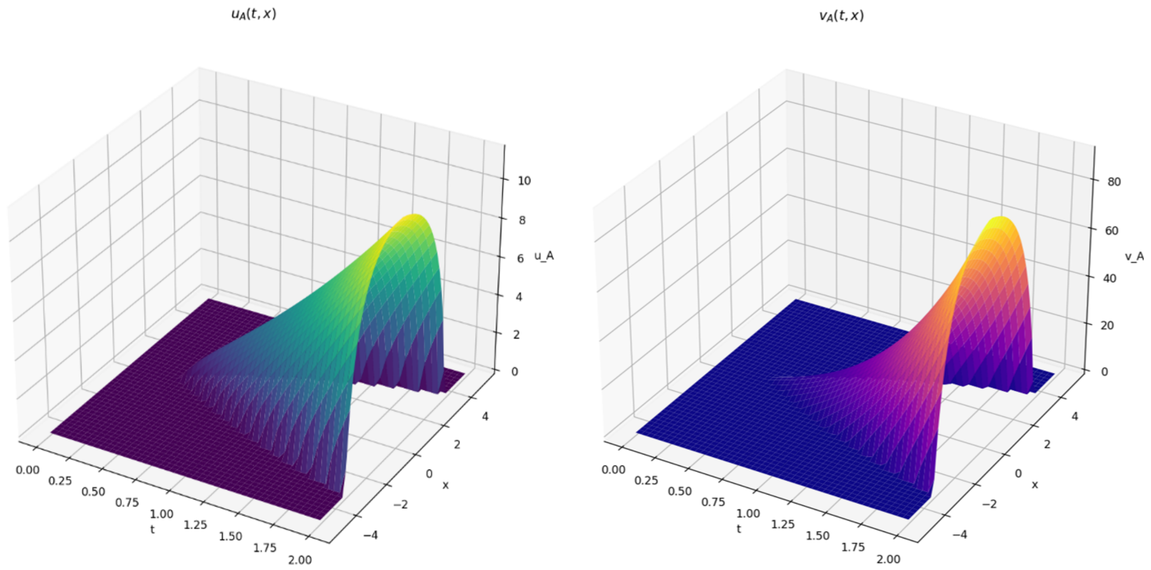

Figure 3. $y_0 = 1.5$, $z_0 = 1.3$, $T = 1.0$, $a = 1.0$, $p = 2.1$, $\alpha = 1.0$, $q_1 = 0.4$, $q_2 = 1.5$, $\sigma_1 = 1.1$, $\sigma_2 = 1.7$

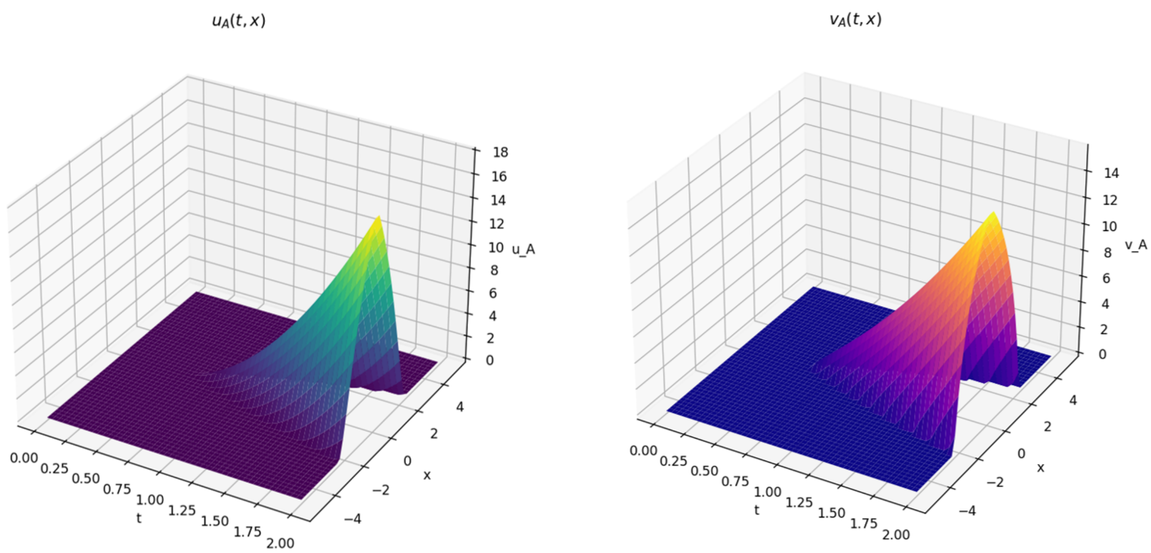

Figure 4. $y_0 = 1.1$, $z_0 = 1.3$, $T = 1.0$, $a = 1.2$, $p = 2.5$, $\alpha = 1.0$, $q_1 = 0.7$, $q_2 = 0.5$, $\sigma_1 = 0.5$, $\sigma_2 = 1.3$

Conclusion

The main novelty of this work is a unified qualitative analysis for a non-homogeneous, doubly nonlinear coupled system under an exponential density $\rho(x) = e^{\alpha x}$. that links three components: comparison estimates, self-similar structure, and computation. The comparison part is based on explicit compactly supported super-solutions constructed from Barenblatt-type profiles and yields global solvability for small initial data in the slow-diffusion range. The self-similar change of variables consistent with the weight converts the PDE system into an auxiliary profile system, which allows us to classify the leading asymptotics of self-similar solutions and to reveal how the parameters control localization and propagation. The resulting Fujita-type criterion provides a clear borderline between the global existence and non-existence of global solutions. Numerical experiments in Python reproduce finite-speed propagation and spatial localization induced by the weight and illustrate the change of qualitative behaviour, in agreement with the theoretical thresholds.

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Author Contributions

All authors contributed equally to this work.

Conflict of Interest

The authors declare no conflict of interest.

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Time-fractional parabolic equation with Zaremba-type boundary conditions: analysis and applications

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This paper investigates a time-fractional parabolic equation with Zaremba-type boundary conditions. The main objective of the present work is to construct reliable and efficient numerical approximations for such problems. To this end, stable finite difference schemes are developed within a consistent analytical framework. A key result is obtaining a coercive stability estimate for the first-order scheme, which guarantees its consistency and supports its practical use in computations. In addition, both first- and second-order schemes are implemented in the one-dimensional case using a modified Gaussian elimination approach. This implementation simplifies the solution process and improves computational reliability when handling the resulting systems. The behavior of the proposed methods is examined through several numerical experiments designed to reflect different parameter choices and settings. The results demonstrate that the schemes achieve the expected levels of accuracy, consistency, and efficiency. An accompanying error analysis explains the observed outcomes and supports the theoretical findings. The numerical results, presented in tables, show strong agreement with the theoretical predictions, thereby confirming the validity and effectiveness of the proposed approach. These conclusions highlight the practical applicability of the proposed numerical schemes for solving this fractional parabolic problem with mixed boundary conditions.

Keywords: Zaremba-type boundary conditions, Riemann–Liouville derivative, time-fractional parabolic equation, boundary value problems, positive operators, modified Gaussian elimination method, difference schemes, stability.

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Introduction

The analysis of fractional partial differential equations has gained considerable attention in recent decades due to their wide applicability in physics, engineering, and mathematical biology [1–3]. Among them, the time-fractional diffusion and parabolic equations [4, 5] are of particular interest because they provide realistic models for anomalous diffusion processes with memory effects. We may refer to [6, 7] and [8–10] for further studies.

The study of boundary value problems for parabolic equations with classical derivatives has a long history. Zaremba [11], following a suggestion by Wirtinger, introduced the mixed Dirichlet–Neumann boundary problem (now known as the Zaremba problem) for the Laplace equation. Modern studies have extended these ideas in various directions, including the analysis of singular interfaces [12].

In [13], fractional powers (FPs) of positive operators (POs) were investigated. The author examined the conditions under which the sum of coercively POs retains positivity and defined their FPs. This work has provided a theoretical foundation for the analysis of fractional operator powers in Banach and Hilbert spaces, which is fundamental for applications in differential and integral equations.

The paper [14] established the well-posedness results for a boundary value problem of fractional parabolic equations (FPEs) with mixed conditions. The authors derived coercive stability estimates

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for the solution associated with the mixed problem and presented stable difference schemes (DSs) for its approximate solution. They considered first and second order accurate DSs in time and first-order accuracy in space, applying a modified Gaussian elimination method for their numerical solution.

Despite these advances, study of time-FPEs with Zaremba-type boundary conditions remains limited. In particular, the development of stable numerical methods and stability estimates for such problems has not been fully addressed in the literature.

This paper is devoted to the study of the time-FPE with Zaremba-type (mixed Dirichlet–Neumann) boundary conditions:

$$\begin{gathered} D_t^{\mu} u(t,\mathrm{z}) - \frac{\partial}{\partial \mathrm{z}} \left(\xi(\mathrm{z})\, u_{\mathrm{z}}(t,\mathrm{z})\right) + \eta\, u(t,\mathrm{z}) = g(t,\mathrm{z}), \quad t \in (0,\mathsf{T}), \quad \mathrm{z} \in (0,L), \\ u_{\mathrm{z}}(t,0) = 0, \quad u(t,L) = 0, \quad t \in [0,\mathsf{T}], \\ u(\mathsf{T},\mathrm{z}) = 0, \quad \mathrm{z} \in [0,L]. \end{gathered} \tag{1}$$

Here, $D_t^{\mu} = D_{\mathsf{T}-}^{\mu}$ stands for the right-sided Riemann–Liouville fractional derivative (FD) of order $\mu \in (0,1)$ [15]. We assume that $\xi(\mathrm{z})$ and $g(t,\mathrm{z})$ are smooth functions for $t \in (0,\mathsf{T})$ and $\mathrm{z} \in (0,L)$, with $\xi(\mathrm{z}) \geq a > 0$ and $\eta > 0$. We construct stable DSs for the approximate solution of the boundary value problem (1) and derive coercive stability estimate for the first-order DS. Additionally, a modified Gaussian elimination method is employed to solve both the first and second order accurate DSs for FPEs.

1 *Preliminaries*

The following statements are established and are used throughout this article.

Let K be a Banach space, and let $A : \mathcal{D}(A) \subset \mathsf{K} \to \mathsf{K}$ be a linear and unbounded operator that is densely defined in K. A is said to be strongly PO in K if its spectrum $\eta(A)$ lies entirely within a sector of angle Θ, with $0 < \Theta < \pi/2$, symmetric w.r.t. the real axis. On the edges of this sector,

$$S_1(\Theta) = \{\rho e^{i\Theta} : 0 \leq \rho < \infty\}, \quad S_2(\Theta) = \{\rho e^{-i\Theta} : 0 \leq \rho < \infty\},$$

and outside the sector, the resolvent $(\lambda I - A)^{-1}$ satisfies the estimate [16]

$$\|(\lambda I - A)^{-1}\| \leq \frac{M}{1 + |\lambda|}.$$

The infimum of all such angles is referred to as the spectral angle $\Theta(A, \mathsf{K})$ of A.

In what follows, M stands for a positive constant, which may take different values in each occurrence. When it is necessary to indicate the dependence of this constant on certain parameters, we write $M(\alpha, \beta, \gamma, \dots)$.

A is a PO in K. For $0 < \gamma < 1$, we define the fractional space $\mathsf{K}_\gamma = \mathsf{K}_\gamma(\mathsf{K}, A)$ as the set of all $u \in \mathsf{K}$ for which the norm

$$\|u\|_{\mathsf{K}_\gamma} = \sup_{\lambda > 0} \lambda^{\gamma} \|(A + \lambda I)^{-1} u\|_{\mathsf{K}} + \|u\|_{\mathsf{K}}$$

is finite.

Theorem 1. [13] If A and B are two commuting POs with $\Theta(A, \mathsf{K}) + \Theta(B, \mathsf{K}) < \pi$, the bounded operator $(A + B)^{-1}$ defined in K exists. In addition, for every $\gamma \in (0,1)$ and every $g \in \mathsf{K}$, the problem $Au + Bu = g$ admits a unique solution $u = u(g)$ and the next estimates hold:

$$\begin{gathered} \|Au\|_{\mathsf{K}_\gamma(\mathsf{K},B)} + \|Bu\|_{\mathsf{K}_\gamma(\mathsf{K},B)} + \|Bu\|_{\mathsf{K}_\gamma(\mathsf{K},A)} \leq M(\gamma) \|g\|_{\mathsf{K}_\gamma(\mathsf{K},B)}, \\ \|Au\|_{\mathsf{K}_\gamma(\mathsf{K},A)} + \|Bu\|_{\mathsf{K}_\gamma(\mathsf{K},A)} + \|Au\|_{\mathsf{K}_\gamma(\mathsf{K},B)} \leq M(\gamma) \|g\|_{\mathsf{K}_\gamma(\mathsf{K},A)}. \end{gathered}$$

Theorem 2. [17] Suppose A is a PO with $\Theta(A,\mathsf{K})<\pi$. Then, for $\gamma\leq\frac{1}{2}$, the FP A^{γ} is also a PO with $\Theta(A^{\gamma},\mathsf{K})<\frac{\pi}{2}$.

Theorem 3. [18] Assume A is the operator in $\mathsf{K}=\mathcal{C}[0,\mathsf{T}]$ given by $(A\,\omega)(t)=-\omega'(t)$, whose domain is $\mathcal{D}(A)=\{\,\omega\in\mathcal{C}[0,\mathsf{T}]:\omega'\in\mathcal{C}[0,\mathsf{T}],\ \omega(\mathsf{T})=0\,\}$. This implies that A is a PO on the space $\mathsf{K}=\mathcal{C}[0,\mathsf{T}]$. Moreover, for every $g\in\mathcal{D}(A)$ and every $\gamma\in(0,1)$, the identity

$$A^{\gamma}g(t)=D^{\gamma}_{\mathsf{T}-}g(t)$$

holds.

Theorem 4. Suppose A and B are POs with $\Theta(A,\mathsf{K})<\pi$ and $\Theta(B,\mathsf{K})<\frac{\pi}{2}$. Consequently, for every $\gamma\leq\frac{1}{2}$, the operator $(D^{\gamma}+B)^{-1}$ exists as a bounded operator on K. Additionally, for each $g\in\mathsf{K}$, there is a unique solution $u=u(g)$ of the equation $D^{\gamma}u+Bu=g$, and the next estimate holds:

$$\|D^{\gamma}u\|_{\mathsf{K}_{\gamma}(\mathsf{K},B)}+\|Bu\|_{\mathsf{K}_{\gamma}(\mathsf{K},B)}\leq M(\gamma)\,\|g\|_{\mathsf{K}_{\gamma}(\mathsf{K},B)}.$$

B^{z}, the differential operator of order two, is defined by

$$B^{\mathrm{z}}u(\mathrm{z})=-\frac{\partial}{\partial\mathrm{z}}(\xi(\mathrm{z})u_{\mathrm{z}}(\mathrm{z}))+\eta u(\mathrm{z}) \tag{2}$$

whose domain is $\mathcal{D}(B^{\mathrm{z}})=\{u\ :\ u,u',u''\in\mathcal{C}[0,L],\ u'(0)=0,\ u(L)=0\}$.

For $\gamma\in(0,1]$, let $C^{\gamma}[0,L]$ be the Banach space of all continuous functions $\psi(\mathrm{z})$ on $[0,L]$ satisfying a Hölder condition, with the norm

$$\|\psi\|_{\mathcal{C}^{\gamma}[0,L]}=\|\psi\|_{\mathcal{C}[0,L]}+\sup_{\mathrm{z}_1\neq\mathrm{z}_2}\frac{|\psi(\mathrm{z}_1)-\psi(\mathrm{z}_2)|}{|\mathrm{z}_1-\mathrm{z}_2|^{\gamma}},$$

where $\mathcal{C}[0,L]$ is the Banach space consisting of continuous functions $\psi(\mathrm{z})$ on $[0,L]$, endowed with the norm $\|\psi\|_{\mathcal{C}[0,L]}=\max_{\mathrm{z}\in[0,L]}|\psi(\mathrm{z})|$.

The positivity of B^{z} in $\mathcal{C}[0,L]$ has been proved. Furthermore, for any $\gamma\in(0,1/2)$, the norms in the spaces $\mathsf{K}_{\gamma}(\mathsf{K},B)$ and $\mathcal{C}^{2\gamma}[0,L]$ are equivalent.

Theorem 5. The norms in the space $\mathsf{K}_{\gamma}(\mathcal{C}[0,L],B^{\mathrm{z}})$ and the Hölder space $\mathcal{C}^{2\gamma}[0,L]$ are equivalent if $\gamma\in(0,1/2)$.

The proof of Theorem 5 relies on the following estimates for the Green function G^{z} of the operator B^{z} defined in (2):

$$|G^{\mathrm{z}}(\mathrm{z},\mathrm{z}_0;\lambda)|\leq\frac{M(\eta,a)}{\sqrt{\eta+\lambda}}\begin{cases}e^{-\frac{1}{2}\sqrt{\frac{\eta+\lambda}{a}}(\mathrm{z}-\mathrm{z}_0)}, & 0\leq\mathrm{z}_0\leq\mathrm{z},\\ e^{-\frac{1}{2}\sqrt{\frac{\eta+\lambda}{a}}(\mathrm{z}_0-\mathrm{z})}, & \mathrm{z}\leq\mathrm{z}_0\leq L,\end{cases}$$

$$|G^{\mathrm{z}}_{\mathrm{z}}(\mathrm{z},\mathrm{z}_0;\lambda)|\leq M(\eta,a)\begin{cases}e^{-\frac{1}{2}\sqrt{\frac{\eta+\lambda}{a}}(\mathrm{z}-\mathrm{z}_0)}, & 0\leq\mathrm{z}_0\leq\mathrm{z},\\ e^{-\frac{1}{2}\sqrt{\frac{\eta+\lambda}{a}}(\mathrm{z}_0-\mathrm{z})}, & \mathrm{z}\leq\mathrm{z}_0\leq L.\end{cases}$$

Theorem 6. For the solution $u(t,\mathrm{z})$ of problem (1), the following coercive SE holds:

$$\max_{0\leq t\leq\mathsf{T}}\|u_{\mathrm{zz}}(t,\cdot)\|_{\mathcal{C}^{\gamma}[0,L]}\leq M(\gamma)\|g(t,\cdot)\|_{\mathcal{C}^{\gamma}[0,L]}$$

for $M(\gamma)$ that is independent of $g(t,\mathrm{z})$ when $t\in[0,\mathsf{T}]$, $\mathrm{z}\in[0,L]$ and $0<\gamma<1$.

The proof of Theorem 6 relies on the positivity of the differential operator B^{z} (see (2)), on Theorem 3 establishing the relationship between FDs and FPs of positive operators, on Theorem 2 concerning the spectral angle of FPs, and on Theorem 4 regarding the FPs of coercively positive sums of two operators.

2 Main results

Problem (1) is discretized in two stages as follows. Firstly, we introduce the discrete mesh

$$[0, L]_h = \{\mathrm{z}_n : \mathrm{z}_n = nh,\ h = L/M,\ 0 \le n \le M\}.$$

To the differential operator B^{z} defined in (2), we associate the difference operator B^{z}_h given by

$$B^{\mathrm{z}}_h u^h(\mathrm{z}) = -\frac{\partial}{\partial \mathrm{z}}(\xi(\mathrm{z})u^h_{\mathrm{z}}(\mathrm{z})) + \eta u^h(\mathrm{z}), \tag{3}$$

which acts in the space of grid functions $u^h(\mathrm{z})$ subject to the boundary conditions

$$D^h u^h(0) = 0, \quad u^h(L) = 0,$$

where $D^h u^h(0)$ denotes the approximation of u_{z} at $\mathrm{z} = 0$.

Using the operator B^{z}_h, we consider the boundary value problem

$$\begin{cases} D^\mu_t \omega^h(t, \mathrm{z}) + B^{\mathrm{z}}_h \omega^h(t, \mathrm{z}) = g^h(t, \mathrm{z}),\ t \in (0, \mathsf{T}),\ \mathrm{z} \in [0, L]_h, \\ \omega^h(\mathsf{T}, \mathrm{z}) = 0,\ \mathrm{z} \in [0, L]_h \end{cases}$$

which represents a finite system of ordinary fractional differential equations.

Secondly, applying the first order of approximation formula [18]

$$D^\mu_\tau u_k = -\frac{1}{\Gamma(1-\mu)} \sum_{r=k}^{N} \frac{\Gamma(r-k-\mu+1)}{\Gamma(r-k+1)} \frac{u_r - u_{r-1}}{\tau^\mu}, \quad 1 \le k \le N$$

for

$$D^\mu_\tau u(t_k) = -\frac{1}{\Gamma(1-\mu)} \int_{t_k}^{\mathsf{T}} (s - t_k)^{-\mu} u'(s)\, ds$$

and applying the first-order accurate stable DS for parabolic equations, the first-order accurate DS in time can be formulated as

$$-\frac{1}{\Gamma(1-\mu)} \sum_{r=k}^{N} \frac{\Gamma(r-k-\mu+1)}{\Gamma(r-k+1)} \frac{u^h_r(\mathrm{z}) - u^h_{r-1}(\mathrm{z})}{\tau^\mu} + B^{\mathrm{z}}_h u^h_k(\mathrm{z}) = g^h_k(\mathrm{z}),$$

$$g^h_k(\mathrm{z}) = g^h(t_k, \mathrm{z}), \quad t_k = k\tau, \quad \tau = \mathsf{T}/N, \quad 1 \le k \le N, \quad \mathrm{z} \in [0, L]_h, \tag{4}$$

$$u^h_N(\mathrm{z}) = 0, \quad \mathrm{z} \in [0, L]_h,$$

which represents the approximate solution of problem (1). In addition, employing the second-order approximation formula for $1 \le k \le N-2$,

$$D^\mu_\tau u_k = d\left\{ w_1 u_{k-1} + w_2 u_k + w_3 u_{k+1} + \sum_{m=k+2}^{N} [a(k,m)u_{m-2} + b(k,m)u_{m-1} + c(k,m)u_m] \right\}, \tag{5}$$

for $k = N-1$,

$$D^{\mu}_{\tau} u_k = d\left[-\frac{3^{1-\mu}}{2^{2-\mu}}\frac{1}{1-\mu} + \frac{3^{2-\mu}}{2^{2-\mu}}\frac{1}{(1-\mu)(2-\mu)}\right] u_{N-2}$$
$$+ d\left[\frac{3^{1-\mu}}{2^{-\mu}}\frac{1}{1-\mu} - \frac{3^{2-\mu}}{2^{1-\mu}}\frac{1}{(1-\mu)(2-\mu)}\right] u_{N-1}$$
$$+ d\left[-\frac{3^{2-\mu}}{2^{2-\mu}}\frac{1}{1-\mu} + \frac{3^{2-\mu}}{2^{2-\mu}}\frac{1}{(1-\mu)(2-\mu)}\right] u_{N},$$

for $k = N$,

$$D^{\mu}_{\tau} u_k = d\left[-\frac{1}{2^{2-\mu}}\frac{1}{1-\mu} + \frac{1}{2^{2-\mu}}\frac{1}{(1-\mu)(2-\mu)}\right] u_{N-2}$$
$$+ d\left[\frac{1}{2^{-\mu}}\frac{1}{1-\mu} - \frac{1}{2^{1-\mu}}\frac{1}{(1-\mu)(2-\mu)}\right] u_{N-1}$$
$$+ d\left[-\frac{3}{2^{2-\mu}}\frac{1}{1-\mu} + \frac{1}{2^{2-\mu}}\frac{1}{(1-\mu)(2-\mu)}\right] u_{N},$$

for

$$D^{\mu}_{\tau} u(t_k - \tau/2) = -\frac{1}{\Gamma(1-\mu)}\int_{t_k-\tau/2}^{\mathsf{T}} (s - t_k + \tau/2)^{-\mu} u'(s)\, ds$$

and employing the Crank–Nicolson DS for parabolic equations, a second-order accurate DS in both t and z can be formulated as

$$D^{\mu}_{\tau} u^h_k(\mathrm{z}) + \frac{1}{2} B^{\mathrm{z}}_h \left(u^h_k(\mathrm{z}) + u^h_{k-1}(\mathrm{z})\right) = g^h_k(\mathrm{z}),$$
$$g^h_k(\mathrm{z}) = g^h(t_k - \tau/2, \mathrm{z}), \quad t_k = k\tau, \quad \tau = \mathsf{T}/N, \quad 1 \le k \le N, \quad \mathrm{z} \in [0, L]_h, \tag{6}$$
$$u^h_N(\mathrm{z}) = 0, \quad \mathrm{z} \in [0, L]_h$$

which represents an approximation for the solution of problem (1).

In (5), it is denoted that

$d = \frac{\tau^{-\mu}}{\Gamma(1-\mu)}, \quad w_1 = -\frac{3^{1-\mu}}{2^{2-\mu}}\frac{1}{1-\mu} + \frac{3^{2-\mu}}{2^{2-\mu}}\frac{1}{(1-\mu)(2-\mu)},$

$w_2 = \frac{3^{1-\mu}}{2^{-\mu}}\frac{1}{1-\mu} - \frac{3^{2-\mu}}{2^{1-\mu}}\frac{1}{(1-\mu)(2-\mu)}, \quad w_3 = -\frac{3^{2-\mu}}{2^{2-\mu}}\frac{1}{1-\mu} + \frac{3^{2-\mu}}{2^{2-\mu}}\frac{1}{(1-\mu)(2-\mu)},$

$q_1(r) = (r + \frac{1}{2})^{1-\mu} - (r - \frac{1}{2})^{1-\mu}, \quad q_2(r) = (r + \frac{1}{2})^{2-\mu} - (r - \frac{1}{2})^{2-\mu}, \quad q_3(r) = (r + \frac{1}{2})^{1-\mu},$

$a(k, m) = \frac{q_1(m-k)}{2-2\mu} - \frac{q_3(m-k)}{1-\mu} + \frac{q_2(m-k)}{(1-\mu)(2-\mu)}, \quad b(k, m) = \frac{2q_3(m-k)}{1-\mu} - \frac{2q_2(m-k)}{(1-\mu)(2-\mu)},$

$c(k, m) = -\frac{q_1(m-k)}{2-2\mu} - \frac{q_3(m-k)}{1-\mu} + \frac{q_2(m-k)}{(1-\mu)(2-\mu)}.$

We note that all computations concerning the problem (6) are conducted for $\mathsf{T} = 1$.

Now, consider the discrete problem

$$B^{\mathrm{z}}_h u^h + \lambda u^h = g^h, \tag{7}$$

in the case $\xi(\mathrm{z}) = 1$.

Lemma 1. Assume $\lambda > 0$. Then, equation (7) admits a unique solution, which is given by the formula

$$u^h = (B_h^z + \lambda I)^{-1} g^h = \left\{ \sum_{j=1}^{M-1} G(j, n; \lambda + \eta) g_j h \right\}_0^M, \tag{8}$$

where

$$G(j, n; \lambda + \eta) = \frac{h(R^n - R^{2M-n})(R^j - R^{2M-j})}{(1 + R^{2M-1})(1 - R^2)} + \frac{h(R^{|n-j|+1} - R^{2M-n-j+1})}{1 - R^2},$$

for $1 \leq j \leq M-1$ and $1 \leq n \leq M$

$$R = \frac{1}{1 + \delta h}, \quad \delta = \frac{h}{2}(\lambda + \eta) + \sqrt{\frac{h^2}{4}(\lambda + \eta)^2 + (\lambda + \eta)}.$$

Here, $G(j, n; \lambda + \eta)$ is said to be the Green's function of the equation (7), for which we derive the next formula

$$\sum_{j=1}^{M-1} G(j, n; \lambda + \eta) h = \frac{1}{\lambda + \eta} - \frac{1}{\lambda + \eta} \frac{R^{M-n} + R^{M+n-1}}{1 + R^{2M-1}}. \tag{9}$$

To demonstrate positivity of B_h^z in the Banach space $\mathcal{C}_h$, we first require the next supplementary lemma.

Lemma 2. The estimates

$$|\delta| \geq \max\left\{ \frac{h}{2} |\lambda + \eta|, \sqrt{|\lambda + \eta|} \right\}, \tag{10}$$

$$|R| \leq \frac{1}{1 + \sqrt{|\lambda + \eta|} h \cos(\theta)} < 1, \quad |\theta| < \frac{\pi}{2} \tag{11}$$

are satisfied.

Theorem 7. $(\lambda I + B_h^z)^{-1}$ defined by (8) holds the next estimate

$$\|(\lambda I + B_h^z)^{-1}\|_{\mathcal{C}_h \to \mathcal{C}_h} \leq \frac{M(\theta, \eta)}{1 + |\lambda|}, \tag{12}$$

for every λ lying in the set $\left\{\lambda : |\arg \lambda| \leq \theta, \, 0 \leq \theta < \frac{\pi}{2}\right\}$.

Proof. Firstly, we consider the operator B_h^z defined by formula (3) for the case $\xi(\mathrm{z}) = 1$. If we set $n = 0$, we get

$$u_0 = \frac{h^2 R \left(1 - R^{2M-2}\right)}{\left(1 + R^{2M-1}\right)(1 - R)} g_1 + \frac{h^2}{\left(1 + R^{2M-1}\right)(1 - R)} \sum_{j=2}^{M-1} \left(R^j - R^{2M-j}\right) g_j.$$

Then it follows that

$$|u_0| \leq 2h^2 \left|\tfrac{R}{1-R}\right| |g_1| + \tfrac{h^2}{1-|R|} \textstyle\sum_{j=2}^{M-1} \left(|R|^j + |R|^{2M-j}\right) |g_j| \leq 2h^2 \|g^h\|_{C_h} \left\{ \left|\tfrac{R}{1-R}\right| + \left(\tfrac{|R|}{1-|R|}\right)^2 \right\}. \tag{13}$$

Now, we estimate $|u_n|$ for $1 \leq n \leq M-1$. Applying the triangle inequality in the formula (8), we achieve

$$|u_n| \leq \frac{2h^2}{|1 - R^2|} \sum_{j=1}^{M-1} 2 |R|^{j+1} |g_j| + \frac{h^2}{|1 - R^2|} \sum_{j=1}^{M-1} 2 |R|^{|n-j|+1} |g_j|.$$

If we compute the geometric series, we obtain

$$|u_n| \le 2h^2 \|g^h\|_{C_h} \left\{ \left(\frac{|R|}{1-|R|} \right)^2 \frac{4}{|1+R|} + \left| \frac{R}{1-R} \right| \frac{1}{|1+R|} \right\}. \tag{14}$$

Then using the estimate (11), we arrive at

$$\left(\frac{|R|}{1-|R|} \right)^2 \le \frac{1}{|\lambda+\eta|\, h^2 \cos^2\theta}. \tag{15}$$

In addition, we have that

$$|\lambda+\eta| = |(|\lambda|\cos\theta+\eta) + i\,|\lambda|\sin\theta| \ge (|\lambda|+\eta)\cos\theta.$$

Therefore, we achieve

$$\frac{1}{|\lambda+\eta|} \le \frac{\frac{1}{\eta\cos\theta}}{1+\frac{1}{\eta}|\lambda|} \le \frac{M(\theta,\eta)}{1+|\lambda|}. \tag{16}$$

Using the estimates (15) and (16), we deduce

$$\left(\frac{|R|}{1-|R|} \right)^2 h^2 \le \frac{\frac{1}{\cos^2\theta}}{|\lambda+\eta|} \le \frac{M(\theta,\eta)}{1+|\lambda|}. \tag{17}$$

By the estimates (10), (16) and the definition of R, we have

$$\left| \frac{R}{1-R} \right| h^2 = \frac{h}{|\delta|} \le \frac{2}{|\lambda+\eta|} \le \frac{M(\theta,\eta)}{1+|\lambda|}. \tag{18}$$

As a result of (13), (14), (17) and (18), we prove that

$$\|u^h\|_{\mathcal{C}_h} \le \frac{M(\theta,\eta)}{1+|\lambda|} \|g^h\|_{\mathcal{C}_h}.$$

Hence, we obtained the estimate (12) when $\xi(\mathrm{z}) = 1$. Additionally, assuming that $\lambda > 0$ is sufficiently large, we employ the fixed-point theorem to derive analogous results for the Green's function (8), thereby completing the proof. □

Theorem 8. Suppose that $\lambda > 0$ and $0 < \gamma < \frac{1}{2}$. The norms in the spaces $\mathsf{K}_\gamma(\mathcal{C}_h,\, B_h^{\mathrm{z}})$ and $\mathcal{C}_h^{2\gamma}$ are equivalent uniformly in h for $0 < h < h_0$.

Proof. It follows from (8) and (9) that

$$\begin{aligned} &\left(\lambda^\gamma B_h^{\mathrm{z}} (B_h^{\mathrm{z}} + \lambda I)^{-1} g^h \right)_n \\ &= \frac{\eta\lambda^\gamma}{\lambda+\eta} g_n + \frac{\lambda^{\gamma+1}}{\lambda+\eta} \frac{R^{M-n} + R^{M+n-1}}{1+R^{2M-1}} g_n + \lambda^{\gamma+1} \sum_{j=1}^{M-1} G(j,n;\lambda+\eta)(g_n - g_j) h. \end{aligned}$$

We apply the triangle inequality to obtain the next estimate

$$\begin{aligned} &\left| \left(\lambda^\gamma B_h^{\mathrm{z}} (B_h^{\mathrm{z}} + \lambda I)^{-1} g^h \right)_n \right| \\ &\le \frac{\eta\lambda^\gamma}{\lambda+\eta} |g_n| + \frac{\lambda^{\gamma+1}}{\lambda+\eta} \frac{|R|^{M-n} + |R|^{M+n-1}}{|1+R^{2M-1}|} |g_n| + \lambda^{\gamma+1} \sum_{j=1}^{M-1} |G(j,n;\lambda+\eta)|\, |g_n - g_j|\, h \\ &\le \left\{ \frac{\eta\lambda^\gamma}{\lambda+\eta} + \frac{\lambda^{\gamma+1}}{\lambda+\eta} + M_1(\eta) \frac{\lambda^{\gamma+1}}{\sqrt{\lambda+\eta}} \sum_{j=1}^{M-1} R^{|n-j|}\, |(n-j)h|^{2\gamma}\, h \right\} \|g^h\|_{\mathcal{C}_h}^{2\gamma} \le M(\eta) \|g^h\|_{\mathcal{C}_h}^{2\gamma} \end{aligned}$$

for $\lambda > 0$ and $\mathrm{z} \in [0, L]$. Thus, we conclude that $g^h \in \mathsf{K}_\gamma(\mathcal{C}_h,\, B_h^{\mathrm{z}})$ and

$$\|g^h\|_{\mathsf{K}_\gamma(\mathcal{C}_h,\, B_h^{\mathrm{z}})} \leq M(\eta)\|g^h\|_{\mathcal{C}_h}^{2\gamma}.$$

Now, we prove the reverse inequality. We have the identity for a PO B_{z}^h as follows:

$$\begin{aligned} g_n &= \int_0^\infty (B_h^{\mathrm{z}} + \lambda I)^{-1} B_h^{\mathrm{z}} (B_h^{\mathrm{z}} + \lambda I)^{-1} g_n d\lambda \\ &= \int_0^\infty \sum_{j=1}^{M-1} G(j, n; \lambda + \eta)\, B_h^{\mathrm{z}} \left(B_h^{\mathrm{z}} + \lambda I\right)^{-1} g_j\, h\, d\lambda. \end{aligned}$$

Then we derive that

$$g_n - g_{n+r} = \int_0^\infty \lambda^{-\gamma} \sum_{j=1}^{M-1} \left(G(j, n; \lambda + \eta) - G(j, n + r; \lambda + \eta)\right) \lambda^\gamma B_h^{\mathrm{z}} \left(B_h^{\mathrm{z}} + \lambda I\right)^{-1} g_j\, h\, d\lambda,$$

from which it follows

$$\|g_n - g_{n+r}\| \leq \int_0^\infty \lambda^{-\gamma} \sum_{j=1}^{M-1} |G(j, n; \lambda + \eta) - G(j, n + r; \lambda + \eta)|\ h\, d\lambda \|g^h\|_{\mathsf{K}_\gamma(\mathcal{C}_h, B_h^{\mathrm{z}})}.$$

If we denote that

$$\mathcal{J}_h = \frac{1}{|rh|^\gamma} \int_0^\infty \lambda^{-\gamma} \sum_{j=1}^{M-1} |G(j, n; \lambda + \eta) - G(j, n + r; \lambda + \eta)|\, h\, d\lambda,$$

then we arrive at

$$\frac{\|g_n - g_{n+r}\|}{|rh|^\gamma} \leq \mathcal{J}_h\, \|g^h\|_{\mathsf{K}_\gamma(\mathcal{C}_h, B_h^{\mathrm{z}})}.$$

Based on the Lemma 2, we get the next estimate

$$\frac{\|g_n - g_{n+r}\|}{|rh|^\gamma} \leq \frac{M}{\gamma(1 - 2\gamma)}\, \|g^h\|_{\mathsf{K}_\gamma(\mathcal{C}_h, B_h^{\mathrm{z}})}, \quad 1 \leq n < n + r \leq M - 1.$$

That is, we deduce

$$\|g^h\|_{\mathcal{C}_h^\gamma} \leq \frac{M}{\gamma(1 - 2\gamma)}\, \|g^h\|_{\mathsf{K}_\gamma(\mathcal{C}_h, B_h^{\mathrm{z}})}.$$

This completes the proof for the case $\xi(\mathrm{z}) = 1$. Now, suppose that $\xi(\mathrm{z})$ is a continuous function, with $\mathrm{z}, \mathrm{z}_0 \in [0, 1]$ being fixed points. Since

$$\left\| \left(B_h^{\mathrm{z}} - B_h^{\mathrm{z}_0}\right) \left(B_h^{\mathrm{z}_0}\right)^{-1} \right\|_{\mathcal{C}_h \to \mathcal{C}_h} \leq M,$$

and, moreover, the following formula holds:

$$\begin{aligned} B_h^{\mathrm{z}} \left(B_h^{\mathrm{z}} + \lambda I\right)^{-1} g^h &= B_h^{\mathrm{z}_0} \left(B_h^{\mathrm{z}_0} + \lambda I\right)^{-1} g^h \\ &\quad + \lambda \left(B_h^{\mathrm{z}} + \lambda I\right)^{-1} \left(B_h^{\mathrm{z}} - B_h^{\mathrm{z}_0}\right) \left(B_h^{\mathrm{z}_0}\right)^{-1} B_h^{\mathrm{z}_0} \left(B_h^{\mathrm{z}_0} + \lambda I\right)^{-1} g^h, \end{aligned}$$

we conclude that

$$\left|\lambda^\gamma B_h^{\mathrm{z}} \left(B_h^{\mathrm{z}} + \lambda I\right)^{-1} g^h\right| \leq \|g^h\|_{\mathsf{K}_\gamma(\mathcal{C}_h, B_h^{\mathrm{z}_0})} + M_1 \lambda \left\|\left(B_h^{\mathrm{z}} + \lambda I\right)^{-1}\right\|_{\mathcal{C}_h \to \mathcal{C}_h} \|g^h\|_{\mathsf{K}_\gamma(\mathcal{C}_h, B_h^{\mathrm{z}_0})}.$$

Therefore,

$$\left|\lambda^{\gamma} B_h^{z} \left(B_h^{z} + \lambda I\right)^{-1} g^h\right| \leq M \left\|g^h\right\|_{\mathsf{K}_{\gamma}(\mathcal{C}_h, B_h^{z_0})}.$$

Thus, we obtain

$$\|g^h\|_{\mathsf{K}_{\gamma}(\mathcal{C}_h, B_h^{z})} \leq M \left\|g^h\right\|_{\mathsf{K}_{\gamma}(\mathcal{C}_h, B_h^{z_0})},$$

which completes the proof of this theorem. □

Theorem 9. [18] Suppose A_τ is an operator in $\mathsf{K}_\tau = \mathcal{C}[a,b]_\tau$ that is given by $A_\tau \omega^\tau = \left\{-\frac{\omega_k - \omega_{k-1}}{\tau}\right\}_1^N$ with $\omega_N = 0$. Then, the operator A_τ is positive in K_τ, and the following relation holds:

$$A_\tau^{\gamma} g^\tau = \left\{-\frac{1}{\Gamma(1-\gamma)} \sum_{m=k}^{N} \frac{\Gamma(m-k-\gamma+1)}{\Gamma(m-k+1)} \frac{g_m - g_{m-1}}{h^\gamma}\right\}_1^N.$$

In addition, the fractional difference derivative is defined as follows:

$$D_\tau^{\gamma} g^\tau := \left\{-\frac{1}{\Gamma(1-\gamma)} \sum_{m=k}^{N} \frac{\Gamma(m-k-\gamma+1)}{\Gamma(m-k+1)} \frac{g_m - g_{m-1}}{h^\gamma}\right\}_1^N.$$

Thus, we arrive at the next theorem.

Theorem 10. Suppose A_τ is an operator in $\mathsf{K}_\tau = \mathcal{C}[a,b]_\tau$ that is given by $A_\tau \omega^\tau = \left\{-\frac{\omega_k - \omega_{k-1}}{\tau}\right\}_1^N$, whose domain is

$$\mathcal{D}(A_\tau) = \left\{\omega^\tau : \frac{\omega_k - \omega_{k-1}}{\tau} \in \mathsf{K}_\tau, \, \omega_N = 0\right\}.$$

Then the operator A_τ is positive in K_τ, and we have

$$A_\tau^{\gamma} g^\tau = D_\tau^{\gamma} g^\tau$$

for all $g^\tau(t) \in \mathcal{D}(A_\tau)$.

Hence, we deduce the next result regarding the coercive stability of the DS (6).

Theorem 11. Let τ and h be sufficiently small positive numbers, and let $0 < \gamma < 1$. Then, the solution of the DS (6) satisfies the coercive stability estimate

$$\max_{1 \leq k \leq N} \left\| \left\{\frac{u_{n+1}^k - 2u_n^k + u_{n-1}^k}{h^2}\right\}_{n=1}^{M-1} \right\|_{\mathcal{C}^{\gamma}[0,L]_h} \leq M(\gamma) \max_{1 \leq k \leq N} \|g_k^h\|_{\mathcal{C}^{\gamma}[0,L]_h},$$

where $M(\gamma)$ is independent of τ, h, and g_k^h for $1 \leq k \leq N$.

The proof of Theorem 11 is based on the positivity of the difference space operator B_h^{z} defined in the formula (3), on Theorem 8 concerning the structure of the fractional space $E_\gamma(C_h, B_h^{z})$, on Theorem 3 regarding the relationship between FDs and FPs of positive operators, on Theorem 2 regarding the spectral angle of FPs of POs, and on Theorem 4 addressing the FPs of coercively positive sums of two operators.

3 Numerical illustrations

For the numerical illustrations, we present the following problem:

$$\begin{gathered}
D_t^{\mu}u(t,\mathrm{z})-\frac{\partial^2 u(t,\mathrm{z})}{\partial \mathrm{z}^2}+u(t,\mathrm{z})=g(t,\mathrm{z}),\\
g(t,\mathrm{z})=\frac{2(1-t)^{2-\mu}}{\Gamma(3-\mu)}\cos^2\left(\frac{\pi \mathrm{z}}{2}\right)+\frac{\pi^2(1-t)^2}{2}\cos(\pi \mathrm{z})\\
+(1-t)^2\cos^2\left(\frac{\pi \mathrm{z}}{2}\right),\\
t\in(0,1),\quad \mathrm{z}\in(0,1),\\
u(1,\mathrm{z})=0,\quad \mathrm{z}\in[0,1],\\
u_{\mathrm{z}}(t,0)=u(t,1)=0,\quad t\in[0,1]
\end{gathered} \tag{19}$$

which represents a one-dimensional FPEs with $0<\mu<1$.

Problem (19) has the exact solution $u(t,\mathrm{z})=(1-t)^2\cos^2\left(\frac{\pi\mathrm{z}}{2}\right)$. Observe that the solution is independent of μ, whereas $g(t,\mathrm{z})$ is dependent explicitly on μ.

Utilizing the DS (4) for the estimate solution of (19), we arrive at

$$-\frac{1}{\Gamma(1-\mu)}\sum_{m=k+1}^{N}\frac{\Gamma(m-k-\mu)}{\Gamma(m-k)}\frac{u_n^m-u_n^{m-1}}{\tau^{\mu}}-\frac{u_{n+1}^k-2u_n^k+u_{n-1}^k}{h^2}+u_n^k=\phi_n^k,$$

$$\begin{gathered}
\phi_n^k=g(t_k,\mathrm{z}_n),\quad t_k=k\tau,\quad 0\le k\le N-1,\quad N\tau=1,\\
\mathrm{z}_n=nh,\quad 1\le n\le M-1,\quad Mh=1,\\
u_n^N=0,\quad 0\le n\le M,\\
u_0^k=u_1^k,\quad u_M^k=0,\quad 0\le k\le N.
\end{gathered}$$

The resulting system of equations can be expressed in matrix form

$$\begin{gathered}
AU_{n+1}+BU_n+CU_{n-1}=D\phi_n,\quad 1\le n\le M-1,\\
U_0-U_1=\tilde{0},\quad U_M=\tilde{0},
\end{gathered} \tag{20}$$

where

$$A=C=\begin{pmatrix}
a_n & 0 & 0 & \cdot & 0 & 0 & 0\\
0 & a_n & 0 & \cdot & 0 & 0 & 0\\
0 & 0 & a_n & \cdot & 0 & 0 & 0\\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot\\
0 & 0 & 0 & \cdot & a_n & 0 & 0\\
0 & 0 & 0 & \cdot & 0 & a_n & 0\\
0 & 0 & 0 & \cdot & 0 & 0 & 0
\end{pmatrix}_{(N+1)\times(N+1)},\quad
D=\begin{pmatrix}
1 & 0 & 0 & \cdot & 0 & 0 & 0\\
0 & 1 & 0 & \cdot & 0 & 0 & 0\\
0 & 0 & 1 & \cdot & 0 & 0 & 0\\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot\\
0 & 0 & 0 & \cdot & 1 & 0 & 0\\
0 & 0 & 0 & \cdot & 0 & 1 & 0\\
0 & 0 & 0 & \cdot & 0 & 0 & 0
\end{pmatrix}_{(N+1)\times(N+1)},$$

$$B=\begin{pmatrix}
b_{11} & b_{12} & b_{13} & \cdot & b_{1,N-1} & b_{1,N} & b_{1,N+1}\\
0 & b_{22} & b_{23} & \cdot & b_{2,N-1} & b_{2,N} & b_{2,N+1}\\
0 & 0 & b_{33} & \cdot & b_{3,N-1} & b_{3,N} & b_{3,N+1}\\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot\\
0 & 0 & 0 & \cdot & b_{N-1,N-1} & b_{N-1,N} & b_{N-1,N+1}\\
0 & 0 & 0 & \cdot & 0 & b_{N,N} & b_{N,N+1}\\
0 & 0 & 0 & \cdot & 0 & 0 & b_{N+1,N+1}
\end{pmatrix}_{(N+1)\times(N+1)},$$

$$\phi_n = \begin{pmatrix} \phi_n^0 \\ \phi_n^1 \\ \phi_n^2 \\ \cdot \\ \phi_n^{N-2} \\ \phi_n^{N-1} \\ \phi_n^N \end{pmatrix}_{(N+1)\times(1)}, \quad U_p = \begin{pmatrix} U_p^0 \\ U_p^1 \\ U_p^2 \\ \cdot \\ U_p^{N-2} \\ U_p^{N-1} \\ U_p^N \end{pmatrix}_{(N+1)\times(1)}, \quad p \in \{n-1, n, n+1\},$$

$$b_{ij} = \begin{cases} 0, & 1 \le j \le i-1, \\ \frac{1}{\tau^\mu} + 1 + \frac{2}{h^2}, & j = i, \\ \frac{\Gamma(2-\mu)-\Gamma(1-\mu)}{\Gamma(1-\mu)\tau^\mu}, & j = i+1, \\ \frac{1}{\Gamma(1-\mu)\tau^\mu}\left(\frac{\Gamma(j-i+1-\mu)}{\Gamma(j-i+1)} - \frac{\Gamma(j-i-\mu)}{\Gamma(j-i)}\right), & i+2 \le j \le N, \\ -\frac{\Gamma(N-i+1-\mu)}{\Gamma(1-\mu)\Gamma(N-i+1)\tau^\mu}, & j = N+1 \end{cases}$$

for $i = 1, 2, \ldots, N-2$, and

$$a_n = -\frac{1}{h^2}, \quad b_{N-1,N-1} = \frac{1}{\tau^\mu} + 1 + \frac{2}{h^2}, \quad b_{N-1,N} = \frac{\Gamma(2-\mu) - \Gamma(1-\mu)}{\Gamma(1-\mu)\tau^\mu},$$

$$b_{N-1,N+1} = -\frac{\Gamma(2-\mu)}{\Gamma(1-\mu)\tau^\mu}, \quad b_{N,N} = \frac{1}{\tau^\mu} + 1 + \frac{2}{h^2}, \quad b_{N,N+1} = -\frac{1}{\tau^\mu}, \quad b_{N+1,N+1} = 1$$

and

$$\phi_n^k = \frac{2(1-k\tau)^{2-\mu}\cos^2\left(\frac{\pi nh}{2}\right)}{\Gamma(3-\mu)} + \frac{\pi^2}{2}(1-k\tau)^2\cos(\pi nh) + (1-k\tau)^2\cos^2\left(\frac{\pi nh}{2}\right).$$

To solve the problem (20), the procedure of modified Gaussian elimination method is utilized. We seek the solution of the matrix equation in the following form:

$$U_j = \alpha_{j-1}U_{j-1} + \beta_{j-1}, \quad j = 1, 2, \ldots, M-1, \quad U_0 = (I - \alpha_0)^{-1}\beta_0,$$

where α_j's are $(N+1)\times(N+1)$ square matrices and β_j's are $(N+1)\times 1$ column matrices defined for $j = M-1, M-2, \cdots, 1$ by

$$\alpha_{j-1} = -\left(B + A\alpha_j\right)^{-1} C,$$
$$\beta_{j-1} = \left(B + A\alpha_j\right)^{-1}\left(D\,\phi_j - A\beta_j\right).$$

Here, α_{M-1} denotes the zero matrix $(N+1)\times(N+1)$, and β_{M-1} denotes the zero matrix $(N+1)\times 1$.

Furthermore, by utilizing the DS (6), we obtain a second-order accurate DS in both t and z. Specifically, the Crank-Nicolson scheme for parabolic equations can be employed to represent a second-order accurate DS with respect to t and z

$$D_\tau^\mu u_n^k - \frac{1}{2}\left[\frac{u_{n+1}^k - 2u_n^k + u_{n-1}^k}{h^2} + \frac{u_{n+1}^{k-1} - 2u_n^{k-1} + u_{n-1}^{k-1}}{h^2}\right] + \frac{1}{2}\left[u_n^k + u_n^{k-1}\right] = \phi_n^k,$$

$$\phi_n^k = g\Big(t_k - \frac{\tau}{2}, \mathrm{z}_n\Big), \quad t_k = k\tau, \quad \mathrm{z}_n = nh, \quad N\tau = 1, \quad Mh = 1,$$
$$1 \le k \le N, \quad 1 \le n \le M-1,$$
$$u_n^N = 0, \quad 0 \le n \le M,$$

$$3u_0^k - 4u_1^k + u_2^k = 0, \quad u_M^k = 0, \quad 0 \le k \le N,$$

where $D_\tau^\mu u_n^k$ is defined by (5). This yields the system of equations in matrix form

$$\begin{gathered} AU_{n+1} + BU_n + CU_{n-1} = D\phi_n, \quad 1 \le n \le M-1, \\ 3U_0 - 4U_1 + U_2 = \tilde{0}, \quad U_M = \tilde{0}, \end{gathered} \tag{21}$$

where

$$A = C = \begin{pmatrix} a_n & a_n & 0 & \cdot & 0 & 0 & 0 \\ 0 & a_n & a_n & \cdot & 0 & 0 & 0 \\ 0 & 0 & a_n & \cdot & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & a_n & a_n & 0 \\ 0 & 0 & 0 & \cdot & 0 & a_n & a_n \\ 0 & 0 & 0 & \cdot & 0 & 0 & 0 \end{pmatrix}_{(N+1)\times(N+1)}, \quad D = \begin{pmatrix} 1 & 0 & 0 & \cdot & 0 & 0 & 0 \\ 0 & 1 & 0 & \cdot & 0 & 0 & 0 \\ 0 & 0 & 1 & \cdot & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & 1 & 0 & 0 \\ 0 & 0 & 0 & \cdot & 0 & 1 & 0 \\ 0 & 0 & 0 & \cdot & 0 & 0 & 0 \end{pmatrix}_{(N+1)\times(N+1)},$$

$$B = \begin{pmatrix} b_{11} & b_{12} & b_{13} & \cdot & b_{1,N-1} & b_{1,N} & b_{1,N+1} \\ 0 & b_{22} & b_{23} & \cdot & b_{2,N-1} & b_{2,N} & b_{2,N+1} \\ 0 & 0 & b_{33} & \cdot & b_{3,N-1} & b_{3,N} & b_{3,N+1} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & b_{N-1,N-1} & b_{N-1,N} & b_{N-1,N+1} \\ 0 & 0 & 0 & \cdot & 0 & b_{N,N} & b_{N,N+1} \\ 0 & 0 & 0 & \cdot & 0 & 0 & b_{N+1,N+1} \end{pmatrix}_{(N+1)\times(N+1)},$$

$$\phi_n = \begin{pmatrix} \phi_n^0 \\ \phi_n^1 \\ \phi_n^2 \\ \cdot \\ \phi_n^{N-2} \\ \phi_n^{N-1} \\ \phi_n^N \end{pmatrix}_{(N+1)\times(1)}, \quad U_p = \begin{pmatrix} U_p^0 \\ U_p^1 \\ U_p^2 \\ \cdot \\ U_p^{N-2} \\ U_p^{N-1} \\ U_p^N \end{pmatrix}_{(N+1)\times(1)}, \quad p \in \{n-1, n, n+1\},$$

$$b_{ij} = \begin{cases} 0, & 1 \le j \le i-1, \\ d.\, w_1 + \frac{1}{h^2} + \frac{1}{2}, & j = i, \\ d\left(a(i,i+2) + w_2\right) + \frac{1}{h^2} + \frac{1}{2}, & j = i+1, \\ d\left(a(i,i+3) + b(i,i+2) + w_3\right), & j = i+2, \\ d\left(a(i,j+1) + b(i,j) + c(i,j-1)\right), & i+3 \le j \le N-1, \\ d\left(b(i,N) + c(i,N-1)\right), & j = N, \\ d.\, c(i,N), & j = N+1 \end{cases}$$

for $i = 1, 2, \ldots, N-4$, and

$$a_n = -\frac{1}{2h^2}, \quad b_{N-3,N-3} = d.\, w_1 + \frac{1}{h^2} + \frac{1}{2},$$

$$b_{N-3,N-2} = d\,(a(N-3, N-1) + w_2) + \frac{1}{h^2} + \frac{1}{2},$$

$$b_{N-3,N-1} = d\,(a(N-3, N) + b(N-3, N-1) + w_3),$$

$$b_{N-3,N} = d\,(b(N-3, N) + c(N-3, N-1)), \quad b_{N-3,N+1} = d.\, c(N-3, N),$$

$$b_{N-2,N-2} = d.\, w_1 + \frac{1}{h^2} + \frac{1}{2}, \quad b_{N-2,N-1} = d\,(a(N-2, N) + w_2) + \frac{1}{h^2} + \frac{1}{2},$$

$$b_{N-2,N} = d\,(b(N-2, N) + w_3), \quad b_{N-2,N+1} = d.\, c(N-2, N),$$

$$b_{N-1,N-1} = d\left(-\frac{3^{1-\mu}}{2^{2-\mu}}\frac{1}{1-\mu} + \frac{3^{2-\mu}}{2^{2-\mu}}\frac{1}{(1-\mu)(2-\mu)}\right) + \frac{1}{h^2} + \frac{1}{2},$$

$$b_{N-1,N} = d\left(\frac{3^{1-\mu}}{2^{-\mu}}\frac{1}{1-\mu} - \frac{3^{2-\mu}}{2^{1-\mu}}\frac{1}{(1-\mu)(2-\mu)}\right) + \frac{1}{h^2} + \frac{1}{2},$$

$$b_{N-1,N+1} = d\left(-\frac{3^{2-\mu}}{2^{2-\mu}}\frac{1}{1-\mu} + \frac{3^{2-\mu}}{2^{2-\mu}}\frac{1}{(1-\mu)(2-\mu)}\right),$$

$$b_{N,N-1} = d\left(-\frac{1}{2^{2-\mu}}\frac{1}{1-\mu} + \frac{1}{2^{2-\mu}}\frac{1}{(1-\mu)(2-\mu)}\right),$$

$$b_{N,N} = d\left(\frac{1}{2^{-\mu}}\frac{1}{1-\mu} - \frac{1}{2^{1-\mu}}\frac{1}{(1-\mu)(2-\mu)}\right) + \frac{1}{h^2} + \frac{1}{2},$$

$$b_{N,N+1} = d\left(-\frac{3}{2^{2-\mu}}\frac{1}{1-\mu} + \frac{1}{2^{2-\mu}}\frac{1}{(1-\mu)(2-\mu)}\right) + \frac{1}{h^2} + \frac{1}{2}, \quad b_{N+1,N+1} = 1$$

and

$$\phi_n^k = \frac{2(1-k\tau+\frac{\tau}{2})^{2-\mu}\cos^2\left(\frac{\pi nh}{2}\right)}{\Gamma(3-\mu)} + \frac{\pi^2}{2}(1-k\tau+\tfrac{\tau}{2})^2\cos(\pi nh) + (1-k\tau+\tfrac{\tau}{2})^2\cos^2\left(\tfrac{\pi nh}{2}\right).$$

To solve the difference problem (21), we use the previous algorithm with

$$U_0 = (3I - 4\alpha_0 + \alpha_1\alpha_0)^{-1}\left((4I - \alpha_1)\,\beta_0 - \beta_1\right).$$

By utilizing the DSs (4) and (6) for the approximate solution of (19), we established the first and the second order accurate DSs. Computational results indicate that the Crank-Nicolson DS exhibits higher accuracy than the first-order scheme. Moreover, all numerical outcomes are independent of the choice of $\mu \in (0, 1)$. To illustrate, Tables 1 and 2 present the results for $\mu = \frac{1}{2}$ and $\mu = \frac{2}{3}$, respectively, for $N = M$ with values $10, 20, 40, 80, 160$.

Table 1

Analysis of errors of first-order and Crank-Nicolson DSs for $\mu = 1/2$

Method	N=M=10	N=M=20	N=M=40	N=M=80	N=M=160
First-order DS	0.1394	0.0676	0.0333	0.0165	0.0082
Crank-Nicolson DS	0.004502	0.000496	0.000111	0.000027	0.000008

Table 2

Analysis of errors of first-order and Crank-Nicolson DSs for $\mu = 2/3$

Method	N=M=10	N=M=20	N=M=40	N=M=80	N=M=160
First-order DS	0.1320	0.0635	0.0311	0.0154	0.0077
Crank-Nicolson DS	0.004611	0.000468	0.000107	0.000026	0.000007

Conclusion

This study established coercive stability estimates for a FPE with Zaremba-type boundary conditions. First- and second-order time-accurate, as well as first-order space-accurate, DSs were analyzed, and their numerical implementation was carried out using a modified Gaussian elimination method. This approach further enables the construction of higher-order schemes in z.

In the future, it would be of interest to investigate alternative definitions of FDs beyond the Riemann–Liouville type, in order to determine if comparable effects arise. Another promising direction of research is the study of various boundary value problems involving nonlocal conditions [19].

Author Contributions

All authors contributed equally to this work.

Conflict of Interest

The authors declare no conflict of interest.

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On integro-differential equations with the highest-order derivative in the integral term

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In this article, a two-point boundary value problem for an integro-differential equation in which the highest-order derivatives appear in the integral term is considered. The Dzhumabaev parametrization method is applied to solve the problem. The original problem is reduced to an equivalent problem for an integro-differential equation with parameters. The resulting problem includes an integro-differential equation with parameters, an initial condition, and an additional relation. Conditions for the existence and uniqueness of a solution to the integro-differential equation with parameters are established in terms of the coefficients and kernels of the equation, as well as the boundary functions. An explicit representation of the solution in terms of the parameters is constructed. The unique solvability of the original two-point boundary value problem is established in terms of the initial data. A special case of the integro-differential equation with the highest-order derivative appearing in the integral term, subject to two-point boundary conditions, is also investigated. The Dzhumabaev parametrization method is used to solve the problem. An explicit form of the solution is obtained.

Keywords: integro-differential equations, highest-order derivative in the integral term, two-point condition, continuous coefficients, Dzhumabaev parametrization method, parametrized problem, functional term, existence and uniqueness, explicit solution.

2020 Mathematics Subject Classification: 34K06, 34K10, 34K60, 45J05.

Introduction

We consider a two-point boundary value problem for the integro-differential equation with higher-order derivatives in the integral term

$$A_1(x)z'(x) + A_0(x)z(x) = F(x) + \int_0^1 \Big\{ K_0(x)L_0(s)z''(s) + K_1(x)L_1(s)z'(s) + K_2(x)L_2(s)z(s) \Big\} ds, \qquad x \in [0,1], \quad (1)$$

$$Bz(0) + Cz(1) = d, \quad (2)$$

where $z(x)$ is the unknown function; the functions $A_0(x)$, $A_1(x)$, and $F(x)$ are continuous on $[0,1]$; the functions $K_j(x)$, $j = 0,1,2$ are continuous on $[0,1]$; the functions $L_1(s)$ and $L_2(s)$ are continuous on $[0,1]$; $L_0(s)$ is continuously differentiable on $[0,1]$; and B, C, and d are constants.

Let $C([0,1],\mathbb{R})$ denote the space of real-valued continuous functions on $[0,1]$, equipped with the norm $||z||_0 = \max_{x\in[0,1]} ||z(x)||$.

