ISSN 2518-7929 (Print)
ISSN 2663-5011 (Online)

BULLETIN
OF THE KARAGANDA UNIVERSITY

MATHEMATICS
SERIES

No. 1(121)/2026

ISSN 2518-7929 (Print)
ISSN 2663–5011(Online)
Индексі 74618
Индекс 74618

ҚАРАҒАНДЫ УНИВЕРСИТЕТІНІҢ **ХАБАРШЫСЫ**

ВЕСТНИК КАРАГАНДИНСКОГО УНИВЕРСИТЕТА

BULLETIN OF THE KARAGANDA UNIVERSITY

МАТЕМАТИКА сериясы

Серия МАТЕМАТИКА

MATHEMATICS Series

No. 1(121)/2026

1996 жылдан бастап шығады
Издается с 1996 года
Founded in 1996

Жылына 4 рет шығады
Выходит 4 раза в год
Published 4 times a year

Қарағанды / Караганда / Karaganda
2026

Postal address: 28, University Str., Karaganda, 100024, Kazakhstan.
E-mail: math.bku@gmail.com; Web-site: mts.buketov.edu.kz

Executive Editor
PhD **G.B. Sarzhanova**

Proofreader
I.N. Murtazina

Computer layout
K.A. Foroastyanova

Bulletin of the Karaganda University. Mathematics Series. — 2026. — No. 1 (121). — 249 p. — ISSN 2518-7929 (Print). ISSN 2663–5011 (Online).

Proprietary: NLC «Karaganda National Research University named after academician Ye.A. Buketov».

Registered by the Ministry of Culture and Information of the Republic of Kazakhstan.
Rediscount certificate No. KZ33VPY00135817 dated 05.12.2025.

Signed in print 30.03.2026. Format 60×84 1/8. Photocopier paper. Volume 31,13 p.sh. Circulation 200 copies.
Price upon request. Order № 21.

Printed in the Publishing house of NLC «Karaganda National Research University named after academician Ye.A. Buketov».
28, University Str., Karaganda, 100024, Kazakhstan. Tel.: (7212) 35-63-16. E-mail: printed@karnu-buketov.edu.kz.

CONTENT

MATHEMATICS

ANNIVERSARIES

MATHEMATICS

https://doi.org/10.31489/2026M1/5-22 *Research article*

Reversed Weighted Hardy-type Inequalities with Negative Indices

K.R. Abdo

University of Technology and Applied Sciences, Al Rustaq, Sultanate of Oman;
Fayoum University, Fayoum, Egypt
(E-mail: khadija.abdo@utas.edu.om, kra00@fayoum.edu.eg)

This research paper presents a comprehensive investigation of novel Hardy-type dynamic inequalities that incorporate two independent weight functions, denoted as $\mathfrak{u}$ and $\mathtt{v}$. A distinctive feature of this work is its focus on time scales calculus with negative parameters, a generalization that unifies and extends discrete and continuous analysis. The basic methodology involves the application of the reverse Hölder's inequality and the Minkowski integral inequality to rigorously deduce all essential results. To illustrate the adaptability of our results, we provide explicit examples of the corresponding discrete and integral analogues for various time scales: when $\mathbb{T} = \mathbb{N}$ (the natural numbers, indicating discrete sequences), $\mathbb{T} = l^{\mathbb{N}_0}$ for $l > 1$ (a quantum time scale), and $\mathbb{T} = \mathbb{R}$ (the real numbers, signifying the classical continuous case). This paper situates its findings within a wider mathematical framework by demonstrating how they contain and extend certain cases of reverse Hardy-type dynamic inequalities previously formulated by distinguished scholars including Prokhorov, Kufner, Yang, Nguyen, and Benaissa. Consequently, this work presents a cohesive framework that broadens the theoretical terrain of Hardy-type inequalities.

Keywords: time scales, dynamic inequalities, reverse Hardy inequality, negative exponents, delta differentiation, Keller's rule, reverse time scale Hölder's inequality, integral time scale Minkowski's inequality.

2020 Mathematics Subject Classification: 26D10, 26D15, 39A12, 34N05, 47A30.

Introduction

In a pivotal paper of Hardy [1], he demonstrated the discrete classical inequality

$$\sum_{\mathfrak{n}=1}^{\infty}\left(\frac{1}{\mathfrak{n}}\sum_{\mathfrak{k}=1}^{\mathfrak{n}}\mathtt{b}^{\frac{1}{\mathtt{r}}}(\mathfrak{k})\right)^{\mathtt{r}} \leq \left(\frac{\mathtt{r}}{\mathtt{r}-1}\right)^{\mathtt{r}}\sum_{\mathfrak{n}=1}^{\infty}\mathtt{b}(\mathfrak{n}), \quad \mathtt{r}>1,$$

that provides an essential implementation for double series of Hilbert's inequality, a widely recognized concept at this time, where $\mathtt{b}(\mathfrak{n}) \geq 0$ for $\mathfrak{n} \geq 1$. Moreover, he [2] discovered the corresponding inequality which affirms that, for $\mathtt{p} > 1$ and $\mathfrak{h}(\mathfrak{x})$ is an integrable positive function that belongs to the weighted space $L^{\mathtt{p}}(\mathbb{R}^+)$, the inequality

$$\|H_{\mathfrak{h}}\|_{L^{\mathtt{p}}(\mathbb{R}^+)} \leq \frac{\mathtt{p}}{\mathtt{p}-1}\|\mathfrak{h}\|_{L^{\mathtt{p}}(\mathbb{R}^+)} \tag{1}$$

Received: 5 January 2025; *Accepted:* 24 November 2025.

holds, where $H_{\mathfrak{h}}(\mathfrak{x}) = \frac{1}{\mathfrak{x}}\int_{\mathtt{a}}^{\mathfrak{x}} \mathfrak{h}\,(t)\,dt$ and $\|\mathfrak{h}\|_{L^{\mathtt{p}}(\mathbb{R}^{+})} = \left(\int_{\mathtt{a}}^{\infty} \mathfrak{h}^{\mathtt{p}}\,(\mathfrak{x})\,d\mathfrak{x}\right)^{1/p}$, with $\mathtt{a} > 0$ where the constant factor $\left(\frac{\mathtt{p}}{\mathtt{p}-1}\right)^{\mathtt{p}}$ is sharp one. Extending inequality (1) over the past few decades has led to the following generic mixed norm inequality

$$\|S_{\mathfrak{h}}\|_{L^{q}_{\mathtt{u}}[\mathtt{a},\mathtt{b})} \leq \mathbf{C}\,\|\mathfrak{h}\|_{L^{\mathtt{p}}_{\mathtt{v}}[\mathtt{a},\mathtt{b})}, \quad \text{where} \quad S_{\mathfrak{h}}\,(\mathfrak{x}) = \int_{\mathtt{a}}^{\mathfrak{x}} \mathfrak{h}(t)\Delta t, \tag{2}$$

that holds with $\mathfrak{h} \in L^{\mathtt{p}}_{\mathtt{v}}[\mathtt{a},\mathtt{b})$ and $S_{\mathfrak{h}} \in L^{q}_{\mathtt{u}}[\mathtt{a},\mathtt{b})$, where the norm of $\mathfrak{h}$ is defined as

$$\|\mathfrak{h}\|_{L^{\mathtt{p}}_{\mathtt{v}}[\mathtt{a},\mathtt{b})} = \left(\int_{\mathtt{a}}^{\mathtt{b}} \mathfrak{h}^{\mathtt{p}}(\mathfrak{x})\mathtt{v}(\mathfrak{x})d\mathfrak{x}\right)^{1/\mathtt{p}},$$

for any given positive measurable weights $\mathtt{u}$ and $\mathtt{v}$, defined on the open interval $(\mathtt{a},\mathtt{b})$ and $1 < \mathtt{p} \leq q < \infty$, for all values of $-\infty \leq \mathtt{a} < \mathtt{b} \leq \infty$. A large body of literature has addressed new generalizations and extensions of the Hardy inequality (2); the interested reader is referred to the relevant papers [3–5] and books [6–8]. Beesack et al. in [9] considered inequality (2), they obtained the sufficient condition for exponents $\mathtt{p}$ and q that are less than one, where $\mathtt{p} \neq 0, q \neq 0$, then for positive weights $\mathtt{u}$ and $\mathtt{v}$ defined on $(\mathtt{a},\mathtt{b})$, they concluded Hardy reverse inequality that has the form

$$\left(\int_{\mathtt{a}}^{\mathtt{b}} (\mathtt{v}\,(\mathfrak{x})\,\mathfrak{h}\,(\mathfrak{x}))^{\mathtt{p}}\,d\mathfrak{x}\right)^{\frac{1}{\mathtt{p}}} \leq \mathbf{C}\left(\int_{\mathtt{a}}^{\mathtt{b}} \left(\mathtt{u}\,(\mathfrak{x})\int_{\mathtt{a}}^{\mathfrak{x}} \mathfrak{h}\,(t)\,dt\right)^{q} d\mathfrak{x}\right)^{\frac{1}{q}}, \tag{3}$$

that is valid if and only if

$$\inf_{s>0}\left(\int_{\mathtt{a}}^{s} \mathtt{u}^{q}\,(\mathfrak{x})\,d\mathfrak{x}\right)^{\frac{1}{q}}\left(\int_{\mathtt{a}}^{s} \mathtt{v}^{-\mathtt{p}'}\,(\mathfrak{x})\,d\mathfrak{x}\right)^{\frac{1}{\mathtt{p}'}} = \mathcal{B}_1 < \infty, \qquad \frac{1}{\mathtt{p}'} + \frac{1}{\mathtt{p}} = 1,$$

where $s > 0$. Moreover, they deduced the dual reverse integral Hardy inequality

$$\left(\int_{\mathtt{a}}^{\mathtt{b}} (\mathtt{v}\,(\mathfrak{x})\,\mathfrak{h}\,(\mathfrak{x}))^{\mathtt{p}}\,d\mathfrak{x}\right)^{\frac{1}{\mathtt{p}}} \leq \mathbf{C}\left(\int_{\mathtt{a}}^{\mathtt{b}} \left(\mathtt{u}\,(\mathfrak{x})\int_{\mathfrak{x}}^{\mathtt{b}} \mathfrak{h}\,(t)\,dt\right)^{q} d\mathfrak{x}\right)^{\frac{1}{q}}, \tag{4}$$

which also asserts if and solely if $\inf\limits_{s>0}\left(\int_{s}^{\mathtt{b}} \mathtt{u}^{q}\,(\mathfrak{x})\,d\mathfrak{x}\right)^{\frac{1}{q}}\left(\int_{s}^{\mathtt{b}} \mathtt{v}^{-\mathtt{p}'}\,(\mathfrak{x})\,d\mathfrak{x}\right)^{\frac{1}{\mathtt{p}'}} = \mathcal{B}_2 < \infty$.

The authors in [10], expanded the reverse Hardy inequalities (3) and (4) with kernels. In specific, they showed that for the non-increasing $\mathbf{K}(\mathfrak{x},y)$ with $q \leq \mathtt{p} < 0$, the resulting inequality

$$\|\mathfrak{h}\|_{L^{\mathtt{p}}_{\mathtt{v}}[0,\infty)} \leq \mathbf{C}\,\|S_{\mathfrak{h}}\|_{L^{q}_{\mathtt{u}}[0,\infty)}, \quad \text{where} \quad S_{\mathfrak{h}} = \left(\int_{0}^{\mathfrak{x}} \mathbf{K}(\mathfrak{x},y)\mathfrak{h}\,(y)\,dy\right)^{q}, \tag{5}$$

that for every $\mathfrak{h} \in L^{\mathtt{p}}_{\mathtt{v}}[0,\infty)$, $S_{\mathfrak{h}} \in L^{q}_{\mathtt{u}}[0,\infty)$ where $\|\mathfrak{h}\|_{L^{\mathtt{p}}_{\mathtt{v}}[0,\infty)} = \left(\int_{0}^{\infty} \mathfrak{h}^{\mathtt{p}}(\mathfrak{x})\mathtt{v}(\mathfrak{x})d\mathfrak{x}\right)^{1/\mathtt{p}}$, then the inequality (5) holds if and only if $\inf\limits_{s} \mathfrak{J}_1(s) = \mathcal{B}_3 > 0$, where $\mathfrak{J}_1(s) = \left(\int_{0}^{s} \mathtt{v}^{1-\mathtt{p}'}\,(\mathfrak{x})\,\mathbf{K}^{p'}(\mathfrak{x},y)dx\right)^{\frac{1}{\mathtt{p}'}}\left(\int_{0}^{s} \mathtt{u}\,(\mathfrak{x})\,d\mathfrak{x}\right)^{\frac{1}{q}}$ is not decreasing and the constant $\mathbf{C}$ satisfies the condition that $1/\mathbf{C} \geq (\mathtt{p}')^{1/\mathtt{p}'}(-\mathtt{p})^{1/q}\mathcal{B}_3$. On the other hand, they deduced the dual reverse integral Hardy inequality

$$\left(\int_{0}^{\infty} \mathtt{v}\,(\mathfrak{x})\,h^{\mathtt{p}}\,(\mathfrak{x})\,d\mathfrak{x}\right)^{\frac{1}{\mathtt{p}}} \leq \mathbf{C}\left(\int_{0}^{\infty} \mathtt{u}\,(\mathfrak{x})\left(\int_{\mathfrak{x}}^{\infty} \mathbf{K}(y,\mathfrak{x})\mathfrak{h}\,(y)\,dy\right)^{q} d\mathfrak{x}\right)^{\frac{1}{q}},$$

which also holds for the non-decreasing $\mathbf{K}(\mathfrak{x}, y)$ if and only if $\inf_s \mathfrak{J}_2(s) = \mathcal{B}_4 > 0$, where $\mathfrak{J}_2(s) = \left(\int_s^{\mathsf{b}} \mathsf{v}^{-\mathsf{p}'}(\mathfrak{x})\, d\mathfrak{x}\right)^{\frac{1}{\mathsf{p}'}} \left(\int_s^{\mathsf{b}} \mathsf{u}^q(\mathfrak{x})\, d\mathfrak{x}\right)^{\frac{1}{q}}$ is non-increasing and $\mathbf{C}$ here satisfies $1/\mathbf{C} \geq (\mathsf{p}')^{1/\mathsf{p}'} (-p)^{1/q} \mathcal{B}_4$.

Since the previous inequalities were discovered, various papers have been published in the literature that dealt with contemporary proofs, generalizations, and extensions. Now, we review some of these results that can stimulate and clarify the content of this paper. Prokhorov in [11] focused on generalizing inequality (3) to the range of $-\infty < q \leq \mathsf{p} < 0$ which has the form

$$\left(\int_{\mathsf{a}}^{\mathsf{b}} \mathfrak{h}^{\mathsf{p}}(\mathfrak{x})\, d\mathfrak{x}\right)^{\frac{1}{\mathsf{p}}} \leq \mathbf{C} \left(\int_{\mathsf{a}}^{\mathsf{b}} \mathsf{u}(\mathfrak{x}) \left(\int_{\mathsf{a}}^{\mathfrak{x}} \mathsf{v}(t)\, \mathfrak{h}(t)\, dt\right)^{q} d\mathfrak{x}\right)^{\frac{1}{q}},$$

and it holds for the condition that $\sup_{\mathsf{a}<t<\mathsf{b}} \left(\int_{\mathsf{a}}^{\mathfrak{x}} \mathsf{v}^{\mathsf{p}'}(t)\, dt\right)^{\frac{-1}{\mathsf{p}'}} \left(\int_{\mathsf{a}}^{\mathfrak{x}} \mathsf{u}^q(t)\, dt\right)^{\frac{-1}{q}} = \mathcal{B}_5 < \infty$, for the lowest suitable constant $\mathbf{C}$, likewise he generated the dual case. Through involving a constraint that $q/\mathsf{p} \geq 1$, Kufner et al. [12] extended inequality (3) to get the best possible estimation that makes the next inequality

$$\left(\int_{\mathsf{a}}^{\mathsf{b}} \mathsf{v}(\mathfrak{x})\, \mathfrak{h}^{\mathsf{p}}(\mathfrak{x})\, d\mathfrak{x}\right)^{\frac{1}{\mathsf{p}}} \leq \mathbf{C} \left(\int_{\mathsf{a}}^{\mathsf{b}} \mathsf{u}(\mathfrak{x}) \left(\int_{\mathsf{a}}^{\mathfrak{x}} \mathfrak{h}(t)\, dt\right)^{q} d\mathfrak{x}\right)^{\frac{1}{q}} \tag{6}$$

holds. They concluded that the best possible constant $\mathbf{C}$ for $-\infty < q \leq \mathsf{p} < 0$, is given by

$$\mathcal{B}_6 \leq \mathbf{C} < 2^{-\frac{1}{q}} \left(\frac{\mathsf{p} + s - 2}{\mathsf{p} - 1}\right)^{\frac{1}{\mathsf{p}'}} \mathcal{B}_6, \quad s \in [\mathsf{p}, 1),$$

where $\mathcal{B}_6 = \sup_{\mathsf{a}<\mathfrak{x}<\mathsf{b}} \left(\int_{\mathsf{a}}^{\mathfrak{x}} V^{\frac{\mathsf{p}-s}{\mathsf{p}} q}(t) \mathsf{u}(t) dt\right)^{\frac{-1}{q}} V^{\frac{1-s}{\mathsf{p}}}(\mathfrak{x})$, they also deduced analogy inequality of (6). In our paper, we will concentrate on the previous inequality (6) and its analogy.

Motivated by developments in the continuous setting, it is natural to investigate weighted discrete Hardy inequalities with negative indices, as demonstrated by Nguyen et al. [13], they proved that the following discrete inequality

$$\sum_{\mathfrak{n}=1}^{\infty} \left(\frac{1}{\mathfrak{n}} \sum_{k=1}^{\mathfrak{n}} \mathsf{b}^r(k)\right)^{\frac{1}{r}} \leq 2^{1-\frac{1}{r}} \frac{r}{r-1} \sum_{\mathfrak{n}=1}^{\infty} \mathsf{b}(\mathfrak{n}), \quad r < -1$$

applies to any non-negative realistic convergent sequence $\{\mathsf{b}(\mathfrak{n})\}$ that for all $\mathfrak{n} \in \mathbb{N}$. Furthermore, the author in [14] deduced that for the case that p and r are less than one where r is not equal to zero, the inequality

$$\sum_{\mathfrak{n}=1}^{\mathbf{N}} \mathsf{a}_{\mathfrak{n}} \left(\frac{1}{A_{\mathfrak{n}}} \sum_{i=1}^{\mathfrak{n}} \mathsf{a}_i \mathsf{b}^r(i)\right)^{1/\mathsf{p}} \leq A_{\mathfrak{n}}^{1-1/r} \left(\sum_{i=1}^{\mathbf{N}} \mathsf{a}_i A^{-r}(i)\right)^{1/r} \times \sum_{\mathfrak{n}=1}^{\mathbf{N}} \left(1 - \frac{\sum_{i=1}^{\mathfrak{n}-1} \mathsf{a}_i A^{-r}(i)}{\sum_{i=1}^{\mathbf{N}} \mathsf{a}_i A^{-r}(i)}\right) \mathsf{b}_{\mathfrak{n}} \mathsf{a}_{\mathfrak{n}}$$

holds for all $\mathbf{N} \in \mathbb{N}$, $\{\mathsf{b}_{\mathfrak{n}}\}$ is a sequence that is not negative and $\{\mathsf{a}_{\mathfrak{n}}\}$ is a sequence of real positive numbers.

Over the past few decades several authors [15–17] have focused on the development of inequality (3) by expanding and generalizing the continuous Hardy's inequality with negative indices, thereby deriving corresponding discrete versions. Due to the significance of this inequality in harmonic analysis and mathematics, we restrict our attention to the representative papers [18–20].

The time scale dynamic inequalities have recently attracted much attention [21–23], as they may serve as examples of discrete and integral inequalities. Time scale calculus consists of three primary instances: differential calculus for $\mathbb{T} = \mathbb{R}$, difference calculus if $\mathbb{T} = \mathbb{N}$, and quantum calculus for

$\mathbb{T} = l^{\mathbb{N}_0} = \{l^t : t \in \mathbb{N}_0,\ l > 1\}$. For the convenience of the reader, we include here several time scale inequalities that are closely related to Hardy-type inequalities. We refer the reader to the previous references [24–26]. For example in [27] authors derived that the condition

$$\sup_{\mathtt{a}<\mathfrak{r}<\mathtt{b}} \left(\int_{\mathfrak{r}}^{\mathtt{b}} \mathtt{u}(t)\,\Delta t \right)^{1/q} \left(\int_{\mathtt{a}}^{\sigma(\mathfrak{r})} \mathtt{v}^{1-\mathtt{p}^*}(t)\,\Delta t \right)^{1/\mathtt{p}^*} = \mathcal{B}_7 < \infty, \qquad \mathtt{p}^* = \frac{\mathtt{p}}{\mathtt{p}-1}$$

is the necessary and sufficient for the generalized dynamic inequality

$$\|R_{\mathfrak{h}}\|_{L^q_{\mathtt{u}}([\mathtt{a},\mathtt{b})_{\mathbb{T}})} \le \mathbf{C}\, \|\mathfrak{h}\|_{L^{\mathtt{p}}_{\mathtt{v}}([\mathtt{a},\mathtt{b})_{\mathbb{T}})}, \quad R_{\mathfrak{h}}(\mathfrak{r}) = \int_{\mathtt{a}}^{\sigma(\mathfrak{r})} \mathfrak{h}(t)\Delta t, \tag{7}$$

where $1 < \mathtt{p} \le q < \infty$ and the constant $\mathbf{C}$ in (7) satisfied the estimation

$$\mathcal{B}_7 \le \mathbf{C} \le \left(1 + \frac{q}{\mathtt{p}^*}\right)^{1/q} \left(1 + \frac{\mathtt{p}^*}{q}\right)^{1/\mathtt{p}^*} \mathcal{B}_7.$$

The natural follow-up question is: Is it possible to define new weight functions on an arbitrary time scale $\mathbb{T}$ that satisfy Hardy-type inequalities with negative indices? The purpose of this paper is to respond affirmatively to this inquiry. To be precise, I will extend inequality (6) and its analogue on time scales, while explaining the associated validity condition in the case that $-\infty < q \le \mathtt{p} < 0$ with the additional requirement that $q/\mathtt{p} \ge 1$. Moreover, we will generate discrete and integral characterizations employing these new characterizations. Our main results will be established using available tools on time scales including reverse Hölder inequality and the integral Minkowski inequality, while developing a novel approach. Following this introduction, Section 2 of the paper consists of a presentation of some fundamental concepts of time scale calculus and auxiliary results that are essential for instituting all main results. Ultimately, Section 3 concludes with some time scale variations of inequality (6). It is worth mentio-ning here that, as an exception case stemmed from our results, for $\mathbb{T} = \mathbb{R}$, $\mathbb{T} = \mathbb{N}$ and $\mathbb{T} = l^{\mathbb{N}_0}$, we will also discuss specific dynamic inequalities of Hardy-type that Prokhorov, Kufner, Yang, Nguyen, and Benaissa have captured in their literature previously.

1 *Preliminaries and Basic Lemmas*

Here we provide a high-level overview of the principles within time scale theory. Furthermore, we include supporting concepts that are required to validate our fundamental results.We recommend consulting two monographs authored by Bohner and Peterson [28–30] for a comprehensive examination of time scale calculus. To keep this work concise, we will just provide the core information that is required to prove our results. The time scale $\mathbb{T}$ is a closed non-empty arbitrary subset of real numbers $\mathbb{R}$.

The operator $\sigma : \mathbb{T} \to \mathbb{T}$ that identified as $\sigma(t) := \inf\{s \in \mathbb{T} : s > t\}$ is called the forward jump operator and that for all $t \in \mathbb{T}$. The graininess function μ is defined as $\mu(t) = \sigma(t) - t$, where $\mu : \mathbb{T} \to [0, \infty)$.

The mapping $\mathfrak{h} : [\mathtt{a}, b] \to \mathbb{R}$ is interpreted as the right-dense continuous function ($rd-$continuous) if it is right continuous at every right-dense point and the left limit is finite and exists at left–dense point, then we can denote the space of all right dense continuous functions as $C_{rd}(\mathbb{T}, \mathbb{R})$.
If the derivative of any function $\mathfrak{h}$ exists then this function is called differentiable. The forward shift of the function $\mathfrak{h}$ is $\mathfrak{h}^\sigma$, where $\mathfrak{h}^\sigma(t) = \mathfrak{h}\circ\sigma(t)$ and the delta derivative of $\mathfrak{h}$ signifies $\mathfrak{h}^\Delta$, where $\mathfrak{h}^\sigma = \mathfrak{h} + \mu\mathfrak{h}^\Delta$.
For every function $\Omega : \mathbb{T} \to \mathbb{R}$, the inscription $\Omega^\sigma(\mathtt{t})$ indicates $\Omega(\sigma(\mathtt{t}))$. For each positive value of ϵ there is a neighbourhood U of $\mathtt{t}$, where

$$|[\Omega(\sigma(\mathtt{t})) - \Omega(s)] - \Omega^\Delta(\mathtt{t})[\sigma(\mathtt{t}) - s]| \le \epsilon|\sigma(\mathtt{t}) - s|,$$

for all $s \in U$ for the existence number $\Omega^{\Delta}(\mathfrak{t})$. Here, we can say that $\Omega^{\Delta}(\mathfrak{t})$ is a delta derivative of Ω at $\mathfrak{t}$ and Ω is a delta differentiable at $\mathfrak{t}$. Recapture the product and quotient formulae for the derivatives of two delta differentiable functions φ and $\imath$, denoted as $\varphi\imath$ and $\varphi/\imath$, respectively

$$(\varphi\imath)^{\Delta} = \varphi^{\Delta}\imath + \varphi^{\sigma}\imath^{\Delta} = \varphi\imath^{\Delta} + \varphi^{\Delta}\imath^{\sigma}, \qquad \left(\frac{\varphi}{\imath}\right)^{\Delta} = \frac{\varphi^{\Delta}\imath - \varphi\imath^{\Delta}}{\imath\imath^{\sigma}}, \qquad \imath\imath^{\sigma} \neq 0.$$

Time scale integration for any delta differentiable function $\jmath : \mathbb{T} \to \mathbb{R}$ is defined as follows: the Cauchy integral of $\jmath^{\Delta}$ is defined as $\int_{\mathsf{a}}^{\mathsf{b}} \jmath^{\Delta}(t)\Delta t = \jmath(b) - \jmath(\mathsf{a})$, for a delta differentiable function $\jmath$, with $\mathsf{a}, b \in \mathbb{T}$, (for more details, see [30]). Observe that for time scale $\mathbb{T} = \mathbb{R}$, we acquire that $t = \sigma(t)$, $\jmath^{\Delta}(t) = \jmath'(t)$ and

$$\int_{\mathsf{a}}^{\mathsf{b}} \jmath(t)\Delta t = \int_{\mathsf{a}}^{\mathsf{b}} \jmath(t)dt,$$

if $\mathbb{T} = \mathbb{Z}$, then $\sigma(t) = 1 + t$, $\jmath^{\Delta}(t) = \Delta\jmath(t)$ and $\int_{\mathsf{a}}^{\mathsf{b}} \jmath(t)\Delta t = \sum_{t=\mathsf{a}}^{b-1} \jmath(t)$.

Further, by setting $\mathbb{T} = \ell^{\mathbb{N}}$, we have $\sigma(t) = \ell t$, $\mu(t) = (\ell - 1)t$ and $\int_{t_0}^{\infty} \jmath(t)\Delta t = \sum_{k=n_0}^{\infty} \jmath(\ell^k)\mu(\ell^k)$, where for every $t \in \mathbb{T}$, $\ell^{\mathbb{N}} = \{t = \ell^k,\ k \in \mathbb{N}_0,\ \ell > 1\}$, and $t_0 = \ell^{n_0}$.

The following two lemmas are mentioned in [30, 31].

Lemma 1. Suppose that η and φ are right-dense continuous mappings that defined on the interval $[0, \infty)_{\mathbb{T}}$. Then, integration by parts formulation is stated as

$$\int_0^{\infty} \eta(t)\varphi^{\Delta}(t)\Delta t = \eta(t)\varphi(t)|_0^{\infty} - \int_0^{\infty} \eta^{\Delta}(t)\varphi^{\sigma}(t)\Delta t. \tag{8}$$

Lemma 2. Assume that the function $\jmath : \mathbb{R} \to \mathbb{R}$ is continuously differentiable function and let $h : \mathbb{T} \to \mathbb{R}$ be a delta-differentiable mapping. Then, the composition of both functions $\jmath$ and h is a delta-differentiable mapping and for every $\zeta \in [t, \sigma(t)]$ the following equation

$$(\jmath h\,(t))^{\Delta} = \jmath'\,(h\,(\zeta))\,h^{\Delta}(t) \tag{9}$$

holds.

The two lemmas that follow are the derived reverse time scale Hölder's inequality and Minkowski's integral inequality, which are crucial to our results [31, 32].

Theorem 1. Suppose η, and φ are two Δ-integrable functions therefore

$$\int_{\mathsf{a}}^{\mathsf{b}} |\eta(t)\varphi(t)|\Delta t \leq \left[\int_{\mathsf{a}}^{\mathsf{b}} |\eta(t)|^{\varepsilon}\Delta t\right]^{\frac{1}{\varepsilon}} \left[\int_{\mathsf{a}}^{\mathsf{b}} |\varphi(t)|^{\varsigma}\Delta t\right]^{\frac{1}{\varsigma}},$$

for every a, and b belongs to $\mathbb{T}$, where $\varepsilon > 1$ and $\frac{1}{\varepsilon} + \frac{1}{\varsigma} = 1$. If we replace ε with negative value $\varepsilon < 0$, we capture the so-called reverse time scale Hölders inequality [33]

$$\int_{\mathsf{a}}^{\mathsf{b}} |\eta(t)\varphi(t)|\Delta t \geq \left[\int_{\mathsf{a}}^{\mathsf{b}} |\eta(t)|^{\varepsilon}\Delta t\right]^{\frac{1}{\varepsilon}} \left[\int_{\mathsf{a}}^{\mathsf{b}} |\varphi(t)|^{\varsigma}\Delta t\right]^{\frac{1}{\varsigma}}. \tag{10}$$

Now, we state Minkowski integral inequality [34, 35].

Theorem 2. Let κ, Θ be rd-continuous and non-negative functions identified on the interval $[\mathsf{a}, \infty)_{\mathbb{T}}$, where a belongs to the time scale $\mathbb{T}$. For $\mathfrak{k} \geq 1$, the inequality

$$\left[\int_{\mathsf{a}}^{\infty} \kappa(s)\left(\int_{\mathsf{a}}^{\sigma(s)} \Theta(t)\Delta t\right)^{\mathfrak{k}} \Delta s\right]^{1/\mathfrak{k}} \leq \int_{\mathsf{a}}^{\infty} \Theta(s)\left(\int_s^{\infty} \kappa(\mathfrak{x})\Delta\mathfrak{x}\right)^{1/\mathfrak{k}} \tag{11}$$

holds. If $0 < \mathfrak{k} \leq 1$, then the direction of inequality (11) is reversed.

We can consider the two next lemmas proven in [35] to be power rules of integration on the time scale $\mathbb{T}$.

Lemma 3. Consider a time scale $\mathbb{T}$, and let $\mathsf{a}, \mathfrak{x} \in \mathbb{T}$ with $\mathfrak{x} \geq a$. Then, the inequality

$$\left(\int_{\mathsf{a}}^{\sigma(\mathfrak{x})} \mathrm{g}(t)\Delta t\right)^{m} \leq m \int_{\mathsf{a}}^{\sigma(\mathfrak{x})} \mathrm{g}(t) \left(\int_{\mathsf{a}}^{\sigma(t)} \mathrm{g}(s)\Delta s\right)^{m-1} \Delta t$$

holds for $m \geq 1$; however, for $0 < m < 1$, the direction of the inequality will reverse.

Lemma 4. Assume that $\mathbb{T}$ is a time scale. Then, for $m \geq 1$, the inequality [36, 37]

$$\left(\int_{\mathfrak{x}}^{b} \mathrm{g}(t)\Delta t\right)^{m} \leq m \int_{\mathfrak{x}}^{b} \mathrm{g}(t) \left(\int_{t}^{b} \mathrm{g}(s)\Delta s\right)^{m-1} \Delta t$$

is valid for every b, where $\mathfrak{x}$ belongs to $\mathbb{T}$ with $\mathfrak{x} \leq b$. While for $0 < m < 1$, the inequality will reverse.

All results in this paper are considered to be based on continuous and non-negative functions.

2 Main Results and Applications

Now, we can explain and validate the main results. Here, it should be mentioned that the integrals in the statements of theorems that follow are assumed to exist. To demonstrate our initial result, we will adopt the reverse of Hölder inequality (10) and the integral inequality of Minkowski (11). In order to make items easier, we use the notations

$$\mathbb{E}_k(\mathfrak{x}, s) = (\Lambda^{\sigma}(\mathfrak{x}))^{\frac{1-s}{p}} \left(\int_{\mathsf{a}}^{\sigma(\mathfrak{x})} \mathfrak{u}(t)\Lambda^{\frac{q(p-s)}{p}}(t)\Delta t\right)^{-1/q}, \quad s \in [p, 1), \tag{12}$$

where the mapping $\Lambda(\mathfrak{x})$ is a non-negative mapping defined on $[\mathsf{a}, b]_{\mathbb{T}}$ and

$$\Lambda(\mathfrak{x}) := \int_{\mathsf{a}}^{\sigma(\mathfrak{x})} \mathrm{v}^{1-p'}(t)\Delta t, \tag{13}$$

where $p' := \frac{p}{p-1}$, further

$$\mathbb{E}_s(\mathfrak{x}, s) = \sup_{\mathfrak{x}\in(\mathsf{a},\ b)_{\mathbb{T}}} (\Lambda^{\sigma}(\mathfrak{x}))^{\frac{1-s}{p}} \left(\int_{\mathsf{a}}^{\sigma(\mathfrak{x})} \mathfrak{u}(t)\Lambda^{\frac{q(p-s)}{p}}(t)\Delta t\right)^{-1/q}, \quad s \in [p, 1), \tag{14}$$

then

$$\mathbb{E}_s := \sup_{\mathfrak{x}\in(\mathsf{a},\ b)_{\mathbb{T}}} \mathbb{E}_k. \tag{15}$$

Theorem 3. Suppose that $\mathbb{T}$ is a time scale and $-\infty < q \leq p < 0$, where $q/p \geq 1$, $\mathfrak{f} \in C_{rd}([\mathsf{a}, b]_{\mathbb{T}}, \mathbb{R})$ is a non-negative function and assume that $\mathfrak{u}$, v are rd-continuous positive functions on the open interval $(\mathsf{a}, b)_{\mathbb{T}}$. Thus, the reverse Hardy inequality

$$\left(\int_{\mathsf{a}}^{b} \mathfrak{f}^{p}(\mathfrak{x})\,\mathrm{v}(\mathfrak{x})\,\Delta\mathfrak{x}\right)^{1/p} \leq \mathbf{C}\left(\int_{\mathsf{a}}^{b}\left(\int_{\mathsf{a}}^{\sigma(\mathfrak{x})} \mathfrak{f}(t)\,\Delta t\right)^{q} \mathfrak{u}(\mathfrak{x})\,\Delta\mathfrak{x}\right)^{1/q} \tag{16}$$

is valid for every function $\mathfrak{f}$, if and only if the following criterion applies,

$$\mathbb{E}_s < \infty. \tag{17}$$

Moreover, the value of $\mathbf{C}$ in (16) possesses the estimation

$$\mathbb{E}_s \leq \mathbf{C} < 2^{-\frac{1}{q}} \left(\frac{p+s-2}{p-1}\right)^{1/p'} \mathbb{E}_s,$$

where the definition of $\mathbb{E}_s$ is mentioned in (14).

Proof. First of all and to facilitate, we reform inequality (16) to take the equivalent form

$$\int_{\mathtt{a}}^{b} \left(\int_{\mathtt{a}}^{\sigma(\mathfrak{x})} \mathfrak{g}^{\frac{1}{p}}(t)\,\varrho(t)\Delta t\right)^{q} \mathfrak{u}(\mathfrak{x})\,\Delta\mathfrak{x} \leq \mathbf{C}^{-q} \left(\int_{\mathtt{a}}^{b} \mathfrak{g}(\mathfrak{x})\,\Delta\mathfrak{x}\right)^{q/p}, \tag{18}$$

and that by setting $\mathfrak{f}(\mathfrak{x}) = \mathfrak{g}^{\frac{1}{p}}(\mathfrak{x})\,w(\mathfrak{x})$ and $\varrho(\mathfrak{x}) = \mathtt{v}^{-1/p}(\mathfrak{x})$. Then, equation (13) can be expressed to

$$\Lambda(\mathfrak{x}) := \int_{\mathtt{a}}^{\sigma(\mathfrak{x})} \varrho^{p'}(t)\Delta t, \quad p' := \frac{p}{p-1}. \tag{19}$$

Now, we prove the sufficiency of condition (17), so we first assume that $\mathbb{E}_s < \infty$, this implies that

$$0 < \Lambda(\mathfrak{x}) := \int_{\mathtt{a}}^{\sigma(\mathfrak{x})} \varrho^{p'}(t)\Delta t < \infty,$$

for every $t \in (\mathtt{a}, b)_{\mathbb{T}}$. Invoking the reverse of Hölder inequality (10) with parameters p and p' on the left-hand side of the inequality (18), we obtain that

$$\begin{aligned}
&\int_{\mathtt{a}}^{b} \left(\int_{\mathtt{a}}^{\sigma(\mathfrak{x})} \mathfrak{g}^{\frac{1}{p}}(t)\,\varrho(t)\mathfrak{u}(\mathfrak{x})\,\Delta t\right)^{q} \Delta\mathfrak{x} \\
&= \int_{\mathtt{a}}^{b} \left(\int_{\mathtt{a}}^{\sigma(\mathfrak{x})} \mathfrak{g}^{\frac{1}{p}}(t)\,\Lambda^{\frac{1-s}{p}}(t)\,\Lambda^{\frac{s-1}{p}}(t)\,\varrho(t)\Delta t\right)^{q} \mathfrak{u}(\mathfrak{x})\,\Delta\mathfrak{x} \\
&\leq \int_{\mathtt{a}}^{b} \left(\int_{\mathtt{a}}^{\sigma(\mathfrak{x})} \mathfrak{g}(t)\,\Lambda^{1-s}(t)\,\Delta t\right)^{q/p} \mathfrak{u}(\mathfrak{x}) \times \left(\int_{\mathtt{a}}^{\sigma(\mathfrak{x})} \Lambda^{\frac{p'(s-1)}{p}}(t)\,\varrho^{p'}(t)\,\Delta t\right)^{q/p'} \Delta\mathfrak{x},
\end{aligned} \tag{20}$$

from the definition of $\Lambda(t)$ that mentioned in (19), we display that

$$\Lambda^{\Delta}(t) \leq \varrho^{p'}(\sigma(t)) \leq \varrho^{p'}(t). \tag{21}$$

Substituting from (21) into (20) and by recalling chain rule (9), we exhibit that

$$\begin{aligned}
&\int_{\mathtt{a}}^{b} \left(\int_{\mathtt{a}}^{\sigma(\mathfrak{x})} \mathfrak{g}^{\frac{1}{p}}(t)\,\varrho(t)\Delta t\right)^{q} \mathfrak{u}(\mathfrak{x})\,\Delta\mathfrak{x} \\
&\leq \int_{\mathtt{a}}^{b} \left(\int_{\mathtt{a}}^{\sigma(\mathfrak{x})} \mathfrak{g}(t)\,\Lambda^{1-s}(t)\,\Delta t\right)^{q/p} \mathfrak{u}(\mathfrak{x}) \times \left(\int_{\mathtt{a}}^{\sigma(\mathfrak{x})} \Lambda^{\frac{p'(s-1)}{p}}(t)\,\Lambda^{\Delta}(t)\Delta t\right)^{q/p'} \Delta\mathfrak{x},
\end{aligned}$$

now by applying the time scale concept of differentiation and the fact that $\Lambda(a) = 0$, so the previous inequality reduces to

$$\begin{aligned}
&\int_{\mathfrak{a}}^{b}\left(\int_{\mathfrak{a}}^{\sigma(\mathfrak{x})}\mathfrak{g}^{\frac{1}{p}}(t)\,\varrho(t)\Delta t\right)^{q}\mathfrak{u}(\mathfrak{x})\,\Delta\mathfrak{x}\\
&\leq\int_{\mathfrak{a}}^{b}\left(\int_{\mathfrak{a}}^{\sigma(\mathfrak{x})}\mathfrak{g}(t)\,\Lambda^{1-s}(t)\,\Delta t\right)^{q/p}\times\left(\frac{\Lambda^{\frac{(s-1)p'}{p}+1}(\sigma(\mathfrak{x}))}{\frac{(s-1)p'}{p}+1}\right)^{q/p'}\mathfrak{u}(\mathfrak{x})\,\Delta\mathfrak{x}\\
&=\int_{\mathfrak{a}}^{b}\left(\int_{\mathfrak{a}}^{\sigma(\mathfrak{x})}\mathfrak{g}(t)\,\Lambda^{1-s}(t)\,\Delta t\right)^{q/p}\times\left(\frac{p-1}{s+p-2}\Lambda^{\frac{p+s-2}{-1+p}}(\sigma(\mathfrak{x}))\right)^{q/p'}\Delta\mathfrak{x}\\
&=\left(\frac{-1+p}{p+s-2}\right)^{q/p'}\int_{\mathfrak{a}}^{b}\mathfrak{u}(\mathfrak{x})\left(\int_{\mathfrak{a}}^{\sigma(\mathfrak{x})}\mathfrak{g}(t)\,\Lambda^{1-s}(t)\,\Delta t\right)^{q/p}\Lambda^{\frac{p+s-2}{-1+p}\cdot\frac{q}{p'}}(\sigma(\mathfrak{x}))\,\mathfrak{u}(\mathfrak{x})\,\Delta\mathfrak{x},
\end{aligned}$$

by simple computations and by interacting the reality that $\mathfrak{x}\leq\sigma(\mathfrak{x})$, we indicate the following

$$\begin{aligned}
&\int_{\mathfrak{a}}^{b}\left(\int_{\mathfrak{a}}^{\sigma(\mathfrak{x})}\mathfrak{g}^{\frac{1}{p}}(t)\,\varrho(t)\Delta t\right)^{q}\mathfrak{u}(\mathfrak{x})\,\Delta\mathfrak{x}\leq\left(\frac{-1+p}{p+s-2}\right)^{q/p'}\\
&\times\left[\left(\int_{\mathfrak{a}}^{b}\left[\left(\int_{\mathfrak{a}}^{\sigma(\mathfrak{x})}\mathfrak{g}(t)\,\Lambda^{1-s}(t)\,\Delta t\right)^{\frac{q}{p}\cdot\frac{p}{q}}\Lambda^{\frac{p+s-2}{-1+p}\cdot\frac{q}{p'}\cdot\frac{p}{q}}(\sigma(\mathfrak{x}))\,\mathfrak{u}^{p/q}(\mathfrak{x})\right]^{q/p}\Delta\mathfrak{x}\right)^{p/q}\right]^{q/p}\\
&\leq\left(\frac{-1+p}{p+s-2}\right)^{q/p'}\times\left[\left(\int_{\mathfrak{a}}^{b}\left[\left(\int_{\mathfrak{a}}^{\sigma(\mathfrak{x})}\mathfrak{g}(t)\,\Lambda^{1-s}(t)\,\Delta t\right)\Lambda^{\frac{s+p-2}{p-1}\cdot\frac{p}{p'}}(\mathfrak{x})\,\mathfrak{u}^{p/q}(\mathfrak{x})\right]^{q/p}\Delta\mathfrak{x}\right)^{p/q}\right]^{q/p}. \qquad (22)
\end{aligned}$$

For $q/p\geq 1$, and by employing Minkowski integral time scale inequality (11) on the inequality (22), we determine I, where

$$\begin{aligned}
I&=\int_{\mathfrak{a}}^{b}\left(\int_{\mathfrak{a}}^{\sigma(\mathfrak{x})}\mathfrak{g}^{\frac{1}{p}}(t)\,\varrho(t)\Delta t\right)^{q}\mathfrak{u}(\mathfrak{x})\,\Delta\mathfrak{x}\\
&\leq\left(\frac{-1+p}{p+s-2}\right)^{q/p'}\left[\left(\int_{\mathfrak{a}}^{b}\left(\int_{t}^{b}\Lambda^{\frac{s+p-2}{p-1}\cdot\frac{q}{p'}}(\mathfrak{x})\,\mathfrak{u}(\mathfrak{x})\,\Delta\mathfrak{x}\right)\mathfrak{g}^{q/p}(t)\,\Lambda^{\frac{q(1-s)}{p}}(t)\,\Delta t\right)^{p/q}\right]^{q/p}\\
&=\left(\frac{-1+p}{p+s-2}\right)^{q/p'}\left[\int_{\mathfrak{a}}^{b}\left(\int_{t}^{b}\Lambda^{\frac{s+p-2}{p}\cdot q}(\mathfrak{x})\,\mathfrak{u}(\mathfrak{x})\,\Delta\mathfrak{x}\right)^{p/q}\mathfrak{g}(t)\,\Lambda^{1-s}(t)\,\Delta t\right]^{q/p}. \qquad (23)
\end{aligned}$$

Setting

$$\mathcal{B}(t)=\left(\int_{t}^{b}\Lambda^{\frac{s+p-2}{p}\cdot q}(\mathfrak{x})\,\mathfrak{u}(\mathfrak{x})\,\Delta\mathfrak{x}\right)^{p/q}\Lambda^{1-s}(t), \qquad (24)$$

in inequality (23), we indicate the following inequality

$$I\leq\left(\frac{-1+p}{p+s-2}\right)^{q/p'}\left(\int_{\mathfrak{a}}^{b}\mathfrak{g}(t)\,\Delta t\right)^{q/p}\mathcal{B}^{q/p},$$

where we denote that $\mathcal{B}=\sup_{t\in(\mathfrak{a},\,b)_{\mathbb{T}}}\mathcal{B}(t)$.