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A function $z(x) \in C([0,1],\mathbb{R})$ is said to be a solution of the two-point problem (1)–(2) if:

1) $z(x)$ possesses derivatives $z'(x)$ and $z''(x)$ at each point $x \in [0,1]$;

2) $z(x)$ satisfies the integro-differential equation (1) for all $x \in [0,1]$;

3) $z(x)$ satisfies the boundary condition (2) at the points $x = 0$ and $x = 1$.

The necessity of mathematical modeling of processes involving singularities and small parameters has stimulated the development of the theory of integro-differential equations with different derivative orders in their differential and integral parts [1,2]. A substantial subclass of such equations consists of first-order integro-differential equations in which higher-order derivatives appear in the integral term.

The existence and uniqueness of solutions to initial value problems for integro-differential equations of various orders, as well as their asymptotic behavior and stability, have been extensively investigated (see the references therein). Integro-differential equations with highest-order derivatives in the integral part were studied in [3–5]. The works [6–8] were devoted to investigate an important class of such equations.

It should be emphasized that integro-differential equations with highest-order derivatives in the integral part are particularly effective for modeling processes with aftereffect phenomena, as well as biological and medical systems involving memory.

If the coefficients $A_0(x)$, $A_1(x)$, the right-hand side $F(x)$, and the kernels $K_j(x)$, $j = 0,1,2$, are continuously differentiable on $[0,1]$, then equation (1), after differentiation of both sides with respect to x, can be reduced to an integro-differential equation of neutral or Fredholm type. In that case, the order of the derivative in the integral term becomes equal to or lower than that in the differential part. Numerous works have been devoted to solvability issues in this setting.

More interesting and technically challenging are the cases where the coefficients $A_0(x)$, $A_1(x)$, the function $F(x)$, and the kernels $K_j(x)$, $j = 0,1,2$, are only continuous on $[0,1]$.

The present paper is devoted precisely to this situation. We investigate the solvability of a two-point boundary value problem for an integro-differential equation in which the integral term contains derivatives of higher order than those appearing in the differential part.

New approaches to solving boundary value problems for systems of integro-differential equations, as well as for loaded differential equations, were proposed in [9–11]. In the works of Dzhumabaev [12–14], coefficient criteria for the unique solvability of boundary value problems for Fredholm systems of integro-differential equations were established. Significant results were also obtained for systems of first-order Fredholm nonlinear integro-differential equations in [15]. Problems with parameters for Fredholm systems of integro-differential equations were investigated in [16,17]. In the works [16,17], also numerical algorithms were proposed, and analysis were carried out for integro-differential systems with integral conditions [18].

In the works of Dzhumabaev [9–11], systems of first-order Fredholm integro-differential equations with two-point conditions for various types of integral kernels were investigated.

The integro-differential equation considered in this paper cannot be reduced to a system of first-order integro-differential equations for two reasons. First, the order of the derivative in the differential part cannot be increased to second order, since the coefficients of the differential part, $A_1(x)$, $A_0(x)$, and the right-hand side $F(x)$ are only continuous functions on the interval $[0,1]$. Second, integro-differential equation (1) cannot be differentiated twice with respect to x.

Thus, we cannot increase the order of the derivative in the differential part to second order. This class of integro-differential equations requires special study and the development of methods for solving boundary value problems for such equations.

In this paper, we study the solvability of the first-order integro-differential equation with highest-order derivatives appearing in the integral part (1)–(2).

1 Scheme of the Parametrization Method and Equivalent Problem

To solve the two-point boundary value problem (1)–(2), we apply the Dzhumabaev parametrization method [19].

Let $\lambda = z(0)$. Introducing a new function $\widetilde{z}(x)$, we perform the change of variables in problem (1)–(2) given by

$$z(x) = \widetilde{z}(x) + \lambda, \qquad x \in [0,1].$$

As a result, we obtain an equivalent problem for an integro-differential equation with a parameter

$$A_1(x)\widetilde{z}'(x) = -A_0(x)\widetilde{z}(x) - A_0(x)\lambda + \int_0^1 K_2(x)L_2(s)ds\lambda + F(x) +$$

$$+\int_0^1 \Big\{K_0(x)L_0(s)\widetilde{z}''(s) + K_1(x)L_1(s)\widetilde{z}'(s) + K_2(x)L_2(s)\widetilde{z}(s)\Big\}ds, \quad x \in [0,1]. \quad (3)$$

$$\widetilde{z}(0) = 0, \quad (4)$$

$$[B + C]\lambda + C\widetilde{z}(1) = d. \quad (5)$$

A pair $(\widetilde{z}(x), \lambda)$ is said to be a solution of the problem for the integro-differential equation with parameter (3)–(5) if:

1) $\widetilde{z}(x)$ possesses derivatives $\widetilde{z}'(x)$ and $\widetilde{z}''(x)$ at each point $x \in [0,1]$;

2) $\widetilde{z}(x)$ and λ satisfy the integro-differential equation (3) for all $x \in [0,1]$;

3) the initial condition (4) is satisfied by $\widetilde{z}(x)$ at the point $x = 0$;

4) condition (5) is satisfied by $\widetilde{z}(x)$ and λ at the point $x = 1$.

For a fixed value of λ, equation (3) together with condition (4) constitutes a Cauchy problem for the integro-differential equation with parameter. The parameter λ is then determined from relation (5).

Next, we describe the construction of a solution to the Cauchy problem for the integro-differential equation with parameter (3), (4).

First, we introduce the following notation

$$\theta_0 = \int_0^1 L_0(s)\widetilde{z}''(s)ds, \qquad \theta_1 = \int_0^1 L_1(s)\widetilde{z}'(s)ds, \qquad \theta_2 = \int_0^1 L_2(s)\widetilde{z}(s)ds.$$

Let $A_1(x) \neq 0$ for all $x \in [0,1]$ and

$$a(x) = -\int_0^x [A_1(s)]^{-1}A_0(s)ds, \qquad x \in [0,1].$$

The solution to the Cauchy problem for the integro-differential equation with parameter (3), (4) can be represented in the following form:

$$\widetilde{z}(x) = e^{a(x)} \int_0^x e^{-a(s)}[A_1(s)]^{-1} F(s)ds -$$

$$- e^{a(x)} \int_0^x e^{-a(s)}[A_1(s)]^{-1} A_0(s)ds\lambda + e^{a(x)} \int_0^x e^{-a(s)}[A_1(s)]^{-1} K_2(s) \int_0^1 L_2(s_1)ds_1 ds\lambda +$$

$$+ e^{a(x)} \int_0^x e^{-a(s)}[A_1(s)]^{-1} K_0(s)ds\theta_0 + e^{a(x)} \int_0^x e^{-a(s)}[A_1(s)]^{-1} K_1(s)ds\theta_1 +$$

$$+ e^{a(x)} \int_0^x e^{-a(s)}[A_1(s)]^{-1} K_2(s)ds\theta_2, \qquad x \in [0,1]. \quad (6)$$

Let

$$\tilde{L} = \int_0^1 L_2(s_1)ds_1, \qquad U(x,Z) = e^{a(x)} \int_0^x e^{-a(s)}[A_1(s)]^{-1} Z(s)ds, \qquad x \in [0,1],$$

where $Z(x)$ is an arbitrary function, continuous on $[0,1]$.

Next, representation (6) can be rewritten in the following form:

$$\widetilde{z}(x) = U(x,F) - U(x,A_0)\lambda + U(x,K_2)\tilde{L}\lambda + U(x,K_0)\theta_0 + U(x,K_1)\theta_1 + U(x,K_2)\theta_2, \qquad x \in [0,1]. \quad (7)$$

Introduce the notations

$$D_i(F) = \int_0^1 L_i(\xi)U(\xi,F)d\xi, \quad D_i(A_0) = \int_0^1 L_i(\xi)U(\xi,A_0)d\xi, \quad D_i(K_j) = \int_0^1 L_i(\xi)U(\xi,K_j)d\xi,$$

$$i = 1,2; \qquad j = 0,1,2.$$

Replacing x by ξ, multiplying both sides of (7) by $L_1(\xi)$ and $L_2(\xi)$, and then integrating over the interval $[0,1]$, we obtain two equations for θ_1 and θ_2:

$$\theta_1 = D_1(F) - D_1(A_0)\lambda + D_1(K_2)\tilde{L}\lambda + D_1(K_0)\theta_0 + D_1(K_1)\theta_1 + D_1(K_2)\theta_2, \quad (8)$$

$$\theta_2 = D_2(F) - D_2(A_0)\lambda + D_2(K_2)\tilde{L}\lambda + D_2(K_0)\theta_0 + D_2(K_1)\theta_1 + D_2(K_2)\theta_2. \quad (9)$$

Suppose that $\Phi_2 = 1 - D_2(K_2) \neq 0$. Then equation (9) uniquely determines θ_2:

$$\theta_2 = -\Phi_2^{-1} D_2(A_0)\lambda + \Phi_2^{-1} D_2(K_2)\tilde{L}\lambda + \Phi_2^{-1} D_2(K_0)\theta_0 + \Phi_2^{-1} D_2(K_1)\theta_1 + \Phi_2^{-1} D_2(F). \quad (10)$$

Substituting the expression for θ_2 into (8), we derive

$$\Big[1 - D_1(K_1) - D_1(K_2)\Phi_2^{-1} D_2(K_1)\Big]\theta_1 = \Big[D_1(K_2)\tilde{L} - D_1(A_0) + D_2(K_2)\Phi_2^{-1}[D_2(K_2)\tilde{L} - D_2(A_0)]\Big]\lambda +$$

$$+ \Big[D_1(K_0) + D_2(K_2)\Phi_2^{-1} D_2(K_0)\Big]\theta_0 + D_1(F) + D_1(K_2)\Phi_2^{-1} D_2(F). \quad (11)$$

Suppose that $\Phi_1 = 1 - D_1(K_1) - D_1(K_2)\Phi_2^{-1} D_2(K_1) \neq 0$. Then equation (11) uniquely determines θ_1:

$$\theta_1 = U_1\lambda + V_1\theta_0 + G_1, \quad (12)$$

where $U_1 = \Phi_1^{-1}\Big[D_1(K_2)\tilde{L} - D_1(A_0) + D_2(K_2)\Phi_2^{-1}[D_2(K_2)\tilde{L} - D_2(A_0)]\Big]$,

$V_1 = \Phi_1^{-1}\Big[D_1(K_0) + D_2(K_2)\Phi_2^{-1}D_2(K_0)\Big]$, $G_1 = \Phi_1^{-1}D_1(F) + \Phi_1^{-1}D_1(K_2)\Phi_2^{-1}D_2(F)$.

Thus, we have expressed θ_1 in terms of λ and θ_0.

Substituting the obtained expression for θ_1 into (10), we obtain

$$\theta_2 = U_2\lambda + V_2\theta_0 + G_2, \tag{13}$$

where $U_2 = \Phi_2^{-1}\Big\{D_2(K_2)\tilde{L} - D_2(A_0) + D_2(K_1)\Phi_1^{-1}\Big[D_1(K_2)\tilde{L} - D_1(A_0)+$

$+D_2(K_2)\Phi_2^{-1}[D_2(K_2)\tilde{L} - D_2(A_0)]\Big]\Big\}$,

$V_2 = \Big\{\Phi_2^{-1}D_2(K_0) + \Phi_2^{-1}D_2(K_1)\Phi_1^{-1}\Big[D_1(K_0) + D_2(K_2)\Phi_2^{-1}D_2(K_0)\Big]\Big\}$,

$G_2 = \Phi_2^{-1}D_2(F) + \Phi_2^{-1}D_2(K_1)\Big[\Phi_1^{-1}D_1(F) + \Phi_1^{-1}D_1(K_2)\Phi_2^{-1}D_2(F)\Big]$.

Consequently, θ_1 and θ_2 are represented in terms of λ and θ_0.

We now proceed to consider

$$\theta_0 = \int_0^1 L_0(s)\widetilde{z}''(s)ds.$$

By integrating the integral in this expression by parts, we obtain

$$\int_0^1 L_0(s)\widetilde{z}''(s)ds = L_0(1)\widetilde{z}'(1) - L_0(0)\widetilde{z}'(0) - \int_0^1 L_0'(s)\widetilde{z}'(s)ds.$$

We have

$$\theta_0 = L_0(1)\widetilde{z}'(1) - L_0(0)\widetilde{z}'(0) - \int_0^1 L_0'(s)\widetilde{z}'(s)ds.$$

Using (3), (4) and (7), we determine $\widetilde{z}'(1)$, $\widetilde{z}'(0)$ and $\int_0^1 L_0'(s)\widetilde{z}'(s)ds$:

$$\widetilde{z}'(1) = -[A_1(1)]^{-1}A_0(1)\widetilde{z}(1) - [A_1(1)]^{-1}A_0(1)\lambda + [A_1(1)]^{-1}K_2(1)\tilde{L}\lambda + [A_1(1)]^{-1}F(1)+$$
$$+ [A_1(1)]^{-1}K_0(1)\theta_0 + [A_1(1)]^{-1}K_1(1)\theta_1 + [A_1(1)]^{-1}K_2(1)\theta_2, \tag{14}$$

$$\widetilde{z}'(0) = -[A_1(0)]^{-1}A_0(0)\lambda + [A_1(0)]^{-1}K_2(0)\tilde{L}\lambda + [A_1(0)]^{-1}F(0)+$$
$$+ [A_1(0)]^{-1}K_0(0)\theta_0 + [A_1(0)]^{-1}K_1(0)\theta_1 + [A_1(0)]^{-1}K_2(0)\theta_2, \tag{15}$$

$$\int_0^1 L_0'(s)\widetilde{z}'(s)ds = -\int_0^1 L_0'(s)[A_1(s)]^{-1}A_0(s)\widetilde{z}(s)ds + [E_1(K_2)\tilde{L} - E_1(A_0)]\lambda+$$
$$+ E_1(F) + E_1(K_0)\theta_0 + E_1(K_1)\theta_1 + E_1(K_2)\theta_2, \tag{16}$$

where

$$E_1(A_0) = \int_0^1 L_0'(s)[A_1(s)]^{-1}A_0(s)ds, \quad E_1(F) = \int_0^1 L_0'(s)[A_1(s)]^{-1}F(s)ds,$$

$$E_1(K_j) = \int_0^1 L_0'(s)[A_1(s)]^{-1} K_j(s) ds, \qquad j = 0, 1, 2.$$

We find

$$\int_0^1 L_0'(s)[A_1(s)]^{-1} A_0(s)\widetilde{z}(s) ds = E_2(F) + E_2(K_2)\tilde{L}\lambda - E_2(A_0)\lambda + \\ + E_2(K_0)\theta_0 + E_2(K_1)\theta_1 + E_2(K_2)\theta_2, \quad (17)$$

where

$$E_2(F) = \int_0^1 L_0'(s)[A_1(s)]^{-1} A_0(s) U(s, F) ds, \quad E_2(A_0) = \int_0^1 L_0'(s)[A_1(s)]^{-1} A_0(s) U(s, A_0) ds,$$

$$E_2(K_j) = \int_0^1 L_0'(s)[A_1(s)]^{-1} A_0(s) U(s, K_j) ds, \quad j = 0, 1, 2.$$

We also find

$$\widetilde{z}(1) = U(1, F) - U(1, A_0)\lambda + U(1, K_2)\tilde{L}\lambda + U(1, K_0)\theta_0 + U(1, K_1)\theta_1 + U(1, K_2)\theta_2. \quad (18)$$

Next, we replace $\widetilde{z}(1)$ in (14) with the expression given in (18):

$$\widetilde{z}'(1) = [A_1(1)]^{-1} F(1) - [A_1(1)]^{-1} A_0(1) U(1, F) + \\ + \Big\{[A_1(1)]^{-1} A_0(1) U(1, A_0) - [A_1(1)]^{-1} A_0(1) U(1, K_2)\tilde{L} - [A_1(1)]^{-1} A_0(1) + [A_1(1)]^{-1} K_2(1)\tilde{L}\Big\}\lambda + \\ + \Big\{[A_1(1)]^{-1} K_0(1) - [A_1(1)]^{-1} A_0(1) U(1, K_0)\Big\}\theta_0 + \\ + \Big\{[A_1(1)]^{-1} K_1(1) - [A_1(1)]^{-1} A_0(1) U(1, K_1)\Big\}\theta_1 + \\ + \Big\{[A_1(1)]^{-1} K_2(1) - [A_1(1)]^{-1} A_0(1) U(1, K_2)\Big\}\theta_2. \quad (19)$$

Finally, using (19), (15) and (16), (17), we find an expression for θ_0:

$$\theta_0 = W(F) + [W(K_2)\tilde{L} - W(A_0)]\lambda + W(K_0)\theta_0 + W(K_1)\theta_1 + W(K_2)\theta_2, \quad (20)$$

$$W(F) = L_0(1)[A_1(1)]^{-1} F(1) - [A_1(1)]^{-1} A_0(1) U(1, F) - L_0(0)[A_1(0)]^{-1} F(0) + E_2(F) - E_1(F),$$

$$W(A_0) = L_0(1)\Big\{[A_1(1)]^{-1} A_0(1) - [A_1(1)]^{-1} A_0(1) U(1, A_0)\Big\} - L_0(0)[A_1(0)]^{-1} A_0(0) + E_2(A_0) - E_1(A_0),$$

$$W(K_0) = L_0(1)\Big\{[A_1(1)]^{-1} K_0(1) - [A_1(1)]^{-1} A_0(1) U(1, K_0)\Big\} - L_0(0)[A_1(0)]^{-1} K_0(0) + E_2(K_0) - E_1(K_0),$$

$$W(K_1) = L_0(1)\Big\{[A_1(1)]^{-1} K_1(1) - [A_1(1)]^{-1} A_0(1) U(1, K_1)\Big\} - L_0(0)[A_1(0)]^{-1} K_1(0) + E_2(K_1) - E_1(K_1),$$

$$W(K_2) = L_0(1)\Big\{[A_1(1)]^{-1} K_2(1) - [A_1(1)]^{-1} A_0(1) U(1, K_2)\Big\} - L_0(0)[A_1(0)]^{-1} K_2(0) + E_2(K_2) - E_1(K_2).$$

Substituting the expression (18) for $\widetilde{z}(1)$ into condition (5), we obtain

$$[B+C]\lambda + C[U(1, K_2)\tilde{L} - U(1, A_0)]\lambda + CU(1, K_0)\theta_0 + CU(1, K_1)\theta_1 + CU(1, K_2)\theta_2 = d - CU(1, F). \quad (21)$$

Using the expressions (12) and (13), and substituting them into (20), we have

$$\Big\{1 - W(K_0) - W(K_1)V_1 - W(K_2)V_2\Big\}\theta_0 = $$
$$= \Big\{W(K_2)\tilde{L} - W(A_0) + W(K_1)U_1 + W(K_2)U_2\Big\}\lambda + W(F) + W(K_1)G_1 + W(K_2)G_2. \quad (22)$$

Let $Q_1 = 1 - W(K_0) - W(K_1)V_1 - W(K_2)V_2$ and assume that $Q_1 \neq 0$.

From (22), we obtain

$$\theta_0 = [Q_1]^{-1}\Big\{W(K_2)\tilde{L} - W(A_0) + W(K_1)U_1 + W(K_2)U_2\Big\}\lambda +$$
$$+ [Q_1]^{-1}\Big\{W(F) + W(K_1)G_1 + W(K_2)G_2\Big\}. \quad (23)$$

Substituting the expression (23) for θ_0 into (12) and (13), we get

$$\theta_1 = U_1\lambda + V_1[Q_1]^{-1}\Big\{W(K_2)\tilde{L} - W(A_0) + W(K_1)U_1 + W(K_2)U_2\Big\}\lambda +$$
$$+ G_1 + [Q_1]^{-1}\Big\{W(F) + W(K_1)G_1 + W(K_2)G_2\Big\}, \quad (24)$$

$$\theta_2 = U_2\lambda + V_2[Q_1]^{-1}\Big\{W(K_2)\tilde{L} - W(A_0) + W(K_1)U_1 + W(K_2)U_2\Big\}\lambda +$$
$$+ G_2 + [Q_1]^{-1}\Big\{W(F) + W(K_1)G_1 + W(K_2)G_2\Big\}. \quad (25)$$

Using the values of θ_0, θ_1, and θ_2 found from (23)—(25), and substituting them into (21), we have

$$\Big\{B + C + C[U(1,K_2)\tilde{L} - U(1,A_0) + U(1,K_1)U_1 + U(1,K_2)U_2] +$$
$$+ C[U(1,K_0) + U(1,K_1)V_1 + U(1,K_2)V_2][Q_1]^{-1}\Big\{W(K_2)\tilde{L} - W(A_0) + W(K_1)U_1 + W(K_2)U_2\Big\}\Big\}\lambda =$$
$$= d - CU(1,F) - CU(1,K_1)G_1 - CU(1,K_2)G_2 -$$
$$- C[U(1,K_0) + U(1,K_1) + U(1,K_2)][Q_1]^{-1}\Big\{W(F) + W(K_1)G_1 + W(K_2)G_2\Big\}. \quad (26)$$

Let

$$Q_2 = B + C + C[U(1,K_2)\tilde{L} - U(1,A_0) + U(1,K_1)U_1 + U(1,K_2)U_2] +$$
$$+ C[U(1,K_0) + U(1,K_1)V_1 + U(1,K_2)V_2][Q_1]^{-1}\Big\{W(K_2)\tilde{L} - W(A_0) + W(K_1)U_1 + W(K_2)U_2\Big\}$$

and assume that $Q_2 \neq 0$.

It then follows from (26) that λ is uniquely determined:

$$\lambda = [Q_2]^{-1}\Big\{d - CU(1,F) - CU(1,K_1)G_1 - CU(1,K_2)G_2\Big\} -$$
$$- [Q_2]^{-1}C[U(1,K_0) + U(1,K_1) + U(1,K_2)][Q_1]^{-1}\Big\{W(F) + W(K_1)G_1 + W(K_2)G_2\Big\}. \quad (27)$$

We have

$$\theta_0 = [Q_1]^{-1}\Big\{W(K_2)\tilde{L} - W(A_0) + W(K_1)U_1 + W(K_2)U_2\Big\}[Q_2]^{-1}\Big\{d - CU(1,F) -$$
$$- CU(1,K_1)G_1 - CU(1,K_2)G_2 - C[U(1,K_0) + U(1,K_1) + U(1,K_2)][Q_1]^{-1} \times$$
$$\times \Big\{W(F) + W(K_1)G_1 + W(K_2)G_2\Big\}\Big\} + [Q_1]^{-1}\Big\{W(F) + W(K_1)G_1 + W(K_2)G_2\Big\}. \quad (28)$$

Finally, using the expression for λ in (27) and for θ_0 in (28), we find an explicit forms for θ_1 and θ_2.

2 *Main Results*

Based on the above, the following statement can be formulated.

Theorem 1. Assume that

a) the functions $A_0(x)$, $A_1(x)$, and $F(x)$ are continuous on $[0,1]$; $A_1(x) \neq 0$ for all $x \in [0,1]$;

b) the functions $K_j(x)$, $j = 0,1,2$ are continuous on $[0,1]$, $L_1(s)$ and $L_2(s)$ are continuous on $[0,1]$; $L_0(s)$ is continuously differentiable on $[0,1]$;

c) B, C and d are constants;

d) $\Phi_2 = 1 - D_2(K_2) \neq 0$ and $\Phi_1 = 1 - D_1(K_1) - D_1(K_2)\Phi_2^{-1}D_2(K_1) \neq 0$;

e) $Q_1 = 1 - W(K_0) - W(K_1)V_1 - W(K_2)V_2 \neq 0$;

f) $Q_2 = B + C + C[U(1,K_2)\tilde{L} - U(1,A_0) + U(1,K_1)U_1 + U(1,K_2)U_2]+$
$+C[U(1,K_0)+U(1,K_1)V_1+U(1,K_2)V_2][Q_1]^{-1}\Big\{W(K_2)\tilde{L} - W(A_0) + W(K_1)U_1 + W(K_2)U_2\Big\} \neq 0$,

where

$$a(x) = -\int_0^x [A_1(s)]^{-1}A_0(s)ds, \quad x \in [0,1]; \ \tilde{L} = \int_0^1 L_2(s)ds,$$

$$D_i(Z) = \int_0^1 L_i(\xi)U(\xi,Z)d\xi, \ \ U(x,Z) = e^{a(x)}\int_0^x e^{-a(s)}[A_1(s)]^{-1}Z(s)ds; \ \ i = 1,2,$$

$$W(Z) = L_0(1)\Big\{[A_1(1)]^{-1}Z(1) - [A_1(1)]^{-1}A_0(1)U(1,Z)\Big\} - L_0(0)[A_1(0)]^{-1}Z(0) + E_2(Z) - E_1(Z),$$

$$V_1 = \Phi_1^{-1}\Big[D_1(K_0) + D_2(K_2)\Phi_2^{-1}D_2(K_0)\Big],$$

$$V_2 = \Big\{\Phi_2^{-1}D_2(K_0) + \Phi_2^{-1}D_2(K_1)\Phi_1^{-1}\Big[D_1(K_0) + D_2(K_2)\Phi_2^{-1}D_2(K_0)\Big]\Big\},$$

$$U_1 = \Phi_1^{-1}\Big[D_1(K_2)\tilde{L} - D_1(A_0) + D_2(K_2)\Phi_2^{-1}[D_2(K_2)\tilde{L} - D_2(A_0)]\Big],$$

$$U_2 = \Phi_2^{-1}\Big\{D_2(K_2)\tilde{L} - D_2(A_0) + D_2(K_1)\Phi_1^{-1}\Big[D_1(K_2)\tilde{L} - D_1(A_0)+$$
$$+D_2(K_2)\Phi_2^{-1}[D_2(K_2)\tilde{L} - D_2(A_0)]\Big]\Big\},$$

$E_1(Z) = \int_0^1 L_0'(s)[A_1(s)]^{-1}Z(s)ds$, $\quad E_2(Z) = \int_0^1 L_0'(s)[A_1(s)]^{-1}A_0(s)U(s,Z)ds$, Z is A_0 or K_j, $j = 0,1,2$.

Then problem for integro-differential equation with parameter (3)–(5) has a unique solution.

Proof. Consider problem (3)–(5).

Using assumptions a)–f) and notations, we construct λ^*, θ_0^*, θ_1^* and θ_2^*:

$$\lambda^* = [Q_2]^{-1}\Big\{d - CU(1,F) - CU(1,K_1)G_1 - CU(1,K_2)G_2\Big\}-$$
$$- [Q_2]^{-1}C[U(1,K_0) + U(1,K_1) + U(1,K_2)][Q_1]^{-1}\Big\{W(F) + W(K_1)G_1 + W(K_2)G_2\Big\}, \quad (29)$$

$$\theta_0^* = [Q_1]^{-1}\Big\{W(K_2)\tilde{L} - W(A_0) + W(K_1)U_1 + W(K_2)U_2\Big\}[Q_2]^{-1}\Big\{d - CU(1,F)-$$
$$- CU(1,K_1)G_1 - CU(1,K_2)G_2 - C[U(1,K_0) + U(1,K_1) + U(1,K_2)][Q_1]^{-1}\times$$
$$\times\Big\{W(F) + W(K_1)G_1 + W(K_2)G_2\Big\}\Big\} + [Q_1]^{-1}\Big\{W(F) + W(K_1)G_1 + W(K_2)G_2\Big\}, \quad (30)$$

$$\theta_1^* = U_1\lambda^* + V_1\theta_0^* + G_1, \quad (31)$$

$$\theta_2^* = U_2\lambda^* + V_2\theta_0^* + G_2, \tag{32}$$

where $G_1 = \Phi_1^{-1}D_1(F) + \Phi_1^{-1}D_1(K_2)\Phi_2^{-1}D_2(F)$,
$G_2 = \Phi_2^{-1}D_2(F) + \Phi_2^{-1}D_2(K_1)\big[\Phi_1^{-1}D_1(F) + \Phi_1^{-1}D_1(K_2)\Phi_2^{-1}D_2(F)\big]$,

$$D_i(F) = \int_0^1 L_i(\xi)U(\xi,F)d\xi, \quad i=1,2, \quad U(x,F) = e^{a(x)}\int_0^x e^{-a(s)}[A_1(s)]^{-1}F(s)ds, \quad x\in[0,1].$$

Then, using the expressions (29)–(32), the unique solution to the Cauchy problem (3)–(4) has the following form

$$\widetilde{z}^*(x) = U(x,F) + [U(x,K_2)\tilde{L} - U(x,A_0)]\lambda^* + U(x,K_0)\theta_0^* + U(x,K_1)\theta_1^* + U(x,K_2)\theta_2^*, \quad x\in[0,1]. \tag{33}$$

The pair $(\widetilde{z}^*(x), \lambda^*)$, determined by (29) and (33), is the unique solution to the problem for the integro-differential equation with parameter (3)–(5).

Theorem 1 is proved. □

Theorem 2. Assume that the conditions a)–f) of Theorem 1 are fulfilled. Then two-point problem for integro-differential equation with higher-order derivatives in integral term (1)–(2) has a unique solution.

Proof. Under the assumptions of Theorem 1, the problem for integro-differential equation with parameter (3)–(5) admits a unique solution, given by the pair $(\widetilde{z}^*(x), \lambda^*)$.

We define

$$z^*(x) = \widetilde{z}^*(x) + \lambda^*, \qquad x\in[0,1].$$

According to the scheme of the method, this function is the unique solution of the two-point boundary value problem for integro-differential equation with higher-order derivatives in integral term (1)–(2).

Theorem 2 is proved. □

3 *Special Case*

We consider special case of the integro-differential equation with second-order derivative in integral term

$$A_0(x)z'(x) + A_1(x)z(x) = F(x) + K(x)\int_0^1 L(s)z''(s)ds, \qquad x\in[0,1], \tag{34}$$

$$Bz(0) + Cz(1) = d, \tag{35}$$

where the function $z(x)$ is the unknown function; $A_i(x)$, $i=0,1$, and $F(x)$ are continuous on $[0,1]$; $K(x)$ is continuous on $[0,1]$; $L(s)$ is continuously differentiable on $[0,1]$; B, C and d are constants.

A function $z(x)\in C([0,1],\mathbb{R})$ is a solution of two-point problem (34)–(35) if:

1) $z(x)$ has derivatives $z'(x), z''(x)$ at each point $x\in[0,1]$;

2) $z(x)$ satisfies to the integro-differential equations (34) on $[0,1]$;

3) the two-point condition (35) is satisfied by $z(x)$ and $z'(x)$ at the points $x=0$, $x=1$.

Integration by parts is applied to the integral in the integro-differential equation (34).

We have

$$K(x)\int_0^1 L(s)z''(s)ds = K(x)L(1)z'(1) - K(x)L(0)z'(0) - K(x)\int_0^1 L'(s)z'(s)ds.$$

Then, the integro-differential equation (34) can be written in the form

$$A_0(x)z'(x) = -A_1(x)z(x) + F(x) + \\ + K(x)L(1)z'(1) - K(x)L(0)z'(0) - K(x)\int_0^1 L'(s)z'(s)ds, \qquad x \in [0,1]. \quad (36)$$

Therefore, we obtain the two-point problem for the neutral integro-differential equation with functional terms (36) and (35). The functional terms include piecewise-constant generalized arguments with respect to the derivative of unknown function at the points $x = 0$ and $x = 1$.

We apply Dzhumabaev parametrization method for solving problem (36)–(35).

Let $\lambda = z(0)$. Introducing the new function $\widetilde{z}(x)$, we perform in problems (36), (35) the change of variables

$$z(x) = \widetilde{z}(x) + \lambda, \qquad x \in [0,1].$$

We obtain the following equivalent problem for the neutral integro-differential equation with parameter and functional terms

$$A_1(x)\widetilde{z}'(x) = -A_0(x)\widetilde{z}(x) - A_0(x)\lambda + F(x) + \\ + K(x)L(1)\widetilde{z}'(1) - K(x)L(0)\widetilde{z}'(0) - K(x)\int_0^1 L'(s)\widetilde{z}'(s)ds, \quad x \in [0,1]. \quad (37)$$

$$\widetilde{z}(0) = 0, \quad (38)$$

$$[B + C]\lambda + C\widetilde{z}(1) = d. \quad (39)$$

A pair $(\widetilde{z}(x), \lambda)$ is a solution to problem for the integro-differential equation with parameter (37)–(39) if:

1) $\widetilde{z}(x)$ has derivatives $\widetilde{z}'(x), \widetilde{z}''(x)$ at each point $x \in [0,1]$;

2) $\widetilde{z}(x)$ and λ satisfy to the integro-differential equation (37) on $[0,1]$;

3) the initial condition (38) is satisfied by $\widetilde{z}(x)$ at the point $x = 0$;

4) the condition (39) is satisfied by $\widetilde{z}(x)$ and λ at the point $x = 1$.

Let $A_1(x) \neq 0$ for all $x \in [0,1]$ and $a(x) = -\int_0^x [A_1(s)]^{-1}A_0(s)ds$. Then we can rewrite the integro-differential equation with parameter and functional terms (37) in the form

$$\widetilde{z}'(x) = -[A_1(x)]^{-1}A_0(x)\widetilde{z}(x) - [A_1(x)]^{-1}A_0(x)\lambda + [A_1(x)]^{-1}F(x) + \\ + [A_1(x)]^{-1}K(x)L(1)\widetilde{z}'(1) - [A_1(x)]^{-1}K(x)L(0)\widetilde{z}'(0) - [A_1(x)]^{-1}K(x)\int_0^1 L'(s)\widetilde{z}'(s)ds, \quad x \in [0,1]. \quad (40)$$

Introduce notations

$$b_p(x) = [A_1(x)]^{-1}K(x)L(p), \qquad p = 0,1; \qquad \theta = \int_0^1 L'(s)\widetilde{z}'(s)ds.$$

Solution to the Cauchy problem for integro-differential equation with parameter and functional terms (37)–(38) can be written in the form

$$\widetilde{z}(x) = -e^{a(x)}\int_0^x e^{-a(s)}[A_1(s)]^{-1}A_0(s)ds\lambda + e^{a(x)}\int_0^x e^{-a(s)}[A_1(s)]^{-1}F(s)ds+ \\ + e^{a(x)}\int_0^x e^{-a(s)}[A_1(s)]^{-1}b_1(s)ds\widetilde{z}'(1) - e^{a(x)}\int_0^x e^{-a(s)}[A_1(s)]^{-1}b_0(s)ds\widetilde{z}'(0)- \\ - e^{a(x)}\int_0^x e^{-a(s)}[A_1(s)]^{-1}K(s)ds\theta, \qquad x\in[0,1]. \quad (41)$$

Let $U(x,Z) = e^{a(x)}\int_0^x e^{-a(s)}[A_1(s)]^{-1}Z(s)ds$; $i = 1,2$; Z is an arbitrary function.

Then we can rewrite the expression (41) in the next form

$$\widetilde{z}(x) = -U(x,A_0)\lambda + U(x,F) + U(x,b_1)\widetilde{z}'(1) - U(x,b_0)\widetilde{z}'(0) - U(x,K)\theta, \qquad x\in[0,1]. \quad (42)$$

From (42) we determine the value $\widetilde{z}(t)$ at $t = 1$:

$$\widetilde{z}(1) = -U(1,A_0)\lambda + U(1,F) + U(1,b_1)\widetilde{z}'(1) - U(1,b_0)\widetilde{z}'(0) - U(1,K)\theta. \quad (43)$$

From (40), and taking into account the notation, we also obtain

$$\widetilde{z}'(x) = -[A_1(x)]^{-1}A_0(x)\widetilde{z}(x) - [A_1(x)]^{-1}A_0(x)\lambda + [A_1(x)]^{-1}F(x)+ \\ + b_1(x)\widetilde{z}'(1) - b_0(x)\widetilde{z}'(0) - [A_1(x)]^{-1}K(x)\theta, \qquad x\in[0,1]. \quad (44)$$

We substitute for $\widetilde{z}(x)$ in (44) the corresponding expression given in (42)

$$\widetilde{z}'(x) = [A_1(x)]^{-1}A_0(x)[U(x,A_0) - 1]\lambda + [A_1(x)]^{-1}[F(x) - A_0(x)U(x,F)]+ \\ + [b_1(x) - [A_1(x)]^{-1}A_0(x)U(x,b_1)]\widetilde{z}'(1) - [b_0(x) - [A_1(x)]^{-1}A_0(x)U(x,b_0)]\widetilde{z}'(0)- \\ - [A_1(x)]^{-1}[K(x) - A_0(x)U(x,K)]\theta, \qquad x\in[0,1]. \quad (45)$$

Changing x by ξ, multiplying both parts of (45) by $L'(\xi)$ and integrating from 0 to 1, we get equation for θ:

$$\theta = \int_0^1 L'(\xi)[A_1(\xi)]^{-1}A_0(\xi)[U(\xi,A_0) - 1]d\xi\lambda + \int_0^1 L'(\xi)[A_1(\xi)]^{-1}[F(\xi) - A_0(\xi)U(\xi,F)]d\xi+ \\ + \int_0^1 L'(\xi)[b_1(\xi) - [A_1(\xi)]^{-1}A_0(\xi)U(\xi,b_1)]d\xi\widetilde{z}'(1) - \int_0^1 L'(\xi)[b_0(\xi) - [A_1(\xi)]^{-1}A_0(\xi)U(\xi,b_0)]d\xi\widetilde{z}'(0)- \\ - \int_0^1 L'(\xi)[A_1(\xi)]^{-1}[K(\xi) - A_0(\xi)U(\xi,K)]d\xi\theta. \quad (46)$$

From (45) we determine equations for finding $\widetilde{z}'(1)$ and $\widetilde{z}'(0)$:

$$[1-b_1(1)+[A_1(1)]^{-1}A_0(1)U(1,b_1)]\widetilde{z}'(1)=[A_1(1)]^{-1}A_0(1)[U(1,A_0)-1]\lambda+ \\ +[A_1(1)]^{-1}[F(1)-A_0(1)U(1,F)]-[b_0(1)-[A_1(1)]^{-1}A_0(1)U(1,b_0)]\widetilde{z}'(0)- \\ -[A_1(1)]^{-1}[K(1)-A_0(1)U(1,K)]\theta, \quad (47)$$

$$\widetilde{z}'(0)=-[A_1(0)]^{-1}A_0(0)\lambda+[A_1(0)]^{-1}F(0)+b_1(0)\widetilde{z}'(1)-b_0(0)\widetilde{z}'(0)-[A_1(0)]^{-1}K(0)\theta. \quad (48)$$

Assume that $B_1=1-b_1(1)+[A_1(1)]^{-1}A_0(1)U(1,b_1)\neq 0$. Then from (47) we uniquely determine $\widetilde{z}'(1)$ through λ, $\widetilde{z}'(0)$ and θ:

$$\widetilde{z}'(1)=B_1^{-1}[A_1(1)]^{-1}A_0(1)[U(1,A_0)-1]\lambda+B_1^{-1}[A_1(1)]^{-1}[F(1)-A_0(1)U(1,F)]- \\ -B_1^{-1}[b_0(1)-[A_1(1)]^{-1}A_0(1)U(1,b_0)]\widetilde{z}'(0)-B_1^{-1}[A_1(1)]^{-1}[K(1)-A_0(1)U(1,K)]\theta. \quad (49)$$

Substituting the found $\widetilde{z}'(1)$ into (48), we obtain

$$\Big[1+b_0(0)+b_1(0)B_1^{-1}[b_0(1)-[A_1(1)]^{-1}A_0(1)U(1,b_0)]\Big]\widetilde{z}'(0)= \\ =\Big[b_1(0)B_1^{-1}[A_1(1)]^{-1}A_0(1)[U(1,A_0)-1]-[A_1(0)]^{-1}A_0(0)\Big]\lambda+ \\ +[A_1(0)]^{-1}F(0)+b_1(0)B_1^{-1}[A_1(1)]^{-1}[F(1)-A_0(1)U(1,F)]- \\ -b_1(0)B_1^{-1}[A_1(1)]^{-1}[K(1)-A_0(1)U(1,K)]\theta-[A_1(0)]^{-1}K(0)\theta. \quad (50)$$

Assume that $B_0=1+b_0(0)+b_1(0)B_1^{-1}[b_0(1)-[A_1(1)]^{-1}A_0(1)U(1,b_0)]\neq 0$. Then from (50) we uniquely determine $\widetilde{z}'(0)$ through λ and θ:

$$\widetilde{z}'(0)=\alpha_0\lambda-\beta_0\theta+\gamma_0, \quad (51)$$

where $\alpha_0=B_0^{-1}\Big[b_1(0)B_1^{-1}[A_1(1)]^{-1}A_0(1)[U(1,A_0)-1]-[A_1(0)]^{-1}A_0(0)\Big]$,
$\beta_0=B_0^{-1}\Big[b_1(0)B_1^{-1}[A_1(1)]^{-1}[K(1)-A_0(1)U(1,K)]+[A_1(0)]^{-1}K(0)\Big]$,
$\gamma_0=B_0^{-1}\Big[[A_1(0)]^{-1}F(0)+b_1(0)B_1^{-1}[A_1(1)]^{-1}[F(1)-A_0(1)U(1,F)]\Big]$.

Substituting the found $\widetilde{z}'(0)$ in (49), we have

$$\widetilde{z}'(1)=\alpha_1\lambda-\beta_1\theta+\gamma_1, \quad (52)$$

where $\alpha_1=B_1^{-1}[A_1(1)]^{-1}A_0(1)[U(1,A_0)-1]-B_1^{-1}[b_0(1)-[A_1(1)]^{-1}A_0(1)U(1,b_0)]\alpha_0$,
$\beta_1=B_1^{-1}[A_1(1)]^{-1}[K(1)-A_0(1)U(1,K)]+B_1^{-1}[b_0(1)-[A_1(1)]^{-1}A_0(1)U(1,b_0)]\beta_0$,
$\gamma_1=B_1^{-1}[A_1(1)]^{-1}[F(1)-A_0(1)U(1,F)]-B_1^{-1}[b_0(1)-[A_1(1)]^{-1}A_0(1)U(1,b_0)]\gamma_0$.

Now, we return to expression (46). We replace $\widetilde{z}'(0)$ and $\widetilde{z}'(1)$ by corresponding representations (51) and (52):

$$[1+\phi_1-\varphi_1\beta_1+\varphi_0\beta_0]\theta=[\phi_2+\varphi_1\alpha_1-\varphi_0\alpha_0]\lambda+\phi_3+\varphi_1\gamma_1-\varphi_0\gamma_0, \quad (53)$$

where $\phi_1=\int\limits_0^1 L'(\xi)[A_1(\xi)]^{-1}[K(\xi)-A_0(\xi)U(\xi,K)]d\xi$,
$\phi_2=\int\limits_0^1 L'(\xi)[A_1(\xi)]^{-1}A_0(\xi)[U(\xi,A_0)-1]d\xi$, $\quad \phi_3=\int\limits_0^1 L'(\xi)[A_1(\xi)]^{-1}[F(\xi)-A_0(\xi)U(\xi,F)]d\xi$,
$\varphi_1=\int\limits_0^1 L'(\xi)[b_1(\xi)-[A_1(\xi)]^{-1}A_0(\xi)U(\xi,b_1)]d\xi$, $\quad \varphi_0=\int\limits_0^1 L'(\xi)[b_0(\xi)-[A_1(\xi)]^{-1}A_0(\xi)U(\xi,b_0)]d\xi$.

Assume that $C_0 = 1 + \phi_1 - \varphi_1\beta_1 + \varphi_0\beta_0 \neq 0$. Then, from (53) it follows

$$\theta = C_0^{-1}[\phi_2 + \varphi_1\alpha_1 - \varphi_0\alpha_0]\lambda + C_0^{-1}[\phi_3 + \varphi_1\gamma_1 - \varphi_0\gamma_0]. \quad (54)$$

Using the expression (43) for $\widetilde{z}(1)$ and taking into account formulas (51), (52), and (54), we get

$$\widetilde{z}(1) = \Big\{-U(1,A_0) + U(1,b_1)\alpha_1 - U(1,b_0)\alpha_0 - \\ - [U(1,b_1)\beta_1 - U(1,b_0)\beta_0 + U(1,K)]C_0^{-1}[\phi_2 + \varphi_1\alpha_1 - \varphi_0\alpha_0]\Big\}\lambda + U(1,F) + U(1,b_1)\gamma_1 - \\ - U(1,b_0)\gamma_0 - [U(1,b_1)\beta_1 - U(1,b_0)\beta_0 + U(1,K)]C_0^{-1}[\phi_3 + \varphi_1\gamma_1 - \varphi_0\gamma_0]. \quad (55)$$

Further, substituting the expression (55) into (39), we obtain

$$\Big[B + C\Big\{1 - U(1,A_0) + U(1,b_1)\alpha_1 - U(1,b_0)\alpha_0 - \\ - [U(1,b_1)\beta_1 - U(1,b_0)\beta_0 + U(1,K)]C_0^{-1}[\phi_2 + \varphi_1\alpha_1 - \varphi_0\alpha_0]\Big\}\Big]\lambda = d - C\Big\{U(1,F) + \\ + U(1,b_1)\gamma_1 - U(1,b_0)\gamma_0 - [U(1,b_1)\beta_1 - U(1,b_0)\beta_0 + U(1,K)]C_0^{-1}[\phi_3 + \varphi_1\gamma_1 - \varphi_0\gamma_0]\Big\}. \quad (56)$$

Assume that

$$Q_0 = \Big[B + C\Big\{1 - U(1,A_0) + U(1,b_1)\alpha_1 - U(1,b_0)\alpha_0 - \\ - [U(1,b_1)\beta_1 - U(1,b_0)\beta_0 + U(1,K)]C_0^{-1}[\phi_2 + \varphi_1\alpha_1 - \varphi_0\alpha_0]\Big\}\Big] \neq 0.$$

Then from (56) we uniquely determine λ:

$$\lambda = [Q_0]^{-1}\Big[d - C\Big\{U(1,F) + U(1,b_1)\gamma_1 - U(1,b_0)\gamma_0 - \\ - [U(1,b_1)\beta_1 - U(1,b_0)\beta_0 + U(1,K)]C_0^{-1}[\phi_3 + \varphi_1\gamma_1 - \varphi_0\gamma_0]\Big\}\Big].$$

Therefore, we found λ, θ, $\widetilde{z}(1)$ and $\widetilde{z}(0)$. Then from (42) we can define the explicit form of solution to the Cauchy problem for integro-differential equation with parameter and functional terms (37), (38).

Theorem 3. Assume that

a) $A_i(x)$, $i = 0,1$, and $F(x)$ are continuous on $[0,1]$; let $A_1(x) \neq 0$ for all $x \in [0,1]$;
b) $K(x)$ is continuous on $[0,1]$; $L(s)$ is continuously differentiable on $[0,1]$; B, C and d are constants;
c) $B_1 = 1 - b_1(1) + [A_1(1)]^{-1}A_0(1)U(1,b_1) \neq 0$ and
$B_0 = 1 + b_0(0) + b_1(0)B_1^{-1}[b_0(1) - [A_1(1)]^{-1}A_0(1)U(1,b_0)] \neq 0$;
d) $C_0 = 1 + \phi_1 - \varphi_1\beta_1 + \varphi_0\beta_0 \neq 0$;
e) $$Q_0 = \Big[B + C\Big\{1 - U(1,A_0) + U(1,b_1)\alpha_1 - U(1,b_0)\alpha_0 - \\ -[U(1,b_1)\beta_1 - U(1,b_0)\beta_0 + U(1,K)]C_0^{-1}[\phi_2 + \varphi_1\alpha_1 - \varphi_0\alpha_0]\Big\}\Big] \neq 0,$$

where $a(x) = -\int_0^x [A_1(s)]^{-1}A_0(s)ds, \quad b_p(x) = [A_1(x)]^{-1}K(x)L(p), \quad p = 0,1, \quad x \in [0,1]$,

$$U(x,Z) = e^{a(x)}\int_0^x e^{-a(s)}[A_1(s)]^{-1}Z(s)ds, \quad x \in [0,1], \ i = 1,2, \ Z \text{ is } A_0,\ b_1,\ b_0 \text{ and } K,$$

$$\alpha_0 = B_0^{-1}\Big[b_1(0)B_1^{-1}[A_1(1)]^{-1}A_0(1)[U(1,A_0) - 1] - [A_1(0)]^{-1}A_0(0)\Big],$$

$\alpha_1 = B_1^{-1}[A_1(1)]^{-1}A_0(1)[U(1,A_0)-1] - B_1^{-1}[b_0(1) - [A_1(1)]^{-1}A_0(1)U(1,b_0)]\alpha_0,$
$\beta_0 = B_0^{-1}\Big[b_1(0)B_1^{-1}[A_1(1)]^{-1}[K(1) - A_0(1)U(1,K)] + [A_1(0)]^{-1}K(0)\Big],$
$\beta_1 = B_1^{-1}[A_1(1)]^{-1}[K(1) - A_0(1)U(1,K)] + B_1^{-1}[b_0(1) - [A_1(1)]^{-1}A_0(1)U(1,b_0)]\beta_0,$
$\phi_1 = \int\limits_0^1 L'(\xi)[A_1(\xi)]^{-1}[K(\xi) - A_0(\xi)U(\xi,K)]d\xi, \quad \phi_2 = \int\limits_0^1 L'(\xi)[A_1(\xi)]^{-1}A_0(\xi)[U(\xi,A_0)-1]d\xi,$
$\varphi_1 = \int\limits_0^1 L'(\xi)[b_1(\xi) - [A_1(\xi)]^{-1}A_0(\xi)U(\xi,b_1)]d\xi, \quad \varphi_0 = \int\limits_0^1 L'(\xi)[b_0(\xi) - [A_1(\xi)]^{-1}A_0(\xi)U(\xi,b_0)]d\xi.$

Then two-point boundary value problem for integro-differential equation (34)–(35) has a unique solution.

Proof. Let's consider problem (34)–(35). We apply the parametrization method and move on to the equivalent problem (37)–(39). Let the conditions of the theorem be satisfied. Then we will uniquely determine the unknowns λ, θ, $\widetilde{z}(0)$, and $\widetilde{z}(1)$.

We have

$$\lambda^* = [Q_0]^{-1}\Big[d - C\Big\{U(1,F) + U(1,b_1)\gamma_1 - U(1,b_0)\gamma_0 - \\ - [U(1,b_1)\beta_1 - U(1,b_0)\beta_0 + U(1,K)]C_0^{-1}[\phi_3 + \varphi_1\gamma_1 - \varphi_0\gamma_0]\Big\}\Big],$$

$$\theta^* = C_0^{-1}[\phi_2 + \varphi_1\alpha_1 - \varphi_0\alpha_0]\lambda^* + C_0^{-1}[\phi_3 + \varphi_1\gamma_1 - \varphi_0\gamma_0],$$

$$\widetilde{z}^{*\prime}(0) = \alpha_0\lambda^* - \beta_0\theta^* + \gamma_0,$$

$$\widetilde{z}^{*\prime}(1) = \alpha_1\lambda^* - \beta_1\theta^* + \gamma_1,$$

where

$\gamma_0 = B_0^{-1}\Big[[A_1(0)]^{-1}F(0) + b_1(0)B_1^{-1}[A_1(1)]^{-1}[F(1) - A_0(1)U(1,F)]\Big],$
$\gamma_1 = B_1^{-1}[A_1(1)]^{-1}[F(1) - A_0(1)U(1,F)] - B_1^{-1}[b_0(1) - [A_1(1)]^{-1}A_0(1)U(1,b_0)]\gamma_0,$
$\phi_3 = \int\limits_0^1 L'(\xi)[A_1(\xi)]^{-1}[F(\xi) - A_0(\xi)U(\xi,F)]d\xi.$

Then, we obtain

$$\widetilde{z}^*(x) = -U(x,A_0)\lambda^* + U(x,F) + U(x,b_1)\widetilde{z}'(1) - U(x,b_0)\widetilde{z}^{*\prime}(0) - U(x,K)\theta^*, \qquad x \in [0,1],$$

where $U(x,F) = e^{a(x)}\int\limits_0^x e^{-a(s)}[A_1(s)]^{-1}F(s)ds.$

Therefore, problem for integro-differential equation with parameter and functional terms (37)–(39) admits a unique solution, given by the pair $(\widetilde{z}^*(x), \lambda^*)$.

We define

$$z^*(x) = \widetilde{z}^*(x) + \lambda^*, \qquad x \in [0,1].$$

According to the scheme of the method, this function is the unique solution of the two-point boundary value problem for integro-differential equation with higher-order derivative in integral term (34)–(35).

Theorem 3 is proved. □

Conclusion

We propose a method for solving boundary value problems for integro-differential equations with higher-order derivatives appearing in the integral term. Unlike classical works and the works of Dzhumabaev, this paper proposes a new approach to solving boundary value problems for integro-differential equations with higher-order derivatives in the integral term, where the coefficients of the equation are assumed to be continuous functions only.

This approach is planned to be extended to problems where the order difference between the differential and integral parts is greater than two, with the goal of refining solvability results for a broader range of boundary value problems.

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Author Contributions

All authors contributed equally to this work.

Conflict of Interest

The authors declare no conflict of interest.

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On the n-inner product spaces from the perspective of its quotient spaces

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In this paper, we investigate several topological properties of n-inner product spaces with respect to the inner products and norms defined on the quotient spaces we constructed. The construction is carried out with respect to a set of n linearly independent vectors, ensuring a consistent analytical framework. This construction was performed in several ways, each resulting in multiple quotient spaces. On each of these quotient spaces, we defined an inner product, along with the corresponding induced norm. Quotient spaces with similar structures are grouped into equivalence classes, thereby yielding several classes of quotient spaces. Within this framework, several topological aspects, including weak convergence, strong convergence, Cauchy sequences, and completeness, are examined with respect to classes of quotient spaces. Consequently, for each aspect, multiple definitions are formulated relative to these classes. We showed that the various definitions associated with a given topological property are equivalent to one another, regardless of the class of quotient spaces we used. Finally, the minimal number of norms within a given class required for an effective investigation of these properties is determined, thereby contributing to a more efficient and non-redundant analytical framework.

Keywords: class of quotient space, inner product, n-inner product space, n-normed space, norm, quotient space, sequence, topology property.

2020 Mathematics Subject Classification: 46B15, 46C50, 46B25.

Introduction

An n-normed space is a generalization of a normed space. This concept was initially introduced by S. Gähler in the 1960's. He subsequently published this concept in a series of papers; see [1] for the concept of 2-normed spaces and [2–4] for generalized metric spaces. Let n be a nonnegative integer and X be a real vector space ($\dim(X) \geq n$). A function $\|\cdot,\dots,\cdot\| : X^n \to \mathbb{R}$ which satisfies the following conditions:

N1. $\|x_1,\dots,x_n\| = 0$ if and only if $x_1,\dots,x_n$ are linearly dependent;

N2. $\|x_1,\dots,x_n\|$ is invariant under permutation;

N3. $\|\alpha x_1,\dots,x_n\| = |\alpha|\|x_1,\dots,x_n\|$ for any $\alpha \in \mathbb{R}$;

N4. $\|x_1 + x_1',\dots,x_n\| \leq \|x_1,\dots,x_n\| + \|x_1',\dots,x_n\|$,

is called an n-norm. The pair $(X, \|\cdot,\dots,\cdot\|)$ is called an n-normed space.

Let $(X, \langle\cdot,\cdot\rangle)$ be an inner product space with $\dim(X) \geq n$,

$$\|x_1 \dots, x_n\|_s = \sqrt{|det(\langle x_i, x_j\rangle)|} = \begin{vmatrix} \langle x_1, x_1\rangle & \langle x_1, x_2\rangle & \cdots & \langle x_1, x_n\rangle \\ \langle x_2, x_1\rangle & \langle x_2, x_2\rangle & \cdots & \langle x_2, x_n\rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle x_n, x_1\rangle & \langle x_n, x_2\rangle & \cdots & \langle x_n, x_n\rangle \end{vmatrix}^{\frac{1}{2}},$$

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defines an n-norm on X. This n-norm is called the standard n-norm. Since then, the theory has been further developed by many researchers; see, for instance, [5–7], which focus on structural and functional analytic aspects of n-normed spaces, and [8–10], which address properties of norms and the topology of n-normed spaces.

On the other hand, we also know that the concept of n-inner product spaces is a generalization of the concept of inner product spaces. For $n = 2$, the concept of 2-inner product space was introduced by Diminnie, Gahler, and White in the 1970s [11]. Misiak then developed the concept of n-inner product spaces for $n \geq 2$ in 1989 [12].

Let n be a nonnegative integer and X be a real vector space with $\dim X \geq n$ ($\dim X$ may be infinite). A function $\langle \cdot, \cdot | \cdot, \dots, \cdot \rangle : X^{n+1} \to \mathbb{R}$ that satisfies

I1. $\langle x_1, x_1 | x_2, \dots, x_n \rangle \geq 0$;
$\langle x_1, x_1 | x_2, \dots, x_n \rangle = 0$, if and only if $x_1, x_2, \dots, x_n$ are linearly dependent;
I2. $\langle x_1, x_1 | x_2, \dots, x_n \rangle = \langle x_{i_1}, x_{i_1} | x_{i_2}, \dots, x_{i_n} \rangle$, for every permutation $(i_1, \dots, i_n)$ of $(1, \dots, n)$;
I3. $\langle x, y | x_2, \dots, x_n \rangle = \langle y, x | x_2, \dots, x_n \rangle$;
I4. $\langle \alpha x, y | x_2, \dots, x_n \rangle = \alpha \langle x, y | x_2, \dots, x_n \rangle$;
I5. $\langle x + x', y | x_2, \dots, x_n \rangle = \langle x, y | x_2, \dots, x_n \rangle + \langle x', y | x_2, \dots, x_n \rangle$,

is called an n-inner product. The pair $(X, \langle \cdot, \cdot | \cdot, \dots, \cdot \rangle)$ is called an n-inner product space.

Let $(X, \langle \cdot, \cdot \rangle)$ be an inner product space with $\dim(X) \geq n$, we define an n-inner product on X as

$$\langle u, v | x_2, \dots, x_n \rangle_S := \begin{vmatrix} \langle u, v \rangle & \langle u, x_2 \rangle & \cdots & \langle u, x_n \rangle \\ \langle x_2, v \rangle & \langle x_2, x_2 \rangle & \cdots & \langle x_2, x_n \rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle x_n, v \rangle & \langle x_n, x_2 \rangle & \cdots & \langle x_n, x_n \rangle \end{vmatrix}$$

we called it the standard n-inner product on X.

Various aspects of n-inner product spaces have been studied by many researchers. The reader may see, for instance, [13–15], which focus on structural and functional analytic properties of n-inner product spaces, and [16–18], which deal with generalized and extended settings, including fuzzy and completion aspects.

On an n-inner product space $(X, \langle \cdot, \cdot | \cdot, \dots, \cdot \rangle)$ we can define an n-norm that induced by the n-inner product, defined by

$$\|x_1, x_2, \dots, x_n\| = \langle x_1, x_1 | x_2, \dots, x_n \rangle^{\frac{1}{2}} \quad \text{for any} \quad x_1, \dots, x_n \in X.$$

One can see that the standard n norm is an induced norm with respect to the standard n-inner product. Moreover, in an n-inner product space $(X, \langle \cdot, \cdot | \cdot, \dots, \cdot \rangle)$, we have Cauchy–Schwarz inequality

$$|\langle x, y | x_2, \dots, x_n \rangle| \leq \|x, x_2, \dots, x_n\| \|y, x_2, \dots, x_n\|,$$

with $\|\cdot, \dots, \cdot\|$ is an induced n-norm.

1 *Results*

1.1 *Construction of Quotient Spaces*

Let $(X, \langle \cdot, \cdot | \cdot, \dots, \cdot \rangle)$ be an n-inner product space. First, choose a linearly independent set, namely $A = \{a_1, a_2, \dots, a_n\}$ in X. Fix a $j \in \{1, 2, \dots, n\}$ and consider a set $A \setminus \{a_j\}$. We define a subspace generated by $A \setminus \{a_j\}$

$$A_j = \text{span } A \setminus \{a_j\} = \left\{ \sum_{i=1, i \neq j}^{n} \gamma_i a_i \ ; \ \gamma_i \in \mathbb{R} \right\}.$$

For any $x \in X$, we define the coset of A_j by

$$\overline{x} = \left\{ x + \sum_{i=1, i \neq j}^{n} \gamma_i a_i \ ; \ \gamma_i \in \mathbb{R} \right\}.$$

We have $\overline{0} = A_j$. Moreover, $\overline{x} = \overline{y}$ if and only if $x - y \in \text{span } A \setminus \{a_j\} = A_j$. Next, we define a quotient space of X as

$$X_j = X/A_j = \{\overline{x} \ : \ x \in X\}.$$

It is easy to see that the addition and scalar multiplication apply in X_j. Consider a function $\langle \cdot, \cdot \rangle_j : X_j^2 \to \mathbb{R}$ defined by

$$\langle \overline{x}, \overline{y} \rangle_j = \langle x, y | a_1, \ldots, a_{j-1}, a_{j+1}, \ldots, a_n \rangle. \tag{1}$$

The following theorem shows that X_j equipped with $\langle x, y \rangle_j$ is an inner product space.

Theorem 1. Let $(X, \langle \cdot, \cdot | \cdot, \ldots, \cdot \rangle)$ be an n-inner product space and $A = \{a_1, a_2, \ldots, a_n\}$ in X be a linearly independent set. Then $(X_j, \langle \cdot, \cdot \rangle_j)$ is an inner product space, with $\langle \cdot, \cdot \rangle_j$ is a function defined in (1).

Proof. We show that the function defined in (1) is an inner product. Using the properties of the n-inner product, we have

$$\langle \overline{x}, \overline{x} \rangle_j = \langle x, x | a_1, \ldots, a_{j-1}, a_{j+1}, \ldots, a_n \rangle \geq 0$$

for all $\overline{x} \in X_j$. Moreover, if $\langle \overline{x}, \overline{x} \rangle_j = 0$, then $\langle x, x | a_1, \ldots, a_{j-1}, a_{j+1}, \ldots, a_n \rangle = 0$. This implies that $x, a_1, \ldots, a_{j-1}, a_{j+1}, \ldots, a_n$ are linearly dependent. As a consequence

$$x = \sum_{i=1, i \neq j}^{n} \gamma_i a_i,$$

since $a_1, a_2, \ldots, a_n$ are linearly independent, we have $x \in A_j$ which leads to $\overline{x} = \overline{0}$ as a result. Conversely, if $\overline{x} = \overline{0}$ it is obvious that

$$\langle \overline{x}, \overline{x} \rangle_j = \langle \overline{0}, \overline{0} \rangle_j = \langle 0, 0 | a_1, \ldots, a_{j-1}, a_{j+1}, \ldots, a_n \rangle = 0.$$

Next, for any $\alpha \in \mathbb{R}$, and $\overline{x}, \overline{y}, \overline{z} \in X_j$, we have

$$\begin{aligned} \langle \alpha \overline{x}, \overline{y} \rangle_j &= \langle \alpha x, y | a_1, \ldots, a_{j-1}, a_{j+1}, \ldots, a_n \rangle \\ &= \alpha \langle x, y | a_1, \ldots, a_{j-1}, a_{j+1}, \ldots, a_n \rangle \\ &= \alpha \langle \overline{x}, \overline{y} \rangle_j. \end{aligned}$$

It is easy to see that using properties of the n-inner product we also have $\langle \overline{x}, \overline{y} \rangle_j = \langle \overline{y}, \overline{x} \rangle_j$ and $\langle \overline{x} + \overline{y}, \overline{z} \rangle_j = \langle \overline{x}, \overline{z} \rangle_j + \langle \overline{y}, \overline{z} \rangle_j$. □

Recall that on an inner product space $(X, \langle \cdot, \cdot \rangle)$ one can define an induced norm from an inner product defined by $\|x\| = \langle x, x \rangle^{\frac{1}{2}}$ for any $x \in X$. As a consequence, we can define a norm on X_j induced by the inner product defined on (1). For all $\overline{x} \in X_j$ we have

$$\begin{aligned} \|\overline{x}\|_j &= \langle x, x \rangle_j^{\frac{1}{2}} \\ &= \langle x, x | a_1, \ldots, a_{j-1}, a_{j+1}, \ldots, a_n \rangle^{\frac{1}{2}} \\ &= \|x, a_1, \ldots, a_{j-1}, a_{j+1}, \ldots, a_n\|. \end{aligned} \tag{2}$$

In an inner product space $(X, \langle\cdot,\cdot\rangle)$, we have Cauchy Schwarz inequality. For any $x, y \in X$

$$|\langle x, y\rangle| \leq \|x\|\|y\|,$$

with $\|\cdot\|$ is the induced norm.

Corollary 1. Let $(X, \langle\cdot,\cdot|\cdot,\ldots,\cdot\rangle)$ be an n-inner product space and $A = \{a_1, a_2, \ldots, a_n\}$ in X be a linearly independent set. Then $(X_j, \|\cdot\|_j)$ is a normed space, with $\|\cdot\|_j$ is an induced norm defined in (2).

The above corollary is a direct corollary from Theorem 1 and equation (2).

The quotient space X_j with a fixed j was constructed with respect to a linearly independent set A by "eliminating" one vector of A. Note that, we can choose any linearly independent set consisting of n vectors to substitute for A. Using this construction, we can get n quotient spaces which are also inner product spaces. We collect these quotient spaces in a set and name it class-1.

Recall that $\overline{x} = \overline{y}$ if and only if $x - y \in \text{span } A \setminus \{a_j\} = A_j$ for a $j \in \{1, \ldots, n\}$. We will examine this case when it applies in all quotient space.

Lemma 1. Let $(X, \langle\cdot,\cdot|\cdot,\ldots,\cdot\rangle)$ be an n-inner product space and $A = \{a_1, a_2, \ldots, a_n\}$ in X be a linearly independent set. Then $x = y$ if and only if $x - y \in \text{span } A \setminus \{a_j\} = A_j$ for all $j \in \{1, \ldots, n\}$.

Proof. Let $x - y \in \text{span } A \setminus \{a_j\} = A_j$ for all $j \in \{1, \ldots, n\}$. We can write

$$x - y = \alpha_{12}a_2 + \alpha_{13}a_3 + \cdots + \alpha_{1n}a_n; \quad (2.1)$$

$$x - y = \alpha_{21}a_1 + \alpha_{23}a_3 + \cdots + \alpha_{2n}a_n; \quad (2.2)$$

$$\vdots$$

$$x - y = \alpha_{n1}a_1 + \alpha_{n2}a_2 + \cdots + \alpha_{n(n-1)}a_{n-1}, \quad (2.n)$$

with $\alpha_{ij} \in \mathbb{R}$, $i, j \in \{1, \ldots, n\}$. Here, we have n equations. By subtracting equation (2.2) from equation (2.1), we obtain

$$0 = -\alpha_{21}a_1 + \alpha_{12}a_2 + (\alpha_{13} - \alpha_{23})a_3 + \cdots + (\alpha_{1n} - \alpha_{1n})a_n.$$

Since $a_1, \ldots, a_n$ is a linearly independent set, it follows that all the coefficients in the above equation must be 0. Especially we have $\alpha_{12} = 0$. Next, by subtracting equation (2.3) from equation (2.1), we obtain

$$0 = -\alpha_{31}a_1 + (\alpha_{12} - \alpha_{32})a_2 + \alpha_{13}a_3 + (\alpha_{14} - \alpha_{22})a_4 + \cdots + (\alpha_{1n} - \alpha_{1n})a_n.$$

Since $a_1, \ldots, a_n$ is a linearly independent set, it follows that all the coefficients in the above equation must be 0. Especially we have $\alpha_{13} = 0$. By repeating this procedure we have $a_{1j} = 0$. This implies $x - y = 0$ or $x = y$. Conversely, if $x = y$, then $x - y = 0 \in \text{span } A \setminus \{a_j\} = A_j$ for all $j \in \{1, \ldots, n\}$. □

Lemma 1 states that, if the condition $x - y \in A_j$, applies for all $j \in \{1, \ldots, n\}$, then $x = y$. We may conclude that, by considering all quotient spaces, we are able to observe the "real" point rather than its representation as a coset.

Example 1. Let $(\mathbb{R}^n, \langle\cdot,\cdot,\cdot,\cdot\rangle)$ be a 4-inner product space, with $n \geq 4$ and $A = \{a_1, a_2, a_3, a_4\}$ be a linearly independent set. The class-1 of $\mathbb{R}^n$ with respect to A consist of four quotient spaces, namely $\mathbb{R}^n_1, \mathbb{R}^n_2, \mathbb{R}^n_3$ and $\mathbb{R}^n_4$. These quotient spaces are inner product spaces and also normed spaces. Their inner product and norm are defined as

$\langle\overline{x}, \overline{y}\rangle_1 = \langle x, y|a_2, a_3, a_4\rangle$ and its induced norm is $\|\overline{x}\|_1 = \|x, a_2, a_3, a_4\|$;
$\langle\overline{x}, \overline{y}\rangle_2 = \langle x, y|a_2, a_3, a_4\rangle$ and its induced norm is $\|\overline{x}\|_2 = \|x, a_1, a_3, a_4\|$;
$\langle\overline{x}, \overline{y}\rangle_3 = \langle x, y|a_1, a_2, a_4\rangle$ and its induced norm is $\|\overline{x}\|_3 = \|x, a_1, a_2, a_4\|$;
$\langle\overline{x}, \overline{y}\rangle_4 = \langle x, y|a_1, a_2, a_3\rangle$ and its induced norm is $\|\overline{x}\|_4 = \|x, a_1, a_2, a_3\|$, respectively.

We will generalize the above construction by 'eliminating' more vectors of A. For an $m \in \{1, 2, \ldots, n\}$ fix some vectors $a_{j_1}, a_{j_2}, \ldots, a_{j_m} \in A$. Consider the set $A \setminus \{a_{j_1}, \ldots, a_{j_m}\}$ and define a subspace generated by $A \setminus \{a_{j_1}, \ldots, a_{j_m}\}$

$$A_{j_1,\ldots,j_m} = \text{span } A \setminus \{a_{j_1}, \ldots, a_{j_m}\} = \left\{ \sum_{\substack{i=1 \\ i \neq j_1,\ldots j_m}}^{n} \beta_i a_i \ : \ \beta_i \in \mathbb{R} \right\}.$$

For any $x \in X$, the corresponding coset of $A_{j_1,\ldots,j_m}$ is

$$\overline{x} = \left\{ x + \sum_{\substack{i=1 \\ i \neq j_1,\ldots j_m}}^{n} \beta_i a_i \ : \ \beta_i \in \mathbb{R} \right\}.$$

One can see that $\overline{0} = A_{j_1,\ldots,j_m}$. Moreover, $\overline{x} = \overline{y}$, if and only if $x - y \in \text{span } A \setminus \{a_{j_1}, \ldots, a_{j_m}\} = A_{j_1,\ldots,j_m}$. We define a quotient space of X (with respect to $A_{j_1,\ldots,j_m}$)

$$X_{j_1,\ldots,j_m} = X / A_{j_1,\ldots,j_m} = \{\overline{x} \ : \ x \in X\}.$$

The addition and scalar multiplication apply in $X_{j_1,\ldots,j_m}$. Next, we define a function $\langle \cdot, \cdot \rangle_{j_1,\ldots,j_m} : X^2_{j_1,\ldots,j_m} \to \mathbb{R}$ defined by

$$\langle \overline{x}, \overline{y} \rangle_{j_1,\ldots,j_m} = \langle x, y | a_1, \ldots, a_{j_1-1}, a_{j_1+1}, \ldots, a_n \rangle + \cdots + \langle x, y | a_1, \ldots, a_{j_m-1}, a_{j_m+1}, \ldots, a_n \rangle \tag{3}$$

We can see that each term of equation (3) is an inner product that is defined on each quotient space of class-1 (see equation (1)). Therefore, equation (3) can be written as

$$\langle \overline{x}, \overline{y} \rangle_{j_1,\ldots,j_m} = \langle \overline{x}, \overline{y} \rangle_{j_1} + \cdots + \langle \overline{x}, \overline{y} \rangle_{j_m}. \tag{4}$$

Moreover, the following theorem states that $\langle \cdot, \cdot \rangle_{j_1,\ldots,j_m}$ is an inner product on $X_{j_1,\ldots,j_m}$.

Theorem 2. Let $(X, \langle \cdot, \cdot | \cdot, \ldots, \cdot \rangle)$ be an n-inner product space and $A = \{a_1, a_2, \ldots, a_n\}$ in X be a linearly independent set. Then $(X_{j_1,\ldots,j_m}, \langle \cdot, \cdot \rangle_{j_1,\ldots,j_m})$ is an inner product space, with $\langle \cdot, \cdot \rangle_{j_1,\ldots,j_m}$ is a function defined in (3).

Proof. We show that $\langle \cdot, \cdot \rangle_{j_1,\ldots,j_m}$ is an inner product on $X_{j_1,\ldots,j_m}$. It is easy to see that

$$\langle \overline{x}, \overline{x} \rangle_{j_1,\ldots,j_m} = \langle \overline{x}, \overline{x} \rangle_{j_1} + \cdots + \langle \overline{x}, \overline{x} \rangle_{j_m} \geq 0.$$

Moreover, if $x = 0$, then it is obvious that $\langle \overline{x}, \overline{x} \rangle_{j_1,\ldots,j_m} = 0$. Conversely, if $\langle \overline{x}, \overline{x} \rangle_{j_1,\ldots,j_m} = 0$, then each term of the above equation is also 0. Based on equation (3), for each term of it we obtain

$$x = \sum_{i=1, i \neq j_1}^{n} \gamma_{1i} a_i, \quad \gamma_{1i} \in \mathbb{R}$$

$$\vdots$$

$$x = \sum_{i=1, i \neq j_m}^{n} \gamma_{mi} a_i, \quad \gamma_{mi} \in \mathbb{R}$$

with $j_1, \ldots, j_m \in \{1, \ldots, n\}$. In other words, x is linearly dependent to $A \setminus \{a_{j_k}\}$ for each $k \in \{1, 2, \ldots, m\}$. It implies x is linearly dependent to $A \setminus \{a_{j_1} \ldots, a_{j_m}\}$. We can write it as

$$\sum_{\substack{i=1 \\ i \neq j_1, \ldots j_m}}^{n} \beta_i a_i \ : \ \beta_i \in \mathbb{R}.$$

This leads to $x \in \text{span } A \setminus \{a_{j_1}, \ldots, a_{j_m}\}$ or $\overline{x} = \overline{0}$.

Moreover, it is straightforward to verify that we also have $\langle \overline{x}, \overline{y} \rangle_{j_1, \ldots, j_m} = \langle \overline{y}, \overline{x} \rangle_{j_1, \ldots, j_m}$, $\langle \alpha \overline{x}, \overline{y} \rangle_{j_1, \ldots, j_m} = \alpha \langle \overline{x}, \overline{y} \rangle_{j_1, \ldots, j_m}$ for any $\alpha \in \mathbb{R}$, and $\langle \overline{x} + \overline{y}, \overline{z} \rangle_{j_1, \ldots, j_m} = \langle \overline{x}, \overline{z} \rangle_{j_1, \ldots, j_m} + \langle \overline{y}, \overline{z} \rangle_{j_1, \ldots, j_m}$. □

Using the relation between an n-inner product and an n-norm on equation (2), we define a norm that is induced by the inner product defined on equation (3)

$$\begin{aligned} \|\overline{x}\|_{j_1, \ldots, j_m} &= \langle \overline{x}, \overline{x} \rangle_{j_1, \ldots, j_m}^{\frac{1}{2}} \\ &= (\langle x, x | a_1, \ldots, a_{j_1 - 1}, a_{j_1 + 1}, \ldots, a_n \rangle + \ldots \\ &\quad + \langle x, x | a_1, \ldots, a_{j_m - 1}, a_{j_m + 1}, \ldots, a_n \rangle)^{\frac{1}{2}} \\ &= \left(\|x, a_1, a_1, \ldots, a_{j_1 - 1}, a_{j_1 + 1}, \ldots, a_n\|^2 + \ldots \right. \\ &\quad \left. + \|x, a_1, \ldots, a_{j_m - 1}, a_{j_m + 1}, \ldots, a_n\|^2 \right)^{\frac{1}{2}} \\ &= \left(\|\overline{x}\|_{j_1}^2 + \cdots + \|\overline{x}\|_{j_m}^2 \right)^{\frac{1}{2}}. \end{aligned} \tag{5}$$

We will have the same result using equation (4). Then we obtain the following corollary.

Corollary 2. Let $(X, \langle \cdot, \cdot | \cdot, \ldots, \cdot \rangle)$ be an n-inner product space and $A = \{a_1, a_2, \ldots, a_n\}$ in X be a linearly independent set. Then $(X_{j_1, \ldots, j_m}, \| \cdot \|_{j_1, \ldots, j_m})$ is a normed space, with $\| \cdot \|_{j_1, \ldots, j_m}$ is an induced norm defined in (5).

The corollary above is a direct corollary from Theorem 2 and equation (5).

Furthermore, using this construction, we can get $\binom{n}{m}$ quotient spaces which are also inner product spaces. We collect these quotient spaces in a set and name it class-m. Since we construct these quotient spaces with respect to a linearly independent set containing n vectors $A = \{a_1, \ldots, a_n\}$, we can have n classes of quotient spaces. These inner products and norms that we defined will be our tools to investigate some aspects in n-inner product spaces.

In particular, for $m = n$ we actually observe an n-inner product space as an inner product space. The class-n will contain itself as a quotient space. The inner product on $(X, \langle \cdot, \cdot | \cdot, \ldots, \cdot \rangle)$ will be defined by

$$\langle \overline{x}, \overline{y} \rangle_{1, \ldots, n} = \sum \langle x, y | a_{j_2}, \ldots, a_{j_n} \rangle,$$

the sum is taken over $\{j_2, \ldots, j_n\} \subset \{1, \ldots, n\}$. Then $(X, \langle \cdot, \cdot \rangle_{1, \ldots, n})$ is an inner product space. This means $(X, \| \cdot \|_{1, \ldots, n})$ is a normed space with the induced norm defined by

$$\|\overline{x}\|_{1, \ldots, n} = \left(\sum \|x, a_{j_2}, \ldots, a_{j_n}\|^2 \right)^{\frac{1}{2}},$$

the sum is taken over $\{j_2, \ldots, j_n\} \subset \{1, \ldots, n\}$.

Lemma 2. Let $(X, \langle \cdot, \cdot | \cdot, \ldots, \cdot \rangle)$ be an n-inner product space and $A = \{a_1, a_2, \ldots, a_n\}$ in X be a linearly independent set. Then $x = y$ if and only if $x - y \in \text{span } A \setminus \{a_{j_1}, \ldots, a_{j_m}\} = A_{j_1, \ldots, j_m}$ for all $\{j_1, \ldots, j_m\} \subset \{1, \ldots, n\}$.

Proof. The proof is similar to proof of Lemma 1. The readers can see that on Lemma 1 we have n equations. Here, we have $\binom{n}{m}$ equations. By subtracting from each of the subsequent equations the first equation, we obtain $x = y$. □

Similar to Lemma 1, if the condition $x - y \in A_{j_1,\ldots,j_m}$ holds for all $\{j_1, \ldots, j_m\} \subset \{1, \ldots, n\}$, then it follows that $x = y$. Therefore, by considering all quotient spaces, we are able to identify the underlying element itself rather than merely its representation as a coset.

Example 2. Let $(\mathbb{R}^n, \langle\cdot,\cdot,\cdot,\cdot\rangle)$ be a 4-inner product space, with $n \geq 4$ and $A = \{a_1, a_2, a_3, a_4\}$ be a linearly independent set. The class-3 of $\mathbb{R}^n$ with respect to A consist of four quotient spaces, namely $\mathbb{R}^n_{1,2,3}, \mathbb{R}^n_{1,2,4}, \mathbb{R}^n_{1,3,4}$ and $\mathbb{R}^n_{2,3,4}$. These quotient spaces are inner product spaces and also normed spaces. Their inner product and norm are defined as

$\langle\overline{x}, \overline{y}\rangle_{1,2,3} = \langle x, y|a_2, a_3, a_4\rangle + \langle x, y|a_1, a_3, a_4\rangle + \langle x, y|a_1, a_2, a_4\rangle$ and its induced norm is $\|x\|_{1,2,3} = \left(\|x, a_2, a_3, a_4\|^2 + \|x, a_1, a_3, a_4\|^2 + \|x, a_1, a_2, a_4\|^2\right)^{\frac{1}{2}}$;

$\langle\overline{x}, \overline{y}\rangle_{1,2,4} = \langle x, y|a_2, a_3, a_4\rangle + \langle x, y|a_1, a_3, a_4\rangle + \langle x, y|a_1, a_2, a_3\rangle$ and its induced norm is $\|x\|_{1,2,4} = \left(\|x, a_2, a_3, a_4\|^2 + \|x, a_1, a_3, a_4\|^2 + \|x, a_1, a_2, a_3\|^2\right)^{\frac{1}{2}}$;

$\langle\overline{x}, \overline{y}\rangle_{1,3,4} = \langle x, y|a_2, a_3, a_4\rangle + \langle x, y|a_1, a_2, a_4\rangle + \langle x, y|a_1, a_2, a_3\rangle$ and its induced norm is $\|x\|_{1,3,4} = \left(\|x, a_2, a_3, a_4\|^2 + \|x, a_1, a_2, a_4\|^2 + \|x, a_1, a_2, a_3\|^2\right)^{\frac{1}{2}}$;

$\langle\overline{x}, \overline{y}\rangle_{2,3,4} = \langle x, y|a_1, a_3, a_4\rangle + \langle x, y|a_1, a_2, a_4\rangle + \langle x, y|a_1, a_2, a_3\rangle$ and its induced norm is $\|x\|_{2,3,4} = \left(\|x, a_1, a_3, a_4\|^2 + \|x, a_1, a_2, a_4\|^2 + \|x, a_1, a_2, a_3\|^2\right)^{\frac{1}{2}}$.

Here one can easily see that each term of the summation in each defined inner product is an inner product of a quotient space in class-1. For example we can write $\langle\overline{x}, \overline{y}\rangle_{1,2,3} = \langle\overline{x}, \overline{y}\rangle_1 + \langle\overline{x}, \overline{y}\rangle_2 + \langle\overline{x}, \overline{y}\rangle_3$, while for the induced norm we can write $\|x\|_{1,2,3} = \left(\|x\|_1^2 + \|x\|_2^2 + \|x\|_3^2\right)^{\frac{1}{2}}$.

Remark 1. We aim to interpret a point in an n-inner product space $(X, \langle\cdot,\cdot|\cdot,\ldots,\cdot\rangle)$ as a "real point" not as a coset. We do so by observing all of its quotient space. For example, if we investigate $0 \in X$ then we can investigate $\overline{0}$ in each $X_{j_1,\ldots,j_m}$ on class-m for all $\{j_1, \ldots, j_m\} \subset \{1, \ldots, n\}$. In other words we observe

$$\bigcap \overline{0}_{j_1,\ldots,j_m}$$

to get 0 as a point, with $\overline{0}_{j_1,\ldots,j_m} \in X_{j_1,\ldots,j_m}$. The intersection is taken by $\{j_1, \ldots, j_m\} \in \{1, \ldots, n\}$. It applies to each vector in X that we observe.

Moreover, we will investigate some aspects of n-inner product space using these inner products and norms of classes of quotient spaces.

As we mentioned before, that all the quotient spaces constructed above are done with respect to a linearly independent set containing n vectors. We can choose any n linearly independent vectors. From here on, we will not mention the set explicitly, unless it is necessary.

1.2 *Some Topology Properties*

We start this section by defining weakly convergent sequences in an n-inner product space using the inner product defined earlier.

Definition 1. Let $(X, \langle\cdot,\cdot|\cdot,\ldots,\cdot\rangle)$ be an n-inner product space. A sequence $\{x_k\} \subset X$ is said to be weakly convergent with respect to class-m to x if

$$\lim_{k\to\infty} \langle\overline{x_k}, \overline{y}\rangle_{j_1,\ldots,j_m} = \langle\overline{x}, \overline{y}\rangle_{j_1,\ldots,j_m}$$

for all $\overline{y} \in X_{j_1,\ldots,j_m}$ and $\{j_1, \ldots, j_m\} \subset \{1, \ldots, n\}$.

The point $x \in X$ is called the limit point of the sequence x_n. One can see that this definition is well defined. We consider the sequence in all quotient spaces. Similar to Lemma 2, this implies that the limit point is identified as the underlying element itself, rather than as a coset.

Moreover, there are n classes of quotient spaces, then the above definition gives n types of weak convergence. The following theorem states that for any two classes we choose ($m \in \{1, \dots, n\}$), the definitions are equivalent.

Theorem 3. Let $(X, \langle \cdot, \cdot | \cdot, \dots, \cdot \rangle)$ be an n-inner product space. A sequence $\{x_k\} \subset X$ is a weakly convergent sequence with respect to class-m_1 if and only if it is a weakly convergent sequence with respect to class-m_2, with $m_1, m_2 \in \{1, \dots, n\}$.

Proof. It suffices to show that a sequence $\{x_k\} \subset X$ is a weakly convergent sequence with respect to class-1 if and only if it is a weakly convergent sequence with respect to class-m, with $m \in \{1, \dots, n\}$. Let $\{x_k\}$ be a weakly convergent sequence with respect to class-1, then for any $\varepsilon > 0$, there is an $N_1 \in \mathbb{N}$ such that for $k \geq N_1$ we have

$$\lim_{k \to \infty} \langle \overline{x_k}, \overline{y} \rangle_1 = \langle \overline{x}, \overline{y} \rangle_1$$

$$\vdots$$

$$\lim_{k \to \infty} \langle \overline{x_k}, \overline{y} \rangle_1 = \langle \overline{x}, \overline{y} \rangle_n$$

from these, we obtain

$$\begin{aligned} \lim_{k \to \infty} \langle \overline{x_k}, \overline{y} \rangle_{j_1, \dots, j_m} &= \lim_{k \to \infty} \langle \overline{x_k}, \overline{y} \rangle_{j_1} + \dots + \lim_{k \to \infty} \langle \overline{x_k}, \overline{y} \rangle_{j_m} \\ &= \langle \overline{x}, \overline{y} \rangle_{j_1} + \dots + \langle \overline{x}, \overline{y} \rangle_{j_m} \\ &= \langle \overline{x}, \overline{y} \rangle_{j_1, \dots, j_m} \end{aligned}$$

for any $\{j_1, \dots, j_m\} \subset \{1, \dots, n\}$. This implies that $\{x_k\}$ is a weakly convergent sequence with respect to class-m. Conversely, let $\{x_k\}$ be a weakly convergent sequence with respect to class-m, then we have

$$\lim_{k \to \infty} \langle \overline{x_k}, \overline{y} \rangle_{j_1, \dots, j_m} = \langle \overline{x}, \overline{y} \rangle_{j_1, \dots, j_m} \tag{6}$$

for any $\{j_1, \dots, j_m\} \subset \{1, \dots, n\}$. From (6) we actually have $\binom{n}{m}$ equations. As a consequence, we have

$$\lim_{k \to \infty} \langle \overline{x_k}, \overline{y} \rangle_i = \langle \overline{x}, \overline{y} \rangle_i$$

for any $i \in \{1, \dots, n\}$. Therefore, $\{x_k\}$ is a weakly convergent sequence with respect to class-1. This ends the proof. □

Note that The conclusion follows by a standard elimination argument on (6). Moreover, the proof of Theorem (3) used class-1 as a bridge to connect all other classes. Later, we will find some theorems that will be proved using the same technique.

Proposition 1. Let x_k be a weakly convergent sequence with respect to class-m, then its limit point is unique.

Proof. Let x, x' be the limit points of x_k, then

$$\langle \overline{x}, \overline{y} \rangle_{j_1, \dots, j_m} = \lim_{k \to \infty} \langle \overline{x_k}, \overline{y} \rangle_{j_1, \dots, j_m} = \langle \overline{x'}, \overline{y} \rangle_{j_1, \dots, j_m}.$$

Here we have

$$\langle \overline{x}, \overline{y} \rangle_{j_1,\ldots,j_m} = \langle \overline{x'}, \overline{y} \rangle_{j_1,\ldots,j_m}$$

for any $\{j_1, \ldots, j_m\} \subset \{1, \ldots, n\}$. This implies

$$\langle \overline{x} - \overline{x'}, \overline{y} \rangle_{j_1,\ldots,j_m} = 0.$$

Since it is applied for all $\overline{y} \in X_{j_1,\ldots,j_m}$ and $\{j_1, \ldots, j_m\} \subset \{1, \ldots, n\}$, we have $x = x'$. We say the limit point is unique. □

In Proposition 1, we consider all quotient spaces simultaneously. This implies that $x - x' = 0$, and hence $x = x'$. Next, we give a definition of a strongly convergent sequence in an n-inner product space with respect to a class of quotient spaces.

Definition 2. Let $(X, \langle \cdot, \cdot | \cdot, \ldots, \cdot \rangle)$ be an n-inner product space. A sequence $\{x_k\} \subset X$ is said to be strongly convergent with respect to class-m to x if

$$\lim_{k\to\infty} \|\overline{x_k} - \overline{x}\|_{j_1,\ldots,j_m} = 0$$

for any $\{j_1, \ldots, j_m\} \subset \{1, \ldots, n\}$.

Definition 2 gives us n types of strongly convergent sequences, since we have n classes of quotient spaces. The following theorem states that all the strongly convergent sequence types are equivalent for any class we choose.

Theorem 4. Let $(X, \langle \cdot, \cdot | \cdot, \ldots, \cdot \rangle)$ be an n-inner product space. A sequence $\{x_k\} \subset X$ is a strongly convergent sequence with respect to class-m_1 if and only if it is a strong convergent sequence with respect to class-m_2, with $m_1, m_2 \in \{1, \ldots, n\}$.

Proof. We prove this theorem by using class-1 as a bridge. Let $\{x_k\} \in X$ is a sequence that strongly converges with respect to class-m to a point x. Based on the definition we have

$$\lim_{k\to\infty} \|\overline{x_k} - \overline{x}\|_{j_1,\ldots,j_m} = \lim_{k\to\infty} \|\overline{x_k} - \overline{x}\|_{j_1} + \cdots + \lim_{k\to\infty} \|\overline{x_k} - \overline{x}\|_{j_m} = 0$$

for any $\{j_1, \ldots, j_m\} \subset \{1, \ldots, n\}$. Therefore, each term of the summation in the above equation equals 0. We write

$$\lim_{k\to\infty} \|\overline{x_k} - \overline{x}\|_i = 0$$

for any $i \in \{1, \ldots, n\}$. This implies, the sequence $\{x_k\}$ is a sequence that strongly converges with respect to class-1 to a point $x \in X$. Conversely, let $\{x_k\} \in X$ is a sequence that strongly converges with respect to class-1 to a point x. We have

$$\lim_{k\to\infty} \|\overline{x_k} - \overline{x}\|_i = 0$$

for any $i, m \in \{1, \ldots, n\}$. Then we have

$$\lim_{k\to\infty} \|\overline{x_k} - \overline{x}\|_{j_1,\ldots,j_m} = \lim_{k\to\infty} \|\overline{x_k} - \overline{x}\|_{j_1} + \cdots + \lim_{k\to\infty} \|\overline{x_k} - \overline{x}\|_{j_m} = 0$$

for any $\{j_1, \ldots, j_m\} \subset \{1, \ldots, n\}$. With this, we know that the sequence $\{x_k\}$ is a sequence that strongly converges with respect to class-m to a point $x \in X$, with $m \in \{1, \ldots, n\}$. This proves the theorem. □

Proposition 2. If $\{x_k\}$ is a strongly convergent sequence with respect to class-m, then its limit point is unique.

Proof. Let x, x' be the limit points of x_k. Then we have

$$\lim_{k\to\infty} \|\overline{x_k} - \overline{x}\|_{j_1,\dots,j_m} = \lim_{k\to\infty} \|\overline{x_k} - \overline{x'}\|_{j_1,\dots,j_m}.$$

One can see that $\overline{x} = \overline{x'}$ this applies in each quotient space $X_{j_1,\dots,j_m}$ for any $\{j_1,\dots,j_m\}$. As a result we have $x = x'$. □

The readers will realize that in the above proof, we use Remark 1 on the last part. Note that we can investigate a convergent sequence (either strongly or weakly) with respect to any class of quotient spaces. The limit point of the sequence will be the same. Next, we will observe a Cauchy sequence on n-normed spaces using tools that we have.

Definition 3. Let $(X, \langle\cdot,\cdot|\cdot,\dots,\cdot\rangle)$ be an n-inner product space. A sequence $\{x_k\} \subset X$ is said to be weakly Cauchy with respect to class-m if

$$\lim_{k,l\to\infty} \langle \overline{x_k} - \overline{x_l}, \overline{y}\rangle_{j_1,\dots,j_m} = 0$$

for all $\overline{y} \in X_{j_1,\dots,j_m}$ and $\{j_1,\dots,j_m\} \subset \{1,\dots,n\}$.

Similar to the weakly convergent sequence, there are n types of the weakly convergent sequence since we have n classes of quotient spaces. We also have all the types of weakly convergent sequence for any class we choose are equivalent. It is stated in the following theorem.

Theorem 5. Let $(X, \langle\cdot,\cdot|\cdot,\dots,\cdot\rangle)$ be an n-inner product space. A sequence $\{x_k\} \subset X$ is a weakly Cauchy sequence with respect to class-m_1 if and only if it is a weakly Cauchy sequence with respect to class-m_2, with $m_1, m_2 \in \{1,\dots,n\}$.

Proof. This proof is analogous to the proof of Theorem 3. □

The readers will easily see that the proof of Theorem 5 also uses class-1 as a bridge. Moreover, the following property establishes that weak convergence implies the weak Cauchy property.

Corollary 3. If $\{x_k\}$ is a weakly convergent sequence with respect to class-m, then it is also weakly Cauchy with respect to class-m for an $m \in \{1,\dots,n\}$.

Proof. Using the Cauchy-Schwarz inequality, we have

$$|\langle \overline{x_k} - \overline{x}, \overline{y}\rangle_{j_1,\dots,j_m}| \le \|\overline{x_k} - \overline{x}\|_{j_1,\dots,j_m} \|\overline{y}\|_{j_1,\dots,j_m}$$

for any $\{j_1,\dots,j_m\} \subset \{1,\dots,n\}$.

Since the right-hand side tends to 0 as k tends to ∞, it applies to the left-hand side. We conclude that weakly convergent implies weakly Cauchy. □

If the converse is true, then the space is called weakly n-Hilbert Space.

Definition 4. Let $(X, \langle\cdot,\cdot|\cdot,\dots,\cdot\rangle)$ be an n-inner product space. A sequence $\{x_k\} \subset X$ is said to be strongly Cauchy with respect to class-m if

$$\lim_{k,l\to\infty} \|\overline{x_k} - \overline{x_l}\|_{j_1,\dots,j_m} = 0$$

for any $\{j_1,\dots,j_m\} \subset \{1,\dots,n\}$.

Since there are n classes of quotient spaces, we also have n types of strongly convergent sequences. Here we give a theorem about relations among all the types of strongly convergent sequence.

Theorem 6. Let $(X, \langle\cdot,\cdot|\cdot,\ldots,\cdot\rangle)$ be an n-inner product space. A sequence $\{x_k\}\subset X$ is a weakly Cauchy sequence with respect to class-m_1 if and only if it is a weakly Cauchy sequence with respect to class-m_2, with $m_1, m_2 \in \{1,\ldots,n\}$.

Proof. This proof is analogous to the proof of Theorem 3. □

One can see that we prove this theorem using the same method, that is using class-1 as a bridge.

Corollary 4. If $\{x_k\}$ is a strongly convergent sequence with respect to class-m then it is also a strongly Cauchy sequence with respect to class-m, for an $m \in \{1,\ldots,n\}$.

Proof. Let $\{x_k\}$ strongly converges with respect to class-m to a point x. Then we have

$$\lim_{k\to\infty}\|\overline{x_k}-\overline{x}\|_{j_1,\ldots,j_m}=0, \text{ and } \lim_{l\to\infty}\|\overline{x_l}-\overline{x}\|_{j_1,\ldots,j_m}=0.$$

On the other hand, we also have

$$\|\overline{x_k}-\overline{x_l}\|_{j_1,\ldots,j_m}\le\|\overline{x_k}-\overline{x}\|_{j_1,\ldots,j_m}+\|\overline{x_l}-\overline{x_l}\|_{j_1,\ldots,j_m}.$$

The right-hand side tends to 0 as $k,l\to\infty$, so does the left-hand side. This leads us to conclude that $\{x_k\}$ is a strongly Cauchy sequence. □

If the converse of the Corollary 4 is true, then the space is called a strong n-Hilbert space.

Remark 2. Since the strongly convergent and Cauchy imply weakly convergent and Cauchy, respectively, the strongly n-Hilbert space implies weakly n-Hilbert space.

Furthermore, in a class of quotient spaces there is more than one quotient space (except for class-n). Do we have to use all the quotient spaces to investigate the above topological properties? To answer this question, we give a simple example below.

Example 3. Let $(\mathbb{R}^4, \langle\cdot,\cdot|\ldots,\cdot\rangle_S)$ be a 4-inner product space with standard n-inner product space defined by

$$\langle u,v|x_2,x_3,x_4\rangle_S := \begin{vmatrix} \langle u,v\rangle & \langle u,x_2\rangle & \langle u,x_3\rangle & \langle u,x_4\rangle \\ \langle x_2,v\rangle & \langle x_2,x_2\rangle & \langle x_2,x_3\rangle & \langle x_2,x_4\rangle \\ \langle x_3,v\rangle & \langle x_3,x_2\rangle & \langle x_3,x_3\rangle & \langle x_3,x_4\rangle \\ \langle x_4,v\rangle & \langle x_4,x_2\rangle & \langle x_4,x_3\rangle & \langle x_4,x_4\rangle \end{vmatrix}.$$

Firstly, we choose $A=\{a_1,a_2,a_3,a_4\}=\{(1,0,0,0),(0,1,0,0),(0,0,1,0),(0,0,0,1)\}$, a standard basis vectors set in $\mathbb{R}^4$. In class-2 we have six quotient spaces, namely $\mathbb{R}^4_{1,2},\mathbb{R}^4_{1,3},\mathbb{R}^4_{1,4},\mathbb{R}^4_{2,3},\mathbb{R}^4_{2,4},\mathbb{R}^4_{3,4}$. Consider a sequence $x_k=\{(\frac{1}{k},\frac{1}{k},\frac{1}{k},\frac{1}{k}):k\in\mathbb{N}\}$. We know that the sequence x_n is a (weakly and strongly) convergent sequence. Let $x=(0,0,0,0)$, consider two quotients spaces in class-2, namely $\mathbb{R}^4_{1,3}$ and $\mathbb{R}^4_{2,4}$ with their inner product. Now, we are investigating the sequence using the quotient spaces. We can see that

$$\begin{aligned}\lim_{k\to\infty}\langle\overline{x_k},\overline{y}\rangle_{1,3} &= \lim_{k\to\infty}\langle x_k,y|a_2,a_3,x_4\rangle_S+\lim_{k\to\infty}\langle x_k,y|a_1,a_2,x_4\rangle_S \\ &= 0 \\ &= \langle\overline{x},\overline{y}\rangle_{1,3},\end{aligned}$$

and

$$\begin{aligned}\lim_{k\to\infty}\langle\overline{x_k},\overline{y}\rangle_{2,4} &= \lim_{k\to\infty}\langle x_k,y|a_1,a_3,x_4\rangle_S+\lim_{k\to\infty}\langle x_k,y|a_1,a_2,x_3\rangle_S \\ &= 0 \\ &= \langle\overline{x},\overline{y}\rangle_{2,4}.\end{aligned}$$

Using these two quotient spaces, it is sufficient to say that x_k weakly converges to x. We will obtain the same conclusion if we choose another two quotient spaces, namely $\mathbb{R}^4_{1,2}$ and $\mathbb{R}^4_{3,4}$, with each inner product to investigate the same sequence.

On the other hand, consider a sequence $x_t = \{(t, 0, 0, 0) : t \in \mathbb{N}\}$. Obviously, this sequence is not a (weakly or strongly) convergent sequence. Recall that the standard n-norm in $\mathbb{R}^4$ is defined by

$$\|x_1, x_2, x_3, x_4\|_S := \begin{vmatrix} \langle x_1, x_1\rangle & \langle x_1, x_2\rangle & \langle x_1, x_3\rangle & \langle x_1, x_4\rangle \\ \langle x_2, x_1\rangle & \langle x_2, x_2\rangle & \langle x_2, x_3\rangle & \langle x_2, x_4\rangle \\ \langle x_3, x_1\rangle & \langle x_3, x_2\rangle & \langle x_3, x_3\rangle & \langle x_3, x_4\rangle \\ \langle x_4, x_1\rangle & \langle x_4, x_2\rangle & \langle x_4, x_3\rangle & \langle x_4, x_4\rangle \end{vmatrix}.$$

Moreover, choose two quotient spaces $\mathbb{R}^4_{2,3}$ and $\mathbb{R}^4_{3,4}$ with their inner product. This time, we investigate the sequence x_t using its quotient spaces of class-2. We have

$$\begin{aligned} \lim_{t\to\infty} \|\overline{x_t} - \overline{x}\|_{2,3} &= \lim_{t\to\infty} \|x_t - x, a_1, a_3, x_4\|_S + \lim_{t\to\infty} \|x_t - x, a_1, a_2, x_4\|_S \\ &= \lim_{t\to\infty} \|x_t, a_1, a_3, x_4\|_S + \lim_{t\to\infty} \|x_t, a_1, a_2, x_4\|_S \\ &= 0. \end{aligned}$$

The result is 0 since x_t is linearly dependent to a_1. This implies $\|x_t, a_1, a_3, x_4\|_S = 0 = \|x_t, a_1, a_2, x_4\|_S$. We also have

$$\begin{aligned} \lim_{t\to\infty} \|\overline{x_t} - \overline{x}\|_{3,4} &= \lim_{t\to\infty} \|x_t - x, a_1, a_2, x_4\|_S + \lim_{t\to\infty} \|x_t - x, a_1, a_2, x_3\|_S \\ &= \lim_{t\to\infty} \|x_t, a_1, a_2, x_4\|_S + \lim_{t\to\infty} \|x_t, a_1, a_2, x_3\|_S \\ &= 0. \end{aligned}$$

The result is 0 since x_t is linearly dependent to a_1. If we only use these two quotient spaces of class-2, we will say that the sequence x_k strongly converges to x. This leads to a false conclusion. But if we add another quotient space of class-2, namely $\mathbb{R}^4_{1,2}$, we have

$$\begin{aligned} \lim_{t\to\infty} \|\overline{x_t} - \overline{x}\|_{1,2} &= \lim_{t\to\infty} \|x_t - x, a_2, a_3, x_4\|_S + \lim_{t\to\infty} \|x_t - x, a_1, a_3, x_4\|_S \\ &= \lim_{t\to\infty} \|x_t, a_2, a_3, x_4\|_S + \lim_{t\to\infty} \|x_t, a_1, a_3, x_4\|_S \\ &= \lim_{t\to\infty} \|x_t, a_2, a_3, x_4\|_S + 0 \\ &= \lim_{t\to\infty} t^2, \end{aligned}$$

which implies the sequence x_t is not a strongly convergent sequence.

From the above example, we can see that there is a condition on how we can choose quotient spaces such that we can investigate the topological properties. We write it in the following remark.

Remark 3. The above topological properties can be investigated by quotient spaces of the class-m ($m \in \mathbb{N}, m \in \{1, \ldots, n\}$) by choosing quotient spaces $X_{j_1,\ldots,j_m}$ such that

$$\bigcup \{j_1, \ldots, j_m\} \supseteq \{1, \ldots, n\}.$$

Moreover, the quotients spaces that we choose on a class-m is at least $\lceil \frac{n}{m} \rceil$. As a consequence, if we use class-1 or class-n of quotient spaces we have to choose all the quotient spaces in it to satisfy the above condition. For other classes, we do not have to choose all quotient spaces.

Conclusion

Aspects of an n-inner product space can be investigated with respect to inner products on norms of its quotient spaces. These inner products and norms are derived from its n-inner product. This approach provides an alternative framework for observing n-inner product spaces. Most researchers viewed some aspects of n-inner product using n-inner product defined on the space (see, for instance [13, 19, 20]). Here we provide a new view point to observe some aspects namely weak and strong convergence, Cauchy sequences, and completeness. We show that these aspects can be defined in several ways, providing additional approaches to study n-inner product spaces. We also show that these definitions are equivalent, so any of them can be used to study the corresponding aspect. We find that we do not need to use all norms or inner products of a class. We give a condition to select minimal norms or inner products to observe an aspect of an n-inner product space. This perspective offers a more flexible and efficient way to study the structure of n-inner product spaces.

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Author Contributions

H. Batkunde constructed the quotient spaces, formulated the concepts and definitions of the topological properties, and proved the main theorems. M. Nur was responsible for verifying the logical flow of the manuscript, checking the proofs, and assisting the first author in preparing the article. M.I. Tilukay contributed to the refinement of the theoretical framework, validation of the results, and improvement of the overall presentation of the manuscript. All authors participated in revising the manuscript and approved the final version for submission. All authors contributed equally to this work.

Conflict of Interest

The authors declare no conflict of interest.

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Marcinkiewicz-type interpolation theorem for discrete net spaces

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In this paper, we investigate the interpolation properties of discrete net spaces $n_{p,q}(M)$ and examine their applications to the analysis of linear operators acting on these spaces. These spaces are characterized by the property that, for monotonically non-increasing sequences, the norm in $n_{p,q}(M)$ coincides with the norm of the discrete Lorentz space $l_{p,q}(M)$. At the same time, unlike Lorentz spaces, these spaces $n_{p,q}(M)$ may contain sequences that do not tend to zero, making them suitable for the study of more general function spaces and operator classes. The main result of this paper is an analogue of Marcinkiewicz-type interpolation theorem for discrete net spaces $n_{p,q}(M)$, which offers a powerful tool to study the boundedness of linear operators within this framework. By extending classical interpolation techniques to discrete nets, the theorem enables researchers to derive strong-type estimates for operators based on weak-type estimates on local nets. Consequently, this approach provides a unified framework for obtaining boundedness results, demonstrating the utility of discrete net spaces in analyzing operators within harmonic analysis. These findings contribute significantly to understanding the structural properties of discrete net spaces. Furthermore, they introduce innovative tools for applications in harmonic analysis, operator theory, and related mathematical fields where such spaces naturally arise, ultimately paving the way for advanced theoretical developments and broader analytical applications.

Keywords: net spaces, discrete net spaces, Lorentz space, Marcinkiewicz-type interpolation theorem, real interpolation method, linear operators, local nets, global nets.

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Introduction

Let S be the set of all finite sets of indices from $\mathbb{Z}^n$. For a fixed set $M \subset S$ we define the space $n_{p,q}(M)$ $(0 < p, q \leq \infty)$ as the set of sequences $a = \{a_m\}_{m\in\mathbb{Z}^n}$ with quasinorm for $0 < p < \infty$, $0 < q < \infty$

$$\|a\|_{n_{p,q}(M)} = \left(\sum_{k=1}^{\infty} k^{\frac{q}{p}-1} (\bar{a}_k(M))^q\right)^{\frac{1}{q}},$$

and for $q = \infty$, $0 < p \leq \infty$

$$\|a\|_{n_{p,\infty}(M)} = \sup_{1\leq k<\infty} k^{\frac{1}{p}} \bar{a}_k(M),$$

where

$$\bar{a}_k(M) = \sup_{\substack{e\in M\\ |e|\geq k}} \frac{1}{|e|} \left|\sum_{m\in e} a_m\right|,$$

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where $|e|$ is the number of indices in e. Here, $\bar{a}_k(M)$ denotes a discrete analogue of the averaging of a function and is defined as the supremum of the average sums of the elements a_m over all $e \in M$.

These spaces, introduced in [1], are called net spaces. Interpolation theory plays a fundamental role in the analysis of operators in functional spaces [2, 3].

The theory of net spaces has been extensively developed in recent decades, finding applications in harmonic analysis, Fourier multipliers, and operator theory. Various embedding and interpolation properties of net spaces on lattices and compact homogeneous manifolds were investigated in [4, 5]. In particular, [4] analyzes $L_p - L_q$ Fourier multipliers on locally compact groups, providing precise boundedness results, while [5] focuses on net spaces on lattices and establishes Hardy–Littlewood type inequalities along with their converses. The general structure and foundational properties of net spaces were systematically developed in [6], providing a theoretical basis for subsequent applications in analysis and operator theory. Net spaces also play a significant role in the study of stochastic processes and interpolation methods. Collectively, these works demonstrate that net spaces offer a unifying and flexible framework for studying inequalities, operator boundedness, interpolation, and stochastic processes. Modern developments in interpolation theory further explore multi-space frameworks with functional parameters, providing tools that can be adapted to generalized sequences, grand net spaces, and other non-standard function spaces, thereby bridging classical analysis with contemporary operator theory.

Recently, generalizations of net spaces, including grand net spaces, have been introduced [7]. These spaces provide a comprehensive framework for analyzing boundedness properties of integral operators and for studying interpolation results in non-standard function spaces [8–10].

In addition, results on trigonometric Fourier series and convolution inequalities in $\lambda_{p,q}$ and anisotropic Lorentz spaces were obtained in [11, 12], which are closely related to interpolation properties of net spaces.

Related problems for Morrey-type spaces and metric interpolation frameworks were studied in [13–15]. These developments emphasize the importance of interpolation methods for operators in generalized function spaces.

Discrete net spaces are closely related to discrete Morrey spaces:

$$m_p^\lambda = \left\{ a = \{a_k\}_{k\in\mathbb{Z}} : \quad \sup_{m\in\mathbb{N}} \sup_{k\in\mathbb{Z}} \frac{1}{m^\lambda} \left(\sum_{r=k}^{k+m} |a_r|^p \right)^{\frac{1}{p}} < \infty \right\}.$$

In the case when $a = \{a_k\}_{k\in\mathbb{Z}},\ a_k \geq 0$, for $\lambda = n\left(1 - \frac{1}{p}\right)$

$$\|a\|_{n_{p,\infty}(M)} \asymp \|a\|_{m_1^\lambda}.$$

Recent investigations consider intermediate and weak discrete Morrey spaces, highlighting new inclusion and interpolation behaviors [16], and generalized local Morrey spaces with applications to Calderon-Zygmund operators [17].

Interpolation properties of Morrey spaces were studied in several works, showing that this scale is not closed under the real interpolation method [18, 19]. Further developments include complex interpolation methods and their applications to Morrey-type and related function spaces [20, 21]. At the same time, Marcinkiewicz-type interpolation theorems were established for Morrey-type spaces [22, 23] and similar ideas were applied to net spaces and their discrete analogues [24, 25].

Given functions F and G, in this paper $F \lesssim G$ means that $F \leq c\, G$ (or $c\, F \geq G$), where c is a positive number, depending only on numerical parameters, that may be different on different occasions. Moreover, $F \asymp G$ means that $F \lesssim G$ and $G \lesssim F$.

1 Main result

Let $b>1$. The parametric family $G_b=\{G_k\}_{k\in\mathbb{N}}$ will be called a local net in $\mathbb{Z}^n$, if

$$G_k \hookrightarrow G_{k+1} \quad \text{and} \quad |G_k| = b^k.$$

Here $|G_k|$ is the number of elements in the set G_k.

The set $F_{G_b}=\{G_k+x\}_{k\in\mathbb{N},x\in\mathbb{Z}^n}$ will be called the global net generated by the net G.

Example. The set of cubes with edge length 2^k, $k\in\mathbb{N}$ in $\mathbb{Z}^n$ is a global net generated by the net of concentric cubes in $\mathbb{Z}^n$ with edge lengths 2^k, $k\in\mathbb{N}$.

We will use the classical Hardy inequalities for discrete sequences, which we formulate as the following lemma.

Lemma 1 (Hardy's inequality). Let $\alpha>0$, $0<q,h\le\infty$ and let the sequence $\{d_k\}_{k\in\mathbb{N}}$ satisfy the following condition for some $\delta>1$

$$\frac{d_{k+1}}{d_k}\ge\delta, \qquad k=2,3,\dots \tag{1}$$

Then the following inequalities hold:

$$\left(\sum_{k=0}^{\infty}\left(d_k^{-\alpha}\left(\sum_{r=0}^{k}|b_r|^h\right)^{\frac{1}{h}}\right)^q\right)^{\frac{1}{q}} \le c_{\alpha,q}\left(\sum_{k=0}^{\infty}\left(d_k^{-\alpha}|b_k|\right)^q\right)^{\frac{1}{q}},$$

$$\left(\sum_{k=0}^{\infty}\left(d_k^{\alpha}\left(\sum_{r=k}^{\infty}|b_r|^h\right)^{\frac{1}{h}}\right)^q\right)^{\frac{1}{q}} \le c_{\alpha,q}\left(\sum_{k=0}^{\infty}\left(d_k^{\alpha}|b_k|\right)^q\right)^{\frac{1}{q}}.$$

Proof. Let $0<h\le q\le\infty$, $0<\varepsilon<\alpha$. We will use Holder's inequality.

$$\left(\sum_{k=0}^{\infty}\left(d_k^{-\alpha}\left(\sum_{r=0}^{k}|b_r|^h\right)^{\frac{1}{h}}\right)^q\right)^{\frac{1}{q}} \le \left(\sum_{k=0}^{\infty}\left(d_k^{-\alpha}\left(\sum_{r=0}^{k}\left(d_r^{-\varepsilon}|b_r|\right)^q\right)^{\frac{1}{q}}\left(\sum_{r=0}^{k}d_r^{\varepsilon\tau}\right)^{\frac{1}{\tau}}\right)^q\right)^{\frac{1}{q}},$$

where $\frac{1}{\tau}=\frac{1}{h}-\frac{1}{q}$. From the condition (1) we have $\sum_{r=0}^{k}d_r^{\varepsilon\tau}\asymp d_k^{\varepsilon\tau}$. Therefore

$$\left(\sum_{k=0}^{\infty}\left(d_k^{-\alpha}\left(\sum_{r=0}^{k}|b_r|^h\right)^{\frac{1}{h}}\right)^q\right)^{\frac{1}{q}} \lesssim \left(\sum_{k=0}^{\infty}d_k^{(\varepsilon-\alpha)q}\sum_{r=0}^{k}\left(d_r^{-\varepsilon}|b_r|\right)^q\right)^{\frac{1}{q}}$$

$$=\left(\sum_{r=0}^{\infty}\left(d_k^{-\varepsilon}|b_r|\right)^q\sum_{k=r}^{\infty}d_r^{(\varepsilon-\alpha)q}\right)^{\frac{1}{q}} \lesssim \left(\sum_{r=0}^{\infty}\left(d_r^{-\alpha}|b_r|\right)^q\right)^{\frac{1}{q}}.$$

Likewise,

$$\left(\sum_{k=0}^{\infty}\left(d_k^{\alpha}\left(\sum_{r=k}^{\infty}|b_r|^h\right)^{\frac{1}{h}}\right)^q\right)^{\frac{1}{q}} \le \left(\sum_{k=0}^{\infty}\left(d_k^{\alpha}\left(\sum_{r=k}^{\infty}(d_r^{\varepsilon}|b_r|)^q\right)^{\frac{1}{q}}\left(\sum_{r=k}^{\infty}d_r^{-\varepsilon\tau}\right)^{\frac{1}{\tau}}\right)^q\right)^{\frac{1}{q}}$$

$$\lesssim \left(\sum_{k=0}^{\infty}d_k^{(-\varepsilon+\alpha)q}\sum_{r=k}^{\infty}(d_r^{\varepsilon}|b_r|)^q\right)^{\frac{1}{q}}$$

$$= \left(\sum_{r=0}^{\infty} (d_k^{\varepsilon}|b_r|)^q \sum_{k=0}^{r} d_r^{(-\varepsilon+\alpha)q} \right)^{\frac{1}{q}} \lesssim \left(\sum_{r=0}^{\infty} (d_r^{\alpha}|b_r|)^q \right)^{\frac{1}{q}}.$$

Now let $0 < q < h \le \infty$. Using Jensen's inequality, we obtain

$$\left(\sum_{k=0}^{\infty} \left(d_k^{-\alpha} \left(\sum_{r=0}^{k} |b_r|^h \right)^{\frac{1}{h}} \right)^q \right)^{\frac{1}{q}} \le \left(\sum_{k=0}^{\infty} d_k^{-\alpha q} \sum_{r=0}^{k} |b_r|^q \right)^{\frac{1}{q}}$$

$$= \left(\sum_{r=0}^{\infty} |b_r|^q \sum_{k=r}^{\infty} d_k^{-\alpha q} \right)^{\frac{1}{q}} \asymp \left(\sum_{r=0}^{\infty} \left(d_r^{-\alpha}|b_r| \right)^q \right)^{\frac{1}{q}}.$$

The second inequality also follows from Jensen's inequality.

Lemma 1 is proved. □

Theorem 1. Let $G = \{G_t\}_{t>0}$ be a local net, and let $F = \bigcup_{x\in\mathbb{Z}^n}(G+x)$ be the global net generated by the net G. Assume that $0 < p_0 < p_1 < \infty$ and $0 < q_0 \le q_1 \le \infty$, $0 < \theta < 1$, $1 \le \tau \le \infty$,

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \qquad \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}.$$

If a linear operator T satisfies the following inequalities for some constants $M_0, M_1 > 0$

$$\|Ta\|_{n_{q_i,\infty}(G+x)} \le M_i \|a\|_{n_{p_i,1}(G+x)}, \quad x \in \mathbb{Z}^n,\ a \in n_{p_i,1}(G+x),\ i = 0,1, \tag{2}$$

then for any $a \in n_{p,\tau}(F)$, the following inequality holds:

$$\|Ta\|nq,\tau(F) \le c \max\{M_0, M_1\} \|a\|np,\tau(F),$$

where the constant $c > 0$ depends only on the parameters $p_0,\ p_1,\ q_0,\ q_1,\ p,\ q,\ \tau,\ \theta$.