Recalling inequality (18), we conclude that

$$\mathbf{C}^{-q} \leq \left(\frac{-1+p}{p+s-2}\right)^{q/p'} \mathcal{B}^{q/p}. \tag{25}$$

Now, by utilizing the power of q/p, we can raise both sides of inequality (24) to the power of q/p, then we derive that

$$\begin{aligned}\mathcal{B}^{q/p} &= \Lambda^{\frac{q(1-s)}{p}}(t)\left(\int_t^b \Lambda^{q\frac{s+p-2}{p}}(\mathfrak{x})\,\mathfrak{u}(\mathfrak{x})\,\Delta\mathfrak{x}\right) \\ &= \Lambda^{\frac{q(1-s)}{p}}(t)\left(\int_t^b \Lambda^{\frac{2q(s-1)}{p}}(\mathfrak{x})\,\Lambda^{\frac{p-s}{p}\cdot q}(\mathfrak{x})\,\mathfrak{u}(\mathfrak{x})\,\Delta\mathfrak{x}\right) \\ &= \Lambda^{\frac{q(1-s)}{p}}(t)\int_t^b \Lambda^{\frac{2q(s-1)}{p}}(\mathfrak{x})\left(\int_t^{\mathfrak{x}} \Lambda^{\frac{p-s}{p}\cdot q}(\tau)\,\mathfrak{u}(\tau)\,\Delta\tau\right)^{\Delta}\Delta\mathfrak{x}. \end{aligned} \tag{26}$$

Putting $L=\int_t^b \Lambda^{\frac{2(s-1)}{p}\cdot q}(\mathfrak{x})\left(\int_t^{\mathfrak{x}} \Lambda^{\frac{p-s}{p}\cdot q}(\tau)\,\mathfrak{u}(\tau)\,\Delta\tau\right)^{\Delta}\Delta\mathfrak{x}$. Then, by applying integration by parts formulation (8) with $\eta(\mathfrak{x})=\Lambda^{\frac{2(s-1)}{p}\cdot q}(\mathfrak{x})$ and $\varphi^{\Delta}(\mathfrak{x})=\left(\int_t^{\mathfrak{x}} \Lambda^{\frac{p-s}{p}\cdot q}(\tau)\,u(\tau)\,\Delta\tau\right)^{\Delta}$, we have that

$$\begin{aligned} L &= \eta(\mathfrak{x})\varphi(\mathfrak{x})\big|_t^b - \int_t^b \eta^{\Delta}(\mathfrak{x})\varphi^{\sigma}(\mathfrak{x})\Delta\mathfrak{x} \\ &= \Lambda^{\frac{2(s-1)}{p}\cdot q}(\mathfrak{x})\left(\int_t^{\mathfrak{x}} \Lambda^{\frac{p-s}{p}\cdot q}(\tau)\,\mathfrak{u}(\tau)\,\Delta\tau\bigg|_t^b\right) - \int_t^b \left(\Lambda^{\frac{2(s-1)}{p}\cdot q}(\mathfrak{x})\right)^{\Delta}\left(\int_t^{\sigma(\mathfrak{x})} \Lambda^{\frac{p-s}{p}\cdot q}(\tau)\,\mathfrak{u}(\tau)\,\Delta\tau\right)\Delta\mathfrak{x}. \end{aligned} \tag{27}$$

From (27), setting

$$J_1 = \Lambda^{\frac{2(s-1)}{p}\cdot q}(\mathfrak{x})\left(\int_t^{\mathfrak{x}} \Lambda^{\frac{p-s}{p}\cdot q}(\tau)\,\mathfrak{u}(\tau)\,\Delta\tau\bigg|_t^b\right) \tag{28}$$

and

$$J_2 = -\int_t^b \left(\int_t^{\sigma(\mathfrak{x})} \Lambda^{\frac{p-s}{p}\cdot q}(\tau)\,\mathfrak{u}(\tau)\,\Delta\tau\right)\left(\Lambda^{\frac{2(s-1)}{p}\cdot q}(\mathfrak{x})\right)^{\Delta}\Delta\mathfrak{x}, \tag{29}$$

by employing equations (12) and (15), the equation (28) can take the form

$$\begin{aligned} J_1 &= \lim_{\mathfrak{x}\to b-} \Lambda^{\frac{2q(s-1)}{p}}(\mathfrak{x})\int_t^{\mathfrak{x}} \Lambda^{\frac{p-s}{p}\cdot q}(\tau)\,\mathfrak{u}(\tau)\,\Delta\tau \\ &\leq \lim_{\mathfrak{x}\to b-} \Lambda^{\frac{2q(s-1)}{p}}(\mathfrak{x})\int_{\mathtt{a}}^{\mathfrak{x}} \Lambda^{\frac{p-s}{p}\cdot q}(\tau)\,\mathfrak{u}(\tau)\,\Delta\tau \\ &= \lim_{\mathfrak{x}\to b-} \Lambda^{q\left(\frac{s-1}{p}+\frac{-1+s}{p}\right)}(\mathfrak{x})\left[\left(\left(\int_{\mathtt{a}}^{\mathfrak{x}} \Lambda^{\frac{p-s}{p}\cdot q}(\tau)\,\mathfrak{u}(\tau)\,\Delta\tau\right)^{-1/q}\right]^{-q} \\ &= \lim_{\mathfrak{x}\to b-} \Lambda^{\frac{s-1}{p}\cdot q}(\mathfrak{x})\,\mathbb{E}_k^{-q}(\mathfrak{x},s) \leq \mathbb{E}_s^{-q}\lim_{\mathfrak{x}\to b-} \Lambda^{\frac{s-1}{p}\cdot q}(\mathfrak{x}) \leq \mathbb{E}_s^{-q}\lim_{\mathfrak{x}\to b-} \Lambda^{\frac{s-1}{p}\cdot q}(\sigma(\mathfrak{x})), \end{aligned} \tag{30}$$

that for $q<0$. In a similar manner, for $t \geq \mathtt{a}$ and by recalling equation (12), where $\mathbb{E}_k(\mathfrak{x},s) \leq \mathbb{E}_s$, we conclude that equation (29) becomes

$$J_2 = -\int_t^b \Lambda^{\frac{q(s-1)}{p}}(\sigma(\mathfrak{x}))\left(\int_t^{\sigma(\mathfrak{x})}\Lambda^{\frac{q(p-s)}{p}}(\tau)\,\mathfrak{u}(\tau)\,\Delta\tau\right)\times\Lambda^{\frac{q(1-s)}{p}}(\sigma(\mathfrak{x}))\left(\Lambda^{\frac{2q(s-1)}{p}}(\mathfrak{x})\right)^{\Delta}\Delta\mathfrak{x}$$
$$\leq -\int_t^b \Lambda^{\frac{q(s-1)}{p}}(\sigma(\mathfrak{x}))\left(\int_{\mathsf{a}}^{\sigma(\mathfrak{x})}\Lambda^{\frac{q(p-s)}{p}}(\tau)\,\mathfrak{u}(\tau)\,\Delta\tau\right)\times\Lambda^{\frac{q(1-s)}{p}}(\sigma(\mathfrak{x}))\left(\Lambda^{\frac{2q(s-1)}{p}}(\mathfrak{x})\right)^{\Delta}\Delta\mathfrak{x}$$
$$\leq -\mathbb{E}_s^{-q}(\mathfrak{x},s)\int_t^b \Lambda^{\frac{q(1-s)}{p}}(\sigma(\mathfrak{x}))\left(\Lambda^{\frac{2q(s-1)}{p}}(\mathfrak{x})\right)^{\Delta}\Delta\mathfrak{x}. \quad (31)$$

Now, by implementing the chain rule (9) where $(\mathfrak{f}(\mathsf{g}(t)))^{\Delta} = \mathfrak{f}'(\mathsf{g}(c))\mathsf{g}^{\Delta}(t)$, with $c \in [\mathfrak{x}, \sigma(\mathfrak{x})]$, we attain that $(\mathbf{U}^{-q}(\mathfrak{x}))^{\Delta} = -q\mathbf{U}^{-q-1}(c)\mathbf{U}^{\Delta}(\mathfrak{x})$. Now, for $c \leq \sigma(\mathfrak{x})$, the non-increasing mapping $\mathbf{U}$ reaches that $\mathbf{U}^{\Delta}(t) \leq 0$ and $\mathbf{U}(c) \geq \mathbf{U}^{\sigma}(\mathfrak{x})$, it follows that

$$\left(\mathbf{U}^{-q}(\mathfrak{x})\right)^{\Delta} \leq -q\mathbf{U}^{-q-1}(\sigma(\mathfrak{x}))\,\mathbf{U}^{\Delta}(\mathfrak{x}). \quad (32)$$

Then, combining relations (31) and (32), we get the inequality

$$\left(\Lambda^{\frac{2q(s-1)}{p}}(\mathfrak{x})\right)^{\Delta} \leq \frac{2q(s-1)}{p}\Lambda^{\frac{2q(s-1)}{p}-1}(\sigma(\mathfrak{x}))\,\Lambda^{\Delta}(\mathfrak{x}), \quad (33)$$

substituting from (33) into (31) and for $q/p \geq 1$, we conclude that

$$J_2 \leq -\mathbb{E}_s^{-q}(\mathfrak{x},s)\int_t^b \Lambda^{\frac{1-s}{p}\cdot q}(\sigma(\mathfrak{x}))\left(\frac{2(s-1)q}{p}\Lambda^{-1+\frac{2(s-1)q}{p}}(\sigma(\mathfrak{x}))\,\Lambda^{\Delta}(\mathfrak{x})\right)\Delta\mathfrak{x}$$
$$= -\frac{2q(s-1)}{p}\mathbb{E}_s^{-q}(\mathfrak{x},s)\int_t^b \Lambda^{\frac{(s-1)q}{p}-1}(\sigma(\mathfrak{x}))\,\Lambda^{\Delta}(\mathfrak{x})\,\Delta\mathfrak{x}$$
$$= -\frac{2q(s-1)}{p}\mathbb{E}_s^{-q}(\mathfrak{x},s)\left(\left.\frac{\Lambda^{\frac{q(s-1)}{p}}(\sigma(\mathfrak{x}))}{\frac{q(s-1)}{p}}\right|_t^b\right)$$
$$= -2\mathbb{E}_s^{-q}(\mathfrak{x},s)\left[\lim_{\mathfrak{x}\to b-}\Lambda^{\frac{s-1}{p}\cdot q}(\sigma(\mathfrak{x})) - \left(\Lambda^{\frac{s-1}{p}\cdot q}(\sigma(t))\right)\right]. \quad (34)$$

Hence from (30) and (34) and by substituting into (26), we obtain that

$$B^{q/p} \leq \mathbb{E}_s^{-q}(\mathfrak{x},s)\,\Lambda^{\frac{1-s}{p}\cdot q}(t)\left[\lim_{\mathfrak{x}\to b-}\Lambda^{\frac{s-1}{p}\cdot q}(\sigma(\mathfrak{x})) - 2\lim_{\mathfrak{x}\to b-}\Lambda^{\frac{s-1}{p}\cdot q}(\sigma(\mathfrak{x})) + 2\Lambda^{\frac{s-1}{p}\cdot q}(\sigma(t))\right]$$
$$= \mathbb{E}_s^{-q}(\mathfrak{x},s)\,\Lambda^{\frac{1-s}{p}\cdot q}(t)\left[-\lim_{\mathfrak{x}\to b-}\Lambda^{\frac{s-1}{p}\cdot q}(\sigma(\mathfrak{x})) + 2\Lambda^{\frac{s-1}{p}\cdot q}(\sigma(t))\right]$$
$$\leq \mathbb{E}_s^{-q}(\mathfrak{x},s)\,\Lambda^{\frac{1-s}{p}\cdot q}(\sigma(t))\left[-\lim_{\mathfrak{x}\to b-}\Lambda^{\frac{s-1}{p}\cdot q}(\sigma(\mathfrak{x})) + 2\Lambda^{\frac{s-1}{p}\cdot q}(\sigma(t))\right] \leq 2\mathbb{E}_s^{-q}(\mathfrak{x},s).$$

Combining with (25), we get that $\mathbf{C}^{-q} \leq 2\left(\frac{-1+p}{p+s-2}\right)^{q/p'}\mathbb{E}_s^{-q}(\mathfrak{x},s)$. Therefore

$$\mathbf{C} \leq 2^{-1/q}\left(\frac{-1+p}{p+s-2}\right)^{-1/p'}\mathbb{E}_s(\mathfrak{x},s),$$

which is the desired sufficient condition. Now, we will show the necessity of condition (17). Just consider that the inequality (18) is valid for every rd-function that is non-negative defined on $[\mathsf{a},b]_{\mathbb{T}}$, then we will prove that $\mathbb{E}_s < \infty$. Firstly, let $s \in (p, 1)$ and for fixed value $t \in (\mathsf{a}, b)$, we formulate the function

$$\mathfrak{g}(\mathfrak{x}) = \begin{cases} \Lambda^{-s}(\mathfrak{x})\varrho^{p'}(\mathfrak{x}), & \mathfrak{x} \in (\mathsf{a}, t)_{\mathbb{T}}\,, \\ 0, & \mathfrak{x} \in [t, b)_{\mathbb{T}}, \end{cases} \tag{35}$$

therefore, we obtain that

$$\left(\int_{\mathsf{a}}^{b} \mathfrak{g}\left(\mathfrak{x}\right)\Delta\mathfrak{x}\right)^{q/p} = \left(\int_{\mathsf{a}}^{\sigma(t)} \Lambda^{-s}(\mathfrak{x})\varrho^{p'}(\mathfrak{x})\Delta\mathfrak{x}\right)^{q/p},$$

recalling the definition of $\Lambda(\mathfrak{x})$ in (19) and the fact that $\Lambda(\mathsf{a}) = 0$, we get that

$$\begin{aligned} \left(\int_{\mathsf{a}}^{b} \mathfrak{g}\left(\mathfrak{x}\right)\Delta\mathfrak{x}\right)^{q/p} &\leq \left(\int_{\mathsf{a}}^{\sigma(t)} \Lambda^{-s}(\mathfrak{x})\Lambda^{\Delta}(\mathfrak{x})\Delta\mathfrak{x}\right)^{q/p} \\ &= \left(\frac{\Lambda^{-s+1}\left(\mathfrak{x}\right)}{-s+1}\right)^{q/p}\Bigg|_{\mathsf{a}}^{\sigma(t)} \leq \left(\frac{1}{-s+1}\right)^{q/p}\Lambda^{\frac{q(-s+1)}{p}}\left(t\right), \end{aligned} \tag{36}$$

on the other hand, by raising both sides of (35) to the power of $1/p$, and by multiplying both sides with the function $\varrho\left(\mathfrak{x}\right)$, we have the equation

$$\mathfrak{g}^{1/p}(\mathfrak{x})\varrho\left(\mathfrak{x}\right) = \begin{cases} \Lambda^{-\frac{s}{p}}(\mathfrak{x})\varrho^{\frac{p'}{p}}(\mathfrak{x}), & \mathfrak{x} \in (\mathsf{a}, t)_{\mathbb{T}}\,, \\ 0, & \mathfrak{x} \in [t, b)_{\mathbb{T}}. \end{cases} \tag{37}$$

By utilizing (37) into (18), we determine the following

$$\begin{aligned} &\int_{\mathsf{a}}^{b}\left(\int_{\mathsf{a}}^{\sigma(\mathfrak{x})} \mathfrak{g}^{\frac{1}{p}}\left(t\right)\varrho(t)\Delta t\right)^{q}\mathfrak{u}\left(\mathfrak{x}\right)\Delta\mathfrak{x} = \int_{\mathsf{a}}^{b}\left(\int_{\mathsf{a}}^{\sigma(\mathfrak{x})} \Lambda^{\frac{-s}{p}}(\tau)\varrho^{\frac{p'}{p}}(\tau)\Delta\tau\right)^{q}\mathfrak{u}\left(\mathfrak{x}\right)\Delta\mathfrak{x} \\ &\geq \int_{\mathsf{a}}^{\sigma(t)}\left(\int_{\mathsf{a}}^{\sigma(\mathfrak{x})} \Lambda^{\frac{-s}{p}}(\tau)\Lambda^{\Delta}(\tau)\Delta\tau\right)^{q}\mathfrak{u}\left(\mathfrak{x}\right)\Delta\mathfrak{x} \\ &= \int_{\mathsf{a}}^{\sigma(t)}\left(\left(\frac{\Lambda^{\frac{-s}{p}+1}\left(\tau\right)}{\frac{-s}{p}+1}\right)^{q}\Bigg|_{\mathsf{a}}^{\sigma(\mathfrak{x})}\right)\mathfrak{u}\left(\mathfrak{x}\right)\Delta\mathfrak{x} \geq \left(\frac{p}{-s+p}\right)^{q}\int_{\mathsf{a}}^{\sigma(t)} \Lambda^{\frac{q(-s+p)}{p}}\mathfrak{u}\left(\mathfrak{x}\right)\Delta\mathfrak{x}. \end{aligned} \tag{38}$$

Combining (36) and (38) implies that

$$\left(\frac{p}{-s+p}\right)^{q}\int_{\mathsf{a}}^{\sigma(t)} \Lambda^{q\left(\frac{p-s}{p}\right)}\left(\mathfrak{x}\right)\mathfrak{u}\left(\mathfrak{x}\right)\Delta\mathfrak{x} \leq \mathbf{C}^{-q}\left(\frac{1}{-s+1}\right)^{q/p}\Lambda^{q\left(\frac{-s+1}{p}\right)}\left(t\right),$$

therefore

$$\left(\frac{p}{-s+p}\right)^{q}(-s+1)^{q/p}\,\Lambda^{q\left(\frac{-1+s}{p}\right)}\left(t\right)\int_{\mathsf{a}}^{\sigma(t)} \Lambda^{q\left(\frac{-s+p}{p}\right)}\left(\mathfrak{x}\right)\mathfrak{u}\left(\mathfrak{x}\right)\Delta\mathfrak{x} \leq \mathbf{C}^{-q}. \tag{39}$$

Raising both sides of (39) to the power of $-1/q$, yields the resultant inequality

$$\left(\frac{-s+p}{p}\right)(-s+1)^{-1/p}\,\Lambda^{\left(\frac{-s+1}{p}\right)}\left(\mathfrak{x}\right)\left(\int_{\mathsf{a}}^{\sigma(t)} \mathfrak{u}\left(\mathfrak{x}\right)\Lambda^{q\left(\frac{-s+p}{p}\right)}\left(\mathfrak{x}\right)\Delta\mathfrak{x}\right)^{-1/q} \leq \mathbf{C},$$

that is identical to the inequality

$$\left(\frac{-s+p}{p}\right)(-s+1)^{-1/p}\,\mathbb{E}_{k}\left(t, s\right) \leq \mathbf{C}.$$

Secondly, for $s = p$ we take the mapping $\mathfrak{g}(\mathfrak{x})$ as the form

$$\mathfrak{g}(\mathfrak{x}) = \begin{cases} \varrho^{p'}(\mathfrak{x}), & \mathfrak{x} \in (\mathtt{a}, t)_{\mathbb{T}}, \\ 0, & \mathfrak{x} \in [t, b)_{\mathbb{T}}, \end{cases} \tag{40}$$

consequently and from (19), we get that

$$\left(\int_{\mathtt{a}}^{b} \mathfrak{g}(\mathfrak{x}) \Delta\mathfrak{x} \right)^{q/p} = \left(\int_{\mathtt{a}}^{\sigma(t)} \varrho^{p'}(\mathfrak{x}) \Delta\mathfrak{x} \right)^{q/p} = \Lambda^{\frac{q}{p}} (\sigma(t)). \tag{41}$$

By substituting from (40) and (41) into (18) we obtain that

$$\int_{\mathtt{a}}^{b} \left(\int_{\mathtt{a}}^{\sigma(t)} \varrho^{p'}(\mathfrak{x}) \mathfrak{u}(\mathfrak{x}) \Delta\mathfrak{x} \right)^{q} \Delta\mathfrak{x} \le \mathbf{C}^{-q} \Lambda^{\frac{q}{p}} (\sigma(t)),$$

then

$$\Lambda^{q} (\sigma(t)) \int_{\mathtt{a}}^{t} \mathfrak{u}(\mathfrak{x}) \Delta\mathfrak{x} \le \Lambda^{q} (\sigma(t)) \int_{\mathtt{a}}^{b} \mathfrak{u}(\mathfrak{x}) \Delta\mathfrak{x} \le \mathbf{C}^{-q} \Lambda^{\frac{q}{p}} (\sigma(t)).$$

Finally, we get that

$$\Lambda^{-\frac{1}{p'}} (\sigma(t)) \left(\int_{\mathtt{a}}^{\sigma(t)} \mathfrak{u}(\mathfrak{x}) \Delta\mathfrak{x} \right)^{-1/q} \le \mathbf{C} < \infty.$$

Thus, $C < \infty$ implies the boundedness of $\mathbb{E}_k(\mathfrak{x}, s)$ in (12). This concludes the theorem's proof. $\square$

Consequently, by employing a similar technique we can easily conclude the duality of (Theorem 3) to obtain the following theorem.

Theorem 4. Assume that $\mathbb{T}$ is a time scale with $-\infty < q \le p < 0$, $\mathfrak{f} \in \mathbf{C}_{rd}([\mathtt{a}, b]_{\mathbb{T}}, \mathbb{R})$ is a non-negative mapping and suppose that $\mathfrak{u}$, $\mathtt{v}$ are rd-continuous positive functions on $(\mathtt{a}, b)_{\mathbb{T}}$. Then the inequality

$$\left(\int_{\mathtt{a}}^{b} \mathfrak{f}^{p}(\mathfrak{x}) \mathtt{v}(\mathfrak{x}) \Delta\mathfrak{x} \right)^{1/p} \le \mathbf{C} \left(\int_{\mathtt{a}}^{b} \left(\int_{\mathfrak{x}}^{b} \mathfrak{f}(t) \mathfrak{u}(\mathfrak{x}) \Delta t \right)^{q} \Delta\mathfrak{x} \right)^{1/q} \tag{42}$$

holds for each mapping $\mathfrak{f}$, if and only if

$$\mathbb{E}_r := \sup_{\mathfrak{x} \in (a,\, b)_{\mathbb{T}}} \Lambda^{\frac{1-s}{p}}(\mathfrak{x}) \left(\int_{\mathfrak{x}}^{b} \Lambda^{q(\frac{p-s}{p})}(t) \mathfrak{u}(t) \Delta t \right)^{-1/q} < \infty.$$

Moreover, the estimation $\mathbf{C}$ in (42) satisfies $\mathbb{E}_r \le \mathbf{C} < 2^{-\frac{1}{q}} \left(\frac{p+s-2}{p-1} \right)^{1/p'} \mathbb{E}_r$, where $\Lambda(\mathfrak{x}) := \int_{\mathfrak{x}}^{b} \mathtt{v}^{1-p'}(t) dt$.

Now, we mention some suitable applications which closely related to Theorem 3 and 4. If we consider $\mathbb{T}$ equals to the set of real numbers $\mathbb{R}$ in Theorem 3, this leads to the following weighted Hardy inequality that is continuous due to Prokhorov [11] and Kufner et al. [12].

Remark 1. If $\mathbb{T} = \mathbb{R}$ for $-\infty < q \le p < 0$ the inequality

$$\left(\int_{\mathtt{a}}^{b} \mathfrak{f}^{p}(\mathfrak{x}) \mathtt{v}(\mathfrak{x}) d\mathfrak{x} \right)^{1/p} \le \mathbf{C} \left(\int_{\mathtt{a}}^{b} \mathfrak{u}(\mathfrak{x}) \left(\int_{\mathtt{a}}^{\mathfrak{x}} \mathfrak{f}(t) dt \right)^{q} d\mathfrak{x} \right)^{1/q}, \tag{43}$$

that holds, if and only if the next condition

$$\sup_{\mathfrak{x}\in(\mathtt{a},\ b)} \Lambda^{\frac{1-s}{p}}(\mathfrak{x}) \left(\int_{\mathtt{a}}^{\mathfrak{x}} \mathfrak{u}\,(t)\,\Lambda^{\frac{p-s}{p}q}(t)dt\right)^{-1/q} < \infty, \quad s \in [p,1)$$

satisfies, where $\Lambda(\mathfrak{x}) := \int_{\mathtt{a}}^{\mathfrak{x}} \mathtt{v}^{1-p'}(t)dt,\ \ p' := \frac{p}{p-1}$.

By using an identical manner, we can put $\mathbb{T} = \mathbb{N}$ in Theorem 3 to obtain the discrete characterization of (43), that can be considered as the discrete result as mentioned in [13] and [14].

Remark 2. For $\mathbb{T} = \mathbb{N}$ and $-\infty < q \leq p < 0$, the inequality

$$\left(\sum_{\mathfrak{n}=0}^{\infty} \mathfrak{f}^p(\mathfrak{n})\mathtt{v}(\mathfrak{n})\right)^{1/p} \leq \mathbf{C}\left(\sum_{\mathfrak{n}=0}^{\infty} \mathfrak{u}(\mathfrak{n})\left(\sum_{\mathbb{k}=0}^{\mathfrak{n}} \mathfrak{f}(\mathbb{k})\right)^q\right)^{1/q}$$

will be contented if and only if $\sup_{\mathfrak{n}}\left(\sum_{\mathfrak{n}=0}^{\infty}\left(\sum_{\mathbb{k}=0}^{\mathfrak{n}} \mathfrak{u}\,(\mathbb{k})\,\Lambda^{\frac{p-s}{p}q}(\mathbb{k})\right)^{-1/q} \Lambda^{\frac{1-s}{p}}(\mathfrak{n})\right) < \infty$, where $\mathfrak{u}\,(\mathfrak{n})$, $\mathtt{v}(\mathfrak{n})$, $\Lambda(\mathfrak{n}) = \sum_{\mathbb{k}=0}^{\mathfrak{n}-1} \mathtt{v}^{1-p'}(\mathbb{k})$ and $\mathfrak{f}(\mathfrak{n})$ are non-negative sequences.

If we consider $\mathbb{T} = l^{\mathbb{N}_0}$, we catch the next inequality.

Remark 3. If $-\infty < q \leq p < 0$ with $\mathbb{T} = l^{\mathbb{N}_0}$ then the inequality

$$\left(\sum_{n=0}^{\infty} \mathtt{v}(l^n)\eta(l^n)\mathfrak{f}^p(l^n)\right)^{1/p} \leq \mathbf{C}\left(\sum_{n=0}^{\infty} \mathfrak{u}(l^n)\eta(l^n)\left(\sum_{\mathbb{k}=0}^{n} \mathfrak{f}(l^{\mathbb{k}})\eta(l^{\mathbb{k}})\right)^q\right)^{1/q}$$

holds if and only if for the sequences $\mathfrak{f}(n)$, $\mathfrak{u}\,(n)$ and $\mathtt{v}(n)$ that are not negative and that for

$$\sup_n\,(l-1)\left(\sum_{n=0}^{\infty}\left(\sum_{\mathbb{k}=0}^{n} \mathfrak{u}(l^{\mathbb{k}})\eta(l^{\mathbb{k}})\Lambda^{\frac{p-s}{p}q}(l^{\mathbb{k}})\right)^{-1/q} \Lambda^{\frac{1-s}{p}}(l^n)\right) < \infty,$$

for $\Lambda(l^n) = (l-1)\sum_{\mathbb{k}=0}^{n-1} \mathtt{v}^{1-p'}(l^{\mathbb{k}})$, and $\eta(l^{\mathbb{k}}) = (l-1)\,l^{\mathbb{k}}$.

In the follow-up, we provide further deductions that demonstrate the useful implications of our main results. Through various substitutions, we can capture the following resultant reverse dynamic inequalities of Hardy-type that have negative parameters p and q.

Corollary 1. Suppose that $\mathbb{T}$ is a time scale, $-\infty < q \leq p < 0$, and the mapping $\mathfrak{f} \in C_{rd}\left([0,\infty)_{\mathbb{T}}, \mathbb{R}\right)$. Consequently, there is a constant $\mathbf{C}$ in the form that Hardy inequality

$$\left(\int_0^{\infty} \mathfrak{f}^p\,(\mathfrak{x})\,\mathfrak{x}^{\beta}\Delta\mathfrak{x}\right)^{1/p} \leq \mathbf{C}\left(\int_0^{\infty}\left(\int_0^{\sigma(\mathfrak{x})} \mathfrak{f}\,(t)\,\Delta t\right)^q \mathfrak{x}^{\alpha}\Delta\mathfrak{x}\right)^{1/q} \tag{44}$$

fulfils for every $\mathfrak{f}$, if and only if $\sup_{\mathfrak{x}\in(0,\ b)_{\mathbb{T}}} \Lambda^{\frac{-s+1}{p}}(\sigma(\mathfrak{x}))\left(\int_0^{\sigma(\mathfrak{x})} \Lambda^{\frac{-s+p}{p}q}(t)t^{\alpha}\Delta t\right)^{-1/q}$, is convergent for $\Lambda(\mathfrak{x}) := \int_0^{\mathfrak{x}} t^{\beta(1-p')}\Delta t$ and α, β have positive values.

Proof. If we put $\mathfrak{u}\,(\mathfrak{x}) = \mathfrak{x}^{\alpha}$ and $\mathtt{v}(\mathfrak{x}) = \mathfrak{x}^{\beta}$ in Theorem 3, we gain the desired result. This concludes the proof. □

Remark 4. In inequality (44), if we set $\mathbb{T}=\mathbb{R}$, our inventory includes the continuous inequality

$$\left(\int_0^\infty \mathfrak{x}^\beta \mathfrak{f}^p(\mathfrak{x})\,d\mathfrak{x}\right)^{1/p} \le \mathbf{C}\left(\int_0^\infty \left(\int_0^{\mathfrak{x}} \mathfrak{f}(t)\,dt\right)^q \mathfrak{x}^\alpha d\mathfrak{x}\right)^{1/q},$$

due to [19], which valid for every function $\mathfrak{f}$, if and only if

$$\sup_{\mathfrak{x}\in(\mathtt{a},\ b)_{\mathbb{T}}} \Lambda^{\frac{-s+1}{p}}(\mathfrak{x})\left(\int_{\mathtt{a}}^{\mathfrak{x}} \Lambda^{\frac{-s+p}{p}q}(t)t^\alpha dt\right)^{-1/q} < \infty,$$

for positive numbers α, β and for $\Lambda(\mathfrak{x}) := \int_{\mathtt{a}}^{\mathfrak{x}} t^{\beta(1-p')}dt$.

Corollary 2. If we take $\mathbb{T}=\mathbb{N}$ in inequality (44), we obtain the discrete inequality

$$\left(\sum_{n=0}^{\infty} n^\alpha \mathfrak{f}^p(n)\right)^{1/p} \le \mathbf{C}\left(\sum_{n=0}^{\infty} n^\beta \left(\sum_{\flat=0}^{n} \mathfrak{f}(\flat)\right)^q\right)^{1/q},$$

which shall be fulfilled if and only if $\sup_n \left(\sum_{n=0}^{\infty}\left(\sum_{\flat=0}^{n} \flat^\alpha \Lambda^{\frac{p-s}{p}q}(\flat)\right)^{-1/q} \Lambda^{\frac{1-s}{p}}(n)\right) < \infty$, which is essentially new.

Corollary 3. Consider that $-\infty < q,\ p < 0$, on the time scale $\mathbb{T}$ with $q=p$, $h \in C_{rd}\left([0,\infty)_{\mathbb{T}}, \mathbb{R}^+\right)$ and the mappings $\mathfrak{u}$ and $\mathtt{v}$ are continuous positive right dense functions on an open interval $(0,\infty)_{\mathbb{T}}$. Further, for $r \neq 1$ suppose that

$$\Xi(\mathfrak{x}) = \begin{cases} \int_0^{\sigma(\mathfrak{x})} h(s)\,\Delta s, & r<1, \\ \int_{\mathfrak{x}}^{\infty} h(s)\,\Delta s, & r>1. \end{cases}$$

If r is less than 1, then the following inequality

$$\int_0^\infty \Xi^p(\mathfrak{x})\,\mathfrak{u}(\mathfrak{x})\,\Delta\mathfrak{x} \le \mathbf{C}_1 \int_0^\infty h^p(\mathfrak{x})\,\mathtt{v}(\mathfrak{x})\,\Delta\mathfrak{x} \tag{45}$$

holds. While the inequality

$$\int_0^\infty \Xi^p(\mathfrak{x})\,\mathfrak{u}(\mathfrak{x})\,\Delta\mathfrak{x} \le \mathbf{C}_2 \int_0^\infty h^p(\mathfrak{x})\,\mathtt{v}(\mathfrak{x})\,\Delta\mathfrak{x} \tag{46}$$

holds for r that is greater than 1.

Proof. When $r<1$, setting $q=p$, $\mathtt{a}=0$, and $b=\infty$ in Theorem 3 and by raising both sides to the power of p, we have the required result in (45). Suppose $r>1$, if we set $\mathtt{a}=0$, $b=\infty$ and $q=p$ in Theorem 4, we obtain the result in (46). This completes the proof. □

Remark 5. In inequality (45), as we consider $\mathbb{T}=\mathbb{R}$, $\mathfrak{u}(\mathfrak{x}) = \mathfrak{x}^{-r}$ and $\mathtt{v}(\mathfrak{x}) = \mathfrak{x}^{-r+p}$, we can establish the continuous inequality

$$\int_0^\infty \left(\int_0^{\mathfrak{x}} \mathfrak{f}(t)\,dt\right)^p \mathfrak{x}^{-r}d\mathfrak{x} \le \mathbf{C}\int_0^\infty \mathfrak{f}^p(\mathfrak{x})\,\mathfrak{x}^{-r+p}d\mathfrak{x},$$

due to [38,39], which holds for each $\mathfrak{f}$, if and only if $\sup_{\mathfrak{x}\in(0,\ \infty)_{\mathbb{T}}} \Lambda^{\frac{-s+1}{p}}(\mathfrak{x})\left(\int_0^{\mathfrak{x}} \mathfrak{x}^{-r}\Lambda^{-s+p}(t)\Delta t\right)^{-1/p} < \infty$, for $q=p$, $r<1$ and $\Lambda(\mathfrak{x}) := \int_0^{\mathfrak{x}} t^{-r+p}dt$.

Remark 6. For $\mathbb{T}=\mathbb{R}$, $\mathtt{u}(\mathfrak{x})=\dfrac{1}{\mathfrak{x}^r}$ and $\mathtt{v}(\mathfrak{x})=\dfrac{1}{\mathfrak{x}^{-p+r}}$ into inequality (46), we derive that

$$\int_0^\infty \left(\int_{\mathfrak{x}}^\infty \mathfrak{f}(t)\,dt\right)^p \frac{1}{\mathfrak{x}^r}d\mathfrak{x} \leq \mathbf{C}\int_0^\infty \frac{1}{\mathfrak{x}^{-p+r}}\mathfrak{f}^p(\mathfrak{x})\,d\mathfrak{x},$$

which verifies for each function $\mathfrak{f}$, regarding to $\sup_{\mathfrak{x}\in(0,\ \infty)_{\mathbb{T}}}(\Lambda^{\frac{-s+1}{p}}(\mathfrak{x}))\left(\int_{\mathtt{a}}^{\mathfrak{x}}\frac{1}{\mathfrak{x}^r}\Lambda^{p-s}(t)\Delta t\right)^{-1/p}<\infty$, that for $q=p$, $\Lambda(\mathfrak{x}):=\int_{\mathfrak{x}}^\infty t^{-r+p}dt$ and $r>1$, as stated in [38, 39].

Example 1. If we consider that $\mathbb{T}=\mathbb{R}$, $\mathtt{v}(\mathfrak{x})=\mathfrak{x}^2$, $\mathtt{u}(\mathfrak{x})=1$ and $p=-1$, $q=-2$, in Theorem 3, where the integration on the interval $\mathtt{a}=1, b=\infty$, then we can directly simplify Theorem 3 to the resultant inequality becomes

$$\left(\int_1^\infty \mathfrak{f}^{-1}(\mathfrak{x})\,\mathfrak{x}^2 d\mathfrak{x}\right)^{-1} \leq \mathbf{C}\left(\int_1^\infty \mathtt{u}(\mathfrak{x})\left(\int_1^{\mathfrak{x}}\mathfrak{f}(t)\,dt\right)^{-2}d\mathfrak{x}\right)^{-1/2},$$

holds for every non-negative function $\mathfrak{f}$ if and only if the following criterion is met:

$$\sup_{\mathfrak{x}\in(1,\infty)}\Lambda^{s-1}(\mathfrak{x})\left(\int_1^{\mathfrak{x}}\Lambda^{-2(s+1)}(t)dt\right)<\infty,$$

that for $\Lambda(t)=\int_1^{\mathfrak{x}}\mathtt{v}^{1-p'}(t)dt$.

Example 2. In Remark 4, assume that $\alpha=-3$, $\beta=-1$ and $p=-2$, $q=-2$, where the integration from $\mathtt{a}=0$ to $b=\infty$, then we can obtain the inequality

$$\left(\int_0^\infty \mathfrak{x}^{-1}\mathfrak{f}^{-2}(\mathfrak{x})\,d\mathfrak{x}\right)^{-1/2} \leq \mathbf{C}\left(\int_0^\infty\left(\int_0^{\mathfrak{x}}\mathfrak{f}(t)\,dt\right)^{-2}\mathfrak{x}^{-3}d\mathfrak{x}\right)^{-1/2},$$

holds for every function $\mathfrak{f}$ if and only if

$$\sup_{\mathfrak{x}\in(0,\infty)}\Lambda^{\frac{s-1}{2}}(\mathfrak{x})\left(\int_0^{\mathfrak{x}}\Lambda^{\frac{s+2}{2}}(t)t^\alpha dt\right)^{1/2}<\infty,$$

that for $\Lambda(t)=\int_0^{\mathfrak{x}}\mathtt{v}^{1-p'}(t)dt$.

Example 3. In Remark 3, if we suppose that $l=2$, where $\mathbb{T}=l^{\mathbb{N}_0}$, then the inequality take the form

$$\left(\sum_{n=0}^\infty \mathtt{v}(2^n)\eta(2^n)\mathfrak{f}^p(2^n)\right)^{1/p} \leq \mathbf{C}\left(\sum_{n=0}^\infty \mathtt{u}(2^n)\eta(2^n)\left(\sum_{\Bbbk=0}^n\mathfrak{f}(2^\Bbbk)\eta(2^\Bbbk)\right)^q\right)^{1/q},$$

and for $p=q=-1$, we capture the inequality

$$\left(\sum_{n=0}^\infty \mathtt{v}(2^n)\eta(2^n)\mathfrak{f}^{-1}(2^n)\right)^{-1} \leq \mathbf{C}\left(\sum_{n=0}^\infty \mathtt{u}(2^n)\eta(2^n)\left(\sum_{\Bbbk=0}^n\mathfrak{f}(2^\Bbbk)\eta(2^\Bbbk)\right)^{-1}\right)^{-1},$$

that holds if and only if

$$\sup_n\,(2-1)\left(\sum_{n=0}^\infty\left(\sum_{\Bbbk=0}^n \mathtt{u}(2^\Bbbk)\eta(2^\Bbbk)\Lambda^{\frac{-1-s}{-1}-1}(2^\Bbbk)\right)\Lambda^{\frac{1-s}{-1}}(2^n)\right)<\infty,$$

for the non-negative sequences $\mathfrak{f}(n)$, $\mathtt{u}(n)$ and $\mathtt{v}(n)$ and for $\Lambda(l^n)=(2-1)\sum_{\Bbbk=0}^{n-1}\mathtt{v}^{1-p'}(2^\Bbbk)$, and $\eta(2^\Bbbk)=(2-1)\,2^\Bbbk$.

Example 4. If we assume that $\mathfrak{u}(\mathfrak{x}) = \chi_{(0,1)}$, $\mathtt{v}(\mathfrak{x}) = \chi_{(0,1)} + \chi_{(1,\infty)}$ and $p = q = -1$, then for $\mathtt{a} = 0, b = \infty$. We can directly simplify Theorem 3 to the resultant inequality that established in [11]

$$\int_0^\infty \left(\mathtt{v}^{-1}(\mathfrak{x})\,\mathfrak{f}(\mathfrak{x})d\mathfrak{x}\right)^{-1} \leq \mathbf{C}\left(\int_0^\infty \left(\int_0^{\mathfrak{x}} \mathfrak{f}(t)dt\right)^{-1} \mathfrak{u}(\mathfrak{x})\,d\mathfrak{x}\right)^{-1},$$

where $\mathbf{C}$ satisfies the following estimate $\sup\limits_{\mathfrak{x}\in(0,\infty)} \Lambda^{-1+s}(\mathfrak{x})\left(\int_0^{\mathfrak{x}} \Lambda^{-1-s}(t)\mathfrak{u}(t)\,dt\right) < \infty$, that for $\Lambda(t) = \int_{\mathtt{a}}^{\mathfrak{x}} \mathtt{v}^{1-p'}(t)dt$ on $\mathbb{T} = \mathbb{R}$.

Conclusion

This research formulates novel reverse Hardy-type inequalities with negative indices on arbitrary time scales, integrating two independent weight functions. Using the reverse Hölder and Minkowski integral inequalities within the time-scale framework, we derive the conditions under which these inequalities hold the obtained results, generalize and consolidate several classical discrete, continuous, and quantum analogies, building upon the foundational works of Prokhorov, Kufner, and others. The main scientific contribution lies in the comprehensive analysis of negative exponents and the formulation of a unified technique across many temporal dimensions. These results expand the theoretical framework of Hardy-type inequalities and suggest possible applications in harmonic analysis and dynamic equations. Future study might investigate multidimensional extensions, applications to partial dynamic equations, or similar outcomes in broader measure spaces.

Conflict of Interest

The authors declare no conflicts of interest.

References

1 Hardy, G.H. (1920). Note on a theorem of Hilbert. *Mathematische Zeitschrift, 6*(3-4), 314–317. https://doi.org/10.1007/BF01199965

2 Hardy, G.H. (1925). Notes on some points in the integral calculus (LX). *An inequality between integrals. Messenger of Mathematics, 54,* 150–156.

3 Anderson, K., & Muckenhoupt, B. (1982). Weighted weak type Hardy inequalities with applications to Hilbert transforms and maximal functions. *Studia Mathematica, 72,* 9–26. https://doi.org/10.4064/sm-72-1-9-26

4 Heinig, H.P. (1985). Weighted norm inequalities for certain integral operators II. *Proceedings of the American Mathematical Society, 95*(3), 387–395. https://doi.org/10.1090/S0002-9939-1985-0806076-3

5 Muckenhoupt, B. (1972). Hardy's inequality with weights. *Studia Mathematica, 44,* 31–38. https://doi.org/10.4064/sm-44-1-31-38

6 Kufner, A., & Persson, L.E. (2003). *Weighted inequalities of Hardy type.* World Scientific. https://doi.org/10.1142/5129

7 Kufner, A., Maligranda, L., & Persson, L.E. (2007). *The Hardy inequalities: About its history and some related results.* Praha: Vydavatelsky Servis Publishing House.

8 Opic, B., & Kufner, A. (1990). *Hardy-type inequalities.* Harlow: Longman Scientific & Technical.

9 Beesack, P.R., & Heinig, H.P. (1981). Hardy's inequalities with indices less than 1. *Proceedings of the American Mathematical Society, 83*(3), 532–536. https://doi.org/10.1090/S0002-9939-1981-0627685-7

10 Andersen, K.F., & Heinig, H.P. (1983). Weighted norm inequalities for certain integral operators. *SIAM Journal on Mathematical Analysis, 14*(4), 834–844. https://doi.org/10.1137/0514064

11 Prokhorov, D.V. (2004). Weighted Hardy's inequalities for negative indices. *Publicacions Matematiques, 48,* 423–443. https://doi.org/10.5565/PUBLMAT_48204_08

12 Kufner, A., & Kuliev, K. (2006). The Hardy inequality with "negative powers". *Advances in Algebraic Analysis, 1*(3), 219–228.

13 Nguyen, T.L., & Nguyen, V.D. (2001). The Carleman's inequality for a negative power number. *Journal of Mathematical Analysis and Applications, 259*(1), 219–225. https://doi.org/10.1006/jmaa.2000.7422

14 Čižmešija, A. (2005). On weighted discrete Hardy's inequality for negative power numbers. *Journal of Mathematical Inequalities and Applications, 8*(2), 273–285. https://doi.org/10.7153/mia-08-26

15 Benaissia, B., & Budak, H. (2021). On Hardy-type integral inequalities with negative parameter. *Turkish Journal of Inequalities, 5*(2), 42–47.

16 Gogatishvili, A., Kuliev, K., & Kulieva, G. (2007). Some conditions characterizing the "reverse" Hardy inequality. *Real Analysis Exchange, 33*(1), 249–258. https://doi.org/10.14321/realanalexch.33.1.0249

17 Heinig, H.P. (1986). Weighted estimates for classical operators. In *Nonlinear analysis, function spaces and applications, 3*, 31–53. Leipzig: Teubner.

18 Kassymov, A., Ruzhansky, M., & Suragan, D. (2022). Reverse integral Hardy inequality on metric measure spaces. *Annales Fennici Mathematici, 47*(1), 39–55. https://doi.org/10.54330/afm.112455

19 Kassymov, A., Ruzhansky, M., & Suragan, D. (2023). Hardy inequalities on metric measure spaces, III: The case $q \leq p \leq 0$ and applications. *Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences, 479*(2269), Article 20220723. https://doi.org/10.1098/rspa.2022.0307

20 Kufner, A., Kuliev, K., Oguntuase, J.A., & Persson, L.-E. (2007). Generalized weighted inequality with negative powers. *Journal of Mathematical Inequalities and Applications, 1*(2), 269–280. https://doi.org/10.7153/JMI-01-24

21 Kufner, A., Kuliev, K., & Kulieva, G. (2008). The Hardy inequality with one negative parameter. *Banach Journal of Mathematical Analysis, 2*(2), 76–84. https://doi.org/10.15352/bjma/1240336294

22 Dattoli, G., & Cesarano, C. (2003). On a new family of Hermite polynomials associated to parabolic cylinder functions. *Applied Mathematics and Computation, 141*(1), 143–149. https://doi.org/10.1016/S0096-3003(02)00328-4

23 Ramírez, W., & Cesarano, C. (2022). Some new classes of degenerated generalized Apostol-Bernoulli, Apostol-Euler and Apostol-Genocchi polynomials. *Carpathian Mathematical Publications, 14*(2), 354–363. https://doi.org/10.15330/cmp.14.2.354-363

24 Kufner, A., & Kuliev, K. (2010). Some characterizing conditions for the Hardy inequality. *Eurasian Mathematical Journal, 1*(2), 86–98.

25 Oguntuase, J.A., Okpoti, C.A., Persson, L., & Allotey, F.K. (2007). Multidimensional Hardy type inequalities for $p < 0$ and $0 < p < 1$. *Journal of Mathematical Inequalities and Applications, 1*(1), 1–11. https://doi.org/10.7153/jmi-01-01

26 Agarwal, R.P., Bohner, M., & Saker, S.H. (2015). Dynamic Littlewood-type inequalities. *Proceedings of the American Mathematical Society, 143*(2), 667–677. https://doi.org/10.1090/proc/12363

27 Saker, S.H., Mahmoud, R.R., & Peterson, A. (2016). Weighted Hardy-type inequalities on time scales with applications. *Mediterranean Journal of Mathematics, 13,* 585–606. https://doi.org/10.1007/s00009-014-0514-y

28 Agarwal, R.P., O'Regan, D., & Saker, S.H. (2016). *Hardy type inequalities on time scales.* Springer. https://doi.org/10.1007/978-3-319-44299-0

29 Bohner, M., & Peterson, A. (2001). *Dynamic equations on time scales: An introduction with applications.* Birkhäuser. https://doi.org/10.1007/978-1-4612-0201-1

30 Bohner, M., & Peterson, A. (Eds.). (2003). *Advances in dynamic equations on time scales.* Birkhäuser. https://doi.org/10.1007/978-0-8176-8230-9

31 Bibi, R., Bohner, M., Pečarić, J., & Varosance, S. (2013). Minkowski and Beckenbach-Dresher inequalities and functionals on time scales. *Journal of Mathematical Inequalities, 7*(3), 299–312. https://doi.org/10.7153/jmi-07-28

32 Saker, S.H., Awwad, E., & Saied, A. (2019). Some new dynamic inequalities involving monotonic functions on time scales. *Journal of Function Spaces, 2019,* Article 7584836. https://doi.org/10.1155/2019/7584836

33 Zhao, C.J., & Cheung, W.S. (2016). Hölder's reverse inequality and its applications. *Publications de l'Institut Mathématique, 99*(113), 211–216. https://doi.org/10.33048/semi.2019.16.048

34 Saker, S.H., Osman, M.M., O'Regan, D., & Agarwal, R.P. (2021). Characterizations of reverse dynamic weighted Hardy-type inequalities with kernels on time scales. *Aequationes Mathematicae, 95*(1), 125–146. https://doi.org/10.1007/s00010-020-00759-6

35 Bohner, M., & Georgiev, S.G. (2016). *Multiple integration on time scales: Multivariable dynamic calculus on time scales.* Springer. https://doi.org/10.1007/978-3-319-47620-9_7

36 Saker, S.H. (2014). Hardy-Leindler type inequalities on time scales. *Applied Mathematics & Information Sciences, 8*(6), 2975–2981. https://doi.org/10.12785/amis/080635

37 Saker, S.H., Osman, M.M., & Abohela, I. (2020). New characterizations of weights in Hardy's type inequalities via Opial's dynamic inequalities. *Journal of Computational Analysis and Applications, 28*(6), 1072–1085.

38 Yang, B. (2007). On a new Hardy type integral inequalities. *International Mathematical Forum, 2*(67), 3317–3322.

39 Benaissia, B., & Sarikaya, M. (2020). Generalization of some Hardy-type integral inequality with negative parameter. *Bulletin of the Transilvania University of Braşov, Series III: Mathematics, Informatics, Physics, 13*(1), 69–76. https://doi.org/10.31926/but.mif.2020.13.62.1.6

*Author Information**

Khadega Reda Abdo — Doctor of Pure Mathematics, Assistant Professor, Department of Mathematics, Rustaq College of Education, University of Technology and Applied Science, Al Rustaq, Sultanate of Oman; e-mail: *khadija.abdo@utas.edu.om*; Lecturer, Department of Mathematics, Faculty of Science, Fayoum University, Fayoum, Egypt; e-mail: *kra00@fayoum.edu.eg*; https://orcid.org/0009-0007-9292-1155

*Authors' names are presented in the following order: first name, middle name (if any), last name.

https://doi.org/10.31489/2026M1/23-36 *Research article*

Stability and existence of multiperiodic solutions for second-order linear equations with a diagonal differentiation operator

G.M. Aitenova[1], Zh.A. Sartabanov[2], B.Zh. Omarova[2], A.Kh. Zhumagaziyev[2,3,*]

[1] *M. Utemisov West Kazakhstan University, Uralsk, Kazakhstan;*
[2] *K. Zhubanov Aktobe Regional University, Aktobe, Kazakhstan;*
[3] *Kh. Dosmukhamedov Atyrau University, Atyrau, Kazakhstan*
(E-mail: gulsezimaitenova@gmail.com, sartabanov42@mail.ru, bibigul.zharbolkyzy@gmail.com, azhumagaziyev@zhubanov.edu.kz)

The stability of differential equations with periodic and quasiperiodic coefficients is a central topic in modern stability theory, with important applications in mechanics, physics, and dynamical systems. A classical result in this area is the Lyapunov integral criterion, which provides stability conditions for linear second-order equations with periodic coefficients. In this paper, we extend this criterion to equations with quasiperiodic coefficients. Our analysis is based on the method of periodic characteristics, which has proven effective in the study of multiperiodic solutions for systems with a diagonal differentiation operator. Within this framework, the multiperiodicity condition is reduced to a functional equation, and a Floquet-type representation of the matricant of the associated system is derived. This representation shows that multiperiodicity of solutions follows from the purely imaginary nature of the characteristic multipliers and the periodicity of the helical characteristics. The obtained results confirm that the Lyapunov integral criterion remains valid for equations with quasiperiodic coefficients. More generally, they demonstrate the effectiveness of the characteristic method for analyzing stability in complex dynamical systems, thereby extending the scope of classical stability theory.

Keywords: Lyapunov integral criterion, stability analysis, periodic coefficients, quasiperiodic coefficients, periodic characteristics method, multiperiodic solutions, Floquet theory, differential equations.

2020 Mathematics Subject Classification: 34D20, 34C25, 34C27, 37C55.

Introduction

The Lyapunov integral criterion for the stability of linear second-order differential equations with periodic coefficients is a fundamental result in stability theory. In this paper, we extend this criterion to the case where the coefficients are quasiperiodic. Thus, our study addresses the stability of linear equations with both periodic and quasiperiodic coefficients.

The analysis is based on the method of periodic characteristics developed in [1, 2], which has been successfully applied to the study of multiperiodic solutions of systems with the diagonal differentiation operator. Classical results on the theory of stability and periodic solutions of differential equations, including the Lyapunov integral criterion, are presented in [3, 4]. Fundamental results on systems of partial differential equations and characteristic methods are discussed in [5]. The theory of almost periodic and almost multiperiodic solutions for ordinary, partial, and evolutionary differential equations is developed in [6–8]. These works provide the theoretical background for the stability analysis and the formulation of multiperiodicity conditions in terms of functional–difference equations.

Unlike traditional approaches, the method proposed in [1, 2] represents the multiperiodicity condition for D-equations as a functional equation, enabling the use of a Floquet representation for the

*Corresponding author. *E-mail: azhumagaziyev@zhubanov.edu.kz*

This research was funded by the Science Committee of the Ministry of Science and Higher Education of the Republic of Kazakhstan (grant No. AP19676629).

Received: 21 August 2025; *Accepted:* 5 January 2026.

matricant of the associated system. The multiperiodicity of solutions then follows from the purely imaginary nature of the multipliers and the periodicity of the helical characteristics.

Research on multiperiodic solutions of systems with differentiation along characteristics has been further advanced in [9–11]. Further developments concerning periodic and impulsive evolution equations are discussed in [12–14]. The existence and multiplicity of periodic and multiperiodic solutions for second-order and parameter-dependent equations are investigated in [15–18]. Methods based on differentiation along characteristics and diagonal operators have received international recognition through publications in leading scientific journals [19, 20]. Related solvability and boundary value problems for parabolic, nonlocal, loaded, and hyperbolic equations are considered in [21–23], as well as in more recent studies [24].

Beyond its theoretical significance, stability analysis of systems with periodic and quasiperiodic coefficients has a wide range of applications. These include oscillatory processes in mechanics, signal propagation in physics and engineering, and models of dynamical systems with multiple interacting frequencies. In such contexts, stability criteria are essential for predicting long-term behavior, preventing undesirable oscillations, and ensuring reliable system performance.

The purpose of this article is twofold: (i) to establish the applicability of the Lyapunov integral criterion to linear second-order equations with quasiperiodic coefficients, and (ii) to demonstrate the effectiveness of the periodic characteristics method in stability analysis of such equations.

1 Main results

Consider the equation with respect to $z = z(\tau, t)$, $\tau \in (-\infty, +\infty) = R$, $t = (t_1, ..., t_m) \in R\times ...\times R = R^m$ of the form

$$D^2 z + a(\tau, t)Dz + b(\tau, t)z = 0, \quad D = \frac{\partial}{\partial \tau} + \sum_{j=1}^{m} \frac{\partial}{\partial t_j}, \tag{1}$$

$$a(\tau + \theta, t + \omega) = a(\tau, t) \in^{\theta,\omega} C^{1,e}_{\tau,t}(R \times R^m), \quad b(\tau + \theta, t + \omega) = b(\tau, t) \in C^{(0,e)}_{\tau,t}(R \times R^m),$$

$$\omega_0 = \theta, \quad \omega = (\omega_1, ..., \omega_m), \quad \omega_i/\omega_j \in Q(i, j = \overline{0, m}), \quad D^2 z = D(Dz),$$

where Q is the set of rational numbers.

${}^{\theta,\omega}C^{\alpha,\beta}_{\tau,t}(R \times R^m)$ is class (θ, ω)-periodic in $(\tau, t) \in R \times R^m$ function smoothness in them (α, β).

1) In the study of any problem for multiperiodic equations with a directional differentiation operator, the Cauchy characteristic method is used. The characteristic equations link the independent variables, and some of them lose their independence. For example, in our case, the variables $t_j, j = \overline{1, m}$ become functions of the variable $\tau : t_j = h_j(\tau)$, $j = \overline{1, m}$. Since the system is θ-periodic with respect to τ, it is desirable that $h_j(\tau)$ have the property of θ-periodicity: $h_j(\tau + \theta) = h_j$, $j = \overline{1, m}$.

If the characteristic equation of the operator D

$$\frac{dt}{d\tau} = e, \quad e = (1, ..., 1), \tag{2}$$

on a manifold of the type of Euclidean space with Cartesian coordinates, then the characteristics do not possess this periodicity property. Consequently, it is necessary to change the type of manifold where the field (2) defines θ-periodic characteristics. It turns out that the typical manifold is a multidimensional cylindrical surface [1, 3].

$$\frac{dt_j}{d\tau} = 1. \tag{3}$$

Consider the surface of a straight circular infinite cylinder with circumference S_θ of length $\theta = 2\pi r$. Then the solutions $t_j = \eta_j + \tau - \xi = \beta_j(\tau - \xi, \eta_j)$ along the cylinder perform a helical motion of the period θ, i.e., the point (τ, t_j) moves along a helical line. Consequently, the direct product of such motions is a motion on the surface m of an $\mathcal{M}^m = \mathcal{M} \times ... \times \mathcal{M}$, $(\tau, t_j) \in \mathcal{M} = R \times S_\theta$, $j = \overline{1, m}$ multidimensional cylindrical surface.