Proof. Let $a = \{a_m\}_{m\in\mathbb{Z}} \in n_{p,\tau}(F)$, $\gamma > 0$. For any $x \in \mathbb{Z}^n$, $s \in \mathbb{N}$ we define the sequences

$$a_{0,s} = a\chi_{(G_s+x)}, \quad a_{1,s} = a(1 - \chi_{(G_s+x)}),$$

where χ_{G_s+x} denotes the characteristic function of the set G_s+x. It is easy to see that $a_{0,s} \in n_{p_0,1}(G+x)$ and $a_{1,s} \in n_{p_1,1}(G+x)$. Then $a = a_{0,s} + a_{1,s}$ and

$$\sup_{\xi\ge t\gamma} \frac{1}{|G_\xi|} \left| \sum_{m\in G_\xi+x} (Ta)_m \right| \le \sup_{\xi\ge t\gamma} \frac{1}{|G_\xi|} \left| \sum_{m\in G_\xi+x} (Ta_{0,s})_m \right|$$

$$+ \sup_{\xi\ge t\gamma} \frac{1}{|G_s|} \left| \sum_{m\in G_\xi+x} (Ta_{1,s})_m \right| = I_1 + I_2.$$

First, let us estimate I_1. According to inequality (2), we have

$$I_1 = \sup_{\xi\ge t\gamma} \frac{1}{|G_\xi|} \left| \sum_{m\in G_\xi+x} (Ta_{0,s})_m \right|$$

$$\le b^{-\frac{t\gamma}{q_0}} \sup_{r\in\mathbb{N}} b^{\frac{r}{q_0}} \sup_{\xi\ge r} \frac{1}{|G_\xi|} \left| \sum_{m\in G_\xi+x} (Ta_{0,s})_m \right| = b^{-\frac{t\gamma}{q_0}} \|Ta_{0,s}\|_{n_{q_0,\infty}(G+x)}$$

$$\leq M_0 b^{-\frac{t\gamma}{q_0}} \|a_{0,s}\|_{n_{p_0,1}(G+x)}$$

$$= M_0 b^{-\frac{t\gamma}{q_0}} \left(\sum_{r=0}^{s} b^{\frac{r}{p_0}} \sup_{\xi \geq r} \frac{1}{|G_\xi|} \left| \sum_{m \in G_\xi + x} (a_{0,s})_m \right| + \sum_{r=s}^{\infty} b^{\frac{r}{p_0}} \sup_{\xi \geq r} \frac{1}{|G_\xi|} \left| \sum_{m \in G_\xi + x} (a_{0,s})_m \right| \right).$$

Let $0 < r \leq s$, if $\xi \leq s$, $m \in G_\xi + x$, we have $a_{0,s}(y) = a_m \chi_{G_s + x} = a_m$, if $\xi > s$, then

$$\left| \sum_{m \in G_\xi + x} (a_{0,s})_m \right| = \left| \sum_{m \in G_\xi + x} a_m \right|.$$

For the first sum, we have the following:

$$\sum_{r=0}^{s} b^{\frac{r}{p_0}} \sup_{\xi \geq r} \frac{1}{|G_\xi|} \left| \sum_{m \in G_\xi + x} (a_{0,s})_m \right|$$

$$\leq \sum_{r=0}^{s} b^{\frac{r}{p_0}} \sup_{\xi \geq r} \frac{1}{|G_\xi} \left| \sum_{m \in G_\xi + x} a_m \right| \leq \sum_{r=0}^{s} b^{\frac{r}{p_0}} \bar{a}(r, F).$$

For the second sum, we have

$$\sum_{r=s}^{\infty} b^{\frac{r}{p_0}} \sup_{\xi \geq r} \frac{1}{|G_\xi|} \left| \sum_{m \in G_\xi + x} (a_{0,s})_m \right| = \sum_{r=s}^{\infty} b^{\frac{r}{p_0}} \sup_{\xi \geq r} \frac{1}{|G_\xi|} \left| \sum_{m \in G_s + x} a_m \right|$$

$$= \left| \sum_{m \in G_s + x} a_m \right| \sum_{r=s}^{\infty} b^{\frac{r}{p_0}} \sup_{\xi \geq r} \frac{1}{|G_\xi|} = \left| \sum_{m \in G_s + x} a_m \right| \sum_{r=s}^{\infty} b^{\frac{r}{p_0} - 1}$$

$$\leq b^{\frac{s}{p_0}} \frac{1}{|G_s|} \left| \sum_{m \in G_s + x} a_m \right| \leq b^{\frac{s}{p_0}} \bar{a}(b^s, F).$$

Thus, we obtain

$$I_1 \lesssim M_0 b^{-\frac{t\gamma}{q_0}} \left(\sum_{r=0}^{s} b^{\frac{r}{p_0}} \bar{a}(r, F) + b^{\frac{s}{p_0}} \bar{a}(s, F) \right).$$

Similarly, we estimate I_2. Applying inequality (2), we obtain

$$I_2 = \sup_{s \geq t\gamma} \frac{1}{|G_s|} \left| \sum_{m \in G_\xi + x} (Ta_{1,s})_m \right|$$

$$\leq b^{-\frac{t\gamma}{q_1}} \sup_{r \in \mathbb{N}} b^{\frac{r}{q_1}} \sup_{s \geq r} \frac{1}{|G_s|} \left| \sum_{m \in G_s + x} (Ta_{1,s})_m \right| = b^{-\frac{t\gamma}{q_1}} \|(Ta_{1,s})_m\|_{n_{q_1,\infty}(G+x)}$$

$$\leq M_1 b^{-\frac{t\gamma}{q_1}} \|a_{1,s}\|_{n_{p_1,1}(G+x)} = M_1 b^{-\frac{t\gamma}{q_1}} \left(\sum_{r=0}^{\infty} b^{\frac{r}{p_1}} \sup_{s \geq r} \frac{1}{|G_s|} \left| \sum_{m \in G_s + x} (a_{1,s})_m \right| \right)$$

$$= M_1 b^{-\frac{t\gamma}{q_1}} \left(\sum_{r=0}^{s} b^{\frac{r}{p_1}} \sup_{\xi \geq r} \frac{1}{|G_\xi|} \left| \sum_{m \in G_\xi + x} (a_{1,s})_m \right| + \sum_{r=s}^{\infty} b^{\frac{r}{p_1}} \sup_{\xi \geq r} \frac{1}{|G_\xi|} \left| \sum_{m \in G_\xi + x} (a_{1,s})_m \right| \right)$$

$$= M_1 b^{-\frac{t\gamma}{q_1}} \left(J_1 + J_2 \right).$$

To estimate J_1 and J_2, we note that

$$\sum_{m\in G_\xi+x}(a_{1,s})_m=\begin{cases}0,\ \xi\le s,\\ \sum_{m\in(G_\xi+x)\setminus(G_s+x)}a_m,\ \xi>s\end{cases}$$

$$=\begin{cases}0\quad\text{for}\quad \xi\le s,\\ \left|\sum_{m\in G_\xi+x}a_m-\sum_{m\in G_s+x}a_m\right|\le\left|\sum_{m\in G_\xi+x}a_m\right|+\left|\sum_{m\in G_s+x}a_m\ \right|\ \text{ for }\ \xi>s.\end{cases}$$

Next, using this estimate, we have

$$J_1\le\sum_{r=0}^{s}b^{\frac{r}{p_1}}\sup_{\xi\ge r}\frac{1}{|G_\xi|}\left(\left|\sum_{m\in}a_m\right|+\left|\sum_{m\in G_s+x}a_m\right|\right)$$

$$\le\sum_{r=0}^{s}b^{\frac{r}{p_1}}\left(\bar a(b^s,F)+\left|\sum_{m\in G_s+x}a_m\right|\sup_{\xi\ge r}\frac{1}{|G_\xi|}\right)\le 2\bar a(b^s,F)\sum_{r=0}^{s}b^{\frac{r}{p_1}}=2p_1b^{\frac{s}{p_1}}\bar a(b^s,F)$$

and

$$J_2\le\sum_{r=s}^{\infty}b^{\frac{r}{p_1}}\sup_{\xi\ge r}\frac{1}{|G_\xi|}\left(\left|\sum_{m\in G_\xi+x}a_m\right|+\left|\sum_{m\in G_s+x}a_m\right|\right)$$

$$\le\sum_{r=s}^{\infty}b^{\frac{r}{p_1}}\left(\bar a(b^s,F)+\left|\sum_{m\in G_s+x}a_m\right|\sup_{\xi\ge r}\frac{1}{|G_\xi|}\right)\le\sum_{r=s}^{\infty}b^{\frac{r}{p_1}}\bar a(b^r,F)$$

$$+\left|\sum_{m\in G_s+x}a_m\right|\sum_{r=s}^{\infty}b^{\frac{r}{p_1}}\sup_{\xi\ge r}\frac{1}{|G_\xi|}\asymp\sum_{r=s}^{\infty}b^{\frac{r}{p_1}}\bar a(b^r,F)+\left|\sum_{m\in G_s+x}a_m\right|b^{\frac{s}{p_1}}$$

$$\lesssim\sum_{r=s}^{\infty}b^{\frac{r}{p_1}}\bar a(b^r,F)+b^{\frac{s}{p_1}}\bar a(b^s,F).$$

Combining the obtained estimates, we obtain the following estimate

$$I_2\lesssim M_1b^{\frac{t\gamma}{q_1}}\left(\sum_{r=s}^{\infty}b^{\frac{r}{p_1}}\bar a(b^r,F)+b^{\frac{s}{p_1}}\bar a(b^s,F)\right).$$

Thus, we got

$$\sup_{\xi\ge t\gamma}\frac{1}{|G_s|}\left|\sum_{m\in G_\xi+x}(Ta)_m\right|\lesssim M_0b^{-\frac{t\gamma}{q_0}}\left(\sum_{r=0}^{s}b^{\frac{r}{p_0}}\bar a(b^r,F)+b^{\frac{s}{p_0}}\bar a(a^s,F)\right)$$

$$+M_1b^{-\frac{t\gamma}{q_1}}\left(\sum_{r=s}^{\infty}b^{\frac{r}{p_1}}\bar a(b^r,F)+b^{\frac{s}{p_1}}\bar a(b^s,F)\right).$$

Let $\gamma=\left(\frac{1}{p_0}-\frac{1}{p_1}\right)\Big/\left(\frac{1}{q_0}-\frac{1}{q_1}\right)$. Using the equivalent normalization for $d=b^\gamma$

$$\|Ta\|_{n_{q,\tau}(F)}\asymp\left(\sum_{t=0}^{\infty}\left(d^{\frac{t}{q}}\sup_{\substack{|G_\xi|\ge d^t\\ x\in\mathbb{Z}^n}}\frac{1}{|G_\xi|}\left|\sum_{m\in G_\xi+x}a_m\right|\right)^{\tau}\right)^{\frac{1}{\tau}}.$$

Since $|G_\xi| = b^\xi$, from the inequality $|G_\xi| \geq d^t$ it follows that $b^\xi \geq d^t = b^{t\gamma}$ which implies $\xi \geq t\gamma$. Let $s = t$, Then, taking into account the obtained inequalities, we have

$$\|Ta\|_{n_{q,\tau}(F)} \asymp \left(\sum_{t=0}^{\infty} \left(d^{\frac{t}{q}} \sup_{\substack{\xi \geq t\gamma \\ x \in \mathbb{Z}^n}} \frac{1}{|G_\xi|} \Big| \sum_{m \in G_\xi + x} a_m \Big| \right)^\tau \right)^{\frac{1}{\tau}}$$

$$\lesssim M_0 A_1 + M_0 A_2 + M_1 A_3 + M_1 A_4,$$

where, taking into account that

$$\gamma \left(\frac{1}{q} - \frac{1}{q_0} \right) = -\theta \left(\frac{1}{p_0} - \frac{1}{p_1} \right)$$

and

$$\gamma \left(\frac{1}{q} - \frac{1}{q_1} \right) = (1-\theta) \left(\frac{1}{p_0} - \frac{1}{p_1} \right).$$

$$A_1 = \left(\sum_{t=0}^{\infty} \left(d^{t(\frac{1}{q} - \frac{1}{q_0})} \sum_{r=0}^{t} b^{\frac{r}{p_0}} \bar{a}(b^r, F) \right)^\tau \right)^{\frac{1}{\tau}} = \left(\sum_{t=0}^{\infty} \left(b^{-\theta t \left(\frac{1}{p_0} - \frac{1}{p_1} \right)} \sum_{r=0}^{t} b^{\frac{r}{p_0}} \bar{a}(b^r, F) \right)^\tau \right)^{\frac{1}{\tau}},$$

$$A_2 = \left(\sum_{t=0}^{\infty} \left(d^{t(\frac{1}{q} - \frac{1}{q_0})} b^{\frac{t}{p_0}} \bar{a}(b^t, F) \right)^\tau \right)^{\frac{1}{\tau}} = \left(\sum_{t=0}^{\infty} \left(b^{-\theta t \left(\frac{1}{p_0} - \frac{1}{p_1} \right)} b^{\frac{t}{p_0}} \bar{a}(b^t, F) \right)^\tau \right)^{\frac{1}{\tau}}$$

$$= \left(\sum_{t=0}^{\infty} \left(b^{\frac{t}{p}} \bar{a}(b^t, F) \right)^\tau \right)^{\frac{1}{\tau}} \asymp \|a\|_{n_{p,\tau}(F)}$$

and

$$A_3 = \left(\sum_{t=0}^{\infty} \left(d^{t(\frac{1}{q} - \frac{1}{q_1})} \sum_{r=t}^{\infty} b^{\frac{r}{p_1}} \bar{a}(b^r, F) \right)^\tau \right)^{\frac{1}{\tau}} = \left(\sum_{t=0}^{\infty} \left(b^{(1-\theta) t \left(\frac{1}{p_0} - \frac{1}{p_1} \right)} \sum_{r=t}^{\infty} b^{\frac{r}{p_1}} \bar{a}(b^r, F) \right)^\tau \right)^{\frac{1}{\tau}},$$

$$A_4 = \left(\sum_{t=0}^{\infty} \left(d^{t(\frac{1}{q} - \frac{1}{q_1})} b^{\frac{t}{p_1}} \bar{a}(b^t, F) \right)^\tau \right)^{\frac{1}{\tau}} = \left(\sum_{t=0}^{\infty} \left(b^{(1-\theta) t \left(\frac{1}{p_0} - \frac{1}{p_1} \right)} b^{\frac{t}{p_1}} \bar{a}(b^t, F) \right)^\tau \right)^{\frac{1}{\tau}}$$

$$= \left(\sum_{t=0}^{\infty} \left(b^{\frac{t}{p}} \bar{a}(b^t, F) \right)^\tau \right)^{\frac{1}{\tau}} \asymp \|a\|_{n_{p,\tau}(F)}.$$

To estimate A_1 and A_3 we use the Hardy inequalities from Lemma 1. Thus, we obtain

$$\|Ta\|_{n_{q,\tau}(F)} \leq c \max\{M_0, M_1\} \|a\|_{n_{p,\tau}(F)},$$

where the constant $c > 0$ depends only on the parameters $p_0,\ p_1,\ q_0,\ q_1,\ p,\ q,\ \tau,\ \theta$.

Consequently, we have obtained the desired estimate. The theorem is proved. □

Conclusion

In this paper, we studied the interpolation properties of discrete net spaces for a broad class of nets. We established an analogue of the Marcinkiewicz-type interpolation theorem for linear operators, extending existing results in the theory of interpolation. Our approach builds upon the ideas developed in [22,23], where alternative analogues of Marcinkiewicz-type interpolation theorems for net spaces were obtained [24, 25].

A comparison of our results with previous works shows that our findings provide a new perspective on interpolation in net spaces, refining and generalizing existing methods. The scientific novelty of this work lies in the further development of interpolation techniques adapted to discrete net spaces, expanding their theoretical framework and potential applications.

The practical significance of our results is reflected in their possible applications to harmonic analysis, operator theory, and stochastic processes, where net spaces serve as a fundamental tool. Future research directions include the extension of interpolation results to nonlinear operators, the exploration of stability properties, and the application of net space interpolation to more complex functional spaces and applied problems.

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Author Contributions

All authors contributed equally to this work. E.D. Nursultanov collected and analyzed data, and led manuscript preparation. A.K. Kalidolday served as the principal investigator of the research grant and supervised the research process. A.N. Sharipova contributed to the analysis and interpretation of results and assisted in manuscript preparation. All authors participated in the revision of the manuscript and approved the final submission.

Conflict of Interest

The authors declare no conflict of interest.

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Three-weight Hardy inequalities with iterated operators and generalized kernels

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The well-known Hardy inequalities, formulated in both continuous and discrete cases, play an important role in mathematical analysis, differential equations and many other branches of mathematics. The original forms of these inequalities were subsequently extended and generalized in various directions, leading to the development of Hardy inequalities as an independent and significant area of research. A central problem in the theory of weighted inequalities is the characterization of conditions under which inequalities involving Hardy-type operators hold. Many cases of weighted estimates for linear integral Hardy-type operators have been considered, and there is a large number of books and scientific articles on this topic. More recently, considerable attention has been given to iterated Hardy-type operators due to their application in Morrey-type spaces. This paper analyzes a class of operators formed by iterating two operators, one of which involves a kernel satisfying conditions that generalize those considered previously. The study examines Hardy-type inequalities associated with these iterated operators and establishes necessary and sufficient conditions for their validity. The characterization of weighted Hardy inequalities involving iterated operators can now be applied to study of bilinear weighted Hardy-type inequalities.

Keywords: integral operator, iterated operator, Hardy-type inequality, weight function, kernel, Lebesgue space, Oinarov condition, Oinarov classes.

2020 Mathematics Subject Classification: 26D10, 47G10, 47B38.

Introduction

For $f \geq 0$, we consider the inequalities

$$\left(\int_0^\infty u(x)\left(\int_0^x\left(\int_t^\infty K(s,t)f(s)\,ds\right)^r w(t)\,dt\right)^{\frac{q}{r}}dx\right)^{\frac{1}{q}} \leq C\left(\int_0^\infty v(x)f^p(x)\,dx\right)^{\frac{1}{p}} \tag{1}$$

and

$$\left(\int_0^\infty u(x)\left(\int_x^\infty\left(\int_0^t K(t,s)f(s)\,ds\right)^r w(t)\,dt\right)^{\frac{q}{r}}dx\right)^{\frac{1}{q}} \leq C\left(\int_0^\infty v(x)f^p(x)\,dx\right)^{\frac{1}{p}}, \tag{2}$$

where u, v, and w are non-negative, measurable, and locally summable weight functions on $I=(0,\infty)$ and $K(\cdot,\cdot)$ is a measurable function referred to as the kernel.

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Define $L_{p,v} \equiv L_{p,v}(I)$, $1 \le p < \infty$, as the Lebesgue space of measurable functions f on I satisfying

$$\|f\|_{p,v} = \left(\int_0^\infty v(x)|f(x)|^p dx \right)^{\frac{1}{p}} < \infty.$$

Then these Hardy-type inequalities (1) and (2) can be rewritten in the shortened form:

$$\|Tf\|_{q,u} \le C\|f\|_{p,v}, \quad 0 \le f \in L_{p,v}, \tag{3}$$

where T is one of the operators

$$T^+ f(x) = \left(\int_0^x \left(\int_t^\infty K(s,t)f(s)ds \right)^r w(t)\, dt \right)^{\frac{1}{r}},$$

$$T^- f(x) = \left(\int_x^\infty \left(\int_0^t K(t,s)f(s)ds \right)^r w(t)\, dt \right)^{\frac{1}{r}}.$$

When $K(\cdot,\cdot) \equiv 1$, inequalities (1) and (2) have been studied in [1–3] and the references therein. Related results on inequalities (1) and (2) can also be found in [4–6]. In [7], the problem of characterizing inequality (2) for $p = 1$ was established. When $K(\cdot,\cdot)$ satisfies the Oinarov condition $\mathcal{O}$, which states that there exists a constant $h \ge 1$ such that

$$\frac{1}{h}\left(K(x,t) + K(t,s)\right) \le K(x,s) \le h\left(K(x,t) + K(t,s)\right) \tag{4}$$

for $x \ge t \ge s > 0$, inequalities (1) and (2) have been treated in [8–10]. In [8], the simplest case $1 < p \le q < \infty$ and $0 < r < \infty$ is addressed. The papers [9, 10] investigate all possible relations between the summation parameters, though their characterizations depend on the use of an auxiliary function. In [11], both cases $1 < p \le q < \infty$ and $1 < q < p < \infty$ are considered, but with the restriction $r < q$.

In this paper, we study inequalities (1) and (2) for both cases $1 < p \le q < \infty$ and $1 < q < p < \infty$, but now allowing any $0 < r < \infty$. Moreover, we assume that the kernels $K(\cdot,\cdot)$ belong to the classes $\mathcal{O}_n^{\pm}$, $n \ge 0$, referred to as *Oinarov classes*, which generalize the class of kernels satisfying condition (4).

The importance of studying inequalities (1) and (2) is highlighted in recent papers [2] and [12], which emphasize that, due to numerous applications, this topic has become highly fashionable in the theory of Hardy inequalities. Since papers [2] and [12] thoroughly reveal all applications of these inequalities, we omit their listing here.

Assume that $\frac{1}{p} + \frac{1}{p'} = 1$, $\frac{1}{q} + \frac{1}{q'} = 1$, and $\frac{1}{\mu} = \frac{1}{q} - \frac{1}{p}$. Let the symbol $A \ll B$ denote that $A \le CB$ for some constant $C > 0$, and let the symbol $A \approx B$ denote that $A \ll B \ll A$.

First, we formulate the characterization of the inequalities (1) and (2) for a kernel that satisfies Oinarov's condition (4), which demonstrates the approaches and main ideas of the proof. This case is a special case of the main results presented in Theorems 3 and 4. It is worse to mention that, in this case, it is also possible to include the case $0 < q < 1 \le p < \infty$, which is excluded in the main results.

Theorem 1. Let $1 < q < p < \infty$ and $0 < r < \infty$. Let the kernel $K(\cdot,\cdot)$ satisfy the Oinarov condition $\mathcal{O}$. Then (1) holds if and only if $\mathcal{A} = \max\{A_{q<p}, A_{00}, A_{01}, A_{11}\} < \infty$, where

$$A_{q<p} = \left(\int_0^\infty u(x) \left(\int_x^\infty u(s)ds \right)^{\frac{\mu}{p}} \left(J_{p,r}^-(0,x)\right)^\mu dx \right)^{\frac{1}{\mu}},$$

$$A_{00} = \left(\int_0^\infty \left(\int_0^t u(s) \left(\int_0^s K^r(s,z)w(z)dz \right)^{\frac{q}{r}} ds \right)^{\frac{\mu}{q}} \left(\int_t^\infty v^{1-p'}(s)\, ds \right)^{\frac{\mu}{q'}} v^{1-p'}(t)dt \right)^{\frac{1}{\mu}},$$

$$A_{01} = \left(\int_0^\infty \left(\int_0^t u(s) \left(\int_0^s w(z)dz \right)^{\frac{q}{r}} K^q(t,s)ds \right)^{\frac{\mu}{q}} \left(\int_t^\infty v^{1-p'}(s)\, ds \right)^{\frac{\mu}{q'}} v^{1-p'}(t)dt \right)^{\frac{1}{\mu}},$$

$$A_{11} = \left(\int_0^\infty \left(\int_0^t u(s) \left(\int_0^s w(z)dz \right)^{\frac{q}{r}} ds \right)^{\frac{\mu}{p}} \left(\int_t^\infty K^{p'}(s,t)v^{1-p'}(s)\, ds \right)^{\frac{\mu}{p'}} u(t) \left(\int_0^t w(s)ds \right)^{\frac{q}{r}} dt \right)^{\frac{1}{\mu}}.$$

Moreover, $\mathcal{A} \approx C$, where C is the best constant in (1).

If we split the interior integral on the left-hand side of inequality (1) into two integrals, we obtain

$$\left(\int_0^\infty u(x) \left(\int_0^x \left(\int_t^\infty K(s,t)f(s)\, ds \right)^r w(t)\, dt \right)^{\frac{q}{r}} dx \right)^{\frac{1}{q}}$$

$$\approx \left(\int_0^\infty u(x) \left(\int_0^x \left(\int_t^x K(s,t)f(s)\, ds \right)^r w(t)\, dt \right)^{\frac{q}{r}} dx \right)^{\frac{1}{q}}$$

$$+ \left(\int_0^\infty u(x) \left(\int_0^x \left(\int_x^\infty K(s,t)f(s)\, ds \right)^r w(t)\, dt \right)^{\frac{q}{r}} dx \right)^{\frac{1}{q}}. \tag{5}$$

Taking into account condition (4), from (5) we derive that the validity of (1) is equivalent to the validity of the following three inequalities:

$$\left(\int_0^\infty u(x) \left(\int_0^x \left(\int_t^x K(s,t)f(s)ds \right)^r w(t)\, dt \right)^{\frac{q}{r}} dx \right)^{\frac{1}{q}} \le C_1 \|f\|_{p,v}, \tag{6}$$

$$\left(\int_0^\infty u(x) \left(\int_0^x K^r(x,t)w(t)\, dt \right)^{\frac{q}{r}} \left(\int_x^\infty f(s)\, ds \right)^q dx \right)^{\frac{1}{q}} \le C_2 \|f\|_{p,v}, \tag{7}$$

$$\left(\int_0^\infty u(x) \left(\int_0^x w(t)\, dt \right)^{\frac{q}{r}} \left(\int_x^\infty K(s,x)f(s)\, ds \right)^q dx \right)^{\frac{1}{q}} \le C_3 \|f\|_{p,v}. \tag{8}$$

Inequality (7) is a standard weighted Hardy inequality, which holds if and only if $A_{00} < \infty$ (see, e.g., [13, Theorem 5]). Inequality (8) is a Hardy-type inequality involving a Volterra-type operator

$K^-f(x) = \int\limits_x^\infty K(s,x)f(s)\,ds$, where $K(\cdot,\cdot)$ satisfies the Oinarov condition $\mathcal{O}$. Its validity follows from the conditions $A_{01} < \infty$ and $A_{11} < \infty$, as established in [14].

In [15], it is proved that inequality (6) holds if and only if $A_{q<p} < \infty$, where

$$J^-_{p,r}(\alpha,\beta) = \sup_{f\geq 0} \frac{\left(\int\limits_\alpha^\beta \left(\int\limits_x^\beta K(s,x)f(s)\,ds\right)^r w(x)\,dx\right)^{\frac{1}{r}}}{\|f\|_{p,v,(\alpha,\beta)}}, \quad 0 \leq \alpha < \beta \leq \infty,$$

as mentioned above, can be found in [14] when $K(\cdot,\cdot)$, the kernel of the Volterra-type operator K^-, satisfies the Oinarov condition $\mathcal{O}$.

Unfortunately, inequality (6) has not received as much attention as inequality (1) due to its fewer applications. Consequently, paper [15] has not garnered as many references as those discussing inequality (1), despite the fact that [15] addresses both cases, $1 \leq p \leq q < \infty$ and $0 < q < p < \infty$, $p \geq 1$, for any $0 < r < \infty$, with the results presented in terms of the quantity $J^-_{p,r}$ without imposing any restrictions on the kernel involved. However, (6), being less popular than inequality (1), serves as the basis for characterizing inequality (1), as demonstrated by the splitting in (5).

Let

$$J^+_{p,r}(\alpha,\beta) = \sup_{f\geq 0} \frac{\left(\int\limits_\alpha^\beta \left(\int\limits_\alpha^t K(t,s)f(s)\,ds\right)^r w(t)\,dt\right)^{\frac{1}{r}}}{\|f\|_{p,v,(\alpha,\beta)}}, \quad 0 \leq \alpha < \beta \leq \infty.$$

Similarly, we can derive characterizations for inequality (2) to hold when the kernel of the operator T^- satisfies condition (4).

Theorem 2. Let $1 < q < p < \infty$ and $0 < r < \infty$. Let the kernel $K(\cdot,\cdot)$ satisfy the Oinarov condition $\mathcal{O}$. Then (2) holds if and only if $\mathcal{B} = \max\{B_{q<p}, B_{00}, B_{10}, B_{11}\} < \infty$, where

$$B_{q<p} = \left(\int\limits_0^\infty u(x)\left(\int\limits_0^x u(s)ds\right)^{\frac{\mu}{p}} \left(J^+_{p,r}(x,\infty)\right)^\mu dx\right)^{\frac{1}{\mu}},$$

$$B_{00} = \left(\int\limits_0^\infty \left(\int\limits_t^\infty u(s)\left(\int\limits_s^\infty K^r(z,s)w(z)dz\right)^{\frac{q}{r}} ds\right)^{\frac{\mu}{q}} \left(\int\limits_0^t v^{1-p'}(s)\,ds\right)^{\frac{\mu}{q'}} v^{1-p'}(t)dt\right)^{\frac{1}{\mu}},$$

$$B_{10} = \left(\int\limits_0^\infty \left(\int\limits_t^\infty u(s)\left(\int\limits_s^\infty w(z)dz\right)^{\frac{q}{r}} K^q(s,t)ds\right)^{\frac{\mu}{q}} \left(\int\limits_0^t v^{1-p'}(s)\,ds\right)^{\frac{\mu}{q'}} v^{1-p'}(t)dt\right)^{\frac{1}{\mu}},$$

$$B_{11} = \left(\int\limits_0^\infty \left(\int\limits_t^\infty u(s)\left(\int\limits_s^\infty w(z)dz\right)^{\frac{q}{r}} ds\right)^{\frac{\mu}{p}} \left(\int\limits_0^t K^{p'}(t,s)v^{1-p'}(s)\,ds\right)^{\frac{\mu}{p'}} u(t)\left(\int\limits_t^\infty w(s)ds\right)^{\frac{q}{r}} dt\right)^{\frac{1}{\mu}}.$$

Moreover, $\mathcal{B} \approx C$, where C is the best constant in (2).

Theorems 1 and 2 are corollaries of the main results presented later in the paper, where inequalities (1) and (2) are established for operators with more general kernels. We chose to present Theorems 1 and 2 in the Introduction because, for over 30 years since the Oinarov condition $\mathcal{O}$ was introduced in 1991, Hardy-type inequalities have been primarily studied for these kernels. The Oinarov classes $\mathcal{O}_n^{\pm}$, $n \geq 0$, were introduced in 2007 in [16]. However, despite the fact that the conditions for belonging to these classes are significantly weaker than the condition $\mathcal{O}$, they have not been as widely used as the Oinarov classes yet.

Theorem 1 examines the case $1 < q < p < \infty$ and $0 < r < \infty$. Since the case $1 < p \leq q < \infty$ and $0 < r < \infty$ was established in [8] using a similar splitting method, it has not been included in the Introduction. However, it is worth noting that the conditions for the validity of standard Hardy inequality (7) in the case $0 < q < 1 \leq p < \infty$ are known from [17]. When the kernel satisfies the Oinarov condition, the missing characterizations for the validity of inequality (8) in the case $0 < q < 1 \leq p < \infty$ were recently provided in [18]. Furthermore, the condition $A_{q<p} < \infty$ is necessary and sufficient for the validity of inequality (6) when $0 < q < p < \infty$, $p \geq 1$, and $0 < r < \infty$. Since the simultaneous validity of inequalities (6), (8), and (7) ensures the validity of inequality (1), we can easily derive its characterizations for the case $0 < q < 1 \leq p < \infty$ and $0 < r < \infty$ when the kernel satisfies the condition $\mathcal{O}$. The same arguments can be applied to determine the conditions for the validity of inequality (2).

The structure of the paper is organized as follows. Section 1 is devoted to Oinarov's classes. In Section 2, we present our first main result, namely the characterization of inequality (1), and in the next section, we present our second main result, namely the characterization of inequality (2).

1 The Oinarov classes of kernels

In the work [16], R. Oinarov introduced the classes of functions $\mathcal{O}_n^{\pm}$, $n \geq 0$. Let us give the definitions of these classes.

Let $\Omega = \{(x,s) : x \geq s > 0\}$. We define the classes $\mathcal{O}_n^{\pm}$, $n \geq 0$, in a recurrent form as a set of functions $K(\cdot,\cdot)$ that are non-negative and measurable on the set Ω and satisfy certain conditions.

Definition 1. The class $\mathcal{O}_0^{+}$ $\left(\mathcal{O}_0^{-}\right)$ consists of functions of the form $K_0(x,s) \equiv r(s)$ $(K_0(x,s) \equiv r(x))$.

Let the classes $\mathcal{O}_i^{\pm}$ be defined for $i = 0,1,...,n-1$, $n \geq 1$.

Definition 2. A function $K(\cdot,\cdot) \equiv K_n(\cdot,\cdot) \in \mathcal{O}_n^{+}$, $n \geq 1$, if it is non-decreasing in the first argument and there exist non-negative measurable on Ω functions $K_i(\cdot,\cdot)$, $K_{n,i}(\cdot,\cdot)$, $i = 0,1,...,n-1$, and a number $h_n \geq 1$ such that $K_i(\cdot,\cdot) \in \mathcal{O}_i^{+}$, $i = 0,1,...,n-1$, and

$$\frac{1}{h_n}\sum_{i=0}^{n} K_{n,i}(x,t)K_i(t,s) \leq K_n(x,s) \leq h_n \sum_{i=0}^{n} K_{n,i}(x,t)K_i(t,s) \tag{9}$$

for all $x \geq t \geq s > 0$, where $K_{n,n}(\cdot,\cdot) \equiv 1$.

Definition 3. A function $K(\cdot,\cdot) \equiv K_n(\cdot,\cdot) \in \mathcal{O}_n^{-}$, $n \geq 1$, if it is non-increasing in the second argument and there exist non-negative measurable on Ω functions $K_i(\cdot,\cdot)$, $K_{i,n}(\cdot,\cdot)$, $i = 0,1,...,n-1$, and a number $\bar{h}_n \geq 1$ such that $K_i(\cdot,\cdot) \in \mathcal{O}_i^{-}$, $i = 0,1,...,n-1$, and

$$\frac{1}{\bar{h}_n}\sum_{i=0}^{n} K_i(x,t)K_{i,n}(t,s) \leq K_n(x,s) \leq \bar{h}_n \sum_{i=0}^{n} K_i(x,t)K_{i,n}(t,s) \tag{10}$$

for all $x \geq t \geq s > 0$, where $K_{n,n}(\cdot,\cdot) \equiv 1$.

Let us present some examples. It is easy to see that the kernel $I_\alpha(x,s)=(x-s)^\alpha$, $\alpha>0$, satisfies the Oinarov condition $\mathcal{O}$. However, if we slightly modify it to the form $K_1(x,s)=(f(x)-g(s))^\alpha$, where $f(\cdot)$ is a non-negative function and $g(\cdot)$ is a non-negative increasing function, then it does not satisfy (5). Indeed, since for all $x\ge t\ge s>0$, we have

$$K_1(x,s)\approx(f(x)-g(t))^\alpha+(g(t)-g(s))^\alpha=K_1(x,t)+K_{0,1}(t,s),$$

where $G(t,s)=(g(t)-g(s))^\alpha$ is taken as $K_{0,1}(t,s)$, i.e., condition (10) holds for $n=1$. Thus, $K_1(x,s)=(f(x)-g(s))^\alpha\in\mathcal{O}_1^-$, but it does not satisfy the Oinarov condition $\mathcal{O}$.

One more kernel $W(x,s)=\int\limits_s^x w(t)dt$ satisfies the Oinarov condition $\mathcal{O}$. Let us modify it by multiplying the weight $w(t)$ by the kernel $K_1(t,s)=(f(t)-g(s))^\alpha$ from the previous example, so that the new kernel takes the form $K_2(x,s)=\int\limits_s^x(f(t)-g(s))^\alpha w(t)\,dt$. Then, for all $x\ge z\ge s>0$, we have

$$K_2(x,s)=\int\limits_s^z(f(t)-g(s))^\alpha w(t)\,dt+\int\limits_z^x(f(t)-g(s))^\alpha w(t)\,dt$$

$$\approx K_2(z,s)+\int\limits_z^x(f(t)-g(z))^\alpha w(t)\,dt+(g(z)-g(s))^\alpha\int\limits_z^x w(t)\,dt$$

$$=K_2(x,z)+W(x,z)K_{1,2}(z,s)+K_2(z,s),$$

where now $G(z,s)=(g(z)-g(s))^\alpha$ is taken as $K_{1,2}(z,s)$, i.e., the condition (10) holds for $n=2$. Therefore, $K_2(x,s)\in\mathcal{O}_2^-$.

In general, it was proved in [19] that the kernel

$$K_w(x,s)=\int\limits_s^x K_n(t,s)w(t)\,dt,\quad x\ge t\ge s>0,$$

belongs to the class $\mathcal{O}_{n+1}^-$ if $K_n(\cdot,\cdot)$ belongs to the class $\mathcal{O}_n^-$, $n\ge0$, and to the class $\mathcal{O}_{n+1}^+$ if $K_n(\cdot,\cdot)$ belongs to the class $\mathcal{O}_n^+$, $n\ge0$.

2 *Characterization of inequality* (1)

Theorem 3. Let $0<r<\infty$ and the kernel $K(\cdot,\cdot)$ belong to the Oinarov class $\mathcal{O}_n^-$, $n\ge1$.

(i) If $1<q<p<\infty$, then (1) holds if and only if $\widehat{A}=\max\{A_{q<p},\widehat{A}_{ji}\}<\infty$, where

$$\widehat{A}_{ji}=\max_{0\le i\le n}\max_{0\le j\le i}\left(\int\limits_0^\infty\left(\int\limits_0^t K_{j,i}^q(t,s)\,u(s)\left(\int\limits_0^s K_{i,n}^r(s,z)w(z)\,dz\right)^{\frac{q}{r}}ds\right)^{\frac{\mu}{q}}\right.$$
$$\left.\times\left(\int\limits_t^\infty K_j^{p'}(s,t)v^{1-p'}(s)ds\right)^{\frac{\mu}{q'}}d\left(-\int\limits_t^\infty K_j^{p'}(s,t)v^{1-p'}(s)ds\right)\right)^{\frac{1}{\mu}}$$

$$\approx \max_{0\le i\le n}\max_{0\le j\le i}\left(\int_0^\infty \left(\int_t^\infty K_j^{p'}(s,t)v^{1-p'}(s)ds\right)^{\frac{\mu}{p'}}\right.$$

$$\times\left(\int_0^t K_{j,i}^q(t,s)\,u(s)\left(\int_0^s K_{i,n}^r(s,z)w(z)\,dz\right)^{\frac{q}{r}}ds\right)^{\frac{\mu}{p}}$$

$$\left.\times d\left(\int_0^t K_{j,i}^q(t,s)\,u(s)\left(\int_0^s K_{i,n}^r(s,z)w(z)\,dz\right)^{\frac{q}{r}}ds\right)\right)^{\frac{1}{\mu}}.$$

Moreover, $\widehat{A}\approx C$, where C is the best constant in (1).

(ii) If $1<p\le q<\infty$, then (1) holds if and only if $\widetilde{A}=\max\{A_{p\le q},\widetilde{A}_{in}\}<\infty$, where

$$A_{p\le q}=\sup_{z>0}\left(\int_z^\infty u(x)\,dx\right)^{\frac{1}{q}} J_{p,r}^-(0,z),$$

$$\widetilde{A}_{in}=\sup_{z>0}\max_{0\le i\le n}\left(\int_z^\infty v^{1-p'}(x)\left(\int_0^z K_i^q(x,s)u(s)\left(\int_0^s K_{i,n}^r(s,t)w(t)\,dt\right)^{\frac{q}{r}}ds\right)^{\frac{p'}{q}}dx\right)^{\frac{1}{p'}}$$

$$\approx\sup_{z>0}\max_{0\le i\le n}\left(\int_0^z\left(\int_z^\infty K_i^{p'}(x,s)v^{1-p'}(x)dx\right)^{\frac{q}{p'}}u(s)\left(\int_0^s K_{i,n}^r(s,t)w(t)\,dt\right)^{\frac{q}{r}}ds\right)^{\frac{1}{q}}.$$

Moreover, $\widetilde{A}\approx C$, where C is the best constant in (1).

Proof. Applying (10) to the second term in (5), we get

$$\left(\int_0^\infty u(x)\left(\int_0^x\left(\int_x^\infty K(s,t)f(s)\,ds\right)^r w(t)\,dt\right)^{\frac{q}{r}}dx\right)^{\frac{1}{q}}$$

$$\approx\sum_{i=0}^n\left(\int_0^\infty u(x)\left(\int_0^x K_{i,n}^r(x,t)w(t)\,dt\right)^{\frac{q}{r}}\left(\int_x^\infty K_i(s,x)f(s)\,ds\right)^q dx\right)^{\frac{1}{q}}.$$

Thus, the validity of inequality (1) is equivalent to the simultaneous validity of inequality (6) and the following $n+1$ inequalities:

$$\left(\int_0^\infty u(x)\left(\int_0^x K_{i,n}^r(x,t)w(t)\,dt\right)^{\frac{q}{r}}\left(\int_x^\infty K_i(s,x)f(s)\,ds\right)^q dx\right)^{\frac{1}{q}}\le C_i'\,\|f\|_{p,v}, \tag{11}$$

where $K_i(\cdot,\cdot)$ belongs to the class O_i^-, $i=0,1,...,n$.

In the case $1 < q < p < \infty$, it follows from [15, Theorem 3.1] that inequality (6) holds if and only if $A_{q<p} < \infty$, while from [20, Theorem 14] it follows that inequalities (11) hold if and only if $\widehat{A}_{ji} < \infty$. Additionally, by combining the best constant C_1 of inequality (6) and the best constants C_i', $i = 0, 1, ..., n$, of inequalities (11), we obtain $\widehat{A} \approx C$.

Similarly, in the case $1 < p \leq q < \infty$, it follows from [15, Theorem 3.1] that inequality (6) holds if and only if $A_{p\leq q} < \infty$, while from [16, Theorem 6] it follows that inequalities (11) hold if and only if $\widetilde{A}_{in} < \infty$. Moreover, in this case, $\widetilde{A} \approx C$. □

3 Characterization of inequality (2)

Theorem 4. Let $0 < r < \infty$ and the kernel $K(\cdot, \cdot)$ belong to the Oinarov class $\mathcal{O}_n^+$, $n \geq 1$.

(i) If $1 < q < p < \infty$, then (2) holds if and only if $\widehat{B} = \max\{B_{q<p}, \widehat{B}_{ij}\} < \infty$, where

$$\widehat{B}_{ij} = \max_{0\leq i\leq n} \max_{0\leq j\leq i} \left(\int_0^\infty \left(\int_t^\infty K_{i,j}^q(s,t)\, u(s) \left(\int_s^\infty K_{n,i}^r(z,s) w(z)\, dz \right)^{\frac{q}{r}} ds \right)^{\frac{\mu}{q}} \right.$$

$$\left. \times \left(\int_0^t K_j^{p'}(t,s) v^{1-p'}(s) ds \right)^{\frac{\mu}{q'}} d \left(\int_0^t K_j^{p'}(t,s) v^{1-p'}(s) ds \right) \right)^{\frac{1}{\mu}}$$

$$\approx \max_{0\leq i\leq n} \max_{0\leq j\leq i} \left(\int_0^\infty \left(\int_0^t K_j^{p'}(t,s) v^{1-p'}(s) ds \right)^{\frac{\mu}{p'}} \right.$$

$$\times \left(\int_t^\infty K_{i,j}^q(s,t)\, u(s) \left(\int_s^\infty K_{n,i}^r(z,s) w(z)\, dz \right)^{\frac{q}{r}} ds \right)^{\frac{\mu}{p}}$$

$$\left. \times d \left(- \int_t^\infty K_{i,j}^q(s,t)\, u(s) \left(\int_s^\infty K_{n,i}^r(z,s) w(z)\, dz \right)^{\frac{q}{r}} ds \right) \right)^{\frac{1}{\mu}}.$$

Moreover, $\widehat{B} \approx C$, where C is the best constant in (2).

(ii) If $1 < p \leq q < \infty$, then (2) holds if and only if $\widetilde{B} = \max\{B_{p\leq q}, \widetilde{B}_{ni}\} < \infty$, where

$$B_{p\leq q} = \sup_{z>0} \left(\int_0^z u(x)\, dx \right)^{\frac{1}{q}} J_{p,r}^+(z, \infty),$$

$$\widetilde{B}_{ni} = \sup_{z>0} \max_{0\leq i\leq n} \left(\int_0^z v^{1-p'}(x) \left(\int_z^\infty K_i^q(s,x) u(s) \left(\int_s^\infty K_{n,i}^r(t,s) w(t)\, dt \right)^{\frac{q}{r}} ds \right)^{\frac{p'}{q}} dx \right)^{\frac{1}{p'}}$$

$$\approx \sup_{z>0} \max_{0\leq i\leq n} \left(\int_z^\infty \left(\int_0^z K_i^{p'}(s,x) v^{1-p'}(x) dx \right)^{\frac{q}{p'}} u(s) \left(\int_s^\infty K_{n,i}^r(t,s) w(t)\, dt \right)^{\frac{q}{r}} ds \right)^{\frac{1}{q}}.$$

Moreover, $\widetilde{B} \approx C$, where C is the best constant in (2).

Proof. Splitting the interior integral on the left-hand side of inequality (2), we get

$$\left(\int_0^\infty u(x)\left(\int_x^\infty\left(\int_0^t K(t,s)f(s)\,ds\right)^r w(t)\,dt\right)^{\frac{q}{r}}dx\right)^{\frac{1}{q}}$$

$$\approx\left(\int_0^\infty u(x)\left(\int_x^\infty\left(\int_0^x K(t,s)f(s)\,ds\right)^r w(t)\,dt\right)^{\frac{q}{r}}dx\right)^{\frac{1}{q}}$$

$$+\left(\int_0^\infty u(x)\left(\int_x^\infty\left(\int_x^t K(t,s)f(s)\,ds\right)^r w(t)\,dt\right)^{\frac{q}{r}}dx\right)^{\frac{1}{q}}. \qquad (12)$$

Applying (9) to the first term in (12), we deduce

$$\left(\int_0^\infty u(x)\left(\int_x^\infty\left(\int_0^x K(t,s)f(s)\,ds\right)^r w(t)\,dt\right)^{\frac{q}{r}}dx\right)^{\frac{1}{q}}$$

$$\approx\sum_{i=0}^n\left(\int_0^\infty u(x)\left(\int_x^\infty K_{n,i}^r(t,x)w(t)\,dt\right)^{\frac{q}{r}}\left(\int_0^x K_i(x,s)f(s)\,ds\right)^q dx\right)^{\frac{1}{q}}.$$

Thus, the validity of inequality (2) is equivalent to the simultaneous validity of inequality

$$\left(\int_0^\infty u(x)\left(\int_x^\infty\left(\int_x^t K(t,s)f(s)\,ds\right)^r w(t)\,dt\right)^{\frac{q}{r}}dx\right)^{\frac{1}{q}}\leq C_1''\,\|f\|_{p,v} \qquad (13)$$

and the following $n+1$ inequalities:

$$\sum_{i=0}^n\left(\int_0^\infty u(x)\left(\int_x^\infty K_{n,i}^r(t,x)w(t)\,dt\right)^{\frac{q}{r}}\left(\int_0^x K_i(x,s)f(s)\,ds\right)^q dx\right)^{\frac{1}{q}}\leq C_i''\,\|f\|_{p,v}, \qquad (14)$$

where $K_i(\cdot,\cdot)$ belongs to the class O_i^+, $i=0,1,...,n$.

In the case $1<q<p<\infty$, it follows from [15, Theorem 3.3] that inequality (13) holds if and only if $B_{q<p}<\infty$, while from [20, Theorem 11] it follows that inequalities (14) hold if and only if $\widehat{B}_{ij}<\infty$. Additionally, by combining the best constant C_1'' of inequality (13) and the best constants C_i'', $i=0,1,...,n$, of inequalities (14), we have $\widehat{B}\approx C$.

Similarly, in the case $1<p\leq q<\infty$, it follows from [15, Theorem 3.3] that inequality (13) holds if and only if $B_{p\leq q}<\infty$, while from [16, Theorem 5] we have that inequalities (14) hold if and only if $\widetilde{B}_{ni}<\infty$. Moreover, in this case, $\widetilde{B}\approx C$.

□

Remark 1. Let us again consider inequality (1). Suppose that the kernel $K(\cdot,\cdot)$ belongs to the Oinarov class $\mathcal{O}_n^+$, $n \geq 1$, but not to the Oinarov class $\mathcal{O}_n^-$, $n \geq 1$, as stated in the condition of Theorem 3. Then, applying (9) to the second term in (5), we obtain the following inequality:

$$\left(\int_0^\infty u(x)\left(\int_0^x K_i^r(x,t)w(t)\,dt\right)^{\frac{q}{r}}\left(\int_x^\infty K_{n,i}(s,x)f(s)\,ds\right)^q dx\right)^{\frac{1}{q}} \leq C_i\,\|f\|_{p,v}$$

instead of (11). The characterization of Hardy-type inequality (3) when the operator T is given by $K^- f(x) = \int_x^\infty K(s,x)f(s)\,ds$, without any restriction on its kernel $K(\cdot,\cdot)$, remains an open problem. Since, by definition of the class $\mathcal{O}_n^+$, $n \geq 1$, there are no restrictions on the non-negative measurable functions $K_{n,i}(\cdot,\cdot)$, we cannot establish the validity of inequality (1) when the kernel $K(\cdot,\cdot)$ belongs to the Oinarov class $\mathcal{O}_n^+$, $n \geq 1$; instead, we can only do so when it belongs to the class $\mathcal{O}_n^-$, $n \geq 1$. A similar situation arises for inequality (2): we can characterize it only if the kernel $K(\cdot,\cdot)$ belongs to the Oinarov class $\mathcal{O}_n^+$, $n \geq 1$.

Conclusion

The necessary and sufficient conditions for the validity of the three-weight Hardy inequalities with iterated operators and generalized kernels belonging to the Oinarov classes were obtained. The obtained results can be used in harmonic analysis, in the theory of differential and difference equations, as well as in other areas of mathematics. Moreover, the characteristics of weighted Hardy inequalities with iterated operators can now be used to study bilinear weighted Hardy-type inequalities.

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Author Contributions

All authors contributed equally to this work.

Conflict of Interest

The authors declare no conflict of interest.

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Ternary Menger hyperalgebras: some algebraic properties

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In this article, we first establish an algebraic hyperstructure called a *ternary Menger hyperalgebra of rank n*, where n is a natural number. The algebraic hyperstructure can be regarded as a novel generalization of ternary semihypergroups. In particular, by setting the natural number n equal to 1, the algebraic hyperstructures of ternary Menger hyperalgebras of rank 1 and ternary semihypergroups are the same. And then, we extend some fundamental results on the ternary semihypergroup theory to study on ternary Menger hyperalgebras of rank n including subhyperalgebras and homomorphisms. Moreover, we investigate some interesting algebraic connections among Menger algebras of rank n, Menger hyperalgebras of rank n, ternary Menger algebras of rank n and ternary Menger hyperalgebras of rank n. In this section, we present that the algebraic hyperstructure of ternary Menger hyperalgebras of rank n can also be considered as an extension of the concepts Menger hyperalgebras of rank n and ternary Menger algebras of rank n. Finally, we use algebraic hyperstructures of ternary Menger hyperalgebras of rank n to construct the so-called diagonal ternary semihypergroups of the ternary Menger hyperalgebras of rank n.

Keywords: Menger algebras, Menger hyperalgebras, ternary Menger algebras, ternary Menger hyperalgebras, semigroups, semihypergroups, ternary semigroups, ternary semihypergroups, diagonal ternary semihypergroups.

2020 Mathematics Subject Classification: 20M10.

Introduction

A unary function, which is defined on a nonempty X (i.e. a mapping from X into X), is known as a transformation of the set X. Later, the idea of *multiplace functions*, which are also called functions of many variables, has been studied. In 1946, Menger [1] presented an algebraic property of the composition of multiplace functions, which is called *superassociative law.* This study has been credited as the first research work concerning on the study of *Menger algebras of rank n*, where n is any positive integer. A Menger algebra (M, o) of rank n is an $(n+1)$-ary algebraic structure such that its $(n+1)$-ary operation o satisfies the superassociative law, i.e.,

$$o(o(x, y_1, \ldots, y_n), z_1, \ldots, z_n) = o(x, o(y_1, z_1, \ldots, z_n), \ldots, o(y_n, z_1, \ldots, z_n)) \tag{1}$$

for all $x, y_1, \ldots, y_n, z_1, \ldots, z_n \in M$. It is easy to see that the superassociative law is an extension of the associative law:

$$o(o(x, y), z) = o(x, o(y, z)) \tag{2}$$

for all $x, y, z \in M$, i.e., the superassociative law (1) is reduced to the associative law (2) if $n = 1$. Furthermore, the Menger algebra (M, o) of rank n is also reduced to a semigroup. As a result, Menger algebras of rank n can be regarded as a natural generalization of semigroups.

Based on the concept of Menger algebras of rank n, there is a number of published papers studied on this structure in various directions and fields, both theoretical and applied mathematics (e.g. [2,3]). The theory of Menger algebras of rank n and its applications have been developed by Dudek and

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Trokhimenko who introduced the idea of principal v-congruences on Menger algebras of rank n, which is an extension of principal right (left) congruences on semigroups [4]. By using the perspective of commutativity, Dudek and Trokhimenko [5–7] investigated some related algebraic properties on Menger algebras of rank n. Moreover, Dudek and Trokhimenko [8] used some operations to characterize the set of multiplace functions. In 1997, Trokhimenko [9] introduced the so-called v-regular Menger algebras of rank n, which can be acted as a generalization of regular semigroups, and also investigated some interesting results on the structures. In 2003, Denecke [10] studied on partial Menger algebras of rank n of terms. Up to 2021, Denecke and Hounnon [11] discussed some properties on Menger algebras of rank n and clones of terms.

The idea of ternary semigroups was first known when Banach (c.f. [12]) demonstrated some examples of ternary semigroups which need not to be reduced to semigroups. In 1932, the algebraic theory of ternary semigroups was firstly introduced by Lehmer [13]. A ternary semigroup $(M, \star)$ consists of a nonempty set M and a ternary operation $\star$ defined on M satisfying the so-called the *ternary associative law*, i.e.,

$$\star(\star(u, v, x), y, z) = \star(u, \star(v, x, y), z) = \star(u, v, \star(x, y, z)) \tag{3}$$

for all $u, v, x, y, z \in M$. According to the significant remark which was demonstrated by Banach, the algebraic properties and applications of ternary semigroups have been extensively studied (c.f. [14]), including ternary algebras and Banach ternary algebras, see [15–17]. Moreover, Chronowski [18] studied some results on ternary linear algebras and topological ternary structures. In [19], Jin et al. investigated some algebraic properties on C^*-ternary algebras. In 1955, Los [12] showed that each ternary semigroup can be embedded into a semigroup. In [20], Santiago and Sri Bala presented a regularity condition of ternary semigroups and studied some properties of regular ternary semigroups. Recently, Nongmanee and Leeratanavalee [21] extended the well-known result, i.e., Cayley's theorem, to study on ternary semigroups.

We note here that Menger hyperalgebras of rank n can be reduced to ternary algebraic structures with its ternary operation satisfying the following identity:

$$o(o(x, y_1, y_2), z_1, z_2) = o(x, o(y_1, z_1, z_2), o(y_2, z_1, z_2)) \tag{4}$$

for all $x, y_1, y_2, z_1, z_2 \in M$ if $n = 2$. It is easy to see that the identities (3) and (4) are not the same. It means that Menger algebras of rank n cannot be a generalization of ternary semigroups. Based on the previous remark, Nongmanee and Leeratanavalee [22] initiated the idea of *ternary Menger algebras of rank n* as a new generalization of ternary semigroups in sense of Menger algebras in 2021. A ternary Menger algebra $(M, \diamond)$ of rank n consists of a nonempty set M and a $(2n+1)$-ary operation which satisfies the so-called the *ternary superassociative law* given as follows:

$$\begin{aligned}
&\diamond(\diamond(u, v_1, \ldots, v_n, x_1, \ldots, x_n), y_1, \ldots, y_n, z_1, \ldots, z_n) \\
&\quad = \diamond(u, \diamond(v_1, x_1, \ldots, x_n, y_1, \ldots, y_n), \ldots, \diamond(v_n, x_1, \ldots, x_n, y_1, \ldots, y_n), z_1, \ldots, z_n) \\
&\quad = \diamond(u, v_1, \ldots, v_n, \diamond(x_1, y_1, \ldots, y_n, z_1, \ldots, z_n), \ldots, \diamond(x_n, y_1, \ldots, y_n, z_1, \ldots, z_n))
\end{aligned} \tag{5}$$

for all $u, v_i, x_i, y_i, z_i \in M,\ i \in \{1, \ldots, n\}$.

Ternary Menger algebras of rank n can be considered as a natural generalization of ternary semigroups by setting the natural number $n = 1$. Then, the ternary superassociative law (5) is reduced to the ternary associative law on ternary semigroups (3) and implies that ternary Menger algebras of rank 1 and ternary semigroups are the same. Nongmanee and Leeratanavalee [23] recently presented the regular condition on ternary Menger algebras of rank n and investigated some algebraic properties of v-regular ternary Menger algebras of rank n.

In 1934, Marty [24] initiated the concept of algebraic hyperstructures when he defined the so-called hypergroups associated with hyperoperations. Such a concept is a new perspective on the classical algebraic theory. That is, on the classical algebras, a composition of two elements (or n elements) of a base set is again an element, but on algebraic hyperstructures, a composition of two elements (or n elements) is a nonempty set. Since then, algebraic properties on hyperstructures have been investigated by many authors and there are many books written on this topic. Corsini and Leoreanu [25] point out the application of hyperstructures to graphs, hypergraphs, cryptography, codes, automata theory, fuzzy set theory, rough set theory and geometry (see also [26, 27]). *Semihypergroups* and *ternary semihypergroups* (sometimes called *hypersemigroups* and *ternary hypersemigroups*, respectively) are also algebraic hyperstructures that can be considered as generalizations of semigroups and ternary semigroups, respectively. The idea of semihypergroups was initiated by Bonansinga and Corsini [28, 29]. There are currently many articles studying semihypergroups and ternary semihypergroups by many mathematicians in different contexts. In 2021, Kumduang and Leeratanavalee [30] combined the concepts of Menger algebras of rank n and hyperstructures to create a generalization of semihypergroups called a *Menger hyperalgebra of rank n*. In their work, they also investigated some fundamental results on the hyperstructures, including the Cayley's theorem and the isomorphism theorem. Recently, Nongmanee and Leeratanavalee [31] studied the algebraic connection between Menger hyperalgebras of rank n and Menger algebras of rank n through regularity.

Inspired by the previous structural and hyperstructural studies, in this article we will start with the following facts: (i) each semihypergroup can be formed into a ternary semihypergroup, while some ternary semihypergroups do not necessarily reduce to an ordinary semihypergroup; (ii) Menger hyperalgebras can be regarded as a generalization of semihypergroups, but it cannot be regarded as a generalization of ternary semihypergroups. Therefore, a potential research question arises as to what is the algebraic structural generalization of ternary semihypergroups in terms of the arbitrary arity of a hyperoperation that satisfies the ternary superassociative law? Thus, in Section 2, the concept of *ternary Menger hyperalgebras of rank n* is introduced by combining the concepts of ternary Menger algebras of rank n and algebraic hyperstructures. In particular, we will examine their basic results and study the algebraic connections between ternary Menger hyperalgebras of rank n and other structures. In Section 3, we use the hyperstructure of ternary Menger algebras of rank n to construct what are called *diagonal ternary semihypergroups*. Moreover, we present some algebraic characteristics of the hyperstructure. In the last section, we will conclude the article by showing the relationship between structures and hyperstructures.

1 *Preliminary background*

To obtain the main results, we first need to recall the initial definitions and results of Menger algebras of rank n, ternary Menger algebras of rank n and Menger hyperalgebras of rank n.

Let (M, o) be a Menger algebra of rank n together with an $(n+1)$-ary operation o given by $(x, y_1, \ldots, y_n) \mapsto o(x, y_1, \ldots, y_n)$. Then, $(M, \diamond)$ forms a ternary Menger algebra of rank n under a $(2n+1)$-ary operation $\diamond$ defined by $(x, y_1, \ldots, y_n, z_1, \ldots, z_n) \mapsto o(o(x, y_1, \ldots, y_n), z_1, \ldots, z_n)$.

The ternary Menger algebra $(M, \diamond)$ of rank n formed by the Menger algebra (M, o) of rank n is called a *ternary Menger algebra of rank n induced by a Menger algebra of rank n*. There are also some examples of ternary Menger algebras of rank n that cannot be reduced to Menger algebras of rank n, which are given in [22, 23].

Let M be a nonempty set and an operation $g : M^{n+1} \longrightarrow P^*(M)$, where $P^*(M)$ is the family of all nonempty subsets of M, be defined by: for any $A, B_i \in P^*(M),\ i \in \{1, \ldots, n\}$,

$$g(A, B_1, \ldots, B_n) = \bigcup_{a \in A, b_i \in B_i, i \in \{1, \ldots, n\}} g(a, b_1, \ldots, b_n).$$

Then g is called a $(2n+1)$-ary hyperoperation. In [30], an $(n+1)$-ary hypergroupoid (M, g) is a pair of a nonempty set M and an $(n+1)$-ary hyperoperation g defined on M. If the $(n+1)$-ary hyperoperation

g satisfies the superassociative law given as in (1), i.e.,

$$\bigcup_{a\in g(x,y_1,\dots,y_n)} g(a,z_1,\dots,z_n) = \bigcup_{b_i\in g(y_i,z_1,\dots,z_n),i\in\{1,\dots,n\}} g(x,b_1,\dots,b_n),$$

for all $a,x,y_1,\dots,y_n,z_1,\dots,z_n \in M$, then the $(n+1)$-ary hypergroupoid (M,g) is called a *Menger hyperalgebra of rank n*. The first observation about Menger hyperalgebras of rank n is the fact that Menger hyperalgebras of rank n immediately reduce to semihypergroups by setting $n=1$. This means that it is a structural generalization of all semihypergroups. The following theorem shows a closed connection between Menger algebras of rank n and Menger hyperalgebras of rank n.

Theorem 1. [30] Let (M,o) be a Menger algebra of rank n. Then, (M,g) forms a Menger hyperalgebra of rank n under an $(n+1)$-ary hyperoperation g defined by $g(x,y_1,\dots,y_n)=\{o(x,y_1,\dots,y_n)\}$ for all $x,y_1,\dots,y_n \in M$.

Example 1. Some examples of Menger hyperalgebras of rank n have already been presented in [30].

(i) The set of all real numbers $\mathbb{R}$ together with an $(n+1)$-ary hyperoperation g defined by

$$g(x,y_1,\dots,y_n)=\left\{x+\tfrac{y_1+\dots+y_n}{n}\right\} \quad \text{for all } x,y_1,\dots,y_n\in\mathbb{R},$$

where $+$ is the usual addition on $\mathbb{R}$, is a Menger hyperalgebra of rank n.

(ii) Define an $(n+1)$-ary hyperoperation g on the unit interval $[0,1]$ as follows:

$$g(x,y_1,\dots,y_n)=\left[0,\tfrac{x\times y_1\times\dots\times y_n}{n+1}\right] \quad \text{for all } x,y_1,\dots,y_n\in[0,1],$$

where $\times$ is the usual multiplication on $[0,1]$. Then, $([0,1],g)$ forms a Menger hyperalgebra of rank n.

(iii) Let $\mathbb{N}$ be the set of all nonnegative integers. We define an $(n+1)$-ary hyperoperation g on $\mathbb{N}$ by

$$g(x,y_1,\dots,y_n)=\{m\in\mathbb{N}\mid m\geq \max\{x,y_1,\dots,y_n\}\} \quad \text{for all } x,y_1,\dots,y_n\in\mathbb{N}.$$

Then, $(\mathbb{N},g)$ is a Menger hyperalgebra of rank n.

To better understand the Menger hyperalgebras of rank n, we refer the reader to [30].

2 *Ternary Menger hyperalgebras of rank n*

In this section, we will first create what is called *ternary Menger hyperalgebras of rank n*, where n is a positive integer. The hyperstructure is a new appropriate generalization of the whole ternary semihypergroup. Then, we will examine some of its interesting algebraic properties, including subhyperalgebras and homomorphisms. The algebraic connections among Menger algebras of rank n, Menger hyperalgebras of rank n, ternary Menger algebras of rank n and ternary Menger hyperalgebras of rank n have also been studied.

Let M be a nonempty set and f be a $(2n+1)$-ary hyperoperation from M into $P^*(M)$. For each $X,Y_i,Z_i\in P^*(M),\ i\in\{1,\dots,n\}$, we define

$$f(X,Y_1,\dots,Y_n,Z_1,\dots,Z_n)=\bigcup_{x\in X,y_i\in Y_i,z_i\in Z_i} f(x,y_1,\dots,y_n,z_1,\dots,z_n).$$

For convenience, throughout this article, the sequence of elements $y_1,\dots,y_n$ is replaced by the symbol $\bar{y}$ and the composition $f(x,y_1,\dots,y_n,z_1,\dots,z_n)$ under the $(2n+1)$-ary hyperoperation f is denoted by the abbreviated notation $x[y_1\dots y_n z_1\dots z_n]$ or $x[\bar{y}\bar{z}]$. In the case where $z_1=\dots=z_n=z$, we write $x[\bar{y}z^n]$ instead of $f(x,y_1,\dots,y_n,z_1,\dots,z_n)$.

Moreover, for any Menger hyperalgebra (M,g) of rank n, we also write $x[\bar{y}]$ instead of $g(x,y_1,\dots,y_n)$. Also, in the case where $y_1=\dots=y_n=y$, the notation $x[y^n]$ is written instead of $g(x,y_1,\dots,y_n)$.

Definition 1. A $(2n+1)$-ary hypergroupoid (M, f) is a $(2n+1)$-ary algebraic hyperstructure consisting of a nonempty set M and a $(2n+1)$-ary hyperoperation f defined on M.

Definition 2. A $(2n+1)$-ary hypergroupoid (M, f) is called a *ternary Menger hyperalgebra of rank n* if its $(2n+1)$-ary hyperoperation f satisfies the ternary superassociative law given in (5), i.e.,

$$u[\bar{v}\bar{x}][\bar{y}\bar{z}] = u[v_1[\bar{x}\bar{y}] \ldots v_n[\bar{x}\bar{y}]\bar{z}] = u[\bar{v}x_1[\bar{y}\bar{z}] \ldots x_n[\bar{y}\bar{z}]]$$

for all $u, \bar{v}, \bar{x}, \bar{y}, \bar{z} \in M$.

For algebraic hyperstructures, the ternary supperassociative law given in (5) means that:

$$\bigcup_{a \in u[\bar{v}\bar{x}]} a[\bar{y}\bar{z}] = \bigcup_{b_i \in v_i[\bar{x}\bar{y}], i \in \{1,\ldots,n\}} u[\bar{b}\bar{z}] = \bigcup_{c_i \in x_i[\bar{y}\bar{z}], i \in \{1,\ldots,n\}} u[\bar{v}\bar{c}]$$

for all $u, \bar{v}, \bar{x}, \bar{y}, \bar{z} \in M$.

By Definition 2, we see that the algebraic hyperstructure of ternary Menger hyperalgebras of rank n can be observed as a new generalization of ternary semihypergroups. This concept differs from the well-known concepts such as: n-ary semihypergroups and ordered n-ary semihypergroups. In particular, in case where $n = 1$, the ternary superassociative law immediately reduces to the ternary associative law presented in (3). Moreover, a ternary Menger hyperalgebra of rank n is immediately reduced to a ternary semihypergroup. However, the n-ary semihypergroup is reduced to a ternary semihypergroup by setting $n = 3$.

For the first result, we show the closed algebraic connection between the algebraic structures of ternary Menger algebras of rank n and the algebraic hyperstructures of ternary Menger hyperalgebras of rank n.

Theorem 2. Any ternary Menger algebra of rank n can form a ternary Menger hyperalgebra of rank n.