Thus, according to [1, 2], the equation

$$t = \beta(\tau - \xi, \eta) \equiv (\beta_1(\tau - \xi, \eta_1), ..., \beta_m(\tau - \xi, \eta_m)) \tag{4}$$

represents a multidimensional helical line defined by the equation (2)-(3), possessing the properties of periodicity and the group

$$\begin{aligned} \beta(\tau + \theta - \xi, \eta) &= \beta(\tau - \xi, \eta), \\ \beta(\tau - \xi, \eta + \omega) &= \beta(\tau - \xi, \eta) + \omega, \\ \beta(\xi - \sigma, \beta(\sigma - \tau, t)) &= \beta(\xi - \tau, t), \end{aligned} \tag{5}$$

where $\eta = (\eta_1, ..., \eta_m)$.

It turns out that, for our purpose of multiperiodic solutions, it is expedient to consider the problem on m-dimensional cylindrical surface $(\tau, t) \in \mathcal{M}^m$.

Note that the equation (4) represents the characteristics of the operator D, and equation

$$\eta = \beta(\xi - \tau, t) \tag{6}$$

there is an equation of the first characteristic integrals:

$$D\beta(\xi - \tau, t) = 0. \tag{7}$$

Usually, the problem is investigated along the characteristics (4), and the results of the investigation are formulated in terms of the first integrals (6). The transition from characteristics to integrals and vice versa is realised by the property of the group from (5).

A sign of the appropriateness of the manifold is the equivalence of (θ, ω)-periodicity of the solution $x(\tau, t)$ of the system according to (τ, t) with (θ, θ, ω)-periodicity of the composition $x(\sigma, \beta(\sigma - \tau, t)) = x(\xi, \eta) \circ (\sigma, \beta(\sigma - \tau, t))$ according to (σ, τ, t) on this manifold at $\sigma \in R$.

It is obvious that this principle holds only on the cylindrical manifold $\mathcal{M}^m$.

2) It is widely known in the literature [3] that Lyapunov's integral criterion for the stability of periodic motions described by a second-order equation with an ordinary differentiation operator $D = \frac{d}{d\tau}$.

An interesting question is whether this property remains valid when the motions are multi-frequency, i.e., quasiperiodic.

This question will be explored below based on a rotational-linear modification [1,2] of Kharasahal's method [6].

To make it easier to control the equation (1), we should reduce the number of determining parameters a and b. To this end, let us set

$$z = e^{-\frac{1}{2}\int\limits_0^\tau a(\sigma, \beta(\sigma - \tau, t))d\sigma} x, \quad (\tau, t) \in \mathcal{M}^m. \tag{8}$$

Taking into account (7), we write the equation (1) as

$$D^2 x + p(\tau, t)x = 0 \tag{9}$$

with a single coefficient $p(\tau, t)$ of the form

$$p(\tau + \theta, t + \omega) = p(\tau, t) = b(\tau, t) - \frac{1}{4}a(\tau, t)^2 - \frac{1}{2}Da(\tau, t) \in C^{(0,e)}_{\tau,t}(\mathcal{M}^m).$$

3) Next, we represent the equation (8) as a system

$$Dx_1 = x_2,$$

$$Dx_2 = -p(\tau,t)x_1, \quad D = \frac{\partial}{\partial\tau} + \sum_{j=1}^{m} \frac{\partial}{\partial t_j}, \tag{10}$$

$$p(\tau+\theta, t+\omega) = p(\tau,t) \in C_{\tau,t}^{(0,e)}(\mathcal{M}^m),$$

where $x = x_1$.

In order to construct a fundamental system of solutions

$$X(\tau,t) = \begin{pmatrix} \varphi(\tau,t) & \psi(\tau,t) \\ D\varphi(\tau,t) & D\psi(\tau,t) \end{pmatrix}. \tag{11}$$

From (10), we will construct solutions $\varphi(\tau,t)$ and $\psi(\tau,t)$ of equation (9) that satisfy the conditions

$$\varphi(\tau,t)|_{\tau=0} = 1, \quad D\varphi(\tau,t)|_{\tau=0} = 0, \tag{12}$$

$$\psi(\tau,t)|_{\tau=0} = 0, \quad D\psi(\tau,t)|_{\tau=0} = 1. \tag{13}$$

We will determine these solutions using the auxiliary equation

$$D^2\widetilde{x} = \mu p(\tau,t)\widetilde{x} \tag{14}$$

in the form of power series with respect to the parameter μ.

In accordance with this, we set

$$\widetilde{\varphi}(\tau,t,\mu) = \sum_{k=0}^{\infty} \varphi_k(\tau,t)\mu^k. \tag{15}$$

Substituting (15) into (14) and equating the coefficients with the same powers of the parameter μ, we have

$$D^2\varphi_0 = 0, \quad \varphi_0(0,t) = 1, \quad D\varphi_0(0,t) = 0, \tag{16}$$

$$D^2\varphi_k(\tau,t) = p(\tau,t)\varphi_{k-1}(\tau,t), \quad \varphi_k(0,t) = D\varphi_k(0,t) = 0, \quad k = 1,2,... \tag{17}$$

Then, from (16) and (17) we have

$$\varphi_0 = 1, \; \varphi_1(\tau,t) = \int_0^{\tau} d\sigma_1 \int_0^{\sigma_1} p(\sigma, \beta(\sigma-\sigma_1, \beta(\sigma_1-\tau,t)))d\sigma =$$

$$= \int_0^{\tau} d\sigma_1 \int_0^{\sigma_1} p(\sigma, \beta(\sigma-\tau,t))d\sigma =$$

$$= \int_0^{\tau} (\tau-\sigma)p(\sigma,\beta(\sigma-\tau,t))d\sigma, ...,$$

$$\varphi_k(\tau,t) = \int_0^{\tau} (\tau-\sigma)p(\sigma,\beta(\sigma-\tau,t)) \cdot \varphi_{k-1}(\sigma,\beta(\sigma-\tau,t))d\sigma, \; k = 1,2,... \tag{18}$$

Evaluating (18) at $|p(\tau,t)| \le \triangle = const \ge 1$, we obtain

$$|\varphi_1(\tau,t)| \le \triangle \left| \int_0^\tau (\tau-\sigma)d\sigma \right| = \frac{\triangle|\tau|^2}{2!},$$

$$|\varphi_2(\tau,t)| \le \left| \int_0^\tau (\tau-\sigma)\triangle \cdot \frac{\triangle\sigma^2}{2!} \right| \le \frac{\triangle}{2!} \left| \left[\frac{\tau\sigma^3}{3} - \frac{\sigma^4}{4} \right] \right|_0^\tau = \frac{\triangle^2\tau^4}{4!}, ...$$

If we set $|\varphi_k(\tau,t)| \le \frac{\triangle^k \tau^{2k}}{2k!}$, then $\varphi_{k+1}(\tau,t)$ satisfies the estimate

$$|\varphi_{k+1}(\tau,t)| \le \left| \int_0^\tau (\tau-\sigma)\triangle \cdot \frac{\triangle^k\sigma^{2k}}{(2k)!} d\sigma \right| \le$$

$$\le \frac{\triangle^{k+1}}{(2k)!} \left| \frac{\tau\sigma^{2k+1}}{2k+1} - \frac{\sigma^{2k+2}}{2k+2} \right| |_0^\tau = \frac{\triangle^{k+1}|\tau|^{2k+2}}{(2k+2)!}, \quad k=0,1,2,... \tag{19}$$

Therefore, by virtue of (16)–(19), the series (15) converges at $|\mu| < \mu_0$, $|\tau| < T$ with any finite constants μ_0 and T.

Then, setting $\mu = -1$, we obtain the solution

$$\varphi(\tau,t) = \widetilde{\varphi}(\tau,t,-1) = \sum_{k=0}^{\infty} (-1)^k \varphi_k(\tau,t) \tag{20}$$

of equations (14) at $\mu = -1$, therefore, equations (10) satisfy the condition (12).

In a similar way, we determine the solution $\psi(\tau,t)$ of the initial problem (10)–(13), setting $\psi(\tau,t) = \widetilde{\psi}(\tau,t,\mu)|_{\mu=-1}$ and

$$\widetilde{\psi}(\tau,t,\mu) = \sum_{k=0}^{\infty} \psi_k(\tau,t)\mu^k, \quad \widetilde{\psi}(\tau,t,\mu)|_{\tau=0} = 0, \quad D\widetilde{\psi}(\tau,t,\mu)|_{\tau=0} = 1, \tag{21}$$

where $\widetilde{\psi}(\tau,t,\mu)$ is the solution to problem (14) with initial condition (21) with coefficients $\psi_k(\tau,t)$, $k=0,1,2,...$, which are determined sequentially by the formulas

$$\psi_0(\tau,t) = \tau, \quad \psi_k(\tau,t) = \int_0^\tau (\tau-\sigma)p(\sigma,\beta(\sigma-\tau,t))\psi_{k-1}(\sigma,\beta(\sigma-\tau,t))d\sigma, \quad k=1,2,... \tag{22}$$

Thus, we have estimates

$$|\psi_0(\tau,t)| \le \frac{|\tau|}{1!}, \quad |\psi_1(\tau,t)| \le \left| \int_0^\tau (\tau-\sigma)\sigma d\sigma \right| \le \triangle \left| \frac{\tau\sigma}{2} - \frac{\sigma^3}{3} \right| |_0^\tau = \frac{\triangle|\tau|^3}{3!}, ...,$$

$$|\psi_{k-1}(\tau,t)| \le \frac{\triangle^{k-1}|\tau|^{2k-1}}{(2k-1)!}, \quad |\psi_k(\tau,t)| \le \triangle \left| \int_0^\tau (\tau-\sigma)\triangle^{k-1}\sigma^{2k-1} d\sigma \right| = \frac{\triangle^k|\tau|^{2k+1}}{(2k+1)!}, ...$$

for which the series (21) converges absolutely and uniformly in finite domains: $|\mu| < \mu_0$, $|\tau| < T$.

Then we obtain the second solution $\psi(\tau,t)$ of equation (10) in the form

$$\psi(\tau,t)=\sum_{k=0}^{\infty}(-1)^k\psi_k(\tau,t), \tag{23}$$

by setting $\mu=-1$ from (21) with coefficients (22).

It is not particularly difficult to show that these solutions (20) and (23) are differentiable.

Thus, we have

$$D\varphi(\tau,t)=\sum_{k=1}^{\infty}(-1)^k D\varphi_k(\tau,t),\quad D\varphi_k(\tau,t)=\int\limits_0^{\tau}p(\sigma,\beta(\sigma-\tau,t))\varphi_{k-1}(\sigma,\beta(\sigma-\tau,t))d\sigma, \tag{24}$$

$$D\psi(\tau,t)=1+\sum_{k=1}^{\infty}(-1)^k D\psi_k(\tau,t),\quad D\psi_k(\tau,t)=\int\limits_0^{\tau}p(\sigma,\beta(\sigma-\tau,t))\psi_{k-1}(\sigma,\beta(\sigma-\tau,t))d\sigma$$

and based on them we obtain the matrix $X(\tau,t)$ of system (10).

4) It is easy to show that the determinant

$$w(\tau,t)=detX(\tau,t)$$

of the matrix $X(\tau,t)$ of system (10) satisfies the equation

$$Dw=SpP(\tau,t)\cdot w,\quad w|_{\tau=0}=detX(0,t),$$

where $SpP(\tau,t)$ is the trace of the matrix. $P(\tau,t)$ of the form

$$P(\tau,t)=\begin{pmatrix}0 & 1\\ -p(\tau,t) & 0\end{pmatrix}$$

of system (10) and is equal to the sum of its diagonal elements:

$$SpP(\tau,t)=0.$$

Therefore,

$$w(\tau,t)=1.$$

The proof of this Ostrogradsky-Jacobi formula is similar to the proof based on linear characteristics [6]. Therefore, we will not dwell on the proof.

Statement 1. For the matricant (11) of system (10), the Ostrogradsky–Jacobi formula holds true

$$detX(\tau,t)=e^{\int\limits_0^{\tau}SpP(\sigma,\beta(\sigma,\tau,t))d\sigma}=1.$$

5) The matricant X of the system

$$Dx=P(\tau,t)x,\quad P(\tau+\theta,t+\omega)=P(\tau,t)\in^{\theta,\omega}C^{0,e}_{\tau,t}(\mathcal{M}^m) \tag{25}$$

has the properties

$$\begin{gathered}DX(\tau,t)=P(\tau,t)X(\tau,t),\quad X(0,t)=E,\\ X(\tau,t+\omega)=X(\tau,t)\in^{0,\omega}C^{(1,e)}_{(\tau,t)}(\mathcal{M}^m),\\ X(\tau+\theta,t)=X(\tau,t)X(\theta,\beta(-\tau,t)),\quad(\tau,t)\in\mathcal{M}^m,\end{gathered} \tag{26}$$

where E is the identity matrix.

Then, by virtue of (26), we have

$$X(\tau + k\theta, t) = X(\tau, t)X^k(\theta, \beta(-\tau, t)), \quad k = 0, 1, 2, ...$$

Hence, setting $\tau = 0$, we obtain

$$X(k\theta, t) = X^k(\theta, t), \quad (k\theta, t) \in \mathcal{M}^m, \quad k = \overline{0, +\infty}.$$

For any fixed value of $t = \eta$, we have the logarithm

$$\frac{1}{k\theta} LnX(k\theta, \eta) = \frac{1}{\theta} LnX(\theta, \eta), \quad k = \overline{1, \infty}.$$

From here, moving to the limit at $\xi = k\theta \longrightarrow \infty$, we obtain

$$\frac{1}{\theta} LnX(\theta, \eta) = \lim_{\xi \longrightarrow \infty} \frac{1}{\xi} LnX(\xi, \eta), \quad (\xi, \eta) \in \mathcal{M}^m.$$

Further, setting

$$LnX(\theta, \eta) = \Lambda(\eta), \quad (\theta, \eta) \in \mathcal{M}^m,$$

we have the representation

$$X(\theta, \eta) = e^{\Lambda(\eta)}, \quad \eta \in S^m_\theta.$$

Due to the arbitrariness of $\eta \in S^m_\theta$, from the properties (26) we obtain

$$\Lambda(\eta + \omega) = \Lambda(\eta) \in^{\omega} C^{(e)}_\eta(S^m_\theta).$$

For the first integral $\eta = \beta(-\tau, t)$ corresponding to the value η from the last two equalities, the properties follow

$$X(\theta, \beta(-\tau, t)) = e^{\Lambda(\beta(-\tau,t))},$$

$$\Lambda(\beta(-\tau - \theta, t)) = \Lambda(\beta(-\tau, t)),$$

$$\Lambda(\beta(-\tau, t + \omega)) = \Lambda(\beta(-\tau, t) + \omega) = \Lambda(\beta(-\tau, t)), \quad (\tau, t) \in \mathcal{M}^m.$$

As a result, due to last equalities from (26), we have the Floquet representation for the system (25) of the form

$$X(\tau, t) = Y(\tau, t)e^{\frac{\tau}{\theta}\Lambda(\beta(-\tau,t))}, \quad (\tau, t) \in \mathcal{M}^m, \tag{27}$$

$$Y(\tau, t) = X(\tau, t)e^{-\frac{\tau}{\theta}\Lambda(\beta(-\tau,t))}, \quad Y(\tau + \theta, t + \omega) = Y(\tau, t) \in^{\theta,\omega} C^{(1,e)}_{\tau,t}(\mathcal{M}^m).$$

Then we can formulate the reducibility theorem.

Theorem 1. The linear system (25) can be reduced to the form

$$x = Y(\tau, t)y, \quad (\tau, t) \in \mathcal{M}^m \tag{28}$$

using the transformation can be reduced to the system

$$Dy = \frac{1}{\theta}\Lambda(\beta(-\tau, t)) \cdot y, \quad (\tau, t) \in \mathcal{M}^m. \tag{29}$$

Corollary 1. A quasiperiodic linear system

$$\frac{dx}{d\tau} = P(\tau, t)x, \quad \frac{dt}{d\tau} = e, \quad P(\tau + \theta, t + \omega) = P(\tau, t) \in {}^{\theta,\omega}C^{(0,e)}_{\tau,t}(R \times R^m) \tag{30}$$

by means of a quasiperiodic substitution

$$x = Y(\tau, \delta(\tau, \eta))y, \quad \tau \in R, \quad \eta \in R^m \tag{31}$$

can be reduced to the system

$$\frac{dy}{d\tau} = \frac{1}{\theta}\Lambda(\eta)y \tag{32}$$

with any fixed $\eta \in R^m$, where η is the value t in the neighbourhood of which the question of reduction is considered, $t = \delta(\tau, \eta)$ is the diagonal characteristic originating from the point $(0, \eta)$.

Theorem 1 remains valid for rectilinear motions, therefore, the characteristic $t = \beta(\tau, \eta)$ can be replaced by $t = \delta(\tau, \eta)$. Then, from the theorem follows consequence 1 regarding equations (30), (32) and transformation (31) in accordance with systems (25), (29) and substitution (28).

Note that if we wrap an isosceles right triangle with side θ around a straight circular cylinder with circumference θ, then the hypotenuse of the triangle becomes a harmonic spiral with pitch $\varphi = 45^0$: $\operatorname{tg}\varphi = \frac{dt_j}{d\tau} = 1$.

$$u = \tau, \quad v = r\sin\frac{2\pi\tau}{\theta} = r\sin\frac{\tau}{r}, \quad \omega = r\cos\frac{2\pi\tau}{\theta} = r\cos\frac{\tau}{r}.$$

$S_{\breve{AC}} = S_{\overline{CE}} = \tau$ the length of the arc $\breve{AC}$ and the length of the rise $\overline{CE}$ are equal to τ.

These geometric interpretations clearly show the connection between rectilinear and circular motions. The measurement of the lengths of circular arcs of a helical line is automatically transferred to the measurement of segment lengths. The main barrier to understanding is psychological in nature and related to the topology of surface lines.

Next, we will examine the question of the stability of the motions described by the system of equations (10) based on the multipliers of the monodromy matrix $X(\theta, \eta)$.

To this end, let us consider the characteristic equation of the matrix $X(\theta, \eta)$ and, based on (10)–(13), (20) and (23), we have

$$det[X(\theta, \eta) - \rho E] = \begin{vmatrix} \varphi(\theta, \eta) - \rho & \psi(\theta, \eta) \\ D\varphi(\theta, \eta) & D\psi(\theta, \eta) - \rho \end{vmatrix} = 0.$$

Obviously, from the characteristic equation $h(\eta, \rho) \equiv det[X(\theta, \eta) - \rho E]$ of the monodromy matrix $X(\theta, \eta)$, it follows that the function $h(\eta, \rho)$ is ω-periodic in η and continuously differentiable with respect to its arguments; moreover, in our case, the roots are distinct: $\rho_1(\eta) \neq \rho_2(\eta)$ and nonzero. When the multiplicities of the roots do not depend on the independent variable η, they can be determined using the implicit function theorem, according to which the properties of the coefficients are inherited by the roots of the equation. Consequently, we obtain $\rho_i(\eta + \omega) = \rho_i(\eta) \in C^{(e)}_\eta(R^m)$, $i = \overline{1, n}$ and the existence of the logarithm of the monodromy matrix $X(\theta, \eta)$.

From this, taking into account Statement 1, we obtain

$$\rho^2 - a\rho + 1 = 0, \tag{33}$$

where $a = a(\eta)$ is determined by the relation

$$a(\eta) = \varphi(\theta, \eta) + D\psi(\theta, \eta), \quad a(\eta + \omega) = a(\eta) \in C^{(e)}_\eta(S^m_\theta), \tag{34}$$

which can be called the Lyapunov's multiperiodic characteristic coefficient with respect to the parameter η for the system (10).

By virtue of (18) from (20) we have

$$\varphi(\theta,\eta)=1-\int\limits_0^\theta(\theta-\sigma_1)p(\sigma_1,\beta(\sigma_1,\eta))d\sigma_1+$$

$$+\int\limits_0^\theta(\theta-\sigma_1)p(\sigma_1,\beta(\sigma_1,\eta))d\sigma_1\int\limits_0^{\sigma_1}(\sigma_1-\sigma_2)p(\sigma_2,\beta(\sigma_2,\eta))d\sigma_2+...+$$

$$+(-1)^k\int\limits_0^\theta d\sigma_1\int\limits_0^{\sigma_1}d\sigma_2...\int\limits_0^{\sigma_{k-1}}(\theta-\sigma_1)(\sigma_1-\sigma_2)...(\sigma_{k-1}-\sigma_k)p(\sigma_1,\beta(\sigma_1,\eta)))...p(\sigma_k,\beta(\sigma_k,\eta))d\sigma_k+...$$

Similarly, due to (23) and (24), we have

$$D\psi(\theta,\eta)=1-\int\limits_0^\theta\sigma_1p(\sigma_1,\beta(\sigma_1,\eta))d\sigma_1+$$

$$+\int\limits_0^\theta p(\sigma_1,\beta(\sigma_1,\eta))d\sigma_1\int\limits_0^{\sigma_1}(\sigma_1-\sigma_2)\sigma_2p(\sigma_2,\beta(\sigma_2,\eta))d\sigma_2+...+(-1)^k\int\limits_0^\theta d\sigma_1\int\limits_0^{\sigma_1}d\sigma_2...$$

$$...\int\limits_0^{\sigma_{k-1}}(\theta-\sigma_1+\sigma_2)(\sigma_1-\sigma_2)...(\sigma_{k-1}-\sigma_k)p(\sigma_1,\beta(\sigma_1,\eta))p(\sigma_2,\beta(\sigma_2,\eta))...p(\sigma_k,\beta(\sigma_k,\eta))d\sigma_k+...$$

Due to last two equalities, from (34) we define the Lyapunov characteristic coefficient of equation (33) for the second-order multiperiodic system (10) in the form

$$a(\eta)=2-\theta\int\limits_0^\theta p(\sigma_1,\beta(\sigma_1,\eta))d\sigma_1+$$

$$+\int\limits_0^\theta d\sigma_1\int\limits_0^{\sigma_1}(\theta-\sigma_1+\sigma_2)(\sigma_1-\sigma_2)p(\sigma_1,\beta(\sigma_1,\eta))p(\sigma_2,\beta(\sigma_2,\eta))d\sigma_2+...+$$

$$+(-1)k\int\limits_0^\theta d\sigma_1\int\limits_0^{\sigma_1}d\sigma_2...\int\limits_0^{\sigma_{k-2}}d\sigma_{k-1}\int\limits_0^{\sigma_{k-1}}(\theta-\sigma_1+\sigma_k)(\sigma_1-\sigma_2)...(\sigma_{k-1}-\sigma_k)p(\sigma_1,\beta(\sigma_1,\eta))...\times$$

$$\times...p(\sigma_k,\beta(\sigma_k,\eta))d\sigma_k+... \tag{35}$$

Next, let us assume that $p(\tau,t)\le 0$ on $\mathcal{M}^m$, i.e., a sign-negative function, and there exists a point $(\xi,\eta)\in\mathcal{M}^m$ such that

$$p(\xi,\eta)<0,\quad(\xi,\eta)\in\mathcal{M}^m. \tag{36}$$

Then $\int\limits_0^\theta p(\sigma,\beta(\sigma,\eta))d\sigma<0$ and

$$(-1)^k\int\limits_0^\theta d\sigma_1\int\limits_0^{\sigma_1}d\sigma_2...\int\limits_0^{\sigma_{k-2}}d\sigma_{k-1}\int\limits_0^{\sigma_{k-1}}(\theta-\sigma_1+\sigma_k)(\sigma_1-\sigma_2)...(\sigma_{k-1}-\sigma_k)p(\sigma_1,\beta(\sigma_1,\eta))...\times$$

$$\times ...p(\sigma_k, \beta(\sigma_k, \eta))d\sigma_k \geq 0, \quad k = 1, 2, ...$$

Consequently, from (35) and (36) we have

$$a(\eta) > 2$$

and the roots $\rho_{1,2}$ of equation (33) are distinct and real, and

$$\rho_1 = \frac{1}{2}(a - \sqrt{a^2 - 4}) < 1, \quad \rho_2 = \frac{1}{2}(a + \sqrt{a^2 - 4}) > 1. \tag{37}$$

Since the solution $x(\tau, t)$ of system (10) with the initial condition

$$x(\tau, t)|_{\tau=\xi} = x^0(t) \in^{\omega} C_t^{(e)}(S_\theta^m)$$

by virtue of (27) at $\eta = \beta(\xi - \tau, t)$, in accordance with Theorem 1, defined by the relation

$$x(\tau, t) = Y(\tau, t)e^{\frac{\tau}{\theta} lnX(\theta, \beta(\xi - \tau, t))}x^0(\beta(\xi - \tau, t)),$$

then from the properties of the multipliers $\rho_{1,2}$ (37) we have the instability of the system (10), and therefore of the equation (9).

Thus, we can formulate Theorem 2.

Theorem 2. Equation (9) with (θ, ω)-periodic $(0, e)$-smooth sign-negative function $p(\tau, t) \neq 0$ at $(\tau, t) \in \mathcal{M}^m$ is unstable, and its multipliers are positive, with one of them greater than unity and the other less than unity.

Now let us consider the case where $p(\tau, t) \neq 0$ is positive, i.e. there exists a point (ξ, η) where $p(\xi, \eta) > 0$.

Next, we estimate the multiperiod Lyapunov characteristic coefficient $a = a(p)$, which depends on the parameter $p \in S_\theta^m$, and we have the inequality

$$I_1 = \theta \int_0^\theta p(\sigma_1, \beta(\sigma_1, \eta))d\sigma_1 > 0, \quad p \in S_\theta^m.$$

Along with this, let us assume that $p(\tau, t)$ satisfied the condition

$$\theta \int_0^\theta p(\sigma_1, \beta(\sigma_1, \eta))d\sigma_1 \leq 4, \quad p \in S_\theta^m. \tag{38}$$

Then, taking into account $0 \leq \sigma_k < \sigma_1$, $\theta - \sigma_1 + \sigma_k < \theta$ and

$$(\theta - \sigma_1 + \sigma_{k+1})(\sigma_k - \sigma_{k+1}) \leq \frac{1}{4}(\theta - \sigma_1 + \sigma_k)^2 < \frac{\theta}{4}(\theta - \sigma_1 + \sigma_k)$$

by virtue of the estimate $xy \leq (\frac{x+y}{2})^2$, we obtain the following estimate:

$$I_{k+1} = \int_0^\theta d\sigma_1 \int_0^{\sigma_1} d\sigma_2 ... \int_0^{\sigma_{k-1}} d\sigma_k \int_0^{\sigma_k} (\theta - \sigma_1 + \sigma_{k+1})(\sigma_1 - \sigma_2)...(\sigma_k - \sigma_{k+1})p(\sigma_1, \beta(\sigma_1, \eta))... \times$$

$$\times ...p(\sigma_{k+1}, \beta(\sigma_{k+1}, \eta))d\sigma_{k+1} = \int_0^\theta d\sigma_1 \int_0^{\sigma_1} d\sigma_2 ... \int_0^{\sigma_{k-2}} d\sigma_{k-1} \int_0^{\sigma_{k-1}} (\theta - \sigma_1 + \sigma_{k+1})(\sigma_1 - \sigma_2)... \times$$

$$\times ...(\sigma_{k-1}-\sigma_k)p(\sigma_1,\beta(\sigma_1,\eta))...p(\sigma_k,\beta(\sigma_k,\eta))d\sigma_k\cdot\int\limits_0^{\sigma_k}(\sigma_k-\sigma_{k+1})p(\sigma_{k+1},\beta(\sigma_{k+1},\eta))d_{k+1}<$$

$$<\int\limits_0^{\theta}(\theta-\sigma_1+\sigma_{k+1})(\sigma_k-\sigma_{k+1})d\sigma_{k+1}\int\limits_0^{\sigma_1}d\sigma_1...\int\limits_0^{\sigma_{k-2}}d\sigma_{k-1}\int\limits_0^{\sigma_{k-1}}(\sigma_1-\sigma_2)...(\sigma_{k-1}-\sigma_k)\times$$

$$\times p(\sigma_1,\beta(\sigma_1,\eta))...p(\sigma_k,\beta(\sigma_k,\eta))d\sigma_k\int\limits_0^{\theta}p(\sigma_{k+1},\beta(\sigma_{k+1},\eta))d\sigma_{k+1}<$$

$$<\int\limits_0^{\theta}d\sigma_1\int\limits_0^{\sigma_1}d\sigma_2...\int\limits_0^{\sigma_{k-2}}d\sigma_{k-1}\int\limits_0^{\sigma_{k-1}}\frac{\theta}{4}(\theta-\sigma_1+\sigma_k)(\sigma_1-\sigma_2)...(\sigma_{k-1}-\sigma_k)p(\sigma_1,\beta(\sigma_1,\eta))...\times$$

$$\times ...p(\sigma_k,\beta(\sigma_k,\eta))d\sigma_k\cdot\int\limits_0^{\theta}p(\sigma,\beta(\sigma,\eta))d\sigma=\frac{\theta}{4}\int\limits_0^{\theta}p(\sigma,\beta(\sigma,\eta))d\sigma\cdot I_k.$$

From this, by virtue of (38), we have

$$0<I_{k+1}(\eta)\le I_k(\eta),\quad I_k(\eta)\to 0\ \ at\ \ k\to\infty.$$

It is obvious that the series

$$a(\eta)=2-I_1(\eta)+I_2(\eta)-I_3(\eta)+...+(-1)^kI_k(\eta)+...,\quad \eta\in S_\theta^m,$$

are series. Leibniz, however, the estimate

$$a(\eta)-\theta\int\limits_0^{\theta}p(\sigma,\beta(\sigma,\eta))d\sigma<a<2,\quad \eta\in S_\theta^m,$$

i.e., $-2<a<2$.

Consequently, the roots $p_{1/2}(\eta)=\frac{1}{2}[a(\eta)\pm\sqrt{a^2(\eta)-4}]$ are distinct and complex conjugate, and $|p_{1/2}(\eta)|=1$.

Thus, the solution of the system (10), and therefore the equations (9), are bounded for all $\eta\in S_\theta^m$, i.e., they are stable.

Theorem 3. Equations (9) with (θ,ω)-periodic smooth positive function $p(\tau,t)\neq 0$ multipliers $\rho_{1,2}(\eta)$ under the condition of Lyapunov

$$0<\theta\int\limits_0^{\theta}p(\sigma,\beta(\sigma,\eta))d\sigma<4,\quad \eta\in S_\theta^m \tag{39}$$

are distinct, complex conjugate, and their moduli $|\rho_{1/2}(\eta)|=1$, and therefore equation (9) is stable.

In conclusion, considering that equation (9) is an extended representation of an ordinary differential equation with a quasi-periodic coefficient of the form

$$\frac{d^2}{d\tau^2}x(\tau,e\tau)+p(\tau,e\tau)x(\tau,e\tau)=0, \tag{40}$$

$$p(\tau+\theta,t+\omega)=p(\tau,t)\in C_{\tau,t}^{(0,e)}(R\times R^m),\quad p(\tau,t)\neq 0,$$

the Theorem 3 is also valid in the case of (40), and we have the following important corollary, where $e\tau=\beta(\tau,0)$.

Corollary 2. If the coefficient $p(\tau, t)$ of equation (40) is positive and satisfies inequality

$$0 < \theta \int_0^\theta p(\sigma, e\sigma) d\sigma < 4,$$

then all solutions of this equation are limited along with their first-order derivatives on the numerical axis R, therefore, equation (40) is stable, and the multipliers $\rho_{1/2}(0)$ are different, complex conjugate and $|\rho_{1/2}(0)| = 1$.

The proof of Corollary 2 follows from Theorem 3 at $\eta = 0$, and last condition is derived from (39) at zero η. The limitation of the first derivatives of solutions is due to the fact that equation (40) is equivalent to a system of two first-order equations.

Conclusion

It should be noted that the Lyapunov method is closely related to the approach proposed by V. Kharasakhal concerning the transition from the ordinary differentiation operator to the D-differentiation operator along the diagonal, defined on a multidimensional cylindrical manifold. Within this framework, it becomes possible to investigate the stability of the Mathieu equation with quasiperiodic coefficients, which represents an important applied aspect of the present study.

If the multiperiodic coefficient of the considered equation is given in a stepwise form with respect to each variable, then, in the purely periodic case, methods of numerical analysis can be effectively employed for qualitative investigations.

Overall, the conducted research provides promising prospects for the further development of the theory of multifrequency oscillations described by Lyapunov-type equations.

Acknowledgments

This research was funded by the Science Committee of the Ministry of Science and Higher Education of the Republic of Kazakhstan (grant No. AP19676629).

Author Contributions

G.M. Aitenova drafted the manuscript and integrated the research findings into a coherent paper. Zh.A. Sartabanov conceived the central idea of the study, secured the research funding, and provided overall supervision of the project. B.Zh. Omarova contributed to data collection and analysis, supporting the validation of results. A.Kh. Zhumagaziyev participated in data acquisition and analysis and coordinated the manuscript revision process. All authors critically reviewed the manuscript, provided intellectual input at various stages of the research, and approved the final version for submission.

Conflict of Interest

The authors declare no conflict of interest.

References

1 Sartabanov, Zh.A. (2023). Periodichnost kharakteristik operatora differentsirovaniia po diagonali [Periodicity of the characteristics of the diagonal differentiation operator]. *Vestnik Kazakhskogo Nationslnogo Pedagogicheskogo Universiteta imeni Abaia. Seriia: Fiziko-matematicheskie nauki*

— *Bulletin of Abai Kazakh National Pedagogical University. Series of Physical and Mathematical sciences, 82*(2), 40–53.

2 Sartabanov, Zh., Omarova, B., Aitenova, G., & Zhumagaziyev, A. (2023). Integrating multiperiodic functions along the periodic characteristics of the diagonal differentiation operator. *Journal of Mathematics, Mechanics and Computer Science, 120*(4), 52–68. https://doi.org/10.26577/JMMCS2023v120i4a6

3 Demidovich, B.P. (1967). *Lektsii po matematcheskoi teorii ustoichivosti [Lectures on the mathematical theory of stability].* Moscow: Izdatelstvo Nauka [in Russian].

4 Petrovsky, G. (1978). *Lectures on partial differential equations.* New York: Interscience Publishers Inc.

5 Rozhdestvenskii, B.L., & Yanenko, N.N. (1983). *Systems of quasilinear equations and their applications to gas dynamics.* Providence: American Mathematical Society. https://doi.org/10.1090/mmono/055

6 Kharasakhal, V.Kh. (1970). *Pochti periodicheskie resheniia obyknovennykh differentsialnykh uravnenii [Almost periodic solutions of ordinary differential equations].* Alma-Ata: Izdatelstvo Nauka [in Russian].

7 Umbetzhanov, D.U. (1979). *Pochti mnogoperiodicheskie resheniia differentsialnykh uravnenii v chastnykh proizvodnykh [Almost Multiperiodic Solutions of Partial Differential Equations].* Alma-Ata: Nauka [in Russian].

8 Umbetzhanov, D.U. (1990). *Pochti periodicheskie resheniia evoliutsionnykh uravnenii [Almost Periodic Solutions of Evolutionary Equations].* Alma-Ata: Nauka [in Russian].

9 Vejvoda, O., Herrmann, L., Lovicar, V., Sova, M., & Straskaba, I. (1982). *Partial differential equations: Time-periodic solutions.* Martinus Nijhoff: The Hague.

10 Feckan, M., Khalladi, M.T., Kostic, M., & Rahmani, A. (2025). Multi-dimensional ρ-almost periodic type functions and applications. *Applicable Analysis, 104*(1), 142–168. https://doi.org/10.1080/00036811.2022.2103678

11 Feckan, M., & Pospisil, M. (2018). On equations with generalized periodic right-hand side. *Ukrainian Mathematical Journal, 70*(2), 288–318. https://doi.org/10.1007/s11253-018-1501-4

12 Liu, K., Feckan, M., & Wang, J. (2022). A Class of (ω, T)-periodic solutions for impulsive evolution equations of Sobolev type. *Bulletin of the Iranian Mathematical Society, 48*(5), 2743–2763. https://doi.org/10.1007/s41980-021-00666-9

13 Ismail, A.I. (2021). Application of a large-parameter technique for solving a singular case of a rigid body. *Advances in Mathematical Physics, 2021*, Article 88422700. https://doi.org/10.1155/2021/8842700

14 Kostic, M. (2022). Multi-dimensional c-almost periodic type functions and applications. *Nonautonomous Dynamical Systems, 8*(1), 1–21. https://doi.org/10.1515/msds-2020-0130

15 Ren, Y., Zhi, Y., Feng. K., Gao, H., & Zhang, J. (2022). A time-frequency representation approach of undersampled signals with multiple periodic FM components. *IEEE Transactions on Aerospace and Electronic Systems, 58*(4), 3624–3632. https://doi.org/10.1109/TAES.2021.3122406

16 Liu, C., & Wang, S. (2023). Multiple periodic solutions of second order parameter-dependent equations via rotation numbers. *AIMS Mathematics, 8*(10), 25195–25219. https://doi.org/10.3934/math.20231285

17 Li, K., (2011). Multiple periodic solutions for asymptotically linear Duffing equations with resonance. *Journal of Mathematical Analysis and Applications, 378*(2), 657–666. https://doi.org/10.1016/j.jmaa.2011.01.049

18 Jebelean, P., & Serban, C. (2022). Multiple periodic solutions for odd perturbations of the discrete relativistic operator. *Mathematics, 10*(9), Article 1595. https://doi.org/10.3390/math10091595

19 Sartabanov, Zh.A., Abdikalikova, G.A., Aitenova, G.M., Omarova, B.Zh., & Zhumagaziyev, A.Kh. (2024). Implementation of the Small Parameter Method for Studying Multiperiodic Solutions of Systems with a Diagonal Differentiation Operator. *Lobachevskii Journal of Mathematics, 45*(12), 6594–6609. https://doi.org/10.1134/S1995080224607197

20 Omarova, B., Sartabanov, Zh., Zhumagaziyev, A., & Aitenova, G. (2025). Multiperiodic solutions of linear systems with a differentiation operator in the directions of the diagonal of the space of independent variables. *Carpathian Journal of Mathematics, 41*(4), 981–998. https://doi.org/10.37193/CJM.2025.04.09

21 Abdikalikova, G. (2026). Solvability of the Boundary Value Problem for a System of Parabolic Equations. *Mathematical Methods in the Applied Sciences, 49*(5), 4328–4339. https://doi.org/10.1002/mma.70348

22 Abdikalikova, G.A. (2024). Solvability of a Nonlocal Boundary Value Problem for One Class of Loaded Partial Differential Equations. *Lobachevskii Journal of Mathematics, 45*(10), 4815–4827. https://doi.org/10.1134/S1995080224606040

23 Abdikalikova, G.A., Assanova, A.T., & Shekerbekova, S.T. (2022). Nonlocal Problem for Fourth-Order Loaded Hyperbolic Equations. *Russian Mathematics, 66*(8), 1–18. https://doi.org/10.3103/S1066369X22080011

24 Bekbauova, A.U. (2025). Constructing a Broad-Sense Solution to Nonlinear First-Order Partial Differential Systems. *Lobachevskii Journal of Mathematics, 46*(1), 369–376. https://doi.org/10.1134/S1995080224608142

*Author Information**

Gulsezim Muratovna Aitenova — PhD, Senior Researcher, M. Utemisov West Kazakhstan University, 162 Nursultan Nazarbayev Ave., Uralsk, 090000, Kazakhstan; e-mail: *gulsezimaitenova@gmail.com*; https://orcid.org/0000-0002-4572-8252

Zhayshylyk Almaganbetovich Sartabanov — Doctor of Physical and Mathematical Sciences, Professor, Chief Researcher, K. Zhubanov Aktobe Regional University, 34 Aliya Moldagulova Ave., Aktobe, 030000, Kazakhstan; e-mail: *sartabanov42@mail.ru*; https://orcid.org/0000-0003-2601-2678

Bibigul Zharbolovna Omarova — PhD, Senior Researcher, K. Zhubanov Aktobe Regional University, 34 Aliya Moldagulova Ave., Aktobe, 030000, Kazakhstan; e-mail: *bibigul.zharbolkyzy@gmail.com*; https://orcid.org/0000-0002-3267-2501

Amire Khaliuly Zhumagaziev (*corresponding author*) — PhD, Senior Researcher, K. Zhubanov Aktobe Regional University, 34 Aliya Moldagulova Ave., Aktobe, 030000; Kh. Dosmukhamedov Atyrau University, Atyrau, 060000, Kazakhstan; e-mail: *azhumagaziyev@zhubanov.edu.kz*; https://orcid.org/0000-0002-6007-3311

*Authors' names are presented in the following order: first name, middle name (if any), last name.

https://doi.org/10.31489/2026M1/37-54 *Research article*

A Boundary Value Problem for a Time-Fractional Diffusion Equation in a Non-Cylindrical Shrinking Domain

A.D. Akhmetshin, M.T. Omarov*, R.Z. Toleukhanova

Buketov Karaganda National Research University, Karaganda, Kazakhstan
(E-mail: aleksandr_050401@mail.ru, madiomarovt@gmail.com, toleuch_r@mail.ru)

This article deals with the fundamental problems in the mathematical theory of fractional differential equations, specifically focusing on the analytical solvability of boundary value problems in time-dependent domains. The relevance of the study implies the necessity of developing methods for equations with non-local operators modeling anomalous diffusion. A one-dimensional diffusion equation containing a Riemann-Liouville fractional derivative with respect to time is examined. The characteristic features of the problem, posed in a non-cylindrical domain bounded by a moving linear boundary and a fixed spatial coordinate, are analyzed. The need to handle inhomogeneous boundary data is identified, and the problem is initially reduced to one with homogeneous conditions. On the basis of the study, the author constructs the fundamental solution in a quarter-plane by means of the bilateral Laplace transform and obtains the Green function for the Dirichlet problem. It is shown that the solution can be expressed through an integral representation in terms of a specific boundary density. This density satisfies a Volterra-type integral equation with a weakly singular kernel. Using the contraction mapping principle, it is proved that this equation has a solution. Consequently, the existence of a regular solution to the original boundary value problem is established.

Keywords: time-fractional diffusion equation, Riemann–Liouville derivative, infinite memory, non-cylindrical domain, shrinking domain, Dirichlet problem, Green function, fundamental solution, Wright function, bilateral Laplace transform, Volterra integral equation, weakly singular kernel.

2020 Mathematics Subject Classification: Primary 35R11; Secondary 26A33, 35K05, 35C15, 45D05, 33E12.

Introduction

Over the last decades, time-fractional diffusion equations have been intensively studied as models of anomalous transport in complex media. For single-term equations on the whole line or in $\mathbb{R}^n$, the Cauchy problem has become classical. Using Laplace and Fourier transforms, the fundamental solutions can be represented in terms of the Wright and M-Wright functions, and their probabilistic interpretation as self-similar densities is well understood [1, 2]. These kernels provide the natural building blocks for Green functions in bounded geometries and for the analysis of qualitative properties such as positivity and scaling.

A large body of work addresses initial-boundary value problems in cylindrical domains with fixed spatial cross-section. For one-dimensional problems with Caputo time-fractional derivatives, a maximum principle and related a priori estimates were established in [3] and later extended to weak solutions and general elliptic operators in [4]. For equations with Riemann–Liouville derivatives, Pskhu developed a Green-function method for one-dimensional boundary value problems: in particular, explicit representations for the first boundary value problem on a finite interval and for related boundary value problems in rectangles were obtained in [5, 6]. These contributions show that, in a fixed domain, boundary data can often be reduced to Volterra integral equations with weakly singular kernels.

*Corresponding author. *E-mail: madiomarovt@gmail.com*

This research was funded by the Science Committee of the Ministry of Science and Higher Education of the Republic of Kazakhstan (Grant no. AP23488729, AP23488740).

Received: 23 January 2025; *Accepted:* 6 January 2026.

Time-fractional diffusion in unbounded or semi-bounded domains has also been investigated. In [7] a time-fractional diffusion equation with mass absorption in a quarter-line was solved under a Dirichlet boundary condition that is harmonic in time; combining Laplace transform in time and sine transform in space yields an explicit solution as a superposition of Wright-type kernels. For numerical purposes, boundary integral equation approaches have been developed: Yao and Wang proposed a boundary-integral scheme for the time-fractional diffusion equation, where the solution is expressed as a single-layer potential and the time dependence is treated in the Laplace domain [8].

The geometry of the spatial domain plays a subtle role once the boundary is allowed to move. A step towards non-cylindrical domains for time-fractional equations was taken by Kubica, Rybka, and Ryszewska, who considered a one-dimensional heat equation with Caputo time derivative in a noncylindrical region and established existence of weak solutions via the Galerkin method [9]. On the other hand, Pskhu solved the first boundary value problem for a diffusion-wave equation with a fractional time derivative of Dzhrbashyan–Nersesyan type in a non-cylindrical domain, derived a Green representation and proved that Hölder regularity of the lateral boundary ensured existence [10].

Moving-boundary problems of Stefan type provide another important class of non-cylindrical configurations. Roscani analysed a one-dimensional time-fractional diffusion equation with a free boundary, employing a fractional weak maximum principle and constructing exact self-similar solutions in terms of Wright functions [11]. Kubica and Ryszewska studied a time-fractional Stefan problem with Riemann–Liouville flux, derived a self-similar formulation, and obtained explicit similarity solutions describing phase-change fronts with memory [12].

A further line of research relevant for the present work concerns equations with fractional derivatives whose lower limit is shifted to minus infinity. Diffusion-wave equations with such derivatives model processes with “infinite memory” and require the introduction of appropriate asymptotic boundary conditions instead of classical initial data. Pskhu and Rekhviashvili constructed Green functions for an asymptotic boundary value problem on the real line, proved solvability in weighted function spaces, and discussed applications to fractional electrodynamics [13]. Their analysis shows that the choice of the lower limit in the fractional derivative substantially affects both the functional setting and the structure of Green representations.

Angular domains create additional difficulties due to corner singularities and the interplay between spatial and temporal degeneracies. For fractional diffusion equations with time-fractional derivatives starting at $t = 0$, a boundary value problem in a curvilinear angle was solved in [14]. The authors proved existence in weighted Hölder spaces, demonstrating that the Hölder continuity of the boundary curve suffices to control the singular behaviour near the vertex. More recently, it was considered the first boundary value problem for a fractional diffusion equation in a degenerate angular domain whose opening shrinks to a point at the initial time [15]. They showed that the associated boundary integral equations are no longer of standard Volterra type; nevertheless, by a careful analysis of Wright kernels and Carleman–Vekua regularisation they obtained existence results in suitable weighted classes.

Loaded fractional diffusion equations, in which the partial differential equation contains additional integral or fractional-differential terms supported on lower-dimensional sets, form another active research direction. Boundary value problems for a diffusion equation with a fractional load supported on a straight ray were studied [16]. Depending on the position of the supporting line and on the order of the fractional operator in the load, the resulting boundary integral equation may be of genuine Volterra type, of pseudo-Volterra type, or even fail to have a unique solution, illustrating the delicate interplay between nonlocality and geometry.

Finally, a number of generalisations replace the Caputo or Riemann–Liouville derivative by more flexible operators. Of particular interest are diffusion models driven by g-fractional derivatives with respect to a monotone function g. Angelani and Garra analysed g-fractional diffusion in bounded domains with absorbing boundaries, obtained explicit series representations of the solution, and studied first-passage time distributions and their dependence on the choice of g [17]. These models demonstrate how non-standard fractional operators can be combined with boundary conditions to generate rich transient behaviours.

Against this background, the present paper addresses a boundary value problem for a time-fractional diffusion equation with Riemann–Liouville derivative of order $0 < \alpha < 1$ whose lower limit is $-\infty$, posed in a non-cylindrical domain

$$\Omega = \{(x,t) : 0 < x < -t,\ -\infty < t < 0\}.$$

The domain degenerates to a point as $t \to 0^-$, and the boundary conditions are prescribed on both the fixed boundary $x = 0$ and the moving boundary $x = -t$. To the best of our knowledge, no previous work combines simultaneously (i) an infinite-memory Riemann–Liouville derivative, (ii) a genuinely non-cylindrical domain whose lateral boundary originates at a single point, and (iii) boundary data given on two intersecting time-like curves.

Methodologically, our approach is close in spirit to the Green-function constructions in [5, 10, 13] but differs in several essential aspects. We first construct a fundamental solution in the quarter-plane $x > 0$, $t < 0$ using the bilateral Laplace transform with respect to time and derive from it a Green function for the Dirichlet problem on the quarter-line, expressed explicitly through Wright kernels [1,2]. This Green function is then employed to represent the solution in the non-cylindrical domain in terms of boundary layer potentials, leading to a Volterra-type integral equation for an unknown boundary density on the moving boundary. The kernel of this equation exhibits only a weak singularity, so the problem is well posed in a weighted Banach space of functions on $(-\infty, 0]$; existence follows from the contraction mapping principle. In this way we obtain an explicit Green representation for the regular solution and identify natural conditions on the right-hand side. The results thus extend the potential-theoretic approach to time-fractional diffusion into a new regime where both nonlocality in time and degeneracy of the spatial domain at the initial moment must be treated simultaneously.

The paper is organized as follows. In the introduction we formulate the boundary value problem for the time-fractional diffusion equation with the Riemann–Liouville derivative on the non-cylindrical domain and introduce the notion of a regular solution. In the next section we reduce the problem with inhomogeneous boundary data to an equivalent problem with homogeneous boundary conditions by an explicit transformation. We then construct in the quarter-plane a fundamental solution of the fractional diffusion equation by means of the bilateral Laplace transform in time and derive the Green function for the Dirichlet problem on the quarter-line. Using this Green function, we obtain in the non-cylindrical domain an integral representation of the solution in terms of volume and boundary layer potentials. The boundary conditions on the moving boundary lead to a Volterra-type integral equation with a weakly singular kernel for an unknown boundary density. Finally, we introduce a suitable weighted Banach space, study the corresponding integral operator, and prove existence of the boundary density, and hence of the regular solution to the original boundary value problem, by the contraction mapping principle.

1 *Problem statement*

Let $0 < \alpha < 1$ and set $\beta = \frac{\alpha}{2} \in \left(0, \frac{1}{2}\right)$. We consider the fractional diffusion equation

$$D^{\alpha}_{-\infty t} u(x,t) - \frac{\partial^2 u}{\partial x^2}(x,t) = f(x,t), \tag{1}$$

in the non-cylindrical domain

$$\Omega = \{(x,t) : 0 < x < -t,\ -\infty < t < 0\}. \tag{2}$$

Here $D^{\alpha}_{-\infty t}$ denotes the Riemann–Liouville fractional derivative of order α with respect to t with lower limit $-\infty$.

We impose boundary conditions on both sides of the space interval:

$$u(0,t) = \varphi(t), \qquad u(-t,t) = \psi(t), \qquad -\infty < t < 0. \tag{3}$$

The right-hand side f and the boundary data φ, ψ will be specified later. Our aim is to construct a representation for the solution of (1), (3) and to obtain conditions on f for the existence of a regular solution.

1.1 *Preliminaries*

For $0 < \alpha < 1$ the Riemann–Liouville derivative $D^{\alpha}_{-\infty t}$ is defined by

$$D^{\alpha}_{-\infty t} u(x,t) = \frac{\partial}{\partial t} D^{\alpha-1}_{-\infty t} u(x,t),$$

where

$$D^{\alpha-1}_{-\infty t} u(x,t) = \frac{1}{\Gamma(1-\alpha)} \int_{-\infty}^{t} (t-\tau)^{-\alpha} u(x,\tau)\, d\tau.$$

We also use the notation

$$h_\gamma(s) = \frac{s^{\gamma-1}}{\Gamma(\gamma)}, \quad s > 0, \quad \gamma > 0,$$

so that $D^{\alpha-1}_{-\infty t} u(x,t) = \int_{-\infty}^{t} u(x,\tau) h_{1-\alpha}(t-\tau)\, d\tau$.

Definition 1 (Regular solution). Let Ω be given by (2). A function $u = u(x,t)$ is called a *regular solution* of (1) in Ω if

- $u \in C(\overline{\Omega})$;
- $\partial^2 u / \partial x^2$ exists and is continuous in Ω;
- for every $(x,t) \in \Omega$ the quantity $D^{\alpha-1}_{-\infty t} u(x,t)$ is well-defined and continuously differentiable in t;
- for every fixed x and every $R < 0$ the function $(R-t)^{-\alpha} u(x,t)$ is integrable on $(-\infty, R)$;
- u satisfies (1) pointwise in Ω and the boundary conditions (3) hold.

Definition 2. The two-sided Laplace transform of a function f is

$$F(s) = \int_{-\infty}^{+\infty} f(t) e^{-st}\, dt$$

and it exists at s if the integral converges absolutely:

$$\int_{-\infty}^{+\infty} |f(t)| e^{-\operatorname{Re} s \cdot t}\, dt < \infty.$$

It is sufficient that:

- f be piecewise continuous on every finite interval;
- $\exists M_1, M_2 > 0, a < b \in \mathbb{R}$ such that

$$|f(t)| \le M_1 e^{at} \quad (t \ge 0), \qquad |f(t)| \le M_2 e^{bt} \quad (t \le 0).$$

Then $F(s)$ exists and is analytic in the entire strip $a < \operatorname{Re} s < b$.

Definition 3. For $\eta \in \mathbb{R}$ and $\beta = \frac{\alpha}{2}$, we define the function $w_\mu(x,y)$ by

$$w_\eta(x,y) = y^{\eta-1}\phi\left(-\beta, \eta; -\frac{x}{y^\beta}\right),$$

where $\phi(\rho,\eta;z)$ denotes the Wright function, given by the series representation

$$\phi(\rho,\eta;z) = \sum_{k=0}^{\infty} \frac{z^k}{k!\Gamma(\eta+\rho k)}, \qquad \rho > -1.$$

In the special case where $\eta = 0$, we denote the function simply as $w(x,y) = w_0(x,y)$.

2 Reduction to homogeneous boundary conditions

We first reduce problem (1), (3) to an equivalent problem with homogeneous boundary conditions. Define

$$u(x,t) = v(x,t) + \varphi(t) - \frac{x}{t}\big(\psi(t) - \varphi(t)\big), \qquad (x,t) \in \Omega. \tag{4}$$

Since $t < 0$ in Ω and $0 < x < -t$, the coefficient x/t is well-defined and satisfies $|x/t| \leq 1$ for all $(x,t) \in \Omega$, in particular it is bounded on compact subsets of Ω.

By direct substitution into the boundary conditions (3) we obtain

$$u(0,t) = v(0,t) + \varphi(t) = \varphi(t) \quad \Rightarrow \quad v(0,t) = 0,$$

and

$$u(-t,t) = v(-t,t) + \varphi(t) - \frac{-t}{t}(\psi(t) - \varphi(t)) = v(-t,t) + \psi(t) \quad \Rightarrow \quad v(-t,t) = 0.$$

Hence v satisfies the homogeneous boundary conditions

$$v(0,t) = v(-t,t) = 0, \qquad -\infty < t < 0. \tag{5}$$

Substituting (4) into the equation (1) we obtain

$$D^\alpha_{-\infty t} v(x,t) - \frac{\partial^2 v}{\partial x^2}(x,t) = f(x,t) - D^\alpha_{-\infty t}\varphi(t) + D^\alpha_{-\infty t}\left[\frac{x}{t}\big(\psi(t) - \varphi(t)\big)\right],$$

because the second derivative with respect to x of the affine function $x \mapsto \varphi(t) - \frac{x}{t}(\psi(t) - \varphi(t))$ vanishes. Thus the function v solves the inhomogeneous problem

$$D^\alpha_{-\infty t} v(x,t) - \frac{\partial^2 v}{\partial x^2}(x,t) = \tilde{f}(x,t), \tag{6}$$

in Ω, with homogeneous boundary conditions (5), where

$$\tilde{f}(x,t) = f(x,t) - D^\alpha_{-\infty t}\varphi(t) + D^\alpha_{-\infty t}\left[\frac{x}{t}\big(\psi(t) - \varphi(t)\big)\right].$$

Therefore it suffices to solve problem (6), (5) and then recover u by (4).

3 Fundamental solution in the quarter-plane

To construct a regular solution, we first partly study the equation

$$D^{\alpha}_{-\infty t} v(x,t) - \frac{\partial^2 v}{\partial x^2}(x,t) = \tilde{f}(x,t), \tag{7}$$

in the quarter-plane

$$\Omega_1 = \{(x,t) : x > 0,\ t < 0\},$$

with the boundary condition

$$v(0,t) = 0, \qquad t < 0. \tag{8}$$

In order to construct the fundamental solution we consider the problem

$$\begin{cases} D^{\alpha}_{-\infty t} v(x,t) - \dfrac{\partial^2 v}{\partial x^2}(x,t) = \delta(x-x_1)\delta(t-t_1), & (x,t) \in \Omega_1, \\ v(0,t) = 0, \quad \lim_{x\to\infty} v(x,t) = 0, & t < 0, \end{cases} \tag{9}$$

where $x_1 > 0$, $t_1 < 0$ are fixed.

3.1 Bilateral Laplace transform in t

We select the solution of (9), i.e.,

$$v(x,t) = 0, \qquad t < t_1. \tag{10}$$

Thus $v(x,t)$ is understood as being defined for all $t \in \mathbb{R}$ by the zero continuation (10). In particular, for every $\operatorname{Re} p > 0$ and every fixed $x > 0$ the bilateral Laplace integral

$$\bar{v}(x,p) = \int_{-\infty}^{+\infty} v(x,t) e^{-pt}\, dt$$

converges absolutely.

Let $v(x,t)$ be a solution of (9) such that, for every $\operatorname{Re} p > 0$

$$\int_{-\infty}^{\infty} |v(x,t)| e^{-pt}\, dt < \infty$$

for all $x > 0$, and assume that $v(x,t)$ is the pre image under the bilateral Laplace transform (this property will be satisfied a posteriori for the explicit fundamental solution constructed below). Consider the bilateral Laplace transform with respect to t,

$$\bar{v}(x,p) = \int_{-\infty}^{\infty} v(x,t) e^{-pt}\, dt, \qquad \operatorname{Re} p > 0.$$

Using integration by parts and the representation of the Riemann–Liouville derivative we obtain

$$\int_{-\infty}^{\infty} D^{\alpha}_{-\infty t} v(x,t) e^{-pt}\, dt = e^{-pt} D^{\alpha-1}_{-\infty t} v(x,t)\Big|_{t=-\infty}^{t=+\infty} + p^{\alpha}\bar{v}(x,p) = p^{\alpha}\bar{v}(x,p),$$

under the aforementioned decay and integrability assumptions the boundary term vanishes. Furthermore,

$$\int_{-\infty}^{\infty} \frac{\partial^2 v}{\partial x^2}(x,t) e^{-pt}\, dt = \frac{\partial^2}{\partial x^2}\bar{v}(x,p).$$

Transforming (9) leads to the ordinary differential equation

$$p^{\alpha}\bar{v}(x,p) - \frac{\partial^2 \bar{v}}{\partial x^2}(x,p) = e^{-pt_1}\delta(x - x_1), \qquad x > 0,\ \operatorname{Re} p > 0.$$

Equivalently,

$$\frac{\partial^2 \bar{v}}{\partial x^2}(x,p) - p^{\alpha}\bar{v}(x,p) = -e^{-pt_1}\delta(x - x_1) =: g(x,p).$$

For $x \neq x_1$ the homogeneous equation $\bar{v}_{xx} - p^{\alpha}\bar{v} = 0$ has the general solution

$$\bar{v}(x,p) = Q_1(p)e^{-p^{\beta}x} + Q_2(p)e^{p^{\beta}x}, \qquad \beta = \frac{\alpha}{2}.$$

To take into account the delta source at $x = x_1$, we write the solution using the method of variation of parameters. We set

$$\bar{v}(x,p) = Q_1(x,p)e^{-p^{\beta}x} + Q_2(x,p)e^{p^{\beta}x}$$

and impose

$$Q_1'(x,p)e^{-p^{\beta}x} + Q_2'(x,p)e^{p^{\beta}x} = 0, \qquad -p^{\beta}Q_1'(x,p)e^{-p^{\beta}x} + p^{\beta}Q_2'(x,p)e^{p^{\beta}x} = g(x,p).$$

Solving this system gives

$$Q_1'(x,p) = -\frac{e^{p^{\beta}x}}{2p^{\beta}}g(x,p), \qquad Q_2'(x,p) = \frac{e^{-p^{\beta}x}}{2p^{\beta}}g(x,p).$$

Consequently,

$$Q_1(x,p) = -\int_0^x \frac{e^{p^{\beta}\xi}}{2p^{\beta}}g(\xi,p)\,d\xi + C_1(p),$$
$$Q_2(x,p) = -\int_x^{\infty} \frac{e^{-p^{\beta}\xi}}{2p^{\beta}}g(\xi,p)\,d\xi + C_2(p).$$

Substituting back yields

$$\begin{aligned}
\bar{v}(x,p) &= Q_1(x,p)e^{-p^{\beta}x} + Q_2(x,p)e^{p^{\beta}x}\\
&= -\int_0^x \frac{g(\xi,p)e^{-p^{\beta}(x-\xi)}}{2p^{\beta}}\,d\xi - \int_x^{\infty} \frac{g(\xi,p)e^{-p^{\beta}(\xi-x)}}{2p^{\beta}}\,d\xi +\\
&\quad + C_1(p)e^{-p^{\beta}x} + C_2(p)e^{p^{\beta}x} =\\
&= -\int_0^{\infty} \frac{g(\xi,p)e^{-p^{\beta}|x-\xi|}}{2p^{\beta}}\,d\xi + C_1(p)e^{-p^{\beta}x} + C_2(p)e^{p^{\beta}x}.
\end{aligned}$$

Using $g(x,p) = -e^{-pt_1}\delta(x - x_1)$, we obtain

$$\bar{v}(x,p) = e^{-pt_1}\frac{e^{-p^{\beta}|x-x_1|}}{2p^{\beta}} + C_1(p)e^{-p^{\beta}x} + C_2(p)e^{p^{\beta}x}.$$

The boundary condition at $x = 0$ reads

$$\bar{v}(0,p) = e^{-pt_1}\frac{e^{-p^{\beta}x_1}}{2p^{\beta}} + C_1(p) + C_2(p) = 0.$$

The boundedness as $x \to \infty$ again implies $C_2(p) = 0$, hence

$$C_1(p) = -e^{-pt_1}\frac{e^{-p^{\beta}x_1}}{2p^{\beta}},$$

and therefore

$$\bar{v}(x,p) = \frac{e^{-pt_1}}{2p^{\beta}}\Big(e^{-p^{\beta}|x-x_1|} - e^{-p^{\beta}(x+x_1)}\Big).$$

3.2 Inverse Laplace transform and the Wright kernels

We now invert the Laplace transform. For this we introduce a family of kernels $w_\eta(x,t)$, $\eta \in \mathbb{R}$, $x \geq 0$, $t > 0$, such that

$$\int_0^\infty e^{-pt} w_\eta(x,t)\,dt = p^{-\eta} e^{-p^\beta x}, \qquad \operatorname{Re} p > 0. \tag{11}$$

Here and below $D^\rho_{\tau t}$ denotes the Riemann–Liouville fractional derivative in t of order ρ with lower limit τ. The kernels w_η are Wright-type functions and satisfy, in particular, the relations

$$D^\rho_{\tau t} w_\eta(x, t-\tau) = w_{\eta-\rho}(x, t-\tau), \tag{12}$$

for all ρ and all admissible η, and

$$D^\alpha_{\tau t} w_\beta(x, t-\tau) = \frac{\partial^2}{\partial x^2} w_\beta(x, t-\tau), \qquad \alpha = 2\beta. \tag{13}$$

From (11) with $\eta = \beta$ we see that the inverse transform of $e^{-p^\beta x}/p^\beta$ is precisely $w_\beta(x,t)$. Hence

$$v(x,t) = \frac{1}{2}\Big(w_\beta(|x-x_1|, t-t_1) - w_\beta(x+x_1, t-t_1)\Big), \qquad t > t_1.$$

We summarize the above in the following lemma.

Lemma 1. Let $x_1 > 0$, $t_1 < 0$ and assume that $v(x,t)$ is bounded in Ω_1, satisfies the boundary condition (8), and solves (9) in the sense of distributions. Then v is given by

$$v(x,t) = \frac{1}{2}\Big(w_\beta(|x-x_1|, t-t_1) - w_\beta(x+x_1, t-t_1)\Big), \qquad t > t_1,$$

and $v(x,t) \equiv 0$ for $t < t_1$.

We define the function

$$G(x,\xi,t,\tau) = \frac{1}{2}\Big(w_\beta(|x-\xi|, t-\tau) - w_\beta(x+\xi, t-\tau)\Big), \tag{14}$$

for $x > 0$, $\xi > 0$, $t > \tau$. This is the candidate for the Green function of the homogeneous problem (7), (8).

4 Fundamental solution and Green function

4.1 Definitions

We formulate the precise definition of a fundamental solution and a Green function for (7), (8).

Definition 4. A function $V(x,\xi,t,\tau)$ is called a *fundamental solution* of (7) if the following conditions hold:

1. For every fixed $\xi > 0$ and $\tau < 0$ the function $V(x,\xi,t,\tau)$ is defined for $t > \tau$ and satisfies

$$\Big(D^\alpha_{\tau t} - \frac{\partial^2}{\partial x^2}\Big) V(x,\xi,t,\tau) = 0, \qquad t > \tau.$$

2. For every interval $[x_1, x_2]$ and every $g \in C[x_1, x_2]$ there holds

$$\lim_{\tau \to t} \int_{x_1}^{x_2} g(\xi)\, D^{\alpha-1}_{\tau t} V(x,\xi,t,\tau)\, d\xi = g(x), \qquad x_1 < x < x_2.$$

Definition 5 (Green function). A function $G(x,\xi,t,\tau)$ is called a *Green function* of the boundary value problem (7), (8) if

- G is a fundamental solution in the sense of Definition 4;
- for every $\xi > 0$, $\tau < 0$ and $t > \tau$ one has $G(0,\xi,t,\tau) = 0$.

4.2 Verification for the kernel G

Lemma 2. Let G be given by (14). Then G is a Green function for the problem (7), (8).

Proof. By (13) we have

$$\Big(D^{\alpha}_{\tau t} - \frac{\partial^2}{\partial x^2}\Big) w_\beta(|x-\xi|, t-\tau) = 0, \qquad \Big(D^{\alpha}_{\tau t} - \frac{\partial^2}{\partial x^2}\Big) w_\beta(x+\xi, t-\tau) = 0,$$

hence

$$\Big(D^{\alpha}_{\tau t} - \frac{\partial^2}{\partial x^2}\Big) G(x,\xi,t,\tau) = 0$$

for $t > \tau$, $x > 0$, $\xi > 0$.

Next, by symmetry in the spatial variable we obtain

$$G(0,\xi,t,\tau) = \frac{1}{2}\Big(w_\beta(|-\xi|, t-\tau) - w_\beta(0+\xi, t-\tau)\Big) = 0,$$

so the boundary condition at $x = 0$ is satisfied.

For the fundamental property, using (12) with $\mu = \beta$ and $\rho = \alpha - 1$ (so that $\alpha = 2\beta$) we obtain

$$D^{\alpha-1}_{\tau t} G(x,\xi,t,\tau) = \frac{1}{2}\Big(w_{1-\beta}(|x-\xi|, t-\tau) - w_{1-\beta}(x+\xi, t-\tau)\Big).$$

Let $g \in C[x_1, x_2]$, $x_1 < x < x_2$. Then

$$\begin{aligned}\int_{x_1}^{x_2} g(\xi) D^{\alpha-1}_{\tau t} G(x,\xi,t,\tau)\, d\xi &= \frac{1}{2}\int_{x_1}^{x_2} g(\xi) w_{1-\beta}(|x-\xi|, t-\tau)\, d\xi - \\ &\quad - \frac{1}{2}\int_{x_1}^{x_2} g(\xi) w_{1-\beta}(x+\xi, t-\tau)\, d\xi =: \\ &=: I_1(\tau) + I_2(\tau).\end{aligned}$$

The first term $I_1(\tau)$ has the structure of a spatial convolution with a kernel that concentrates near $\xi = x$ as $t-\tau \to 0^+$. Under the standard estimates for Wright-type kernels one shows that $w_{1-\beta}(\cdot, t-\tau)$ forms an approximate identity, hence $\lim_{\tau\to t} I_1(\tau) = g(x)$. For $I_2(\tau)$ one uses the decay of $w_{1-\beta}(x+\xi, t-\tau)$ in ξ and the boundedness of g to obtain $\lim_{\tau\to t} I_2(\tau) = 0$. Combining these limits we obtain the desired relation, and the lemma follows. □

5 Representation formula in the non-cylindrical domain

We now return to the original domain Ω given by (2) and the homogeneous boundary value problem

$$D^{\alpha}_{-\infty t} v(x,t) - \frac{\partial^2 v}{\partial x^2}(x,t) = \tilde{f}(x,t), \qquad (x,t) \in \Omega, \tag{15}$$

$$v(0,t) = 0, \qquad v(-t,t) = 0, \qquad -\infty < t < 0. \tag{16}$$

To construct the solution we use the Green function (14) in the quarter-plane and apply a Green-type representation argument.

We introduce the auxiliary functions

$$\Phi_1(x,t) = \int_{-\infty}^{t} w_0(x, t-\tau)\, \mu(\tau)\, d\tau, \tag{17}$$

$$\Phi_2(x,t) = \frac{1}{2}\int_{-\infty}^{t} \nu(\tau)\big(w_0(x-\tau, t-\tau) - w_0(-\tau-x, t-\tau)\big)\, d\tau, \tag{18}$$

$$F(x,t) = \frac{1}{2}\int_{-\infty}^{t}\int_{0}^{-\tau} \tilde{f}(\xi,\tau)\big(w_\beta(|x-\xi|, t-\tau) - w_\beta(x+\xi, t-\tau)\big)\, d\xi\, d\tau. \tag{19}$$

Here $\mu(t)$ and $\nu(t)$ are unknown boundary densities to be chosen so that

$$v(x,t) = \Phi_1(x,t) + \Phi_2(x,t) + F(x,t)$$

satisfies the boundary conditions (16) and the equation (15).

5.1 *Some properties of* Φ_1 *and* Φ_2

We use the derivative relation (12). Under suitable decay assumptions on μ, ν the following identities hold.

Lemma 3. Assume that $\mu(t) \in C((-\infty,0]) \cap L_1((-\infty,0))$ and that

$$\lim_{t\to-\infty} (-t)^{\delta_1}\mu(t) = 0$$

for some $\delta_1 > 1-2\beta$. Then

$$D^{\alpha}_{-\infty t}\Phi_1(x,t) = \frac{\partial^2}{\partial x^2}\Phi_1(x,t), \qquad (x,t)\in\Omega.$$

Similarly, if ν satisfies

$$\nu(t) \in C((-\infty,0]) \cap L_1((-\infty,0)), \qquad \lim_{t\to-\infty}(-t)^{\delta_2}\nu(t) = 0, \quad \delta_2 > 1-2\beta,$$

then

$$D^{\alpha}_{-\infty t}\Phi_2(x,t) = \frac{\partial^2}{\partial x^2}\Phi_2(x,t), \qquad (x,t)\in\Omega.$$

Proof. Using (12) with $\eta = 0$ and $\rho = \alpha$, we obtain

$$D^{\alpha}_{-\infty t}\Phi_1(x,t) = \int_{-\infty}^{t}\mu(\tau)D^{\alpha}_{\tau t}w_0(x,t-\tau)\,d\tau = \int_{-\infty}^{t}\mu(\tau)w_{-\alpha}(x,t-\tau)\,d\tau = \int_{-\infty}^{t}\mu(\tau)w_{-2\beta}(x,t-\tau)\,d\tau,$$

since $\alpha = 2\beta$. From (11) it follows that $w_{-2\beta} = \partial_x^2 w_0$, so

$$D^{\alpha}_{-\infty t}\Phi_1(x,t) = \int_{-\infty}^{t}\mu(\tau)\frac{\partial^2}{\partial x^2}w_0(x,t-\tau)\,d\tau = \frac{\partial^2}{\partial x^2}\Phi_1(x,t).$$

The argument for Φ_2 is analogous, using linearity, symmetry of the kernels and the same derivative identity. Justification of bringing the derivative under the integral sign is based on the decay assumptions on μ, ν and the bounds on w_η; these are standard and follow from the known estimates for Wright-type kernels. $\square$

We also need an approximation property of w_0.

Lemma 4. Let $g \in C(-\infty,T) \cap L_1(-\infty,T-\varepsilon)$ for every $T\in\mathbb{R}$ and $\varepsilon>0$. Then

$$\lim_{x\to0}\int_{-\infty}^{t} g(\tau)w_0(x,t-\tau)\,d\tau = g(t). \tag{20}$$

Proof. We write

$$\int_{-\infty}^{t} g(\tau)w_0(x,t-\tau)d\tau = \int_{-\infty}^{t}(g(\tau)-g(t))w_0(x,t-\tau)d\tau + g(t)\int_{-\infty}^{t}w_0(x,t-\tau)d\tau =$$
$$= \left(\int_{-\infty}^{t-\varepsilon} + \int_{t-\varepsilon}^{t}\right)(g(\tau)-g(t))w_0(x,t-\tau)d\tau + g(t)\lim_{\tau\to-\infty}w_1(x,t-\tau) =$$
$$= \left(\int_{-\infty}^{t-\varepsilon} + \int_{t-\varepsilon}^{t}\right)[g(\tau)-g(t)]w_0(x,t-\tau)d\tau + g(t) = I_1 + I_2 + g(t).$$

Fix $\varepsilon > 0$, using integrability of $g(\tau) - g(t)$ on $(-\infty, t-\varepsilon)$ and the bound

$$|w_0(x,s)| \le C x^{-\theta} s^{\beta\theta - 1}, \qquad \theta \ge -1, \quad C = C(\beta, \theta),$$

one shows that I_1 tends to zero as $x \to 0$. For I_2 we use continuity of g near t and the fact that $\int_0^\varepsilon w_0(x,s)\,ds$ is uniformly bounded in x to obtain

$$\lim_{x\to 0} |I_2(x,t)| \le \sup_{\tau\in(t-\varepsilon,t)} |g(\tau) - g(t)|,$$

which can be made arbitrarily small by choosing ε small. Combining these, we obtain (20). □

5.2 Boundary limits and the integral equation for ν

We now compute the limits of Φ_1, Φ_2, F as $x \to 0$ and $x \to -t$ to enforce the boundary conditions (16). We outline the main steps.

From Lemma 4, $\lim_{x\to 0} \Phi_1(x,t) = \mu(t)$. Using symmetry and derivative estimates for w_0 one shows that $\lim_{x\to 0} \Phi_2(x,t) = 0$, so that $\lim_{x\to 0} v(x,t) = \mu(t) + \lim_{x\to 0} F(x,t)$. Imposing the boundary condition $v(0,t) = 0$ leads to $\mu(t) = -\lim_{x\to 0} F(x,t) = 0$. Consequently,

$$\Phi_1(x,t) \equiv 0.$$

Next, consider the limit $x \to -t$ (approaching the moving boundary from inside Ω). Using again the estimates for w_η and by virtue of the Lemma 4 one shows that

$$\lim_{x\to -t} \Phi_2(x,t) = \frac{1}{2}\int_{-\infty}^{t} \nu(\tau)\big(w_0(-t-\tau, t-\tau) - w_0(t-\tau, t-\tau)\big)\,d\tau - \frac{1}{2}\nu(t).$$

Moreover $F(-t,t) \in C(-\infty, 0]$ is well defined and satisfies a decay estimate of the form,

$$|F(-t,t)| \le C(-t)^{2-\theta-\sigma_3+\beta\theta}, \quad \theta \in (0,1), \quad t < 0,$$

under assumptions on $\tilde{f}$ to be stated below.

Consequently, the boundary condition $v(-t,t) = 0$ leads to the integral equation

$$\nu(t) - \int_{-\infty}^{t} \nu(\tau)\big(w_0(-t-\tau, t-\tau) - w_0(t-\tau, t-\tau)\big)\,d\tau = 2F(-t,t), \tag{21}$$

where $F(-t,t)$ is given by (19). It is this Volterra-type equation that we will solve in a suitable function space.

6 Assumptions on the data and main existence result

6.1 Assumptions on the boundary data and the compatibility condition

Since $\Omega = \{(x,t) : 0 < x < -t,\ -\infty < t < 0\}$ degenerates to the single point $(0,0)$ as $t \to 0^-$, the boundary values prescribed on $x = 0$ and on $x = -t$ must be compatible at the vertex.

We assume that the boundary data φ, ψ satisfy the following conditions.

(B1)

$$\varphi, \psi \in C((-\infty, 0]) \quad \text{and} \quad \lim_{t\to 0^-} \varphi(t) = \lim_{t\to 0^-} \psi(t) =: \varphi_0.$$

This condition is necessary if one requires $u \in C(\overline{\Omega})$ in Definition 1; one may then set $u(0,0) = \varphi_0$.

(B2)

$$\varphi, \psi \in C((-\infty, 0]) \cap L_1(-\infty, 0),$$

Because the domain Ω shrinks to the vertex $(0,0)$ as $t \to 0^-$, continuity of a regular solution on $\overline{\Omega}$ forces a compatibility condition at the intersection point of the lateral boundaries. It is also sufficient for the boundary correction in the reduction (4) to extend continuously to the vertex, because $|x/t| \leq 1$ in Ω and $\psi(t) - \varphi(t) \to 0$ as $t \to 0^-$.

Condition **(B2)** ensures that the boundary traces have the required integrability at $t = -\infty$ and that the boundary correction term in (4) belongs to the same integrability class.

More precisely, the following elementary fact will be used repeatedly.

Lemma 5. Let $0 < \alpha < 1$ and $g \in C((-\infty, 0]) \cap L_1((-\infty, 0))$. Then for each $t < 0$ the fractional integral

$$(I^{1-\alpha}_{-\infty} g)(t) := \frac{1}{\Gamma(1-\alpha)} \int_{-\infty}^{t} (t-\tau)^{-\alpha} g(\tau)\, d\tau$$

is well defined (finite). Moreover, $I^{1-\alpha}_{-\infty} g$ is continuous on $(-\infty, 0]$, and for every $R < 0$ the function $(R-t)^{-\alpha} g(t)$ is integrable on $(-\infty, R)$.

Proof. Fix $t < 0$ and split the integral into $(-\infty, t-1] \cup [t-1, t]$. On $(-\infty, t-1]$ one has $(t-\tau)^{-\alpha} \leq 1$, hence

$$\int_{-\infty}^{t-1} (t-\tau)^{-\alpha} |g(\tau)|\, d\tau \leq \int_{-\infty}^{t-1} |g(\tau)|\, d\tau < \infty$$

by $g \in L_1$. On $[t-1, t]$ we use boundedness of g and $\int_0^1 s^{-\alpha} ds < \infty$ (since $\alpha < 1$), which yields finiteness. Continuity in t follows from the same decomposition and dominated convergence.

Finally, for any $R < 0$ we split $\int_{-\infty}^{R} (R-t)^{-\alpha} |g(t)| dt$ into $(-\infty, R-1] \cup [R-1, R]$; on $(-\infty, R-1]$ the weight is bounded by 1, while on $[R-1, R]$ the weight is integrable because $\alpha < 1$. $\square$

We emphasize that the additional smoothness needed for the quantities $D^{\alpha}_{-\infty t}\varphi(t)$ and $D^{\alpha}_{-\infty t}\big[\frac{x}{t}(\psi(t) - \varphi(t))\big]$ in the definition of $\tilde{f}$ is imposed directly through the assumption $\tilde{f} \in C(\overline{\Omega})$.

6.2 Assumptions on the right-hand side

We assume that $f, \tilde{f} \in C(\overline{\Omega})$ and that the following conditions are satisfied: there exists $q \in (0, 1]$ and $\sigma_4 > 1 + q$ such that

$$|\tilde{f}(x,t) - \tilde{f}(\xi,t)| \leq C(-t)^{-\sigma_4} |x - \xi|^q, \qquad x, \xi \geq 0, \quad t < 0.$$

There exists constant σ_3, such that

$$\sup_{x \geq 0,\, t<0} |\tilde{f}(x,t)|\, (-t)^{\sigma_3} < \infty, \qquad \sigma_3 > 2 + \beta.$$

Under these assumptions one can show that $F(x,t)$ defined by (19) is continuous in $\overline{\Omega}$ and satisfies suitable decay estimates as $t \to -\infty$.

6.3 The function space for ν

Let $T_1 < 0$ be fixed. We define the Banach space

$$Q = Q(T_1) = \left\{ g:\ (-\infty, T_1] \to \mathbb{R}:\ g \in C((-\infty, T_1]),\ g \in L_1((-\infty, T_1)),\ \lim_{t \to -\infty} (-t)^{\delta_2} g(t) = 0 \right\},$$

where $\delta_2 > 1 - 2\beta$ is fixed. The norm in Q is

$$\|g\|_Q = \sup_{t \le T_1} |(-t)^{\delta_2} g(t)| + \int_{-\infty}^{T_1} |g(t)|\, dt. \tag{22}$$

For convenience, we split the norm (22) into two components:

$$\|g\|_1 := \sup_{t \le T_1} |(-t)^{\delta_2} g(t)|, \qquad \|g\|_2 := \int_{-\infty}^{T_1} |g(t)|\, dt,$$

so that $\|g\|_Q = \|g\|_1 + \|g\|_2$. In the contraction estimate for the Volterra operator A we control $\|\cdot\|_1$ and $\|\cdot\|_2$ separately: the weighted supremum norm is used to propagate decay at $t \to -\infty$, while the L_1 part allows us to estimate $\int_{-\infty}^{T_1} |Ag(t)|\, dt$ by Fubini's theorem.

Lemma 6. The space $(Q, \|\cdot\|_Q)$ is a Banach space.

Proof. It is straightforward to check that (22) defines a norm. Let (f_n) be a Cauchy sequence in Q. Then both

$$\sup_{t \le T_1} |(-t)^{\delta_2} (f_n(t) - f_m(t))| \to 0, \qquad \int_{-\infty}^{T_1} |f_n(t) - f_m(t)|\, dt \to 0$$

as $n, m \to \infty$. Hence (f_n) is Cauchy in the weighted sup-norm and in $L_1(-\infty, T_1)$. It follows that there exist $g_1 \in C((-\infty, T_1])$ and $g_2 \in L_1(-\infty, T_1)$ such that $f_n \to g_1$ uniformly on $(-\infty, T_1]$ with weight $(-t)^{\delta_2}$ and $f_n \to g_2$ in L_1. A standard subsequence argument shows that $g_1 = g_2$ a.e., hence $g := g_1 = g_2 \in Q$ and $f_n \to g$ in $\|\cdot\|_Q$. The property $\lim_{t \to -\infty} (-t)^{\delta_2} g(t) = 0$ follows from the uniform convergence with respect to the weighted sup-norm. Thus Q is complete. □

6.4 The integral operator and contraction

Lemma 7. Let $0 < \beta < \frac{1}{2}$. Then:

1. For every $\theta \in [-1, 1]$ there exists $C = C(\beta, \theta) > 0$ such that

$$0 \le w_0(x, t) \le C\, x^{-\theta}\, t^{\beta\theta - 1}, \qquad x > 0, \quad t > 0. \tag{23}$$

2. Along the diagonal one has

$$w_0(s, s) = -\frac{\beta}{1 - \beta} \frac{d}{ds} w_1(s, s), \qquad s > 0, \tag{24}$$

and consequently

$$\int_0^\infty w_0(s, s)\, ds = \frac{\beta}{1 - \beta}. \tag{25}$$

Proof. Estimate (23) is a standard bound for Wright-type kernels; see, e.g., [1, 2].

To prove (24), write the series expansions on the diagonal $x = t = s$:

$$w_1(s, s) = \sum_{k=0}^{\infty} \frac{(-1)^k s^{(1-\beta)k}}{k!\, \Gamma(1 - \beta k)}, \qquad w_0(s, s) = \sum_{k=1}^{\infty} \frac{(-1)^k s^{(1-\beta)k - 1}}{k!\, \Gamma(-\beta k)}.$$

Differentiate $w_1(s, s)$ termwise and use $\Gamma(1 - \beta k) = (-\beta k)\Gamma(-\beta k)$ to obtain (24). Integrating (24) over $(0, \infty)$ yields (25), because $w_1(0, 0) = 1$ and $w_1(s, s) \to 0$ as $s \to \infty$ (see [1, 2]). □

We rewrite equation (21) as

$$\nu(t) = A\nu(t) + 2F(-t,t),$$

where

$$A\nu(t) = \int_{-\infty}^{t} K(t,\tau)\nu(\tau)\,d\tau, \tag{26}$$

and

$$K(t,\tau) = w_0(-t-\tau, t-\tau) - w_0(t-\tau, t-\tau). \tag{27}$$

Proposition 1. Let $0 < \beta < \frac{1}{2}$ and let $Q(T_1)$ be the Banach space defined above. Consider the operator (26), (27) for $t \le T_1 < 0$. Then:

1) A maps $Q(T_1)$ into itself;
2) there exist $\theta \in (0,1]$, $C > 0$ and $T_1 < 0$ such that

$$\|A\nu_1 - A\nu_2\|_Q \le q(T_1)\|\nu_1 - \nu_2\|_Q, \qquad q(T_1) = \frac{\beta}{1-\beta} + C(-T_1)^{-\theta(1-\beta)} < 1.$$

In particular, A is a contraction on $Q(T_1)$.

Proof. Fix $\nu_1, \nu_2 \in Q(T_1)$ and set $\delta\nu = \nu_1 - \nu_2$. Since $w_0 \ge 0$, we have

$$|K(t,\tau)| \le w_0(t-\tau, t-\tau) + w_0(-t-\tau, t-\tau) =: w_-(t-\tau) + w_+(t,\tau).$$

For $t \le T_1$ and $\tau \le t$ we have $(-t)^{\delta_2}(-\tau)^{-\delta_2} \le 1$, hence

$$(-t)^{\delta_2}|A\delta\nu(t)| \le \sup_{\tau \le t}(-\tau)^{\delta_2}|\delta\nu(\tau)| \int_{-\infty}^{t} |K(t,\tau)|\,d\tau \le \|\delta\nu\|_1 \int_{-\infty}^{t} \big(w_- + w_+\big)\,d\tau.$$

By Lemma 7 we have

$$\int_{-\infty}^{t} w_-(t-\tau)\,d\tau = \int_0^{\infty} w_0(s,s)\,ds = \frac{\beta}{1-\beta}.$$

Next, using (23) with any fixed $\theta \in (0,1]$ and the substitution $s = t - \tau > 0$,

$$\int_{-\infty}^{t} w_+(t,\tau)\,d\tau = \int_0^{\infty} w_0(s-2t, s)\,ds \le C\int_0^{\infty} (s-2t)^{-\theta} s^{\beta\theta - 1}\,ds \le C_1(-t)^{-\theta(1-\beta)}.$$

Therefore,

$$\|A\delta\nu\|_1 \le \Big(\frac{\beta}{1-\beta} + C_1(-T_1)^{-\theta(1-\beta)}\Big)\|\delta\nu\|_1.$$

By Fubini,

$$\begin{aligned}
\|A\delta\nu\|_2 &= \int_{-\infty}^{T_1}\Big|\int_{-\infty}^{t} K(t,\tau)\delta\nu(\tau)\,d\tau\Big|\,dt \\
&\le \int_{-\infty}^{T_1}\int_{-\infty}^{t} |K(t,\tau)|\,|\delta\nu(\tau)|\,d\tau\,dt = \int_{-\infty}^{T_1} |\delta\nu(\tau)| \int_{\tau}^{T_1} |K(t,\tau)|\,dt\,d\tau.
\end{aligned}$$

For the w_- part we again use Lemma 7:

$$\int_{\tau}^{T_1} w_-(t-\tau)\,dt = \int_0^{T_1-\tau} w_0(s,s)\,ds \le \int_0^{\infty} w_0(s,s)\,ds = \frac{\beta}{1-\beta}.$$

For the w_+ part, by (23) and the fact that $-t-\tau \geq -T_1-\tau$ for $\tau \leq t \leq T_1$, we obtain

$$\int_{\tau}^{T_1} w_+(t,\tau)\,dt = \int_0^{T_1-\tau} w_0(-T_1-\tau+(T_1-t),s)\,ds \leq C(-T_1-\tau)^{-\theta}\int_0^{T_1-\tau} s^{\beta\theta-1}\,ds \leq C_2(-T_1-\tau)^{-\theta(1-\beta)}.$$

Since $\tau \leq T_1$ implies $-T_1-\tau \geq -2T_1$, we get

$$\int_{\tau}^{T_1} |K(t,\tau)|\,dt \leq \frac{\beta}{1-\beta} + C_2(T_1)^{-\theta(1-\beta)}.$$

Hence,

$$\|A\delta\nu\|_2 \leq \Big(\frac{\beta}{1-\beta} + C_2(T_1)^{-\theta(1-\beta)}\Big)\|\delta\nu\|_2.$$

Putting the bounds for $\|\cdot\|_1$ and $\|\cdot\|_2$ together and recalling $\|\cdot\|_Q = \|\cdot\|_1 + \|\cdot\|_2$, we obtain

$$\|A\nu_1 - A\nu_2\|_Q = \|A\delta\nu\|_1 + \|A\delta\nu\|_2 \leq q(T_1)\big(\|\delta\nu\|_1 + \|\delta\nu\|_2\big) = q(T_1)\|\nu_1-\nu_2\|_Q,$$

where $q(T_1) = \frac{\beta}{1-\beta} + C(-T_1)^{-\theta(1-\beta)}$ with $C = \max\{C_1, C_2\}$. Since $\beta < \frac{1}{2}$, we have $\frac{\beta}{1-\beta} < 1$, and choosing $T_1 < 0$ with sufficiently large $|T_1|$ makes $C(-T_1)^{-\theta(1-\beta)}$ arbitrarily small, hence $q(T_1) < 1$.

The above estimates show $\|A\nu\|_1 < \infty$ and $\|A\nu\|_2 < \infty$ for $\nu \in Q(T_1)$. Moreover, since

$$\lim_{t\to-\infty}\sup_{\tau\leq t}(-\tau)^{\delta_2}|\nu(\tau)| = 0$$

and $\int_{-\infty}^{t}|K(t,\tau)|d\tau$ is uniformly bounded for $t \leq T_1$, we obtain $\lim_{t\to-\infty}(-t)^{\delta_2}(A\nu)(t) = 0$. Continuity of $A\nu$ follows from continuity of K away from the diagonal and dominated convergence. Thus $A\nu \in Q(T_1)$.

The proposition is proved. □

A detailed analysis shows that there exists $T_1 < 0$ depending on the parameters of the problem such that $A : Q(T_1) \to Q(T_1)$ and

$$\|Av_1 - Av_2\|_Q \leq q_1\|v_1 - v_2\|_Q, \qquad v_1, v_2 \in Q(T_1),$$

with some $q_1 \in (0,1)$. Hence A is a contraction on $Q(T_1)$. By the Banach fixed point theorem there exists a unique $v \in Q(T_1)$ such that

$$\nu(t) = A\nu(t) + 2F(-t,t), \qquad t \leq T_1.$$

For $t \in [T_1, 0]$ the integral equation (21) can be rewritten in the form

$$\nu(t) - \int_{T_1}^{t}\frac{H(t,\tau)}{(t-\tau)^{\beta}}\nu(\tau)\,d\tau = \mathcal{F}_2(t),$$

where

$$\mathcal{F}_2(t) = 2F(-t,t) + \int_{-\infty}^{T_1} K(t,\tau)\nu(\tau)d\tau,$$

$H(t,\tau) = (t-\tau)^{\beta}K(t,\tau)$ is continuous in the triangular domain $\{(t,\tau) : T_1 \leq \tau < t \leq 0\}$ and $\mathcal{F}_2(t)$ is continuous as well. This is a Volterra equation of the second kind with a weakly singular kernel of order $\beta \in (0, 1/2)$, and standard results on such equations ensure that there exists a unique continuous solution ν on $[T_1, 0]$. Combining this with the solution on $(-\infty, T_1]$ we obtain a function ν defined on $(-\infty, 0]$ which satisfies (21).

We summarize the above in the main theorem.

Theorem 1. Let $0 < \alpha < 1$, $\beta = \alpha/2$. The regular solution u of the boundary value problem (1), (3) is given by

$$u(x,t) = \varphi(t) - \frac{x}{t}\big(\psi(t) - \varphi(t)\big) +$$
$$+\frac{1}{2}\int_{-\infty}^{t} \nu(\tau)\big(w_0(x-\tau, t-\tau) - w_0(-\tau - x, t-\tau)\big)\, d\tau +$$
$$+\frac{1}{2}\int_{-\infty}^{t}\int_{0}^{-\tau} \tilde{f}(\xi,\tau)\big(w_\beta(|x-\xi|, t-\tau) - w_\beta(x+\xi, t-\tau)\big)\, d\xi\, d\tau,$$

$\nu(t)$ being the solution

$$\nu(t) - \int_{-\infty}^{t} \nu(\tau)\big(w_0(-t-\tau, t-\tau) - w_0(t-\tau, t-\tau)\big)\, d\tau = 2F(-t,t),$$

$$F(x,t) = \frac{1}{2}\int_{-\infty}^{t}\int_{0}^{-\tau} \tilde{f}(\xi,\tau)\big(w_\beta(|x-\xi|, t-\tau) - w_\beta(x+\xi, t-\tau)\big)\, d\xi\, d\tau,$$

where

$$\tilde{f}(x,t) := f(x,t) - D^{\alpha}_{-\infty t}\varphi(t) + D^{\alpha}_{-\infty t}\Big[\frac{x}{t}\big(\psi(t) - \varphi(t)\big)\Big],$$

φ, ψ satisfy **(B1)-(B2)**, $f, \tilde{f} \in C(\overline{\Omega})$ and the following estimates hold:

- There exist $q \in (0,1]$ and $\sigma_4 > 1 + q$ such that

$$|\tilde{f}(x,t) - \tilde{f}(\xi,t)| \le C(-t)^{-\sigma_4}|x-\xi|^q, \qquad 0 \le x, \xi \le -t, \quad t < 0.$$

- There exists $\sigma_3 > 2 + \beta$ such that

$$\sup_{(x,t)\in\Omega} (-t)^{\sigma_3}\, |\tilde{f}(x,t)| < \infty.$$

Remark 1. The main technical work in the proof consists of establishing the estimates that justify the passage of the fractional derivative under the integral sign in (17)–(19), the boundary limits as $x \to 0$ and $x \to -t$, and the contraction property of the operator A in $Q(T_1)$. These estimates rely on detailed bounds for the kernels $w_\eta(x,t)$, which are Wright-type functions [10], and on the precise choice of the exponents $\sigma_3, \sigma_4, \delta_1, \delta_2$. The structure of the argument follows the detailed derivation in the original text but is presented here in a streamlined, self-contained form.

Conclusion

In this work, we have analysed a boundary value problem for a one-dimensional time-fractional diffusion equation with a Riemann–Liouville derivative of order $0 < \alpha < 1$ with respect to time, posed in a non-cylindrical domain whose spatial cross-section degenerates as $t \to 0^-$. By reducing the inhomogeneous boundary conditions to homogeneous ones, we reformulated the problem in a form amenable to potential-theoretic techniques. The fundamental solution in the quarter-plane was constructed by the bilateral Laplace transform in time, and the corresponding Green function for the Dirichlet problem on the quarter-line was obtained explicitly in terms of Wright-type kernels.

On the basis of this Green function, we derived an integral representation of the solution in the non-cylindrical domain as a sum of volume and boundary potentials. The boundary condition on the moving boundary naturally leads to a Volterra integral equation with a weakly singular kernel for an unknown boundary density. We showed that, under appropriate growth and regularity assumptions on the right-hand side and the boundary data, this integral equation is well posed in a weighted Banach space of functions on $(-\infty, 0]$, and that the associated integral operator is a contraction. As a consequence, the existence of a regular solution to the original boundary value problem follows from the Banach fixed-point theorem.

The analysis developed here demonstrates that the Green-function approach can be successfully extended to time–fractional diffusion equations with infinite temporal memory in non-cylindrical, degenerate domains. Possible directions for future research include higher-dimensional generalizations, other types of fractional time derivatives and nonlocal operators, numerical methods based on the obtained representation formulas, as well as the study of related inverse and free-boundary problems in similar geometries. The question of uniqueness will be addressed in forthcoming work by the authors.

Author Contributions

All authors contributed equally to this work.

Conflict of Interest

The authors declare no conflict of interest.

References

1 Mainardi, F., & Pagnini, G. (2003). The Wright functions as solutions of the time-fractional diffusion equation. *Applied Mathematics and Computation, 141*(1), 51–62. https://doi.org/10.1016/S0096-3003(02)00320-X

2 Gorenflo, R., Luchko, Y., & Mainardi, F. (2000). Wright functions as scale-invariant solutions of the diffusion-wave equation. *Journal of Computational and Applied Mathematics, 118*(1-2), 175–191. https://doi.org/10.1016/S0377-0427(00)00288-0

3 Luchko, Y. (2011). Maximum principle and its application for the time-fractional diffusion equations. *Fractional Calculus and Applied Analysis, 14*(1), 110–124. https://doi.org/10.2478/s13540-011-0008-6

4 Luchko, Y., & Yamamoto, M. (2017). On the maximum principle for a time-fractional diffusion equation. *Fractional Calculus and Applied Analysis, 20*(5), 1131–1145. https://doi.org/10.1515/fca-2017-0060

5 Pskhu, A.V. (2003). Solution of boundary value problems for the fractional diffusion equation by the Green function method. *Differential Equations, 39*(10), 1509–1513. https://doi.org/10.1023/B:DIEQ.0000017925.68789.e9

6 Pskhu, A.V. (2003). Solution of the first boundary value problem for a fractional-order diffusion equation. *Differential Equations, 39*(9), 1359–1363. https://doi.org/10.1023/B:DIEQ.0000012703.45373.aa

7 Povstenko, Y., & Kyrylych, T. (2018). Time-fractional diffusion with mass absorption in a half-line domain due to boundary value of concentration varying harmonically in time. *Entropy, 20*(5), 346. https://doi.org/10.3390/e20050346

8 Yao, Q., & Wang, H. (2022). A numerical scheme for the time-fractional diffusion equation by layer potentials. *Journal of Computational Physics, 448*, Article 110763. https://doi.org/10.1016/j.jcp.2021.110763

9 Kubica, A., Rybka, P., & Ryszewska, K. (2017). Weak solutions of fractional differential equations in noncylindrical domains. *Nonlinear Analysis: Real World Applications, 36*, 154–182. https://doi.org/10.1016/j.nonrwa.2017.01.005

10 Pskhu, A.V. (2017). The first boundary-value problem for a fractional diffusion-wave equation in a non-cylindrical domain. *Izvestiia: Mathematics, 81*(6), 1212–1233. https://doi.org/10.1070/IM8520

11 Roscani, S.D. (2017). Moving-boundary problems for the time-fractional diffusion equation. *Electronic Journal of Differential Equations, 2017*(44), 1–12.

12 Kubica, A., & Ryszewska, K. (2021). A self-similar solution to the time-fractional Stefan problem. *Mathematical Methods in the Applied Sciences, 44*(6), 4245–4275. https://doi.org/10.1002/mma.7028

13 Pskhu, A.V., & Rekhviashvili, S. (2020). Fractional diffusion-wave equation with application in electrodynamics. *Mathematics, 8*(11), Article 2086. https://doi.org/10.3390/math8112086

14 Pskhu, A.V., Ramazanov, M.I., Gulmanov, N.K., & Iskakov, S.A. (2022). Boundary value problem for fractional diffusion equation in a curvilinear angle domain. *Bulletin of the Karaganda University. Mathematics Series,* 1(105), 83–95. https://doi.org/10.31489/2022M1/83-95

15 Ramazanov, M.I., Pskhu, A.V., & Omarov, M.T. (2024). The first boundary value problem for the fractional diffusion equation in a degenerate angular domain. *Bulletin of the Karaganda University. Mathematics Series,* 1(113), 162–173. https://doi.org/10.31489/2024M1/162-173

16 Pskhu, A.V., Ramazanov, M.I., & Kosmakova, M. (2023). Boundary value problem for a loaded fractional diffusion equation. *Turkish Journal of Mathematics, 47*(5), 1585–1594. https://doi.org/10.55730/1300-0098.3450

17 Angelani, L., & Garra, R. (2023). *g*-fractional diffusion models in bounded domains. *Physical Review E, 107*(1), Article 014127. https://doi.org/10.1103/PhysRevE.107.014127

*Author Information**

Aleksandr Dmitrievich Akhmetshin — Doctoral student, Karaganda National Research University named after academician Ye.A. Buketov, 28 Universitetskaya St., Karaganda, 100028, Kazakhstan; e-mail: *aleksandr_050401@mail.ru*; https://orcid.org/0000-0003-2970-0804

Madi Tulegenovich Omarov (*corresponding author*) — Doctoral student, Karaganda National Research University named after academician Ye.A. Buketov, 28 Universitetskaya St., Karaganda, 100028, Kazakhstan; e-mail: *madiomarovt@gmail.com*; https://orcid.org/0000-0002-9026-5912

Raushan Zhumagazikyzy Toleukhanova — Senior Lecturer, Karaganda National Research University named after academician Ye.A. Buketov, 28 Universitetskaya St., Karaganda, 100028, Kazakhstan; e-mail: *toleuch_r@mail.ru*; https://orcid.org/0009-0002-5504-4732

*Authors' names are presented in the following order: first name, middle name (if any), last name.

Bulletin of the Karaganda University. Mathematics Series, No. 1(121), 2026, pp. 55–66

https://doi.org/10.31489/2026M1/55-66 *Research article*

Local derivation on the Schrödinger Lie algebra in $(n+1)$-dimensional space-time

A.K. Alauadinov[1,2], B.B. Yusupov[1,3,*]

[1] *V.I. Romanovskiy Institute of Mathematics Uzbekistan Academy of Sciences, Tashkent, Uzbekistan;*
[2] *Karakalpak State University, Nukus, Uzbekistan;*
[3] *Urgench State University, Urgench, Uzbekistan*
(E-mail: amir_t85@mail.ru, baxtiyor_yusupov_93@mail.ru)

This paper investigates local derivations on the Schrödinger Lie algebra $\mathfrak{s}_n$, the Lie algebra of the $(n+1)$-dimensional space-time Schrödinger group. As a finite-dimensional Lie algebra that is neither semisimple nor solvable, the Schrödinger algebra plays an important role in mathematical physics, particularly as the symmetry algebra of the free Schrödinger equation. While local derivations are well understood for semisimple, solvable, and certain infinite-dimensional Lie algebras, much less is known for non-semisimple algebras. We prove that for all integers $n \geq 3$, every local derivation on $\mathfrak{s}_n$ is a derivation. Our approach uses the explicit structure of the Schrödinger algebra together with a detailed description of its derivation algebra. First, we reduce the problem to derivations that act trivially on the semisimple part, and then we perform a coefficient-wise analysis in a fixed basis. This shows that every local derivation is an ordinary derivation. Moreover, such derivations decompose in the usual way into inner derivations and the known outer derivations. Our result extends earlier low-dimensional cases and shows a uniform rigidity phenomenon for all higher-dimensional Schrödinger algebras.

Keywords: Lie algebras, semisimple Lie algebra, solvable Lie algebra, nilpotent Lie algebra, Schrödinger algebras, inner derivations, derivations, local derivations.

2020 Mathematics Subject Classification: 17A32, 17B30, 17B10.

Introduction

Local derivations are useful tools in studying the structure of rings and algebras, where there are still many related unsolved problems. R.V. Kadison, D.R. Larson, and A.R. Sourour first introduced the notion of local derivations on algebras in their remarkable paper [1,2]. Since then, many researchers have been studying local derivations of different types of algebras (e.g., see [3–5]). In [6] the authors proved that every local derivation on a finite-dimensional semisimple Lie algebra $\mathcal{L}$ over an algebraically closed field of characteristic zero is a derivation.