Proof. Let $(M, \diamond)$ be a ternary Menger algebra of rank n. On the base set M, we define a $(2n+1)$-ary hyperoperation f by

$$x[\bar{y}\bar{z}] = \{\diamond(x, \bar{y}, \bar{z})\} \quad \text{for all } x, \bar{y}, \bar{z} \in M. \tag{6}$$

Indeed, for each $u, \bar{v}, \bar{x}, \bar{y}, \bar{z} \in M$, we have

$$\begin{aligned}
u[\bar{v}\bar{x}][\bar{y}\bar{z}] &= \diamond(u, \bar{v}, \bar{x})[\bar{y}\bar{z}] \\
&= \{\diamond(\diamond(u, \bar{v}, \bar{x}), \bar{y}, \bar{z})\} \\
&= \{\diamond(u, \diamond(v_1, \bar{x}, \bar{y}), \ldots, \diamond(v_n, \bar{x}, \bar{y}), \bar{z})\} \\
&= u[\{\diamond(v_1, \bar{x}, \bar{y})\} \ldots \{\diamond(v_n, \bar{x}, \bar{y})\}\bar{z}] \\
&= u[v_1[\bar{x}\bar{y}] \ldots v_n[\bar{x}\bar{y}]\bar{z}], \\
u[\bar{v}x_1[\bar{y}\bar{z}] \ldots x_n[\bar{y}\bar{z}]] &= u[\bar{v} \diamond (x_1, \bar{y}, \bar{z}) \ldots \diamond (x_n, \bar{y}, \bar{z})] \\
&= \{\diamond(u, \bar{v}, \diamond(x_1, \bar{y}, \bar{z}) \ldots \diamond (x_n, \bar{y}, \bar{z}))\} \\
&= \{\diamond(\diamond(u, \bar{v}, \bar{x}), \bar{y}, \bar{z})\} \\
&= \diamond(u, \bar{v}, \bar{x})[\bar{y}\bar{z}] \\
&= u[\bar{v}\bar{x}][\bar{y}\bar{z}].
\end{aligned}$$

It follows that f satisfies the ternary superassociative law. Thus, (M, f) forms a ternary Menger hyperalgebra of rank n. □

We call (M, f), where f is defined in (6) of Theorem 2, a *ternary Menger hyperalgebra of rank n induced by a ternary Menger algebra of rank n* or a *trivial ternary Menger hyperalgebra of rank n*. To better understand the nature of the ternary Menger hyperalgebras of rank n, the following example is needed.

Example 2. (i) Let M be a nonempty set. Define a $(2n+1)$-ary hyperoperation $f : M^{2n+1} \longrightarrow P^*(M)$ by

$$x[\bar{y}\bar{z}] = \{x\} \quad \text{for all } x, \bar{y}, \bar{z} \in M.$$

Therefore, (M, f) forms a ternary Menger hyperalgebra of rank n.

(ii) Let $\mathbb{Z}$ be the set of all integers. A $(2n+1)$-ary hypergroupoid $(\mathbb{Z} \times \mathbb{Z}, f)$, together with a $(2n+1)$-ary hyperoperation f defined by:

$$(a,b)[(u_1,v_1)\ldots(u_n,v_n)(x_1,y_1)\ldots(x_n,y_n)] = \{(a,y_n)\} \quad \text{for all } a,b,\bar{u},\bar{v},\bar{x},\bar{y} \in \mathbb{Z}$$

forms a ternary Menger hyperalgebra of rank n.

(iii) Define a $(2n+1)$-ary hyperoperation f on the set $\mathbb{R}$ of all real numbers as follows:

$$x[\bar{y}\bar{z}] = \{x + \tfrac{y_1+\cdots+y_n+z_1+\cdot+z_n}{n}\} \quad \text{for all } x, \bar{y}, \bar{z} \in \mathbb{R}.$$

Therefore, $(\mathbb{R}, f)$ is a ternary Menger hyperalgebra of rank n.

(iv) On the set $\mathbb{R}_+$ of all positive real numbers, we define a $(2n+1)$-ary hyperoperation f as follows:

$$x[\bar{y}\bar{z}] = \{x \times \sqrt[n]{y_1 \times \cdots \times y_n \times z_1 \times \cdots \times z_n}\} \quad \text{for all } x, \bar{y}, \bar{z} \in \mathbb{R}_+$$

Then, $(\mathbb{R}_+, f)$ is a ternary Menger hyperalgebra of rank n.

Example 3. Let $\mathbb{N}$ be the set of all natural numbers. Define a $(2n+1)$-ary hyperoperation f by

$$x[\bar{y}\bar{z}] = \{m \in \mathbb{N} \mid m \geq \max\{x, \bar{y}, \bar{z}\}\} \quad \text{for all } x, \bar{y}, \bar{z} \in \mathbb{N}.$$

Then, $(\mathbb{N}, f)$ is a ternary Menger hyperalgebra of rank n.

Example 4. Let M be the unit interval $[0,1]$. Define a $(2n+1)$-ary hyperoperation f by

$$x[\bar{y}\bar{z}] = [m, 1] \quad \text{for all } x, \bar{y}, \bar{z} \in M,$$

where $m = \min\{x, \bar{y}, \bar{z}\}$. Hence, the $(2n+1)$-ary hypergroupoid (M, f) forms a ternary Menger hyperalgebra of rank n.

Example 5. On the unit interval $[0,1]$, we define a $(2n+1)$-ary hyperoperation f by

$$x[\bar{y}\bar{z}] = [0, \tfrac{x \times y_1 \times \cdots \times y_n \times z_1 \times \cdots \times z_n}{2n+1}] \quad \text{for all } x, \bar{y}, \bar{z} \in [0,1],$$

where $\times$ is the usual multiplication on $[0,1]$. Then the $(2n+1)$-ary hypergroupoid (M, f) forms a ternary Menger hyperalgebra of rank n.

To obtain our main result, some special elements of ternary Menger hyperalgebras of rank n are defined in a manner similar to semihypergroup theory and ternary semihypergroup theory.

Definition 3. Let $(M.f)$ be a ternary Menger hyperalgebra of rank n. An element $e \in M$ is said to be

(i) a (scalar) left identity element if

$$x \in e[e^n x^n] \ (\{x\} = e[e^n x^n]) \quad \text{for all } x \in M;$$

(ii) a (scalar) right identity element if

$$x \in x[e^n e^n] \ (\{x\} = x[e^n e^n]) \quad \text{for all } x \in M;$$

(iii) a (scalar) lateral identity element if

$$x \in e[x^n e^n] \ (\{x\} = e[x^n e^n]) \quad \text{for all } x \in M;$$

(iv) an (scalar) identity element if

$$x \in e[e^n x^n] \cap e[x^n e^n] \cap x[e^n e^n] \ (\{x\} = e[e^n x^n] = e[x^n e^n] = x[e^n e^n]) \quad \text{for all } x \in M.$$

Next, we provide basic methods, namely subhyperalgebras and homomorphic images, to construct new ternary Menger hyperalgebras of rank n by using the given ternary Menger hyperalgebras of rank n.

Definition 4. Let (M, f) be a ternary Menger hyperalgebra of rank n and S be a nonempty subset of M. A $(2n+1)$-ary algebraic hyperstructure (S, f) is called a *ternary Menger subhyperalgebra of rank n* of (M, f) if S is closed under the $(2n+1)$-ary hyperoperation f, i.e., if $x, \bar{y}, \bar{z} \in S$, then $x[\bar{y}\bar{z}] \subseteq S$. If (S, f) is a ternary Menger subhyperalgebra of rank n of (M, f), then we write $(S, f) \leq (M, f)$.

Example 6. (i) Consider the ternary Menger hyperalgebra (M, f) of rank n in Example 4. Let $S_p = [p, 1]$, where $m \leq p \leq 1$. Then, $(S_p, f) \leq (M, f)$.

(ii) Consider the ternary Menger hyperalgebra (M, f) of rank n in Example 5. Let $S_q = [0, q]$, where $0 \leq q \leq 1$. Then $(Sq, f) \leq (M, f)$.

Using the concept of ternary Menger subhyperalgebras of rank n, we will examine some basic results about the $(2n+1)$-ary algebraic hyperstructures.

Theorem 3. Let $(S_1, f), (S_2, f)$ and (S_3, f) be ternary Menger subhyperalgebras of rank n of a ternary Menger hyperalgebra (M, f) of rank n. Then, the following assertions hold:

(i) if $(S_1, f) \leq (S_2, f)$ and $(S_2, f) \leq (S_3, f)$, then $(S_1, f) \leq (S_3, f)$;

(ii) if $S_1 \subseteq S_2 \subseteq S_3$, $(S_1, f) \leq (S_3, f)$ and $(S_2, f) \leq (S_3, f)$, then $(S_1, f) \leq (S_2, f)$.

Proof. The proof is straightforward. □

Theorem 4. Let I be an indexed set and let (S_i, f), where $i \in I$, be ternary Menger subhyperalgebras of rank n of a ternary Menger hyperalgebra (M, f) of rank n. Then, $(\bigcap_{i \in I} S_i, f) \leq (M, f)$.

Proof. The proof is straightforward. □

Let (M, f) be a ternary Menger hyperalgebra of rank n and S be a nonempty subset of M. Then, there is at least one ternary Menger hyperalgebra of rank n containing the subset S. That is, (M, f) is a ternary Menger hyperalgebra of rank n containing S. By Theorem 4, we immediately obtain that the intersection of all ternary Menger subhyperalgebras of rank n of (M, f) which contain the subset S forms a ternary Menger subhyperalgebra of rank n of (M, f). We denote it by $(\hat{S}, f)$. Using the previous fact, the following algebraic property is examined.

Proposition 1. Let (M, f) be a ternary Menger hyperalgebra of rank n and S be a nonempty subset of M. Then,

(i) $S \subseteq \hat{S}$;

(ii) if (T, f) is a ternary Menger subhyperalgebra of rank n of (M, f) which contains the subset S, then $(\hat{S}, f) \leq (T, f)$.

Proof. The proof is straightforward. □

The next purpose of this section is to demonstrate other basic results of ternary Menger hyperalgebras of rank n related to the concepts of homomorphism and ternary Menger subhyperalgebra of rank n. In order to obtain the result, it is necessary to introduce the concept of several types of homomorphism in ternary Menger hyperalgebras of rank n.

Definition 5. Let (M, f) and (N, g) be ternary Menger hyperalgebras of rank n. A mapping $\phi : (M, f) \longrightarrow (N, g)$ is called:

(i) a *homomorphism* if for each $x, \bar{y}, \bar{z} \in M$,

$$\phi(x[\bar{y}\bar{z}]) \subseteq \phi(x)[\phi(y_1)\dots\phi(y_n)\phi(z_1)\dots\phi(z_n)];$$

(ii) a *strong homomorphism* if for each $x, \bar{y}, \bar{z} \in M$,

$$\phi(x[\bar{y}\bar{z}]) = \phi(x)[\phi(y_1)\dots\phi(y_n)\phi(z_1)\dots\phi(z_n)].$$

By Definition 5, if the homomorphism ϕ is injective, it is called a *monomorphism* (or an *embedding*). Moreover, if the strong homomorphism ϕ is both injective and surjective, it is called an *isomorphism* from (M, f) onto (N, g). In this particular situation, we say that (M, f) is isomorphic to (N, g) and denote by $(M, f) \cong (N, g)$.

Theorem 5. Let (M, f) and (N, g) be ternary Menger hyperalgebras of rank n and let $\phi : (M, f) \longrightarrow (N, g)$ be a homomorphism. Then,
(i) if $(S, f) \leq (M, f)$ then $(\phi(S), g) \leq (N, g)$;
(ii) if $(T, g) \leq (N, g)$ and $\phi^{-1}(T) \neq \emptyset$ then $(\phi^{-1}(T), f) \leq (M, f)$,
where $\phi(S) = \{t \in N \mid \phi(s) = t \text{ for some } s \in S\}$ and $\phi^{-1}(T) = \{s \in M \mid \phi(s) = t \text{ for some } t \in T\}$.

Proof. (i) Assume that $(S, f) \leq (M, f)$. Let $x, \bar{y}, \bar{z} \in \phi(S)$. Then, there exist $a, \bar{b}, \bar{c} \in S$ such that $\phi(a) = x, \phi(b_i) = y_i$ and $\phi(c_i) = z_i$, where $i \in \{1, \dots, n\}$.

By our assumption, we get $a[\bar{b}\bar{c}] \in S$ and hence, $\phi(a[\bar{b}\bar{c}]) \in \phi(S)$. Since ϕ is a homomorphism from (M, f) into (N, g), we have

$$x[\bar{y}\bar{z}] = \phi(a)[\phi(b_1)\dots\phi(b_n)\phi(c_1)\dots\phi(c_n)] = \phi(a[\bar{b}\bar{c}]) \in \phi(S).$$

It follows that $(\phi(S), g)$ is a ternary Menger subhyperalgebra of rank n of (N, g).

(ii) The proof is similar to the argument of the assertion (i). □

We next conclude this section by examining the algebraic connections among Menger algebras of rank n, Menger hyperalgebras of rank n, ternary Menger algebras of rank n and ternary Menger hyperalgebras of rank n.

Theorem 6. Any Menger hyperalgebra of rank n can form a ternary Menger hyperalgebra of rank n.

Proof. Let (M, g) be a Menger hyperalgebra of rank n. We define a $(2n+1)$-ary hyperoperation f on M by

$$x[\bar{y}\bar{z}] = x[\bar{y}][\bar{z}] \quad \text{for all } x, \bar{y}, \bar{z} \in M. \tag{7}$$

Let $u, \bar{v}, \bar{x}, \bar{y}, \bar{z} \in M$. Using the definition of f and the superassociativy of g, we obtain

$$\begin{aligned}
u[\bar{v}\bar{x}][\bar{y}\bar{z}] &= u[\bar{v}][\bar{x}][\bar{y}][\bar{z}] \\
&= u[v_1[\bar{x}]\dots v_n[\bar{x}]][\bar{y}][\bar{z}] \\
&= u[v_1[\bar{x}][\bar{y}]\dots v_n[\bar{x}][\bar{y}]][\bar{z}] \\
&= u[v_1[\bar{x}\bar{y}]\dots v_n[\bar{x}\bar{y}]][\bar{z}] \\
&= u[v_1[\bar{x}\bar{y}]\dots v_n[\bar{x}\bar{y}]\bar{z}], \\
u[\bar{v}x_1[\bar{y}\bar{z}]\dots x_n[\bar{y}\bar{z}]] &= u[\bar{v}][x_1[\bar{y}\bar{z}]\dots x_n[\bar{y}\bar{z}]] \\
&= u[\bar{v}][x_1[\bar{y}][\bar{z}]\dots x_n[\bar{y}][\bar{z}]] \\
&= u[\bar{v}][x_1[\bar{y}]\dots x_n[\bar{y}]][\bar{z}] \\
&= u[\bar{v}][\bar{x}][\bar{y}][\bar{z}] \\
&= u[\bar{v}\bar{x}][\bar{y}\bar{z}].
\end{aligned}$$

It implies that

$$u[\bar{v}\bar{x}][\bar{y}\bar{z}] = u[v_1[\bar{x}\bar{y}]\dots v_n[\bar{x}\bar{y}]\bar{z}] = u[\bar{v}x_1[\bar{y}\bar{z}]\dots x_n[\bar{y}\bar{z}]] \quad \text{for all } u, \bar{v}, \bar{x}, \bar{y}, \bar{z} \in M.$$

Thus, the $(2n+1)$-ary hyperoperation f satisfies the ternary superassociative law. Thus, (M, f) forms a ternary Menger hyperalgebra of rank n. □

However, there are some ternary Menger hyperalgebras of rank n that do not need to be reduced to Menger hyperalgebras of rank n. To illustrate the previous statement, the following related examples are given.

Example 7. (i) Let $\mathbb{Z}$ be the set of all integers. Define a ternary hyperoperation $\cdot$ on $\mathbb{Z}$ by

$$\cdot(x, y, z) = \{x - y + z\} \quad \text{for all } x, y, z \in \mathbb{Z},$$

where $-$ and $+$ are the usual subtraction and the usual addition, respectively. Then, $(\mathbb{Z}, \cdot)$ forms a ternary semihypergroup that is not reduced to a semihypegroup. Next, we define a $(2n+1)$-ary hyperoperation f on $\mathbb{Z}$ as follows:

$$x[\bar{y}\bar{z}] = \{\cdot(x, y_1, z_1)\} \quad \text{for all } x, \bar{y}, \bar{z} \in \mathbb{Z}.$$

Using the ternary associativity of $\cdot$ on $\mathbb{Z}$, the $(2n+1)$-ary hyperoperation f also satisfies the ternary superassociative law. Thus, $(\mathbb{Z}, f)$ is a ternary Menger hyperalgebra of rank n that is not reduced to a Menger hyperalgebra of rank n.

(ii) Let $M = \{1, -1, i, -i\}$, where i is the imaginary unit, i.e., $i^2 = -1$. Define a $(2n+1)$-ary hyperoperation f_p, where $p \in \{1, \ldots, n\}$, on M by

$$x[\bar{y}\bar{z}] = \{x \times y_p \times z_p\} \quad \text{for all } x, y, z \in M,$$

where $\times$ is the usual multiplication. Then, (M, f_p) forms a ternary Menger hyperalgebra of rank n for all $p \in \{1, \ldots, n\}$. It also does not reduce to a Menger hyperalgebra of rank n.

Proposition 2. Every ternary Menger algebra of rank n induced by a Menger algebra of rank n can be formed into a ternary Menger hyperalgebra of rank n induced by a Menger hyperalgebra of rank n.

Proof. Let $(M, \diamond)$ be a ternary Menger algebra of rank n induced by a Menger algebra (M, o) of rank n, where a $(2n+1)$-ary operation $\diamond$ is defined by

$$\diamond(x, \bar{y}, \bar{z}) = o(o(x, \bar{y}), \bar{z}) \quad \text{for all } x, \bar{y}, \bar{z} \in M.$$

By Theorem 1, we obtain that (M, g) is a Menger hyperalgebra of rank n under an $(n+1)$-ary hyperoperation g defined by

$$x[\bar{y}] = \{o(x, \bar{y})\} \quad \text{for all } x, \bar{y} \in M.$$

By Theorem 2, (M, f) forms a ternary Menger hyperalgebra of rank n together with a $(2n+1)$-ary hyperoperation f defined as in (6), i.e.,

$$x[\bar{y}\bar{z}] = \{\diamond(x, \bar{y}, \bar{z})\} \quad \text{for all } x, \bar{y}, \bar{z} \in M.$$

Then, for each $x, \bar{y}, \bar{z} \in M$, we have

$$x[\bar{y}\bar{z}] = \{\diamond(x, \bar{y}, \bar{z})\} = \{o(o(x, \bar{y}), \bar{z})\} = \bigcup_{a \in \{o(x,\bar{y})\}} \{o(a, \bar{z})\} = \bigcup_{a \in x[\bar{y}]} a[\bar{z}] = x[\bar{y}][\bar{z}].$$

So, we get $x[\bar{y}\bar{z}] = x[\bar{y}][\bar{z}]$ for all $x, \bar{y}, \bar{z} \in M$. It means that (M, f) forms a ternary Menger hyperalgebra of rank n induced by the Menger hyperalgebra (M, g) of rank n. □

Proposition 3. Let e be a scalar identity element of a Menger hyperalgebra (M, g) of rank n, i.e., $\{x\} = x[e^n] = e[x^n]$ for all $x \in M$. Then, the element is again a scalar identity element of a ternary Menger hyperalgebra (M, f) of rank n induced by (M, g).

Proof. Let (M, g) be a Menger hyperalgebra of rank n consisting of a scalar identity element e. By Theorem 6, we immediately obtain that (M, f) forms a ternary Menger hyperalgebra of rank n under a $(2n+1)$-ary hyperoperation f defined as (7), i.e., $x[\bar{y}\bar{z}] = x[\bar{y}][\bar{z}]$ for all $x, \bar{y}, \bar{z}$.

In fact, for each $x \in M$ we have $x[e^n e^n] = x[e^n][e^n] = x[e^n] = \{x\}$, $e[x^n e^n] = e[x^n][e^n] = x[e^n] = \{x\}$ and $e[e^n x^n] = e[e^n][x^n] = e[x^n] = \{x\}$.

Hence, the element e is again a scalar identity element of (M, f). □

3 Diagonal ternary semihypergroups

In this section, we use the $(2n+1)$-ary algebraic hyperstructures of ternary Menger hyperalgebras of rank n to construct new ternary semihypergroups called *diagonal ternary semihypergroups.* We then study the algebraic connections between ternary Menger hyperalgebras of rank n and its diagonal ternary semihypergroups.

Lemma 1. Let (M, f) be a ternary Menger hyperalgebra of rank n. Then, $(M, \bullet_f)$ forms a ternary semihypergroup with respect to a ternary hyperoperation $\bullet_f$ which is defined via a $(2n+1)$-ary hyperoperation f on M by

$$\bullet_f\,(x, y, z) = x[y^n z^n] \quad \text{for all } x, y, z \in M. \tag{8}$$

Proof. Let (M, f) be a ternary Menger hyperalgebra of rank n. Define a ternary hyperoperation $\bullet_f$ via a $(2n+1)$-ary hyperoperation f on M as (8). Next, we show that $\bullet_f$ satisfies the ternary associative law.

Using the ternary superassociativity of f on M, we obtain that

$$\begin{aligned}
\bullet_f(\bullet_f(u, v, x), y, z) &= u[v^n x^n][y^n z^n] \\
&= u[(v[x^n y^n])^n z^n] \\
&= \bullet_f(u, \bullet_f(v, x, y), z), \\
\bullet_f(u, v, \bullet_f(x, y, z)) &= u[v^n (x[y^n z^n])^n] \\
&= u[v^n x^n][y^n z^n] \\
&= \bullet_f(\bullet_f(u, v, x), y, z)
\end{aligned}$$

for all $u, v, x, y, z \in M$. It immediately implies that $\bullet_f$ satisfies the ternary associative law. Thus, $(M, \bullet_f)$ forms a ternary semihypergroup. □

As in Lemma 1, the important point is to know that each ternary Menger hyperalgebra (M, f) of rank n can be used to form a ternary semihypergroup $(M, \bullet_f)$ with a ternary hyperoperation $\bullet_f$ defined as (8). In this article, $(M, \bullet_f)$ is called a *diagonal ternary semihypergroup of a ternary Menger hyperalgebra* (M, f) *of rank n.*

According to the definition of the ternary hyperoperation $\bullet_f$, we immediately get the following result.

Proposition 4. Let (M, f) be a ternary Menger hyperalgebra of rank n. Then, a diagonal ternary semihypergroup of a ternary Menger hyperalgebra (M, f) of rank n is unique.

Proof. The proof is straightforward. □

Remark 1. Non-isomorphic ternary Menger hyperalgebras of rank n can have the same diagonal ternary semihypergroup.

The following example is provided to demonstrate that the statement in Remark 1 is true.

Example 8. Let $M = \{a, b, c, d, e\}$. Define a ternary hyperoperation $\cdot$ on M by

$$\cdot(x, y, z) = \{x, y, z\} \quad \text{for all } x, y, z \in M.$$

It is easily to show that $(M, \cdot)$ forms a ternary semihypergroup. Next, we define $(2n+1)$-ary hyperoperations f_1 and f_2 on M as follows:

$$f_1(x, \bar{y}, \bar{z}) = \cdot(x, y_1, y_1) \text{ and } f_2(x, \bar{y}, \bar{z}) = \cdot(x, y_2, y_2) \text{ for all } x, \bar{y}, \bar{z} \in M.$$

Then, we can see that (M, f_1) and (M, f_2) are ternary Menger hyperalgebras of rank n. Suppose that (M, f_1) and (M, f_2) are isomorphic via an isomorphism ϕ. Consider, $\phi(f_1(a, a, b^{n-1}, a^n)) = \phi(\cdot(a, a, a)) = \phi(a)$ and $f_2(\phi(a), \phi(a), (\phi(b))^{n-1}, (\phi(a))^n) = \cdot(\phi(a), \phi(b), \phi(b)) = \{\phi(a), \phi(b)\}$.

It implies that $\phi(f_1(a, a, b^{n-1}, a^n)) \neq f_2(\phi(a), \phi(a), (\phi(b))^{n-1}, (\phi(a))^n)$. This means that ϕ is not an isomorphism. Thus, (M, f_1) and (M, f_2) are non-isomorphic ternary Menger hyperalgebras of rank n. We also obtain that $(M, \bullet_{f_1})$ and $(M, \bullet_{f_2})$ are diagonal ternary semihypergroups of (M, f_1) and (M, f_2), respectively, where $\bullet_{f_1}$ and $\bullet_{f_2}$ are defined by

$$\bullet_{f_1}(x, y, z) = x[y^n z^n] \text{ and } \bullet_{f_2}(x, y, z) = x[y^n z^n] \quad \text{for all } x, y, z \in M.$$

Furthermore, we can also see that $(M, \bullet_{f_1})$ and $(M, \bullet_{f_2})$ are the same.

Proposition 5. Let (M, f) be a ternary Menger hyperalgebra of rank n containing a scalar right (resp. left, lateral) identity element e. If there exists $m \in M$ satisfying the property

$$e[e^n m^n] = \{e\} \text{ (resp. } m[e^n e^n] = \{e\}, e[m^n e^n] = \{e\}),$$

then m is also a scalar right (resp. left, lateral) identity element.

Proof. Let e be a scalar right identity element of a ternary Menger hyperalgebra (M, f) of rank n. Then, $x[e^n e^n] = \{x\}$ for all $x \in M$. Assume that $m \in M$ satisfies the property $e[e^n m^n] = \{e\}$. Consider, for each $x \in M$,

$$\begin{aligned} x[m^n m^n] &= x[(m[e^n e^n])^n m^n] \\ &= x[m^n (e[e^n m^n])^n] \\ &= x[e^n e^n][m^n (e[e^n m^n])^n] \\ &= x[e^n e^n][m^n e^n] \\ &= x[(e[e^n m^n])^n e^n] \\ &= x[e^n e^n] \\ &= \{x\}. \end{aligned}$$

This means that m is also a scalar right identity element. Similar to the previous argument, we can prove the rest. □

Corollary 1. Let (M, f) be a ternary Menger hyperalgebra of rank n containing a scalar identity element e. If there exists $m \in M$ satisfying the property

$$\{e\} = m[e^n e^n] = e[m^n e^n] = e[e^n m^n],$$

then m is also a scalar identity element.

Proposition 6. Let (M, f) be a ternary Menger hyperalgebra of rank n containing a scalar right (left) identity element e. If there exists $m \in M$ satisfying the property

$$e[m^n m^n] = \{e\} \ (\ m[m^n e^n] = \{e\}),$$

then m is also a scalar right (left) identity element.

Proof. Let (M, f) be a ternary Menger hyperalgebra of rank n containing a scalar right identity element e. So, we get $x[e^n e^n] = \{x\}$ for all $x \in M$. Assume that $m \in M$ satisfies the property $e[m^n m^n] = \{e\}$. Indeed, for each $x \in M$, we have

$$x[m^n m^n] = x[e^n e^n][m^n m^n] = x[e^n (e[m^n m^n])^n] = x[e^n e^n] = \{x\}.$$

This yields that m is a scalar right identity element. Analogously, we can show the rest. □

According to Proposition 5, Corollary 1, and Proposition 6, we show that a scalar (right, left, or lateral) identity element of a ternary Menger hyperalgebra of rank n need not be unique. In Example 7 (ii), we see that 1 and -1 are two distinct scalar identity elements of the ternary Menger hyperalgebra (M, f_p).

Proposition 7. Let (M, f) be a ternary Menger hyperalgebra of rank n containing a scalar left identity element e, and for each $x, y \in M$ there is $m \in M$ satisfying the property that $\{e\} = m[x^n y^n]$. Then, a diagonal ternary semihypergroup $(M, \bullet_f)$ of (M, f) is left cancellative, i.e.,

$$\bullet_f(x, y, u) = \bullet_f(x, y, v) \Longrightarrow u = v \quad \text{for all } u, v, x, y \in M.$$

Proof. Assume that $\bullet_f(x, y, u) = \bullet_f(x, y, v)$ for all $u, v, x, y \in M$. By the definition of $\bullet_f$, we have $x[y^n u^n] = x[y^n v^n]$. Consider,

$$\begin{aligned}\{u\} &= e[e^n u^n] \\ &= e[(m[x^n y^n])^n u^n] \\ &= e[m^n (x[y^n u^n])^n] \\ &= e[m^n (x[y^n v^n])^n] \\ &= e[(m[x^n y^n])^n v^n] \\ &= e[e^n v^n] \\ &= \{v\}.\end{aligned}$$

Hence, $\{u\} = \{v\}$ and then, $u = v$. It follows that $(M, \bullet_f)$ is left cancellative. □

Proposition 8. Let $(M, \bullet_f)$ be a diagonal ternary semihypergroup of a ternary Menger hyperalgebra (M, f) of rank n. Then,

$$\bullet_f(x, y, z)[\bar{u}\bar{v}] = \bullet_f(x, y, z[\bar{u}\bar{v}]) \quad \text{for all } \bar{u}, \bar{v}, x, y, z \in M.$$

Proof. The proposition is true by the ternary superassociativity of f on M. □

Previously, we presented interesting results showing several algebraic connections among ternary Menger hyperalgebras of rank n and their diagonal ternary semihypergroups. In fact, if (M, f) is a ternary Menger hyperalgebra of rank n, then its diagonal ternary semihypergroup can be constructed immediately. Conversely, we found that if a ternary semihypergroup satisfies certain significant conditions, it can act as a diagonal ternary semihypergroup of some ternary Menger hyperalgebras of rank n.

Theorem 7. Let $(M, \circ)$ be a ternary semihypergroup. If there exists an n-ary operation h defined on M that satisfies the following properties:
(i) $h(x^n) = x$ for all $x \in M$;
(ii) $\circ(h(\bar{x}), y, z) = h(\circ(x_1, y, z), \ldots, \circ(x_n, y, z))$ for all $\bar{x}, y, z \in M$,
then $(M, \circ)$ forms a diagonal ternary semihypergroup of some ternary Menger hyperalgebras of rank n.

Proof. Assume that there exists an n-ary operation h defined on M that satisfies the above properties (i) and (ii). We define a $(2n+1)$-ary hyperoperation on M by

$$x[\bar{y}\bar{z}] = \circ(x, h(\bar{y}), h(\bar{z})) \quad \text{for all } x, \bar{y}, \bar{z} \in M. \tag{9}$$

In fact, for every $u, \bar{v}, \bar{x}, \bar{y}, \bar{z} \in M$, we have

$$\begin{aligned}
u[\bar{v}\bar{x}][\bar{y}\bar{z}] &= \circ(\circ(u, h(\bar{v}), h(\bar{x})), h(\bar{y}), h(\bar{z})) \\
&= \circ(u, \circ(h(\bar{v}), h(\bar{x}), h(\bar{y})), h(\bar{z})) \\
&= \circ(u, h(\circ(v_1, h(\bar{x}), h(\bar{y})), \ldots, \circ(v_n, h(\bar{x}), h(\bar{y}))), h(\bar{z})) \\
&= \circ(u, h(v_1[\bar{x}\bar{y}], \ldots, v_n[\bar{x}\bar{y}]), h(\bar{z})) \\
&= u[v_1[\bar{x}\bar{y}] \ldots v_n[\bar{x}\bar{y}]\bar{z}], \\
u[\bar{v}x_1[\bar{y}\bar{z}] \ldots x_n[\bar{y}\bar{z}]] &= \circ(u, h(\bar{v}), h(x_1[\bar{y}\bar{z}], \ldots, x_n[\bar{y}\bar{z}])) \\
&= \circ(u, h(\bar{v}), h(\circ(x_1, h(\bar{y}), h(\bar{z})), \ldots, \circ(x_n, h(\bar{y}), h(\bar{z})))) \\
&= \circ(u, h(\bar{v}), \circ(h(\bar{x}), h(\bar{y}), h(\bar{z}))) \\
&= \circ(\circ(u, h(\bar{v}), h(\bar{x})), h(\bar{y}), h(\bar{z})) \\
&= u[\bar{v}\bar{x}][\bar{y}\bar{z}].
\end{aligned}$$

This implies that the $(2n+1)$-ary hyperoperation f satisfies the ternary superassociative law.

Finally, we assume that $(M, \bullet_f)$ is a diagonal ternary semihypergroup of ternary Menger hyperalgebra (M, f) of rank n. Then, we obtain $\bullet_f(x, y, z) = x[y^n z^n]$ for all $x, y, z \in M$.

In fact, for each $x, y, z \in M$, we obtain

$$\circ(x, y, z) = \circ(x, h(y^n), h(z^n)) = x[y^n z^n] = \bullet_f(x, y, z).$$

So, the ternary hyperoperation $\circ$ and $\bullet_f$ are the same, which means that $(M, \circ)$ forms a diagonal ternary semihypergroup of the ternary Menger hyperalgebra (M, f) of rank n. □

Theorem 8. Let $(M, \circ)$ be a ternary semihypergroup containing a scalar left identity element. Then, $(M, \circ)$ is a diagonal ternary semihypergroup of the ternary Menger hyperalgebra (M, f) of rank n containing a scalar left identity element if and only if there exists an n-ary operation h defined on M satisfying the given properties (i) and (ii) in Theorem 7.

Proof. ($\Longrightarrow$) Firstly, we assume that $(M, \circ)$ is a diagonal ternary semihypergroup of the ternary Menger hyperalgebra (M, f) of rank n containing a scalar left identity element e.

By the assumption, we have $\{x\} = e[e^n x^n]$ for all $x \in M$. We define an n-ary operation h on M by

$$h(\bar{x}) = e[e^n \bar{x}] \quad \text{for all } \bar{x} \in M.$$

Next, let $x, \bar{x}, y, z \in M$. Then, we have

$$\begin{aligned}
h(x^n) &= e[e^n x^n] \\
&= \{x\}, \\
\circ(h(\bar{x}), y, z) &= \circ(e[e^n \bar{x}], y, z) \\
&= e[e^n \bar{x}][y^n z^n] \\
&= e[e^n x_1[y^n z^n] \ldots x_n[y^n z^n]] \\
&= h(x_1[y^n z^n], \ldots, x_n[y^n z^n]) \\
&= h(\circ(x_1, y, z), \ldots, \circ(x_n, y, z)).
\end{aligned}$$

Therefore, the n-ary operation h satisfies the properties (i) and (ii) in Theorem 7.

$(\Longleftarrow)$ Assume that there exists an n-ary operation h defined on M satisfies the given properties (i) and (ii) in Theorem 7. By Theorem 7, $(M, \circ)$ forms a diagonal ternary semihypergroup of a ternary Menger hyperalgebra (M, f) of rank n, where a $(2n+1)$-ary hyperoperation f is given as (9), i.e.,

$$x[\bar{y}\bar{z}] = \circ(x, h(\bar{y}), h(\bar{z})) \quad \text{for all } x, \bar{y}, \bar{z} \in M.$$

Now, let $e \in M$ be a scalar left identity element in a ternary semihypergroup $(M, \circ)$. Then, we obtain $\circ(e, e, x) = \{x\}$ for all $x \in M$. Since $(M, \circ)$ is also a diagonal ternary semihypergroup of the ternary Menger hyperalgebra (M, f) of rank n, we obtain

$$e[e^n x^n] = \circ(e, e, x) = \{x\} \quad \text{for all } x \in M.$$

Then, the element $e \in M$ is again a scalar left identity element of the ternary Menger hyperalgebra (M, f) of rank n.

Therefore, the proof is completed. □

To illustrate Theorem 7 and Theorem 8, the following example is required.

Example 9. Let $\mathbb{N}$ be the set of all natural numbers. Then, $(\mathbb{N}, \circ)$ forms a ternary semihypergroup under a ternary hyperoperation $\circ$ defined by

$$\circ(x, y, z) = \{x\} \quad \text{for all } x, y, z \in \mathbb{N}.$$

Moreover, each element in $\mathbb{N}$ is a scalar identity element. We define an n-ary operation h on $\mathbb{N}$ by

$$h(\bar{x}) = \min\{x_1, \ldots, x_n\} \quad \text{for all } \bar{x} \in \mathbb{N}.$$

In fact, for each $x, \bar{x}, y, z \in \mathbb{N}$,

$$\begin{aligned} h(x^n) &= \min\{x, \ldots, x\} = x, \\ \circ(h(\bar{x}), y, z) &= h(\bar{x}) \\ &= \min\{x_1, \ldots, x_n\} \\ &= \min\{\circ(x_1, y, z), \ldots, \circ(x_n, y, z)\} \\ &= h(\circ(x_1, y, z), \ldots, \circ(x_1, y, z)). \end{aligned}$$

This implies that the n-ary operation h satisfies the properties (i) and (ii) in Theorem 7. By Theorem 7, the ternary semihypergroup $(M, \circ)$ forms as a diagonal ternary semihypergroup of a ternary Menger hyperalgebra (M, f) of rank n such that a $(2n+1)$-ary hyperoperation f is defined as (9).

Conclusion

In this article, we begin with the fact that the concept of Menger hyperalgebras of rank n, where n is a fixed integer, can be considered as a generalization of semihypergroups, but not a generalization of ternary semihypergroups. Based on the previous observation, an interesting research question arises: what is a natural generalization of ternary semihypergroup? Since this is an important question, in this article, we start by establishing the so-called *a ternary Menger hyperalgebra of rank n* using the concept of Menger hyperalgebras of rank n. In particular, it is a $(2n+1)$-ary algebraic hyperstructure that consists of a nonempty set together with a $(2n+1)$-ary hyperoperation satisfying the *ternary superassociative law.* In case $n = 1$, the ternary Menger hyperalgebras of rank n are immediately reduced to ternary semihypergroups. Thus, the concept of ternary Menger hyperalgebras of rank n can be considered as an appropriate generalization of ternary semihypergroups. Moreover, it differs from the former generalizations of ternary semihypergroups such as n-ary semihypergroups and ordered

n-ary semihypergroups. That is, an n-ary semihypergroup can be reduced to a ternary semihypergroup if $n = 3$.

In Section 2, we presented some basic algebraic results on ternary Menger hyperalgebras of rank n through the concepts of subhyperalgebra and homomorphism. Then, we also show the algebraic connections between Menger algebras of rank n, ternary Menger algebras of rank n, Menger hyperalgebras of rank n and ternary Menger hyperalgebras of rank n. We have also proven that any Menger hyperalgebra of rank n can be induced to a ternary Menger hyperalgebra of rank n, while there are some ternary Menger hyperalgebras of rank n that do not necessarily reduce to Menger hyperalgebras of rank n.

Based on the above results, we show some algebraic connections as shown in Figure 1.

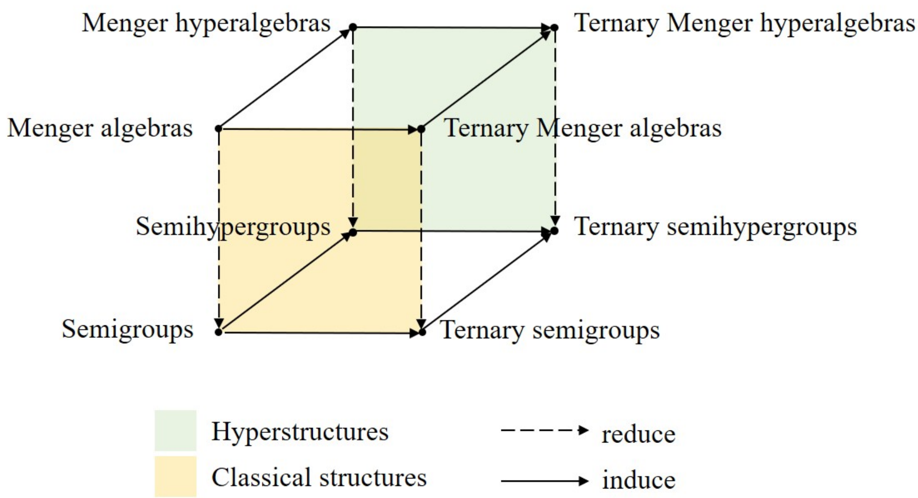

Figure 1. Algebraic connections among classical structures and hyperstructures

From Figure 1, we can briefly conclude that

- From a reductionist perspective, the concept of ternary Menger hyperalgebras of rank n can be regarded as a new generalization of ternary semihypergroups.
- From another perspective, the concept of ternary Menger hyperalgebras of rank n can also be considered as an extension of Menger hyperalgebras of rank n and ternary Menger algebras of rank n.

Finally in Section 3, we constructed the so-called *diagonal ternary semihypergroups* using base sets and $(2n + 1)$-ary hyperoperation of ternary Menger hyperalgebras of rank n. Then, some interesting properties related to the connections between ternary Menger hyperalgebras of rank n and their diagonal ternary semihypergroups were examined. In the final, we characterize that some ternary semihypergroups that satisfy the special conditions can be diagonal ternary semihypergroups of some ternary Menger hyperalgebras of rank n.

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Author Contributions

All authors participated in the revision of the manuscript and approved the final submission. All authors contributed equally to this work.

Conflict of Interest

The authors declare no conflict of interest.

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Analytical, numerical, and biomedical aspects of boundary value problems for third-order elliptic-type equations with singular coefficients

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Mathematical modeling of various real-world processes frequently leads to boundary value problems (BVPs) for third-order partial differential equations of mixed and composite types, which have no classical analogues in mathematical physics. Foundational studies by A.V. Bitsadze and M.S. Salakhitdinov first addressed well-posed boundary value problems for degenerate equations of third-order mixed and mixed-composite types. A key approach in their work involved representing the general solution of a composite-type equation as a sum of functions, which proved essential for operators constructed as products of permuted differential operators. Following these foundational contributions, the study of third-order partial differential equations involving Lavrentiev-Bitsadze, Gellerstedt, heat conduction, and string-type operators has been further advanced by both international and Uzbek mathematicians. Despite these developments, boundary value problems for third-order equations of parabolic-hyperbolic and elliptic-hyperbolic types with singular coefficients remain largely unexplored. In this article, we formulate and investigate boundary value problems for a third-order elliptic equation with a singular coefficient. The existence and uniqueness of classical solutions are rigorously proved. A new extremum principle for third-order equations is developed and applied to establish uniqueness. The existence of a solution is reduced to a singular integral equation of normal type, which is regularized using the classical Carleman-Vekua method, leading to an equivalent Fredholm equation of the second kind. The analytical framework is complemented by a numerical scheme that verifies the theoretical results and illustrates the qualitative behavior of solutions near the degenerate boundary. Furthermore, a numerical illustration is provided to demonstrate the stability and smoothness of the obtained solutions even in the presence of singular coefficients. Finally, the potential biomedical relevance of the model is discussed through its application to steady-state diffusion processes in tumor tissues.

Keywords: Analogue of the Dirichlet problem, representation of the general solution, third-order elliptic equation, extremum principle, regularization method, singular equation of normal type, finite-difference scheme, stability analysis, degenerate boundary, diffusion in tumor tissue

AMS Mathematics Subject Classification: 35J70, 35J25, 35J75, 65N12, 35B40, 92C50.

1 Introduction

The first fundamental studies for model equations of composite and mixed-composite types were conducted by A.V. Bitsadze and M.S. Salakhitdinov [1] in the early 1960s. Correct boundary value problems for third-order elliptic-hyperbolic and parabolic-hyperbolic type equations, where the principal part of the operator contains a derivative with respect to x or y, were first investigated in the works of A.V. Bitsadze, M.S. Salakhitdinov [1], T.D. Djuraev [2], and M.S. Salakhitdinov [3]. In these

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studies, the general solution of a mixed-composite type equation was represented as a sum of functions, a key idea for operators constructed as products of commutative differential operators. This direction was further developed in subsequent works devoted to various classes of third-order partial differential equations [4–6].

In works [7, 8] methods were proposed for solving third-order parabolic-hyperbolic and elliptic-hyperbolic equations by reducing them to inverse problems for second-order mixed-type equations with unknown right-hand sides. Numerical approaches to related boundary value problems were considered in [9]. It was also established that the coefficients of lower-order terms have a significant influence on the formulation and solvability of boundary value problems [2, 10]. Inverse problems for second-order mixed-type equations with unknown right-hand sides in various domains were examined in [11, 12]. Other types of problems for third-order mixed-composite equations were addressed in [13], while works [14, 15] considered boundary value problems for loaded parabolic-hyperbolic equations of third order in different domains.

Despite these developments, boundary value problems for third-order elliptic-type equations with singular coefficients have not been systematically studied so far.

In the present work, we study boundary value problems for a third-order elliptic-type equation with a singular coefficient. Theorems on the existence and uniqueness of classical solutions are proved based on the extremum principle and the theory of singular and Fredholm integral equations. An explicit Green's function is constructed using the method of double-layer potentials, and the integral representation of the solution is obtained.

In addition to the theoretical analysis, a numerical approach is developed to illustrate the behavior and stability of solutions near the singular boundary. The computational results confirm the analytical predictions and demonstrate the influence of the singular coefficient on the solution profile. Furthermore, the obtained results have potential applications in modeling steady-state diffusion and transport phenomena in heterogeneous biological media, including tumor tissues, where singular or degenerate coefficients naturally arise due to spatially variable diffusivity or irregular geometry.

Novelty and Contribution

In contrast to the well-known studies devoted to third-order mixed-type or elliptic equations with regular coefficients, this paper addresses boundary value problems for a third-order elliptic-type equation containing a singular coefficient. New formulations of Dirichlet- and Neumann-type problems are proposed for this class of equations. The corresponding Green's function is constructed for the first time, and the existence and uniqueness theorems for the classical solution are established using the extremum principle together with the theory of singular and Fredholm integral equations.

In addition, the paper complements the analytical framework with numerical simulations and an applied interpretation relevant to biomedical contexts, such as diffusion processes in tumor microenvironments. These results extend and generalize the classical works of Bitsadze and Salakhitdinov to a broader class of singular elliptic operators, providing both theoretical and applied contributions to the study of degenerate diffusion-type equations.

2 Problem statement

We consider the following third-order partial differential equation:

$$\frac{\partial}{\partial x}\left(y^m u_{xx} + u_{yy} + \frac{\beta_0}{y} u_y\right) = 0, \tag{1}$$

where $(x, y) \in \Omega$, $\Omega \subset \mathbb{R}^2$ is a bounded domain described below, and the parameters satisfy

$$m > 0, \quad -\frac{m}{2} < \beta_0 < 1. \tag{2}$$

Here and throughout the paper we use the notation $\beta = \frac{m+2\beta_0}{2(m+2)}$.

Note that the condition $m > 0$ ensures the degeneracy of ellipticity near the boundary $y = 0$. The inequality $-\frac{m}{2} < \beta_0 < 1$ is independent and characterizes the strength of the singular term $\frac{\beta_0}{y} u_y$.

Remark 1. Equation (1) is written in a divergence form with respect to the variable x. Integrating with respect to x formally leads to a second-order equation with an unknown function of y in the right-hand side.

However, the third-order formulation is essential since it allows:
(i) a natural representation of the solution in the form $u(x, y) = v(x, y) + \omega(y)$;
(ii) a correct formulation of boundary conditions involving derivatives along the curve σ;
(iii) the application of extremum principles adapted to higher-order operators.

Thus, equation (1) serves as the primary model, while an equivalent reduced second-order equation is introduced in the subsequent analysis.

The curve σ is a smooth Jordan curve located in the upper half-plane $y \geq 0$, connecting the points $A(-1, 0)$ and $B(1, 0)$.

We assume that the curve σ satisfies the following conditions:

1) The curve σ intersects any line $x = const$ at only one point.

2) The functions $x(s)$, $y(s)$, which provide the parametric equation of the curve σ, have continuous derivatives $x'(s)$, $y'(s)$, which do not simultaneously vanish, and have second derivatives satisfying a Hölder condition of order κ, $0 < \kappa < 1$, in the interval $0 \leq s \leq l$.

3) The curve ends with arbitrarily small arcs BB' and AA' of the normal contour σ_0 :

$$x^2 + \frac{4}{(m+2)^2} y^{m+2} = 1, \quad (y \geq 0)\,. \tag{3}$$

4) Each line $y = c,\ 0 \leq c < h$, intersects σ at two points, and the line $y = h$ has a single common point $N(0, h)$ (the point of tangency) with σ. We denote the parts of the arc σ AN and BN by σ_1 and σ_2 respectively.

Here Ω is the domain bounded by the curve σ and the segment $J = \{(x, 0) : -1 \leq x \leq 1\}$.

Problem AD (analogous to the Dirichlet problem). It is required to find a function $u(x, y)$, with the following properties:

1) $u(x, y) \in C\left(\bar{\Omega}\right) \cup C^1\left(\Omega \cup \sigma_1 \cup J\right)$, where $u_x(u_y)$ may approach infinity of an order less than $(1 - 2\beta)$ at points $A\,(-1, 0)$ and $B\,(1, 0)$;

2) *u(x,y)* is a twice continuously differentiable solution of equation (1) in region Ω;

3) $u(x, y)$ satisfies the conditions

$$u(x, y)|_{\sigma} = \varphi(s), \quad s \in [0, l]\,, u\,(x, +0) = \tau\,(x)\,, \quad (x, 0) \in \bar{J}, \tag{4}$$

$$\left.\frac{\partial u(x, y)}{\partial n}\right|_{\sigma_1} = g(s), \quad s \in \left[\frac{l}{2}, l\right], \tag{5}$$

where n is the inward normal, s is curve arc length σ, measured from the point $B\,(1, 0)$, l is curve length σ, $\varphi\,(s)\,,\ g\,(s)\,,\ \tau\,(x)$ are given functions, wherein

$$\varphi(l) = \tau(-1), \quad \varphi(0) = \tau(1), \tag{6}$$

$$\varphi(\xi(s), \eta(s)) = \eta^2(s)\varphi_1(\xi(s), \eta(s)), \quad \varphi_1(s) \in C\,[0, l]\,, \quad g(s) \in C^2\left(\frac{l}{2}, l\right), \tag{7}$$

$$\tau\,(x) \in C(\bar{J}) \cap C^2(J), \tag{8}$$

the function $g(s)$ may have singularities of order less than one at $s \to l$, but at $s \to \frac{l}{2}$ the function is bounded.

Problem AN (analogous to the Neumann problem). It is required to find a function $u(x, y)$, with the following properties:

1) $u(x, y) \in C\left(\bar{\Omega}\right) \cup C^1\left(\Omega \cup \sigma_1 \cup J\right)$, where $u_x(u_y)$ may approach infinity of an order less than $(1-2\beta)$ at points $A\,(-1,0)$ and $B\,(1,0)$;

2) *u(x,y)* is a twice continuously differentiable solution of equation (1) in region Ω;

3) $u(x, y)$ satisfies the conditions (4)–(6) and

$$u(x, y)|_\sigma = \varphi(s), \quad s \in [0, l]\,, \quad u_y(x, y)|_{y=+0} = \nu\,(x)\,, \quad (x, 0) \in J,$$

where $\nu(x)$ is a given function, where

$$\nu\,(x) \in C^2(J), \tag{9}$$

and function $\nu(x)$ may have singularities of order less than $1-2\beta$ at the endpoints of J.

Without loss of generality, we can assume

$$u(A) = u(B) = u'(A) = u'(B) = 0, \tag{10}$$

where derivatives are taken along the tangent to σ [16]. Due to these conditions, the function $\varphi(s)$ admits a suitable representation.

3 Investigation of the problem AD

Any regular solution of equation (1) in the region Ω can be represented in the form [16]:

$$u(x, y) = \upsilon(x, y) + \omega(y), \tag{11}$$

here $\upsilon\,(x, y)$ is a regular solution of the equation

$$L\upsilon \equiv y^m \upsilon_{xx} + \upsilon_{yy} + \frac{\beta_0}{y}\upsilon_y = 0, \quad (x, y) \in D, \tag{12}$$

and $\omega(y)$ is an arbitrary twice continuously differentiable function, and without loss of generality, we can assume that the function $\omega(y)$ satisfies the conditions

$$\omega(0) \;= \omega(h) = 0. \tag{13}$$

We will assume that everywhere along the arc σ_1 except at point $N(0, h)$ the following condition is satisfied (see condition (10))

$$\frac{dx}{dn} \neq 0. \tag{14}$$

Consequently, the problem AD reduces to the problem AD^* of determining in the region Ω a regular solution $\upsilon(x, y)$ of equation (12) that satisfies the conditions

$$\upsilon|_\sigma = \varphi(s) - \omega\left(y(s)\right), \quad s \in [0, l]\,, \upsilon|_{y=0} = \tau(x), \quad -1 \le x \le 1, \tag{15}$$

$$\left.\frac{\partial \upsilon(x, y)}{\partial n}\right|_{\sigma_1} = g(s) - \omega'(y(s))\frac{dy}{dn}, \quad s \in \left(\frac{l}{2}, l\right). \tag{16}$$

4 Uniqueness of solution to the problem AD

Theorem 1. If there exists a solution to problem AD, then it is unique if and only if conditions (13) and (14) are satisfied.

Proof. Let $\tau(x) \equiv \varphi(s) \equiv g(s) \equiv 0$, then by virtue of (15), in region Ω the regular solution $\upsilon(x,y)$ of equation (12) satisfies the conditions

$$\upsilon|_{\sigma} = -\omega(y(s)),\ s \in [0,l]\,,\ \upsilon|_{y=0} = 0,\ x \in [-1,1],\ \left.\frac{\partial \upsilon(x,y)}{\partial n}\right|_{\sigma_1} = -\omega'(y)\frac{dy}{dn},\ s \in \left(\frac{l}{2}, l\right). \tag{17}$$

A regular solution $\upsilon(x,y)$ of equation (12) inside the domain Ω cannot attain a positive maximum and a negative minimum [16, 17]. By the extremum principle for elliptic equations [18], it follows that the solution $\upsilon(x,y)$ of equation (12) in $\bar{\Omega}$ reaches on $\overline{AB}\bigcup\bar{\sigma}$ its positive maximum and negative minimum. From (13) and the second condition of (17), it follows that $\upsilon(x,y)$ does not attain a positive maximum and negative minimum at point $N(0,h)$ and on the segment AB. Therefore, considering (14), we prove that the function $\upsilon(x,y)$ cannot attain a positive maximum and negative minimum on the open arcs σ_1 and σ_2. Suppose the opposite, let $\upsilon(x,y)$ attain its positive maximum (negative minimum) at some interior point s_0 of the arc σ_1. Then, since $\upsilon(x,y)$ on σ_1 takes on the values of the function $\omega(y(s))$, the necessary condition for an extremum gives at this point

$$\left.\frac{\partial \omega(y(s))}{\partial s}\right|_{\sigma_1} = \omega'(y(s))\frac{dy}{ds} = -\omega'(y(s))\frac{dx}{dn}.$$

From here, based on (14), we obtain $\omega'(y(s_0)) = 0$. Then, due to the last condition (17) with regard to (13), at the considered point we have $\frac{\partial \upsilon}{\partial n} = 0$. But this last equality contradicts the known property of harmonic functions, namely, that at a boundary point of a positive maximum (negative minimum), it is $\frac{\partial \upsilon}{\partial n} < 0$ $\left(\frac{\partial \upsilon}{\partial n} > 0\right)$ [19]. Therefore, $\upsilon(x,y)$ cannot attain a positive maximum and negative minimum on the arcs σ_1. By virtue of the first condition in (17), we conclude that the function $z\,(x,y)$ cannot attain an extremum other than zero on the arc σ_2.

Hence, based on the extremum principle [4] and equalities (13), we conclude that $\upsilon(x,y) \equiv \omega(y) \equiv 0$ in $\bar{\Omega}$. Therefore, from (11), we have $u(x,y) \equiv 0,\ (x,y) \in \bar{\Omega}$. This proves the uniqueness of the solution to problem AD. □

5 Existence of a solution to the problem AD

Let the curve σ satisfy conditions 1)–4).

Definition 1. The Green's function for the Dirichlet problem for equation (12) with conditions (15) is called the function, $G(\xi,\eta;x,y)$ which:

1) is a regular solution of equation (12) everywhere in the domain Ω, except for the point (x,y);
2) satisfies the boundary condition

$$G\,(\xi,\eta;x,y)|_{\sigma\cup\bar{J}} = 0, \qquad (x,y) \in \Omega; \tag{18}$$

3) can be represented as

$$G(\xi,\eta;x,y) = g(\xi,\eta;x,y) + \vartheta(\xi,\eta;x,y). \tag{19}$$

Here

$$g(\xi,\eta;x,y) = k\left(\frac{4}{m+2}\right)^{4\beta-2}\frac{\left(1-\rho^2\right)^{1-2\beta}}{\left(r_1{}^2\right)^{\beta}}F\left(1-\beta,\ 1-\beta,\ 2-2\beta;\ 1-\rho^2\right),$$

where $F(a,b,c;z)$ denotes the Gauss hypergeometric function, $g(\xi,\eta;x,y)$ is the fundamental solution of equation (12), and $\vartheta(\xi,\eta;x,y)$ is a regular solution of equation (12) everywhere inside the region Ω, satisfying the conditions

$$\vartheta\left(\xi(s),\eta(s);x,y\right)\big|_{\sigma}=-g\left(\xi(s),\eta(s),x,y\right)\big|_{\sigma},\ (x,y)\in\Omega,\quad \vartheta\left(\xi,\eta;x,y\right)\big|_{\eta=0}=0, \tag{20}$$

$$\left.\begin{matrix} r^2 \\ {r_1}^2 \end{matrix}\right\}=(x-\xi)^2+\frac{4}{(m+2)^2}\left(y^{(m+2)/2}\mp\eta^{(m+2)/2}\right)^2,\quad \rho^2=\frac{r^2}{{r_1}^2},$$

$$k=\frac{1}{4\pi}\left(\frac{4}{m+2}\right)^{2-2\beta}\frac{\Gamma^2\left(1-\beta\right)}{\Gamma\left(2-2\beta\right)},\quad \beta=\frac{m+2\beta_0}{2(m+2)}.$$

According to a well-known formula [20], it follows that when $r\to 0$ and $\rho\to 0$ $(y>0)$ the function $g(\xi,\eta;x,y)$ has a logarithmic singularity [17] and satisfies condition $g(\xi,0;x,y)=0$ for all ξ.

Let us construct the Green's function using the double-layer potential

$$W(x,y)=\int\limits_0^l \mu(t)\ \eta^{\beta_0}(t)A_t\left[g\left(\xi(t),\eta(t);x,y\right)\right]dt, \tag{21}$$

where $\mu(t)\in C\left[0,l\right]$, and $A_t\left[g\left(\xi,\eta;x,y\right)\right]=\eta^m\frac{\partial g}{\partial\xi}\cdot\frac{d\eta}{dt}-\frac{\partial g}{\partial\eta}\cdot\frac{d\xi}{dt}$ is normal derivative [17].

Lemma 1. If the curve σ satisfies conditions 1)–2) and the density of $\mu(t)\in C\left[0,l\right]$, then the following formulas are valid

$$W_i(s)=-\frac{1}{2}\mu(s)+\int\limits_0^l \mu(t)K(s,t)dt, \tag{22}$$

$$W_e(s)=\frac{1}{2}\mu(s)+\int\limits_0^l \mu(t)K(s,t)dt. \tag{23}$$

Here $W_i(s)$ and $W_e(s)$ denote the interior and exterior boundary limits of the double-layer potential $W(x,y)$, respectively, and the kernel $K(s,t)$ is defined by

$$K(s,t)=\eta^{\beta_0}(t)A_t\big[g(\xi(t),\eta(t);x(s),\eta(s))\big],\quad (\xi(t),\eta(t))\in\sigma.$$

Proof. The proof of the lemma is carried out in the same way as in the works [17, 20]. □

Note that the potential $W(x,y)$ is a regular solution of equation (12) in each part of the upper half-plane that does not intersect either with curve σ, or with the x-axis. The double-layer potential $W(x,y)$ is defined for all points in the upper half-plane.

Let Ω_0 be a normal domain bounded by segment $[-1,1]$ of the x-axis and the normal curve (3), then the Green's function for the Dirichlet problem for equation (12) with conditions (15) is explicitly written as

$$G_0(\xi,\eta;x,y)=g(\xi,\eta;x,y)-R^{-2\beta}g(\xi,\eta;\bar{x},\bar{y}), \tag{24}$$

where $R^2=x^2+\frac{4}{(m+2)^2}y^{m+2}$, $\bar{x}=\frac{x}{R^2}$, $\bar{y}^{\frac{m+2}{2}}=y^{\frac{m+2}{2}}/R^2$.

The Green's function for an arbitrary domain, as in the case of the Gellerstedt equation [20], is constructed on the basis of the representation (24) using the double-layer potential (22) and satisfies the boundary condition (18). It can be represented as

$$G(\xi,\eta;x,y)=G_0(\xi,\eta;x,y)+H(\xi,\eta;x,y),$$

where

$$H(\xi,\eta;x,y)=\int\limits_0^l \lambda(t;\xi,\eta)\,G_0(\xi(t),\eta(t);x,y)dt.$$

Now we find the density $\lambda(t;x,y)$. According to representation (19), we seek the function $\vartheta\,(\xi,\eta;x,y)$ in the form of a double-layer potential (25):

$$\vartheta(\xi,\eta;x,y)=\int\limits_0^l \lambda(t;x,y)\eta^{\beta_0}(t)A_t\left[g\left(\xi(t),\eta(t);\xi,\eta\right)\right]dt. \tag{25}$$

Taking into account equalities (22), (23) and the boundary conditions (20), we obtain an integral equation for the density $\lambda\,(t,\,x,\,y)$:

$$\lambda(s;\,x,\,y)-2\int\limits_0^l K(s,t)\lambda\,(t;\,x,\,y)\,dt=2g\left(\xi(s),\,\eta(s);x,\,y\right). \tag{26}$$

The right-hand side of the equation is a continuous function of s(point $(x,\,y)\in\Omega$), kernel $K\,(s,t)$ has a weak singularity, and the number 2 is not its eigenvalue [17]. Therefore, equation (26) is solvable, and its continuous solution can be written as

$$\lambda\,(s;\,x,\,y)=2g\left(\xi(s),\eta(s);\,x,\,y\right)+4\int\limits_0^l R\,(s,t;2)g\left(\xi(t),\eta(t);\,x,\,y\right)dt, \tag{27}$$

where $R\,(s,t;2)$ the kernel resolvent $K\,(s,t)\,;\,(\xi(s),\,\eta(s))\in\sigma$.

Substituting (27) into (25), we find the function $\vartheta(\xi,\eta;x,y)$.

Theorem 2. If conditions (2), (3), (7)–(9) are satisfied, then in region Ω a solution to problem AD exists.

Proof. Using the Green's function in region Ω the solution to problem (15) for equation (12) can be represented as follows (see, (21)):

$$\upsilon(x,y)=\int\limits_{-1}^{1}\tau(\xi)\left[\eta^{\beta_0}(\xi)\frac{\partial G(\xi,\eta;x,y)}{\partial\eta}\right]_{\eta=0}d\xi+\int\limits_{\sigma}\left[\varphi(s)-\omega(\eta(s))\right]A_s\left[G(\xi(s),\,\eta(s);\,x,y)\right]ds, \tag{28}$$

where $\omega(\eta(s))$ is an unknown function to be determined.

I. Let the curve σ coincide with the normal contour σ_0, then we find the unknown function $\omega(y)$. Due to (25) and the form of the function $g(\xi,\eta;x,y)$ from (28), we find

$$\upsilon(x,y)=k(1-\beta_0)y^{1-\beta_0}\int\limits_{-1}^{1}\tau(t)\left\{\left[(x-t)^2+4y^{m+2}/(m+2)^2\,\right]^{\beta-1}-\right.$$

$$\left.-\left[(1-xt)^2+4t^2y^{m+2}/(m+2)^2\,\right]^{\beta-1}\right\}dt-k(1-\beta)(m+2)(1-R^2)y^{1-\beta_0}\times$$

$$\times\int\limits_{\sigma_0}\left[\varphi(s)-\omega(\eta(s))\right]\;\left(r_1^2\right)^{\beta-2}F(1-\beta,2-\beta,2-2\beta;1-\rho^2)ds. \tag{29}$$

Transitioning to polar coordinates $x = \cos\theta,\ y = ((m+2)\sin\theta/2\ )^{2/(m+2)}$ in formula (29), we have

$$\upsilon\left(R,\theta_0\right) = k\left(\frac{m+2}{2}R\sin\theta_0\right)^{1-2\beta}\int\limits_{-1}^{1}\tau(\xi)\left[\left(R^2-2R\xi\cos\theta_0+\xi^2\right)^{\beta-1}-\right.$$

$$\left.-\left(1-2R\xi\cos\theta_0 R^2\xi^2\right)^{\beta-1}\right]d\xi + k(1-\beta)(m+2)\left(\frac{m+2}{2}R\sin\theta_0\right)^{1-2\beta}\left(1-R^2\right)\int\limits_{0}^{\pi}\{\varphi\left(\theta\right)-$$

$$-\omega\left[\left(\frac{m+2}{2}R\sin\theta\right)^{1-2\beta}\right]\Bigg\}\left(r_1^2\right)^{\beta-2}F\left(1-\beta,2-\beta,2-2\beta,1-\rho^2\right)\sin\theta\,d\theta. \tag{30}$$

To satisfy condition (16), we first compute the derivatives of the function $\upsilon\left(R,\theta_0\right)$ defined by formula (30) with respect to R and θ_0, and then, after integrating the obtained expressions by parts, we take the limit as $R\to 1\ \left(\frac{\pi}{2}\leq\theta\leq\pi\right)$, obtaining

$$\gamma\left(\theta_0\right)\sin\theta_0 + \int\limits_{0}^{\pi}\gamma\left(\theta\right)\left[M_1\left(\theta_0,\theta\right)+M_2\left(\theta_0,\theta\right)\right]d\theta = f\left(\theta_0\right), \tag{31}$$

where

$$\gamma\left(\theta_0\right) = \omega'\left[\left((m+2)\sin\theta_0/2\ \right)^{1-2\beta}\right].$$

Due to conditions (7)–(9), we conclude that the kernel $M_1\left(\theta_0,\theta\right)$ at $\theta=\theta_0$ diverges to first-order infinity, i.e., it is singular, while $M_2\left(\theta_0,\theta\right)$ has a weak singularity and is a Fredholm kernel. $f(\theta_0)\in C^2\left(\frac{\pi}{2},\pi\right)$ and $f(\theta_0)$ are bounded at $\theta_0\to\ \frac{\pi}{2}$, but at $\theta_0\to\pi$ are unbounded.

In (31), we will split the integral from 0 to π into two parts: from 0 to $\frac{\pi}{2}$ and from $\frac{\pi}{2}$ to π. In the first part, we will substitute the variable according to the formula $\theta=\pi-\theta_1$, and then instead of θ_1 we will again write θ. Then, due to $\gamma\left(\theta\right)=\gamma\left(\pi-\theta\right)$ equation (32) can be written as

$$\gamma(\theta_0)\sin\theta_0 + \frac{\cos\theta_0}{2\pi}\int\limits_{\frac{\pi}{2}}^{\pi}\gamma(\theta)\left(ctg\frac{\theta-\theta_0}{2}-ctg\frac{\theta+\theta_0}{2}\right)d\theta +$$

$$+\int\limits_{\frac{\pi}{2}}^{\pi}M(\theta_0,\theta)\,\gamma(\theta)d\theta = f(\theta_0),\quad \theta_0\in\left(\frac{\pi}{2},\pi\right), \tag{32}$$

where the kernel $M(\theta_0,\theta)$ has a weak singularity [3].

Performing the substitution $t=e^{i\,\theta},\ t_0=e^{i\,\theta_0}$ in (32) taking into account

$$\frac{1}{2}\left(ctg\frac{\theta-\theta_0}{2}-ctg\frac{\theta+\theta_0}{2}\right)d\theta = \left(\frac{1}{t-t_0}-\frac{1}{1-t\,t_0}\right)dt + \frac{1+t_0}{1-t\,t_0}dt,$$

$$\sin\theta_0 = \frac{t_0^2-1}{2\,i\,t_0},\ \ \cos\theta_0 = \frac{t_0^2+1}{2t_0},\ \ \theta = \frac{1}{i}\ln t,\ \ \theta_0 = \frac{1}{i}\ln t_0,\ \ d\theta = \frac{1}{i}\cdot\frac{dt}{t},$$

we obtain

$$a(t_0)\gamma_1(t_0) + \frac{b(t_0)}{\pi i}\int\limits_{c_0}\gamma_1(t)\left(\frac{1}{t-t_0}-\frac{1}{1-t\,t_0}\right)dt + \int\limits_{c_0}N(t_0,t)\ \gamma_1(t)dt = f_1(t_0), \tag{33}$$

where c_0 is the contour of integration representing a quarter of a circle θ_0,

$$\gamma_1(t_0) = \gamma\left(\frac{1}{i}\ \ln t_0\right), \quad a(t_0) = 1 - t_0^2, \quad b(t_0) = 1 + t_0^2,$$

$$f_1(t_0) = -2i\, t_0 f\left(\frac{1}{i}\ lnt_0\right), \quad f_1(e^{i\,\theta_0}) = -2i\, e^{i\,\theta_0} f(\theta_0),$$

$$N\,(t_0, t) = \frac{(1+t_0)\left(1+t_0^2\right)}{\pi i\,(1 - t\, t_0)} - \frac{2t_0}{t} M\left(\frac{1}{i}\ln t_0;\ \frac{1}{i}\ln t\right).$$

Transitioning to the question of the solvability of the singular integral equation (33), first of all, we note that it is an equation of the normal type [18], i.e. $a^2(t_0) + b^2(t_0) = 2(1 + t_0^4) \neq 0$. Its index is zero in the class h [5], which is bounded at $t_0 \to i$, but at $t_0 \to -1$ unbounded.

The singular integral equation (33) will be reduced by the well-known Carleman–Vekua regularization method [18], to an equivalent second-kind Fredholm equation, and returning to the old variables, we have

$$\chi(\theta_0) + \int\limits_{\pi/2}^{\pi} \nu_1(\theta_0, \theta)\chi(\theta)d\theta = f_2(\theta_0), \quad \theta_0 \in \left(\frac{\pi}{2}, \pi\right). \tag{34}$$

From the class h and the properties of the Cauchy-type integral [18], we conclude that the right-hand side of equation (34) is bounded at $\theta_0 \to \pi/2$ and can diverge to infinity of order not exceeding ε_2 at $\theta_0 \to \pi$, while the kernel $N_1(\theta_0, \theta)$ has a weak singularity [2].

According to the theory of Fredholm integral equations [18] and from the uniqueness of the solution to problem AD (see Theorem 1), we conclude that the integral equation (34) is uniquely solvable in class $C^2\,(\pi/2\,;\pi)$, moreover $\gamma(\theta_0)$ may have a singularity of order less than ε_2 at $\theta_0 \to \pi$, and at $\theta_0 \to \pi/2$ it is bounded, and its solution is given by the formula:

$$\gamma(\theta_0) = f_2(\theta_0) - \int\limits_{\pi/2}^{\pi} \nu_1^*(\theta_0, \theta)\, f_2(\theta)d\theta, \quad \theta_0 \in \left(\frac{\pi}{2}, \pi\right), \tag{35}$$

here $N_1^*(\theta_0, \theta)$ is the kernel resolvent $N_1(\theta_0, \theta)$.

From (35) we find the unknown function $\omega(y)$. Then, the solution to equation (12) satisfying conditions (15) is determined by formula (29). From this and from the general representation of (11), we find the solution to the problem AD. This proves the existence of a solution to the problem AD in the case when σ coincides with σ_0.

II. The proof for a general curve σ follows the same scheme as above, with appropriate modifications related to the geometry of the boundary, and therefore is omitted for brevity. □

Theorem 3. If conditions (2), (3), (7), (20) are satisfied, then in region Ω there exists a unique regular solution to the problem AN.

Theorem 3 is proven using the extremum principle and the method of integral equations using the property of the Green's function.

6 Numerical Illustration

To verify the analytical results established in the previous sections and to illustrate the qualitative behavior of solutions, we consider the corresponding reduced problem for the function $v(x, y)$ governed by equation (12), which is obtained from the original third-order equation (1) via the representation of the solution. Numerical simulations not only confirm the correctness of the proposed analytical framework but also provide valuable insight into the behavior of solutions, especially near the degenerate boundary where analytical evaluation is challenging [21–23].

$$y^m u_{xx} + u_{yy} + \frac{\beta_0}{y} u_y = 0, \quad (x, y) \in \Omega = [-1, 1] \times [0, 1], \tag{36}$$

subject to the boundary conditions

$$u(-1, y) = 0, \quad u(1, y) = 0, \quad u(x, 0) = 0, \quad u(x, 1) = \sin(\pi x).$$

The boundary function $\sin(\pi x)$ at $y = 1$ ensures smoothness and compatibility with the homogeneous conditions at $x = \pm 1$, producing a nontrivial, well-behaved solution suitable for numerical analysis.

Finite-Difference Scheme and Regularization

Regularization techniques for degenerate or singular diffusion models are widely used in numerical analysis [24, 25], while analytical estimates are studied in [26]. A discrete analogue of (36) was constructed using second-order central finite differences in both spatial directions:

$$y_j^m \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h_x^2} + \frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{h_y^2} + \frac{\beta_0}{y_j} \frac{u_{i,j+1} - u_{i,j-1}}{2h_y} = 0,$$

for $i = 1, \ldots, N_x - 1$, $j = 1, \ldots, N_y - 1$, with grid steps $h_x = h_y = 0.05$.

The term $\frac{\beta_0}{y} u_y$ contains a singularity at $y = 0$. To avoid numerical instability, a regularization was introduced by replacing y with $\max(y_j, \varepsilon)$, where $\varepsilon = 10^{-4}$. This modification preserves the accuracy of the approximation while preventing division by zero, and is fully consistent with the boundary condition $u(x, 0) = 0$.

The resulting sparse linear algebraic system was symmetric and diagonally dominant, allowing the use of the conjugate gradient method for efficient numerical solution.

Mesh Convergence and Stability Analysis

To examine the convergence of the finite-difference scheme, computations were performed on uniform grids with $h_x = h_y = \{0.1, 0.05, 0.025\}$. The discrete L_2-error was estimated by comparing solutions on successive meshes. The error decreased by approximately a factor of four when the grid step was halved, confirming the second-order accuracy of the scheme. No oscillations or instabilities were observed even near the degenerate line $y = 0$, demonstrating the numerical stability of the regularized formulation.

Effect of Parameters m and β_0

The parameters m and β_0 control the degree of elliptic degeneracy and the strength of the singular term, respectively. Table 1 shows representative values of the computed maximum and minimum of the solution for several parameter sets.

Table 1

Influence of parameters m and β_0 on the solution profile

(m, β_0)	$\max u(x, y)$	$\min u(x, y)$	Qualitative behavior
(1.0, 0.2)	0.42	-0.38	smooth, symmetric profile
(2.0, 0.5)	0.55	-0.53	sharper gradients near $y = 0$
(3.0, 0.8)	0.67	-0.61	strong localization near upper boundary

An increase in either m or β_0 enhances the anisotropy of diffusion and causes the solution to flatten near $y = 0$, illustrating the physical effect of singular diffusion suppression in heterogeneous media.

Numerical Results

Table 2 provides representative numerical values of the computed solution for $m = 2$, $\beta_0 = 0.5$.

Table 2

Selected numerical values of $u(x, y)$ for $m = 2$, $\beta_0 = 0.5$

$y \backslash x$	-1.0	-0.5	0.0	0.5	1.0
0.0	0.000	0.000	0.000	0.000	0.000
0.2	-0.182	-0.094	0.000	0.108	0.194
0.4	-0.361	-0.189	0.000	0.215	0.381
0.6	-0.533	-0.279	0.000	0.319	0.546
0.8	-0.712	-0.366	0.000	0.420	0.702
1.0	0.000	-1.000	0.000	1.000	0.000

Figures 1 and 2 show the resulting solution surfaces and contour maps, revealing the smooth and symmetric nature of $u(x, y)$.

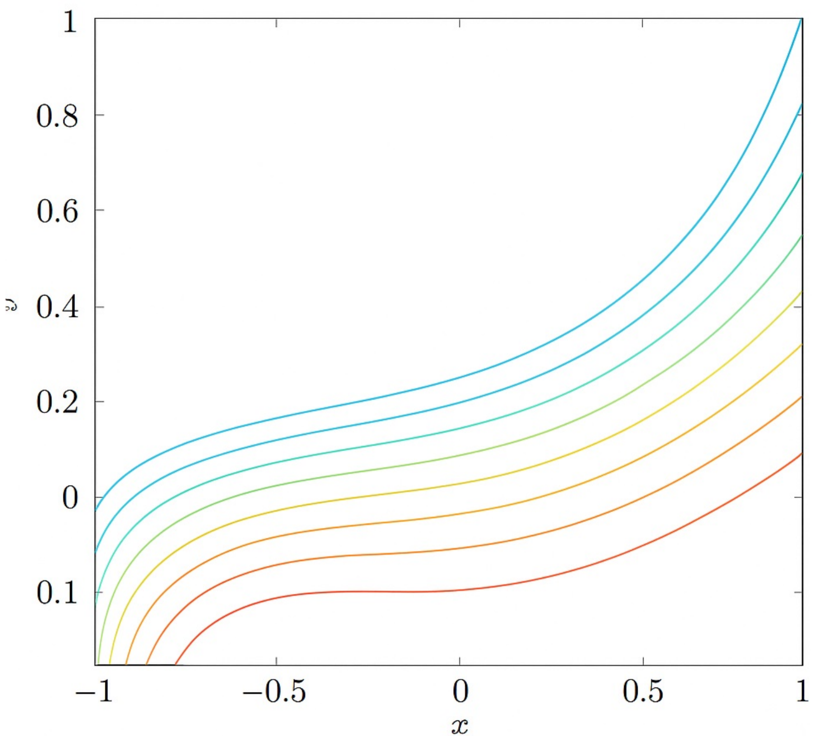

Figure 1. Contour plot of the numerical solution $u(x, y)$ for $m = 2$, $\beta_0 = 0.5$

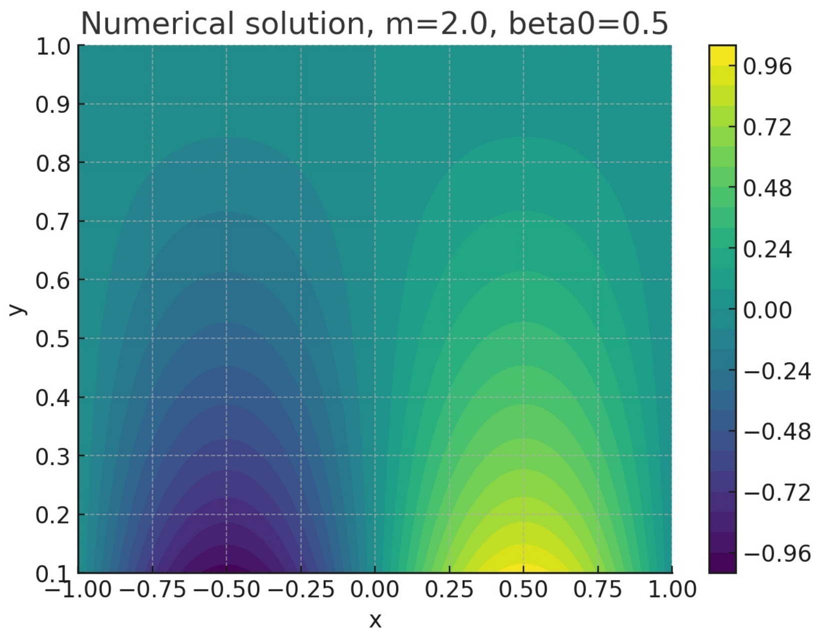

Figure 2. Color map of the numerical solution $u(x, y)$ for $m = 2$, $\beta_0 = 0.5$

Discussion and Interpretation

The numerical simulations confirm the theoretical predictions on the stability and smoothness of the classical solution, even in the presence of a singular coefficient. The results also demonstrate that the singularity primarily influences the diffusion rate in the y-direction, suppressing the amplitude of $u(x, y)$ near the degenerate line.

From a physical perspective, this effect is analogous to diffusion slow-down in heterogeneous biological media with variable permeability. Such models are relevant in biomedical contexts, including tumor growth and drug transport, where nonuniform tissue properties naturally give rise to singular or degenerate diffusion coefficients.

Conclusion of Numerical Analysis

The extended numerical analysis demonstrates that:
- the finite-difference scheme is second-order accurate and stable under mild regularization;
- the parameters m and β_0 significantly affect the localization and anisotropy of the solution;
- the computational results fully support the analytical existence and uniqueness theorems established earlier.

Hence, the combination of theoretical and numerical analyses provides a consistent and comprehensive understanding of third-order elliptic-type equations with singular coefficients, establishing a foundation for further research on their applications in complex physical and biomedical systems.

Applications in Oncology

Boundary value problems for degenerate elliptic-type equations with singular coefficients have recently gained considerable attention in the modeling of biological processes such as tumor growth, nutrient diffusion, and drug transport in cancerous tissues [27–29].

In this biomedical context, the governing equation

$$y^m u_{xx} + u_{yy} + \frac{\beta_0}{y} u_y = 0$$

can be interpreted as a steady-state diffusion model describing the spatial concentration $u(x, y)$ of oxygen, nutrients, or therapeutic agents within a heterogeneous tumor domain.

The coefficient y^m characterizes anisotropic or spatially varying diffusivity, which decreases toward necrotic regions ($y \to 0$), while the singular term $\frac{\beta_0}{y} u_y$ accounts for enhanced gradients and transport resistance near the tumor core or vascular interfaces. Such singular terms naturally arise in multilayer diffusion models, where permeability changes sharply across tumor boundaries, or in reduced radial formulations of spherical tumor growth, where the coordinate y represents the distance from the tumor center. The singular point $y = 0$ corresponds to a low-activity necrotic zone, where both diffusion and metabolic processes are significantly reduced.

Biophysical interpretation of parameters. The exponent m defines the degree of spatial inhomogeneity of diffusion, reflecting the structural heterogeneity of tumor tissues ($m \in [1, 3]$ for dense or fibrotic tumors). The parameter β_0 quantifies transport resistance or gradient intensity near the tumor center; larger β_0 values correspond to stronger attenuation of concentration fluxes in poorly vascularized regions.

Connection with numerical results. The numerical experiments presented in Section 6 demonstrated that for typical biological parameters ($m = 2$, $\beta_0 = 0.5$), the computed solutions preserve stability and smoothness even in the presence of singularities. This confirms that the proposed model adequately captures realistic diffusion behavior in heterogeneous media, where diffusion slows down but remains continuous as it approaches necrotic or impermeable regions.

Practical relevance. The analytical methods developed in this paper based on the Green's function construction and the degenerate double-layer potential enable accurate modeling of concentration fields in domains with irregular boundaries and spatially varying diffusivity. Such formulations are particularly relevant to:

- simulating oxygen and nutrient distribution in avascular tumor spheroids;
- analyzing the penetration of chemotherapeutic agents through layered tumor tissues with variable permeability;
- investigating steady-state profiles of diffusive signaling molecules influencing tumor-host interactions and microenvironmental feedbacks.

Thus, the combined analytical and numerical framework provides a rigorous mathematical foundation for describing stationary diffusion and transport processes in tumor tissues with spatial heterogeneity. These results contribute to the development of realistic and interpretable cancer models that can later be extended by incorporating reaction-diffusion or proliferation mechanisms to capture tumor evolution dynamics.

Conclusion

In this study, we examined boundary value problems for a class of degenerate elliptic-type equations with singular coefficients, where degeneracy occurs along a portion of the boundary according to a power-type law. A constructive analytical approach was proposed for obtaining the Green's function using the theory of degenerate double-layer potentials, guaranteeing the existence and uniqueness of classical solutions. The resulting integral representation explicitly reflects both the boundary geometry and the degeneracy structure of the problem.

Compared with standard elliptic models [30–32], the inclusion of a singular coefficient introduces essential mathematical challenges: the loss of uniform ellipticity near the degenerate boundary requires weighted functional formulations and special handling of singular kernels. The present approach extends previous studies on degenerate and related elliptic operators [33,34] by accommodating irregular and biologically motivated geometries, which enhances its practical applicability.

A distinctive aspect of this work lies in the introduction of a *degenerate double-layer potential*, whose analytical properties make it particularly suitable for modeling diffusion phenomena in heterogeneous biological media. Such structures are characteristic of tumor-host systems, where diffusivity can vanish or vary drastically near necrotic or vascular regions [35, 36]. Both analytical and numerical results confirm that the developed formulation preserves stability and smoothness even in the presence of singular coefficients, consistent with physical and biological expectations.

Beyond biomedical applications, the proposed framework can be extended to other applied fields such as porous media flow, heat transfer in nonuniform materials, and population dynamics, where similar singularities and degeneracies naturally occur.

Future research will focus on refining numerical solvers for the associated integral equations, incorporating nonlocal and time-fractional effects, and validating the model through comparison with experimental or clinical data. Overall, the developed theoretical and computational methodology provides a solid mathematical basis for analyzing degenerate diffusion processes in complex and heterogeneous environments.

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Author Contributions

D.B. Eshmamatova: Conceptualization, supervision, and analytical methodology.

N.K. Ochilova: Formal analysis, computations, and manuscript preparation.

All authors reviewed and approved the final manuscript.

Conflicts of interest

The authors declare no conflict of interest.

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Cauchy problem for an essentially loaded fractional diffusion equation

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In this paper, we solve the Cauchy problem for a loaded fractional diffusion equation in an infinite strip. The loaded term is defined as the trace of the fractional derivative of the desired solution on a continuous curve lying inside the domain. We consider all three cases of possible distribution of the order of differentiation in the loaded term (μ) and the order of the time-fractional derivative in the principal differential part of the equation (α). In the first case considered ($\alpha > \mu$), the problem under study is reduced to an integral equation. In the second case ($\alpha = \mu$), we obtain a functional equation. In the third case ($\alpha < \mu$), we are dealing with a differential equation. We show that the condition $\alpha > \mu$ ensures the unique solvability of the problem under consideration. In the case of an essentially loaded equation ($\alpha \leq \mu$), the problem may lose both uniqueness and solvability. In particular, it is shown that if $\alpha < \mu$, then the problem under consideration ceases to be uniquely solvable, and the corresponding homogeneous problem has infinitely many nontrivial solutions. Moreover, in this case, the solvability requires additional conditions that narrow the set of admissible input data.

Keywords: loaded differential equation, Cauchy problem, essentially loaded equation, fractional diffusion equation, non-uniqueness, moving load, Wright function, Mittag-Leffler function.

2020 Mathematics Subject Classification: 45D05, 35K20.

Introduction

Consider the equation

$$\left(D_{0y}^{\alpha} - \frac{\partial^2}{\partial x^2}\right) u(x,y) = \lambda \left[D_{0y}^{\mu} u(x,y)\right]_{x=z(y)}, \tag{1}$$

where D_{0y}^{σ} denotes the Riemann–Liouville fractional derivative (integral) of order σ with respect to y with origin at the point $y = 0$ [1]; $\lambda \in \mathbb{R}$ and $z(y)$ is a given continuous function, $z : (0,T) \to \mathbb{R}$. Here we assume that $\alpha \in (0,1]$ and $\mu \in \mathbb{R}$.

Equation (1) belongs to the class of loaded differential equations [2–4]. Loaded equations are an important and actively developing section of the modern theory of differential equations [5, 6]. Boundary value problems for loaded equations are considered for equations of parabolic [7], hyperbolic [8–10] type, as well as integro-differential equations [11] and equations of mixed type [12]. Moreover, loaded equations arise in the theory of inverse problems [13], in control problems [14], in numerical methods [15], in modeling [16], etc.

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The main part of equation (1) is the fractional diffusion operator [17–19]. The loaded term is given in the form of a trace of the derivative of the desired solution on line $x = z(y)$. Loads of this type are usually called moving loads. Another peculiarity of this equation is that μ can be greater than α, that is, the order of differentiation in the loaded term can be greater than the order of the fractional derivative in the principal differential part of the equation. It turns out that this feature affects the unique solvability of problems for the equation (1). In particular, the uniqueness of solution may be violated, and the parameter λ may play the role of a spectral parameter.

This effect of an essentially loaded term was discovered in [20–22]. These works considered parabolic equations with loads differentiated with respect to spatial variables. It has been shown that in problems for equations with a load of this type, a spectrum appears with respect to the coefficient of the load. Among the works close to the present work, we also cite articles [23–25], in which various issues of solvability of problems for loaded heat and fractional diffusion equations were considered. In this paper, we consider the loaded term in the form of a fractional derivative with respect to the time variable. We solve the Cauchy problem for the equation (1) in all three cases of possible distribution of α and μ. We show that the condition $\alpha > \mu$ ensures the unique solvability of the problem under consideration (Theorem 1). In the case of essentially loaded equation ($\alpha \leq \mu$), the problem may lose uniqueness or solvability. If $\alpha = \mu$, then the problem ceases to be solvable for $\lambda = 1$ (Theorem 2). When $\alpha < \mu$, the problem loses uniqueness of solution (Section 7). Moreover, in this case, the solvability requires additional conditions that narrow the set of admissible input data of the problem (Theorem 3, Remark 1).

1 Problem statement

For a positive σ, the Riemann–Liouville fractional derivative of order σ with origin at $y = 0$ is defined by

$$D_{0y}^{\sigma} g(y) := \frac{\partial^n}{\partial y^n} D_{0y}^{\sigma-n} g(y) \qquad (n-1 < \sigma \leq n, \quad n \in \mathbb{N}),$$

where

$$D_{0y}^{-\gamma} g(y) := \frac{1}{\Gamma(\gamma)} \int_0^y g(t)(y-t)^{\gamma-1} dt \qquad (\gamma > 0) \tag{2}$$

is the Riemann–Liouville fractional integral of order γ; it is also assumed that $D_{0y}^{0} g(y) := g(y)$.

In what follows we use the notations

$$\Omega := \{(x, y) : x \in \mathbb{R},\ y \in (0, T)\} = \mathbb{R} \times (0, T)$$

and

$$\Omega_0 := \{(x, y) : x \in \mathbb{R},\ y \in [0, T)\} = \mathbb{R} \times [0, T).$$

As usual, $AC[0, T)$ stands for the space of absolutely continuous functions on the segment $[0, c]$ for any $c \in (0, T)$.

A function $u(x, y)$ is called a *regular solution* of the equation (1) in Ω if $y^{1-\varepsilon} u(x, y) \in C(\Omega_0)$ for some $\varepsilon > 0$; $D_{0y}^{\alpha-1} u(x, y)$ belongs to $AC[0, T)$ as a function of y for every fixed $x \in \mathbb{R}$; $u(x, y)$ is twice continuously differentiable with respect to $x \in \mathbb{R}$ for every $y \in (0, T)$; and $u(x, y)$ satisfies the equation (1) for all $(x, y) \in \Omega$.

The Cauchy problem for the equation (1) is formulated as follows: *find a regular solution of the equation* (1) *in* Ω *satisfying the initial condition*

$$\lim_{y \to 0} D_{0y}^{\alpha-1} u(x, y) = \tau(x), \qquad x \in \mathbb{R}. \tag{3}$$

2 Auxiliary statements

Consider the function

$$w_\delta(x,y) := \frac{y^{\delta-1}}{2}\phi\left(-\beta,\delta;-\frac{|x|}{y^\beta}\right), \tag{4}$$

where

$$\phi\,(a,b;\,z) := \sum_{k=0}^{\infty}\frac{z^k}{k!\,\Gamma(ak+b)} \qquad (a>-1)$$

is the Wright function [26, 27]. Also, here and in what follows, β means $\frac{\alpha}{2}$, i.e.

$$\beta := \frac{\alpha}{2}. \tag{5}$$

It is known that [27]

$$\phi(-\beta,\delta;-t) = C\,t^{\frac{1-2\delta}{2(1-\beta)}}\exp\left(-\beta^{\frac{\beta}{1-\beta}}(1-\beta)t^{\frac{1}{1-\beta}}\right)\left[1+O\left(t^{-\frac{1}{1-\beta}}\right)\right] \quad (t\to\infty,\quad 0<\beta<1), \tag{6}$$

$$\frac{d}{dx}\,\phi(-\beta,\,\delta;\,x) = \phi(-\beta,\,\delta-\beta;\,x), \tag{7}$$

$$D^\nu_{0y}\left[y^{\delta-1}\phi\left(-\beta,\,\delta;\,-\frac{c}{y^\beta}\right)\right] = y^{\delta-\nu-1}\phi\left(-\beta,\,\delta-\nu;\,-\frac{c}{y^\beta}\right) \quad (c>0), \tag{8}$$

and

$$\int_0^\infty \phi\,(-\beta,\,\delta;\,-x)\;dx = \frac{1}{\Gamma(\beta+\delta)}. \tag{9}$$

The asymptotic expansion (6) gives that

$$\left|y^{\delta-1}\phi\left(-\beta,\,\delta;\,-\frac{x}{y^\beta}\right)\right| \le C\,x^{-\theta}y^{\beta\theta+\delta-1}, \tag{10}$$

where

$$\theta \ge \begin{cases} 0, & (-\delta)\notin\mathbb{N}\cup\{0\},\\ -1, & (-\delta)\in\mathbb{N}\cup\{0\}.\end{cases}$$

Here and in what follows, the letter C denotes positive constants, which are assumed to be different in different cases. When necessary, the parameters on which they may depend will be indicated in parentheses: $C = C(\alpha,\beta,...)$.

The formulas (7), (8), and (9) yield that

$$D^\nu_{0y}w_\delta(x,y) = w_{\delta-\nu}(x,y), \qquad \lim_{\varepsilon\to0+}\left[\frac{\partial}{\partial x}w_\delta(x-s,y)\right]_{s=x-\varepsilon}^{s=x+\varepsilon} = \frac{y^{\delta-\beta-1}}{\Gamma(\delta-\beta)},$$

$$\left(D^\alpha_{0y}-\frac{\partial^2}{\partial x^2}\right)w_\delta(x-s,y)=0, \quad \left(D^\alpha_{0y}-\frac{\partial^2}{\partial s^2}\right)w_\delta(x-s,y)=0 \quad (x\ne s),$$

and

$$\int_{-\infty}^{\infty} w_\delta(x-s,y)ds = \frac{y^{\delta+\beta-1}}{\Gamma(\mu+\beta)}. \tag{11}$$

In what follows, we will need the solution of the Cauchy problem for the fractional diffusion equation in a particular case. Now we recall it.

From now on, by $\mathbf{T}_{\alpha,\omega}$ we will denote the set of continuous functions that grow no faster than $\exp\left(\omega x^{\frac{2}{2-\alpha}}\right)$ as $|x| \to \infty$ for given α and ω, i.e.

$$\mathbf{T}_{\alpha,\omega} := \left\{ g(x) \in C(\mathbb{R}) : \lim_{|x|\to\infty} g(x) \exp\left(-\omega |x|^{\frac{2}{2-\alpha}}\right) = 0 \right\}.$$

As usual, $L_{\text{loc}}(J)$ denotes a set of locally integrable functions on J, that is the set of functions that are integrable on any compact subset of J. In particular,

$$L_{\text{loc}}[0,T) := \{g(x) \in L(0,c),\ \forall c \in (0,T)\}.$$

Lemma 1. Let $v(y) \in L_{\text{loc}}[0,T)$ and $\tau(x) \in \mathbf{T}_{\alpha,\omega}$ for some $\omega < (1-\beta)\left(\frac{\beta}{T}\right)^{\frac{\beta}{1-\beta}}$. Then the function

$$u(x,y) = D_{0y}^{-\alpha} v(y) + \int_{-\infty}^{\infty} \tau(s)\, w_\beta(x-s,y)\, ds \tag{12}$$

is a regular solution of the equation

$$\left(D_{0y}^{\alpha} - \frac{\partial^2}{\partial x^2} \right) u(x,y) = v(y), \tag{13}$$

and satisfies the condition (3).

Moreover, the problem (13) and (3) has at most one solution in the class of functions satisfying the condition

$$\sup_{y\in(0,T)} y^{1-\alpha} u(x,y) \in \mathbf{T}_{\alpha,\rho} \tag{14}$$

for some $\rho > 0$.

Proof. It follows from [28, Theorem 2] that a regular solution of problem (13) and (3) has the form

$$u(x,y) = \int_{-\infty}^{\infty}\int_0^y v(t)\, w_\beta(x-s,y)\, dt\, ds + \int_{-\infty}^{\infty} \tau(s)\, w_\beta(x-s,y)\, ds. \tag{15}$$

Given (2), (5), and (11), the first term on the right-hand side of (15) can be written as follows:

$$\int_{-\infty}^{\infty}\int_0^y v(t)\, w_\beta(x-s,y)\, dt\, ds = \int_0^y v(t) \int_{-\infty}^{\infty} w_\beta(x-s,y)\, ds\, dt = \int_0^y v(t) \frac{(y-s)^{\alpha-1}}{\Gamma(\alpha)}\, dt = D_{0y}^{-\alpha} v(y).$$

This gives (12) and proves, in particular, the uniqueness of the problem.

Further, a direct verification shows that the conditions of the lemma guarantee that the function (12) is a solution of the equation (13) and satisfies the initial condition (3). □

Consider the operator H^δ, which acts on the function $g(x) \in \mathbf{T}_{\alpha,\omega}$ as follows:

$$\left(H^\delta g\right)(x,y) := \int_{-\infty}^{\infty} g(s)\, w_\delta(x-s,y)\, ds. \tag{16}$$

Lemma 2. Let $g(x) \in \mathbf{T}_{\alpha,\omega}$ for some $\omega < (1-\beta)\left(\frac{\beta}{T}\right)^{\frac{\beta}{1-\beta}}$. Then

$$\left(H^\delta g\right)(x,y) \in C^\infty(\Omega), \tag{17}$$

$$y^{1-\beta-\delta}\left(H^{\delta}g\right)(x,y)\in C(\Omega_0) \tag{18}$$

and

$$D_{0y}^{\sigma}\left(H^{\delta}g\right)=\left(H^{\delta+\sigma}g\right)(x,y) \tag{19}$$

for $\delta>-\beta$ if $\sigma\notin\mathbb{N}$, and for $\delta\in\mathbb{R}$ if $\sigma\in\mathbb{N}$.

Proof. The inclusion (17) follows from the formulas (6), (7) and (8).

Next, by (4) we can write

$$\left(H^{\delta}g\right)(x,y)=y^{\delta-1}\int_{-\infty}^{\infty}g(s)\,\phi\left(-\beta,\delta;\,-\frac{|x-s|}{y^{\beta}}\right)ds=y^{\delta-1}\int_{-\infty}^{\infty}g(x+s)\,\phi\left(-\beta,\delta;\,-\frac{|s|}{y^{\beta}}\right)ds=$$

$$=y^{\delta+\beta-1}\int_{-\infty}^{\infty}g(x+y^{\beta}s)\,\phi\left(-\beta,\delta;\,-|s|\right)ds.$$

Given (6), this proves (18).

Taking into account (6) and (10), by (8) we get (19). □

3 Reduction to an integro-differential equation

Let $u(x,y)$ be a regular solution of the problem (1) and (3) that satisfies (14) and let $v(y)$ be the loaded term in (1), i.e.

$$v(y)=\left[D_{0y}^{\mu}u(x,y)\right]_{x=z(y)}.$$

We will assume that $\tau(x)$ and $v(y)$ satisfy the conditions imposed in Lemma 1. Moreover, in the case $\mu>\alpha$ we will assume that $D_{0y}^{\mu-\alpha-m}v(y)\in AC^{m}[0,T)$, where

$$m=\lceil\mu-\alpha\rceil:=\min\{n\in\mathbb{N}:\,\mu-\alpha\le n\} \tag{20}$$

is the floor of the number $\mu-\alpha$.

Under the above assumptions, taking into account (12) and (16), we can write

$$u(x,y)=\lambda\,D_{0y}^{-\alpha}v(y)+\left(H^{\beta}\tau\right)(x,y). \tag{21}$$

Acting with D_{0y}^{μ} on both sides of (21) and substituting $x=z(y)$, using (19), we obtain

$$v(y)=\lambda\,D_{0y}^{\mu-\alpha}v(y)+f_{\tau,z}(y), \tag{22}$$

where

$$f_{\tau,z}(y):=\left(H^{\beta-\mu}\tau\right)(z(y),y). \tag{23}$$

Thus, the question of the solvability of the problem (1), (3) is reduced to the question of the solvability of the equation (22). This equation is integral if $\mu<\alpha$, functional if $\mu=\alpha$, and differential if $\mu>\alpha$. Below we will consider all three of these cases.

4 The case of integral equation ($\mu < \alpha$)

Let $\mu < \alpha$. In this case, (22) is an integral equation. In accordance with (18) and (23) we have that $f_{\tau,z}(y) \in L_{\text{loc}}[0,T)$. Thus, we can conclude that the equation (22) has a unique solution $v(y) \in L_{\text{loc}}[0,T)$, and this solution can be written as (see, e.g., [1,29])

$$v(y) = f_{\tau,z}(y) + \lambda \int_0^y f_{\tau,z}(t)\,(y-t)^{\alpha-\mu-1} E_{\alpha-\mu,\alpha-\mu}\left(\lambda(y-t)^{\alpha-\mu}\right) dt, \tag{24}$$

where

$$E_{\xi,\eta}(z) := \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\xi k+\eta)}$$

is the Mittag-Leffler function.

Now we can formulate a theorem on the solvability of the problem (1), (3) in the case $\mu < \alpha$.

Theorem 1. Let $\mu < \alpha$, $z(y) \in C[0,T)$ and $\tau(x) \in \mathbf{T}_{\alpha,\omega}$ for some $\omega < (1-\beta)\left(\frac{\beta}{T}\right)^{\frac{\beta}{1-\beta}}$. Then there exists a unique regular solution of the problem (1), (3) in the class of functions satisfying the condition (14). The solution has the form

$$u(x,y) = \lambda \int_0^y f_{\tau,z}(t)\,(y-t)^{\alpha-1} E_{\alpha-\mu,\alpha}\left(\lambda(y-t)^{\alpha-\mu}\right) dt + \left(H^{\beta}\tau\right)(x,y), \tag{25}$$

where H^{β} and $f_{\tau,z}(x)$ are defined by (16) and (23), respectively.

Proof. As shown in Section 3, if $u(x,y)$ is a regular solution of the problem (1), (3), then it has the form (21), where $v(x)$ should be found from (22). It was obtained above that in the case $\mu < \alpha$, which we are now considering, $v(x)$ is given by the formula (24). Taking this into account, we transform the first term on the right-hand side of the equation (21). Using (2) and the formulas (see, e.g., [1,30,31])

$$D_{0y}^{\gamma} y^{\eta-1} E_{\xi,\eta}\left(c\,y^{\xi}\right) = y^{\eta-\gamma-1} E_{\xi,\eta-\gamma}\left(c\,y^{\xi}\right) \qquad (\gamma \in \mathbb{R},\ \eta > 0), \tag{26}$$

$$E_{\xi,\eta}(z) = \frac{1}{\Gamma(\eta)} + z\,E_{\xi,\eta+\xi}(z) \tag{27}$$

and

$$D_{0y}^{-\gamma} \int_0^y g(t)h(y-t)\,dt = \int_0^y \left(D_{0y}^{-\gamma} g\right)(t)h(y-t)\,dt = \int_0^y g(t)\left(D_{0y}^{-\gamma} h\right)(y-t)\,dt \qquad (\gamma > 0), \tag{28}$$

we get

$$D_{0y}^{-\alpha} v(x) = D_{0y}^{-\alpha}\left[f_{\tau,z}(y) + \lambda \int_0^y f_{\tau,z}(t)\,(y-t)^{\alpha-\mu-1} E_{\alpha-\mu,\alpha-\mu}\left(\lambda(y-t)^{\alpha-\mu}\right) dt\right] =$$

$$= D_{0y}^{-\alpha} f_{\tau,z}(y) + \lambda \int_0^y f_{\tau,z}(t)\,(y-t)^{2\alpha-\mu-1} E_{2\alpha-\mu,\alpha-\mu}\left(\lambda(y-t)^{\alpha-\mu}\right) dt =$$

$$= \int_0^y f_{\tau,z}(t)\left[\frac{(y-t)^{\alpha-1}}{\Gamma(\alpha)} + \lambda\,(y-t)^{\alpha-\mu-1} E_{\alpha-\mu,\alpha-\mu}\left(\lambda(y-t)^{\alpha-\mu}\right)\right] dt =$$

$$= \int_0^y f_{\tau,z}(t)\,(y-t)^{\alpha-1} E_{\alpha-\mu,\alpha}\left(\lambda(y-t)^{\alpha-\mu}\right) dt.$$

This equality and (21) prove (25). In particular, the representation (25) yields the uniqueness of the problem (1), (3). Indeed, for the corresponding homogeneous problem ($\tau(x) \equiv 0$), we have $f_{\tau,z}(y) \equiv 0$.

Therefore, the difference between two different solutions with the same initial function $\tau(x)$ must be equal to zero. Due to the linearity of the problem under consideration, this is equivalent to the uniqueness of its solution.

The fact that the function (25) is a regular solution to the problem under consideration can be easily verified by direct computation. □

5 *The case of functional equation* $(\mu = \alpha)$

Now let us consider the case when $\mu = \alpha$. In this case the equation (22) takes the form

$$v(y) = \lambda\, v(y) + f_{\tau,z}(y). \tag{29}$$

It is easy to see that if $\lambda = 1$, then the equation (29) has a solution only for $f_{\tau,z}(y) \equiv 0$, and, moreover, then any function defined on the interval $(0,T)$ is a solution of this equation. Therefore, the solvability condition for (29) is the inequality $\lambda \neq 1$. If $\lambda \neq 1$, then

$$v(y) = \frac{f_{\tau,z}(y)}{1-\lambda}. \tag{30}$$

Now the solution of the problem (1), (3) can be found from (21). It should be noted that the function $v(y)$ must be integrable, therefore the function $f_{\tau,z}(y)$ must also be integrable.

Theorem 2. Let $\mu = \alpha$, $\lambda \neq 1$, $z(y) \in C[0,T)$, $\tau(x) \in \mathbf{T}_{\alpha,\omega}$ for some $\omega < (1-\beta)\left(\frac{\beta}{T}\right)^{\frac{\beta}{1-\beta}}$, and let $\tau(x)$ be locally Hölder continuous. Then there exists a unique regular solution of the problem (1), (3) in the class of functions satisfying the condition (14). The solution is of the form

$$u(x,y) = \frac{\lambda}{1-\lambda}\, D_{0y}^{-\alpha}\, f_{\tau,z}(y) + \left(H^{\beta}\tau\right)(x,y). \tag{31}$$

Proof. The representation (31) follows from (21) and (30). In fact, to complete the proof it remains to show that the conditions of the theorem guarantee the inclusion $f_{\tau,z}(y) \in L_{\mathrm{loc}}[0,T)$. Let us check this. By (16), we can write

$$\left(H^{-\beta}\tau\right)(x,y) = \int_{-\infty}^{\infty} [\tau(s) - \tau(x)]\; w_{-\beta}(x-s,y)\,ds + \tau(x)\int_{-\infty}^{\infty} w_{-\beta}(x-s,y)\,ds.$$

Taking into account (6), (9) and (10), we get

$$\left|\left(H^{-\beta}\tau\right)(x,y)\right| \leq \int_{-\infty}^{\infty} |(x-s)^{\varepsilon}\, w_{-\beta}(x-s,y)|\,ds \leq C y^{\beta\delta-1} \tag{32}$$

for any $\delta \in (0,\varepsilon)$ and $|x| < r$, where $C = C(\omega,\delta,r)$ and ε is the Hölder exponent for $\tau(x)$. Next, in accordance with (5), (23) and (32), we obtain

$$f_{\tau,z}(y) = \left(H^{-\beta}\tau\right)(z(y),y) \in L_{\mathrm{loc}}[0,T).$$

This completes the proof. □

6 The case of differtial equation ($\mu > \alpha$)

Now it remains to consider the last case, when $\mu > \alpha$. In this case, the order of differentiation in the loaded term exceeds the order of differentiation in the principal part of the equation. Equations with such a load are called essentially loaded [20] (as well as equations with $\mu = \alpha$ discussed in the previous section).

Let $u(x, y)$ be a solution of the problem (1) and (3). Then, as shown above, $u(x, y)$ can be represented as (21), where $v(y)$ is a solution of the equation (22), which in the case under consideration is a differential equation. The conditions $\lambda \neq 0$ and $f_{\tau,z}(y) \in L_{\mathrm{loc}}[0, T)$ guarantee (see, e.g., [31, 32]) that every solution of (22) can be given by

$$v(y) = \sum_{k=0}^{m-1} c_k \, G_{\mu-\alpha-k}(y) - \frac{1}{\lambda} \int_0^y f_{\tau,z}(t) \, G_{\mu-\alpha}(y - t) \, dt \tag{33}$$

for some set of numbers c_k, $k = 0, 1, \ldots, m - 1$. Here m is defined by (20) and

$$G_\sigma(y) := y^{\sigma-1} E_{\mu-\alpha,\sigma} \left(-\frac{1}{\lambda} y^{\mu-\alpha} \right). \tag{34}$$

Substituting (33) into (21) yields

$$u(x, y) = \lambda \, D_{0y}^{-\alpha} \left[\sum_{k=0}^{m-1} c_k \, G_{\mu-\alpha-k}(y) - \frac{1}{\lambda} \int_0^y f_{\tau,z}(t) \, G_{\mu-\alpha}(y - t) \, dt \right] + \left(H^\beta \tau \right)(x, y).$$

Using (26), (28) and (34), we get that if $u(x, y)$ is a solution of the problem (1) and (3) then it has the form

$$u(x, y) = \lambda \sum_{k=0}^{m-1} c_k \, G_{\mu-k}(y) - \int_0^y f_{\tau,z}(t) \, G_\mu(y - t) \, dt + \left(H^\beta \tau \right)(x, y) \tag{35}$$

for some c_k, $k = 0, 1, \ldots, m - 1$, where $f_{\tau,z}(y)$ is defined by (23).

Next, let us prove the converse statement: if the conditions $\lambda \neq 0$ and $f_{\tau,z}(y) \in L_{\mathrm{loc}}[0, T)$ are met, then the function (35) is a solution of the problem (1) and (3) for any set of c_k, $k = 0, 1, \ldots, m - 1$.

Indeed, using Lemma 1 and the formulas (19), (26), (27), (28), (34) and assuming that $u(x, y)$ is given by (35), we can write

$$\left(D_{0y}^\alpha - \frac{\partial^2}{\partial x^2} \right) u(x, y) = \lambda \sum_{k=0}^{m-1} c_k \, G_{\mu-\alpha-k}(y) - \int_0^y f_{\tau,z}(t) \, G_{\mu-\alpha}(y - t) \, dt, \tag{36}$$

$$D_{0y}^{\alpha-1} u(x, y) = \lambda \sum_{k=0}^{m-1} c_k \, G_{\mu-\alpha-k+1}(y) - \int_0^y f_{\tau,z}(t) \, G_{\mu-\alpha+1}(y - t) \, dt + \left(H^{1-\beta} \tau \right)(x, y), \tag{37}$$

$$D_{0y}^\mu u(x, y) = \sum_{k=0}^{m-1} c_k \, G_{\mu-\alpha-k}(y) - f_{\tau,z}(y) - \frac{1}{\lambda} \int_0^y f_{\tau,z}(t) \, G_{\mu-\alpha}(y - t) \, dt + \left(H^{\beta-\mu} \tau \right)(x, y),$$

and, by (23),

$$\left[D_{0y}^\mu u(x, y) \right]_{x=z(y)} = \sum_{k=0}^{m-1} c_k \, G_{\mu-\alpha-k}(y) - \frac{1}{\lambda} \int_0^y f_{\tau,z}(t) \, G_{\mu-\alpha}(y - t) \, dt. \tag{38}$$

By (18), (36), (37) and (38), we get

$$\left(D_{0y}^{\alpha}-\frac{\partial^2}{\partial x^2}\right)u(x,y)-\lambda\left[D_{0y}^{\mu}u(x,y)\right]_{x=z(y)}=0$$

and

$$\lim_{y\to 0}D_{0y}^{\alpha-1}u(x,y)=\tau(x).$$

This proves that $u(x,y)$, defined by (35), is really a solution of the problem (1) and (3).

Thus, the problem (1) and (3) is solvable for those functions $\tau(x)$ and $z(y)$ that guarantee the validity of inclusion $f_{\tau,z}(y)\in L_{\mathrm{loc}}[0,T)$ (as everywhere, $f_{\tau,z}(y)$ is defined by the equality (23)).

Let us formulate what was proved above in the following rigorous statement.

Theorem 3. Let $\lambda\neq 0$, $\mu>\alpha$, $z(y)\in C[0,T)$, $\tau(x)\in\mathbf{T}_{\alpha,\omega}$ for some $\omega<(1-\beta)\left(\frac{\beta}{T}\right)^{\frac{\beta}{1-\beta}}$, and let

$$f_{\tau,z}(y)\in L_{\mathrm{loc}}[0,T). \tag{39}$$

Then every regular solution of the problem (1) and (3) from the class of functions satisfying the condition (14) has the form (35) for some set of c_k, $k=0,1,\ldots,m-1$.

Conversely, any function $u(x,y)$ defined as (35) is a regular solution of the problem (1) and (3) for any set of c_k, $k=0,1,\ldots,m-1$.

Remark 1. The condition (39) is essential for the solvability of the problem (1) and (3). However, for fairly simple functions $\tau(x)$ and $z(y)$ this condition may not be met. For example, if we take $\tau(x)=const$, then by (4), (11), (16) and (23), we obtain

$$f_{\tau,z}(y)=const\int_{-\infty}^{\infty}w_{\beta-\mu}(z(y)-s,y)\,ds=const\frac{y^{\alpha-\mu-1}}{\Gamma(\alpha-\mu)}.$$

This means that $f_{\tau,z}(y)\notin L_{\mathrm{loc}}[0,T)$ for any $\mu>\alpha$ regardless of the choice of $z(y)$.

The question arises: whether there exist functions $\tau(x)$ and $z(y)$ for which this condition is satisfied. Let us show that they do exist. Let $z(y)=y^{\beta}$ and $\tau(x)=|x|^{\varepsilon}$. Then, by (4), (16) and (23), we get

$$f_{\tau,z}(y)=\int_{-\infty}^{\infty}|s|^{\varepsilon}w_{\beta-\mu}(y^{\beta}-s,y)\,ds=\int_{-\infty}^{\infty}|y^{\beta}-s|^{\varepsilon}w_{\beta-\mu}(s,y)\,ds=$$

$$=y^{\beta-\mu+\beta\varepsilon-1}\int_{-\infty}^{\infty}|1-s|^{\varepsilon}w_{\beta-\mu}(s,1)\,ds.$$

This means that

$$|f_{\tau,z}(y)|\leq Cy^{\beta-\mu+\beta\varepsilon-1}.$$

Thus, $f_{\tau,z}(y)\in L_{\mathrm{loc}}[0,T)$ for every $\varepsilon>\frac{\mu-\beta}{\beta}$.

7 *Non-uniqueness of solution in the problem with an essential load*

In the case $\mu>\alpha$, which is considered in the previous section, the problem (1) and (3) ceases to be uniquely solvable. Indeed, consider the function

$$u_0(x,y)=\sum_{k=0}^{m-1}c_k\,G_{\mu-k}(y), \tag{40}$$

where c_k, $k = 0, 1, \dots, m-1$ are constants, at least one of which is not equal to zero; and m is defined by (20). Using (26), (27) and (34), we can write

$$\left(D_{0y}^{\alpha} - \frac{\partial^2}{\partial x^2}\right) u_0(x,y) = \sum_{k=0}^{m-1} c_k\, G_{\mu-\alpha-k}(y), \qquad \Big[D_{0y}^{\mu} u_0(x,y)\Big]_{x=z(y)} = \frac{1}{\lambda} \sum_{k=0}^{m-1} c_k\, G_{\mu-\alpha-k}(y)$$

and

$$\lim_{y\to 0} D_{0y}^{\alpha-1} u_0(x,y) = \lim_{y\to 0} \sum_{k=0}^{m-1} c_k\, G_{\mu-\alpha-k+1}(y) = 0.$$

This gives that $u_0(x,y)$ is a regular solution of the equation (1) and satisfies the homogeneous initial value condition

$$\lim_{y\to 0} D_{0y}^{\alpha-1} u_0(x,y) = 0, \qquad x \in \mathbb{R}.$$

Due to the linearity, this means that solution of the problem (1) and (3) is not unique. If $u(x,y)$ is a regular solution of (1) and (3), then the function $u(x,y) + u_0(x,y)$ will also be a regular solution of this problem.

Conclusion

Thus, we considered the issue of solvability of the Cauchy problem (3) for the loaded equation (1) in all three cases of possible mutual distribution of α and μ ($\alpha > \mu$, $\alpha = \mu$ or $\alpha < \mu$).

It is shown that the condition $\alpha > \mu$ guarantees the unique solvability of the problem (1) and (3) (see Theorem 1). When $\alpha \le \mu$ the equation (1) becomes essentially loaded, and the problem under consideration may lose uniqueness or solvability. If $\alpha = \mu$, then the problem ceases to be solvable for $\lambda = 1$ (see Theorem 2). In the case $\alpha < \mu$, the problem (1) and (3) loses uniqueness of solution: the corresponding homogeneous problem has infinitely many non-trivial solutions (see Section 7) given in the form (40). Moreover, for the problem to be solvable in this case, it is necessary to impose an additional non-trivial condition (39) (see Theorem 3). This condition narrows the set of acceptable initial data, namely the initial value $\tau(x)$, as well as the function $z(y)$ that specifies the loaded term (see Remark 1).

In this last case, the questions arise: is it possible to achieve uniqueness of the solution by imposing additional conditions? And if so, what are these conditions? Also, can we equivalently reformulate the condition (39) in terms of $\tau(x)$ and $z(y)$? Further research is needed to answer these questions.

Author Contributions

All authors contributed equally to this work.

Conflict of Interest

The authors declare no conflict of interest.

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A risk model for insurance companies on time scales

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This article deals with the problems of constructing and analyzing a collective risk model for an insurance company when the time evolution is defined on a general time scale. The relevance of the study is determined by the need to describe premium accumulation and claim payments occurring at discrete or irregular time instants within a unified analytical framework. The characteristic features of the classical risk model and its extension to time scales are analyzed, and the need to investigate the behavior of the non-ruin probability under such a generalization is identified and justified. On the basis of the study, the authors construct an analogue of the classical model on time scales and derive a dynamic equation for the distribution of the number of claims. An integral equation on a time scale for the non-ruin probability is formulated. Conditions ensuring the correctness of the constructed model are established. It is proved that the non-ruin probability defined on a family of time scales converges pointwise to the corresponding probability in the classical continuous-time risk model as the graininess function tends to zero. It is shown that the proposed approach provides a rigorous justification of the transition from discrete to continuous risk models.

Keywords: time scales, risk process, non-ruin probability, graininess function, weak convergence, local asymptotic stability, dynamic equation, integral equation, claim number distribution.

2020 Mathematics Subject Classification: 34N05, 34B15.

Introduction

In modeling the operation of an insurance company, the classical risk model is well known (see, e.g., [1, 2]). According to this model, premium income from policyholders accumulates linearly over time as ct, where $c > 0$ denotes the premium intensity. At random times $\tau_1, \tau_2, \dots, \tau_n, \dots$, insurance claims are paid out, with claim sizes described by a sequence of independent and identically distributed random variables $Y_1, Y_2, \dots, Y_n, \dots$. It is assumed that the claim arrival times form a Poisson process with intensity $\alpha > 0$, that is, the stochastic process N_t, representing the number of claims occurring in the interval $[0, t)$, is a Poisson process. The total capital (surplus) of the insurance company at time t is given by

$$U_t = u + ct - S_t,$$

where u denotes the initial capital of the company and S_t is the aggregate claims up to time t. Clearly,

$$S_t = \sum_{k=1}^{N_t} Y_k.$$

The main quantitative characteristic of this model is the function $\varphi(u)$, which represents the non-ruin probability, defined by

$$\varphi(u) = P\{U_t \geq 0 \text{ for all } t \geq 0\}. \tag{1}$$

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It is well known that this function satisfies the following integro-differential equation:

$$\varphi'(u) = \frac{\alpha}{c}\varphi(u) - \frac{\alpha}{c}\int_0^u \varphi(u-z)\,dF(z),$$

where $F(z)$ is the distribution function of the claim sizes Y_k.

However, in practical applications, premium income does not accumulate continuously according to the linear law ct, but rather occurs at discrete or irregular time instants. Such time structures can be naturally described within the framework of time scale calculus.

The theory of dynamic equations on time scales, introduced in the pioneering work of S. Hilger [3], provides a unified approach to the study of continuous and discrete dynamical systems. Comprehensive treatments of the theory can be found in the monographs [4–6]. Recent developments and applications of dynamic equations on time scales are discussed, for example, in [7–9].

In this paper, we construct and study a mathematical model of an insurance company operating on time scales. First, under suitable assumptions, we develop an analogue of the classical risk model in this setting. Next, we derive a dynamic equation on a time scale for the function $P_n(t)$, defined by

$$P_n(t) = P\{N_t = n\}. \tag{2}$$

At a subsequent stage, we derive an integral equation on a time scale for the non-ruin probability $\varphi(u)$.

The aim of this work is to show that the non-ruin probability $\varphi_\lambda(u)$, defined on a family of time scales $\mathbb{T}_\lambda$, converges pointwise to the function $\varphi(u)$ as the graininess function $\mu_\lambda(t)$ tends to zero. Here, $\varphi(u)$ denotes the probability of non-ruin in the classical continuous-time risk model.

Noteworthy results concerning dynamic equations on time scales with complex topological structures (for example, Cantor-type sets) were obtained in [10, 11]. It should be noted that quantum calculus (or q-calculus) can be viewed as a particular case of the theory of dynamic equations on time scales, corresponding to time scales of the form $\mathbb{T} = q^{\mathbb{N}_0}$ or their modifications. Recent investigations of differential equations in the framework of q-calculus include, for example, [12–14].

Various properties of solutions have been investigated in the context of transitions between differential and dynamic equations. In particular, optimal control problems for ordinary differential equations and dynamic equations on time scales have been studied in [15–17], which is important for applications. Related questions concerning qualitative properties and boundary-value problems for dynamic equations on time scales were also studied in [18–20].

Applications of dynamic equations on time scales arise in various fields. In [21–23], a population model of Beverton–Holt type is investigated. Economic applications include the Solow growth model studied in [24, 25]. Furthermore, in [26, 27], an Arrow–Pratt type model on time scales is considered in the context of optimal insurance decisions.

Despite the extensive development of both risk theory and time scale calculus, collective risk models describing the operation of insurance companies within the framework of time scales have not been sufficiently studied. The present work aims to address this gap.

The paper is organized as follows. In Section 1, we introduce basic notions related to time scales and present auxiliary results. In Section 2, we construct a mathematical model of an insurance company operating on a time scale. Section 3 contains the main result concerning the convergence of the non-ruin probability $\varphi_\lambda(u)$ for models defined on a family of time scales to the non-ruin probability $\varphi(u)$ of the classical risk model.

1 Preliminaries

1.1 Time scales and basic definitions

We begin by recalling some basic notions from the theory of time scales (see, e.g., [4, 5]). A *time scale* $\mathbb{T}$ is a nonempty closed subset of $\mathbb{R}$. For a given set $A \subset \mathbb{R}$, we define $A_{\mathbb{T}} := A \cap \mathbb{T}$.

The *forward jump operator* $\sigma : \mathbb{T} \to \mathbb{T}$ is defined by

$$\sigma(t) := \inf\{s \in \mathbb{T} : s > t\}.$$

Similarly, the *backward jump operator* $\rho : \mathbb{T} \to \mathbb{T}$ is given by

$$\rho(t) := \sup\{s \in \mathbb{T} : s < t\}.$$

A point $t \in \mathbb{T}$ is called *left-dense (LD)*, *left-scattered (LS)*, *right-dense (RD)*, or *right-scattered (RS)* if $\rho(t) = t$, $\rho(t) < t$, $\sigma(t) = t$, or $\sigma(t) > t$, respectively.