In [4], local derivations on solvable Lie algebras are studied. It is shown that within this class, there exist solvable Lie algebras admitting local derivations that are not derivations, as well as solvable Lie algebras for which every local derivation is a derivation. Moreover, it is proved that every local derivation on a finite-dimensional solvable Lie algebra with a model nilradical and a complementary space of maximal dimension is a derivation. In [5], the authors proved that every local derivations on solvable Lie algebras whose nilradical has maximal rank is a derivation. In [3], the authors proved that every local derivation on the conformal Galilei algebra is a derivation.

We note that the aforementioned algebras are finite-dimensional algebras. In the infinite-dimensional case, the authors of [7–9] proved that every local derivation on some class of locally simple Lie algebras, generalized Witt algebras, Witt algebras, and Witt algebras over a field of prime characteristic is a derivation.

[*]Corresponding author. *E-mail: baxtiyor_yusupov_93@mail.ru*
Received: 30 October 2024; *Accepted:* 27 December 2025.

The Schrödinger Lie group is the symmetry group of the free-particle Schrödinger equation (see [10]). The Lie algebra $\mathfrak{s}_n$ in $(n+1)$-dimensional space-time of the Schrödinger Lie group is called the Schrödinger algebra, see [11, 12]. The Schrödinger algebra $\mathfrak{s}_n$ is a non-semisimple Lie algebra and plays an important role in mathematical physics. Recently there was a series of papers on studying the structure and representation theory of the Schrödinger algebra $\mathfrak{s}_1$ in the case of $(1+1)$-dimensional space-time, see [13–15].

In this paper, we generalize our previous result to all integers $n > 2$. In [16], we proved that for $n = 1, 2$, every local derivation on the Schrödinger algebra $\mathfrak{s}_n$ (in $(n+1)$-dimensional space–time) is a derivation. Hence, the same result holds for all $n \in \mathbb{N}$.

1 Preliminaries

In this section, we first recall the definition of $\mathfrak{s}_n$ from [11] in a different form. We know that the general linear Lie algebra $\mathfrak{gl}_{2n}$ has the natural representation on $\mathbb{C}^{2n}$ by left matrix multiplication. Let $\{e_1, e_2, \cdots, e_{2n}\}$ be the standard basis of $\mathbb{C}^{2n}$.

The Heisenberg Lie algebra $\mathfrak{h}_n = \mathbb{C}^{2n} \oplus \mathbb{C}z$ is the Lie algebra with Lie bracket given by

$$[e_i, e_{n+i}] = z, \quad [z, \mathfrak{h}_n] = 0, \quad 1 \le i \le n.$$

Recall that the Schrödinger Lie algebra $\mathfrak{s}_n$ is the semidirect product Lie algebra

$$\mathfrak{s}_n = (\mathfrak{sl}_2 \oplus \mathfrak{so}_n) \ltimes \mathfrak{h}_n,$$

where $\mathfrak{sl}_2$ is embedded in $\mathfrak{gl}_{2n}$ by the mapping

$$\begin{pmatrix} a & b \\ c & -a \end{pmatrix} \mapsto \begin{pmatrix} aI_n & bI_n \\ cI_n & -aI_n \end{pmatrix}$$

and $\mathfrak{so}_n$ is embedded in $\mathfrak{gl}_{2n}$ by

$$A \in \mathfrak{so}_n \mapsto \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix}.$$

Here I_n is the $n \times n$ identity matrix, $\mathfrak{sl}_2 \oplus \mathfrak{so}_n$ acts on $\mathfrak{h}_n$ by matrix multiplication, and $[z, \mathfrak{s}_n] = 0$.

Next, we will introduce a basis of $\mathfrak{s}_n$. Let

$$\begin{aligned}
&h = \begin{pmatrix} I_n & 0 \\ 0 & -I_n \end{pmatrix}, \quad e = \begin{pmatrix} 0 & I_n \\ 0 & 0 \end{pmatrix}, \quad f = \begin{pmatrix} 0 & 0 \\ I_n & 0 \end{pmatrix}, \\
&s_{ij} = \begin{pmatrix} e_{ij} - e_{ji} & 0 \\ 0 & e_{ij} - e_{ji} \end{pmatrix}, \quad 1 \le i < j \le n, \\
&u_k = e_k, v_k = e_{n+k}, \quad 1 \le k \le n,
\end{aligned}$$

where $e_{i,j}$ $(1 \le i, j \le n)$ the $n \times n$ matrix with zeros everywhere except a 1 on position (i, j).

The Schrödinger algebra $\mathfrak{s}_n$ is a Lie algebra with a $\mathbb{C}$-basis

$$\{e, f, h, z, u_i, v_i, s_{jk}(= -s_{kj}) \mid 1 \leqslant i \leqslant n,\ 1 \leqslant j < k \leqslant n\}$$

equipped with the following non-trivial commutation relations:

$$\begin{aligned}
&[h, e] = 2e, \quad [h, f] = -2f, \quad [e, f] = h, \\
&[u_i, v_i] = z, \quad [h, u_i] = u_i, \quad [h, v_i] = -v_i, \\
&[e, v_i] = u_i, \quad [f, u_i] = v_i, \\
&[s_{kl}, u_i] = \delta_{li} u_k - \delta_{ki} u_l, \quad [s_{kl}, v_i] = \delta_{li} v_k - \delta_{ki} v_l, \\
&[s_{ij}, s_{kl}] = \delta_{kj} s_{il} + \delta_{il} s_{jk} + \delta_{lj} s_{ki} + \delta_{ki} s_{lj},
\end{aligned}$$

where δ_{ij} is the Kronecker Delta defined as 1 for $i=j$ and as 0 otherwise.

We fix an order on the basis as follows:

$$\{e, f, h, z, u_i, v_i, s_{jk}(=-s_{kj}) \mid 1 \le i \le n,\ 1 \le j < k \le n\}.$$

The Schrödinger algebra $\mathfrak{s}_n$ is a finite-dimensional Lie algebra that is neither semisimple nor solvable. It can be realized as the semidirect product

$$\mathfrak{s}_n = (\mathfrak{sl}_2 \oplus \mathfrak{so}_n) \ltimes \mathfrak{h}_n,$$

where $\mathfrak{sl}_2 = \mathrm{Span}_{\mathbb{C}}\{e, f, h\}$ is the 3-dimensional simple Lie algebra, $\mathfrak{so}_n = \mathrm{Span}_{\mathbb{C}}\{s_{kl} \mid 1 \le k < l \le n\}$ is the orthogonal Lie algebra, and $\mathfrak{h}_n = \mathrm{Span}_{\mathbb{C}}\{z, u_i, v_i \mid 1 \le i \le n\}$ is the Heisenberg Lie algebra.

A derivation on a Lie algebra $\mathcal{L}$ is a linear map $D : \mathcal{L} \to \mathcal{L}$ which satisfies the Leibniz rule:

$$D([x, y]) = [D(x), y] + [x, D(y)], \quad \text{for any} \quad x, y \in \mathcal{L}.$$

For any element $y \in \mathcal{L}$ the operator of right multiplication $\mathrm{ad}_y : \mathcal{L} \to \mathcal{L}$, defined as $\mathrm{ad}_y(x) = [y, x]$ is a derivation, and derivations of this form are called inner derivations. The set of all inner derivations of $\mathcal{L}$, denoted by $\mathrm{Inn}(\mathcal{L})$, is an ideal in $\mathrm{Der}(\mathcal{L})$.

Definition 1. A linear operator Δ is called a local derivation if for any $x \in \mathcal{L}$, there exists a derivation $D_x : \mathcal{L} \to \mathcal{L}$ (depending on x) such that $\Delta(x) = D_x(x)$. The set of all local derivations on $\mathcal{L}$ we denote by $\mathrm{LocDer}(\mathcal{L})$.

We use the following definition given in [17].

Definition 2. The following derivations are outer derivations of $\mathfrak{s}_n$.

- When $n \ge 2$, the derivation $\sigma : \mathfrak{s}_n \to \mathfrak{s}_n$ is given by

$$\sigma(e) = \sigma(f) = \sigma(h) = \sigma(s_{kl}) = 0,\ \ \sigma(z) = z,\ \ \sigma(u_i) = \frac{1}{2}u_i,\ \ \sigma(v_i) = \frac{1}{2}v_i,$$

  for all $1 \le i \le n,\ 1 \le k < l \le n$.
- When $n = 2$, the derivation $\tau : \mathfrak{s}_2 \to \mathfrak{s}_2$ is given by

$$\tau(e) = \tau(f) = \tau(h) = \tau(z) = \tau(u_i) = \tau(v_i) = 0,\ \ \tau(s_{12}) = z,\ \ i = 1, 2.$$

- When $n = 1$, the derivation $\sigma_1 : \mathfrak{s}_1 \to \mathfrak{s}_1$ is given by

$$\sigma_1(e) = \sigma_1(f) = \sigma_1(h) = 0,\ \ \sigma_1(z) = z,\ \ \sigma_1(u_1) = \frac{1}{2}u_1,\ \ \sigma_1(v_1) = \frac{1}{2}v_1.$$

The following theorem is proved in [17].

Theorem 1. The derivations of the Schrödinger algebra $\mathfrak{s}_n$ are given by

$$\mathrm{Der}(\mathfrak{s}_n) = \begin{cases} \mathrm{Inn}(\mathfrak{s}_1) \oplus \mathbb{C}\sigma_1, & n = 1, \\ \mathrm{Inn}(\mathfrak{s}_2) \oplus \mathbb{C}\sigma \oplus \mathbb{C}\tau, & n = 2, \\ \mathrm{Inn}(\mathfrak{s}_n) \oplus \mathbb{C}\sigma, & n > 2, \end{cases}$$

where σ_1, τ, σ are given by Definition 2.

For any $x \in \mathfrak{s}_n,\ n \ge 3$, there exists

$$a = a_e e + a_f f + a_h h + a_z z + \sum_{i=1}^{n} a_{u_i} u_i + \sum_{i=1}^{n} a_{v_i} v_i + \sum_{1 \le k < l \le n} a_{s_{k,l}} s_{k,l}$$

and $\lambda \in \mathbb{C}$ such that, by Theorem 1, we can write

$$D(x) = [a, x] + \lambda\sigma(x).$$

Now consider

$$\begin{aligned}
D(e) &= [a, e] + \lambda\sigma(e) \\
&= \left[a_e e + a_f f + a_h h + a_z z + \sum_{i=1}^{n} a_{u_i} u_i + \sum_{i=1}^{n} a_{v_i} v_i + \sum_{1\le p<q\le n} a_{s_{p,q}} s_{p,q}, e\right] \\
&= 2a_h e - a_f h - \sum_{i=1}^{n} a_{v_i} u_i, \\
D(f) &= [a, f] + \lambda\sigma(f) = -2a_h f + a_e h - \sum_{i=1}^{n} a_{u_i} v_i, \\
D(h) &= [a, h] + \lambda\sigma(h) = -2a_e e + 2a_f f - \sum_{i=1}^{n} a_{u_i} u_i + \sum_{i=1}^{n} a_{v_i} v_i, \\
D(u_i) &= [a, u_i] + \lambda\sigma(u_i) = a_f v_i + \left(a_h + \frac{\lambda}{2}\right) u_i - a_{v_i} z + \sum_{1\le p<i} a_{s_{p,i}} u_p - \sum_{i<q\le n} a_{s_{i,q}} u_q, \\
D(v_i) &= [a, v_i] + \lambda\sigma(v_i) = \left(-a_h + \frac{\lambda}{2}\right) v_i + a_e u_i + a_{u_i} z + \sum_{1\le p<i} a_{s_{p,i},} v_p - \sum_{i<q\le n} a_{s_{i,q}} v_q, \\
D(s_{k,l}) &= [a, s_{k,l}] + \lambda\sigma(s_{k,l}) = -a_{u_l} u_k + a_{u_k} u_l - a_{v_l} v_k + a_{v_k} v_l + \\
&+ \sum_{1\le p<k,\ p\ne l} a_{s_{p,k}} s_{p,l} + \sum_{l<q\le n,\ q\ne k} a_{s_{l,q}} s_{q,k} + \\
&+ \sum_{1\le p<l,\ p\ne k} a_{s_{p,l}} s_{k,p} + \sum_{k<q\le n,\ l\ne q} a_{s_{k,q}} s_{l,q}.
\end{aligned}$$

2 Main results

In this section, we will prove that every local derivation on the Schrödinger algebra $\mathfrak{s}_n$ is a derivation.

Theorem 2. Every local derivation on the Schrödinger algebra $\mathfrak{s}_n,\ n \ge 3$ is a derivation.

To obtain this result, we first prove several lemmas.

Lemma 1. Let Δ be a local derivation on $\mathfrak{s}_n$ and $D \in \mathrm{Der}(\mathfrak{s}_n)$. Define $\Delta' = \Delta - D$. Then

$$\Delta'(x) \in \mathfrak{h}_n \quad \text{for all } x \in \mathfrak{s}_n. \tag{1}$$

Proof. By Theorem 1, every derivation $D \in \mathrm{Der}(\mathfrak{s}_n)$ can be written in the form

$$D(y) = [a, y] + \lambda\,\sigma(y)$$

for some $a \in \mathfrak{s}_n$ and $\lambda \in \mathbb{C}$, where σ is the outer derivation from Definition 2. Since Δ is a local derivation, for each $x \in \mathfrak{s}_n$ there exists a derivation $D_x \in \mathrm{Der}(\mathfrak{s}_n)$ of the same type such that

$$\Delta(x) = D_x(x) = [a(x), x] + \lambda(x)\,\sigma(x).$$

We use the theorem of Ayupov and Kudaybergenov [6], which asserts that any local derivation on a finite-dimensional semisimple Lie algebra is a derivation. Since $\mathfrak{sl}_2$ and $\mathfrak{so}_n$ are semisimple, their restrictions $\Delta|_{\mathfrak{sl}_2}$ and $\Delta|_{\mathfrak{so}_n}$ are derivations, and hence such a D exists.

We choose a derivation $D \in \mathrm{Der}(\mathfrak{s}_n)$ satisfying

$$D|_{\mathfrak{sl}_2} = \Delta|_{\mathfrak{sl}_2}, \quad D|_{\mathfrak{so}_n} = \Delta|_{\mathfrak{so}_n}.$$

For arbitrary $x \in \mathfrak{s}_n$ we have

$$\Delta'(x) = \Delta(x) - D(x) = [a(x) - a, x] + (\lambda(x) - \lambda)\, \sigma(x). \tag{2}$$

Using (2) and the fact that $a(s) - a \in \mathfrak{h}_n \oplus \mathbb{C}z$, we obtain $\Delta'(s) = 0$ for all $s \in \mathfrak{sl}_2 \cup \mathfrak{so}_n$.

Consequently, $a(x) - a \in \mathfrak{h}_n \oplus \mathbb{C}z$ for any $x \in \mathfrak{s}_n$. Because $\mathfrak{h}_n$ is an ideal of $\mathfrak{s}_n$ and $[\mathbb{C}z, \mathfrak{s}_n] = 0$, we have $[a(x) - a, x] \in \mathfrak{h}_n$. Moreover, by Definition 2, $\sigma(x)$ maps $\mathfrak{s}_n$ into $\mathfrak{h}_n$. Therefore, both terms in (2) belong to $\mathfrak{h}_n$, and hence

$$\Delta'(x) \in \mathfrak{h}_n \quad \text{for all} \quad x \in \mathfrak{s}_n.$$ □

For each $x \in \mathfrak{s}_n$, there exist an element $a = a(x) \in \mathfrak{s}_n$ of the form

$$a = a_e e + a_f f + a_h h + a_z z + \sum_{i=1}^{n} a_{u_i} u_i + \sum_{i=1}^{n} a_{v_i} v_i + \sum_{1 \le k < l \le n} a_{s_{k,l}}\, s_{k,l},$$

and a scalar $\lambda = \lambda(x) \in \mathbb{C}$ such that, by Theorem 1,

$$\Delta'(x) = [a, x] + \lambda\, \sigma(x).$$

Here $a_e, a_f, a_h, a_z, a_{u_i}, a_{v_i}, a_{s_{k,l}}$, and λ are complex numbers depending on x.

By applying (1) to $x = h$ and $x = z$, we get

$$\begin{aligned}\Delta'(h) &= -\sum_{i=1}^{n} a_{u_i}^{(h)} u_i + \sum_{i=1}^{n} a_{v_i}^{(h)} v_i,\\ \Delta'(z) &= \lambda^{(z)} z.\end{aligned}$$

Let $x_0 = \sum\limits_{i=1}^{n} a_{u_i}^{(h)} u_i + \sum\limits_{i=1}^{n} a_{v_i}^{(h)} v_i$. Consider the following statement

$$\Delta'' = \Delta' - \mathrm{ad}(x_0) - \lambda^{(h)} \sigma.$$

Then Δ'' is a local derivation. By direct verification we have

$$\Delta''(x) \in \mathfrak{h}_n, \quad \text{for all} \quad x \in \mathfrak{s}_n, \tag{3}$$

and

$$\Delta''(h) = \Delta''(z) = 0.$$

Considering (3), we find the values of the operator Δ'' in the basis elements:

$$\begin{aligned}
\Delta''(f) &= -\sum_{i=1}^{n} a_{u_i}^{(f)} v_i,\\
\Delta''(e) &= -\sum_{i=1}^{n} a_{v_i}^{(e)} u_i,\\
\Delta''(u_i) &= a_f^{(u_i)} v_i + \left(a_h^{(u_i)} + \frac{\lambda^{(u_i)}}{2}\right) u_i - a_{v_i}^{(u_i)} z + \sum_{1 \le k < i} a_{s_{k,i}}^{(u_i)} u_k - \sum_{i < l \le n} a_{s_{i,l}}^{(u_i)} u_l,\\
\Delta''(v_i) &= \left(\frac{\lambda^{(v_i)}}{2} - a_h^{(v_i)}\right) v_i + a_e^{(v_i)} u_i + a_{u_i}^{(v_i)} z + \sum_{1 \le k < i} a_{s_{k,i}}^{(v_i)} v_k - \sum_{i < l \le n} a_{s_{i,l}}^{(v_i)} v_l,\\
\Delta''(s_{k,l}) &= -a_{u_l}^{(s_{k,l})} u_k + a_{u_k}^{(s_{k,l})} u_l - a_{v_l}^{(s_{k,l})} v_k + a_{v_k}^{(s_{k,l})} v_l.
\end{aligned} \tag{4}$$

We take an element $b = b_e e + b_f f + b_h h + b_z z + \sum_{i=1}^{n} b_{u_i} u_i + \sum_{i=1}^{n} b_{v_i} v_i + \sum_{1\le k<l\le n} b_{s_{k,l}} s_{k,l}$ and $\mu \in \mathbb{C}$, where $b \in \mathfrak{s}_n$, $b_e, b_f, b_h, b_z, b_{u_i}, b_{v_i}, b_{s_{k,l}}, \mu$ are complex numbers depending on b.

Lemma 2. Coefficients $a_f^{(u_i)}$ and $a_e^{(v_i)}$, $(1 \le i \le n)$ in the formula (4) are equal to zero.

Proof. Fix i with $1 \le i \le n$ and set $x = e + u_i$. By the definition of a local derivation, there exist $b = b(x) \in \mathfrak{s}_n$ and $\mu = \mu(x) \in \mathbb{C}$ such that

$$\Delta''(x) = [b, x] + \mu\,\sigma(x).$$

Hence

$$\begin{aligned}\Delta''(x) = \Delta''(e+u_i) &= [b, e+u_i] + \mu\,\sigma(e+u_i)\\ &= \Big[b_e e + b_f f + b_h h + b_z z + \sum_{j=1}^{n} b_{u_j} u_j + \sum_{j=1}^{n} b_{v_j} v_j + \sum_{1\le k<l\le n} b_{s_{k,l}} s_{k,l},\ e+u_i\Big] + \mu\,\sigma(e+u_i)\\ &= -b_f h + b_f v_i + *e + \sum_{j=1}^{n} *u_j + *z.\end{aligned}$$

On the other hand, based on (4), we calculate the following equality:

$$\Delta''(x) = \Delta''(e+u_i) = \Delta''(e) + \Delta''(u_i) = a_f^{(u_i)} v_i + \sum_{j=1}^{n} *u_j + *z.$$

Comparing the coefficients at the basis elements h and v_i, we get $b_f = 0,\ b_f = a_f^{(u_i)}$, which implies

$$a_f^{(u_i)} = 0.$$

Now, consider the element $x = f + v_i$ for fixed $1 \le i \le n$,

$$\Delta''(x) = \Delta''(f+v_i) = [b, f+v_i] + \mu\sigma(f+v_i) = b_e h + b_e u_i + *f + \sum_{j=1}^{n} *v_j + *z.$$

On the other hand,

$$\Delta''(x) = \Delta''(f+v_i) = \Delta''(f) + \Delta''(v_i) = a_e^{(v_i)} u_i + \sum_{j=1}^{n} *v_j + *z.$$

Comparing the coefficients at the basis elements h and u_i, we get $b_e = 0,\ b_e = a_e^{(v_i)}$, which implies

$$a_e^{(v_i)} = 0. \qquad \square$$

Lemma 3. Coefficients $a_{v_i}^{(u_i)}$ and $a_{u_i}^{(v_i)}$, $1 \le i \le n$ in the formula (4) are equal to zero.

Proof. Fix i with $1 \le i \le n$ and set $x = h + u_i$. Then

$$\begin{aligned}\Delta''(x) &= [b, h+u_i] + \mu\,\sigma(h+u_i)\\ &= 2b_f f + \sum_{j=1}^{n} b_{v_j} v_j + b_f v_i - b_{v_i} z + *e + \sum_{j=1}^{n} *u_j.\end{aligned}$$

On the other hand,

$$\Delta''(x) = \Delta''(h + u_i) = \Delta''(h) + \Delta''(u_i) = -a_{v_i}^{(u_i)} z + \sum_{j=1}^{n} * u_j.$$

Comparing the coefficients at the basis elements $f,\ z$ and v_i, we get $b_f = b_{v_i} = 0,\ b_{v_i} = a_{v_i}^{(u_i)}$, which implies

$$a_{v_i}^{(u_i)} = 0.$$

Fix i with $1 \le i \le n$ and set $x = h + v_i$. Then

$$\begin{aligned}\Delta''(x) &= \Delta''(h + v_i) = [b, h + v_i] + \mu\sigma(h + v_i)\\ &= -2b_e e - \sum_{j=1}^{n} b_{u_j} u_j + b_e u_i + b_{u_i} z + *f + \sum_{j=1}^{n} * v_j.\end{aligned}$$

On the other hand,

$$\Delta''(x) = \Delta''(h + v_i) = \Delta''(h) + \Delta''(v_i) = a_{u_i}^{(v_i)} z + \sum_{j=1}^{n} * v_j.$$

Comparing the coefficients at the basis elements $e,\ z$ and u_i, we get $b_e = b_{u_i} = 0,\ b_{u_i} = a_{u_i}^{(v_i)}$, which implies

$$a_{u_i}^{(v_i)} = 0. \qquad \square$$

Lemma 4. $\Delta''(f) = 0$ and $a_h^{(v_i)} - \frac{\lambda^{(v_i)}}{2} = 0$ in the formula (4).

Proof. Take an element $x = f - \frac{1}{2}z + v_i$ $(1 \le i \le n)$. Then

$$\begin{aligned}\Delta''(x) &= \left[b,\ f - \tfrac{1}{2}z + v_i\right] + \mu\,\sigma\left(f - \tfrac{1}{2}z + v_i\right)\\ &= -2b_h f + b_e h - \sum_{j=1}^{n} b_{u_j} v_j - \frac{\mu}{2} z - b_h v_i + b_e u_i + b_{u_i} z\\ &\quad + \sum_{1\le k<i} b_{s_{k,i}} v_k - \sum_{i<l\le n} b_{s_{i,l}} v_l + \frac{\mu}{2} v_i.\end{aligned} \tag{5}$$

On the other hand,

$$\begin{aligned}\Delta''(x) &= \Delta''(f) - \Delta''(\frac{z}{2}) + \Delta''(v_i) = -\sum_{i=1}^{n} a_{u_i}^{(f)} v_i - a_h^{(v_i)} v_i\\ &\quad + \sum_{1\le k<i} a_{s_{k,i}}^{(v_i)} v_k - \sum_{i<l\le n} a_{s_{i,l}}^{(v_i)} v_l + \frac{\lambda^{(v_i)}}{2} v_i.\end{aligned} \tag{6}$$

Comparing the coefficients at the basis elements $f,\ z$ and v_i, (5) and (6), we get

$$\begin{cases} -2b_h & = 0,\\ -\frac{\mu}{2} + b_{u_i} & = 0,\\ -b_{u_i} - b_h + \frac{\mu}{2} & = -a_{u_i}^{(f)} - a_h^{v_i} + \frac{\lambda^{v_i}}{2}, \end{cases}$$

which implies

$$a_{u_i}^{(f)} = -a_h^{(v_i)} + \frac{\lambda^{(v_i)}}{2}. \tag{7}$$

Take an element $x = f - \frac{1}{2}z - v_i$. Then

$$\begin{aligned}\Delta''(x) = \left[b, f - \frac{1}{2}z - v_i\right] + \mu\sigma(f - \frac{1}{2}z - v_i) = -2b_h f + b_e h - \sum_{j=1}^{n} b_{u_j} v_j \\ - \frac{\mu}{2}z + b_h v_i - b_e u_i - b_{u_i} z - \sum_{1\le k<i} b_{s_{k,i}} v_k + \sum_{i<l\le n} b_{s_{i,l}} v_l - \frac{\mu}{2} v_i.\end{aligned} \tag{8}$$

On the other hand,

$$\begin{aligned}\Delta''(x) = \Delta''(f) - \Delta''(\frac{z}{2}) - \Delta''(v_i) = -\sum_{j=1}^{n} a_{u_j}^{(f)} v_j + a_h^{(v_i)} v_i \\ - \sum_{1\le k<i} a_{s_{k,i}}^{(v_i)} v_k + \sum_{i<l\le n} a_{s_{i,l}}^{(v_i)} v_l - \frac{\lambda^{(v_i)}}{2} v_i.\end{aligned} \tag{9}$$

Comparing the coefficients at the basis elements $f,\ z$ and v_i, (8) and (9), we get

$$\begin{cases} -2b_h & = 0, \\ -\frac{\mu}{2} - b_{u_i} & = 0, \\ -b_{u_i} + b_h - \frac{\mu}{2} & = -a_{u_i}^{(f)} + a_h^{v_i} - \frac{\lambda^{v_i}}{2}, \end{cases}$$

which implies

$$a_{u_i}^{(f)} = a_h^{(v_i)} - \frac{\lambda^{(v_i)}}{2}. \tag{10}$$

Comparing (7) and (10), we obtain that

$$a_{u_i}^{(f)} = 0, \quad a_h^{(v_i)} = \frac{\lambda^{(v_i)}}{2}.$$

So, $\Delta''(f) = 0$ follows from equality (4). Thus, the coefficients satisfy the relation

$$a_h^{(v_i)} - \frac{\lambda^{(v_i)}}{2} = 0. \qquad \square$$

Lemma 5. $\Delta''(e) = 0$ and $a_h^{(u_i)} + \frac{\lambda^{(u_i)}}{2} = 0$ in the formula (4).

Proof. Take an element $x = e + \frac{1}{2}z + u_i$. Then

$$\begin{aligned}\Delta''(x) &= [b, e + \frac{1}{2}z + u_i] + \mu\sigma(e + \frac{1}{2}z + u_i) \\ &= 2b_h e - b_f h - \sum_{j=1}^{n} b_{v_j} u_j + \frac{\mu}{2} z \\ &+ b_f v_i + b_h u_i - b_{v_i} z + \sum_{1\le k<i} b_{s_{k,i}} u_k - \sum_{i<l\le n} b_{s_{i,l}} u_l + \frac{\mu}{2} u_i.\end{aligned} \tag{11}$$

On the other hand,

$$\begin{aligned}\Delta''(x) = \Delta''(e) + \Delta''(\frac{1}{2}z) + \Delta''(u_i) = -\sum_{j=1}^{n} a_{v_j}^{(e)} u_j + a_h^{(u_i)} u_i \\ + \sum_{1\leq k<i} a_{s_{k,i}}^{(u_i)} u_k - \sum_{i<l\leq n} a_{s_{i,l}}^{(u_i)} u_l + \frac{\lambda^{(u_i)}}{2} u_i.\end{aligned} \tag{12}$$

Comparing the coefficients at the basis elements $e,\ z$ and u_i, (11) and (12), we obtain that

$$\begin{cases} 2b_h & = 0, \\ \frac{\mu}{2} - b_{v_i} & = 0, \\ -b_{v_i} + b_h + \frac{\mu}{2} & = -a_{v_i}^{(e)} + a_h^{u_i} + \frac{\lambda^{u_i}}{2}, \end{cases}$$

which implies

$$a_{v_i}^{(e)} = a_h^{(u_i)} + \frac{\lambda^{(u_i)}}{2}. \tag{13}$$

Next, we take an element $x = e + \frac{1}{2}z - u_i$, then

$$\begin{aligned}\Delta''(x) = [b, e + \frac{1}{2}z - u_i] + \mu\sigma(e + \frac{1}{2}z - u_i) = 2b_h e - b_f h - \sum_{j=1}^{n} b_{v_j} u_j + \frac{\mu}{2} z \\ - b_f v_i - b_h u_i + b_{v_i} z - \sum_{1\leq k<i} b_{s_{k,i}} u_k + \sum_{i<l\leq n} b_{s_{i,l}} u_l - \frac{\mu}{2} u_i.\end{aligned} \tag{14}$$

On the other hand,

$$\begin{aligned}\Delta''(x) = \Delta''(e) + \Delta''(\frac{z}{2}) - \Delta''(u_i) = -\sum_{j=1}^{n} a_{v_j}^{(e)} u_j - a_h^{(u_i)} u_i \\ - \sum_{1\leq k<i} a_{s_{k,i}}^{(u_i)} u_k + \sum_{i<l\leq n} a_{s_{i,l}}^{(u_i)} u_l - \frac{\lambda^{(u_i)}}{2} u_i.\end{aligned} \tag{15}$$

Comparing the coefficients at the basis elements $e,\ z$ and u_i, (14) and (15), we get

$$\begin{cases} 2b_h & = 0, \\ \frac{\mu}{2} + b_{v_i} & = 0, \\ -b_{v_i} - b_h - \frac{\mu}{2} & = -a_{v_i}^{(e)} - a_h^{u_i} - \frac{\lambda^{u_i}}{2}, \end{cases}$$

which implies

$$a_{v_i}^{(e)} = -a_h^{(u_i)} - \frac{\lambda^{(u_i)}}{2}. \tag{16}$$

Comparing (13) and (16), we obtain that

$$a_{v_i}^{(e)} = 0, \quad a_h^{(u_i)} = -\frac{\lambda^{(u_i)}}{2}.$$

Thus, $\Delta''(e) = 0$ follows from equality (4). We have the following connection

$$a_h^{(u_i)} + \frac{\lambda^{(u_i)}}{2} = 0$$

between the coefficients. □

Lemma 6. $\Delta''(\mathfrak{so}_n) = \{0\}$ and $\Delta''(\mathfrak{h}_n) = \{0\}$.

Proof. Let k, l $(k \neq l)$ fixed numbers in the set $\{1, 2, ..., n\}$. Next, set $x = u_k + s_{k,l}$ (if $l < k$, then $s_{k,l} = -s_{l,k}$). Then

$$\begin{aligned} \Delta''(x) &= [b, u_k + s_{k,l}] + \mu\sigma(u_k + s_{k,l}) \\ &= b_f v_k + b_h u_k - b_{v_k} z + \sum_{1 \le p < k} b_{s_{p,k}} u_p \\ &- \sum_{k < q \le n} b_{s_{k,q}} u_q + \frac{\mu}{2} u_k - b_{u_l} u_k + b_{u_k} u_l - b_{v_l} v_k + b_{v_k} v_l \\ &+ \sum_{1 \le p < k,\ p \neq l} b_{s_{p,k}} s_{p,l} + \sum_{l < q \le n,\ q \neq k} b_{s_{l,q}} s_{q,k} \\ &+ \sum_{1 \le p < l,\ p \neq k} b_{s_{p,l}} s_{k,p} + \sum_{k < q \le n,\ l \neq q} b_{s_{k,q}} s_{l,q}. \end{aligned} \tag{17}$$

On the other hand,

$$\begin{aligned} \Delta''(x) = \Delta''(u_k) + \Delta(s_{k,l}) = \sum_{1 \le j < k} a_{s_{j,k}}^{(u_k)} u_j - \sum_{k < j \le n} a_{s_{k,j}}^{(u_i)} u_j \\ - a_{u_l}^{(s_{k,l})} u_k + a_{u_k}^{(s_{k,l})} u_l - a_{v_l}^{(s_{k,l})} v_k + a_{v_k}^{(s_{k,l})} v_l. \end{aligned} \tag{18}$$

Comparing the coefficients at the basis elements z and v_l, (17) and (18), we obtain that

$$\begin{cases} b_{v_k} &= 0, \\ b_{v_k} &= a_{v_k}^{(s_{k,l})}, \end{cases}$$

which implies

$$a_{v_k}^{(s_{k,l})} = 0.$$

Similarly, from equality:

$$\begin{aligned} \Delta''(u_l + s_{l,k}) = \Delta''(u_k) + \Delta''(s_{l,k}) \text{ we obtain } a_{v_l}^{(s_{k,l})} = 0; \\ \Delta''(v_k + s_{k,l}) = \Delta''(v_k) + \Delta''(s_{k,l}) \text{ we obtain } a_{u_k}^{(s_{k,l})} = 0; \\ \Delta''(v_l + s_{l,k}) = \Delta''(v_k) + \Delta''(s_{l,k}) \text{ we obtain } a_{u_l}^{(s_{k,l})} = 0. \end{aligned}$$

If we substitute the above four results into (4), we get $\Delta''(s_{l,k}) = 0,\ 1 \le l < k \le n$.
Then equation (18) can be rewritten as

$$\Delta''(x) = \Delta''(u_k) + \Delta(s_{k,l}) = \sum_{1 \le j < k} a_{s_{j,k}}^{(u_k)} u_j - \sum_{k < j \le n} a_{s_{k,j}}^{(u_k)} u_j. \tag{19}$$

Comparing the coefficients at the basis elements u_j $(j \neq k, j \neq l)$ and $s_{j,l}$, (17) and (19), we obtain that

- if $j < k$, then $b_{s_{j,k}} = 0$ and $b_{s_{j,k}} = a_{s_{j,k}}^{(u_k)}$;
- if $j > k$, then $b_{s_{k,j}} = 0$ and $b_{s_{k,j}} = a_{s_{k,j}}^{(u_k)}$;

we get

$$a^{(u_k)}_{s_{k,j}} = 0, \quad (j \neq k). \tag{20}$$

Take a number i such that $i \neq k$ and $i \neq l$.

$$\begin{aligned} \Delta''(u_k + s_{k,i}) &= \Delta''(u_k) + \Delta''(s_{k,i}) \;\Rightarrow\; a^{(u_k)}_{s_{k,i}} = 0, \\ \Delta''(v_k + s_{k,j}) &= \Delta''(v_k) + \Delta''(s_{k,j}) \;\Rightarrow\; a^{(v_k)}_{s_{k,j}} = 0, \quad (j \neq k), \\ \Delta''(v_k + s_{k,i}) &= \Delta''(v_k) + \Delta''(s_{k,i}) \;\Rightarrow\; a^{(v_k)}_{s_{k,i}} = 0. \end{aligned} \tag{21}$$

Thus, according to (4), (20) and (21), we have

$$\Delta''(\mathfrak{so}_n) = \{0\} \quad \text{and} \quad \Delta''(\mathfrak{h}_n) = \{0\}.$$ □

Now we are in position to prove Theorem 2.

Proof of Theorem 2. From (2) and Lemmas 2–6 we obtain

$$\Delta'' = 0. \tag{22}$$

Together (2) and (22) give

$$\Delta' = \mathrm{ad}(x_0) + \lambda^{(h)}\sigma. \tag{23}$$

Together (1) and (23) give

$$\Delta = D + \mathrm{ad}(x_0) + \lambda^{(h)}\sigma.$$

Hence, any local derivation of the algebra $\mathfrak{s}_n$ $(n \geq 3)$ is a derivation. □

Conclusion

We study local derivations on the Schrödinger algebra $\mathfrak{s}_n$ in $(n+1)$-dimensional space-time of Schrödinger Lie groups for any integer n. We prove that every local derivations on the Schrödinger algebra $\mathfrak{s}_n$ in $(n+1)$-dimensional space-time are derivations.

Author Contributions

All authors contributed equally to this work.

Conflict of Interest

The authors declare no conflict of interest.

References

1 Kadison, R.V. (1990). Local derivations. *Journal of Algebra, 130*(2), 494–509. https://doi.org/10.1016/0021-8693(90)90095-6

2 Larson, D.R., & Sourour, A.R. (1990). Local derivations and local automorphisms of $B(X)$. *Proceedings of Symposia in Pure Mathematics, 51*, 187–194.

3 Alauadinov, A.K., & Yusupov, B.B. (2024). Local derivations of conformal Galilei algebra. *Communications in Algebra, 52*(6), 2489–2508. https://doi.org/10.1080/00927872.2023.2301539

4 Ayupov, Sh., & Khudoyberdiyev, A. (2021). Local derivations on solvable Lie algebras. *Linear and Multilinear Algebra, 69*(7), 1286–1301. https://doi.org/10.1080/03081087.2019.1626336

5 Kudaybergenov, K.K., Omirov, B.A., & Kurbanbaev, T.K. (2022). Local derivations on solvable Lie algebras of maximal rank. *Communications in Algebra, 50*(9), 1–11. https://doi.org/10.1080/00927872.2022.2045494

6 Ayupov, Sh.A., & Kudaybergenov, K.K. (2016). Local derivations on finite-dimensional Lie algebras. *Linear Algebra and its Applications, 493*, 381–398. https://doi.org/10.1016/j.laa.2015.11.034

7 Ayupov, Sh.A., Kudaybergenov, K.K., & Yusupov, B.B. (2024). Local and 2-Local Derivations of Locally Simple Lie Algebras. *Journal of Mathematical Sciences, 278*(4), 613–622. https://doi.org/10.1007/s10958-024-06943-2

8 Chen, Y., Zhao, K., & Zhao, Y. (2022). Local derivations on Witt algebras. *Linear and multilinear algebra, 70*(6), 1159–1172. https://doi.org/10.1080/03081087.2020.1754750

9 Yao, Y.-F. (2020). Local derivations on the Witt algebra in prime characteristic. *Linear and multilinear algebra, 70*(15), 1–15. https://doi.org/10.1080/03081087.2020.1819189

10 Perroud, M. (1997). Projective representations of the Schrödinger group. *Helvetica Physica Acta, 50*, 233–252.

11 Dobrev, V.K., Doebner, H.D., & Mrugalla, C. (1997). Lowest weight representations of the Schrödinger algebra and generalized heat equations. *Reports on Mathematical Physics, 39*(2), 201–218. https://doi.org/10.1016/S0034-4877(97)88001-9

12 Liu, G., Li, Y., & Wang, K. (2021). Irreducible weight modules over the Schrödinger Lie algebra in (n+1) dimensional space-time. *Journal of Algebra, 575*, 1–13. https://doi.org/10.1016/j.jalgebra.2021.01.034

13 Aizawa, N., & Dobrev, V. (2010). Intertwining operator realization of non-relativistic holography. *Nuclear Physics B, 828*(3), 581–593. https://doi.org/10.1016/j.nuclphysb.2009.10.019

14 Yang Y., & Tang, X. (2018). Derivation of the Schrödinger algebra and their applications. *Journal of Applied Mathematics and Computing, 58*, 567–576. https://doi.org/10.1007/s12190-017-1157-5

15 Zhang, X., & Cheng, Y. (2015). Simple Schrödinger modules which are locally finite over the positive part. *Journal of Pure and Applied Algebra, 219*(7), 2799–2815. https://doi.org/10.1016/j.jpaa.2014.09.029

16 Alauadinov, A.K., & Yusupov, B.B. (2025). Local derivations of the Schrödinger algebras. *Algebra Colloquium, 32*(3), 467–480. https://doi.org/10.1142/S1005386725000343

17 Wu, Q., & Tang, X. (2022). Derivations and Biderivations of the Schrodinger algebra in $(n+1)$-dimensional space-time. *Linear and Multilinear Algebra, 71*(7), 1073–1097. https://doi.org/10.1080/03081087.2022.2050884

Author Information[*]

Amir Kadirbergenovich Alauadinov — PhD in Physical and Mathematical Sciences, Associate Professor, Department of Mathematics, Karakalpak State University, 1 Ch. Abdirov St., Nukus, 230113, Uzbekistan; e-mail: *amir_t85@mail.ru*; https://orcid.org/0000-0002-1492-4939

Bakhtiyor Bakhrambek ugli Yusupov (*corresponding author*) — PhD in Physical and Mathematical Sciences, Senior Researcher, V.I. Romanovskiy Institute of Mathematics Uzbekistan Academy of Sciences, 9 University St., Tashkent, 100174, Uzbekistan; e-mail: *baxtiyor_yusupov_93@mail.ru*; https://orcid.org/0000-0001-9400-5811

[*]Authors' names are presented in the following order: first name, middle name (if any), last name.

https://doi.org/10.31489/2026M1/67-81 *Research article*

On certain identities of generalized derivations of semirings with involution

L. Ali*, M. Aslam

Government College University, Lahore, Pakistan
(E-mail: rehmani.pk786@gmail.com, aslam298@gcu.edu.pk)

MA-semirings form a proper subclass of inverse semirings that properly contains both the class of rings and the class of distributive lattices with the least element. In this paper, we study generalized derivations satisfying certain algebraic identities of MA-semirings with involution. The main objective of this research is to investigate identities involving three, two, one generalized derivation in MA-semirings with involution, ensuring commutativity. Hermitian and skew-Hermitian elements are primarily used to formulate the basic tools for the development of this paper and these notions are the fundamental units of the second kind involution. Involution of the second kind plays a key role not only for proving the main results (see Theorems 1, 3, 5) but also it enables us to observe more results from their proofs (see Theorems 2, 4, 6). Since every derivation is a generalized derivation, the results obtained naturally extend a variety of results on derivations. Moreover, several well-established results on derivations of MA-semirings and rings under the similar environment can be concluded as special cases.

Keywords: semirings, MA-semirings, prime semirings, Hermitian elements, skew-Hermitian elements, involution, second kind involution, derivations, generalized derivations.

2020 Mathematics Subject Classification: 16Y60, 16W25, 16W10.

Introduction

The theory of semirings has tremendous and direct applications in the sciences. For instance, idempotent analysis based on additive inverse semirings has interesting applications in quantum physics (see [1,2]), and the same algebraic structure is used to develop the formal languages [3,4] and automata theory [4–6]. One can find the applications of semirings in other fields of science and mathematics such as theoretical computer sciences and engineering, parallel computational systems, optimization theory, combinatorics, functional analysis, topology, graph theory, Euclidean geometry, and mathematical modeling of quantum physics (see [7–9]). Moreover, semirings have some notable applications in cryptography (see [10, 11]). B^*-algebras as well as C^*-algebras are well-known examples of rings with involution (see [12–14]) in the canvass of functional analysis, which is indeed a primitive source of motivation for ring theorists. For the ring's theoretical background, we would like to refer to [15–17].

The class of MA-semirings [18] has a significant potential to accommodate the study of derivations and generalized derivations satisfying different identities on semirings with involution [19–21] and without involution [22–24] for exploring commuting conditions and other features. In the present paper, we generalize a few results of [25] in the framework MA-semirings with second-kind involution.

In the next section, we include some necessary preliminaries for the sake of completeness and examples for exploring the features of this paper.

*Corresponding author. *E-mail: rehmani.pk786@gmail.com*
Received: 15 October 2024; *Accepted:* 12 December 2025.

1 Preliminaries and Examples

An additive commutative inverse semiring $(S,+,.)$ with absorbing zero 0 and center $Z(S)$ is said to be an MA-semiring [18] if $w+w' \in Z(S)$ for all $w \in S$, where w' denotes the pseudo inverse of w, which is indeed unique (see [8, 9]). Throughout the paragraph, by S we mean an MA-semiring. A mapping $\varrho : S \longrightarrow S$ is a derivation if $\varrho(w+v) = \varrho(w)+\varrho(v)$ and $\varrho(wv) = \varrho(w)v + w\varrho(v)$. An additive mapping $F_\varrho : S \longrightarrow S$ is said to be a generalized derivation associated with a derivation ϱ, if $F_\varrho(uv) = F_\varrho(u)v + u\varrho(v)$. The commutator and anti-commutator of $w, v \in S$ are respectively defined as $[w,v] = wv+v'w$ and $w\circ v = wv+vw$. Involution is an additive mapping $* : S \longrightarrow S$ satisfying $(w^*)^* = w$ and $(vw)^* = w^*v^*$, for all $v, w \in S$. $\mathbb{H}(S) = \{w \in S : w^* = w\}$ and $\mathbb{K}(S) = \{w \in S : w^* = w'\}$ respectively represents the sets of Hermitian and skew Hermitian elements of S. Involution of the second kind was introduced in [26] in the framework of MA-semirings. If $Z(S) \nsubseteq \mathbb{H}(S)$, then involution is of second kind otherwise it is of first kind.

Example 1. [27] Let $(\mathbb{Z},+,.)$ be the ring of integers and $I(\mathbb{Z})$ be the collection of all ideals of $\mathbb{Z}$. Consider the set $S = M_2(\mathbb{Z}) \times I(\mathbb{Z})$ and let $u = (A_1, I), v = (A_2, J) \in S$. Define addition $\oplus$ and multiplication $\odot$ by $u \oplus v = (A_1 + A_2, I + J)$ and $u \odot v = (A_1A_2, IJ)$. Then $(S,\oplus,\odot)$ is an example of a proper MA-semiring. Furthermore, define a mapping $* : S \longrightarrow S$ by $(A, I)^* = (A^T, I)$, where A^T is the transpose of A. Then $*$ defines an involution on S. We further see that $Z(S) \subseteq \mathbb{H}(S)$, therefore $*$ is an involution of first kind.

Example 2. [27] Let $\mathbb{Z}$ be the set of integers, $\mathbb{Z}_0^+$ be the set of all non-negative integers and $R = \mathbb{Z}\times\mathbb{Z}_0^+$. Define addition $\oplus$ and multiplication $\odot$ by $(u_1, v_1) \oplus (u_2, v_2) = (u_1 + u_2, v_1 \vee v_2)$ and $(u_1, v_1) \odot (u_2, v_2) = (u_1.u_2, v_1.v_2)$, where $v_1 \vee v_2 = max\{v_1, v_2\}$. Then the triplet $(R,\oplus,\odot)$ forms an MA-semiring which is not a ring. One can observe that

$$M_R = \left\{ \begin{bmatrix} w & v & u & x \\ 0 & w & 0 & u \\ 0 & 0 & w & v' \\ 0 & 0 & 0 & w \end{bmatrix} : u, v, w, x \in R \right\}$$

(where v' is the pseudo inverse of v) is an MA-semiring under matrix addition and multiplication. Next, we define a mapping $* : M_R \longrightarrow M_R$ by

$$\begin{bmatrix} w & v & u & x \\ 0 & w & 0 & u \\ 0 & 0 & w & v' \\ 0 & 0 & 0 & w \end{bmatrix}^* = \begin{bmatrix} w & v & u & x' \\ 0 & w & 0 & u \\ 0 & 0 & w & v' \\ 0 & 0 & 0 & w \end{bmatrix}.$$

The mapping $*$ defines a second kind involution on M_R.

Example 3. [28] Let $(R,+,.)$ be a ring and $I(R)$ be the collection of all ideals of R. Consider the set $S = R \times I(R)$ and let $u = (r_1, I), v = (r_2, J) \in S$. Define addition $\oplus$ and multiplication $\odot$ by $u \oplus v = (r_1 + r_2, I + J)$ and $u \odot v = (r_1r_2, IJ)$. Then $(S,\oplus,\odot)$ forms an MA-semiring which is not a ring.

Throughout the sequel by a semiring S, we mean an MA-semiring S unless mentioned otherwise. Furthermore, we take $h_z, \tilde{h}_z \in \mathbb{H}(S) \cap Z(S)$ and $k_z \in \mathbb{K}(S) \cap Z(S)$, for the sake of convenience.

Following results are indeed useful to establish the main results of this paper.

Lemma 1. [18] Let S be a semiring and ϱ be a derivation of S. Then for all $u, v, w \in S,\ z \in Z(S)$, we have

(i) $[w, wu] = w[w, u]$,
(ii) $[w, uv] = [w, u]v + u[w, v]$,
(iii) $[wu, v] = w[u, v] + [w, v]u$,
(iv) $(wu)' = w'u = wu'$,
(v) $[w, u] + [u, w] = u(w + w') = w(u + u')$,
(vi) $[w, u]' = [w, u'] = [w', u] = [u, w]$,
(vii) $w \circ (u + v) = w \circ u + w \circ v$,
(viii) $\varrho(w') = (\varrho(w))'$,
(ix) $(w')^* = ((w)^*)'$,
(x) $[w, uz] = z[w, u] = [w, u]z$,
(xi) $[w, w] = [w, w]'$,
(xii) $w + u = 0 \ \Rightarrow w = u'$, however the converse may not hold in general.

For more such identities, one can see [28–30].

Lemma 2. [31] Let S be a semiprime semiring with involution $*$ of second kind. Then $\mathbb{K}(S) \cap Z(S) \neq \{0\}$ and hence $\mathbb{H}(S) \cap Z(S) \neq \{0\}$.

The following lemma is readily discernible from the definitions of Hermitian and the skew Hermitian elements of a semiring with second kind involution.

Lemma 3. [27] If S is a semiring with second kind involution $*$, then for any $k \in \mathbb{K}(S)$ and $h \in \mathbb{H}(S)$, we have

(i) $k^2 \in \mathbb{H}(S)$,
(ii) $hh_z \in \mathbb{H}(S)$,
(iii) $kk_z \in \mathbb{H}(S)$,
(iv) $hk_z \in \mathbb{K}(S)$.

Lemma 4. [27] Let ϱ be a derivation of a 2-torsion free prime semiring S with involution $*$ of second kind. If $\varrho(h_z) = 0$, then $\varrho(k_z) = 0$.

Lemma 5. [27] Let ϱ be a derivation of a 2-torsion free prime semiring S with involution $*$ of second kind. If $\varrho(h_z) = 0$, then $\varrho(z) = 0$, for all $z \in Z(S)$.

Following lemma is a special case of Theorem 2.2 of [31].

Lemma 6. [31] Let ϱ be a nonzero derivation of a prime semiring S such that $[\varrho(w), w] = 0$, for all $w \in S$. Then S is commutative.

Lemma 7. Let F_σ be a generalized derivation associated with a nonzero derivation σ of a prime semiring S. If $[F_\sigma(w), w] = 0$, for all $w \in S$, then S is commutative.

Proof. The hypothesis states that
$$[F_\sigma(w), w] = 0. \tag{1}$$
Linearizing (1) and again using (1), we get
$$[F_\sigma(w), s] + [F_\sigma(s), w] = 0. \tag{2}$$
Substituting ws for w in (2), we find
$$[F_\sigma(w), s]s + [w\sigma(s), s] + [F_\sigma(s), w]s + w[F_\sigma(s), s] = 0.$$
Using (1) and (2) in the last expression, we get
$$[w\sigma(s), s] = 0. \tag{3}$$
In (3) substituting rw for w, and using Lemma 1, we obtain
$$0 = [rw\sigma(s), s] = r[w\sigma(s), s] + [r, s]w\sigma(s),$$

and using (3) again, we obtain
$$[r, s]w\sigma(s) = 0. \tag{4}$$
Multiplying (4) by s from the right, we get
$$[r, s]w\sigma(s)s = 0. \tag{5}$$
In (4) writing ws' for w, we obtain
$$[r, s]ws'\sigma(s) = 0. \tag{6}$$
Adding (5) and (6) and then substituting $\sigma(s)$ for r, we get $[\sigma(s), s]S[\sigma(s), s] = 0$. As S is prime, we can write $[\sigma(s), s] = 0$ for all $s \in S$. By Lemma 6, S is commutative. □

2 *Main Results*

In this section, we present the key results of this research article. We investigate several identities involving generalized derivations working in pairs and triplets with a key role of involution of the second kind. Through out this section by a prime semiring we mean a prime MA-semiring unless mentioned otherwise.