If $\mathbb{T}$ has a left-scattered maximum M, then we define $\mathbb{T}^\kappa := \mathbb{T} \setminus \{M\}$; otherwise, we set $\mathbb{T}^\kappa := \mathbb{T}$.

The *graininess function* $\mu : \mathbb{T} \to [0, \infty)$ is defined by

$$\mu(t) := \sigma(t) - t,$$

and characterizes the local structure of the time scale.

A function $f : \mathbb{T} \to \mathbb{R}^d$ is said to be Δ-*differentiable* at a point $t \in \mathbb{T}^\kappa$ if the limit

$$f^\Delta(t) := \lim_{s \to t} \frac{f(\sigma(t)) - f(s)}{\sigma(t) - s}$$

exists in $\mathbb{R}^d$. In this case, $f^\Delta(t)$ is called the Δ-*derivative* of f at t.

If f is continuous at t and t is right-scattered, then f is Δ-differentiable at t and

$$f^\Delta(t) = \frac{f(\sigma(t)) - f(t)}{\mu(t)}.$$

If t is right-dense, then f is Δ-differentiable at t provided that the limit

$$\lim_{s \to t} \frac{f(t) - f(s)}{t - s}$$

exists, and in this case

$$f^\Delta(t) = \lim_{s \to t} \frac{f(t) - f(s)}{t - s}.$$

Analogously to the classical approach based on the Carathéodory construction, one can introduce the notion of the Lebesgue Δ-measure on a time scale $\mathbb{T}$, which is denoted by μ_Δ.

The following properties hold:

1) for any $t_0 \in \mathbb{T}^\kappa$, one has

$$\mu_\Delta(\{t_0\}) = \mu(t_0);$$

2) if $a, b \in \mathbb{T}$ and $a \le b$, then

$$\mu_\Delta([a, b)) = b - a, \quad \mu_\Delta((a, b)) = b - \sigma(a), \quad \mu_\Delta((a, b]) = \sigma(b) - \sigma(a), \quad \mu_\Delta([a, b]) = \sigma(b) - a.$$

The Lebesgue integral associated with the measure μ_Δ is called the *Lebesgue* Δ-*integral* on the time scale $\mathbb{T}$.

For a Δ-measurable set $E \subset \mathbb{T}$ and a function $f : E \to \mathbb{R}$, the corresponding integral of f over E is denoted by

$$\int_E f(t)\, \Delta t.$$

Consequently, all results of the general theory of Lebesgue integration, including theorems concerning limit processes, are valid for the Lebesgue Δ-integral on $\mathbb{T}$.

Let $A \subset \mathbb{T} \setminus \{\sup \mathbb{T}\}$ be a μ_Δ-measurable set. Consider a function f, defined Δ-almost everywhere on A, with values in $\mathbb{R}^d$.

Define the set

$$\widetilde{A} := A \cup \bigcup_{r \in A \cap RS} (r, \sigma(r)),$$

and introduce the function $\widetilde{f}$, which is an extension of f defined almost everywhere on $\widetilde{A}$, by

$$\widetilde{f}(t) = \begin{cases} f(t), & t \in A, \\ f(r), & t \in (r, \sigma(r)), \quad r \in A \cap RS. \end{cases}$$

Note that the function f is μ_Δ-measurable on A if and only if the function $\widetilde{f}$ is Lebesgue measurable on $\widetilde{A}$. The following result holds.

Theorem 1. [28] The function f is Δ-integrable on A if and only if the function $\widetilde{f}$ is Lebesgue integrable on $\widetilde{A}$, and

$$\int_A f(t)\,\Delta t = \int_{\widetilde{A}} \widetilde{f}(t)\,dt.$$

In what follows, the exponential function on a time scale, denoted by $e_P(t,s)$, will play an essential role. Specifically, the function $e_P(\cdot, s)$ is defined as the unique solution of the matrix initial value problem

$$x^\Delta(t) = P(t)x(t), \quad x(s) = E,$$

where $P(t)$ is a matrix-valued function and E denotes the identity matrix of size $d \times d$.

Throughout the paper, $|x|$ denotes the Euclidean norm of a vector $x \in \mathbb{R}^d$, and $\|A\|$ denotes the matrix norm of $A \in \mathbb{R}^{d \times d}$ induced by the Euclidean vector norm.

1.2 On the correspondence between solutions of dynamic equations on time scales and ordinary differential equations

For the subsequent analysis, we require several results concerning the relationship between the properties of solutions of dynamic equations on time scales and solutions of the corresponding ordinary differential equations.

We consider the system of ordinary differential equations on the semi-axis $t \geq 0$ given by

$$\frac{dx}{dt} = \dot{x} = f(x), \quad x(0) = x_0, \tag{3}$$

and the corresponding family of initial value problems for dynamic equations on time scales $\mathbb{T}_\lambda$, where $\lambda \in \Lambda \subset \mathbb{R}$ and $\lambda = 0$ is a limit point of the set Λ.

We denote

$$[0,T]_\lambda := [0,T] \cap \mathbb{T}_\lambda$$

and assume that the points 0 and T belong to all time scales $\mathbb{T}_\lambda$. The corresponding dynamic initial value problems are of the form

$$x_\lambda^\Delta = f(x), \quad x_\lambda(0) = x_0. \tag{4}$$

Here $x \in B_r$, $B_r := \{x \in \mathbb{R}^d : |x| \leq r\}$, $r > 0$, and $f : B_r \to \mathbb{R}^d$. Let $\mu_\lambda(t) : \mathbb{T}_\lambda \to [0, \infty)$ denote the graininess function of the time scale $\mathbb{T}_\lambda$. We set

$$\mu_\lambda := \sup_{t \in \mathbb{T}_\lambda} \mu_\lambda(t),$$

and assume that $\mu_\lambda \to 0$ as $\lambda \to 0$.

Assume further that the function f is continuously differentiable on B_r. That is, there exists a constant $C > 0$ such that

$$|f(x)| + \left\| \frac{\partial f(x)}{\partial x} \right\| \leq C \tag{5}$$

for all $x \in B_r$.

We will need a result on the preservation of exponential stability when passing from ordinary differential equations to dynamic equations on time scales. In our opinion, this result is also of independent interest.

We assume that the function $f(x)$ is defined for $x \in B_r$ and that condition (5) holds on this set. Moreover, we impose the following assumptions:

(a1) $\sup \mathbb{T}_\lambda = \infty$ for all $\lambda \in \Lambda$;

(a2) $\mu_\lambda \to 0$ as $\lambda \to 0$.

We also assume that (3) and (4) admit the trivial solution, i.e., $f(0) = 0$.

Definition 1. A function $\beta : [0, r_0) \times [0, \infty) \to [0, \infty)$ is said to belong to the class $\mathcal{K}$ if the following conditions hold:

1) the function $\beta(r, t)$ is continuous with respect to the variables $r \in [0, r_0)$ and $t \geq 0$;
2) for every $t \geq 0$, the function $\beta(\cdot, t)$ is strictly increasing with $\beta(0, t) = 0$, and for every $r \in [0, r_0)$, the function $\beta(r, \cdot)$ is strictly decreasing and tends to zero as $t \to \infty$.

Definition 2. Systems (3) and (4) are called *locally asymptotically stable (LAS)* if there exists a function $\beta \in \mathcal{K}$ such that

$$|x(t, t_0, x_0)| \leq \beta(|x_0|, t - t_0), \qquad \beta(|x_0|, t - t_0) \to 0 \quad \text{as } t \to \infty,$$

and

$$|x_\lambda(t, t_0, x_0)| \leq \beta(|x_0|, t - t_0), \qquad \beta(|x_0|, t - t_0) \to 0 \quad \text{as } t \to \infty, \quad t, t_0 \in \mathbb{T}_\lambda.$$

Here $x(t, t_0, x_0)$ and $x_\lambda(t, t_0, x_0)$ denote the solutions of the corresponding Cauchy problems with the initial condition

$$x(t_0, t_0, x_0) = x_\lambda(t_0, t_0, x_0) = x_0.$$

The given notion, as well as the more general concept of input-to-state stability (ISS), was introduced by E. Sontag [29] to characterize the robust stabilization of nonlinear systems. In this context, the dependence of the function β on $|x_0|$ describes the rate at which solutions converge to zero.

For example, for an exponentially stable system, one can take

$$\beta(|x_0|, t - t_0) = |x_0| e^{-\alpha(t - t_0)}, \quad \alpha > 0.$$

For the nonlinear equation $\dot{x} = -x^3$, we have

$$\beta(|x_0|, t - t_0) = \frac{|x_0|}{\sqrt{2x_0^2(t - t_0) + 1}}.$$

In the general case, ISS is characterized in terms of Lyapunov functions (see, e.g., [30, 31]).

Concerning the relationship between the local asymptotic stability of systems (3) and (4), the following theorem holds.

Theorem 2. Suppose that the following conditions hold:

(A1) System (3) is locally asymptotically stable for $x_0 \in D$.

(A2) There exists $\varepsilon > 0$ such that the trivial solution $x \equiv 0$ of (3) is exponentially stable in B_ε, i.e., there exist constants $L > 0$ and $\gamma > 0$ such that

$$|x(t, t_0, x_0)| \leq L e^{-\gamma(t - t_0)} |x_0|, \qquad t \geq t_0, \quad |x_0| \leq \varepsilon.$$

A3) For $|x| \leq r$, $x \neq 0$, the inequality
$$\big(f(x), x\big) < 0$$
holds, where $(\cdot, \cdot)$ denotes the scalar product in $\mathbb{R}^d$.

Moreover, conditions (a1) and (a2) are satisfied.

Then there exists $\lambda_0 \in \Lambda$ such that for all $\lambda \leq \lambda_0$ system (4) is also locally asymptotically stable for $x_0 \in D$.

Proof. Without loss of generality, we set $t_0 = 0 \in \mathbb{T}_\lambda$ for all $\lambda \in \Lambda$.

From condition (A3) it follows that $\frac{d}{dt}|x(t,x_0)|^2 = 2\big(x(t,x_0), f(x(t,x_0))\big) \leq 0$. Hence, the function $|x(t,x_0)|$ is non-increasing and therefore $|x(t,x_0)| \leq |x_0|$ for $t \geq 0$.

Moreover, by condition (A3) and the compactness of the set $A := \{x \in \mathbb{R}^d : \varepsilon \leq |x| \leq r\}$, there exists a constant $\alpha > 0$ such that $\big(f(x), x\big) \leq -\alpha$ for all $x \in A$.

Now consider a solution $x_\lambda(t,x_0)$ of system (4) with $x_0 \in A$ on the interval $[0, t_\lambda^*)_{\mathbb{T}_\lambda}$, where t_λ^* denotes the first exit time of the solution from the set A.

For $t \in [0, t_\lambda^*)_{\mathbb{T}_\lambda}$, we compute the Δ-derivative
$$\begin{aligned}\frac{d^\Delta}{dt}\big|x_\lambda(t,x_0)\big|^2 &= \left(\int_0^1 2\big(x_\lambda(t,x_0) + h\mu_\lambda(t) f(x_\lambda(t,x_0))\big)dh,\ f(x_\lambda(t,x_0))\right) \\ &= 2\big(x_\lambda(t,x_0), f(x_\lambda(t,x_0))\big) + \mu_\lambda(t)\,\big|f(x_\lambda(t,x_0))\big|^2 \leq -2\alpha + \mu_\lambda \sup_{x\in A}|f(x)|^2.\end{aligned}$$

Choose $\lambda_1 \in \Lambda$ such that
$$-2\alpha + \mu_\lambda \sup_{x\in A}|f(x)|^2 \leq 0 \quad \text{for all} \quad \lambda \leq \lambda_1.$$

Then $|x_\lambda(t,x_0)| \leq r$ for all $t \in \mathbb{T}_\lambda$, $\lambda \leq \lambda_1$. Therefore, for all $t \geq 0$ with $t \in \mathbb{T}_\lambda$, the solutions of systems (3) and (4) remain in the ball B_r, provided that $|x_0| \leq r$ and $\lambda \leq \lambda_1$.

From condition (A1) it follows that there exists a function $\beta \in \mathcal{K}$ such that
$$|x(t,x_0)| \leq \beta(|x_0|, t) \leq \beta(r,t) \to 0, \qquad t \to \infty.$$

Hence, there exists $T_1 > 0$ such that
$$|x(t,x_0)| < \frac{\varepsilon}{4}, \qquad t \geq T_1.$$

We choose $\lambda_2 \in \Lambda$, $\lambda_2 \leq \lambda_1$, such that the interval $[T_1, T_1+1]$ contains a point $T_{1,\lambda} \in \mathbb{T}_\lambda$ for all $\lambda \leq \lambda_2$.

Next, we need the following proposition.

Proposition 1. Let $t_0 \in \mathbb{T}_\lambda$, and let $x(t)$ and $x_\lambda(t)$ be the solutions of systems (3) and (4), respectively, satisfying the initial condition $x(t_0) = x_\lambda(t_0)$. Assume that
$$x(t) \in B_r \quad \text{for } t \in [t_0, t_0+T], \qquad x_\lambda(t) \in B_r \quad \text{for } t \in [t_0, t_0+T]_{\mathbb{T}_\lambda}.$$

Then, for all $t \in [t_0, t_0+T]_{\mathbb{T}_\lambda}$, the estimate
$$|x(t) - x_\lambda(t)| \leq \mu_\lambda\, K(T, C_r) \max_{s\in[t_0,t_0+T]} |f(x(s))| \tag{6}$$
holds, where $C_r := \max_{x\in B_r}\left\{|f(x)|,\ \left\|\frac{\partial f(x)}{\partial x}\right\|\right\}$.

Proof. The proof follows from the proof of Lemma 2.1 in [32, p. 2102], provided that we observe that the constant C_1 in inequality (2.12) of [32, p. 2103] can be chosen as $C_1^2 = \max_{t\in[t_0,t_0+T]} \left|\frac{\partial f(x(t))}{\partial x} f(x(t))\right|$, which completes the proof of Proposition 1. □

We now proceed with the proof of the theorem. Since $f(0) = 0$, it follows that

$$|f(x)| \le \sup_{x\in B_r} \left\|\frac{\partial f(x)}{\partial x}\right\| |x|. \tag{7}$$

Then, from assumption (A3), inequalities (6) and (7), we obtain

$$|x(t) - x_\lambda(t)| \le \mu_\lambda\, K(T, C_r)\, |x(t_0)|.$$

Hence, for $t \in [0, T_{1,\lambda}]_{\mathbb{T}_\lambda}$,

$$|x_\lambda(t)| \le |x(t)| + |x(t) - x_\lambda(t)| \le \beta(|x_0|, t) + \mu_\lambda K|x_0|. \tag{8}$$

In particular,

$$|x_\lambda(T_{1,\lambda})| \le \beta(|x_0|, T_{1,\lambda}) + \mu_\lambda K|x_0| \le \frac{\varepsilon}{4} + \mu_\lambda K|x_0|.$$

Now choose $\lambda_3 \le \lambda_2$ such that for all $\lambda \in (0, \lambda_3]$,

$$\mu_\lambda K r \le \frac{\varepsilon}{4}. \tag{9}$$

Then, from (8) and (9), we obtain

$$|x_\lambda(t)| \le \beta(|x_0|, t) + \mu_\lambda K|x_0| \quad \text{and} \quad |x_\lambda(T_{1,\lambda})| \le \beta(|x_0|, T_{1,\lambda}) + \frac{\varepsilon}{4} \le \frac{\varepsilon}{2}.$$

Next, by condition (A2), we obtain

$$|x(t, x_0)| \le \frac{|x(t_0)|}{8}, \qquad \text{for } t - t_0 \ge T := \frac{1}{\gamma}\ln(8L).$$

Let $x_T(t)$ be the solution of system (3) such that $x_T(T_{1,\lambda}) = x_\lambda(T_{1,\lambda})$.

Consider the interval $[T_{1,\lambda}, T_{2,\lambda}]_{\mathbb{T}_\lambda}$, where $T_{2,\lambda} \in [T_{1,\lambda} + T,\ T_{1,\lambda} + T + 1]$.

From the condition $\mu_\lambda \to 0$ as $\lambda \to 0$, it follows that for all sufficiently small $\lambda \le \lambda_4 \le \lambda_3$ such a point $T_{2,\lambda}$ exists. Therefore, for $t \in [T_{1,\lambda}, T_{2,\lambda}]$, we obtain

$$|x_T(t)| \le |x_\lambda(T_{1,\lambda})| \le \beta(|x_0|, T_{1,\lambda}) + \mu_\lambda K|x_0|.$$

Now choose $\lambda_5 \le \lambda_4$ such that for all $\lambda \le \lambda_5$

$$\mu_\lambda K(T+1, C_r) \le \frac{1}{8}.$$

Then

$$|x_T(t)| \le \beta(|x_0|, T_{1,\lambda}) + \frac{|x_0|}{8}.$$

Hence,

$$|x_\lambda(t)| \le |x_T(T_{1,\lambda})| + \frac{1}{8}|x_\lambda(T_{1,\lambda})| = \frac{9}{8}|x_\lambda(T_{1,\lambda})| \le \frac{9}{8}\big(\beta(|x_0|, T_{1,\lambda}) + |x_0|\big), \tag{10}$$

and

$$|x_\lambda(T_{2,\lambda})| \le \frac{1}{4}\big|x_\lambda(T_{1,\lambda})\big| \le \frac{1}{4}\big(\beta(|x_0|, T_{1,\lambda}) + |x_0|\big). \tag{11}$$

Next, consider the interval $[T_{2,\lambda}, T_{3,\lambda}]$, where $T_{3,\lambda} \in \mathbb{T}_\lambda$ and $T_{3,\lambda} \in [T_{1,\lambda} + 2T,\ T_{1,\lambda} + 2T + 1]$.

Let $x_{2T}(t)$ be the solution of system (3) such that $x_{2T}(T_{2,\lambda}) = x_\lambda(T_{2,\lambda})$. Then, from (10), we obtain

$$|x_\lambda(t)| \le \frac{9}{8}|x_\lambda(T_{2,\lambda})| \le \frac{9}{8}\cdot\frac{1}{4}\big(\beta(|x_0|, T_{1,\lambda}) + |x_0|\big),$$

and from (11) we get

$$|x_\lambda(T_{3,\lambda})| \le \frac{1}{4}|x_\lambda(T_{2,\lambda})| \le \left(\frac{1}{4}\right)^2\big(\beta(|x_0|, T_{1,\lambda}) + |x_0|\big). \tag{12}$$

Proceeding inductively, on the interval $[T_{k,\lambda}, T_{k+1,\lambda}]_{\mathbb{T}_\lambda}$, we similarly obtain

$$|x_\lambda(t)| \le \frac{9}{8}\left(\frac{1}{4}\right)^{k-1}\big(\beta(|x_0|, T_{1,\lambda}) + |x_0|\big), \quad |x_\lambda(T_{k+1,\lambda})| \le \left(\frac{1}{4}\right)^{k}\big(\beta(|x_0|, T_{1,\lambda}) + |x_0|\big). \tag{13}$$

This completes the proof of the theorem. □

Remark 1. It follows from the proof of Theorem 2 (see estimates (12) and (13)) that if the trivial solution of system (3) is globally exponentially stable, then for sufficiently small λ the trivial solution of system (4) is also globally exponentially stable.

2 Construction of the mathematical model

Let $\mathbb{T}$ be a time scale satisfying $\sup \mathbb{T} = +\infty$ and $\mu(t) \le 1$ for all $t \in \mathbb{T}$. In this section, we construct a mathematical model of an insurance company whose dynamics evolve on the time scale $\mathbb{T}$.

We assume that the insurance company starts its operation at time $t = 0$ (with the initial moment included, $0 \in \mathbb{T}$) with an initial capital $u \ge 0$. Premiums are assumed to be received continuously according to a linear law ct, where $c > 0$ denotes the premium intensity. Insurance claims occur at random time instants $\tau_1, \tau_2, \ldots, \tau_n, \ldots$. Let $\overline{\tau}_1, \overline{\tau}_2, \ldots, \overline{\tau}_n, \ldots$ denote the points of the time scale $\mathbb{T}$ immediately to the right of $\tau_1, \tau_2, \ldots, \tau_n, \ldots$, at which insurance payments (claims) are made.

The corresponding claim amounts $Y_1, Y_2, \ldots, Y_n, \ldots$ form a sequence of independent and identically distributed random variables defined on a probability space $(\Omega, \mathcal{F}, \mathcal{P})$. Obviously, $Y_n \ge 0$ almost surely.

Let $F(y)$ denote the distribution function of Y_n, with $F(0) = 0$. Moreover, the first two moments are assumed to exist: $\mathbb{E}Y_n = \mu$ and $\mathrm{Var}(Y_n) = \sigma^2$. We further assume that the claim sizes $\{Y_k\}$ are independent of the claim arrival times $\{\tau_k\}$. Let $N(t)$ denote, as above, the number of insurance claims occurring on the interval $[0, t)$.

With respect to the claim arrival process, we impose the following assumptions:
1. The numbers of claims occurring on disjoint time intervals are independent random variables.
2. (a) If t is a right-scattered point of the time scale $\mathbb{T}$, then the probability that exactly one insurance claim occurs at time t equals $\alpha\,\mu(t)$, where $\alpha \in (0,1)$ is the claim intensity. The probability that more than one claim occurs at such a point is equal to zero.
   (b) If t is a right-dense point of $\mathbb{T}$, then the probability that at least one insurance claim occurs on the interval $[t, t+h)_{\mathbb{T}}$ is equal to $\alpha h + o(h)$ as $h \to 0$, while the probability that more than one claim occurs on $[t, t+h)_{\mathbb{T}}$ is $o(h)$ as $h \to 0$. Here $h \in V_t := \{\beta \geq 0 : t + \beta \in \mathbb{T}\}$. Moreover, the distribution of the number of claims on $[t, t+h)_{\mathbb{T}}$ depends only on h.

Recall that we have previously introduced the risk process as a random process equal to the total capital (surplus) of the insurance company on the interval $[0, t)_{\mathbb{T}}$, namely,

$$U_t = u + ct - S_t,$$

where $S_t = \sum_{k=1}^{N_t} Y_k$. Here it is assumed that $\sum_{k=1}^{0} Y_k = 0$.

Recall also that the non-ruin probability in this risk model is defined by formula (1).

Similarly to the classical risk model, we now derive an equation for the function $P_n(t)$ defined by (2). We begin with the function $P_0(t)$, which is the probability that no insurance claims occur on the interval $[0, t)_{\mathbb{T}}$.

Let t be a right-scattered point of the time scale $\mathbb{T}$. Then the event

$$A = \{\text{no insurance claims occur on } [0, \sigma(t))_{\mathbb{T}}\}$$

can be represented as the intersection of the two events

$$A_1 = \{\text{no insurance claims occur on } [0, t)_{\mathbb{T}}\} \quad \text{and} \quad A_2 = \{\text{no claim occurs at time } t\}.$$

By assumption 1 on the claim arrival process, the events A_1 and A_2 are independent.

Therefore,

$$P(A) = P(A_1) \cdot P(A_2).$$

By the definition of $P_0(t)$, it follows that $P_0(\sigma(t)) = P_0(t)\big(1 - \alpha\,\mu(t)\big)$, or, equivalently,

$$\frac{P_0(\sigma(t)) - P_0(t)}{\mu(t)} = -\alpha\, P_0(t).$$

Thus, in this case we obtain

$$P_0^{\Delta}(t) = -\alpha\, P_0(t).$$

If the point t is right-dense, then the same arguments yield

$$P_0(t+h) = P_0(t)\big(1 - \alpha h + o(h)\big), \qquad h \to 0.$$

Hence, $P_0^{\Delta}(t) = -\alpha\, P_0(t)$. Therefore, the function $P_0(t)$ satisfies the dynamic equation

$$P_0^{\Delta}(t) = -\alpha\, P_0(t), \tag{14}$$

with the obvious initial condition $P_0(0) = 1$.

The solution of this Cauchy problem is given by the exponential function on time scales, $e_{-\alpha}(t, 0) = e_{-\alpha}(t)$. Consequently,

$$P_0(t) = e_{-\alpha}(t). \tag{15}$$

Example 1. Let $\mathbb{T} = \mathbb{R}$. Then the exponential function on time scales coincides with the classical exponential, i.e., $e_{-\alpha}(t) = e^{-\alpha t}$, $t \in \mathbb{R}$.

Example 2. Let $\mathbb{T} = h\mathbb{Z}$ with $h > 0$. Then the exponential function on this time scale is given by

$$e_{-\alpha}(t) = (1 - \alpha h)^{\frac{t}{h}}, \qquad t \in \mathbb{T}. \tag{16}$$

Let now $n \geq 1$. Assume that t is a right-scattered point of the time scale $\mathbb{T}$. Then the event

$$A = \{\text{exactly } n \text{ insurance claims occur on } [0, \sigma(t))_{\mathbb{T}}\}$$

can be represented as the union of two mutually exclusive events $A = A_1 \cup A_2$.

The event A_1 is the intersection of two independent events $A_1 = B_{11} \cap B_{12}$, where

$$B_{11} = \{\text{exactly } n \text{ insurance claims occur on } [0, t)_{\mathbb{T}}\}, \quad B_{12} = \{\text{no insurance claim occurs at time } t\}.$$

Similarly, the event A_2 is the intersection of two independent events $A_2 = B_{21} \cap B_{22}$, where

$$B_{21} = \{\text{exactly } n-1 \text{ insurance claims occur on } [0, t)_{\mathbb{T}}\},$$

$$B_{22} = \{\text{exactly one insurance claim occurs at time } t\}.$$

Consequently,

$$P_n(\sigma(t)) = P(A) = P(B_{11})P(B_{12}) + P(B_{21})P(B_{22}) = P_n(t)\big(1 - \alpha\mu(t)\big) + P_{n-1}(t)\, \alpha\mu(t).$$

Hence,

$$P_n^{\Delta}(t) = -\alpha P_n(t) + \alpha P_{n-1}(t), \qquad n \geq 1. \tag{17}$$

If the point t is right-dense, then by assumption 2 on the claim arrival process, analogous arguments show that the same relation holds. Indeed, the event

$$A = \{\text{exactly } n \text{ insurance claims occur on } [0, t+h)_{\mathbb{T}}\},$$

with $h \in V_t$, can be represented as a union of mutually exclusive events $A = \bigcup_{i=0}^{n} A_i$, where each event A_i is the intersection of two independent events, $A_i = B_{i1} \cap B_{i2}$, with

$$B_{i1} = \{\text{exactly } i \text{ insurance claims occur on } [0, t)_{\mathbb{T}}\},$$

$$B_{i2} = \{\text{exactly } n-i \text{ insurance claims occur on } [t, t+h)_{\mathbb{T}}\}.$$

By the assumptions on the claim arrival process, we have $P(B_{i1}) = P_i(t)$, $P(B_{i2}) = P_{n-i}(h)$.

Therefore,

$$P_n(t+h) = P_n(t)P_0(h) + P_{n-1}(t)P_1(h) + \sum_{k=2}^{n} P_{n-k}(t)P_k(h). \tag{18}$$

However, by assumption 2b we have $\sum_{k=2}^{n} P_{n-k}(t)P_k(h) = o(h)$ as $h \to 0$. Hence, from (18) it follows that

$$\frac{P_n(t+h) - P_n(t)}{h} = -\alpha P_n(t) + \alpha P_{n-1}(t) + \frac{o(h)}{h}.$$

Passing to the limit as $h \to 0$, we obtain

$$P_n^{\Delta}(t) = -\alpha P_n(t) + \alpha P_{n-1}(t), \qquad n = 1, 2, \ldots. \tag{19}$$

Hence, taking into account (17) and (19), the function $P_n(t)$ satisfies the following infinite system of linear dynamic equations:

$$P_n^{\Delta}(t) = -\alpha\, P_n(t) + \alpha\, P_{n-1}(t), \qquad t \in \mathbb{T}, \quad n = 1, 2, \ldots \tag{20}$$

with the initial conditions

$$P_n(0) = 0, \qquad n = 1, 2, \ldots$$

The system (20) can be simplified by introducing the substitution

$$P_n(t) = e_{-\alpha}(t)\, Q_n(t). \tag{21}$$

According to the product rule for the Δ-derivative, we obtain

$$P_n^{\Delta}(t) = -\alpha\, e_{-\alpha}(t)\, Q_n(t) + e_{-\alpha}(\sigma(t))\, Q_n^{\Delta}(t).$$

Since the exponential function on time scales satisfies $e_{-\alpha}(\sigma(t)) = \big(1 - \alpha\,\mu(t)\big)e_{-\alpha}(t)$, from (21) and (20) we obtain

$$Q_n^{\Delta}(t) = \frac{\alpha}{1 - \alpha\,\mu(t)}\, Q_{n-1}(t).$$

Note that $Q_0(0) = 1$ and $Q_n(0) = 0$, $n \geq 1$. Therefore,

$$Q_n(t) = \int_0^t \frac{\alpha}{1 - \alpha\,\mu(s)}\, Q_{n-1}(s)\,\Delta s,$$

and hence

$$P_n(t) = e_{-\alpha}(t) \int_0^t \frac{\alpha}{1 - \alpha\,\mu(s)}\, Q_{n-1}(s)\,\Delta s. \tag{22}$$

Remark 2. Expression (22) takes a particularly simple form in the case $\mathbb{T} = \mathbb{R}$. Indeed, in this case the exponential function on time scales reduces to the classical exponential, $e_{-\alpha}(t) = e^{-\alpha t}$, and we have $Q_0(t) \equiv 1$, while $Q_n(t) = \int_0^t \alpha\, Q_{n-1}(s)\, ds$. Therefore,

$$P_n(t) = e^{-\alpha t}\frac{(\alpha t)^n}{n!}, \qquad n = 0, 1, 2, \ldots$$

3 *Non-ruin probability*

As noted above, the principal quantitative characteristic used to assess risk is the non-ruin probability function $\varphi(u)$, defined by $\varphi(u) = P\{U_t \geq 0,\ t \geq 0\}$.

In this section, similarly to the classical risk model, we derive an integral equation for this function and study the convergence of the family of non-ruin probability functions $\{\varphi_\lambda(u)\}$ for risk models defined on a family of time scales, under the assumption that the graininess function tends to zero.

To derive an equation for $\varphi(u)$, we employ an integral analogue of the law of total probability. First, we condition on the time $\overline{\tau}_1$ of the first insurance payment, and then on the corresponding claim size Y_1.

Since $\varphi(u) = P\{U(t) \geq 0,\ \forall\, t \geq 0\} = P(A)$, where the event A means that ruin does not occur for the insurance company with initial capital u, by the law of total probability, conditioning first on the time $\overline{\tau}_1$ of the first insurance payment and then on the corresponding claim size Y_1, we obtain

$$\begin{aligned}\varphi(u) = P(A) = \int_0^\infty P\{A \mid \overline{\tau}_1 = s\}\, d^{\Delta}F_{\overline{\tau}_1}(s) &= \int_0^\infty \int_0^\infty P\{A \mid \overline{\tau}_1 = s,\ Y_1 = z\}\, dF(z)\, d^{\Delta}F_{\overline{\tau}_1}(s)\\ &= \int_0^\infty \int_0^{u+cs} P\{A \mid \overline{\tau}_1 = s,\ Y_1 = z\}\, dF(z)\, d^{\Delta}F_{\overline{\tau}_1}(s),\end{aligned}$$

where $F_{\overline{\tau}_1}(s)$ denotes the distribution function of the first insurance payment time.

By the assumptions of the model on the independence of $\overline{\tau}_1$ and Y_1, it follows from the previous formula that

$$\varphi(u)=\int_0^{\infty}\int_0^{u+cs}\varphi(u+cs-z)\,dF(z)\,d^{\Delta}F_{\overline{\tau}_1}(s). \tag{23}$$

Since

$$F_{\overline{\tau}_1}(s)=P\{\overline{\tau}_1<s\}=P\{N(s)\geq 1\}=1-P\{N(s)=0\}=1-e_{-\alpha}(s),$$

according to (15), we obtain $d^{\Delta}F_{\overline{\tau}_1}(s)=\alpha\, e_{-\alpha}(s)\,\Delta s$. Substituting this expression into (23), we obtain

$$\varphi(u)=\alpha\int_0^{\infty}\int_0^{u+cs}\varphi(u+cs-z)\,dF(z)\,e_{-\alpha}(s)\,\Delta s. \tag{24}$$

Equation (24) is the non-ruin probability equation on a time scale. It is a linear dynamic integral equation defined on $\mathbb{T}$.

Theorem 3. If the distribution function $F(x)$ is continuous, then equation (24) has a unique solution in the class of functions continuous on $[0,\infty)$.

Proof. Clearly, $\varphi(u)\in[0,1]$ for all $u\geq 0$. In the metric space $\mathcal{B}=C([0,\infty),\mathbb{R})$, we consider the closed unit ball $B_1(0):=\{\varphi\in\mathcal{B}:\max_{u\geq 0}|\varphi(u)|\leq 1\}$.

Define the mapping H by

$$H(\varphi)=\alpha\int_0^{\infty}\int_0^{u+cs}\varphi(u+cs-z)\,dF(z)\,e_{-\alpha}(s)\,\Delta s.$$

We show that $H(B_1(0))\subset B_1(0)$ and that H is a contraction mapping.

We establish the continuity of $H(\varphi)(u)$. Suppose that $u\to u_0$ and $u\leq u_0$. We have

$$\begin{aligned}
\lim_{u\to u_0}H(\varphi)(u)&=\lim_{u\to u_0}\alpha\int_0^{\infty}\int_0^{u+cs}\varphi(u+cs-z)\,dF(z)\,e_{-\alpha}(s)\,\Delta s\\
&=\lim_{u\to u_0}\alpha\int_0^{\infty}\int_0^{u_0+cs}\varphi(u+cs-z)\,dF(z)\,e_{-\alpha}(s)\,\Delta s\\
&\quad+\lim_{u\to u_0}\alpha\int_0^{\infty}\left[\int_0^{u+cs}\varphi(u+cs-z)\,dF(z)-\int_0^{u_0+cs}\varphi(u+cs-z)\,dF(z)\right]e_{-\alpha}(s)\,\Delta s\\
&=:\lim_{u\to u_0}I_1(u)+\lim_{u\to u_0}I_2(u).
\end{aligned} \tag{25}$$

Since

$$\int_0^{\infty}\int_0^{u+cs}\varphi(u+cs-z)\,dF(z)\,e_{-\alpha}(s)\,\Delta s\leq\int_0^{\infty}\int_0^{\infty}dF(z)\,e_{-\alpha}(s)\,\Delta s<\infty,$$

the continuity of the first term in (25) follows from the Lebesgue dominated convergence theorem, i.e.,

$$\lim_{u\to u_0}I_1(u)=I_1(u_0).$$

For the second term in (25) we have

$$\begin{aligned}
&\left|\int_0^{\infty}\left[\int_0^{u+cs}\varphi(u+cs-z)\,dF(z)-\int_0^{u_0+cs}\varphi(u+cs-z)\,dF(z)\right]e_{-\alpha}(s)\,\Delta s\right|\\
&\leq\int_0^{\infty}\left|\int_{u+cs}^{u_0+cs}\varphi(u+cs-z)\,dF(z)\right|e_{-\alpha}(s)\,\Delta s\leq\int_0^{\infty}|F(u_0+cs)-F(u+cs)|\,e_{-\alpha}(s)\,\Delta s.
\end{aligned} \tag{26}$$

Since the distribution function $F(t)$ is continuous, we have $F(u_0 + cs) - F(u + cs) \to 0$ as $u \to u_0$. Hence, the convergence of expression (26) to zero again follows from the Lebesgue dominated convergence theorem.

The case $u \geq u_0$ can be treated analogously.

Next, we have

$$\sup_{u \geq 0} |H(\varphi)(u)| \leq \alpha \int_0^\infty \int_0^\infty dF(z)\, e_{-\alpha}(s)\, \Delta s \leq \alpha \leq 1.$$

Therefore, the mapping H maps $B_1(0)$ into itself.

We now prove that H is a contraction mapping. We have

$$\sup_{u \geq 0} |H(\varphi_1)(u) - H(\varphi_2)(u)| \leq \alpha \int_0^\infty \int_0^\infty \sup_{u \geq 0} |\varphi_1(u) - \varphi_2(u)|\, dF(z)\, e_{-\alpha}(s)\, \Delta s \leq \alpha \sup_{u \geq 0} |\varphi_1(u) - \varphi_2(u)|\,.$$

Since $\alpha \in (0, 1)$, the contraction property is proved. Hence, by the Banach fixed-point theorem, equation (24) has a unique solution. The theorem is proved.

□

Let $\{\mathbb{T}_\lambda\}$ be a family of time scales such that $\sup \mathbb{T}_\lambda = \infty$ for all $\lambda \in \Lambda$, and $\sup_{t \in \mathbb{T}_\lambda} \mu_\lambda(t) = \mu_\lambda \to 0$ as $\lambda \to 0$. On each time scale $\mathbb{T}_\lambda$, we consider the risk model described above and denote by $\varphi_\lambda(u)$ the corresponding non-ruin probability. We assume that the parameters of the risk process c, α, and the distribution function $F(t)$ are the same for all time scales.

Theorem 4. If the distribution function $F(t)$ is continuous, then

$$\varphi_\lambda(u) \to \varphi(u) \quad \text{for all } u \geq 0, \quad \text{as } \lambda \to 0,$$

where $\varphi(u)$ denotes the non-ruin probability in the classical risk model.

Proof. Clearly, $\int_0^\infty |\varphi_\lambda(x)|^2\, dF(x) \leq 1$. Thus, the family $\{\varphi_\lambda(u)\}$ is weakly compact in $L^2(\mathbb{R}, F(x))$ and, consequently, contains a weakly convergent subsequence $\{\varphi_{\lambda_n}(u)\}$ such that

$$\varphi_{\lambda_n} \xrightarrow{w} \psi \quad \text{in } L^2(\mathbb{R}, F(x)). \tag{27}$$

On the other hand, by (24), each function $\varphi_{\lambda_n}(u)$ satisfies the equation

$$\varphi_{\lambda_n}(u) = \alpha \int_0^\infty \int_0^{u+cs} \varphi_{\lambda_n}(u + cs - z)\, dF(z)\, e_{-\alpha}^{(\lambda_n)}(s)\, \Delta_{\lambda_n} s. \tag{28}$$

Here $e_{-\alpha}^{(\lambda_n)}(s)$ denotes the exponential function on the time scale $\mathbb{T}_{\lambda_n}$. From (27) we have

$$\int_0^{u+cs} \varphi_{\lambda_n}(u + cs - z)\, dF(z) \;\longrightarrow\; \int_0^{u+cs} \psi(u + cs - z)\, dF(z), \qquad \text{as } \lambda_n \to 0. \tag{29}$$

The functions $e_{-\alpha}^{(\lambda_n)}(s)$ are solutions of equation (14), which satisfies all the assumptions of Theorem 2. Moreover, the limiting equation corresponding to (14) has the form

$$\dot{x} = -\alpha x, \qquad x(0) = 1,$$

with the obvious solution.

According to Proposition 1, we have

$$e_{-\alpha}^{(\lambda_n)}(s) - e_{-\alpha}(s) \to 0, \quad \text{as } \lambda_n \to 0, \quad s \in \mathbb{T}_{\lambda_n}. \tag{30}$$

Moreover, analyzing the proof of Theorem 2, we note that the constant T in the present case is equal to $\frac{\ln 8}{\alpha}$.

Next, we have

$$\int_0^\infty e_{-\alpha}^{(\lambda_n)}(s)\,\Delta_{\lambda_n} s = \sum_{k=1}^\infty \int_{T_{k,\lambda}}^{T_{k+1,\lambda}} e_{-\alpha}^{(\lambda_n)}(s)\,\Delta_{\lambda_n} s, \tag{31}$$

where we used the decomposition of $[0,\infty)_{\mathbb{T}_{\lambda_n}}$ into the intervals $[T_{k,\lambda}, T_{k+1,\lambda}]_{\mathbb{T}_{\lambda_n}}$ introduced in the proof of Theorem 2 and the additivity of the Δ-integral.

Note that, according to the choice of the points $T_{k,\lambda}$, we have

$$T_{k+1,\lambda} - T_{k,\lambda} \le \max\{T, T_1\} + 1,$$

where T and T_1 are fixed constants. Therefore, from formula (31) we obtain

$$\sum_{k=1}^\infty \int_{T_{k,\lambda}}^{T_{k+1,\lambda}} e_{-\alpha}^{(\lambda_n)}(s)\,\Delta_{\lambda_n} s \le \sum_{k=1}^\infty \left(\frac{1}{4}\right)^{k-1} \left(e^{-\alpha T_1} + 1\right)(\max\{T, T_1\} + 1) < \infty. \tag{32}$$

Thus, from (29), (30), and (32), taking into account the Lebesgue dominated convergence theorem, it follows from (28) that

$$\varphi(u) = \alpha \int_0^\infty \int_0^{u+cs} \psi(u + cs - z)\,dF(z)\,e^{-\alpha s}\,ds. \tag{33}$$

By the uniqueness of the solution to equation (33), we conclude that $\psi(u) = \varphi(u)$. Consequently, $\varphi_\lambda(u) \to \varphi(u)$ for all $u \ge 0$ as $\lambda \to 0$.

□

Example 3. It is well known that in the classical risk model, under the assumption that the claim sizes are exponentially distributed,

$$F_{Y_k}(z) = \begin{cases} 1 - e^{-z/\mu}, & z \ge 0, \\ 0, & z < 0, \end{cases}$$

the probability of non-ruin can be found explicitly: $\varphi(u) = 1 - \frac{1}{1+\rho} e^{-\frac{\rho}{(1+\rho)\mu}u}$, where $\rho = \frac{c}{\alpha\mu} - 1$.

Now consider a time scale of the form $\mathbb{T} = \{kh \mid t = kh,\ k = 0, 1, 2, \dots\}$, for which the exponential function is given by (16). Accordingly, equation (24) takes the form

$$\varphi_h(u) = \frac{\alpha}{\mu} h \sum_{k=0}^\infty \int_0^{u+ckh} \varphi_h(u + ckh - z)\,e^{-z/\mu}\,dz\,(1 - \alpha h)^k.$$

By Theorem 4, it follows that for every $u \ge 0$,

$$\varphi_h(u) \to 1 - \frac{1}{1+\rho} e^{-\frac{\rho}{(1+\rho)\mu}u}, \quad \text{as } h \to 0.$$

Conclusion

We developed a collective risk model for an insurance company on time scales, deriving a dynamic equation for the probabilities $P_n(t)$ and an integral equation for the non-ruin probability. The main result proves the pointwise convergence of the non-ruin probability $\varphi_\lambda(u)$ to the classical probability $\varphi(u)$ as the graininess function tends to zero. This rigorously justifies the classical risk model as a limiting case of models defined on arbitrary time scales. The approach extends time scale calculus to actuarial mathematics and opens perspectives for studying more general claim processes, stochastic perturbations, and optimal control problems on non-uniform time scales.

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Author Contributions

All authors contributed equally to this work.

Conflict of Interest

The authors declare no conflict of interest.

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Characterization of weighted inequalities for superpositions of integral and supremal operators

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In this paper, we provide a characterization of the boundedness of positive sublinear operators that are superposition of three operators: Copson, Hardy, and Tandori (supremal) operators defined on the half-axis from a weighted Lebesgue space $L_p(v)$ to another weighted Lebesgue space $L_1(w)$, where v and w are weight functions on the half-axis $(0, \infty)$, and $1 < p < \infty$. Our characterization is entirely different from existing results in the literature. The motivation for investigating such inequalities stems from the problem of finding a minimal rearrangement invariant space that contains the cones of non-increasing rearrangement of the functions represented by generalized fractional maximal function acting on functions from weighted Lorentz function spaces. More specifically, by obtaining two-sided estimates for the best constant in the corresponding inequality, we derive a characterization of the associate space of minimal rearrangement invariant spaces containing cones of non-increasing rearrangement of generalized fractional maximal function. To achieve this goal, we are using discretization and anti-discretization methods. In particular, we extend existing discretization techniques to handle the operators formed by iterating the Copson, Hardy, and Tandori operators. We first establish a discrete characterization in Theorem 3. Then, applying anti-discretization techniques, we derive a continuous characterization in Theorem 1.

Keywords: weighted inequality, function spaces, supremal operator, Copson operator, Hardy operator, Tandori operator, best constant, superposition of operators, discretization.

2020 Mathematics Subject Classification: 42B25, 46E30, 47L07, 47B38.

Introduction

In this paper, we study weighted one-dimensional inequalities for operators generated by iterating the Copson, Hardy, and Tandori operators. We do this by developing the methods of discretization and anti-discretization from the paper [1].

In the paper [2], it was shown that the boundedness of the fractional maximal function M_γ between classical Lorentz spaces is equivalent to the following weighted inequality

$$\left(\int_0^\infty \left(\operatorname*{ess\,sup}_{t<s<\infty} s^{\frac{\gamma}{n}-1} \int_0^s f(y)dy \right)^q w(t)dt \right)^{\frac{1}{q}} \le C \left(\int_0^\infty f^p(t)v(t)dt \right)^{\frac{1}{p}}$$

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for all non-increasing functions f on $(0,\infty)$, $0 < p, q < \infty$ and $\gamma \in (0, n)$, $n \in \mathbb{N}$. The last estimate can be interpreted as the boundedness of non-linear operators T_γ, defined by

$$(T_\gamma f)(t) = \operatorname*{ess\,sup}_{t<s<\infty} s^{\frac{\gamma}{n}-1} \int_0^s f(y)dy$$

on the cone of non-increasing functions defined on $(0,\infty)$. The operator T_γ was studied in [3], and it was named as *a Hardy-type operator involving suprema.* Various aspects of these operators were studied in [4–6] and their extensions and applications were further considered in [7–9]. More recent results, including weighted estimates for iterated operators, can be found in [10–12]. Additional results are given in [13, 14], as well as in [15, 16]. A systematic exposition of the theory is presented in the monograph [17].

The weighted inequalities with quasilinear integral operators restricted on the cone of monotone functions arise in connection with the studies of boundedness of the operators of harmonic analysis in Lorentz spaces. In the papers mentioned above, the authors use the reduction method for quasilinear operators of iterated type. The inequality restricted to the cone of monotone functions is reduced to the inequalities on all non-negative measurable functions. In this case, reduced inequalities contain the iteration of three operators.

To find a minimal rearrangement invariant space for generalized Besov, Sobolev, and Caldéron spaces is reduced to finding a minimal rearrangement invariant space for cones consisting of decreasing functions of Riesz and Bessel potential of functions from function spaces (see, [18]). This approach was used in the papers [19] and [20] to obtain a characterization of minimal rearrangement invariant spaces for generalized Sobolev and Bessel spaces. In a recent paper [21], general methods were developed to describe a minimal rearrangement invariant space for the cone of decreasing functions related to the generalized fractional maximal functions M_φ. To achieve this goal, we need to obtain the characterization of iterated inequality containing Copson, Hardy, and Tandori(supremal) operators.

Let $1 < p < \infty$. Let w and v be weight functions i.e. non-negative measurable functions on $(0,\infty)$. Let φ be a non-negative, non-decreasing quasi-concave function on $(0,\infty)$. Our goal in this article is to find the conditions on v, w, and φ under which the inequality

$$\int_0^\infty \left(\operatorname*{ess\,sup}_{t<s<\infty} \frac{1}{\varphi(s)} \int_0^s \left(\int_\tau^\infty h(y)dy \right) d\tau \right) w(t)dt \le C \left(\int_0^\infty h^p(t)v(t)dt \right)^{\frac{1}{p}}$$

is satisfied for all non-negative measurable functions h on $(0,\infty)$. Using Fubini's Theorem, we obtain an equivalent form of this inequality as follows:

$$\int_0^\infty \operatorname*{ess\,sup}_{t<s<\infty} \frac{1}{\varphi(s)} \left(\int_0^s \tau h(\tau)d\tau + s\int_s^\infty h(\tau)d\tau \right) w(t)dt \le C \left(\int_0^\infty h^p(t)v(t)dt \right)^{\frac{1}{p}}. \tag{1}$$

As we already mentioned above, the related inequalities

$$\int_0^\infty \operatorname*{ess\,sup}_{t<s<\infty} \frac{1}{\varphi(s)} \left(\int_0^s \tau h(\tau)d\tau \right) w(t)dt \le C \left(\int_0^\infty h^p(t)v(t)dt \right)^{\frac{1}{p}}, \tag{2}$$

$$\int_0^\infty \operatorname*{ess\,sup}_{t<s<\infty} \frac{s}{\varphi(s)} \left(\int_s^\infty h(\tau)d\tau \right) w(t)dt \le C \left(\int_0^\infty h^p(t)v(t)dt \right)^{\frac{1}{p}} \tag{3}$$

have been studied intensively over the last twenty years. The characterization of inequality (1) is, in fact, equivalent to the characterization of inequalities (2) and (3). Each of the latter yields two separate conditions, resulting in a total of four conditions. However, such a formulation is not convenient for future applications, where a single, unified condition is preferable. To achieve this, we employ discretization techniques as developed in [1] and further elaborated in the monograph [17].

Throughout the paper, for $p > 1$, $p' = p/(p-1)$. We always denote by c or C a positive constant which is independent of the main parameters, but it may vary from line to line. We write $A \lesssim B$ (or $A \gtrsim B$) if $A \le cB$ (or $cA \ge B$) for some positive constant c independent of appropriate quantities involved in the expressions A and B, and $A \approx B$ if $A \lesssim B$ and $A \gtrsim B$.

To formulate our main result, we need some definitions.

Definition 1. [1, Definition 2.2] Let φ be a continuous strictly increasing function on $[0, \infty)$ such that $\varphi(0) = 0$ and $\lim_{t\to\infty} \varphi(t) = \infty$. Then we say that φ is admissible.

Let φ be an admissible function. We say that a function h is φ-quasiconcave if h is equivalent to an increasing function on $[0, \infty)$ and $\frac{h}{\varphi}$ is equivalent to a decreasing function on $(0, \infty)$. We say that a φ-quasiconcave function h is non-degenerate if

$$\lim_{t\to 0+} h(t) = \lim_{t\to\infty} \frac{1}{h(t)} = \lim_{t\to\infty} \frac{h(t)}{\varphi(t)} = \lim_{t\to 0+} \frac{\varphi(t)}{h(t)} = 0.$$

The family of non-degenerate φ-quasi-concave functions will be denoted by Ω_φ. If $\varphi(t) = t$ we say that h is quasi-concave. If h is a non-degenerate quasi-concave function, we write $h \in \Omega_1$.

Definition 2. [1, Definition 2.9] Let φ be an admissible function and let ν be a non-negative Borel measure on $[0, \infty)$. We say that the function h, defined as

$$h(t) = \varphi(t) \int_{[0,\infty)} \frac{d\nu(s)}{\varphi(s) + \varphi(t)}, \quad t \in (0, \infty),$$

is the fundamental function of the measure ν with respect to φ. We will also say that ν is a representation measure of h with respect to φ.

We say that ν is a non-degenerate measure with respect to φ if the following conditions are satisfied for every $t \in (0, \infty)$:

$$\int_{[0,\infty)} \frac{d\nu(s)}{\varphi(s) + \varphi(t)} < \infty, \ t \in (0, \infty) \quad \text{and} \quad \int_{[0,1]} \frac{d\nu(s)}{\varphi(s)} = \int_{[1,\infty)} d\nu(s) = \infty.$$

The main result of this paper is presented in the following theorem. To this end, define

$$H(t) := \varphi^{p'}(t) \operatorname*{ess\,sup}_{t<s<\infty} \frac{1}{\varphi^{p'}(s)} \left(\int_0^\infty \frac{\tau^{p'} s^{p'}}{s^{p'} + \tau^{p'}} v^{1-p'}(\tau) d\tau \right). \tag{4}$$

Observe that $H \in \Omega_{\varphi^{p'}}$. There exist a representation measure ν of H with respect to $\varphi^{p'}$, (see, Lemma 1), i.e.,

$$H(t) = \int_0^\infty \frac{\varphi^{p'}(t)}{\varphi^{p'}(s) + \varphi^{p'}(t)} d\,\nu(s). \tag{5}$$

Theorem 1. Let $1 < p < \infty$. Assume that v and w are non-negative measurable functions on $(0, \infty)$ and that φ is a quasi-concave admissible function on $(0, \infty)$. Then following statements are equivalent:

(i) There exists a constant $C > 0$ such that inequality (1) holds for all non-negative measurable functions h on $(0, \infty)$.

(ii) $C_1 < \infty$, where

$$C_1 := \left(\int_0^\infty \left(\int_0^\infty \frac{w(s) ds}{\varphi(s) + \varphi(t)} \right)^{p'} d\nu(t) \right)^{\frac{1}{p'}},$$

and ν is a representation measure of H (defined in (4)) with respect to $\varphi^{p'}$, i.e, satisfies (5).

(iii) $C_2 < \infty$, where

$$C_2 := \left(\int_0^\infty \left(\int_0^\infty \frac{w(s)ds}{\varphi(s)+\varphi(t)} \right)^{p'-1} \operatorname*{ess\,sup}_{t<s<\infty} \frac{1}{\varphi^{p'}(s)} \left(\int_0^\infty \frac{\tau^{p'} s^{p'}}{s^{p'}+\tau^{p'}} v^{1-p'}(\tau) d\tau \right) w(t) \varphi^{p'-1}(t) dt \right)^{\frac{1}{p'}}.$$

Moreover, the best constant C in (1) satisfies $C \approx C_1 \approx C_2$.

The results obtained may have important applications in the theory of integral operators, interpolation theory and nonlinear analysis, especially in problems related to three-weight estimates and non-standard function spaces.

The paper is organized as follows. Section 1 provides several auxiliary definitions and preliminary lemmas. In Section 2, we develop a discrete characterization of inequality (1). Section 3 contains the proof of the main result, namely, the proof of Theorem 1.

1 *Definitions and Preliminaries*

This section provides the necessary definitions and supporting known statements that are used in the proofs of the main theorems.

Definition 3. [1, Definition 2.1] Let $\{a_k\}_{k\in\mathbb{Z}}$ be a sequence of positive numbers. We say that $\{a_k\}_{k\in\mathbb{Z}}$ is strongly increasing or strongly decreasing and write $a_k \upuparrows$ or $a_k \downdownarrows$ if

$$\inf_{k\in\mathbb{Z}} \frac{a_{k+1}}{a_k} > 1 \quad \text{or} \quad \sup_{k\in\mathbb{Z}} \frac{a_{k+1}}{a_k} < 1,$$

respectively.

Definition 4. [1, Definition 2.4] Assume that φ is an admissible function and $h \in \Omega_\varphi$. We say that $\{\mu_k\}_{k\in\mathbb{Z}}$ is a discretization sequence for h with respect to φ if

(i) $\mu_0 = 1$ and $\varphi(\mu_k) \upuparrows$;

(ii) $h(\mu_k) \upuparrows$ and $\frac{h(\mu_k)}{\varphi(\mu_k)} \downdownarrows$;

(iii) there is a decomposition $\mathbb{Z} = \mathbb{Z}_1 \cup \mathbb{Z}_2$ such that $\mathbb{Z}_1 \cap \mathbb{Z}_2 = \emptyset$ and for every $t \in [\mu_k, \mu_{k+1}]$,

$$h(\mu_k) \approx h(t) \quad \text{if} \quad k \in \mathbb{Z}_1,$$
$$\frac{h(\mu_k)}{\varphi(\mu_k)} \approx \frac{h(t)}{\varphi(t)} \quad \text{if} \quad k \in \mathbb{Z}_2.$$

By Lemma [1, Lemma 2.7], for every admissible function φ and $h \in \Omega_\varphi$, there exists, a discretization sequence for h with respect to φ.

Lemma 1. [1, Lemma 2.8] Let φ be an admissible function. Then the following statements are equivalent:

(i) $h \in \Omega_\varphi$.

(ii) There exists a non-negative, a non-degenerate Borel measure ν on $[0,\infty)$ with respect to φ such that

$$h(t) \approx \varphi(t) \int_{[0,\infty)} \frac{d\nu(s)}{\varphi(s)+\varphi(t)}, \quad t \in (0,\infty).$$

Let us now present special cases of some well-known results concerning the discretization of integral expressions, as found in [1, 17].

Lemma 2. [1, Corollary 2.13] Assume that φ is an admissible function, $f \in \Omega_\varphi$, ν is a non-negative non-degenerate Borel measure on $[0,\infty)$ and h is the fundamental function of ν with respect to φ. If $\{x_k\}_{k\in\mathbb{Z}}$ is a discretization sequence for h with respect to φ, then

$$\int_{[0,\infty)} \frac{f(t)}{\varphi(t)} d\nu(t) \approx \sum_{k\in Z} \frac{f(x_k)}{\varphi(x_k)} h(x_k).$$

Lemma 3. [1, Lemma 3.5] Let $1 < p < \infty$. Assume that φ is an admissible function, $f \in \Omega_\varphi$ and $g \in \Omega_{\varphi^p}$. If $\{x_k\}_{k\in\mathbb{Z}}$ is a discretization sequence for f with respect to φ and $\{\lambda_k\}_{k\in\mathbb{Z}}$ is a discretization sequence of g with respect to φ^p. Then

$$\sum_{k\in\mathbb{Z}} \frac{f(x_k)^{p'}}{g(x_k)^{\frac{p'}{p}}} \approx \sum_{\ell\in\mathbb{Z}} \frac{f(\lambda_\ell)^{p'}}{g(\lambda_\ell)^{\frac{p'}{p}}}$$

and

$$\sup_{t\in(0,\infty)} \frac{f(t)}{g(t)^{\frac{1}{p}}} \approx \sup_{k\in Z} \frac{f(x_k)}{g(x_k)^{\frac{1}{p}}} \approx \sup_{\ell\in Z} \frac{f(\lambda_\ell)}{g(\lambda_\ell)^{\frac{1}{p}}}.$$

Lemma 4. [1, Lemma 3.6] Let $0 < r < \infty$. Assume that φ is an admissible function, ν is a non-degenerate positive Borel measure on $[0,\infty)$ measure with respect to φ^r, h is the fundamental function of ν with respect to φ^r and f is a measurable function on $[0,\infty)$. If $\{x_k\}_{k\in\mathbb{Z}}$ is a discretization sequence for h with respect to φ^r. Then

$$\int_0^\infty \left(\int_0^\infty \frac{|f(t)|dt}{\varphi(t)+\varphi(x)} \right)^r d\nu(x) \approx \sum_{k\in\mathbb{Z}} \left(\int_0^\infty \frac{|f(t)|dt}{\varphi(t)+\varphi(x_k)} \right)^r h(x_k).$$

Lemma 5. [1, Lemma 3.7] Assume that φ is an admissible function, ν is a non-degenerate non-negative Borel measure on $[0,\infty)$, h is the fundamental function of ν with respect to φ and f is a measurable function on $[0,\infty)$. If $\{x_k\}_{k\in\mathbb{Z}}$ is a discretization sequence for h with respect to φ, then

$$\int_{[0,\infty)} \operatorname*{ess\,sup}_{y\in(0,\infty)} \frac{|f(y)|}{\varphi(x)+\varphi(y)} d\nu(x) \approx \sum_{k\in\mathbb{Z}} \operatorname*{ess\,sup}_{x_k\le y<x_{k+1}} |f(y)|\varphi^{-1}(y)h(y).$$

Lemma 6. [1, Lemma 3.1] (see also [22] for related results). Let $\{a_k\}_{k\in\mathbb{Z}}$, $\{b_k\}_{k\in\mathbb{Z}}$ and $\{\sigma_k\}_{k\in\mathbb{Z}}$ be sequences of non-negative numbers.

If $\tau_k \downarrow\downarrow$, then

$$\sum_{k\in\mathbb{Z}} \left(\sum_{m=-\infty}^{k} a_m \right)^q \tau_k \approx \sum_{k\in\mathbb{Z}} a_k^q \tau_k. \tag{6}$$

If $\sigma_k \uparrow\uparrow$, then

$$\sum_{k\in\mathbb{Z}} \left(\sum_{m=k}^{\infty} a_m \right)^q \sigma_k \approx \sum_{k\in\mathbb{Z}} a_k^q \sigma_k. \tag{7}$$

Lemma 7. [1, Lemma 3.2] Let $\{a_k\}_{k\in\mathbb{Z}}$, $\{b_k\}_{k\in\mathbb{Z}}$ and $\{\sigma_k\}_{k\in\mathbb{Z}}$ be sequences of non-negative numbers. If $\sigma_k \uparrow\uparrow$, then

$$\sum_{k\in\mathbb{Z}} \left(\sup_{k\le m<\infty} a_m \right) \sigma_k \approx \sum_{k\in\mathbb{Z}} a_k \sigma_k. \tag{8}$$

Lemma 8. [1, Proposition 4.1, (ii)] Let $1 < p < \infty$ and $\{v_k\}_{k\in\mathbb{Z}}$ be a sequence of positive numbers. Then inequality

$$\sum_{k\in\mathbb{Z}} a_k v_k \le c\bigg(\sum_{k\in\mathbb{Z}} a_k^p\bigg)^{1/p}$$

is satisfied for every sequence $\{a_k\}_{k\in\mathbb{Z}}$ of non-negative numbers if and only if

$$B_1 := \bigg(\sum_{k\in\mathbb{Z}} v_k^{p'}\bigg)^{1/p'} < \infty.$$

Moreover, the best constant c in inequality (8) satisfies $c \approx B_1$.

Theorem 2. [17, Theorem 2.4.3] Let $0 < r < \infty$. Assume that φ is an admissible function and f is a non-negative measurable function on $(0,\infty)$. Then

$$\bigg(\int_0^\infty \frac{\varphi(t)\varphi(x)|f(t)|dt}{\varphi(t)+\varphi(x)}\bigg)^r \approx \int_0^\infty \min\Big(\varphi^r(s), \varphi^r(x)\Big)\bigg(\int_0^\infty \frac{\varphi(t)|f(t)|dt}{\varphi(t)+\varphi(s)}\bigg)^{r-1}|f(s)|ds. \tag{9}$$

2 Auxiliary results and Discrete characterization

In this section, we present the necessary lemmas, propositions, and proofs to establish the main theorems. The results presented here play a crucial role in the analysis of weighted inequalities and their discretized forms. We rely on known properties of integral operators to establish a connection between the continuous case and its discrete counterpart.