Following result describes an identity involving three generalized derivations, which leads to the commutativity of a semiring and this result is an extended version of Theorem 2.2 of [25].

Theorem 1. For a 2-torsion free prime semiring S with involution $*$ of second kind, let F_σ,G_ϱ and D_δ be generalized derivations respectively associated with derivations σ,ϱ and δ such that $\sigma \neq 0$ and $\varrho \neq 0$. If
$$[F_\sigma(w)G_\varrho(w^*) + D_\delta(ww^*), t] = 0, \tag{7}$$
for all $t, w \in S$, then S is commutative.

Proof. Linearizing (7) and using (7) again, we obtain
$$[F_\sigma(w)G_\varrho(s^*) + F_\sigma(s)G_\varrho(w^*) + D_\delta(ws^*) + D_\delta(sw^*), t] = 0. \tag{8}$$
Writing $s\tilde{h}_z$ for s in (8) and hence after the rearrangement of terms, we obtain
$$\begin{aligned}&[F_\sigma(w)G_\varrho(s^*) + F_\sigma(s)G_\varrho(w^*) + D_\delta(ws^*) + D_\delta(sw^*), t]\tilde{h}_z \\ &\qquad + [F_\sigma(w)s^*\varrho(\tilde{h}_z) + s\sigma(\tilde{h}_z)G_\varrho(w^*) + (ws^* + sw^*)\delta(\tilde{h}_z), t] = 0.\end{aligned}$$
Using (8) again, we obtain
$$[F_\sigma(w)s^*\varrho(\tilde{h}_z) + s\sigma(\tilde{h}_z)G_\varrho(w^*) + (ws^*)\delta(\tilde{h}_z) + (sw^*)\delta(\tilde{h}_z), t] = 0. \tag{9}$$
Substituting sk_z for s in (9), we obtain
$$[(F_\sigma(w)s^*\varrho(\tilde{h}_z))' + s\sigma(\tilde{h}_z)G_\varrho(w^*) + (ws^*)'\delta(\tilde{h}_z) + sw^*\delta(\tilde{h}_z), t]Sk_z = \{0\}.$$
Because of the primeness of S, we have
$$[(F_\sigma(w)s^*\varrho(\tilde{h}_z))' + s\sigma(\tilde{h}_z)G_\varrho(w^*) + (ws^*)'\delta(\tilde{h}_z) + sw^*\delta(\tilde{h}_z), t] = 0.$$
Using assertion (xii) of Lemma 1, we get
$$[F_\sigma(w)s^*\varrho(\tilde{h}_z) + ws^*\delta(\tilde{h}_z), t] = [s\sigma(\tilde{h}_z)G_\varrho(w^*) + sw^*\delta(\tilde{h}_z), t]. \tag{10}$$

In view of the 2-torsion freeness of S, using (10) in (9), we find

$$[s(\sigma(\tilde{h}_z)G_\varrho(w^*) + w^*\delta(\tilde{h}_z)), t] = 0. \tag{11}$$

Writing rs for s in (11) and again using (11), we can write

$$[r,t]S(\sigma(\tilde{h}_z)G_\varrho(w^*) + w^*\delta(\tilde{h}_z)) = \{0\}$$

and by the primeness of S, we have either S is commutative or $\sigma(\tilde{h}_z)G_\varrho(w^*) + w^*\delta(\tilde{h}_z) = 0$, which further implies

$$\sigma(\tilde{h}_z)G_\varrho(w) + w\delta(\tilde{h}_z) = 0. \tag{12}$$

Writing wh_z for w in (12), we get $(\sigma(\tilde{h}_z)G_\varrho(w) + w\delta(\tilde{h}_z))h_z + \sigma(\tilde{h}_z)w\varrho(h_z) = 0$ and by the use of (12) again, we further get $\sigma(\tilde{h}_z)S\varrho(h_z) = \{0\}$. Due to the primeness of S, we have either $\sigma(\tilde{h}_z) = 0$, for all $\tilde{h}_z \in \mathbb{H}(S) \cap Z(S)$ or $\varrho(h_z) = 0$, for all $h_z \in \mathbb{H}(S) \cap Z(S)$. Assume that $\sigma(\tilde{h}_z) = 0$, then from (12), we obtain $w\delta(\tilde{h}_z) = 0$, which further implies $\delta(\tilde{h}_z) = 0$. Now since $\sigma(\tilde{h}_z) = 0 = \delta(\tilde{h}_z)$, from (9) we obtain $[F_\sigma(w)s^*\varrho(\tilde{h}_z), t] = 0$ and therefore

$$[F_\sigma(w)s\varrho(\tilde{h}_z), t] = 0. \tag{13}$$

Substituting wp for w in (13), we obtain $[F_\sigma(w)ps\varrho(\tilde{h}_z), t] + [w\sigma(p)s\varrho(\tilde{h}_z), t] = 0$ and using (13) again, we get

$$[w\sigma(p)s\varrho(\tilde{h}_z), t] = 0. \tag{14}$$

On replacement of w by rw in (14) and making use of (14) again, we obtain

$$[r,t]S\sigma(p)s\varrho(\tilde{h}_z) = \{0\}$$

and by the primeness of S, we have either S is commutative or $\sigma(p)S\varrho(\tilde{h}_z) = \{0\}$. From the second possibility, since $\sigma \neq 0$, we have $\varrho(\tilde{h}_z) = 0$. Hence we conclude that $\sigma(h_z) = \varrho(h_z) = \delta(h_z) = 0$, for all $h_z \in \mathbb{H}(S) \cap Z(S)$. By Lemma 4, we have $\sigma(k_z) = \varrho(k_z) = \delta(k_z) = 0$, for all $k_z \in \mathbb{K}(S) \cap Z(S)$ and by Lemma 5, we have $\sigma(z) = \varrho(z) = \delta(z) = 0$, for all $z \in Z(S)$. Substituting sk_z for s in (8) and using the assumption that $\sigma(k_z) = \varrho(k_z) = \delta(k_z) = 0$, we get

$$[(F_\sigma(w)G_\varrho(s^*))' + F_\sigma(s)G_\varrho(w^*) + D_\delta(ws^*))' + D_\delta(sw^*), t]k_z = 0.$$

and by the primeness, we have

$$[(F_\sigma(w)G_\varrho(s^*))' + F_\sigma(s)G_\varrho(w^*) + D_\delta(ws^*))' + D_\delta(sw^*), t] = 0.$$

As $s + t = 0$ implies $s = t'$ for all $s, t \in S$, therefore from the last identity, we can write

$$[F_\sigma(s)G_\varrho(w^*) + D_\delta(sw^*), t] = [F_\sigma(w)G_\varrho(s^*) + D_\delta(ws^*), t]. \tag{15}$$

In view of the 2-torsion freeness of S, using (15) in (8), and then substituting w^* for w, we have

$$[F_\sigma(s)G_\varrho(w) + D_\delta(sw), t] = 0. \tag{16}$$

Replacing w by wt in (16), we get $[F_\sigma(s)G_\varrho(w)t + F_\sigma(s)w\varrho(t) + D_\delta(sw)t + sw\delta(t), t] = 0$, which further implies by using Lemma 1 that $[F_\sigma(s)G_\varrho(w) + D_\delta(sw), t]t + [F_\sigma(s)w\varrho(t) + sw\delta(t), t] = 0$ and using (16) again, we obtain

$$[F_\sigma(s)w\varrho(t) + sw\delta(t), t] = 0. \tag{17}$$

In (17) writing sp in place of s, we find

$$[F_\sigma(s)pw\varrho(t) + s\sigma(p)w\varrho(t) + spw\delta(t), t] = 0. \tag{18}$$

In (17) replacing w by pw, we get

$$[F_\sigma(s)pw\varrho(t) + spw\delta(t), t] = 0. \tag{19}$$

Using (19) in (18), we get

$$[s\sigma(p)w\varrho(t), t] = 0. \tag{20}$$

From (20), we have $s\sigma(p)w\varrho(t)t + t's\sigma(p)w\varrho(t) = 0$ and by Lemma 1, we can further write $s\sigma(p)w\varrho(t)t = ts\sigma(p)w\varrho(t) = 0$. In (20), replacing s by sr, we get $[sr\sigma(p)w\varrho(t), t] = 0$, and by Lemma 1, we can write $[s,t]r\sigma(p)w\varrho(t) = 0$, and therefore

$$[s,t]S\sigma(p)w\varrho(t) = \{0\}.$$

We consider the subsets $S_1 = \{t \in S : [s,t] = 0, \text{ for all } s \in S\}$ and $S_2 = \{t \in S : \sigma(p)S\varrho(t) = \{0\}, \text{ for all } p \in S\}$. We see that $S = S_1 \cup S_2$. Our claim is that either $S = S_1$ or $S = S_2$. For this we prove that either $S_1 \subseteq S_2$ or $S_2 \subseteq S_1$. On the contrary, let $t_1 \in S_1 \setminus S_2$ and $t_2 \in S_2 \setminus S_1$. Then $t_1 + t_2 \in S_1 + S_2 \subseteq S_1 \cup S_2 = S$. If $t_1 + t_2 \in S_1$, then $0 = [r, t_1 + t_2] = [r, t_1] + [r, t_2] = [r, t_2]$, which implies $t_2 \in S_1$, a contradiction. On the other hand if $t_1 + t_2 \in S_2$, then $\{0\} = \sigma(p)S\varrho(t_1 + t_2) = \sigma(p)S\varrho(t_1) + \sigma(p)S\varrho(t_2) = \sigma(p)S\varrho(t_1)$, therefore $t_1 \in S_2$, a contradiction. Hence we have either $S_1 \subseteq S_2$ or $S_2 \subseteq S_1$ and therefore we respectively have $S_1 = S$ or $S_2 = S$. Firstly if $S_1 = S$, then S is commutative. Secondly if $S_2 = S$, then $\sigma(p)S\varrho(t) = \{0\}$ for all $p, t \in S$, then by the primeness of S, we have either $\sigma = 0$ or $\varrho = 0$ which contradicts the hypothesis. This completes the proof. □

From the proof of Theorem 1, one can obtain the following result.

Theorem 2. Let F_σ, G_ϱ and D_δ be generalized derivations respectively associated with the nonzero derivations σ, ϱ and a derivation δ of a 2-torsion free prime semiring S with involution $*$ of second kind. If

$$[F_\sigma(w)G_\varrho(s) + D_\delta(ws), t] = 0$$

for all $t, w, s \in S$, then S is commutative.

If $D_\delta = 0$, then we can obtain the following result from Theorem 1.

Corollary 1. For a 2-torsion free prime semiring S with involution $*$ of second kind, let F_σ and G_ϱ be generalized derivations respectively associated with derivations σ and ϱ such that $\sigma \neq 0$ and $\varrho \neq 0$. If

$$[F_\sigma(w)G_\varrho(w^*), t] = 0,$$

for all $t, w \in S$, then S is commutative.

A generalized version of Theorem 2.4 of [25] is given in the following theorem.

Theorem 3. Let S be a 2-torsion free prime semiring with involution $*$ of second kind. Let F_σ be a nonzero generalized derivation associated with a derivation σ that satisfies one of the statements below:

1. $F_\sigma[w, w^*] + [(w^*), \sigma(w)] = 0$,
2. $[F_\sigma(w), w^*] + \sigma[(w^*), w] = 0$

for all $w \in S$. Then S is commutative.

Proof. 1. The hypothesis states that

$$F_\sigma[w, w^*] + [(w^*), \sigma(w)] = 0, \tag{21}$$

for all $w \in S$. Linearizing (21) and using (21) again, we get

$$F_\sigma[w, s^*] + F_\sigma[s, w^*] + [(w^*), \sigma(s)] + [(s^*), \sigma(w)] = 0. \tag{22}$$

If $\sigma = 0$, then from (22) and then replacing s by s^*, we have

$$F_\sigma[w,s] + F_\sigma[s^*,w^*] = 0. \tag{23}$$

Writing sk_z for s in (23), we obtain $(F_\sigma[w,s] + F_\sigma[s^*,w^*]')Sk_z = \{0\}$ and since S is prime, we have $F_\sigma[w,s] + F_\sigma[s^*,w^*]' = 0$ and since $u + v = 0$ implies $u = v'$ for all $u, v \in S$, therefore we can write

$$F_\sigma[w,s] = F_\sigma[s^*,w^*]. \tag{24}$$

Using (24) in (23) and then using 2-torsion freeness of S, we get

$$F_\sigma[w,s] = 0. \tag{25}$$

In (25) substituting wp for w and using Lemma 1, we get $F_\sigma([w,s]p + w[p,s]) = 0$, which further implies that $F_\sigma([w,s])p + [w,s]\sigma(p) + F_\sigma(w)[p,s] + w\sigma[p,s] = 0$ and using (25), we obtain

$$F_\sigma(w)[p,s] = 0. \tag{26}$$

In (26) replacing p by rp and using (26) again, we obtain $F_\sigma(w)S[p,s] = \{0\}$. As $F_\sigma \neq 0$, by the primeness of S is commutative.

We now consider the case, when $\sigma \neq 0$. In (22) for each $h_z \in \mathbb{H}(S) \cap Z(S)$, replacing s by sh_z, we get

$$\begin{aligned}(F_\sigma[w,s^*] + F_\sigma[s,w^*] + [(w^*),\sigma(s)] + [(s^*),\sigma(w)])h_z \\ + [w,s^*]\sigma(h_z) + [s,w^*]\sigma(h_z) + [(w^*),s\sigma(h_z)] = 0\end{aligned}$$

and using (22) again, we get

$$[w,s^*]\sigma(h_z) + [s,w^*]\sigma(h_z) + [w^*,s\sigma(h_z)] = 0. \tag{27}$$

Replacing s by sk_z in (27), we obtain

$$([w,s^*]'\sigma(h_z) + [s,w^*]\sigma(h_z) + [w^*,sd(h_z)])Sk_z = \{0\}.$$

Due to the primeness of S, we $[w,s^*]'\sigma(h_z) + [s,w^*]\sigma(h_z) + [w^*,s\sigma(h_z)] = 0$ and using Lemma 1, we further get

$$[s,w^*]\sigma(h_z) + [w^*,s\sigma(h_z)] = [w,s^*]\sigma(h_z). \tag{28}$$

Using (28) in (27) and then using 2-torsion freeness of S, we obtain $[w,s^*]\sigma(h_z) = 0$ and replacing s by s^*, we get

$$[w,s]\sigma(h_z) = 0. \tag{29}$$

In (29) substituting sr for s and again using (29), we obtain $[w,s]S\sigma(h_z) = \{0\}$. By the primeness of S, either S is commutative, or $\sigma(h_z) = 0$, for all $h_z \in \mathbb{H}(S) \cap Z(S)$.

Assume that $\sigma(h_z) = 0$. By Lemma 4, we have $\sigma(k_z) = 0$ for all $k_z \in \mathbb{K}(S) \cap Z(S)$. For each $k_z \in \mathbb{K}(S) \cap Z(S)$, replacing s by sk_z in (22) and using the assumption that $\sigma(k_z) = 0$, we obtain

$$(F_\sigma[w,s^*]' + F_\sigma[s,w^*] + [(w^*),\sigma(s)] + [(s^*),\sigma(w)]')Sk_z = \{0\}.$$

Due to the primeness of S, we obtain

$$F_\sigma[w,s^*]' + F_\sigma[s,w^*] + [(w^*),\sigma(s)] + [(s^*),\sigma(w)]' = 0,$$

and using Lemma 1, we have

$$F_\sigma[w,s^*]+[(s^*),\sigma(w)]=F_\sigma[s,w^*]+[(w^*),\sigma(s)]. \tag{30}$$

Using (30) in (22) and using 2-torsion freeness of S, we get

$$F_\sigma[w,s^*]+[(s^*),\sigma(w)]=0$$

and making substitution of s by s^*, we get

$$F_\sigma[w,s]+[s,\sigma(w)]=0. \tag{31}$$

In (31) replacing w by ws, we obtain

$$F_\sigma([w,s]s)+[s,\sigma(ws)]=0. \tag{32}$$

Using semiring identities from Lemma 1 and rearranging the terms, we have

$$F_\sigma([w,s]s)+[s,\sigma(ws)]=(F_\sigma[w,s]+[s,\sigma(w)])s+[w,s]\sigma(s)+[s,w\sigma(s)]$$

and using (31) and rearranging terms, we obtain

$$\begin{aligned}F_\sigma([w,s]s)+[s,\sigma(ws)]&=[w,s]\sigma(s)+[s,w\sigma(s)]\\&=ws\sigma(s)+(s'+s)w\sigma(s)+w\sigma(s)s'.\end{aligned}$$

As S is a semiring, $s+s'\in Z(S)$ and therefore

$$\begin{aligned}F_\sigma([w,s]s)+[s,\sigma(ws)]&=ws\sigma(s)+w\sigma(s)(s'+s)+w\sigma(s)s'\\&=ws\sigma(s)+w\sigma(s)(s'+s+s').\end{aligned}$$

As $s'+s+s'=s'$ and $s+s'+s=s$, therefore

$$F_\sigma([w,s]s)+[s,\sigma(ws)]=ws\sigma(s)+w\sigma(s)s'=w[s,\sigma(s)].$$

Therefore from (32) we can write $w[s,\sigma(s)]=0$ and on replacement of w by $[s,\sigma(s)]w$, it further implies $[s,\sigma(s)]S[s,\sigma(s)]=\{0\}$. Due to the primeness of S the last relation gives $[s,\sigma(s)]=0, \forall s\in S$. Hence by Lemma 6, S is commutative.

2. We have

$$[F_\sigma(w),w^*]+\sigma[(w^*),w]=0, \tag{33}$$

for all $w\in S$. Linearizing (33) and again using (33), we get

$$[F_\sigma(w),s^*]+[F_\sigma(s),w^*]+\sigma[(w^*),s]+\sigma[(s^*),w]=0. \tag{34}$$

If $\sigma=0$, then from (34), we have $[F_\sigma(w),s^*]+[F_\sigma(s),w^*]=0$ and replacing s by s^*, we get

$$[F_\sigma(w),s]+[F_\sigma(s^*),w^*]=0. \tag{35}$$

Writing sk_z for s in (35), we obtain $([F_\sigma(w),s]+[F_\sigma(s^*),w^*]')Sk_z=\{0\}$ and by the primeness, we obtain $[F_\sigma(w),s]+[F_\sigma(s^*),w^*]'=0$. Making use of the last equation, we obtain

$$[F_\sigma(w),s]=[F_\sigma(s^*),w^*]. \tag{36}$$

Using (36) in (35) and then by the 2-torsion freeness of S, we have

$$[F_\sigma(w), s] = 0. \tag{37}$$

In (37) replacing w by wr and using (37) again, we get $F_\sigma(w)[r, s] = 0$ and therefore $F_\sigma(w)S[r, s] = \{0\}$. As $F_\sigma \neq 0$, by the primeness of S, we have $[r, t] = 0$ which implies that S is commutative.

We now consider the case when $\sigma \neq 0$. In (34) substituting s^* for s, we find

$$[F_\sigma(w), s] + [F_\sigma(s^*), w^*] + \sigma[(w^*), s^*] + \sigma[s, w] = 0. \tag{38}$$

In (38) replacing s by sh_z for each $h_z \in \mathbb{H}(S) \cap Z(S)$, we obtain

$$\begin{aligned}[F_\sigma(w), s]h_z + [F_\sigma(s^*), w^*]h_z + [s^*\sigma(h_z), w^*] + \sigma[w^*, s^*]h_z \\ + [w^*, s^*]\sigma(h_z) + \sigma[s, w]h_z + [s, w]\sigma(h_z) = 0.\end{aligned}$$

After the rearrangement of terms, we get

$$\begin{aligned}([F_\sigma(w), s] + [F_\sigma(s^*), w^*] + \sigma[w^*, s^*] + \sigma[s, w])h_z \\ + [s^*\sigma(h_z), w^*] + [w^*, s^*]\sigma(h_z) + [s, w]\sigma(h_z) = 0.\end{aligned}$$

Using (38) again, we obtain

$$[s^*\sigma(h_z), w^*] + [w^*, s^*]\sigma(h_z) + [s, w]\sigma(h_z) = 0. \tag{39}$$

In (39), writing sk_z in place of s, we obtain

$$([s^*\sigma(h_z), w^*]' + [w^*, s^*]'\sigma(h_z) + [s, w]\sigma(h_z))Sk_z = \{0\}.$$

Using primeness, after the rearrangement of terms, we obtain

$$[s^*\sigma(h_z), w^*]' + [w^*, s^*]'\sigma(h_z) + [s, w]\sigma(h_z) = 0.$$

Since $u + v = 0$ implies $u = v'$, for all $u, v \in S$, therefore from the last identity, we can write

$$[s^*\sigma(h_z), w^*] + [w^*, s^*]\sigma(h_z) = [s, w]\sigma(h_z). \tag{40}$$

Using (40) in (39), we obtain $[s, w]\sigma(h_z) = 0$ and therefore $[s, w]Sd(h_z) = \{0\}$. Because of the primeness, we have either S is commutative or $\sigma(h_z) = 0, h_z \in \mathbb{H}(S) \cap Z(S)$. Assume that $\sigma(h_z) = 0$, then by Lemma 4, $\sigma(k_z) = 0$ for all $k_z \in \mathbb{K}(S) \cap Z(S)$. Substituting sk_z for s in (34), and then using the assumption that $\sigma(k_z) = 0$, we obtain

$$([F_\sigma(w), s^*]' + [F_\sigma(s), w^*] + \sigma[(w^*), s] + \sigma[(s^*), w]')Sk_z = \{0\}.$$

Due to the primeness of S, we have

$$[F_\sigma(w), s^*]' + [F_\sigma(s), w^*] + \sigma[(w^*), s] + \sigma[(s^*), w]' = 0,$$

which further implies

$$[F_\sigma(w), s^*] + \sigma[(s^*), w] = [F_\sigma(s), w^*] + \sigma[(w^*), s]. \tag{41}$$

Using (41) in (34) and the 2-torsion freeness of S, we obtain $[F_\sigma(w), s^*] + \sigma[(s^*), w] = 0$ and therefore

$$[F_\sigma(w), s] + \sigma[s, w] = 0. \tag{42}$$

In (42) taking $s = w$, we have $[F_\sigma(w), w] + \sigma[w, w] = 0$. As $[w, w] = [w, w]'$, therefore $[F_\sigma(w), w] + \sigma[w, w]' = 0$ and hence

$$[F_\sigma(w), w] = \sigma[w, w]. \tag{43}$$

Using (42) and (43), we obtain $[F_\sigma(w), w] = 0, \forall w \in S$. By Lemma 7, we conclude that S is commutative. □

From the proof of Theorem 3, one can obtain the following result.

Theorem 4. Let F_σ be nonzero generalized derivation associated with a derivation σ of a 2-torsion free prime semiring S with involution $*$ of second kind. If one of the following holds:
1. $F_\sigma[w,s] + [s,\sigma(w)] = 0$,
2. $[F_\sigma(w),s] + \sigma[(s),w] = 0$

for all $s,w \in S$, then S is commutative.

Following result is an extended form of Theorem 2.6 of [25].

Theorem 5. Let S be a 2-torsion free prime semiring S with involution $*$ of second kind. Then there is no nonzero generalized derivation F_σ satisfying one of the following statements:
1. $F_\sigma(w) \circ w^* + \sigma(w^* \circ w)' = 0$,
2. $F_\sigma(w \circ w^*) + \sigma(w^*) \circ w' = 0$

for all $w \in S$.

Proof. (1). Assume that F_σ is a nonzero generalized derivation satisfying

$$F_\sigma(w) \circ w^* + \sigma(w^* \circ w)' = 0. \tag{44}$$

Linearizing (44) and using (44) again, we get

$$F_\sigma(w) \circ s^* + F_\sigma(s) \circ w^* + \sigma(w^* \circ s)' + \sigma(s^* \circ w)' = 0. \tag{45}$$

Substituting sh_z for s in (45), we get

$$\begin{aligned}(F_\sigma(w) \circ s^*)h_z + (F_\sigma(s) \circ w^*)h_z + ((s\sigma(h_z)) \circ w^*)&\\ + \sigma(w^* \circ s)'h_z + (w^* \circ s)'\sigma(h_z) + \sigma(s^* \circ w)'h_z + (s^* \circ w)'\sigma(h_z) &= 0\end{aligned}$$

and therefore

$$\begin{aligned}((F_\sigma(w) \circ s^*) + (F_\sigma(s) \circ w^*) + \sigma(w^* \circ s)' + \sigma(s^* \circ w)')h_z&\\ + ((s\sigma(h_z)) \circ w^*) + (w^* \circ s)'\sigma(h_z) + (s^* \circ w)'\sigma(h_z) &= 0.\end{aligned}$$

Using (45), we obtain

$$((s\sigma(h_z)) \circ w^*) + (w^* \circ s)'\sigma(h_z) + (s^* \circ w)'\sigma(h_z) = 0.$$

From the last equation, we can write

$$s\sigma(h_z)w^* + w^*s\sigma(h_z) + w^*s'\sigma(h_z) + s'w^*\sigma(h_z) + (s^* \circ w)'\sigma(h_z) = 0$$

and therefore

$$s\sigma(h_z)w^* + w^*(s+s')\sigma(h_z) + s'w^*\sigma(h_z) + (s^* \circ w)'\sigma(h_z) = 0.$$

As $s + s' \in Z(S)$, therefore

$$s\sigma(h_z)w^* + (s' + s + s')w^*\sigma(h_z) + (s^* \circ w)'\sigma(h_z) = 0$$

and since $s' + s + s' = s'$, therefore

$$s\sigma(h_z)w^* + s'w^*\sigma(h_z) + (s^* \circ w)'\sigma(h_z) = 0,$$

which further implies

$$s[\sigma(h_z), w^*] + (s^* \circ w)'\sigma(h_z) = 0. \tag{46}$$

By Lemma 1, from (46), we can write

$$s[\sigma(h_z), w^*] = (s^* \circ w)\sigma(h_z). \tag{47}$$

In (46), replacing s by sk_z, we obtain $(s[\sigma(h_z), w^*] + (s^* \circ w)\sigma(h_z))k_z = 0$, which further implies $(s[\sigma(h_z), w^*] + (s^* \circ w)\sigma(h_z))Sk_z = \{0\}$. In view of Lemma 2, by the primeness of S, we can find

$$s[\sigma(h_z), w^*] + (s^* \circ w)\sigma(h_z) = 0. \tag{48}$$

Using (47) in (48), we obtain $2(s^* \circ w)\sigma(h_z) = 0$, and then by the 2-torsion freeness, we have $(s^* \circ w)\sigma(h_z) = 0$, and replacing s by s^*, we get

$$(s \circ w)\sigma(h_z) = 0. \tag{49}$$

Using Lemma 1 in (49), we can write

$$sw\sigma(h_z) = w's\sigma(h_z). \tag{50}$$

In (49), substituting rw for w, we get

$$srw\sigma(h_z) + rws\sigma(h_z) = 0. \tag{51}$$

Using (50) in (51), we get $srw\sigma(h_z) + r'sw\sigma(h_z) = 0$ and so $[s, r]w\sigma(h_z) = 0$ i.e $[s, r]S\sigma(h_z) = \{0\}$. Due to the primeness, either S is commutative or $\sigma(h_z) = 0$ for all $h_z \in \mathbb{H}(S) \cap Z(S)$.

Assume that $\sigma(h_z) = 0$. By Lemma 4, $\sigma(k_z) = 0$ for all $k_z \in \mathbb{K}(S) \cap Z(S)$. In view of the fact that $\sigma(k_z) = 0$ for all $k_z \in \mathbb{K}(S) \cap Z(S)$, replacing s by $sk_z, k_z \in \mathbb{K}(S) \cap Z(S)$ in (45), we obtain

$$((F_\sigma(w) \circ s^*)' + F_\sigma(s) \circ w^* + \sigma(w^* \circ s)' + \sigma(s^* \circ w))Sk_z = \{0\}$$

and by the primeness, we obtain

$$(F_\sigma(w) \circ s^*)' + F_\sigma(s) \circ w^* + \sigma(w^* \circ s)' + \sigma(s^* \circ w) = 0,$$

which further implies

$$F_\sigma(s) \circ w^* + \sigma(w^* \circ s)' = F_\sigma(w) \circ s^* + \sigma(s^* \circ w)'. \tag{52}$$

As S is 2-torsion free, using (52) in (45), we get $F_\sigma(w) \circ s^* + \sigma(s^* \circ w)' = 0$ and making substitution of s by s^*, we get

$$F_\sigma(w) \circ s + \sigma(s \circ w)' = 0. \tag{53}$$

In (53), replacing s by h_z and using the assumption that $\sigma(h_z) = 0$, we get $(F_\sigma(w) + (\sigma(w))')Sh_z = \{0\}$. Because of the primeness, $F_\sigma(w) + (\sigma(w))' = 0$ for all $w \in S$ which implies that $F = \sigma$ and therefore from (53) becomes

$$\sigma(w) \circ s + \sigma(s \circ w)' = 0. \tag{54}$$

By the definition of Jordan product, we can write

$$\sigma(w) \circ s + \sigma(s \circ w)' = \sigma(w)s + s\sigma(w) + s'\sigma(w) + \sigma(s)w' + w'\sigma(s) + \sigma(w)s'.$$

Rearranging the terms, we have

$$\sigma(w) \circ s + \sigma(s \circ w))' = \sigma(w)(s + s') + (s + s')\sigma(w) + \sigma(s)w)' + w'\sigma(s)).$$

By the definition of MA-semiring $w + w' \in Z(S)$, therefore

$$\sigma(w) \circ s + (\sigma(s \circ w))' = (s + s' + s + s')\sigma(w) + \sigma(s)w' + w'\sigma(s).$$

By the definition of pseudo inverse, we have

$$\sigma(w) \circ s + \sigma(s \circ w))' = (s + s')\sigma(w) + \sigma(s)w' + w'\sigma(s).$$

Therefore (54) becomes

$$(s + s')\sigma(w) + (\sigma(s) \circ w') = 0. \tag{55}$$

As $s + s' = (s + s')'$, therefore from (55), we have $(s + s')'\sigma(w) + (\sigma(s) \circ w') = 0$, which further implies

$$(\sigma(s) \circ w') = (s + s')\sigma(w) \tag{56}$$

In view of the 2-torsion freeness of S, using (56) in (55), we obtain

$$\sigma(s) \circ w = \sigma(s)w + w\sigma(s) = 0. \tag{57}$$

From (57), we can write

$$\sigma(s)w = w'\sigma(s). \tag{58}$$

In (57) substituting wt for w, we obtain $\sigma(s)wt + wt\sigma(s) = 0$ and using (58), we further get $\sigma(s)wt + w'\sigma(s)t = 0$ that is $[\sigma(s), w]t = 0$. From the last relation, we can write $[\sigma(s), w]S[\sigma(s), w] = \{0\}$ and by the primeness of S, we have $[\sigma(s), w] = 0$ and so $\sigma(s) \in Z(S)$. Therefore, from (57) we have $2w\sigma(s) = 0$ and by 2-torsion freeness of S, we have $w\sigma(s) = 0$. Substituting wr for w in the last equation, we get $wS\sigma(s) = \{0\}$. As S is prime, we have $\sigma = 0$ and hence $F_\sigma = 0$, a contradiction. On the other hand if S is commutative, then $\sigma(w) \in Z(S)$. Repeating the same arguments as above we obtain $F_\sigma = 0$, a contradiction. This completes the proof.

(2). Assume that F_σ is a nonzero generalized derivation associated with a derivation σ such that

$$F_\sigma(w \circ w^*) + \sigma(w^*) \circ w' = 0, \tag{59}$$

for all $w \in S$. Linearizing (59) and again using (59), we obtain

$$F_\sigma(w \circ s^*) + F_\sigma(s \circ w^*) + \sigma(w^*) \circ s' + \sigma(s^*) \circ w' = 0. \tag{60}$$

For each $h_z \in \mathbb{H}(S) \cap Z(S)$, substituting sh_z for s in (60), we get

$$F_\sigma(w \circ (sh_z)^*) + F_\sigma((sh_z) \circ w^*) + \sigma(w^*) \circ (sh_z)' + \sigma((sh_z)^*) \circ w' = 0,$$

which implies

$$F_\sigma(w \circ (s^* h_z)) + F_\sigma((sh_z) \circ w^*) + \sigma(w^*) \circ (sh_z)' + \sigma((s^* h_z)) \circ w' = 0$$

and therefore, we can write

$$F_\sigma((w \circ s^*)h_z) + F_\sigma((s \circ w^*)h_z) + \sigma(w^*) \circ s')h_z + \sigma((s^* h_z)) \circ w' = 0.$$

As F_σ is a generalized derivation and σ is a derivation, therefore

$$\begin{aligned}\{F_\sigma(w \circ s^*)h_z + (w \circ s^*)\sigma(h_z)\} + \{F_\sigma(s \circ w^*)h_z + (s \circ w^*)\sigma(h_z)\}\\ + \{(\sigma(w^*) \circ s')h_z\} + \{\sigma(s^*)h_z + s^*\sigma(h_z)\} \circ w' = 0,\end{aligned}$$

which further implies

$$F_\sigma(w\circ s^*)h_z+(w\circ s^*)\sigma(h_z)+F_\sigma(s\circ w^*)h_z+(s\circ w^*)\sigma(h_z)$$
$$+(\sigma(w^*)\circ s')h_z+(\sigma(s^*)\circ w')h_z+(s^*\sigma(h_z))\circ w'=0.$$

Rearranging terms of the last expression, we obtain

$$\{F_\sigma(w\circ s^*)+F_\sigma(s\circ w^*)+(\sigma(w^*)\circ s')+(\sigma(s^*)\circ w')\}h_z$$
$$+(w\circ s^*)\sigma(h_z)+(s\circ w^*)\sigma(h_z)+(s^*\sigma(h_z))\circ w'=0.$$

Using (60), we obtain

$$(w\circ s^*)\sigma(h_z)+(s\circ w^*)\sigma(h_z)+(s^*\sigma(h_z))\circ w'=0. \tag{61}$$

Substituting sk_z for s in (61), we obtain

$$((w\circ s^*)'\sigma(h_z)+(s\circ w^*)\sigma(h_z)+(s^*\sigma(h_z))\circ w)Sk_z=\{0\}.$$

Since S is prime and $\mathbb{K}(S)\cap Z(S)\neq\{0\}$, we get

$$((w\circ s^*)'\sigma(h_z)+(s\circ w^*)\sigma(h_z)+(s^*\sigma(h_z))\circ w)=0$$

and therefore by Lemma 1, we obtain

$$(w\circ s^*)\sigma(h_z)+(s^*\sigma(h_z))\circ w'=(s\circ w^*)\sigma(h_z). \tag{62}$$

Using (62) in (61) and then using 2-torsion freeness, we get $(s\circ w^*)\sigma(h_z)=0$, which further gives

$$(s\circ w)\sigma(h_z)=0. \tag{63}$$

Equation (63) is the same as (49). Therefore, using similar arguments, we obtain that either $\sigma(h_z)=0$ or S is commutative. Firstly assume that $\sigma(h_z)=0$. By Lemma 4, we have $\sigma(k_z)=0$, for all $k_z\in\mathbb{K}(S)\cap Z(S)$ and by Lemma 5, $\sigma(z)=0$ for all $z\in Z(S)$. Substituting sk_z for s in (60), we obtain

$$(F_\sigma(w\circ s^*)'+F_\sigma(s\circ w^*)+\sigma(w^*)\circ s'+\sigma(s^*)\circ w)Sk_z=\{0\}.$$

Due to the primeness of S, we have

$$F_\sigma(w\circ s^*)'+F_\sigma(s\circ w^*)+\sigma(w^*)\circ s'+\sigma(s^*)\circ w=0.$$

By Lemma 1, since $u+v=0$ implies $u=v'$, therefore from the last identity, we can write

$$F_\sigma(w\circ s^*)+\sigma(s^*)\circ w'=F_\sigma(s\circ w^*)+\sigma(w^*)\circ s'. \tag{64}$$

Using (64) in (60) and the 2-torsion freeness of S, we obtain

$$F_\sigma(w\circ s)+\sigma(s)\circ w'=0. \tag{65}$$

In (65), replacing w by $z\in Z$ and using $\sigma(z)=0$, we obtain $(F_\sigma(s)+\sigma(s)')Sz=\{0\}$. Using the primeness of S, we have $F_\sigma(s)+\sigma(s)'=0$ and therefore $F_\sigma(s)=\sigma(s)$ and hence $F_\sigma=\sigma$. Therefore (65) becomes

$$\sigma(w\circ s)+\sigma(s)\circ w'=0. \tag{66}$$

Equation (66) is same as (54), therefore by the similar arguments, we can conclude that $F_\sigma=0$, a contradiction. On the other hand, if S is commutative, then by the similar arguments, we can again conclude that $F_\sigma=0$, a contradiction and this completes the proof. □

From the proof of Theorem 5, one can obtain the following result.

Theorem 6. Let S be a 2-torsion free prime semiring S with involution $*$ of second kind. Then there is no nonzero generalized derivation F_σ satisfying one of the following statements:

1. $F_\sigma(w) \circ s + \sigma(s \circ w)' = 0$,
2. $F_\sigma(w \circ s) + \sigma(s) \circ w' = 0$

for all $w, s \in S$.

Conclusion

The research work presented in this paper provides a motivation to investigate the results for semiprime MA-semirings with similar or different environment.

Acknowledgments

We are thankful to the referee for the insightful suggestions for the improvement of this paper.

Author Contributions

All authors contributed equally to this work.

Conflict of Interest

The authors declare no conflict of interest.

References

1 Kolokoltsov, V.N., & Maslov, V.P. (1997). *Idempotent analysis and its applications.* Springer Dordrecht. https://doi.org/10.1007/978-94-015-8901-7

2 Maslov, V.P., & Samborskiĭ, S.N. (Eds.). (1992). *Idempotent Analysis.* Advances in Soviet Mathematics, Vol. 13. Providence: American Mathematical Society.

3 Bistarelli, S. (2004). *Semirings for soft constraint solving and programming.* Berlin: Springer-Verlag. https://doi.org/10.1007/B95712

4 Eilenberg, S. (1974). *Automata, Languages, and Machines.* New York: Academic Press.

5 Esik, Z., & Kuich, W. (2012). *Modern Automata Theory.* Institut fur Diskrete Mathematik und Geometrie. https://doi.org/10.34726/2481

6 Wirth, L. (2022). *Weighted Automata, Formal Power Series and Weighted Logic.* Wiesbaden: Springer Spektrum. https://doi.org/10.1007/978-3-658-39323-6

7 Barvinok, A.I. (1993). Combinatorial optimization and computations in the ring of polynomials. *DIMACS Technical Report*, 93–103.

8 Golan, J.S. (2013). *Semirings and their applications.* Springer. https://doi.org/10.1007/978-94-015-9333-5

9 Hebisch, U., & Weinert, H.J. (1998). *Semirings: Algebraic theory and applications in computer science.* World Scientific. https://doi.org/10.1142/3903

10 Huang, H., Jiang, X., Peng, C., & Pan, G. (2024). *A new semiring and its cryptographic applications. AIMS Mathematics, 9*(8), 20677–20691. https://doi.org/10.3934/math.20241005

11 Durcheva, M. (2020). *Semirings as building blocks in cryptography.* Cambridge: Cambridge Scholars Publishing.

12 Davidson, K.R. (2025). *Functional analysis and operator algebras.* Cham: Springer. https://doi.org/10.1007/978-3-031-63665-3

13 Strung, K.R. (2020). *An introduction to C*-Algebras and the classification program.* Birkhauser Cham. https://doi.org/10.1007/978-3-030-47465-2

14 Yang, D., & Wen, Y. (2025). *Selected lectures on functional analysis: Spectral theory of operators, Banach algebras, and semigroups of operators.* World Scientific. https://doi.org/10.1142/14097

15 Ali, S., Ashraf, M., De Filippis, V., Oukhtite, L., & Rehman, N.U. (2025). *Differential identities in rings and algebras and their applications.* Boca Raton: CRC Press. https://doi.org/10.1201/9781003504573

16 Elliot, J. (2019). *Rings, modules, and closure operations.* Springer. https://doi.org/10.1007/978-3-030-24401-9

17 Khattar, D., & Agrawal, N. (2023). *Ring Theory.* Cham: Springer. https://doi.org/10.1007/978-3-031-29440-2

18 Javed, M.A., Aslam, M., & Hussain, M. (2012). On condition (A_2) of Bandlet and Petrich for inverse semirings. *International Mathematical Forum, 7*(59), 2903–2914.

19 Ali, L., Khan, Y.A., Mousa, A.A., Khalek, S.A., & Farid, G. (2021). Some differential identities of MA-semirings with involution. *AIMS Mathematics, 6*(3), 2304–2314. https://doi.org/10.3934/math.2021139

20 Ali, L., Aslam, M., Khan, Y.A., & Farid, G. (2020). On generalized derivations of semirings with involution. *Journal of Mechanics of Continua and Mathematical Sciences, 15*(4), 138–152. https://doi.org/10.26782/jmcms.2020.04.00011

21 Dadhwal, M., & Devi, G. (2024). A study on derivations of inverse semirings with involution. *Proyecciones (Antofagasta), 43*(2), 383–400. https://doi.org/10.22199/issn.0717-6279-5627

22 Ahmed, Y., & Dudek, W.A. (2022). On generalised reverse derivations in semirings. *Bulletin of the Iranian Mathematical Society, 48*(2), 895–904. https://doi.org/10.1007/s41980-021-00552-4

23 Ahmed, Y., & Dudek, W.A. (2021). Left Jordan derivations on certain semirings. *Hacettepe Journal of Mathematics and Statistics, 50*(3), 624–633. https://doi.org/10.15672/hujms.491343

24 Sara, S., & Uzma, R. (2024). Some results on dependent elements in semirings. *Discussiones Mathematicae – General Algebra and Applications, 44*(1), 93–99. https://doi.org/10.7151/dmgaa.1445

25 Oukhtite, L., & Ait Zemzami, O. (2021). A study of differential prime rings with involution. *Georgian Mathematical Journal, 28*(1), 133–139. https://doi.org/10.1515/gmj-2019-2061

26 Ali, L., Aslam, M., & Ahmed Khan, Y. (2020). On Jordan ideals of inverse semirings with involution. *Indian Journal of Science and Technology, 13*(04), 430–438. https://doi.org/10.17485/ijst/2020/v13i04/149311

27 Ali, L., Aslam, M., Elamin, M., Ahamd, H.U.M., Yahia, N., & Rathour, L. (2024). On commuting conditions of semirings with involution. *Journal of Applied Mathematics & Informatics, 42*(2), 417–432. https://doi.org/10.14317/jami.2024.417

28 Sara, S. (2019). A study of MA-Semirings. *Doctor's thesis.* Government College University, Lahore.

29 Sara, S. & Aslam, M. (2017). On Jordan mappings of inverse semirings. *Open Mathematics, 15*(1), 1123–1131. https://doi.org/10.1515/math-2017-0088

30 Nadeem, M. (2020). Some criteria of commutativity of semirings. *Journal of Mechanics of Continua and Mathematical Sciences, 15*(5), 49–55. https://doi.org/10.26782/jmcms.2020.05.00004

31 Ali, L., Aslam, M., Khan, Y.A. (2020). Some results on commutativity of MA-semirings. *Indian Journal of Science and Technology, 13*(31), 3198–3203. https://doi.org/10.17485/IJST/v13i31.1022

Author Information[*]

Liaqat Ali (*corresponding author*) — Doctor of Philosophy of Mathematics, Department of Mathematics, Government College University, Kachehry Road, Lahore, 54000, Pakistan; e-mail: *rehmani.pk786@gmail.com*; https://orcid.org/0000-0003-2352-1127

Muhammad Aslam — Doctor of Philosophy of Mathematics, Department of Mathematics, Government College University, Kachehry Road, Lahore, 54000, Pakistan; e-mail: *aslam298@gcu.edu.pk*; https://orcid.org/0000-0001-9429-625X

[*]Authors' names are presented in the following order: first name, middle name (if any), last name.

https://doi.org/10.31489/2026M1/82-86 *Research article*

On Triggers of Order

Ye.R. Baissalov*, J.A. Tussupov

L.N. Gumilyov Eurasian National University, Astana, Kazakhstan
(E-mail: baisalov_ yer@enu.kz, tussupov@mail.ru)

It is shown that the concepts of heir and coheir, introduced by D. Lascar and B. Poizat, play a fundamental role in model theory, particularly in classification theory. The related notions of proper heir and proper coheir are introduced, containing important constructs within themselves. Poizat's lemma on the existence of a proper heir of any non-definable type over a model is presented as an important fact of existence in unstable theories. The concept of an order trigger in a model is then introduced as the skeleton of an algorithmic device that produces ω-evidence of the order property in it. This evidence is constructed using a method very similar to the "back and forth" method of classical model theory, where at each step two possibilities for choosing elements are alternated. As an example of use, a simplified proof of the characterization theorem of the class of unstable theories using these concepts is explained. It is pointed out that applications of more advanced constructions, such as order triggers, can help in solving problems related to the classification of small, countable and minimal models of unstable theories.

Keywords: proper heir, proper coheir, strong heir, unstable theory, non-definable type, order property, trigger of order, ω-evidence.

2020 Mathematics Subject Classification: 03C45.

Introduction

A new philosophy of model theory based on the notions of heir and coheir of a type was presented in [1]; it was shown that for types over models of stable theories these notions and one of non-forking extension coincide. The study of the properties of heirs and coheirs in a general context was continued in [2], which outlined their possible applications.

Definition 1. [1] Let $\mathcal{M}$ be a model, let $\mathcal{N}$ be an elementary extension of it, and let $q(x)$ be a complete type over N and $p(x) = q \upharpoonright M$. Then

(1) q is an *heir* of p if for any formula $\varphi(x, y)$ over $\mathcal{M}$ with $\varphi(x, a) \in q$ there exists $b \in M$ with $\varphi(x, b) \in p$;

(2) q is a *coheir* of p if any formula in q is realized in $\mathcal{M}$.

As can be seen, a coheir does not speak about the relationship between a type and its restriction, but rather about the relationship between a type (as a set of formulas) and a model. Therefore, the concept is often used, especially when specifying the relationship between a type and a model, under the name "finitely satisfiable": instead of "a type is a coheir of its restriction over $\mathcal{M}$", one can say "a type is finitely satisfiable in $\mathcal{M}$" [2].

Coheirs have been well accepted by the model theory community because types that are finitely satisfiable in $\mathcal{M}$ are M-invariant, and they also allow one to construct Morley sequences (see e.g. [3,4]). Heirs are also used (see [5]), but, being more abstract, much less frequently than coheirs.

*Corresponding author. *E-mail: baisalov_ yer@enu.kz*

The research was supported by the Science Committee of the Ministry of Science and Higher Education of the Republic of Kazakhstan (AP19677451, AP26194247).

Received: 10 July 2025; *Accepted:* 29 November 2025.

In the hierarchy of concepts in model theory (and classification theory), what role do heirs and coheirs play?

This note aims to address answer this question.

In the note, we consider infinite models of the first-order language L with equality. As usual, we live in a universe (i.e., a fairly saturated and homogeneous model of L), which we do not designate in any way; all objects (sets, relations, tuples, etc.) are taken from it, models are its elementary submodels. Unless otherwise stated, the satisfiability of formulas and sentences, which is denoted by $\models$, also applies to the universe.

Models are denoted by uppercase calligraphic Latin letters, and their basic sets are denoted by the corresponding ordinary letters. Tuples of elements and variables are denoted by lowercase Latin letters (without dashes); Greek letters φ, ψ, θ are used to denote formulas in L. The space of complete types with variable x over a set A is denoted by $S^x(A)$. Similarly, we write M^x instead of $M^{l(x)}$, where $l(x)$ is the length of the tuple x. The type of a tuple a over a set A is denoted by $tp(a/A)$.

1 *Poizat's Lemma*

In [1], the following properties of a non-definable type p over a model $\mathcal{M}$ were shown:

(1) p can have arbitrarily many distinct heirs over some elementary extension of $\mathcal{M}$;

(2) the number of coheirs of type p over any elementary extension of $\mathcal{M}$ is bounded by some fixed cardinal depending only on the cardinality of $\mathcal{M}$.

Definition 2. Let p, q be complete types over models and let q be an extension of p. Then q is a *proper heir* of p if it is heir of p, but not coheir of p. Likewise, q is a *proper coheir* of p if it is coheir of p, but not heir of p.

From the above properties (1), (2) it almost directly follows that any non-definable type over a model has a proper heir.

In [2, p. 300] B. Poizat gave an elegant proof of the latter fact using the following lemma: any non-definable type over a model $\mathcal{M}$ has a (strong) heir that is not M-invariant; the proof relied on primary model-theoretic methods of definability dating back to E.W. Beth, A. Robinson, L. Svenonius and others.

For completeness, we present here Lemma 1, which is a weakened version of Poizat's lemma and is sufficient for our purposes.

Let $\mathcal{M}$ be a model of the language L and $p \in S^x(M)$. For each formula of the form $\varphi(x, y)$ in the language L, we define the relation R_φ as a subset of M^y:

$$a \in R_\varphi \Leftrightarrow \varphi(x, a) \in p. \tag{1}$$

The enrichment of the model $\mathcal{M}$ to the language $L^* = L \cup \{R_\theta\}_{\theta(x,\cdot)\in L}$ is denoted by $\mathcal{M}^*$.

Definition 3. [2] If $\mathcal{N}^*$ is an elementary extension of $\mathcal{M}^*$, then the relations $\{R_\theta\}_{\theta(x,\cdot)\in L}$ define a type over N that is an heir of p, it is called a *strong heir* of p .

There may also exist non-strong heirs of p over N: if $\mathcal{N}$ is an elementary extension of a model $\mathcal{M}$ that does not admit enrichment to an elementary extension of $\mathcal{M}^*$, then it is clear that there is no strong heir of p over N [2].

Lemma 1. If a formula $\varphi(x, y)$ of a language L is such that R_φ from (1) is not definable in $\mathcal{M}$. Then there exist an elementary extension $\mathcal{N}$ of $\mathcal{M}$, type $q \in S^x(N)$ and tuples $a, b \in N^y$ of the same type over M such that q is a strong heir of p and $\varphi(x, a) \wedge \neg\varphi(x, b) \in q$.

Proof. Let $\varphi(x,y)$ be a formula in L and R_φ is not definable in $\mathcal{M}$. We will first show that the following theory is then consistent:

$$T^* = Th(\mathcal{M}^*) \cup \{\psi(c_1) \leftrightarrow \psi(c_2)\}_{\psi(y)\in L(M)} \cup \{R_\varphi(c_1), \neg R_\varphi(c_2)\},$$

where $\mathcal{M}^*$ is the enrichment of $\mathcal{M}$ described before Definition 3, and c_1, c_2 are new constants.

On the contrary, let us assume that T^* is inconsistent. Then, by compactness, there exists a finite set $\{\psi_1(y), \dots, \psi_n(y)\}$ of $L(M)$-formulas such that

$$\forall yz \left(\bigwedge_{i=1}^{n} (\psi_i(y) \leftrightarrow \psi_i(z)) \to (R_\varphi(y) \leftrightarrow R_\varphi(z)) \right) \in Th(\mathcal{M}^*).$$

This means that there is a Boolean function $f : \{0,1\}^n \to \{0,1\}$ such that if $\varepsilon_1, \dots \varepsilon_n, \varepsilon \in \{0,1\}$ and $f(\varepsilon_1, \dots \varepsilon_n) = \varepsilon$ then

$$\forall y \left(\bigwedge_{i=1}^{n} \psi_i^{\varepsilon_i}(y) \to R_\varphi^{\varepsilon}(y) \right) \in Th(\mathcal{M}^*).$$

Here, as usual, for any formula θ we put $\theta^1 = \theta$ and $\theta^0 = \neg\theta$. Now it is obvious that the $L(M)$-formula

$$\bigvee_{f(\varepsilon_1,\dots\varepsilon_n)=1} \left(\bigwedge_{i=1}^{n} \psi_i^{\varepsilon_i}(y) \right)$$

defines R_φ, which is a contradiction.

So, T^* is consistent. Let $\mathcal{N}^*$ be its model, where the constants c_1 and c_2 are interpreted by tuples a and b, respectively. The set of formulas $\{R_\theta\}_{\theta(x,\cdot)\in L}$ defines a type q over N, which is a strong heir of p. It is now easy to see that $\mathcal{N}$ and q, together with a and b, satisfy the statement of the lemma. □

Since the formula $\varphi(x,a) \wedge \neg\varphi(x,b)$ from Lemma 1 is not realized in $\mathcal{M}$, we obtain the following:

Corollary 1. [2] Any non-definable type over a model has a strong proper heir.

On the other hand, a proper heir of a type over a model may exist even if all types over that model are definable.

Example 1. Over a model $\mathcal{M} = \langle\omega; =, \in\rangle$, where $L = \{=, \in\}$, all types are definable, and there exists a unique non-algebraic 1-type over it, which we denote by ∞. Over any proper elementary extension of a model $\mathcal{M}$, the heir of ∞ will be proper.

2 Trigger of Order

We will start this section with a definition.

Definition 4. A triple $\langle a, b, \varphi(x,y)\rangle$ is called a *trigger of order* in $\mathcal{M}$ if it satisfies the following conditions:

(1) $\models \varphi(a,b)$,
(2) $\varphi(x,b)$ is not realized in $\mathcal{M}$,
(3) $tp(b/Ma)$ is finitely satisfiable in $\mathcal{M}$ (i.e. $tp(b/Ma)$ is a coheir of $tp(b/M)$).

In this definition, the word "order" comes from the order property (OP) defined in [6] (see also the theorem below).

Obviously, a proper heir of a type over a model $\mathcal{M}$ provides us with a trigger of order in $\mathcal{M}$: take as a the realization of this proper heir and its formula $\varphi(x,b)$, which is not realized in $\mathcal{M}$. Likewise, any proper coheir produces a trigger of order.

Theorem 1. If there is a trigger of order in a model, then the model contains ω-evidence of the order property.

Proof. Let $\langle a, b, \varphi(x, y)\rangle$ be an order trigger in $\mathcal{M}$ as in Definition 4. By condition (2) of Definition 4 it follows that for each $m \in M^x$ we have $\neg\varphi(m, y) \in r := tp(b/Ma)$.

Let $B := \varphi(a, \mathcal{M})$. By induction on $i \in \omega$ we determine $a_i \in M^x$ and $b_i \in B$. We choose an arbitrary $a_0 \in M^x$. Since $\neg\varphi(a_0, y) \in r$, by the property (3) of Definition 4 one can find $b_0 \in B$ such that $\models \neg\varphi(a_0, b_0)$.

Let us assume that the elements a_i, b_i are defined for all $i < k$. We choose $a_k \in M^x$ realizing $\bigwedge_{i<k} \varphi(x, b_i)$ (note that the last formula is realized by a). Finally, we choose $b_k \in B$ realizing $\bigwedge_{i\leq k} \neg\varphi(a_i, y)$, which is again possible by property (3) of Definition 4.

From the construction it is clear that for any $m, n \in \omega$ the following holds:

$$\models \varphi(a_m, b_n) \Leftrightarrow n < m,$$

which means that the formula $\neg\varphi(x, y)$ has the order property [6]. $\square$

To our knowledge, [7] was the first paper to give an example of using proper heirs to discover structure in a (minimal) model; it inspired us to write [8] and the present note. We also noticed that [9] considered very closely related issues. There are many unsolved problems about small (i.e., countable, minimal, etc.) models of countable unstable theories [7,10]. Theorem 1 provides a method for analyzing the structures of such models.

Let us note one more consequence of Theorem 1. After it, the equivalence of the following properties of unstable theories can be easily proved along the line (a)$\Rightarrow$(b)$\Rightarrow$(c)$\Rightarrow$(d)$\Rightarrow$(e)$\Rightarrow$(a), using only the arithmetic of cardinals and the compactness theorem:

(a) for some formula φ, there are too many φ-types over some small set,

(b) there exists a non-definable φ-type over a model,

(c) there exists a proper heir,

(d) there exists an order trigger,

(e) the theory has the order property.

Conclusion

The fact that heirs and coheirs incorporate combinatorics has been known for a long time (e.g. Morley sequences). In this note we show another confirmation of this fact, as a result of a simplest use of proper heirs. It is clear that more advanced applications of such concepts will yield deeper, more meaningful results. It is also certain that constructions like order triggers can be used to classify small, countable and minimal models of unstable theories.

Acknowledgments

The authors would like to thank ALL participants in the Franco–Kazakh colloquiums held from 1990 to 2025, who served as a constant source of inspiration for our research in model theory. This modest note also emerged as a result of this long-term collaboration.

Author Contributions

All authors contributed equally to this work.

Conflict of Interest

The authors declare no conflict of interest.

References

1 Lascar, D., & Poizat, B. (1979). An introduction to forking. *The Journal of Symbolic Logic, 44*, 330–350. https://doi.org/10.2307/2273127

2 Poizat, B. (1985). *Cours de théorie des modèles [Course in model theory]*. Villeurbanne: Nur al-Mantiq wal-Ma'rifah [in French].

3 Chernikov, A., & Ramsey, N. (2016). On model-theoretic tree properties. *Journal of Mathematical Logic, 16*(2), Article 1650009. https://doi.org/10.1142/S0219061316500094

4 Kaplan, I., & Ramsey, N. (2021). Transitivity of Kim-independence. *Advances in Mathematics, 379*, Article 107573. https://doi.org/10.1016/j.aim.2021.107573

5 Kaplan, I., Ramsey, N., & Shelah, S. (2019). Local character of Kim-independence. *Proceedings of the American Mathematical Society, 147*(4), 1719–1732. https://doi.org/10.1090/proc/14305

6 Shelah, S. (1990). *Classification theory and the number of nonisomorphic models.* Amsterdam; New York: North-Holland.

7 Krupiński, K., Tanović, P., & Wagner, F.O. (2018). Around Podewski's Conjecture. *arXiv: 1201.5709v2 [math.LO]*. https://doi.org/10.48550/arXiv.1201.5709

8 Baissalov, Y., & Tussupov, J. (2022). Formulas over minimal structures. *Materials of the VII Franco–Kazakh Colloquium in Model Theory*, November 14–18, Lyon.

9 Khanaki, K., & Pillay, A. (2018). Remarks on the NIP in a model. *Mathematical Logic Quarterly, 64*(6), 429–434. https://doi.org/10.1002/malq.201700070

10 Tanović, P., & Pillay, A. (2025). The number of countable models of first-order theories. *arXiv: 2508.06854v1 [math.LO]*. https://doi.org/10.48550/arXiv.2508.06854

*Author Information**

Yerzhan Rakhmettollaevich Baissalov (*corresponding author*) — Candidate of Physical and Mathematical Sciences, Associate Professor of Department of Cryptology, L.N. Gumilyov Eurasian National University, 2 Satbaev St., Astana, 10000, Kazakhstan; e-mail: *baisalov_yer@enu.kz*; https://orcid.org/0000-0002-3874-2726

Jamalbek Aliaskarovich Tussupov — Doctor of Physical and Mathematical Sciences, Professor of Department of Information Systems, L.N. Gumilyov Eurasian National University, 2 Satbaev St., Astana, 10000, Kazakhstan; e-mail: *tussupov@mail.ru*; https://orcid.org/0000-0002-9179-0428

*Authors' names are presented in the following order: first name, middle name (if any), last name.

Bulletin of the Karaganda University. Mathematics Series, No. 1(121), 2026, pp. 87–102

https://doi.org/10.31489/2026M1/87-102 *Research article*

Boundedness of the pseudo-differential operators generated by 1D-Dunkl operator

B. Bekbolat[1,2,*], N. Tokmagambetov[2]

[1] *SDU University, Kaskelen, Kazakhstan;*
[2] *Institute of Mathematics and Mathematical Modeling, Almaty, Kazakhstan*
(E-mail: bayan.bekbolat@sdu.edu.kz, tokmagambetov@math.kz)

This article is devoted to the study of pseudo-differential operators generated by Dunkl operators, focusing primarily on their boundedness properties. We establish that, under a set of suitable assumptions on the symbols and the underlying function spaces, these operators are bounded on specific Banach spaces. In addition, we define the composition of pseudo-differential operators generated by Dunkl operators and rigorously prove that this composition also inherits boundedness properties under appropriate conditions. The analysis is carried out using techniques based on the Dunkl transform, which provides a powerful tool for handling operators associated with reflection groups and allows for the derivation of precise estimates. Beyond the theoretical development, we illustrate an application of the obtained results, demonstrating how these boundedness properties can be employed to address complex problems in mathematical physics and harmonic analysis. Overall, the work contributes to a deeper understanding of Dunkl analysis and the structure of pseudo-differential operators in this context. The results presented not only consolidate existing knowledge but also open new perspectives for further investigations in the field, providing a solid foundation for future research on Dunkl operators and their applications in both theoretical and applied analysis.

Keywords: Dunkl analysis, Dunkl operator, Dunkl kernel, Dunkl transform, inverse Dunkl transform, pseudo-differential operators, composition of pseudo-differential operators, boundedness results.

2020 Mathematics Subject Classification: Primary 47G30; Secondary 44A15.

Introduction

Pseudo-differential operators T_a (Definition 3), generated by the Dunkl operator were introduced by A. Dachraoui in 2001 in [1]. In his paper, the author defined two classes of symbols, S_0^m and S^m for $m \in \mathbb{R}$, with $S^m \subset S_0^m$, and introduced Sobolev-type spaces $W_\alpha^{s,p}(\mathbb{R}, d\mu_\alpha)$, where $s \in \mathbb{R}$, $p \in [1, +\infty]$, and $\alpha \geq -1/2$ (definitions are provided below). He proved that the pseudo-differential operator T_a, generated by the Dunkl operator with symbol $a \in S^m$, is continuous from $W_\alpha^{\frac{m}{2},1}(\mathbb{R}, d\mu_\alpha)$ to $W_\alpha^{0,\infty}(\mathbb{R}, d\mu_\alpha)$, and from $W_\alpha^{\frac{m}{2},p}(\mathbb{R}, d\mu_\alpha)$ to $W_\alpha^{0,p}(\mathbb{R}, d\mu_\alpha)$ for $p \geq 1$.

Definition 1. Let m be a real number. The function $a : \mathbb{R} \times \mathbb{C} \to \mathbb{C}$ is called a symbol in the class S_0^m, if it satisfies

- for a fixed x in $\mathbb{R}$, the function $\lambda \mapsto a(x, \lambda)$ is a smooth function on $\mathbb{R}$;
- for a fixed λ in $\mathbb{R}$, the function $x \mapsto a(x, \lambda)$ is a smooth function on $\mathbb{R}$;
- for all $k, n \in \mathbb{N}$, there exists $C_{k,n,m} > 0$, such that

$$\left| \frac{\partial^k}{\partial x^k} \frac{\partial^n}{\partial \lambda^n} a(x, \lambda) \right| \leq C_{k,n,m} (1 + |\lambda|^2)^{\frac{m-n}{2}},$$

for all $x \in \mathbb{R}$ and $\lambda \in \mathbb{R}$.

*Corresponding author. *E-mail: bekbolat@math.kz*

This research was funded by the Committee of Science of the Ministry of Science and Higher Education of the Republic of Kazakhstan (Grant No. AP23487589).

Received: 12 November 2024; *Accepted:* 29 December 2025.

Definition 2. Let m be a real number. The function $a : \mathbb{R} \times \mathbb{C} \to \mathbb{C}$ is called a symbol in the class S^m, if it satisfies

- for a fixed x in $\mathbb{R}$, the function $\lambda \mapsto a(x, \lambda)$ is a smooth function on $\mathbb{R}$;
- for a fixed λ in $\mathbb{R}$, the function $x \mapsto a(x, \lambda)$ is a smooth function on $\mathbb{R}$;
- for all $k, \ell, n \in \mathbb{N}$, there exists $C_{k,\ell,n,m} > 0$, such that

$$\left|(1 + |x|^2)^{\ell} \frac{\partial^k}{\partial x^k} \frac{\partial^n}{\partial \lambda^n} a(x, \lambda)\right| \leq C_{k,\ell,n,m}(1 + |\lambda|^2)^{\frac{m-n}{2}},$$

for all $x \in \mathbb{R}$ and $\lambda \in \mathbb{R}$.

Definition 3. Let $a \in S_0^m$ and $\alpha \geq -1/2$. The pseudo-differential operator associated with a symbol a is defined on $\mathcal{S}(\mathbb{R})$ by

$$T_a f(x) = \int_{\mathbb{R}} E_\alpha(x, \lambda) a(x, \lambda) \mathcal{F}_\alpha[f](\lambda) d\mu_\alpha(\lambda),$$

where E_α is the Dunkl kernel defined by

$$E_\alpha(x, \lambda) = j_\alpha(x\lambda) + i\frac{x\lambda}{2(\alpha + 1)} j_{\alpha+1}(x\lambda), \tag{1}$$

j_α is the normalized Bessel function of the first kind, $\mathcal{F}_\alpha[f]$ is the Dunkl transform given by

$$\mathcal{F}_\alpha[f](\lambda) = \int_{\mathbb{R}} E_\alpha(-x, \lambda) f(x) d\mu_\alpha(x), \tag{2}$$

and

$$d\mu_\alpha(x) = \frac{|x|^{2\alpha+1}}{2^{\alpha+1}\Gamma(\alpha + 1)} dx, \tag{3}$$

Γ is a Gamma function.

Definition 4. The space $W_\alpha^{s,p}(\mathbb{R}, d\mu_\alpha)$, where s is real number, and $1 \leq p \leq +\infty$, is defined as the closure of the space of C^∞-functions on $\mathbb{R}$ with compact supports, with respect to the norms

$$\|f\|_{W_\alpha^{s,p}} := \|(1 + \lambda^2)^{s/2} \mathcal{F}_\alpha[f]\|_{p,\alpha}, \quad \text{if} \quad 1 \leq p < +\infty,$$

and

$$\|f\|_{W_\alpha^{s,\infty}} := \sup_{\lambda \in \mathbb{R}} (1 + \lambda^2)^{s/2} |\mathcal{F}_\alpha[f](\lambda)| \quad \text{if} \quad p = +\infty,$$

where

$$\|f\|_{p,\alpha} = \sqrt[p]{\int_{\mathbb{R}} |f(x)|^p d\mu_\alpha(x)}.$$

Then L^2 and L^p-boundedness of the pseudo-differential operators T_a associated with the Dunkl operators were studied by the authors C. Abdelkefi, B. Amri, and M. Sifi [2] for classes of symbols $S_{1,0}^0$, or simply S^0, which contains symbols with property

$$\left|\partial_\lambda^n \partial_x^k a(x, \lambda)\right| \leq \frac{C_{k,n}}{(1 + |\lambda|)^n},$$

for all $k, n \in \mathbb{N}$ and $x, \lambda \in \mathbb{R}$.

After, B. Amri, S. Mustapha, and M. Sifi [3] have extended L^2-theorem of Calderón–Vaillancourt to the pseudo-differential operator T_a associated with the Dunkl operator on $\mathbb{R}$.

Theorem 1 (Calderón–Vaillancourt). Assume that $0 \leq \rho < 1$ and $a \in S^0_{\rho,\rho}$, which is $a \in C^\infty(\mathbb{R} \times \mathbb{R})$ and satisfies

$$\left|\partial_x^k \partial_\lambda^n a(x,\lambda)\right| \leq C_{n,k}(1+|\lambda|)^{\rho(k-n)},$$

for all $n, k \in \mathbb{N}$ and all $x, \lambda \in \mathbb{R}$. Then T_a can be extended to a bounded operator on $L^2(\mathbb{R}, d\mu_\alpha)$.

In [3], the L^p-boundedness of the operator T_a with symbols in $S^0_{1,\delta}$, $0 \leq \delta < 1$, was established. A symbol a is said to belong to the class $S^0_{1,\delta}$, $0 \leq \delta < 1$, if $a \in C^\infty(\mathbb{R} \times \mathbb{R})$ and satisfies

$$\left|\partial_x^k \partial_\lambda^n a(x,\lambda)\right| \leq \frac{C_{n,k}}{(1+|\lambda|)^{n-\delta k}},$$

for all $n, k \in \mathbb{N}$ and all $x, \lambda \in \mathbb{R}$.

Theorem 2. Let $a \in S^0_{1,\delta}$, $0 \leq \delta < 1$. Then T_a can be extended to a bounded operator from $L^p(\mathbb{R}, d\mu_\alpha)$ into itself for all $1 < p < +\infty$.

In this paper, we establish several boundedness results for pseudo-differential operators generated by the Dunkl operators on the space $L(\mathbb{R}, d\mu_\alpha)$ (Definition 6) under certain assumptions. The techniques used in this paper are adapted from [4] and [5].

For a comprehensive overview of recent developments in this area, the reader is referred to the work [6].

1 Preliminaries

In this section, we recall some basic definitions from Dunkl analysis. The Dunkl operator

$$D_\alpha : C^1(\mathbb{R}) \to C(\mathbb{R}), \quad \alpha \geq -\frac{1}{2},$$

associated with the reflection group $\mathbb{Z}_2$ on $\mathbb{R}$, is defined by

$$D_\alpha f(x) := \frac{d}{dx} f(x) + \left(\alpha + \frac{1}{2}\right) \frac{f(x) - f(-x)}{x}.$$

Following the definition of the Dunkl operator, we note that this operator was firstly introduced by C.F. Dunkl [7].

Note that if $\alpha = -1/2$, then the Dunkl operator D_α is a first order differential operator and operator is well defined on other important function spaces, some of them listed below.

Lemma 1. [8, Lemma 2.2, p. 6] If $f \in C^m(\mathbb{R})$ with $m \geq 1$, then we have $D_\alpha f \in C^{m-1}(\mathbb{R})$.

Lemma 2. [9, Proposition 3.4, p. 28] The Dunkl operators leaves invariant

$$C^\infty(\mathbb{R}), \quad C_c^\infty(\mathbb{R}) \quad \text{and} \quad \mathcal{S}(\mathbb{R}).$$

As we frequently work with the Schwartz space, let us recall its definition.

Definition 5 (Schwartz space $\mathcal{S}(\mathbb{R})$). The Schwartz space $\mathcal{S}(\mathbb{R})$ is the topological vector space of functions $f : \mathbb{R} \longrightarrow \mathbb{R}$ such that $f \in C^\infty(\mathbb{R})$ and

$$x^k \frac{d^n}{dx^n} f(x) \to 0 \quad \text{as} \quad |x| \to \infty$$

for all $n, k \in \mathbb{N}$.

Let $\alpha \geq -1/2$ and $\lambda \in \mathbb{R}$. The equation

$$D_\alpha f(x) = i\lambda f(x)$$

with initial condition $f(0) = 1$ has a unique solution defined by (1). In the literature the function $E_\alpha(x, \lambda)$ is called the Dunkl kernel. The Dunkl kernel has following properties:

- for all $\lambda \in \mathbb{C}$, the function $x \longmapsto E_\alpha(x, \lambda)$ is a C^∞-function on $\mathbb{R}$;
- for all $x \in \mathbb{R}$, the function $\lambda \longmapsto E_\alpha(x, \lambda)$ is an entire function on $\mathbb{C}$;
- $E_\alpha(x, \lambda) = E_\alpha(\lambda, x)$;
- $E_\alpha(\xi x, \lambda) = E_\alpha(x, \xi\lambda), \quad \xi \in \mathbb{C}$;
- $\overline{E_\alpha(x, \lambda)} = E_\alpha(-x, \overline{\lambda}) = E_\alpha(x, -\overline{\lambda})$;
- $|E_\alpha(x, \lambda)| \leq 1, \quad \lambda \in \mathbb{R}$.

Note that the Dunkl kernel is the exponential function $\exp(ix\lambda)$ when $\alpha = -1/2$. The Dunkl kernel leads to the Dunkl transform. Before introducing the Dunkl transform, we need to consider the L^p space with the $d\mu_\alpha$ measure (defined below).

The space $L^p(\mathbb{R}, d\mu_\alpha)$, $1 \leq p \leq +\infty$, is the space of measurable functions f on $\mathbb{R}$, for which norms defined as

$$\|f\|_{p,\alpha} = \sqrt[p]{\int_{\mathbb{R}} |f(x)|^p d\mu_\alpha(x)} < +\infty, \quad \text{if} \quad 1 \leq p < +\infty,$$

and

$$\|f\|_\infty = \operatorname*{ess\,sup}_{x\in\mathbb{R}} |f(x)| < +\infty, \quad \text{if} \quad p = +\infty,$$

where $d\mu_\alpha$ is defined by (3).

Using property of the Dunkl kernel, we obtain

$$\|\mathcal{F}_\alpha[f]\|_\infty \leq \|f\|_{1,\alpha},$$

where the Dunkl transform $\mathcal{F}_\alpha$ is defined by (2).

The inverse Dunkl transform is defined by

$$\mathcal{F}_\alpha^{-1}[f](\lambda) = \mathcal{F}_\alpha[f](-\lambda) = \int_{\mathbb{R}} E_\alpha(x, \lambda) f(x) d\mu_\alpha(x), \quad \lambda \in \mathbb{R}.$$

If $\alpha = -1/2$, then we obtain the classical Fourier transform and inverse Fourier transform. The Dunkl transform has following important properties:

Theorem 3. (1) (Plancherel theorem) [10, Theorem 4.26, p. 160] The Dunkl transform has a unique extension to an isometric isomorphism of $L^2(\mathbb{R}, d\mu_\alpha)$, i.e.,

$$\|\mathcal{F}_\alpha[f]\|_{2,\alpha} = \|f\|_{2,\alpha},$$

for all $f \in L^2(\mathbb{R}, d\mu_\alpha)$.

(2) [10, Corollary 4.22, p. 159] The Dunkl transform is a homeomorphism of $\mathcal{S}(\mathbb{R})$.

(3) (Inverse Dunkl transform) [10, Theorem 4.20, p. 159] Let $f \in L^1(\mathbb{R}, d\mu_\alpha)$ and $\mathcal{F}_\alpha[f] \in L^1(\mathbb{R}, d\mu_\alpha)$, then we have

$$f(x) = \mathcal{F}_\alpha^{-1}[\mathcal{F}_\alpha[f]](x) \quad \text{a.e.}$$

We have the following product formula for the function $j_\alpha(x\lambda)$ with $\alpha > -\frac{1}{2}$ and parameter $\lambda \in \mathbb{C}$ [11, Formula 1.II.23, p. 13]:

$$j_\alpha(x\lambda) j_\alpha(y\lambda) = \int_0^{+\infty} j_\alpha(z\lambda) k_\alpha(x, y, z) z^{2\alpha+1} dz,$$

for $x, y > 0$, where

$$k_\alpha(x,y,z) = 2^{2\alpha-1}\frac{\Gamma(\alpha+1)}{\Gamma(\alpha+\frac{1}{2})\Gamma(\frac{1}{2})}\frac{\Delta(x,y,z)^{2\alpha-1}}{(xyz)^{2\alpha}} \cdot 1_{[|x-y|,x+y]}(z).$$

Here 1_A is the indicator function of A and

$$\Delta(x,y,z) := \frac{1}{4}\sqrt{(x+y+z)(x+y-z)(x-y+z)(y+z-x)}$$

denotes the area of the triangle with sides $x, y, z > 0$. The function $k_\alpha(x,y,z)$ satisfies the following properties [11, p. 13-14]:

- for all $z > 0$, we have $k_\alpha(x,y,z) \geq 0$;
- for $x, y > 0$, we have
$$\int_0^{+\infty} k_\alpha(x,y,z)z^{2\alpha+1}dz = 1;$$
- for all $x, y, z > 0$, we have
$$k_\alpha(x,y,z) = k_\alpha(y,x,z) \quad \text{and} \quad k_\alpha(x,y,z) = k_\alpha(x,z,y).$$

For our convenience, we fix some notations. For all $x, y, z \in \mathbb{R}$, we put

$$b_{x,y,z} := \begin{cases} \frac{x^2+y^2-z^2}{2xy}, & \text{if} \quad x, y \neq 0, \\ 0, & \text{otherwise}, \end{cases}$$

and

$$\rho(x,y,z) := \frac{1}{2}(1 - b_{x,y,z} + b_{z,x,y} + b_{z,y,x}).$$

Theorem 4. [12, Theorem 2.4, p. 5] (1) Let $\alpha > -\frac{1}{2}$ and $\lambda \in \mathbb{C}$. The Dunkl kernel satisfies the following product formula:

$$E_\alpha(x,\lambda)E_\alpha(y,\lambda) = \int_{\mathbb{R}} E_\alpha(z,\lambda)d\nu_{x,y}(z)$$

for $x, y \in \mathbb{R}$, where

$$d\nu_{x,y}(z) := \begin{cases} W_\alpha(x,y,z)|z|^{2\alpha+1}\, dz, & \text{if} \quad x, y \neq 0, \\ d\delta_x(z), & \text{if} \quad y = 0, \\ d\delta_y(z), & \text{if} \quad x = 0, \end{cases}$$

with kernel

$$W_\alpha(x,y,z) = k_\alpha(|x|,|y|,|z|)\rho(x,y,z).$$

(2) The measures $\nu_{x,y}$ have the following properties:

- $\mathrm{supp}\nu_{x,y} = [-|x|-|y|, -||x|-|y||] \cup [||x|-|y||, |x|+|y|]$ for $x, y \neq 0$;
- $\|\nu_{x,y}\| := \int_{\mathbb{R}} W_\alpha(x,y,z)|z|^{2\alpha+1}dz \leq 4$ for all $x, y \in \mathbb{R}$.

Remark 1. In Theorem 4, δ_x is the Dirac measure. So, we have

- if $y = 0$, then
$$E_\alpha(x,\lambda) = E_\alpha(x,\lambda)E_\alpha(0,\lambda) = \int_{\mathbb{R}} E_\alpha(z,\lambda)d\delta_x(z) = E_\alpha(x,\lambda),$$
- if $x = 0$, then
$$E_\alpha(y,\lambda) = E_\alpha(0,\lambda)E_\alpha(y,\lambda) = \int_{\mathbb{R}} E_\alpha(z,\lambda)d\delta_y(z) = E_\alpha(y,\lambda).$$

Remark 2. Let $x, y \neq 0$. Then from

$$E_\alpha(x,\lambda)E_\alpha(y,\lambda) = \int_{\mathbb{R}} E_\alpha(z,\lambda)W_\alpha(x,y,z)|z|^{2\alpha+1}dz$$
$$= 2^{\alpha+1}\Gamma(\alpha+1)\int_{\mathbb{R}} E_\alpha(z,\lambda)W_\alpha(x,y,z)d\mu_\alpha(z),$$

we obtain

$$W_\alpha(x,y,z) = \frac{1}{2^{\alpha+1}\Gamma(\alpha+1)}\int_{\mathbb{R}} E_\alpha(-z,\lambda)E_\alpha(x,\lambda)E_\alpha(y,\lambda)d\mu_\alpha(\lambda). \tag{4}$$

Lemma 3. Let $x, y, z \in \mathbb{R}$. Then

$$W_\alpha(x,-y,z) = W_\alpha(x,-z,y).$$

Furthermore, we have

$$W_\alpha(x,-y,z)|z|^{2\alpha+1}dzd\mu_\alpha(y) = W_\alpha(x,-z,y)|y|^{2\alpha+1}dyd\mu_\alpha(z).$$

Proof. For any $x, y, z \in \mathbb{R}$ a short calculation gives us the following equalities:

$$b_{x,-y,z} = \frac{x^2+(-y)^2-z^2}{2x(-y)} = -\frac{x^2+y^2-z^2}{2xy},$$

$$b_{z,x,-y} = \frac{z^2+x^2-(-y)^2}{2zx} = \frac{z^2+x^2-y^2}{2zx},$$

$$b_{z,-y,x} = \frac{z^2+(-y)^2-x^2}{2z(-y)} = -\frac{z^2+y^2-x^2}{2zy},$$

and

$$\begin{aligned}\rho(x,-y,z) &= \frac{1}{2}(1-b_{x,-y,z}+b_{z,x,-y}+b_{z,-y,x})\\ &= \frac{1}{2}\left(1+\frac{x^2+y^2-z^2}{2xy}+\frac{z^2+x^2-y^2}{2zx}-\frac{z^2+y^2-x^2}{2zy}\right)\\ &= \frac{1}{2}\left(1+\frac{x^2+z^2-y^2}{2zx}+\frac{y^2+x^2-z^2}{2xy}-\frac{y^2+z^2-x^2}{2zy}\right)\\ &= \frac{1}{2}(1-b_{x,-z,y}+b_{y,x,-z}+b_{y,-z,x})\\ &= \rho(x,-z,y).\end{aligned}$$

Then using property of the function $k_\alpha(x,y,z)$, we obtain

$$\begin{aligned}W_\alpha(x,-y,z) &= k_\alpha(|x|,|-y|,|z|)\rho(x,-y,z)\\ &= k_\alpha(|x|,|-z|,|y|)\rho(x,-z,y)\\ &= W_\alpha(x,-z,y).\end{aligned}$$

Thus, we have

$$W_\alpha(x,-y,z)|z|^{2\alpha+1}dzd\mu_\alpha(y) = W_\alpha(x,-z,y)\frac{|y|^{2\alpha+1}}{2^{\alpha+1}\Gamma(\alpha+1)}dy|z|^{2\alpha+1}dz$$
$$= W_\alpha(x,-z,y)|y|^{2\alpha+1}dyd\mu_\alpha(z).$$

□

For all $x, y \in \mathbb{R}$ and f, continuous function on $\mathbb{R}$, we define

$$\tau_x f(y) := \int_{\mathbb{R}} f(z) d\nu_{x,y}(z).$$

The operators $\tau_x, x \in \mathbb{R}$ are called Dunkl translation operators on real line.

Proposition 1. [13, Proposition 2, p. 20] The operators $\tau_x, x \in \mathbb{R}$ have the following properties:

- for all $x \in \mathbb{R}$ and $f \in L^p(\mathbb{R}, d\mu_\alpha)$, $p \in [1, +\infty]$, we have

$$\|\tau_x f\|_{p,\alpha} \le 4\|f\|_{p,\alpha};$$

- for all $\lambda, x \in \mathbb{R}$ and $f \in L^1(\mathbb{R}, d\mu_\alpha)$, we obtain

$$\mathcal{F}_\alpha[\tau_x f](\lambda) = E_\alpha(x, \lambda)\mathcal{F}_\alpha[f](\lambda).$$

For two continuous functions f and g on $\mathbb{R}$ with compact supports, we define the convolution product $*_\alpha$ by

$$(f *_\alpha g)(x) := \int_{\mathbb{R}} \tau_x f(-y) g(y) d\mu_\alpha(y), \quad x \in \mathbb{R},$$

where $\tau_x, x \in \mathbb{R}$ is the Dunkl translation operators on $\mathbb{R}$.

Remark 3. Note that $*_{-\frac{1}{2}}$ is the standard convolution $*$.

Proposition 2. [13, Proposition 3, p. 21] (i) Let $p, q, r \in [1, \infty]$ and satisfy $\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}$. Then the map $(f, g) \mapsto f *_\alpha g$ can be extended to a continuous map from $L^p(\mathbb{R}, d\mu_\alpha) \times L^q(\mathbb{R}, d\mu_\alpha)$ to $L^r(\mathbb{R}, d\mu_\alpha)$, and

$$\|f *_\alpha g\|_{r,\alpha} \le 4\|f\|_{p,\alpha}\|g\|_{q,\alpha}.$$

(ii) For any $f \in L^1(\mathbb{R}, d\mu_\alpha)$ and $g \in L^2(\mathbb{R}, d\mu_\alpha)$, we have

$$\mathcal{F}_\alpha[f *_\alpha g](\lambda) = \mathcal{F}_\alpha[f](\lambda)\mathcal{F}_\alpha[g](\lambda), \quad \lambda \in \mathbb{R}.$$

Now we define our $L(\mathbb{R}, d\mu_\alpha)$ space, as following:

Definition 6. Let us define the space $L(\mathbb{R}, d\mu_\alpha)$, as following:

$$L(\mathbb{R}, d\mu_\alpha) := \{f \in L^1(\mathbb{R}, d\mu_\alpha) \,:\, \mathcal{F}_\alpha[f] \in L^1(\mathbb{R}, d\mu_\alpha)\}$$

with the norm

$$\|f\|_L := \|\mathcal{F}_\alpha[f]\|_{1,\alpha} = \int_{\mathbb{R}} |\mathcal{F}_\alpha[f](\lambda)| \, d\mu_\alpha(\lambda).$$

Remark 4. The space $L(\mathbb{R}, d\mu_\alpha)$ is a subspace of $L^1(\mathbb{R}, d\mu_\alpha)$ and the norm $\|\cdot\|_L$ is defined equivalently via the Dunkl transform. Additionally, the space $(L(\mathbb{R}, d\mu_\alpha), \|\cdot\|_L)$ is Banach space.

2 *Main results*

In this section, we obtain some boundedness results for pseudo-differential operators and the composition of pseudo-differential operators generated by the Dunkl operator on the space $L(\mathbb{R}, d\mu_\alpha)$.

Assumption 1. We assume the symbol $a \in S^m_{\rho,\delta}(\mathbb{R} \times \mathbb{R})$ is defined as:

$$a(x, \lambda) = \int_{\mathbb{R}} E_\alpha(x, \xi) V(\xi, \lambda) d\mu_\alpha(\xi),$$

where $V(\xi, \lambda)$ is a complex-valued measurable function on $\mathbb{R} \times \mathbb{R}$, such that

$$|V(\xi, \lambda)| \le K(\xi),$$

for all $\xi, \lambda \in \mathbb{R}$, and $K \in L^1(\mathbb{R}, d\mu_\alpha)$ is a continuous function.

Remark 5. The integral (1) exists, because

$$|a(x,\lambda)| \leq \int_{\mathbb{R}} |E_\alpha(x,\xi)V(\xi,\lambda)|d\mu_\alpha(\xi) \leq \int_{\mathbb{R}} |K(\xi)|d\mu_\alpha(\xi) < +\infty.$$

Theorem 5. Let $f \in \mathcal{S}(\mathbb{R})$. Then the pseudo-differential operator

$$T_a f(x) = \int_{\mathbb{R}} E_\alpha(x,\lambda)a(x,\lambda)\mathcal{F}_\alpha[f](\lambda)d\mu_\alpha(\lambda)$$

is a bounded linear operator under Assumption 1 on $L(\mathbb{R}, d\mu_\alpha)$, i.e.,

$$\|T_a f\|_L \leq 4\|K\|_{1,\alpha}\|f\|_L.$$

Remark 6. In the statements of theorems, the operators are formulated as bounded operators on the space $L(\mathbb{R}, d\mu_\alpha)$. However, in the proofs, the corresponding estimates are first established for functions from $\mathcal{S}(\mathbb{R})$. This is sufficient, since the Schwartz space $\mathcal{S}(\mathbb{R})$ is dense in $L(\mathbb{R}, d\mu_\alpha)$, and the operators under consideration are continuous with respect to the $\|\cdot\|_L$ norm. Therefore, by a standard density argument, each operator admits a unique continuous extension from $\mathcal{S}(\mathbb{R})$ to the whole space $L(\mathbb{R}, d\mu_\alpha)$, and all estimates obtained for Schwartz functions remain valid for arbitrary functions in $L(\mathbb{R}, d\mu_\alpha)$.

Proof. Let $f \in \mathcal{S}(\mathbb{R})$. Then by the definition of the pseudo-differential operator we obtain

$$\begin{aligned}
&\int_{\mathbb{R}} E_\alpha(x,\lambda)a(x,\lambda)\mathcal{F}_\alpha[f](\lambda)d\mu_\alpha(\lambda)\\
&= \int_{\mathbb{R}} E_\alpha(x,\lambda)\left(\int_{\mathbb{R}} E_\alpha(x,\xi)V(\xi,\lambda)d\mu_\alpha(\xi)\right)\mathcal{F}_\alpha[f](\lambda)d\mu_\alpha(\lambda)\\
&= \int_{\mathbb{R}}\int_{\mathbb{R}} E_\alpha(x,\lambda)E_\alpha(x,\xi)V(\xi,\lambda)\mathcal{F}_\alpha[f](\lambda)d\mu_\alpha(\xi)d\mu_\alpha(\lambda)\\
&= \int_{\mathbb{R}}\int_{\mathbb{R}}\left(\int_{\mathbb{R}} E_\alpha(x,\eta)d\nu_{\lambda,\xi}(\eta)\right)V(\xi,\lambda)\mathcal{F}_\alpha[f](\lambda)d\mu_\alpha(\xi)d\mu_\alpha(\lambda)\\
&= \int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}} E_\alpha(x,\eta)V(\xi,\lambda)\mathcal{F}_\alpha[f](\lambda)W_\alpha(\lambda,\xi,\eta)|\eta|^{2\alpha+1}d\eta d\mu_\alpha(\xi)d\mu_\alpha(\lambda)\\
&= \int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}} E_\alpha(x,\eta)V(\xi,\lambda)\mathcal{F}_\alpha[f](\lambda)W_\alpha(-\lambda,\eta,\xi)|\xi|^{2\alpha+1}d\xi d\mu_\alpha(\eta)d\mu_\alpha(\lambda)\\
&= \int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}} E_\alpha(x,\eta)V(\xi,\lambda)\mathcal{F}_\alpha[f](\lambda)d\nu_{-\lambda,\eta}(\xi)d\mu_\alpha(\eta)d\mu_\alpha(\lambda)\\
&= \int_{\mathbb{R}} E_\alpha(x,\eta)\left(\int_{\mathbb{R}}\int_{\mathbb{R}} V(\xi,\lambda)\mathcal{F}_\alpha[f](\lambda)d\nu_{-\lambda,\eta}(\xi)d\mu_\alpha(\lambda)\right)d\mu_\alpha(\eta),
\end{aligned}$$

using above assumption and Fubini's theorem. After applying Dunkl transform $\mathcal{F}_\alpha$ to the both sides of the equation, we have

$$\begin{aligned}
\mathcal{F}_\alpha[T_a f](\eta) &= \int_{\mathbb{R}}\int_{\mathbb{R}} V(\xi,\lambda)\mathcal{F}_\alpha[f](\lambda)d\nu_{-\lambda,\eta}(\xi)d\mu_\alpha(\lambda)\\
&= \int_{\mathbb{R}}\int_{\mathbb{R}} V(\xi,\lambda)\mathcal{F}_\alpha[f](\lambda)W_\alpha(-\lambda,\eta,\xi)|\xi|^{2\alpha+1}d\xi d\mu_\alpha(\lambda).
\end{aligned}$$

Hence, taking integral from both sides, we obtain

$$\begin{aligned}
&\int_{\mathbb{R}} |\mathcal{F}_\alpha[T_a f](\eta)| d\mu_\alpha(\eta) \\
&\leq \int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}} |V(\xi,\lambda)\mathcal{F}_\alpha[f](\lambda)W_\alpha(-\lambda,\eta,\xi)||\xi|^{2\alpha+1} d\xi d\mu_\alpha(\lambda) d\mu_\alpha(\eta) \\
&\leq \int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}} K(\xi)|\mathcal{F}_\alpha[f](\lambda)W_\alpha(-\lambda,\eta,\xi)||\xi|^{2\alpha+1} d\xi d\mu_\alpha(\lambda) d\mu_\alpha(\eta) \\
&= \int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}} K(\xi)|\mathcal{F}_\alpha[f](\lambda)W_\alpha(\lambda,\xi,\eta)||\eta|^{2\alpha+1} d\eta d\mu_\alpha(\xi) d\mu_\alpha(\lambda) \\
&\leq 4\int_{\mathbb{R}}\int_{\mathbb{R}} K(\xi)|\mathcal{F}_\alpha[f](\lambda)| d\mu_\alpha(\xi) d\mu_\alpha(\lambda) \\
&\leq 4\|K\|_{1,\alpha} \int_{\mathbb{R}} |\mathcal{F}_\alpha[f](\lambda)| d\mu_\alpha(\lambda).
\end{aligned}$$

This completes proof of the theorem. □

Let $f, g \in \mathcal{S}(\mathbb{R})$. The composition of two pseudo-differential operators

$$T_a g(x) = \int_{\mathbb{R}}\int_{\mathbb{R}} E_\alpha(x,\lambda)E_\alpha(-y,\lambda)a(x,\lambda)g(y) d\mu_\alpha(y) d\mu_\alpha(\lambda)$$

and

$$T_b f(y) = \int_{\mathbb{R}}\int_{\mathbb{R}} E_\alpha(y,\xi)E_\alpha(-z,\xi)b(y,\xi)f(z) d\mu_\alpha(z) d\mu_\alpha(\xi),$$

with the symbols $a(x,\lambda)$ and $b(y,\xi)$ respectively, is

$$\begin{aligned}
T_a(T_b f)(x) &= \int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}} E_\alpha(x,\lambda)E_\alpha(-y,\lambda)a(x,\lambda)E_\alpha(y,\xi)E_\alpha(-z,\xi)b(y,\xi)f(z) \\
&\times d\mu_\alpha(z) d\mu_\alpha(\xi) d\mu_\alpha(y) d\mu_\alpha(\lambda) \\
&= \int_{\mathbb{R}}\int_{\mathbb{R}} E_\alpha(x,\xi)E_\alpha(-z,\xi)c(x,\xi)f(z) d\mu_\alpha(z) d\mu_\alpha(\xi) \\
&= \int_{\mathbb{R}} E_\alpha(x,\xi)c(x,\xi)\mathcal{F}_\alpha[f](\xi) d\mu_\alpha(\xi),
\end{aligned}$$

where

$$c(x,\xi) = \frac{1}{E_\alpha(x,\xi)} \int_{\mathbb{R}}\int_{\mathbb{R}} E_\alpha(x,\lambda)E_\alpha(-y,\lambda)E_\alpha(y,\xi)a(x,\lambda)b(y,\xi) d\mu_\alpha(y) d\mu_\alpha(\lambda).$$

Thus,

$$T_c f(x) = T_a(T_b f)(x) = \int_{\mathbb{R}} E_\alpha(x,\xi)c(x,\xi)\mathcal{F}_\alpha[f](\xi) d\mu_\alpha(\xi)$$

is a pseudo-differential operator with symbol

$$c(x,\xi) = \frac{1}{E_\alpha(x,\xi)} \int_{\mathbb{R}}\int_{\mathbb{R}} E_\alpha(x,\lambda)E_\alpha(-y,\lambda)E_\alpha(y,\xi)a(x,\lambda)b(y,\xi) d\mu_\alpha(y) d\mu_\alpha(\lambda).$$

Now, let us discuss the existence of such an integral under Assumption 1. Let us have

$$a(x,\lambda) = \int_{\mathbb{R}} E_\alpha(x,\eta)V_a(\eta,\lambda) d\mu_\alpha(\eta) \tag{5}$$

and

$$b(y,\xi) = \int_{\mathbb{R}} E_\alpha(y,\sigma) V_b(\sigma,\xi) d\mu_\alpha(\sigma), \tag{6}$$

where $V_a(\eta,\lambda)$ and $V_b(\sigma,\xi)$ are complex-valued measurable functions on $\mathbb{R}\times\mathbb{R}$, such that

$$|V_a(\eta,\lambda)| \le K_a(\eta) \quad \text{and} \quad |V_b(\sigma,\xi)| \le K_b(\sigma)$$

for all $\eta,\lambda,\sigma,\xi \in \mathbb{R}$, and $K_a, K_b \in L^1(\mathbb{R}, d\mu_\alpha)$ are continuous functions. Then by using integral expressions (5) and (6) of $a(x,\lambda)$ and $b(y,\xi)$ respectively, we obtain

$$\begin{aligned}
c(x,\xi) &= \frac{1}{E_\alpha(x,\xi)} \int_{\mathbb{R}}\int_{\mathbb{R}} E_\alpha(x,\lambda)E_\alpha(-y,\lambda)E_\alpha(y,\xi)a(x,\lambda)b(y,\xi)d\mu_\alpha(y)d\mu_\alpha(\lambda) \\
&= \frac{1}{E_\alpha(x,\xi)} \int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}} E_\alpha(x,\lambda)E_\alpha(-y,\lambda)E_\alpha(y,\xi)E_\alpha(x,\eta)E_\alpha(y,\sigma) \\
&\times V_a(\eta,\lambda)V_b(\sigma,\xi)d\mu_\alpha(\sigma)d\mu_\alpha(\eta)d\mu_\alpha(y)d\mu_\alpha(\lambda) \\
&= \frac{1}{E_\alpha(x,\xi)} \int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}} E_\alpha(x,\lambda)\left(\int_{\mathbb{R}} E_\alpha(-y,\lambda)E_\alpha(y,\xi)E_\alpha(y,\sigma)d\mu_\alpha(y)\right) \\
&\times E_\alpha(x,\eta)V_a(\eta,\lambda)V_b(\sigma,\xi)d\mu_\alpha(\sigma)d\mu_\alpha(\eta)d\mu_\alpha(\lambda) \\
&= \frac{1}{E_\alpha(x,\xi)} \int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}} E_\alpha(x,\lambda)W_\alpha(\xi,\sigma,\lambda)E_\alpha(x,\eta)V_a(\eta,\lambda)V_b(\sigma,\xi)\, d\mu_\alpha(\sigma)d\mu_\alpha(\eta)d\mu_\alpha(\lambda).
\end{aligned}$$

After taking absolute value from both sides of the equation, as following:

$$\begin{aligned}
|c(x,\xi)| &\le \int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}} |W_\alpha(\xi,\sigma,\lambda)V_a(\eta,\lambda)V_b(\sigma,\xi)|d\mu_\alpha(\sigma)d\mu_\alpha(\eta)d\mu_\alpha(\lambda) \\
&\le 4\int_{\mathbb{R}}\int_{\mathbb{R}} K_a(\eta)K_b(\sigma)d\mu_\alpha(\sigma)d\mu_\alpha(\eta) \\
&\le 4\|K_a\|_{1,\alpha}\|K_b\|_{1,\alpha},
\end{aligned}$$

we can see that the $c(x,\xi)$ is a bounded function.

Corollary 1. Let T_a and T_b are pseudo-differential operators with symbols a and b, respectively. Then under Assumption 1 their composition is a pseudo-differential operator $T_a \circ T_b$, which is continuous linear map on $\mathcal{S}(\mathbb{R})$.

Corollary 2. Let $f \in \mathcal{S}(\mathbb{R})$. Then under Assumption 1 the composition of pseudo-differential operators T_a and T_b is a bounded linear operator on $L(\mathbb{R}, d\mu_\alpha)$, i.e.,

$$\|T_a(T_b f)\|_L \le \frac{16}{2^{\alpha+1}\Gamma(\alpha+1)}\|K_a\|_{1,\alpha}\|K_b\|_{1,\alpha}\|f\|_L.$$

Proof. Let $f \in \mathcal{S}(\mathbb{R})$. Then we have

$$\begin{aligned}
T_a(T_b f)(x) &= \int_{\mathbb{R}} E_\alpha(x,\lambda)a(x,\lambda)\mathcal{F}_\alpha[T_b f](\lambda)d\mu_\alpha(\lambda) \\
&= \int_{\mathbb{R}} E_\alpha(x,\lambda)\left(\int_{\mathbb{R}} E_\alpha(x,\xi)V_a(\xi,\lambda)d\mu_\alpha(\xi)\right)\mathcal{F}_\alpha[T_b f](\lambda)d\mu_\alpha(\lambda)
\end{aligned}$$

and

$$\begin{aligned}
&\mathcal{F}_\alpha[T_a(T_b f)](\eta) \\
&= \int_{\mathbb{R}} E_\alpha(-x,\eta)T_a(T_b f)(x)d\mu_\alpha(x) \\
&= \int_{\mathbb{R}} E_\alpha(-x,\eta)\left(\int_{\mathbb{R}} E_\alpha(x,\lambda)\left(\int_{\mathbb{R}} E_\alpha(x,\xi)V_a(\xi,\lambda)d\mu_\alpha(\xi)\right)\mathcal{F}_\alpha[T_b f](\lambda)d\mu_\alpha(\lambda)\right)d\mu_\alpha(x)
\end{aligned}$$

$$= \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}} E_\alpha(-x,\eta) E_\alpha(x,\lambda) E_\alpha(x,\xi) V_a(\xi,\lambda) \mathcal{F}_\alpha[T_b f](\lambda) d\mu_\alpha(\xi) d\mu_\alpha(\lambda) d\mu_\alpha(x)$$
$$= \int_{\mathbb{R}} \int_{\mathbb{R}} \left(\int_{\mathbb{R}} E_\alpha(-x,\eta) E_\alpha(x,\lambda) E_\alpha(x,\xi) d\mu_\alpha(x) \right) V_a(\xi,\lambda) \mathcal{F}_\alpha[T_b f](\lambda) d\mu_\alpha(\xi) d\mu_\alpha(\lambda)$$
$$= \frac{1}{2^{\alpha+1}\Gamma(\alpha+1)} \int_{\mathbb{R}} \int_{\mathbb{R}} W_\alpha(\lambda,\xi,\eta) V_a(\xi,\lambda) \mathcal{F}_\alpha[T_b f](\lambda) d\mu_\alpha(\xi) d\mu_\alpha(\lambda),$$

where we have used (4). Then taking absolute value and integrating, we have

$$\int_{\mathbb{R}} |\mathcal{F}_\alpha[T_a(T_b f)](\eta)| \, d\mu_\alpha(\eta)$$
$$\leq \frac{1}{2^{\alpha+1}\Gamma(\alpha+1)} \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}} |W_\alpha(\lambda,\xi,\eta) V_a(\xi,\lambda) \mathcal{F}_\alpha[T_b f](\lambda)| \, d\mu_\alpha(\xi) d\mu_\alpha(\lambda) d\mu_\alpha(\eta)$$

$$\leq \frac{4}{2^{\alpha+1}\Gamma(\alpha+1)} \int_{\mathbb{R}} \int_{\mathbb{R}} K_a(\xi) \, |\mathcal{F}_\alpha[T_b f](\lambda)| \, d\mu_\alpha(\xi) d\mu_\alpha(\lambda)$$
$$\leq \frac{4}{2^{\alpha+1}\Gamma(\alpha+1)} \|K_a\|_{1,\alpha} \int_{\mathbb{R}} |\mathcal{F}_\alpha[T_b f](\lambda)| \, d\mu_\alpha(\lambda)$$
$$\leq \frac{16}{2^{\alpha+1}\Gamma(\alpha+1)} \|K_a\|_{1,\alpha} \|K_b\|_{1,\alpha} \int_{\mathbb{R}} |\mathcal{F}_\alpha[f](\lambda)| \, d\mu_\alpha(\lambda).$$

□

Assumption 2. We assume the symbol $a \in S^m_{\rho,\delta}(\mathbb{R} \times \mathbb{R})$ is defined by

$$a(x,\lambda) = \int_{\mathbb{R}} E_\alpha(x,\xi) V(\xi,\lambda) d\mu_\alpha(\xi)$$

satisfies

$$a(x,\lambda) = \int_{\mathbb{R}} E_\alpha(x,\xi) V_1(\xi) V_2(\lambda) d\mu_\alpha(\xi) = V_2(\lambda) \int_{\mathbb{R}} E_\alpha(x,\xi) V_1(\xi) d\mu_\alpha(\xi),$$

where $V_1 \in L^1(\mathbb{R}, d\mu_\alpha)$ is a continuous function.