The following lemma provides a discretized formulation of the left-hand side of inequality (1). To formulate the next result, define

$$G(t) := \varphi(t)\int_0^\infty \frac{w(s)ds}{\varphi(t)+\varphi(s)}. \tag{10}$$

Observe that by Lemma 1, $G \in \Omega_\varphi$.

Lemma 9. Let h and w be non-negative measurable functions on $(0,\infty)$ and φ be a quasi-concave admissible function on $(0,\infty)$. Assume that G is as defined in (10) and let $\{x_k\}_{k\in\mathbb{Z}}$ be a discretization sequence of G with respect to φ. Then

$$\begin{aligned}\int_0^\infty \operatorname*{ess\,sup}_{t<s<\infty} \frac{1}{\varphi(s)}\bigg(\int_0^s \tau h(\tau)d\tau + s\int_s^\infty h(\tau)d\tau\bigg)w(t)dt \\ \approx \sum_{k\in\mathbb{Z}} \operatorname*{ess\,sup}_{x_k<s<x_{k+1}} \frac{G(s)}{\varphi(s)}\bigg(\int_{x_k}^s \tau h(\tau)d\tau + s\int_s^{x_{k+1}} h(\tau)d\tau\bigg).\end{aligned} \tag{11}$$

Proof. As $\big(\int_0^s \tau h(\tau)d\tau + s\int_s^\infty h(\tau)d\tau\big)$ is increasing, we have

$$\begin{aligned}&\operatorname*{ess\,sup}_{t<s<\infty} \frac{1}{\varphi(s)}\bigg(\int_0^s \tau h(\tau)d\tau + s\int_s^\infty h(\tau)d\tau\bigg) \\ &\quad= \max\bigg\{\frac{1}{\varphi(t)}\operatorname*{ess\,sup}_{0<s<t}\bigg(\int_0^s \tau h(\tau)d\tau + s\int_s^\infty h(\tau)d\tau\bigg), \\ &\operatorname*{ess\,sup}_{t<s<\infty} \frac{1}{\varphi(s)^r}\bigg(\int_0^s \tau h(\tau)d\tau + s\int_s^\infty h(\tau)d\tau\bigg)\bigg\} \\ &\quad= \operatorname*{ess\,sup}_{0<s<\infty} \min\bigg\{\frac{1}{\varphi(t)}, \frac{1}{\varphi(s)}\bigg\}\bigg(\int_0^s \tau h(\tau)d\tau + s\int_s^\infty h(\tau)d\tau\bigg).\end{aligned}$$

Moreover, using the elementary inequality

$$\frac{1}{\varphi(s)+\varphi(t)} \le \min\left\{\frac{1}{\varphi(t)}, \frac{1}{\varphi(s)}\right\} \le \frac{2}{\varphi(s)+\varphi(t)}, \tag{12}$$

we get

$$\begin{aligned}&\operatorname*{ess\,sup}_{0<s<\infty} \min\left\{\frac{1}{\varphi(t)}, \frac{1}{\varphi(s)}\right\}\left(\int_0^s \tau h(\tau)d\tau + s\int_s^\infty h(\tau)d\tau\right)\\ &\qquad\approx \operatorname*{ess\,sup}_{0<s<\infty} \frac{1}{\varphi(s)+\varphi(t)}\left(\int_0^s \tau h(\tau)d\tau + s\int_s^\infty h(\tau)d\tau\right).\end{aligned}$$

Therefore,

$$\begin{aligned}&\int_0^\infty \operatorname*{ess\,sup}_{t<s<\infty} \frac{1}{\varphi(s)}\left(\int_0^s \tau h(\tau)d\tau + s\int_s^\infty h(\tau)d\tau\right)w(t)dt\\ &\quad\approx \int_0^\infty \operatorname*{ess\,sup}_{0<s<\infty} \frac{1}{\varphi(t)+\varphi(s)}\left(\int_0^s \tau h(\tau)d\tau + s\int_s^\infty h(\tau)d\tau\right)w(t)dt.\end{aligned}$$

Let $\{x_k\}_{k\in\mathbb{Z}}$ be a discretization sequence for G with respect to φ. Applying Lemma 5 with a function $f(s) = \int_0^s \tau h(\tau)d\tau + s\int_s^\infty h(\tau)d\tau$, and measure $d\,\nu(x) = w(x)\,dx$, we get

$$\begin{aligned}&\int_0^\infty \operatorname*{ess\,sup}_{t<s<\infty} \frac{1}{\varphi(s)}\left(\int_0^s \tau h(\tau)d\tau + s\int_s^\infty h(\tau)d\tau\right)w(t)dt\\ &\quad\approx \sum_{k\in\mathbb{Z}} \sup_{x_k<s<x_{k+1}} \left(\int_0^s \tau h(\tau)d\tau + s\int_s^\infty h(\tau)d\tau\right)\varphi^{-1}(s)G(s).\end{aligned}$$

Since $x_k < s < x_{k+1}$, by decomposing the integrals as $\int_0^s = \int_0^{x_k} + \int_{x_k}^s$ and $\int_s^\infty = \int_s^{x_{k+1}} + \int_{x_{k+1}}^\infty$, we obtain

$$\begin{aligned}&\int_0^\infty \operatorname*{ess\,sup}_{t<s<\infty} \frac{1}{\varphi(s)}\left(\int_0^s \tau h(\tau)d\tau + s\int_s^\infty h(\tau)d\tau\right)w(t)dt\\ &\quad\approx \sum_{k\in\mathbb{Z}} \sup_{x_k<s<x_{k+1}} \frac{G(s)}{\varphi(s)}\left(\int_{x_k}^s \tau h(\tau)d\tau + s\int_s^{x_{k+1}} h(\tau)d\tau\right)\\ &\qquad+\sum_{k\in\mathbb{Z}} \sup_{x_k<s<x_{k+1}} \frac{G(s)}{\varphi(s)}\int_0^{x_k} \tau h(\tau)d\tau + \sum_{k\in\mathbb{Z}} \sup_{x_k<s<x_{k+1}} \frac{G(s)}{\varphi(s)} s\int_{x_{k+1}}^\infty h(\tau)d\tau\\ &\quad=: I_1 + I_2 + I_3.\end{aligned} \tag{13}$$

Since $\frac{G}{\varphi}\downarrow$ and $\frac{G(x_k)}{\varphi(x_k)}\downarrow\downarrow$, according to (6), we have

$$\begin{aligned}I_2 &= \sum_{k\in\mathbb{Z}} \frac{G(x_k)}{\varphi(x_k)}\int_0^{x_k} \tau h(\tau)d\tau\\ &= \sum_{k\in\mathbb{Z}} \frac{G(x_k)}{\varphi(x_k)}\sum_{m=-\infty}^{k}\int_{x_{m-1}}^{x_m} \tau h(\tau)d\tau\\ &\approx \sum_{k\in\mathbb{Z}} \frac{G(x_k)}{\varphi(x_k)}\int_{x_{k-1}}^{x_k} \tau h(\tau)d\tau\\ &\le \sum_{k\in\mathbb{Z}} \sup_{x_{k-1}<s<x_k} \frac{G(s)}{\varphi(s)}\int_{x_{k-1}}^{s} \tau h(\tau)d\tau\\ &\le I_1.\end{aligned}$$

Next, as $\frac{x_{k+1}}{\varphi(x_{k+1})} \uparrow$ and $G(x_{k+1}) \upuparrows$, it follows that $\frac{G(x_{k+1})x_{k+1}}{\varphi(x_{k+1})} \upuparrows$. Applying (7), we get

$$\begin{aligned} I_3 &= \sum_{k\in\mathbb{Z}} \frac{G(x_{k+1})}{\varphi(x_{k+1})} x_{k+1} \int_{x_{k+1}}^{\infty} h(\tau)d\tau \\ &= \sum_{k\in\mathbb{Z}} \frac{G(x_{k+1})}{\varphi(x_{k+1})} x_{k+1} \sum_{m=k+1}^{\infty} \int_{x_m}^{x_{m+1}} h(\tau)d\tau \\ &\approx \sum_{k\in\mathbb{Z}} \frac{G(x_{k+1})}{\varphi(x_{k+1})} x_{k+1} \int_{x_{k+1}}^{x_{k+2}} h(\tau)d\tau \\ &\le \sum_{k\in\mathbb{Z}} \sup_{x_{k+1}<s<x_{k+2}} \frac{G(s)s}{\varphi(s)} \int_s^{x_{k+2}} h(\tau)d\tau \\ &\le I_1. \end{aligned}$$

Therefore,

$$I_1 + I_2 + I_3 \approx I_1$$

holds, and from (13) follows (11). □

Proposition 1. Let $1 < p < \infty$, w, v be non-negative measurable functions on $(0,\infty)$ and φ be a quasi-concave admissible function on $(0,\infty)$. Assume that G is as defined in (10) and let $\{x_k\}_{k\in\mathbb{Z}}$ be a discretization sequence of G with respect to φ. Then, there exists a positive constant C such that (1) holds for all non-negative measurable functions h on $(0,\infty)$ if and only if there exists a positive constant $\widetilde{C}$ such that

$$\sum_{k\in\mathbb{Z}} \operatorname*{ess\,sup}_{x_k<s<x_{k+1}} \frac{G(s)}{\varphi(s)} \left(\int_{x_k}^{s} \tau h(\tau)d\tau + s \int_s^{x_{k+1}} h(\tau)d\tau \right) \le \widetilde{C} \left(\sum_{k\in\mathbb{Z}} \int_{x_k}^{x_{k+1}} h^p(\tau)v(\tau)d\tau \right)^{\frac{1}{p}} \tag{14}$$

holds for all non-negative measurable functions h on $(0,\infty)$. Moreover the best constants C and $\widetilde{C}$, respectively in (1) and (14) satisfy $C \approx \widetilde{C}$.

Proof. Let $\{x_k\}_{k\in\mathbb{Z}}$ be a discretization sequence of G. Applying (11), for the left hand side of (1), we have

$$\begin{aligned} &\int_0^{\infty} \operatorname*{ess\,sup}_{t<s<\infty} \frac{1}{\varphi(s)} \left(\int_0^s \tau h(\tau)d\tau + s \int_s^{\infty} h(\tau)d\tau \right) w(t)dt \\ &\qquad \approx \sum_{k\in\mathbb{Z}} \operatorname*{ess\,sup}_{x_k<s<x_{k+1}} \frac{G(s)}{\varphi(s)} \left(\int_{x_k}^{s} \tau h(\tau)d\tau + s \int_s^{x_{k+1}} h(\tau)d\tau \right), \end{aligned}$$

on the other hand, for the right-hand side of (1), we have

$$\left(\int_0^{\infty} h^p(\tau)v(\tau)d\tau \right)^{\frac{1}{p}} = \left(\sum_{k\in\mathbb{Z}} \int_{x_k}^{x_{k+1}} h^p(\tau)v(\tau)d\tau \right)^{\frac{1}{p}}.$$

Then, it is clear that there exists a positive constant C such that inequality (1) holds for all non-negative measurable functions h on $(0,\infty)$ if and only if there exists a positive constant $\widetilde{C}$ such that (14) holds for all non-negative measurable functions h on $(0,\infty)$. Moreover, $C \approx \widetilde{C}$. □

The next proposition establishes an inequality that is equivalent to inequality (11). For this purpose, denote by

$$M(x_k, x_{k+1}) := \sup_{h\geq 0} \frac{\operatorname*{ess\,sup}\limits_{x_k<t<x_{k+1}} \dfrac{G(t)}{\varphi(t)} \left(\displaystyle\int_{x_k}^{t} \tau h(\tau) d\tau + t \int_{t}^{x_{k+1}} h(\tau) d\tau \right)}{\left(\int_{x_k}^{x_{k+1}} h^p(\tau) v(\tau) d\tau \right)^{\frac{1}{p}}}, \tag{15}$$

where $1 < p < \infty$, and the functions φ and G are as specified in Proposition 1.

Note that, using the characterizations of weighted iterated Copson and Hardy inequalities (see [17, Theorem 5.3.1]), we have

$$M(x_k, x_{k+1}) \approx \operatorname*{ess\,sup}_{x_k<t<x_{k+1}} \frac{G(t)}{\varphi(t)} \left(\int_{x_k}^{t} \tau^{p'} v^{1-p'}(\tau) d\tau + t^{p'} \int_{t}^{x_{k+1}} v^{1-p'}(\tau) d(\tau) \right)^{\frac{1}{p'}}. \tag{16}$$

Proposition 2. Let $1 < p < \infty$, w, v be non-negative measurable functions on $(0, \infty)$ and φ be a quasi-concave admissible function on $(0, \infty)$. Assume that G is as defined in (10) and let $\{x_k\}_{k\in\mathbb{Z}}$ be a discretization sequence of G with respect to φ. Then, there exist, a positive constant $\widetilde{C}$ such that inequality (14) holds for all non-negative measurable h on $(0, \infty)$ if and only if there exists a positive constant $\mathcal{C}$ such that

$$\sum_{k\in\mathbb{Z}} a_k M(x_k, x_{k+1}) \leq \mathcal{C} \left(\sum_{k\in\mathbb{Z}} a_k^p \right)^{\frac{1}{p}}, \tag{17}$$

holds for every sequence of non-negative numbers $\{a_k\}_{k\in\mathbb{Z}}$, where $M(x_k, x_{k+1})$ is defined in (15). Moreover, the best constants $\widetilde{C}$ and $\mathcal{C}$ in (14) and (17), respectively satisfy $\widetilde{C} \approx \mathcal{C}$.

Proof. Suppose that inequality (14) holds for all non-negative measurable h on $(0, \infty)$. By the definition of $M(x_k, x_{k+1})$ in (15), there exist functions $h_k \geq 0$, such that

$$\operatorname{supp} h_k \subset [x_k, x_{k+1}], \quad \left(\int_{x_k}^{x_{k+1}} h_k^p(\tau) v(\tau) d\tau \right)^{\frac{1}{p}} = 1 \tag{18}$$

and

$$\operatorname*{ess\,sup}_{x_k<t<x_{k+1}} \frac{G(t)}{\varphi(t)} \left(\int_{x_k}^{t} \tau h_k(\tau) d\tau + t \int_{t}^{x_{k+1}} h_k(\tau) d\tau \right) \geq \frac{1}{2} M(x_k, x_{k+1}). \tag{19}$$

For any sequence of non-negative numbers $\{a_k\}_{k\in\mathbb{Z}}$ define

$$h(\tau) = \sum_{j\in Z} a_j h_j(\tau).$$

Testing inequality (14) with h, and using (19), we get

$$\begin{aligned} \sum_{k\in\mathbb{Z}} \sup_{x_k<s<x_{k+1}} \frac{G(s)}{\varphi(s)} &\left(\int_{x_k}^{s} \tau h(\tau) d\tau + s \int_{s}^{x_{k+1}} h(\tau) d\tau \right) \\ &= \sum_{k\in\mathbb{Z}} a_k \sup_{x_k<s<x_{k+1}} \frac{G(s)}{\varphi(s)} \left(\int_{x_k}^{s} \tau h_k(\tau) d\tau + s \int_{s}^{x_{k+1}} h_k(\tau) d\tau \right) \\ &\geq \frac{1}{2} \sum_{k\in\mathbb{Z}} a_k M(x_k, x_{k+1}) \end{aligned}$$

and by (18) we have

$$\left(\sum_{k\in\mathbb{Z}}\int_{x_k}^{x_{k+1}} h^p(\tau)v(\tau)d\tau\right)^{\frac{1}{p}} = \left(\sum_{k\in\mathbb{Z}} a_k^p\int_{x_k}^{x_{k+1}} h_k^p(\tau)v(\tau)d\tau\right)^{\frac{1}{p}} = \left(\sum_{k\in\mathbb{Z}} a_k^p\right)^{\frac{1}{p}}.$$

Thus, (17) holds with $\mathcal{C} \lesssim \widetilde{C}$.

Conversely, observe that for each non-negative measurable h

$$M(x_k,x_{k+1}) \geq \frac{\operatorname*{ess\,sup}\limits_{x_k<t<x_{k+1}} \frac{G(t)}{\varphi(t)}\left(\int_{x_k}^{t}\tau h(\tau)d\tau + t\int_t^{x_{k+1}} h(\tau)d(\tau)\right)}{\left(\int_{x_k}^{x_{k+1}} h^p(\tau)v(\tau)d\tau\right)^{\frac{1}{p}}}$$

holds. Then (14) follows by inserting

$$a_k = \left(\int_{x_k}^{x_{k+1}} h^p(\tau)v(\tau)d\tau\right)^{\frac{1}{p}}$$

in (17) and $\widetilde{C} \leq \mathcal{C}$. Therefore, we obtain $\widetilde{C} \approx \mathcal{C}$. □

Theorem 3. Let $1 < p < \infty$, w, v be non-negative measurable functions on $(0,\infty)$ and φ be a quasi-concave admissible function on $(0,\infty)$. Assume that G is as defined in (10) and let $\{x_k\}_{k\in\mathbb{Z}}$ be a discretization sequence of G with respect to φ. Then there exists a positive constant C such that inequality (1) holds for all non-negative measurable functions f on $(0,\infty)$ if and only if

$$A := \left(\sum_{k\in\mathbb{Z}} \operatorname*{ess\,sup}_{x_k<t<x_{k+1}} \frac{G^{p'}(t)}{\varphi^{p'}(t)}\left(\int_{x_k}^{t} s^{p'}v^{1-p'}(s)ds + t^{p'}\int_t^{x_{k+1}} v^{1-p'}(s)ds\right)\right)^{\frac{1}{p'}} < \infty. \tag{20}$$

Moreover, the best constant C in (1) satisfies $C \approx A$.

Proof. Using Proposition 1 and Proposition 2, the best constant in (1) satifies $C \approx \mathcal{C}$, where $\mathcal{C}$ is the best constant in inequality (17). Therefore, applying Lemma 8 and using (16), the result follows. □

3 *Proof of Theorem 1*

Proof of Theorem 1 (i)⇔ (ii). By Theorem 3, it is enough to show that $C_1 \approx A$. Let H be defined as in (4) and G be defined as in (10). Assume that $\{x_k\}_{k\in\mathbb{Z}}$ is a discretization sequence of G with respect to φ and $\{y_k\}_{k\in Z}$ is a discretization sequence of H with respect to $\varphi^{p'}$.

Using Lemma 4 and the definition of G (see (10)), we have

$$C_1^{p'} \approx \sum_{k\in\mathbb{Z}}\left(\int_0^\infty \frac{w(s)ds}{\varphi(y_k)+\varphi(s)}\right)^{p'} H(y_k) = \sum_{k\in\mathbb{Z}} \frac{G^{p'}(y_k)}{\left(\varphi^p(y_k)H(y_k)^{-\frac{p}{p'}}\right)^{\frac{p'}{p}}}.$$

It is easy to check that $g = \varphi^p H^{-\frac{p}{p'}} \in \Omega_{\varphi^p}$. As $\{y_k\}_{k\in\mathbb{Z}}$ is a discretization sequence of H with respect to $\varphi^{p'}$, it is also a discretization sequence for g with respect to φ^p. Now, applying Lemma 3 for function $f = G$ and g and using the definition of H (see (4)), we obtain

$$C_1^{p'} \approx \sum_{k\in\mathbb{Z}} \frac{G^{p'}(x_k)}{\left(\varphi^p(x_k)H^{-\frac{p}{p'}}(x_k)\right)^{\frac{p'}{p}}} = \sum_{k\in\mathbb{Z}} G^{p'}(x_k) \operatorname*{ess\,sup}_{x_k<t<\infty} \frac{1}{\varphi^{p'}(t)}\left(\int_0^\infty \frac{s^{p'}t^{p'}}{s^{p'}+t^{p'}} v^{1-p'}(s)ds\right).$$

On the other hand, observe that

$$\int_0^\infty \frac{t^{p'} s^{p'}}{s^{p'}+t^{p'}} v^{1-p'}(s)ds \approx \int_0^t s^{p'} v^{1-p'}(s)ds + t^{p'} \int_t^\infty v^{1-p'}(s)ds.$$

Then,

$$\begin{aligned} C_1^{p'} &\approx \sum_{k\in Z} G^{p'}(x_k) \operatorname*{ess\,sup}_{x_k<t<\infty} \frac{1}{\varphi^{p'}(t)} \left(\int_0^t s^{p'} v^{1-p'}(s)ds + t^{p'} \int_t^\infty v^{1-p'}(s)ds \right) \\ &= \sum_{k\in \mathbb{Z}} G^{p'}(x_k) \sup_{k\le m} \operatorname*{ess\,sup}_{x_m<t<x_{m+1}} \frac{1}{\varphi^{p'}(t)} \left(\int_0^t s^{p'} v^{1-p'}(s)ds + t^{p'} \int_t^\infty v^{1-p'}(s)ds \right). \end{aligned}$$

Since $G^{p'}(x_k) \uparrow\uparrow$, applying (8), we obtain

$$C_1^{p'} \approx \sum_{k\in \mathbb{Z}} G^{p'}(x_k) \operatorname*{ess\,sup}_{x_k<t<x_{k+1}} \frac{1}{\varphi^{p'}(t)} \left(\int_0^t s^{p'} v^{1-p'}(s)ds + t^{p'} \int_t^\infty v^{1-p'}(s)ds \right).$$

Next, we want to show that

$$\begin{aligned} &\sum_{k\in \mathbb{Z}} G^{p'}(x_k) \operatorname*{ess\,sup}_{x_k<t<x_{k+1}} \frac{1}{\varphi^{p'}(t)} \left(\int_0^t s^{p'} v^{1-p'}(s)ds + t^{p'} \int_t^\infty v^{1-p'}(s)ds \right) \\ &\qquad \approx \sum_{k\in \mathbb{Z}} \operatorname*{ess\,sup}_{x_k<t<x_{k+1}} \frac{G^{p'}(t)}{\varphi^{p'}(t)} \left(\int_0^t s^{p'} v^{1-p'}(s)ds + t^{p'} \int_t^\infty v^{1-p'}(s)ds \right). \end{aligned} \tag{21}$$

As the function $G^{p'}$ is increasing, the upper estimate is trivial. Conversely,

$$\begin{aligned} &\sum_{k\in \mathbb{Z}} \operatorname*{ess\,sup}_{x_k<t<x_{k+1}} \frac{G^{p'}(t)}{\varphi^{p'}(t)} \left(\int_0^t s^{p'} v^{1-p'}(s)ds + t^{p'} \int_t^\infty v^{1-p'}(s)ds \right) \\ &\quad \approx \sum_{k\in \mathbb{Z}_1} \operatorname*{ess\,sup}_{x_k<t<x_{k+1}} \frac{G^{p'}(t)}{\varphi^{p'}(t)} \left(\int_0^t s^{p'} v^{1-p'}(s)ds + t^{p'} \int_t^\infty v^{1-p'}(s)ds \right) \\ &\qquad + \sum_{k\in \mathbb{Z}_2} \operatorname*{ess\,sup}_{x_k<t<x_{k+1}} \frac{G^{p'}(t)}{\varphi^{p'}(t)} \left(\int_0^t s^{p'} v^{1-p'}(s)ds + t^{p'} \int_t^\infty v^{1-p'}(s)ds \right) \\ &\quad \approx \sum_{k\in \mathbb{Z}_1} G^{p'}(x_k) \operatorname*{ess\,sup}_{x_k<t<x_{k+1}} \frac{1}{\varphi^{p'}(t)} \left(\int_0^t s^{p'} v^{1-p'}(s)ds + t^{p'} \int_t^\infty v^{1-p'}(s)ds \right) \\ &\qquad + \sum_{k\in \mathbb{Z}_2} \frac{G^{p'}(x_{k+1})}{\varphi^{p'}(x_{k+1})} \left(\int_0^{x_{k+1}} s^{p'} v^{1-p'}(s)ds + x_{k+1}^{p'} \int_{x_{k+1}}^\infty v^{1-p'}(s)ds \right) \\ &\quad \lesssim \sum_{k\in \mathbb{Z}} G^{p'}(x_k) \operatorname*{ess\,sup}_{x_k<t<x_{k+1}} \frac{1}{\varphi^{p'}(t)} \left(\int_0^t s^{p'} v^{1-p'}(s)ds + t^{p'} \int_t^\infty v^{1-p'}(s)ds \right). \end{aligned}$$

Therefore (21) holds. Thus

$$C_1^{p'} \approx \sum_{k\in \mathbb{Z}} \operatorname*{ess\,sup}_{x_k<t<x_{k+1}} \frac{G^{p'}(t)}{\varphi(t)^{p'}} \left(\int_0^t s^{p'} v^{1-p'}(s)ds + t^{p'} \int_t^\infty v^{1-p'}(s)ds \right).$$

Next, decomposing the integrals as $\int_0^t = \int_0^{x_k} + \int_{x_k}^t$ and $\int_t^\infty = \int_t^{x_{k+1}} + \int_{x_{k+1}}^\infty$ and using fact that $G^{p'}(x_k) \uparrow\uparrow$ and $\frac{t}{\varphi(t)}$ is increasing and therefore $\frac{G^{p'}(x_k)x_{k+1}^{p'}}{\varphi^{p'}(x_{k+1})} \uparrow\uparrow$ and applying Lemma 6, by (20) we obtain

$$\begin{aligned}
C_1^{p'} &\approx \sum_{k\in\mathbb{Z}} \operatorname*{ess\,sup}_{x_k<t<x_{k+1}} \frac{G^{p'}(t)}{\varphi^{p'}(t)} \left(\int_{x_k}^t s^{p'} v^{1-p'}(s)ds + t^{p'} \int_t^{x_{k+1}} v^{1-p'}(s)ds \right) \\
&\quad + \sum_{k\in\mathbb{Z}} \frac{G^{p'}(x_k)}{\varphi^{p'}(x_k)} \int_0^{x_k} s^{p'} v^{1-p'}(s)ds + \sum_{k\in\mathbb{Z}} \frac{G^{p'}(x_{k+1})x_{k+1}^{p'}}{\varphi^{p'}(x_{k+1})} \int_{x_{k+1}}^\infty v^{1-p'}(s)ds \\
&\approx \sum_{k\in\mathbb{Z}} \operatorname*{ess\,sup}_{x_k<t<x_{k+1}} \frac{G^{p'}(t)}{\varphi^{p'}(t)} \left(\int_{x_k}^t s^{p'} v^{1-p'}(s)ds + t^{p'} \int_t^{x_{k+1}} v^{1-p'}(s)ds \right) \\
&\quad + \sum_{k\in\mathbb{Z}} \frac{G^{p'}(x_k)}{\varphi^{p'}(x_k)} \int_{x_{k-1}}^{x_k} s^{p'} v^{1-p'}(s)ds + \sum_{k\in\mathbb{Z}} \frac{G^{p'}(x_{k+1})x_{k+1}^{p'}}{\varphi^{p'}(x_{k+1})} \int_{x_{k+1}}^{x_{k+2}} v^{1-p'}(s)ds \\
&\approx \sum_{k\in\mathbb{Z}} \operatorname*{ess\,sup}_{x_k<t<x_{k+1}} \frac{G^{p'}(t)}{\varphi^{p'}(t)} \left(\int_{x_k}^t s^{p'} v^{1-p'}(s)ds + t^{p'} \int_t^{x_{k+1}} v^{1-p'}(s)ds \right) \\
&= A^{p'}.
\end{aligned}$$

Consequently, $C_1 \approx A$, which is the desired estimate.

(ii)$\Leftrightarrow$ (iii). Using (9) for $f = \varphi^{-1}w$ and $r = p'$, (12), Fubini theorem and (5), we obtain

$$\begin{aligned}
C_1^{p'} &\approx \int_0^\infty \left(\int_0^\infty \left(\int_0^\infty \frac{w(s)ds}{\varphi(s)+\varphi(t)} \right)^{p'-1} \frac{\varphi(t)^{p'-1}w(t)}{\varphi(x)^{p'}+\varphi(t)^{p'}} dt \right) d\nu(x) \\
&= \int_0^\infty \left(\int_0^\infty \frac{w(s)ds}{\varphi(s)+\varphi(t)} \right)^{p'-1} \left(\int_0^\infty \frac{d\nu(x)}{\varphi(x)^{p'}+\varphi(t)^{p'}} \right) \varphi(t)^{p'-1} w(t)dt \\
&\approx \left(\int_0^\infty \left(\int_0^\infty \frac{w(s)ds}{\varphi(s)+\varphi(t)} \right)^{p'-1} \operatorname*{ess\,sup}_{t<s<\infty} \frac{1}{\varphi^{p'}(s)} \left(\int_0^\infty \frac{\tau^{p'}s^{p'}}{s^{p'}+\tau^{p'}} v^{1-p'}(\tau)d\tau \right) w(t)\varphi^{p'-1}(t)dt \right)^{\frac{1}{p'}} \\
&= C_2^{p'}.
\end{aligned}$$

The theorem is proved.

Remark 1. Note that the case when $p = 1$ was considered in [23].

Conclusion

In this paper, we investigate the conditions on the weight functions v, w ensuring that inequality (1) holds for all non-negative measurable functions. Inequality (1) holds if and only if a certain condition involving the weight functions v, w and the quasi-concave function φ holds.

The main result, presented in Theorem 1, confirms that the inequality under consideration holds for all non-negative measurable functions h when the associated integral condition is finite. To derive this result, we first prove Theorem 3, which provides a discrete characterization of inequality (1). This discrete formulation provides a detailed understanding of the best constant C in inequality (1), and shows conditions under which this constant can be determined. Finally, by applying antidiscretization techniques, we obtain the proof of the main result, Theorem 1.

The results obtained in this paper may be useful for further study of inequalities in weighted spaces, especially in analysing the interactions between weight functions and quasi-concave functions.

In particular, characterizations of the embeddings of the cones generated by the non-increasing rearrangements of the generalized fractional maximal functions [21]. They may also have significant applications in various areas of analysis, where such inequalities play a crucial role in understanding the behaviour of integrals involving multiple weight functions and functional operators.

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Author Contributions

All authors contributed equally to this work. They also participated in revising the manuscript and approved the final submission.

Conflict of Interest

The authors declare no conflict of interest.

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Fractal completeness and compactness in first-order model theory

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This article develops a formal framework for fractal structures within classical first-order model theory. The notion of fractality is reformulated in purely logical terms by replacing metric self-similarity with logical self-similarity induced by elementary endomorphisms of structures. A hierarchy of morphisms is introduced, including endomorphisms, elementary endomorphisms, and endiks, which preserve the truth of formulas in one or both directions. Based on these morphisms, fractal subsets and fractal models are defined via finite families of elementary self-maps. On the syntactic level, fractality is expressed through finite systems of T-elementary syntactic endomorphisms generating a stabilization process called the fractal corridor (a sequence of theories generated by iterated application of syntactic endomorphisms). A compatibility condition between syntactic and semantic fractality is formulated and proved. Using a Henkin-type construction, syntactic operators are lifted to semantic self-maps of a canonical model, yielding fractal completeness. A corresponding compactness theorem is also established. All constructions are carried out within standard first-order logic.

Keywords: first-order logic, fractal model, elementary endomorphism, syntactic fractality, semantic fractality, fractal completeness, fractal compactness, model theory.

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Introduction

Fractal structures originally arose in geometry as self-similar sets constructed via finite systems of contractions. In classical model theory, no metric structure is available; therefore, if fractality is to be formulated in logical terms, self-similarity must be expressed through structure-preserving self-maps. Traditionally, fractal structures are investigated within the framework of geometry as self-similar sets formed through iterated function systems of contractions, as detailed in the classical works of [1–3] and [4], as well as in the context of hierarchical and tree-like structures [5]. However, in classical model theory, the metric structure is absent, necessitating a reformulation of the notion of self-similarity in logical terms. In this paper, this transition is achieved by replacing metric self-similarity with logical self-similarity induced by elementary endomorphisms of structures. This approach allows fractality to be viewed as an internal structural condition imposed on theories themselves, based on the fundamental principles of constructing first-order models as laid out by [6] and [7]. The guiding principle of this paper is the replacement of metric self-similarity by logical self-similarity. Instead of contractions, we consider elementary endomorphisms of structures [7, 8]. Using finite families of such maps, fractal subsets and fractal models are defined. On the syntactic side, fractality is encoded via finite systems of T-elementary syntactic endomorphisms generating a stabilization mechanism called the fractal corridor [9].

A compatibility condition between semantic and syntactic systems is introduced. Under this condition, fractal-proof coincides with truth in compatible fractal models. A Henkin-type construction

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[6, 10] yields a canonical fractal model, establishing fractal completeness. Compactness follows by a finitary reduction argument.

The paper is organized as follows. Section 1 introduces the morphisms underlying fractal self-similarity. Section 2 defines fractal subsets and models. Section 3 develops syntactic fractality and the fractal corridor. Section 4 establishes the agreement theorem. Sections 5 and 6 contain the completeness and compactness of the fractal.

1 Morphisms in fractal model theory

Let L be a first-order signature and M an L-structure. A crucial aspect of formalizing fractality is the introduction of a hierarchy of mappings: endomorphisms, elementary endomorphisms, and "endiks". The use of the latter ensures the preservation of formula truth in both directions, which is fully consistent with the modern theory of morphisms by [7]. The proposed method extends the research on the semantic properties of models and fragments of Jonsson theories presented in the works of [8] and [11]. Within this paradigm, a fractal model is defined as a structure that constitutes a finite union of elementary images of itself. Conceptually, this correlates with the Iterated Function Systems (IFS) of [3] and [4], but implemented at the level of logical semantics.

Definition 1. [6] A mapping $f : M \to M$ is an endomorphism if it preserves functions and relations in the forward direction.

Definition 2. [6, 7] An endomorphism f is elementary if for every formula $\varphi(\bar{x})$ and tuple $\bar{a}$, $M \models \varphi(\bar{a}) \Rightarrow M \models \varphi(f(\bar{a}))$.

Definition 3. An elementary endomorphism is an *endik* if $M \models \varphi(\bar{a}) \iff M \models \varphi(f(\bar{a}))$.

Proposition 1. If f is an endik, then $f[M]$ is an elementary substructure of M and f is an isomorphism onto its image.

Example 1. In $(\mathbb{Q}, <)$, the maps $f_0(x) = x/2,\ f_1(x) = (x + 1)/2$ are elementary endomorphisms.

Example 2. (An endomorphism which is not elementary) Let $M = (Z, +)$ and define $f(x) = 2x$. Then f is an endomorphism of M, but not elementary.

Consider the formula

$$\psi(x) = \exists y(y + y = x).$$

Then $M \models \neg\psi(1)$, while $M \models \psi(f(1))$, since $f(1) = 2 = 1 + 1$. Thus, truth is not preserved under f, so f is not elementary.

Example 3. (Elementary endomorphism with proper image) Let V be an infinite-dimensional vector space over a field K, considered in the language of K-vector spaces. Fix a basis $(e_i)_{i\in\mathbb{N}}$ and define the shift

$$s(e_i) = e_{i+1},$$

extended linearly. Then s is an injective endomorphism with proper image. Since the theory of vector spaces over K admits quantifier elimination [6], s is elementary.

Example 4. (Endiks in the pure equality language) Let $M = (U; =)$ be an infinite structure in the language consisting only of equality. Every injective self-map $f : U \to U$ is an endik, since all formulas reduce to Boolean combinations of equalities among variables.

These examples show that [7, 8]

$$\text{endomorphism} \subsetneq \text{elementary endomorphism} \subsetneq \text{endik}.$$

2 Fractal subsets and fractal models

Let $n \neq 1$.

Definition 4. A subset $X \subseteq M$ is *fractal* if

$$X = \bigcup_{i=1}^{n} f_i[X]$$

for a finite family $\{f_1, \dots, f_n\}$ of elementary endomorphisms of M where $n \geq 1$ [3,4].
If $n = 1$, the map f_1 must not be an automorphism.

Definition 5. A structure M is called *fractal* if

$$M = \bigcup_{i=1}^{n} f_i[M]$$

for a finite family $\{f_1, \dots, f_n\}$ of elementary endomorphisms of M where $n \geq 1$ [3,4].
If $n = 1$, the map f_1 must not be an automorphism.

Remark 1. Degenerate one-map case. In Definitions 4 and 5, if the family consists of a single map ($n = 1$), this map must not be an automorphism. Otherwise, the equalities

$$X = f[X] \quad \text{or} \quad M = f[M]$$

become tautological and do not express genuine fractal self-similarity.

Proposition 2. If each f_i is an endik, then each image $f_i[M]$ is an elementary substructure of M [6,7].

3 Syntactic fractality and the fractal corridor

Let T be a first-order L-theory.
Convention. All syntactic constructions are considered relative to a fixed finite family

$$\Psi = \{\sigma_1, \dots, \sigma_m\}.$$

If $m = 1$, the operator σ_1 must be non-trivial, i.e. it must not act as an identity or as a purely automorphic symmetry preserving all sentences tautologically.

At the syntactic level, fractality generates a sequence of theories called the “fractal corridor”. Unlike geometric fractals, where the key parameters are measured and set dimension [2,3,5], stabilization in logical systems is considered on fragments of limited quantifier depth. The analysis of formula complexity and its fragments is a standard in modern model theory, allowing the application of finitary proof methods to infinite syntactic chains [9,12].

Let Form_L denote the set of all L-formulas.

Definition 6. A mapping $\sigma : L \to L$ is called a syntactic endomorphism if it induces a map on terms and on quantifier-free formulas such that:

1. For every function symbol f, $\sigma(f(t_1, \dots, t_n)) = f(\sigma(t_1), \dots, \sigma(t_n))$.
2. For every predicate symbol P, $\sigma(P(t_1, \dots, t_n)) = P(\sigma(t_1), \dots, \sigma(t_n))$.
3. Boolean structure is preserved: $\sigma(\neg\varphi) = \neg\sigma(\varphi)$, $\quad \sigma(\varphi \wedge \psi) = \sigma(\varphi) \wedge \sigma(\psi)$, and similarly for other connectives.

Definition 7. A syntactic endomorphism σ is called elementary if it extends to all formulas of L by the rules

$$\sigma(\forall x\,\varphi) = \forall x\,\sigma(\varphi), \quad \sigma(\exists x\,\varphi) = \exists x\,\sigma(\varphi),$$

and therefore acts on the whole set of sentences of L.

Definition 8. Let T be a theory in L. An elementary syntactic endomorphism σ is called T-elementary if for every sentence φ

$$T \vdash \varphi \quad \Longleftrightarrow \quad T \vdash \sigma(\varphi).$$

This condition is the syntactic analogue of endik symmetry [7, 8].

Definition 9. Let T be a theory in a first-order language L, and let

$$\Psi = \{\sigma_1, \ldots, \sigma_m\}$$

be a finite family of T-elementary syntactic endomorphisms. Define the fractal corridor recursively by

$$T[0] = T, \quad T[k+1] = \mathrm{Th}\Big(\bigcup_{j=1}^{m} \sigma_j(T[k])\Big), \quad k \geq 0.$$

The sequence T[k] is called the fractal corridor generated by Ψ.

A theory T is called syntactically fractal with respect to Ψ if there exists $n \geq 1$ such that $T[n] = T$.

Thus, every sentence of T is obtained as the image of some sentence of T under one of the operators in Ψ.

Definition 10. A sentence φ is called *fractal-proofed* from T relative to Ψ if there exists $k \in N$ such that $\varphi \in T[k]$.

Lemma 1. (Monotonicity dichotomy). Let $\Psi = \{\sigma_1, \ldots, \sigma_m\}$ be a finite family of syntactic endomorphisms and define $\{T[k]\}$ by

$$T[0] = T, \quad T[k+1] = \mathrm{Th}\Big(\bigcup_{j=1}^{m} \sigma_j(T[k])\Big).$$

(1) If the identity operator belongs to Ψ, then

$$T[k] \subseteq T[k+1] \quad \text{for all } k.$$

(2) If the identity operator does not belong to Ψ, monotonicity may fail; however, each $T[k]$ is a theory.

Proof. (1) Assume $id \in \Psi$. Let $\phi \in T[k]$. Then

$$\phi = id(\phi) \in \bigcup_{j=1}^{m} \sigma_j(T[k]).$$

Hence

$$\phi \in \mathrm{Th}\Big(\bigcup_{j=1}^{m} \sigma_j(T[k])\Big) = T[k+1],$$

so $T[k] \subseteq T[k+1]$.

(2) For the general case, by definition

$$T[k+1] = \mathrm{Th}\Big(\bigcup_{j=1}^{m} \sigma_j(T[k])\Big),$$

so $T[k+1]$ is closed deductively. Since $T[0] = T$ is a theory, by induction it follows that every $T[k]$ is a theory. $\square$

This completes the syntactic framework. In the next section, we establish the agreement theorem connecting syntactic fractality with semantic fractal models.

Lemma 2. Let σ be a T-elementary syntactic endomorphism. If $T \vdash \varphi$, then $T \vdash \sigma(\varphi)$.

Proof. If $T \vdash \varphi$, then $T \vdash \forall \bar{x}\,(\varphi \leftrightarrow \top)$. Since σ respects T-equivalence, $T \vdash \forall \bar{x}\,(\sigma(\varphi) \leftrightarrow \sigma(\top))$. But $\sigma(\top) = \top$, hence $T \vdash \sigma(\varphi)$. □

Proposition 3. For every $k \in \mathbb{N}$, the set $T[k]$ is a theory.

Proof. This follows immediately from the definition of $T[k+1]$ as a deductive closure. □

Recall that the corridor levels are theories defined by

$$T[0] = T, \qquad T[k+1] = \mathrm{Th}\Big(\bigcup_{j=1}^{m} \sigma_j(T[k])\Big).$$

We define the limit set of sentences

$$T[\infty] := \bigcup_{k \in \mathbb{N}} T[k].$$

Lemma 3. Assume that each $T[k]$ is consistent. Then $T[\infty]$ is consistent.

Proof. If $T[\infty]$ were inconsistent, then some finite set $\Delta \subseteq T[\infty]$ would be inconsistent. Since Δ is finite, there exists k_0 such that $\Delta \subseteq T[k_0]$. Hence $T[k_0]$ is inconsistent, which contradicts the assumption. □

Remark 2. Global stabilization of the entire chain $\{T[k]\}$ does not need to be in first-order logic. The correct finitary substitute is stabilization on bounded syntactic fragments [9], which is captured by Theorem 1. Thus, Theorem 2 provides finitary control of the corridor, while $T[\infty]$ serves as the global limit object used in the Henkin construction.

Corollary 1. If each $T[k]$ is consistent, then $T[\infty]$ admits a Henkin extension $\widehat{T} \supseteq T[\infty]$ and therefore has a term model $M_{\widehat{T}}$.

Proof. By Lemma 3, $T[\infty]$ is consistent. The standard Henkin [6, 10] extension procedure applies to any consistent set of sentences. □

To formulate a strict stabilization statement, we work with the quantifier depth $\mathrm{qd}(\varphi)$ of formulas.

Definition 11. For $N \in \mathbb{N}$, let $\mathrm{Sent}_{\le N}$ be the set of all L-sentences φ with $\mathrm{qd}(\varphi) \le N$. Define the N-fragment of $T[k]$ by

$$T[k]_{\le N} := T[k] \cap \mathrm{Sent}_{\le N}.$$

Lemma 4. For every $j \le m$ and every sentence φ, $\mathrm{qd}(\sigma_j(\varphi)) = \mathrm{qd}(\varphi)$.

Proof. By Definition 6, each σ_j commutes with $\exists$ and $\forall$ and preserves logical form. Hence quantifier nesting is preserved. □

Theorem 1. (Finite stabilization on bounded depth). For every $N \in \mathbb{N}$ there exists $k(N) \in \mathbb{N}$ such that

$$T[k(N)]_{\le N} = T[k(N)+1]_{\le N}.$$

Proof. (By induction on N).

Base $N = 0$. Sentences of quantifier depth 0 are Boolean combinations of atomic sentences. By Definition 6, each σ_j preserves Boolean structure and does not introduce quantifiers, hence acts inside Sent≤ 0. Therefore the sequence $T[k]{\leq 0}$ is an ascending chain of subsets of Sent$_{\leq 0}$. Since at each step we apply only finitely many operators to already obtained sentences (and do not increase depth), there is a stage $k(0)$ after which no new depth-0 sentences appear. Hence stabilization holds.

Inductive step. Assume the claim holds for N. Consider depth $N+1$. Any sentence φ of depth $\leq N+1$ is built from:

- Boolean combinations of sentences of depth $\leq N+1$, and
- quantified sentences $\exists x\,\psi(x)$ or $\forall x\,\psi(x)$, where $\mathrm{qd}(\psi) \leq N$.

By Lemma 4, each σ_j preserves quantifier depth and commutes with quantifiers:

$$\sigma_j(\exists x\,\psi) = \exists x\,\sigma_j(\psi), \qquad \sigma_j(\forall x\,\psi) = \forall x\,\sigma_j(\psi).$$

Hence, to generate new sentences of depth $\leq N+1$ at stage $k+1$, it suffices to generate new inner formulas of depth $\leq N$ at stage k. By the inductive hypothesis, there exists $k(N)$ such that $T[k(N)]{\leq N} = T[k(N)+1]{\leq N}$. From that stage onward, applying σ_j cannot produce new depth$\leq N$ components, hence cannot produce genuinely new quantified sentences of depth $\leq N+1$, nor new Boolean combinations thereof. Therefore stabilization holds at depth $N+1$ for some $k(N+1)$. □

4 Agreement between syntactic and semantic fractality

Let T be a first-order theory and $M \models T$.

Let

$$\Phi = \{f_1, \ldots, f_n\}$$

be a finite family of elementary endomorphisms of M [6, 7], and

$$\Psi = \{\sigma_1, \ldots, \sigma_m\}$$

a finite family of T-elementary syntactic endomorphisms.

The purpose of this section is to formulate a precise compatibility condition under which syntactic fractality corresponds to semantic fractality.

We first formalize the connection between Φ and Ψ.

Definition 12. The pair (Φ, Ψ) is called compatible if for every $j \leq m$ there exists $i \leq n$ such that for every formula $\varphi(\bar{x})$ and every tuple $\bar{a} \in M$,

$$M \models \sigma_j(\varphi)(\bar{a}) \iff M \models \varphi(f_i(\bar{a})).$$

Thus each syntactic operator corresponds to evaluation along a semantic endomorphism [7, 13].

Let the fractal corridor $T[k]$ be defined relative to Ψ.

Lemma 5. Assume (Φ, Ψ) is compatible and $M \models T$. Then for every k and every sentence $\varphi \in T[k]$, $M \models \varphi$.

Proof. We argue by induction on k. For $k = 0$, the claim holds since $M \models T$.

Assume the claim holds for k. Let $\varphi \in T[k+1]$ be a *sentence*. By Definition 10,

$$T[k+1] = \mathrm{Th}\Big(\bigcup_{j=1}^{m} \sigma_j(T[k])\Big),$$

hence φ is a logical consequence of finitely many sentences from $\bigcup_{j=1}^{m} \sigma_j(T[k])$.

By the induction hypothesis, $M \models \psi$ for each sentence $\psi \in T[k]$. By compatibility of the syntactic and semantic operators used in the paper, truth is preserved along the corresponding images, and therefore $M \models \sigma_j(\psi)$ for all such ψ and all $j \leq m$. Consequently, $M \models \varphi$. □

We now obtain the main result of this section.

Theorem 2. (Agreement Theorem). Let T be syntactically fractal with respect to Ψ, and let M be fractal with respect to Φ. Assume (Φ, Ψ) is compatible. Then:

1. Every fractal-proofed sentence from T is true in M.
2. The stabilized corridor $T[\infty]$ is contained in $\mathrm{Th}(M)$.

Proof. (1) Follows immediately from Lemma 5.

(2) By definition,

$$T[\infty] = \bigcup_{k \in N} T[k].$$

By Lemma 5, every sentence in each $T[k]$ is true in M. Hence every sentence in $T[\infty]$ is true in M, and therefore $T[\infty] \subseteq Th(M)$. □

Under compatibility, the syntactic corridor describes iterated evaluation along the finite system of elementary endomorphisms Φ. Stabilization therefore, corresponds to semantic closure under logical self-similarity [8, 11].

This establishes the precise bridge between syntactic and semantic fractality.

5 *Fractal completeness*

In this section we establish a completeness theorem corresponding to the fractal proof mechanism generated by the corridor $T[k]$. The proof is based on a Henkin-type construction in which the finite syntactic system Ψ induces a finite semantic system of elementary self-maps.

The proof of the fractal completeness theorem is based on a modified Henkin construction, detailed by [6, 10]. Within this framework, syntactic operators are canonically lifted to semantic mappings of the term model. This enables the establishment of an equivalence between syntactic fractal consistency and the existence of a corresponding fractal model. The resulting fractal compactness theorem confirms that the developed framework preserves the key properties of classical first-order logic [6, 7], adapting them to the requirements of the theory of self-similar structures [3, 4].

Throughout this section T is a first-order L-theory syntactically fractal with respect to a fixed finite family

$$\Psi = \{\sigma_1, \ldots, \sigma_m\}$$

of T-elementary syntactic endomorphisms.

We begin with the basic preservation property.

Lemma 6. If T is consistent, then each level $T[k]$ is consistent.

Proof. Assume $T[k]$ is consistent. Suppose for contradiction that $T[k+1]$ is inconsistent. Then there exist sentences $\varphi_1, \ldots, \varphi_r \in T[k+1]$ such that

$$\vdash (\varphi_1 \wedge \cdots \wedge \varphi_r) \to \bot.$$

By definition of $T[k+1]$, for each $\ell \leq r$ there exist $j(\ell) \leq m$ and $\psi_\ell \in T[k]$ such that

$$\varphi_\ell = \sigma_{j(\ell)}(\psi_\ell).$$

Consider the conjunction $\psi := \psi_1 \wedge \cdots \wedge \psi_r$. Since each σ_j preserves conjunctions,

$$\sigma_{j(\ell)}(\psi) = \sigma_{j(\ell)}(\psi_1) \wedge \cdots \wedge \sigma_{j(\ell)}(\psi_r).$$

Using the inconsistency of $\varphi_1, \ldots, \varphi_r$ and T-elementarity of the operators, we obtain a contradiction to consistency of $T[k]$. Hence $T[k+1]$ is consistent. The base case $T[0] = T$ is consistent by assumption. □

Let L_H be the Henkin expansion of L: for every formula $\exists x\, \varphi(x, \bar{y})$ we add a new function symbol $c_\varphi(\bar{y})$ intended to witness existence.

Let T_H be the corresponding Henkin extension of $T[\infty]$ obtained by adding all Henkin axioms.

$$\forall \bar{y}\, \big(\exists x\, \varphi(x, \bar{y}) \to \varphi(c_\varphi(\bar{y}), \bar{y})\big).$$

Lemma 7. If $T[\infty]$ is consistent, then the Henkin extension T_H is consistent.

Proof. Standard Henkin construction [6, 10]. □

Let Tm be the set of L_H-terms. Define an equivalence relation:

$$t \equiv s \quad \Longleftrightarrow \quad T_H \vdash t = s.$$

Let M be the term model whose universe is $\mathrm{Tm}/\equiv$, with interpretation given by

$$F^M([t_1], \ldots, [t_n]) := [F(t_1, \ldots, t_n)].$$

By the standard Henkin argument, $M \models T_H$, hence $M \models T[\infty]$, and in particular $M \models T$.

We now lift the syntactic operators to semantic self-maps of the term model. For each $\sigma_j \in \Psi$ define a mapping $f_j : M \to M$ by $f_j([t]) := [\sigma_j(t)]$.

Lemma 8. (Well-definedness). Each f_j is well-defined.

Proof. Assume $[t] = [s]$, i.e. $T_H \vdash t = s$. Then $T_H \vdash (t = s) \leftrightarrow (s = s)$. Since σ_j is T-elementary and preserves equality, it respects provable equivalence, hence $T_H \vdash \sigma_j(t) = \sigma_j(s)$, so $[\sigma_j(t)] = [\sigma_j(s)]$. □

Lemma 9. Each f_j is an elementary endomorphism of M.

Proof. Let $\varphi(\bar{x})$ be any L-formula and $\bar{t}$ a tuple of terms. By the Truth Lemma for the term model [6],

$$M \models \varphi([\bar{t}]) \quad \Longleftrightarrow \quad T_H \vdash \varphi(\bar{t}).$$

Applying σ_j and using preservation of logical form,

$$T_H \vdash \varphi(\bar{t}) \;\Rightarrow\; T_H \vdash \sigma_j(\varphi(\bar{t})) = \sigma_j(\varphi)(\sigma_j(\bar{t})).$$

Hence

$$M \models \sigma_j(\varphi)(f_j([\bar{t}])).$$

This is precisely the elementarity condition for f_j. □

We now show that the resulting family $\Phi = \{f_1, \ldots, f_m\}$ yields a fractal covering of M.

Proposition 4. The term model M is a fractal model with respect to $\Phi = \{f_1, \ldots, f_m\}$, and the pair (Φ, Ψ) is compatible.

Proof. Compatibility follows from the construction:

$$M \models \sigma_j(\varphi)(\bar{a}) \quad \Longleftrightarrow \quad M \models \varphi(f_j(\bar{a})).$$

To show covering, let $[t] \in M$. By construction of the term model, every term is obtained by applying finitely many syntactic operators from Ψ to initial terms. Hence, there exists $j \leq m$ and a term s such that $t = \sigma_j(s)$. Therefore

$$[t] = [\sigma_j(s)] = f_j([s]) \in f_j[M].$$

Thus

$$M = \bigcup_{j=1}^{m} f_j[M],$$

and M is fractal. □

We can now state the completeness theorem.

Theorem 3. (Fractal Completeness). Let T be syntactically fractal with respect to Ψ. For every sentence φ, if

$$M \models \varphi \quad \text{for every fractal model } M \models T \text{ compatible with } \Psi,$$

then φ is fractal-proofed from T.

Equivalently,

$$\varphi \notin \bigcup_k T[k] \quad \Longrightarrow \quad \exists M \models T \text{ fractal and compatible, such that } M \not\models \varphi.$$

Proof. Assume φ is not fractal-proofed from T. Then $T[\infty] \cup \{\neg\varphi\}$ is consistent. By Lemma 7 we obtain a Henkin extension and hence a term model $M \models T[\infty] \cup \{\neg\varphi\}$. By Proposition 4 this model is fractal and compatible. Thus $M \not\models \varphi$. □

6 *Fractal compactness*

In this section we derive a compactness theorem corresponding to the fractal proof system introduced above. The argument follows the classical scheme but uses the fractal-proof mechanism instead of ordinary derivability.

Throughout this section T is syntactically fractal with respect to a fixed finite family Ψ.

Definition 13. A set of sentences Σ is called finitely fractal-consistent relative to T and Ψ if for every finite subset $\Sigma_0 \subseteq \Sigma$,

$$\bot \notin \bigcup_k (T \cup \Sigma_0)[k],$$

where the corridor is defined relative to Ψ.

Theorem 4. (Fractal Compactness).

Let Σ be a set of sentences. If Σ is finitely fractal-consistent relative to T and Ψ, then there exists a fractal model $M \models T \cup \Sigma$ compatible with Ψ.

Proof. Assume Σ is finitely fractal-consistent. Suppose for contradiction that $T \cup \Sigma$ is fractal-inconsistent, i.e. $\bot \in \bigcup_k (T \cup \Sigma)[k]$. Then there exists k such that

$$\bot \in (T \cup \Sigma)[k].$$

Since the corridor construction is finitary at each step and each level depends only on finitely many applications of operators from Ψ, there exists a finite subset $\Sigma_0 \subseteq \Sigma$ such that

$$\perp \in \bigcup_k (T \cup \Sigma_0)[k].$$

This contradicts finite fractal-consistency. Hence $T \cup \Sigma$ is fractal-consistent. By Theorem 3 (Fractal Completeness), there exists a fractal model $M \models T \cup \Sigma$. Compatibility follows from the construction of Section 5. □

We therefore obtain:

Corollary 2. Fractal-proof over T satisfies compactness:

$$\varphi \text{ is fractal-proofed from } T \quad \Longleftrightarrow \quad \text{there exists a finite } T_0 \subseteq T \text{ such that } \varphi \text{ is fractal-proofed from } T_0.$$

This completes the proof of fractal compactness.

Conclusion

We have introduced a formal framework for fractality within classical first-order model theory. The central idea consists in replacing metric self-similarity by logical self-similarity generated by finite systems of elementary self-maps. On the semantic side, fractal models are defined as finite unions of elementary images of themselves. On the syntactic side, fractality is expressed through finite systems of T-elementary syntactic endomorphisms generating the fractal corridor.

A precise compatibility condition between the semantic system Φ and the syntactic system Ψ establishes a bridge between model-theoretic self-embeddings and syntactic stabilization. Under this condition, fractal-proof coincides with truth in compatible fractal models, yielding a soundness theorem. Using a Henkin-type construction, syntactic operators are lifted canonically to elementary endomorphisms of the term model, producing a fractal model and establishing fractal completeness. Compactness follows by a finitary reduction argument inside the corridor mechanism.

All constructions remain entirely within standard first-order logic. No extension of the language or modification of classical proof theory is required. Fractality appears as an internal structural condition imposed on theories and models rather than as a new logical system.

This completes the development of fractal completeness and compactness within first-order model theory.

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Author Contributions

All authors contributed equally to this work.

Conflict of Interest

The authors declare no conflict of interest.

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ANNIVERSARIES

Professor Yeshkeyev's Life in Science and Logic: Celebrating the Scholar's 70th Anniversary

In the age of digitalisation and rapid technological progress, it is easy to forget that every computer program, every algorithm, and the very structure of our thinking are founded on mathematical logic. In this abstract yet vitally important field, Kazakhstan has its own recognised authority.

In May 2026, Aibat Rafhatovich Yeshkeyev marks his 70th birthday. He is a Doctor of Physical and Mathematical Sciences, Professor, and Professor-Researcher at the Karaganda National Research University named after academician Ye.A. Buketov, a man whose work in model theory has built a bridge between Kazakhstani science and the international academic community.

Born in Karaganda on 17 May 1956, Aibat Rafhatovich has devoted his life to the pursuit of truth through rigorous mathematical reasoning. His academic path is an example of dedication to his vocation and to his alma mater.

A graduate of Karaganda State University in 1978, he quickly moved from teaching to serious research.

A crucial stage was his postgraduate and doctoral studies at the Institute of Mathematics of the Siberian Branch of the Russian Academy of Sciences in Novosibirsk's Akademgorodok, one of the major centres of mathematical thought.

In 1995, he defended his Candidate of Sciences thesis on "Jonsson Theories" at the Institute of Theoretical and Applied Mathematics of the National Academy of Sciences of the Republic of Kazakhstan (Almaty). His supervisor was Professor T. G. Mustafin. In 2010, he defended his doctoral thesis on "The Structure of Perfect Positive Jonsson Theories" at Eurasian National University.

Mathematics is an international science, yet it also includes some of the most demanding and elite areas of research. Model theory, in which Professor Yeshkeyev specialises, is considered one of the most complex and intellectually demanding disciplines. Jonsson theories occupy a special place in his research. Professor Yeshkeyev is a recognised expert on Jonsson theories. His research on the structure of perfect positive theories has become a classic in modern logic.

Aibat Rafhatovich has succeeded in developing and enriching this field by proposing new approaches to the classification of models and the study of their properties.

His international recognition is not merely formal. It is grounded in publications in high-impact journals and presentations at scientific symposia, where his ideas are evaluated by leading experts in the field. He has published over 300 scientific papers, including articles in international peer-reviewed journals indexed in the Web of Science and Scopus databases. His h-index is 14 in Web of Science and 12 in Scopus.

A.R. Yeshkeyev is actively involved in international scientific activities. The results of his research have been presented at scientific conferences and seminars in a number of countries, including France, the United Kingdom, Germany, the Czech Republic, China, South Korea, Turkey, Italy and others. He has delivered plenary lectures at international conferences dedicated to topical issues in model theory.

A.R. Yeshkeyev has led and participated in a number of research and grant-funded projects supported by the Ministry of Education and Science of the Republic of Kazakhstan. He is currently leading a research project dedicated to the study of fragments of definable closures in semantic models of Jonsson theories.

A.R. Yeshkeyev is a member of the Association for Symbolic Logic. He is actively involved in research and teaching, supervising undergraduate, master's, and doctoral theses.

Among international colleagues, the name Yeshkeyev is firmly associated with the "Karaganda School of Logic".

Aibat Rafhatovich's entire professional life has been inextricably linked with Karaganda National Research University. Here, he rose from a young researcher to a professor, becoming a living legend of the faculty.

For the university, he is not merely a theoretical scholar, but also a powerful source of ideas.

Under his leadership, a research school has taken shape, focusing on current issues in model theory and algebraic systems. More than 30 master's theses and 11 PhD dissertations have been completed under his supervision, and the university has strengthened its position as one of the leading centres of mathematical education in Central Asia.

For his contribution to the development of science and education, he has been awarded a number of state and departmental honours, including the badge "For Merit in the Development of Science of the Republic of Kazakhstan", the title "Best University Lecturer" (2016), as well as commemorative medals marking the independence of the Republic of Kazakhstan.

Professor Yeshkeyev's students know that his exam is a serious intellectual test. But behind his outward strictness lies a sincere desire to teach young people the most important thing: a culture of thought. Aibat Rafhatovich is convinced that a mathematician is not merely someone who performs calculations, but someone who performs construct flawless logical chains and see beauty in the rigour of proofs.

At the age of 70, Aibat Rafhatovich retains an enviable capacity for work and a sharp mind. He continues to publish actively, review his colleagues' work, and participate in the life of the scientific community. His life journey is an example of how dedication to a single idea and a single science allows one to reach heights recognised far beyond the borders of one's country.

His anniversary is a celebration not only for his family and colleagues, but for the entire scientific community of Kazakhstan. For as long as scholars of such calibre continue to teach at our universities, we can be confident that the intellectual potential of the country remains in reliable hands.

The staff of the Faculty of Mathematics and Information Technologies
of the Karaganda National Research University named after Academician Ye.A. Buketov
and the editorial board of the journal
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