Theorem 6. Let $f \in \mathcal{S}(\mathbb{R})$. Then the pseudo-differential operator T_a with symbol $a(x,\lambda)$, which satisfies Assumption 2, has a representation

$$T_a f(x) = 2^{\alpha+1}\Gamma(\alpha+1) \mathcal{F}_\alpha^{-1}(V_1 *_\alpha V_2 \mathcal{F}_\alpha[f])(x)$$

and satisfies following inequality

$$\|T_a f\|_L \leq 2^{\alpha+3}\Gamma(\alpha+1) \|V_1\|_{1,\alpha} \|V_2 \mathcal{F}_\alpha[f]\|_{1,\alpha}.$$

Proof. By using Assumption 2, we have

$$\int_{\mathbb{R}} E_\alpha(x,\lambda) a(x,\lambda) \mathcal{F}_\alpha[f](\lambda) d\mu_\alpha(\lambda)$$
$$= \int_{\mathbb{R}} E_\alpha(x,\lambda) \left(V_2(\lambda) \int_{\mathbb{R}} E_\alpha(x,\xi) V_1(\xi) d\mu_\alpha(\xi) \right) \mathcal{F}_\alpha[f](\lambda) d\mu_\alpha(\lambda)$$
$$= \int_{\mathbb{R}} \int_{\mathbb{R}} E_\alpha(x,\lambda) E_\alpha(x,\xi) V_2(\lambda) V_1(\xi) \mathcal{F}_\alpha[f](\lambda) d\mu_\alpha(\xi) d\mu_\alpha(\lambda)$$

$$= \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}} E_\alpha(x,\eta) V_2(\lambda) V_1(\xi) \mathcal{F}_\alpha[f](\lambda) d\nu_{\lambda,\xi}(\eta) d\mu_\alpha(\xi) d\mu_\alpha(\lambda)$$
$$= \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}} E_\alpha(x,\eta) V_2(\lambda) V_1(\xi) \mathcal{F}_\alpha[f](\lambda) W_\alpha(\lambda,\xi,\eta) |\eta|^{2\alpha+1} d\eta d\mu_\alpha(\xi) d\mu_\alpha(\lambda)$$
$$= \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}} E_\alpha(x,\eta) V_2(\lambda) V_1(\xi) \mathcal{F}_\alpha[f](\lambda) W_\alpha(-\lambda,\eta,\xi) |\xi|^{2\alpha+1} d\xi d\mu_\alpha(\eta) d\mu_\alpha(\lambda)$$

$$= 2^{\alpha+1}\Gamma(\alpha+1) \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}} E_\alpha(x,\eta) V_2(\lambda) V_1(\xi) \mathcal{F}_\alpha[f](\lambda) d\nu_{-\lambda,\eta}(\xi) d\mu_\alpha(\eta) d\mu_\alpha(\lambda)$$
$$= 2^{\alpha+1}\Gamma(\alpha+1) \int_{\mathbb{R}} E_\alpha(x,\eta) \left(\int_{\mathbb{R}} \int_{\mathbb{R}} V_2(\lambda) V_1(\xi) \mathcal{F}_\alpha[f](\lambda) d\nu_{-\lambda,\eta}(\xi) d\mu_\alpha(\lambda) \right) d\mu_\alpha(\eta)$$
$$= 2^{\alpha+1}\Gamma(\alpha+1) \int_{\mathbb{R}} E_\alpha(x,\eta) \left(\int_{\mathbb{R}} \tau_\eta V_1(-\lambda) V_2(\lambda) \mathcal{F}_\alpha[f](\lambda) d\mu_\alpha(\lambda) \right) d\mu_\alpha(\eta)$$
$$= 2^{\alpha+1}\Gamma(\alpha+1) \int_{\mathbb{R}} E_\alpha(x,\eta) (V_1 *_\alpha V_2 \mathcal{F}_\alpha[f])(\eta) d\mu_\alpha(\eta)$$
$$= 2^{\alpha+1}\Gamma(\alpha+1) \mathcal{F}_\alpha^{-1}(V_1 *_\alpha V_2 \mathcal{F}_\alpha[f])(x).$$

Thus, applying the Dunkl transform, we obtain

$$\mathcal{F}_\alpha[T_a f](\eta) = 2^{\alpha+1}\Gamma(\alpha+1)(V_1 *_\alpha V_2 \mathcal{F}_\alpha[f])(\eta). \tag{7}$$

By taking the integral of both sides of the above equation, we are able to calculate

$$\int_{\mathbb{R}} |\mathcal{F}_\alpha[T_a f](\eta)| \, d\mu_\alpha(\eta) = 2^{\alpha+1}\Gamma(\alpha+1) \int_{\mathbb{R}} |(V_1 *_\alpha V_2 \mathcal{F}_\alpha[f])(\eta)| \, d\mu_\alpha(\eta)$$

and

$$\|\mathcal{F}_\alpha[T_a f]\|_{1,\alpha} = 2^{\alpha+1}\Gamma(\alpha+1)\|V_1 *_\alpha V_2 \mathcal{F}_\alpha[f]\|_{1,\alpha} \leq 2^{\alpha+3}\Gamma(\alpha+1)\|V_1\|_{1,\alpha}\|V_2\mathcal{F}_\alpha[f]\|_{1,\alpha},$$

where we have used the Proposition 2. Furthermore, by using the Definition of the Sobolev-type space, it can be written as

$$\|T_a f\|_L \leq 2^{\alpha+3}\Gamma(\alpha+1)\|V_1\|_{1,\alpha}\|V_2\mathcal{F}_\alpha[f]\|_{1,\alpha}.$$

□

Assumption 3. We assume the symbol $a \in S^m_{\rho,\delta}(\mathbb{R}\times\mathbb{R})$ is defined by

$$a(x,\lambda) = \int_{\mathbb{R}} E_\alpha(x,\xi) V(\xi,\lambda) d\mu_\alpha(\xi),$$

and satisfies

$$a(x,\lambda) = \int_{\mathbb{R}} E_\alpha(x,\xi) V_1(\xi) V_2(\lambda) d\mu_\alpha(\xi) = V_2(\lambda) \int_{\mathbb{R}} E_\alpha(x,\xi) V_1(\xi) d\mu_\alpha(\xi),$$

where $V_1 \in L^1(\mathbb{R}, d\mu_\alpha)$ is a continuous function and $V_2(\lambda) = A$ is a constant. So we have

$$a(x,\lambda) = A \int_{\mathbb{R}} E_\alpha(x,\xi) V_1(\xi) d\mu_\alpha(\xi).$$

Theorem 7. Let $f \in \mathcal{S}(\mathbb{R})$. Then the composition of the pseudo-differential operators T_a and T_b with symbols a and b, which satisfy Assumption 3, has a representation

$$T_a(T_b f)(x) = \left(2^{\alpha+1}\Gamma(\alpha+1)\right)^2 A \cdot \mathcal{F}_\alpha^{-1}[V_1 *_\alpha (W_1 *_\alpha B \cdot \mathcal{F}_\alpha[f])](x)$$

and satisfies the following inequality

$$\|T_a(T_b f)\|_L \leq 16 \left(2^{\alpha+1}\Gamma(\alpha+1)\right)^2 AB\|V_1\|_{1,\alpha}\|W_1\|_{1,\alpha}\|f\|_L.$$

Proof. Let $f \in \mathcal{S}(\mathbb{R})$. Then we have

$$\begin{aligned}
&T_a(T_b f)(x)\\
&= \int_{\mathbb{R}} E_\alpha(x,\lambda)a(x,\lambda)\mathcal{F}_\alpha[T_b f](\lambda)d\mu_\alpha(\lambda)\\
&= \int_{\mathbb{R}} E_\alpha(x,\lambda)\left(\int_{\mathbb{R}} E_\alpha(x,\xi)V_a(\xi,\lambda)d\mu_\alpha(\xi)\right)\mathcal{F}_\alpha[T_b f](\lambda)d\mu_\alpha(\lambda)\\
&= A\int_{\mathbb{R}}\int_{\mathbb{R}} E_\alpha(x,\lambda)E_\alpha(x,\xi)V_1(\xi)\mathcal{F}_\alpha[T_b f](\lambda)d\mu_\alpha(\xi)d\mu_\alpha(\lambda)\\
&= A\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}} E_\alpha(x,\eta)V_1(\xi)\mathcal{F}_\alpha[T_b f](\lambda)d\nu_{\lambda,\xi}(\eta)d\mu_\alpha(\xi)d\mu_\alpha(\lambda)\\
&= 2^{\alpha+1}\Gamma(\alpha+1)A\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}} E_\alpha(x,\eta)V_1(\xi)\mathcal{F}_\alpha[T_b f](\lambda)d\nu_{-\lambda,\eta}(\xi)d\mu_\alpha(\eta)d\mu_\alpha(\lambda)\\
&= 2^{\alpha+1}\Gamma(\alpha+1)A\int_{\mathbb{R}} E_\alpha(x,\eta)\left(\int_{\mathbb{R}}\int_{\mathbb{R}} V_1(\xi)\mathcal{F}_\alpha[T_b f](\lambda)d\nu_{-\lambda,\eta}(\xi)d\mu_\alpha(\lambda)\right)d\mu_\alpha(\eta).
\end{aligned}$$

Now, applying the Dunkl transform, we obtain

$$\begin{aligned}
\mathcal{F}_\alpha[T_a(T_b f)](\eta) &= 2^{\alpha+1}\Gamma(\alpha+1)A\int_{\mathbb{R}} \tau_\eta V_1(-\lambda)\mathcal{F}_\alpha[T_b f](\lambda)d\mu_\alpha(\lambda)\\
&= 2^{\alpha+1}\Gamma(\alpha+1)A(V_1 *_\alpha \mathcal{F}_\alpha[T_b f])(\eta).
\end{aligned}$$

Then by using (7), we obtain

$$\mathcal{F}_\alpha[T_a(T_b f)](\eta) = \left(2^{\alpha+1}\Gamma(\alpha+1)\right)^2 A(V_1 *_\alpha (W_1 *_\alpha B\mathcal{F}_\alpha[f]))(\eta),$$

so that

$$\int_{\mathbb{R}} |\mathcal{F}_\alpha[T_a(T_b f)](\eta)|\, d\mu_\alpha(\eta) = \left(2^{\alpha+1}\Gamma(\alpha+1)\right)^2 AB\int_{\mathbb{R}} |(V_1 *_\alpha (W_1 *_\alpha \mathcal{F}_\alpha[f]))(\eta)|\, d\mu_\alpha(\eta).$$

Thus, we have

$$\begin{aligned}
\|T_a(T_b f)\|_L &= \left(2^{\alpha+1}\Gamma(\alpha+1)\right)^2 AB\|V_1 *_\alpha (W_1 *_\alpha \mathcal{F}_\alpha[f])\|_{1,\alpha}\\
&\leq 4\left(2^{\alpha+1}\Gamma(\alpha+1)\right)^2 AB\|V_1\|_{1,\alpha}\|W_1 *_\alpha \mathcal{F}_\alpha[f]\|_{1,\alpha}\\
&\leq 16\left(2^{\alpha+1}\Gamma(\alpha+1)\right)^2 AB\|V_1\|_{1,\alpha}\|W_1\|_{1,\alpha}\|\mathcal{F}_\alpha[f]\|_{1,\alpha}\\
&= 16\left(2^{\alpha+1}\Gamma(\alpha+1)\right)^2 AB\|V_1\|_{1,\alpha}\|W_1\|_{1,\alpha}\|f\|_L.
\end{aligned}$$

This completes proof of the theorem. □

3 Application

In this section, we present an application from the previous section. We also considered other applications of the Dunkl analysis to inverse source problems, as discussed in [14, 15].

Corollary 3. Let

$$a_k(x) = \int_{\mathbb{R}} E_\alpha(x,\xi) V_1^k(\xi) d\mu_\alpha(\xi),$$

where $V_1^k \in L^1(\mathbb{R}, d\mu_\alpha)$ is a continuous function for all k. Then the operator

$$\begin{cases} P_{n,\alpha} = \sum_{k=0}^{n} a_k(x) D_\alpha^k, \\ Dom(P_{n,\alpha}) = \mathcal{S}(\mathbb{R}) \end{cases}$$

is a continuous linear operator from $\mathcal{S}(\mathbb{R})$ to $L(\mathbb{R}, d\mu_\alpha)$. Moreover, we have

$$\|P_{n,\alpha} f\|_L \leq \sum_{k=0}^{n} 2^{\alpha+3} \Gamma(\alpha+1) \|V_1^k\|_{1,\alpha} \|V_2^k \mathcal{F}_\alpha[f]\|_{1,\alpha},$$

where $V_2^k(\lambda) = (i\lambda)^k$.

Remark 7. Here, the functions V_1^k and V_2^k are chosen to satisfy Assumption 2.

Proof. Let $f \in \mathcal{S}(\mathbb{R})$. Then

$$f(x) = \int_{\mathbb{R}} E_\alpha(x,\lambda) \mathcal{F}_\alpha[f](\lambda) d\mu_\alpha(\lambda)$$

and

$$\begin{aligned} P_{n,\alpha} f(x) &= \sum_{k=0}^{n} \int_{\mathbb{R}} a_k(x) D_\alpha^k E_\alpha(x,\lambda) \mathcal{F}_\alpha[f](\lambda) d\mu_\alpha(\lambda) \\ &= \sum_{k=0}^{n} \int_{\mathbb{R}} E_\alpha(x,\lambda) a_k(x) (i\lambda)^k \mathcal{F}_\alpha[f](\lambda) d\mu_\alpha(\lambda). \end{aligned}$$

Hence, symbol of the pseudo-differential operator $P_{n,\alpha}$ expressed by the form

$$a(x,\lambda) = \sum_{k=0}^{n} a_k(x,\lambda) = \sum_{k=0}^{n} a_k(x)(i\lambda)^k = \sum_{k=0}^{n} (i\lambda)^k \int_{\mathbb{R}} E_\alpha(x,\xi) V_1^k(\xi) d\mu_\alpha(\xi).$$

Then by applying Theorem 6, we obtain

$$\|P_{n,\alpha} f\|_L \leq \sum_{k=0}^{n} 2^{\alpha+3} \Gamma(\alpha+1) \|V_1^k\|_{1,\alpha} \|V_2^k \mathcal{F}_\alpha[f]\|_{1,\alpha},$$

where $V_2^k(\lambda) = (i\lambda)^k$. □

Conclusion

In this research paper, our aim is to obtain some boundedness results for pseudo-differential operator generated by the Dunkl operator. We obtain the following main results:

- Let $f \in \mathcal{S}(\mathbb{R})$. Then, under Assumption 1 the pseudo-differential operator

$$T_a f(x) = \int_{\mathbb{R}} E_\alpha(x,\lambda) a(x,\lambda) \mathcal{F}_\alpha[f](\lambda) d\mu_\alpha(\lambda)$$

  is a bounded linear operator on $L(\mathbb{R}, d\mu_\alpha)$.
- Let $f \in \mathcal{S}(\mathbb{R})$. Then, under Assumption 1, the composition of pseudo-differential operators T_a and T_b is a bounded linear operator on $L(\mathbb{R}, d\mu_\alpha)$.

These results are obtained under the assumption that the symbol has an integral representation. Future improvements could focus on obtaining these results without this restriction.

Acknowledgments

This research was funded by the Committee of Science of the Ministry of Science and Higher Education of the Republic of Kazakhstan (Grant No. AP23487589).

The authors would like to thank the anonymous reviewer for the thorough evaluation of the manuscript and for the insightful comments and valuable suggestions that greatly improved the quality of this paper.

Author Contributions

All authors contributed equally to this work.

Conflict of Interest

The authors declare no conflict of interest.

References

1 Dachraoui, A. (2001). Pseudodifferential-difference operators associated with Dunkl operators. *Integral Transforms and Special Functions, 12*(2), 161–178. https://doi.org/10.1080/10652460108819342

2 Abdelkefi, C., Amri, B., & Sifi, M. (2007). Pseudo-differential operator associated with the Dunkl operator. *Differential and Integral Equations, 20*(9), 1035–1051. https://doi.org/10.57262/die/1356039310

3 Amri, B., Mustapha, S., & Sifi, M. (2010). On the boundedness of pseudo-differential operators associated with the Dunkl transform on the real line. *Advances in Pure and Applied Mathematics, 2*(1), 89–107. https://doi.org/10.1515/apam.2010.028

4 Prasad, A., & Singh, M.K. (2019). Composition of Pseudo-Differential Operators Associated with Jacobi Differential Operator. *Proceedings of the National Academy of Sciences, India Section A: Physical Sciences, 89*, 509–516. https://doi.org/10.1007/s40010-018-0484-8

5 Prasad, A., & Singh, M.K. (2015). Pseudo-differential operators associated with the Jacobi differential operator and Fourier-cosine wavelet transform. *Asian-European Journal of Mathematics, 8*(1), Article 1550010, 1–16. https://doi.org/10.1142/S1793557115500102

6 Verma, R.K., & Prasad, A. (2025). SG pseudo-differential operators in Dunkl setting. *Georgian Mathematical Journal Open source preview, 32*(3), 501–513. https://doi.org/10.1515/gmj-2024-2072

7 Dunkl, C.F. (1988). Reflection groups and orthogonal polynomials on the sphere. *Mathematische Zeitschrift, 197*(1), 33–60. https://doi.org/10.1007/BF01161629

8 Rösler, M. (2003). Dunkl operators: Theory and applications. In *Orthogonal Polynomials and Special Functions*. Leuven, 93–135.

9 Anker, J.-Ph. (2017). An introduction to Dunkl theory and its analytic aspects. In *Analytic, Algebraic and Geometric Aspects of Differential Equations.* Trends in Mathematics. Cham: Birkhäuser, 3–58.

10 de Jeu, M.F.E. (1993). The Dunkl transform. *Inventiones mathematicae, 113*(1), 147–162. https://doi.org/10.1007/BF01244305

11 Trimèche, K. (2001). *Generalized Harmonic Analysis and Wavelet Packets: An Elementary Treatment of Theory and Applications.* Boca Raton: CRC Press. https://doi.org/10.1201/9781482283174

12 Rösler, M. (1995). Bessel-type signed hypergroups on $\mathbb{R}$. In H. Heyer, A. Mukherjea (Eds.), *Proceedings of the XI Probability Measures on Groups and Related Structures.* Singapore: World Scientific, 292–304.

13 Soltani, F. (2004). L^p-Fourier multipliers for the Dunkl operator on the real line. *Journal of Function Analysis, 209*(1), 16–35. https://doi.org/10.1016/j.jfa.2003.11.009

14 Bekbolat, B., & Tokmagambetov, N. (2025). The heat equation for singular Dunkl–Laplacian operator. *Georgian Mathematical Journal, 32*(5), 789–798. https://doi.org/10.1515/gmj-2025-2019

15 Bekbolat, B., Nalzhupbayeva, G., & Tokmagambetov, N. (2024). The Diffusion Equation Involving the Dunkl–Laplacian Operator in a Punctured Domain. *Fractal and Fractional, 8*(12), Article 696. https://doi.org/10.3390/fractalfract8120696

*Author Information**

Bayan Bekbolat — PhD, Assistant Professor, SDU University, 1/1 Abylai Khan St., 040900, Kaskelen, Kazakhstan; e-mail: *bayan.bekbolat@sdu.edu.kz*; Senior Researcher, Institute of Mathematics and Mathematical Modeling, 28 Shevchenko St., Almaty, 050010, Kazakhstan; e-mail: *bekbolat@math.kz*; https://orcid.org/0000-0003-2347-0630

Niyaz Tokmagambetov — PhD, Leading Researcher, Institute of Mathematics and Mathematical Modeling, 28 Shevchenko St., Almaty, 050010, Kazakhstan; e-mail: *tokmagambetov@math.kz*; https://orcid.org/0000-0001-5725-9740

*Authors' names are presented in the following order: first name, middle name (if any), last name.

Bulletin of the Karaganda University. Mathematics Series, No. 1(121), 2026, pp. 103–109

https://doi.org/10.31489/2026M1/103-109 *Research article*

Criterion for a formula-definable quasivariety

M.I. Bekenov[1], A. Mamyraly[1,*], A. Kassatova[2], A. Kabidenov[1]

[1]*L.N. Gumilyov Eurasian National University, Astana, Kazakhstan;*
[2]*Abai Kazakh National Pedagogical University, Almaty, Kazakhstan*
(E-mail: makhsut.bekenov@gmail.com, aliyamamyraly@gmail.com, kasatova_ aida@mail.ru, kabiden@gmail.com)

In this paper, we study classes of models of a first-order language L with a countable signature σ. For a model A, let $\mathrm{Th}(A)$ denote the set of all sentences of L that are true in A, called the elementary type of A. The cardinality of the set T of all elementary types of the signature σ does not exceed the continuum. The product of elementary types of models A and B is defined by $\mathrm{Th}(A) \cdot \mathrm{Th}(B) = \mathrm{Th}(A \times B)$, where $A \times B$ is the Cartesian product of A and B. Infinite products, ultraproducts, and ultrapowers of elementary types with respect to an ultrafilter D are defined analogously. This yields an algebra $\langle T, \cdot \rangle$, which is a commutative semigroup with identity and zero. A binary absorption (recognition) relation is introduced in this semigroup. An elementary type N absorbs an elementary type M if $N \cdot M = N$. This notion leads to the concept of a formula-definable class of models. Formula-definable classes are closed under ultraproducts as well as finite and infinite direct products; they are idempotently formula-definable and axiomatizable. Varieties and quasivarieties are also considered. All varieties form formula-definable classes of models. Examples of a formula-definable class of models and of a class that is not formula-definable are given. An example of a formula-definable quasivariety that is not a variety is presented. It is shown that not all quasivarieties are formula-definable. Criteria are obtained for a quasivariety to be formula-definable and for a formula-definable class of models to be a quasivariety.

Keywords: model, identities, quasi-identities, variety, quasi-variety, Cartesian product of theories, elementary type, h-quasi-identities, equivalence relation, Boolean algebra.

2020 Mathematics Subject Classification: 03C05, 03C07, 03C10, 03C13, 03C20.

Introduction

Let L be a first-order language with a countable signature σ. For any model A (that is, an algebraic structure of signature σ) of the language L, denote by $\mathrm{Th}(A)$ the set of all sentences (closed formulas) of L that are true in the model A. We call the theory $\mathrm{Th}(A)$ the elementary type of the model A.

The abstract class of all models of a countable signature σ of the language L is partitioned into classes by the relation of elementary equivalence of models (A. Tarski [1]).

Thus we obtain Th_L, the set of all elementary types of the signature σ in the language L. The cardinality of the set Th_L of all elementary types of a countable signature σ of the language L does not exceed 2^ω.

In what follows, let $T \in Th_L$ denote an elementary type of the signature σ of the language L of some model.

If K is a class of models of the signature σ of the language L, then the set of elementary types of all models from K is denoted by $Th_L(K)$. That is, if V is a quasivariety, then $Th_L(V)$ is the set of elementary types of all models of the quasivariety V.

If H is a set of elementary types of signature σ of language L, then K_H is the class of all models of all elementary types from the set H.

*Corresponding author. *E-mail: aliyamamyraly@gmail.com*

This research has been funded by the Science Committee of the Ministry of Science and Higher Education of the Republic of Kazakhstan (AP26194247).

Received: 3 August 2025; *Accepted:* 10 December 2025.

1 *Formula-Definable classes of models*

Definition 1. [2] The product of the elementary type T_1 of a model A and the elementary type T_2 of a model B, both of signature σ of the language L, is defined as

$$T_1 \cdot T_2 = Th(A \times B),$$

where $A \times B$ is the Cartesian product of the models A and B. Similarly, one defines the infinite product $\prod_{i \in I} T_i$, the ultraproduct $\prod_{i \in I} T_i/D$, the ultrapower T^I/D with respect to an ultrafilter D, and the filtered product of elementary types.

The algebra $\langle Th_L, \cdot \rangle$ is a commutative semigroup with identity and zero [2]. In this semigroup we introduce a binary absorption ("recognition") relation.

Definition 2. [2] An elementary type T_2 absorbs an elementary type T_1 if

$$T_1 \cdot T_2 = T_2.$$

An elementary type T is called *idempotent* if $T \cdot T = T$. A model A absorbs a model B when

$$Th(A \times B) = Th(A).$$

A model A is called *idempotent* if

$$Th(A \times A) = Th(A).$$

Theorem 1. [2] Let T_1, T_2, T_3 be elementary types from Th_L. If

$$T_1 \cdot T_2 \cdot T_3 = T_3,$$

then

$$T_1 \cdot T_3 = T_3.$$

Definition 3. [2] A class of models K is called a *formula-definable class of models* if there exists a model A of signature σ such that for any model B of signature σ,

$$B \in K \quad \text{if and only if} \quad Th(A \times B) = Th(A).$$

In this case, the model A is called a *determiner* of the class K. If, in addition, the model A is idempotent, then the class K is called an *idempotently formula-definable class of models.*

In other words, in an idempotently formula-definable class of models, there exists an idempotent model that absorbs ("recognizes") only the models of this class.

Examples:

1. The class of models of a single equivalence relation is formula-definable. A determiner of this class of models is a model with an infinite number of equivalence classes, each of which is infinite [3,4].

2. The class of all ω-stable models, the class of all superstable models, the class of all stable models, and the class of all unstable elementary types are not formula-definable sets of elementary types [2,5,6].

The main point of the subsequent discussion is that one moves from studying the properties of classes of models to studying the properties of the sets of elementary types of these classes. This approach allows us to consider the semigroup $\langle Th_L, \cdot \rangle$ and the properties of its subsemigroups [7,8], as well as to discover some new properties of classes of models using the operation of the direct product of models [9,10].

Theorem 2. [2] A formula-definable class of models is closed under ultraproducts, finite and infinite direct products. Moreover, it is an idempotently formula-definable class of models and an axiomatizable class of models.

2 Formula-Definable Quasivarieties

Formulas of the form

$$P\big(f_1(x_1,\ldots,x_m),\ldots,f_n(x_1,\ldots,x_m)\big), \qquad f(x_1,\ldots,x_m)=g(x_1,\ldots,x_m),$$

where $f, g, f_1, \ldots, f_n$ are terms of signature Ω and $P \in \Omega$, are called *quasi-atomic formulas* of signature Ω in the variables $x_1, \ldots, x_m$.

If

$$A_1(x_1,\ldots,x_m),\ldots,A_{s+1}(x_1,\ldots,x_m)$$

are some quasi-atomic formulas of the signature Ω (including equality) in the variables $x_1, \ldots, x_m$, then formulas of the form

$$(\forall x_1 \ldots x_m)\, A_1(x_1,\ldots,x_m)$$

are called *identities*, and formulas of the form

$$(\forall x_1 \ldots x_m)\big(A_1(x_1,\ldots,x_m) \wedge \cdots \wedge A_s(x_1,\ldots,x_m) \to A_{s+1}(x_1,\ldots,x_m)\big)$$

are called *quasi-identities.*

Every identity

$$(\forall x_1 \ldots x_m)\, A_1(x_1,\ldots,x_m)$$

is equivalent to the quasi-identity

$$(\forall x_1 \ldots x_m)\,\big(x_1 = x_1 \to A_1(x_1,\ldots,x_m)\big).$$

A class K of systems of signature Ω is called a *variety* (respectively, a *quasivariety*) if there exists a set Σ of identities (respectively, quasi-identities) of signature Ω such that K consists precisely of those systems of signature Ω in which all formulas from Σ are true [1]. In this case, the set Σ is called the defining set of identities (respectively, quasi-identities) of the variety (quasivariety) K.

Theorem 3. [1] A class of models V is a quasivariety if and only if it is:

1) closed under ultraproducts;

2) hereditary;

3) multiplicatively closed;

4) contains the trivial model.

Theorem 4. A formula-definable class of models is a quasivariety if and only if it is hereditary and contains the trivial model.

Proof. This follows from Theorem 2 and Theorem 3. □

Thus, the notion of a formula-definable quasivariety is effectively introduced here.

Theorem 5. [2] Every variety is an idempotently formula-definable quasivariety.

The example given above (Example 1) is an idempotently formula-definable quasivariety, but it is not a variety.

Theorem 6. A quasivariety defined by the quasi-identity

$$\forall x\,(\,P_1(x) \to P_2(x)\,)$$

is not a formula-definable quasivariety. However, a quasivariety defined by the quasi-identity

$$\forall x\,(\,x = a \to P(x)\,),$$

where a is a constant in the signature, *and is* a formula-definable quasivariety.

Proof. Let the quasivariety V be defined by the quasi-identity

$$\forall x\,(P_1(x) \to P_2(x)).$$

Then the quasivariety V also contains a model M in which $P_1(x)$ is false for all $x \in M$.

Consider the direct product of all countable models of the quasivariety V until an idempotent model is obtained; such an idempotent model belonging to V exists because V is closed under Cartesian products and because the set of elementary types has bounded cardinality (the language has a countable signature). Denote this model by $N \in V$.

By Theorem 1, the model N absorbs all models of the quasivariety V. However, in the model N, the formula $P_1(x)$ is false for all $x \in N$. Hence N also absorbs a model S in which the quasi-identity

$$\forall x\,(P_1(x) \to P_2(x))$$

is false.

Therefore, V is not a formula-definable quasivariety. □

Now let V be a quasivariety defined by the quasi-identity

$$\forall x\,(x = a \to P(x)),$$

where a is a constant in the signature. By the same construction as in the previous case, we obtain an idempotent model $N \in V$ that absorbs all models of the quasivariety V. It is clear that the model N is a determiner of the quasivariety V.

Definition 4. Let

$$A_1(x_1,\dots,x_m),\dots,A_{s+1}(x_1,\dots,x_m)$$

be quasi-atomic formulas. Formulas of the form

$$\exists x_1\dots x_m\,\big(A_1(x_1,\dots,x_m)\wedge\dots\wedge A_s(x_1,\dots,x_m)\big)\wedge$$

$$(\forall x_1\dots x_m)\,\big(A_1(x_1,\dots,x_m)\wedge\dots\wedge A_s(x_1,\dots,x_m)\to A_{s+1}(x_1,\dots,x_m)\big)$$

are called *h-quasi-identities.*

Thus, an h-quasi-identity can be viewed as a certain restriction of the corresponding quasi-identity.

For example, the quasi-identity

$$\forall x\,(x = a \to P(x))$$

can be written as

$$\exists x\,(x = a)\ \wedge\ \forall x\,(x = a \to P(x)).$$

Similarly, as in Example 1, the quasi-identity

$$\forall x\,\forall y\,\forall z\,(xEy \wedge yEz \to xEz)$$

can be written as the h-quasi-identity

$$\exists x\,(xEx)\ \wedge\ \forall x\,\forall y\,\forall z\,(xEy \wedge yEz \to xEz).$$

Every identity can be represented as an h-quasi-identity.

We now give a criterion for when a quasivariety is a formula-definable quasivariety.

Theorem 7. A quasivariety V is a formula-definable quasivariety if and only if all quasi-identities defining this quasivariety are bounded by the corresponding h-quasi-identities.

Proof. Let the quasivariety V be formula-definable, and suppose that some quasi-identity

$$(\forall x_1 \dots x_m)\big(A_1(x_1, \dots, x_m) \wedge \dots \wedge A_s(x_1, \dots, x_m) \rightarrow A_{s+1}(x_1, \dots, x_m)\big)$$

defining this quasivariety is not bounded by the corresponding h-quasi-identity. Then there exists a model in V in which the premise of this quasi-identity is false for all $x_1, \dots, x_m$.

Consider the Cartesian product of all models of this quasivariety until we obtain an idempotent model that absorbs all the models of V. In this resulting model, the premise is as follows.

$$A_1(x_1, \dots, x_m) \wedge \dots \wedge A_s(x_1, \dots, x_m)$$

is also false for all $x_1, \dots, x_m$.

It is clear that this model will absorb a model in which the quasi-identity

$$(\forall x_1 \dots x_m)\big(A_1(x_1, \dots, x_m) \wedge \dots \wedge A_s(x_1, \dots, x_m) \rightarrow A_{s+1}(x_1, \dots, x_m)\big)$$

is false. Therefore, the quasivariety V is not formula-definable.

Now, suppose that all quasi-identities defining the quasivariety V are bounded by the corresponding h-quasi-identities. Then, by taking the product of all models of this quasivariety, we obtain an idempotent model that absorbs all models of V. It is clear that this model does not absorb any model in which the corresponding quasi-identities are false. Therefore, the quasivariety V is formula-definable. □

By combining different sets of quasi-identities and h-quasi-identities, one can construct various examples of formula-definable and non-formula-definable quasivarieties.

Conclusion

In a number of papers published between 2020 and 2025 by Kazakhstan [11–13] and foreign [14,15] authors, various relations between elementary types have been actively studied. In the present paper, an algebraic structure is defined as a semigroup of elementary types with respect to the direct product. By introducing a binary absorption (recognition) relation, this structure is studied as an algebraic system. As a result, it becomes possible to investigate formula-definable classes of algebraic systems, which turn out to be axiomatizable classes. As noted above, all varieties form formula-definable classes. However, not all quasivarieties are formula-definable. The main result of the paper is a criterion for the formula-definability of a quasivariety.

Acknowledgments

This research has been funded by the Science Committee of the Ministry of Science and Higher Education of the Republic of Kazakhstan (AP26194247).

Author Contributions

All authors participated in the revision of the manuscript and approved the final submission. All authors contributed equally to this work.

Conflict of Interest

The authors declare no conflict of interest.

References

1 Maltsev, A.I. (1970). *Algebraicheskie sistemy [Algebraic systems].* Moscow: Nauka [in Russian].

2 Bekenov, M., Kassatova, A., & Nurakunov, A. (2024). On absorption's formula definable semigroups of complete theories. *Archive for Mathematical Logic, 64*, 107–116. https://doi.org/10.1007/s00153-024-00937-2

3 Weinstein, J.M. (1965). First-order properties preserved by direct product. *Doctor's Thesis.* University of Wisconsin–Madison.

4 Bekenov, M.I. (2018). Properties of Elementary Embeddability in Model Theory. *Journal of Mathematical Sciences, 230,* 10–13. https://doi.org/10.1007/s10958-018-3721-4

5 Wierzejewski, J. (1976). On stability and products. *Fundamenta Mathematicae, 93*, 81–95. https://doi.org/10.4064/fm-93-2-81-95

6 Stepanova, A.A., & Krasitskaya, A.I. (2021). P-stability of some classes S-acts. *Siberian Mathematical Journal, 62*(2), 357–363. https://doi.org/10.1134/S0037446621020142

7 Kulpeshov, B.Sh., Pavlyuk, I.I., & Sudoplatov, S.V. (2025). Pseudo-countably-categorical formulas and theories. *Mathematical Notes, 117*(3), 442–457. https://doi.org/10.1134/S0001434625030095

8 Shvidefski, M.V. (2020). On a class of subsemigroup lattices. *Siberian Mathematical Journal, 61*(5), 1177–1193. https://doi.org/10.33048/smzh.2020.61.517

9 D'Aquino, P., & Macintyre, A. (2024). Products of pseudofinite structures. *arXiv preprint* arXiv:2411.08808 [math.LO]. https://doi.org/10.48550/arXiv.2411.08808

10 Garcia, D., Macpherson, D., & Steinhorn, C. (2015). Pseudofinite structures and simplicity. *Journal of Mathematical Logic, 15*(1), Article 1550002. https://doi.org/10.1142/S0219061315500026

11 Baizhanov, B., Orynbassarov, D., & Verbovskiy, V. (2020). Some remarks on definability of types and conservative extensions. *Kazakh Mathematical Journal, 20*(3), 92–100.

12 Yeshkeyev, A.R., & Ulbrikht, O.I. (2016). JSp-kosemantichnost i JSB svoistvo abelevykh grupp [JSp-cosemanticness and JSB property of Abelian groups]. *Sibirskie elektronnye matematicheskie izvestiia — Siberian Electronic Mathematical Reports, 13*, 861–874 [in Russian]. https://doi.org/10.17377/semi.2016.13.068

13 Markhabatov, N.D. (2022). Approximations of acyclic graphs. *The Bulletin of Irkutsk State University. Series Mathematics, 40*, 104–111. https://doi.org/10.26516/1997-7670.2022.40.104

14 Macpherson, D., & Tent, K. ω-categorical pseudofinite groups. *arXiv preprint*, [math.LO]. https://doi.org/10.48550/arXiv.2403.17684

15 Palchunov, D.E. (2022). Model Theory of Subject Domains. I. *Algebra and Logic, 61*(2), 166–173. https://doi.org/10.1007/s10469-022-09685-9

*Author Information**

Mahsut Iskanderovich Bekenov — Candidate of Physical and Mathematical Sciences, Professor of the Algebra and Geometry Department, L.N. Gumilyov Eurasian National University, 13 Kazhymukan St., Astana, 010008, Kazakhstan; *makhsut.bekenov@gmail.com*; https://orcid.org/0009-0007-4511-5476

Aliya Mamyraly *(corresponding author)* — MSc, PhD student, L.N. Gumilyov Eurasian National University, 13 Kazhymukan St., Astana, 010008, Kazakhstan; e-mail: *aliyamamyraly@gmail.com*; https://orcid.org/0009-0008-1930-1985

*Authors' names are presented in the following order: first name, middle name (if any), last name.

Aida Kassatova — MSc, Head of Digitalization Management, Abai Kazakh National Pedagogical University, 13 Dostyk Ave., Almaty, 050010, Kazakhstan; e-mail: *kasatova_aida@mail.ru*; https://orcid.org/0000-0002-4603-819X

Anuar Kabidenov — PhD, L.N. Gumilyov Eurasian National University, 13 Kazhymukan St., Astana, 010008, Kazakhstan; e-mail: *kabiden@gmail.com*; https://orcid.org/0009-0002-6756-5711

Bulletin of the Karaganda University. Mathematics Series, No. 1(121), 2026, pp. 110–121

https://doi.org/10.31489/2026M1/110-121 *Research article*

Measures and Stability in a Model, revisited

C.M.B.J. D'Elbée[1,2], K.A.C. Gannon[3,*]

[1] *University of the Basque Country, Leioa, Spain;*
[2] *Institute for Logic, Cognition, Language and Information, Donostia-San Sebastián, Spain;*
[3] *Peking University, Beijing International Center for Mathematical Research (BICMR), Beijing, China*
(E-mail: christian.delbee@ehu.eus, kgannon@bicmr.pku.edu.cn)

This article is written in honor of the 8th Kazakh–French Logical Colloquium. We expand on an unpublished research note of the second author. We record some results concerning local Keisler measures with respect to a formula which is *stable in a model*. We prove that in this context, every local Keisler measure on the associated local type space is a weighted sum of (at most countably many) local types. Using this observation, we give an elementary proof of the commutativity of the Morley product in this context. We then give a functional analytic proof that the double limit property lifts to the appropriate evaluation map on pairs of local measures. We conclude with observations regarding the NOP and local Keisler measures in the (properly) stable context. Finally, we provide two proofs that the evaluation map on pairs of local Keisler measures is stable (in continuous logic). The first follows almost immediately from the work of Ben Yaacov and Keisler on the randomization; the other proof follows from the VC theorem.

Keywords: model theory, Keisler measures, stability in a model, stability, Morley product, double limit, randomization, VC theory, Krein-Smulian, functional analysis.

2020 Mathematics Subject Classification: 03C45, 03C66.

Introduction

The Franco–Kazakh connections in mathematical logic date back to the late 1980s with the collaboration between Tolendi Mustafin and Bruno Poizat, which led to the first Soviet–French Colloquium in Model Theory held at Karaganda State University in Kazakhstan in 1990. Since then, the determination to maintain and strengthen these mathematical links has persisted and was rekindled in recent years, first in Lyon in 2022, and then in Astana in 2025. The scope of those connections naturally go far beyond France and Kazakhstan, as they witness the fruitful transfer of knowledge and cross-fertilization of ideas among researchers from Europe, Russia, the United States, and China. The present paper embodies this transversality, and presents work stemming (in part) from the authors' participation in the *8th Kazakh-French Logical Colloquium*. The authors are very grateful to the organizers of that meeting. They hope that their contribution to the Kazakh mathematical library will help strengthen the preexisting intellectual and social relationships between our research groups.

After Ben Yaacov's original article connecting stability theory with some of Grothendieck's functional analytic work [1], the concepts of *stability in a model* and *NIP in a model* were studied by a myriad of researchers in the field. In particular, it was the subject of several intense discussions at the Notre Dame model theory seminar in the Spring of 2017 (e.g., see [2]; in the context of a group [3];

*Corresponding author. *E-mail: kgannon@bicmr.pku.edu.cn*

D'Elbée was fully supported by the Ramon Cajal grant RYC2023-042677-I funded by MICIU/AEI/10.13039/501100011033 and by ESF+. Gannon was supported by the Chateaubriand Fellowship of the Office for Science and Technology of the Embassy of France in the United States; by the Fundamental Research Funds for the Central Universities, Peking University, grant no. 7100604835; by the National Natural Science Fund of China, grant no. 12501001.

Received: 28 July 2025; *Accepted:* 24 December 2025.

the NIP variant [4]; a localized history of the subject in Persian [5]). This research arose from that localized frenzy of activity.

It turns out that fundamental results in local stability theory can be generalized to the context of *stability in a model* – in particular via a reinterpretation of some of Grothendieck's work on functional analysis. From our perspective, it is natural to ask, *What does the theory of Keisler measures look like in this setting?* We show that it also closely resembles the picture in the stable context. First, we show that if $\varphi(x, y)$ is stable in M, then every φ-measure on M (finitely additive probability measure on φ-definable subsets of M^x) is the sum of (at most countably many) weighted φ-types. As a consequence, all φ-measures on M are φ^{opp}-definable. Thus evaluating the Morley product between arbitrary pairs of φ and φ^{opp}-measures on the formula $\varphi(x, y)$ is well-defined. In [1], Ben Yaacov demonstrates that the *fundamental theorem of stability theory* extends to the *stable in a model* context; it then follows from the observations above that the Morley product evaluated at $\varphi(x, y)$ commutes on appropriate pairs of measures. In other words, if $\varphi(x, y)$ is stable in M, $\mu \in \mathfrak{M}_\varphi(M)$, and $\nu \in \mathfrak{M}_{\varphi^{\mathrm{opp}}}(M)$, then

$$(\mu_x \otimes \nu_y)(\varphi(x, y)) = (\nu_y \otimes \mu_x)(\varphi(x, y)).$$

We are also interested in the evaluation map itself. We consider the function $E_\varphi : \mathfrak{M}_\varphi(M) \times \mathfrak{M}_{\varphi^{\mathrm{opp}}}(M) \to [0, 1]$ by

$$E_\varphi(\mu, \nu) = (\mu_x \otimes \nu_y)(\varphi(x, y)).$$

We show that if $\varphi(x, y)$ is stable in M, then E_φ also witnesses the appropriate variant of the *non-order property*. This follows more or less directly from results in functional analysis, i.e., Grothendieck's double limit theorem and the Krein–Smulian theorem. Finally, we consider the context, where $\varphi(x, y)$ is a stable formula (i.e., $\varphi(x, y)$ does not have the k-order property for some fixed k). We prove that the map E_φ is (r, ϵ)-stable for any choice of $r \in (0, 1)$ and $\epsilon > 0$. We remark that this follows implicitly by a result of Ben Yaacov and Keisler, namely the fact that the randomization of a stable formula remains stable ([6, Theorem 5.14]). We exposit why this is true and then provide a different proof which follows from the VC-theorem, mimicking techniques involving measures in NIP theories (i.e., see the proof of [7, Theorem 3.12]). This latter method implies the existence of bounds, yet we leave analysis along these lines open.

1 Preliminaries

Throughout the article, fix a language $\mathcal{L}$ and an $\mathcal{L}$-structure, M. We use the letters $x, y, z, \ldots$ to denote finite tuples of variables. The formula $\varphi(x, y)$ is a partitioned $\mathcal{L}$-formula with *variable* tuple x and *parameter* tuple y. We let $\varphi^{\mathrm{opp}}(y, x)$ be the same formula as $\varphi(x, y)$, but with exchanged roles for the variables and parameters. We let $S_\varphi(M)$ be the space of φ-types with parameters from M. We let $\mathrm{Def}_\varphi(M)$ be the Boolean algebra of definable subsets of M generated by $\{\varphi(x, b) : b \in M\}$. We will routinely identify definable sets with the formulas which define them. A φ-formula is an element of $\mathrm{Def}_\varphi(M)$. Likewise, we have analogous definitions for $S_{\varphi^{\mathrm{opp}}}(M)$ and $\mathrm{Def}_{\varphi^{\mathrm{opp}}}(M)$. A φ^{opp}-definition for a type p in $S_\varphi(M)$ is a φ^{opp}-formula, $d_p^{\varphi^{\mathrm{opp}}}(y)$, such that for each $b \in M^y$, $\varphi(x, b) \in p$ if and only if $M \models d_p^{\varphi^{\mathrm{opp}}}(b)$. Finally, we let $\mathfrak{M}_\varphi(M)$ and $\mathfrak{M}_{\varphi^{\mathrm{opp}}}(M)$ denote the spaces of finitely additive probability measures on $\mathrm{Def}_\varphi(M)$ and $\mathrm{Def}_{\varphi^{\mathrm{opp}}}(M)$ respectively. We recall that we can identify a measure in each of these spaces canonically with a regular Borel probability measure on their corresponding type space, e.g. $\mathfrak{M}_\varphi(M)$ is in canonical correspondence with regular Borel probability measures on $S_\varphi(M)$. For some helpful background on topics including continuous logic and stability, we refer the reader to [8].

Definition 1 (Double Limit Property). Let X and Y be sets and let $f : X \times Y \to [0, 1]$. We say that f has the *double limit property* if for any two sequence $(a_i)_{i \in \mathbb{N}}$, $(b_j)_{j \in \mathbb{N}}$ with $a_i \in X$ and $b_j \in Y$,

$$\lim_i \lim_j f(a_i, b_j) = \lim_j \lim_i f(a_i, b_j)$$

provided limits on both sides exist.

The definition of *stability in a model* given in [1] restricted to discrete structures is as follows:

Definition 2. A formula $\varphi(x, y)$ is *stable in* M if $\varphi : M^x \times M^y \to \{0, 1\}$ has the double limit property, where $\varphi(a, b) = 1$ if $M \models \varphi(a, b)$ and $\varphi(a, b) = 0$ otherwise.

We first remark that clearly $\varphi(x, y)$ is stable in M if and only if $\varphi^{\mathrm{opp}}(y, x)$ is stable in M. We also remark that if $\varphi(x, y)$ is stable, then $\varphi(x, y)$ is stable in any model of the underlying theory. On the other hand, if $\varphi(x, y)$ is stable in a model M, it does not imply that it is stable in an elementary extension of M. An example of a formula which is stable in a model but not stable is the edge relation in the graph constructed by taking the disjoint union of all finite graphs on subsets of $\mathbb{N}$.

In [1], Ben Yaacov established a surprising connection between functional analysis and local stability. In particular, he gave a proof of the *fundamental theorem of stability* using Grothendieck's *double limit theorem* [9]. Before stating the theorem, let us briefly recall some basic functional analysis.

Definition 3 (Weak Topology)*.* Let Y be a Banach space over a field F ($F = \mathbb{R}$ or $\mathbb{C}$). Let Y^* be the space of continuous linear functionals from Y to F. Then, the *weak topology on* Y is the coarsest topology such that each element of Y^* remains a continuous function from Y to F.

Definition 4 (Relatively Weakly Compact)*.* Let Y be a Banach space and let $A \subset Y$. We say that A is *weakly compact* if A is a compact subset of Y under the weak topology. Furthermore, we say that A is *relatively weakly compact* if the closure of A under the weak topology is weakly compact.

Let X be a topological space. Then $C_b(X)$ denotes the Banach space of bounded, continuous, complex-valued functions on X, equipped with the uniform norm, $|| \cdot ||_\infty$. We say that a set A is $|| \cdot ||_\infty$-bounded if there exists c in $\mathbb{R}$ such that for all f in A, $||f||_\infty < c$. Grothendieck's theorem (as formulated in [1]) is as follows:

Theorem 1 (Grothendieck [9])*.* Let X be an arbitrary topological space, $X_0 \subseteq X$ a dense subset. Then the following are equivalent for $A \subset C_b(X)$:
(i) The set A is relatively weakly compact in $C_b(X)$.
(ii) The set A is $|| \cdot ||_\infty$-bounded, and whenever $f_n \in A$ and $x_n \in X_0$ form two sequences we have that
$$\lim_n \lim_m f_n(x_m) = \lim_m \lim_n f_n(x_m),$$
provided both limits exist.

Via the double limit theorem above, Ben Yaacov derived the following (among other results):

Theorem 2. Assume that $\varphi(x, y)$ is stable in M, $p \in S_\varphi(M)$, and $q \in S_{\varphi^{\mathrm{opp}}}(M)$. Then p has a φ^{opp}-definition $d_p^\varphi(y)$, q has a φ-definition $d_q^{\varphi^{\mathrm{opp}}}(x)$, and $d_p^\varphi(y) \in q$ if and only if $d_q^{\varphi^{\mathrm{opp}}}(x) \in p$.

Finally, we recall the following fact.

Fact 1. The following are equivalent:
(i) $\varphi(x, y)$ is stable in M.
(ii) There does not exist $(a_i, b_j)_{(i,j)\in\omega\times\omega}$ from $M^x \times M^y$such that
  (a) either for every $i \neq j$, $M \models \varphi(a_i, b_j)$ if and only if $i < j$,
  (b) or for all $i \neq j$, $M \models \varphi(a_i, b_j)$ if and only if $i > j$.
(iii) The map $\chi_\varphi : S_\varphi(M) \times S_{\varphi^{\mathrm{opp}}}(M) \to \{0, 1\}$ has the double limit property where
$$\chi_\varphi(p, q) = 1 \iff d_p^\varphi(y) \in q \iff d_q^{\varphi^{\mathrm{opp}}}(x) \in p.$$

In the above fact, the equivalence of (*i*) and (*ii*) can be found in [2, Proposition 2.3]. Clearly, (*iii*) implies (*i*) and (*i*) implies (*iii*) follows from Theorem 1 together with Theorem 2. We remark that a proof of these equivalences can also be found in Starchenko's unpublished research note on the topic.

2 Local Keisler measures

In this section, we prove that if $\varphi(x, y)$ is stable in M, then all φ-measures are (at most) countable sums of weighted φ-types. The proof of this theorem uses the Sobczyk–Hammer decomposition theorem for positive, bounded charges. We recall this theorem in the case of finitely additive probability measures. But first, we need to recall two different kinds of finitely additive measures.

Definition 5. Let $\mathbb{B}$ be a Boolean algebra of subsets of X (containing both X and $\emptyset$) and μ be a finitely additive probability measure on $\mathbb{B}$.

1. We say that μ is *strongly continuous* on $\mathbb{B}$ if for all $\epsilon > 0$ there exist $F_1, ..., F_n \in \mathbb{B}$ such that $\{F_i\}_{i=1}^n$ forms a partition of X and $\mu(F_i) < \epsilon$, for each $i \leq n$.
2. We say that μ is $\{0, 1\}$*-valued* on $\mathbb{B}$ if for every F in $\mathbb{B}$, $\mu(F) \in \{0, 1\}$.

We refer the reader to [10, Theorem 5.2.7, p. 146] for a proof of the following theorem.

Theorem 3 (Sobczyk–Hammer Decomposition Theorem). Let $\mathbb{B}$ be a Boolean algebra on X (containing $\emptyset$ and X) and μ be a finitely additive probability measure on $\mathbb{B}$. Then, there exists an initial segment I of $\mathbb{N}$, a sequence of distinct finitely additive probability measures $(\mu_i)_{i \in I}$, and a sequence of non-negative real numbers $(r_i)_{i \in I}$, with the following properties:

(i) μ_0 is strongly continuous on $\mathbb{B}$,
(ii) μ_i is $\{0, 1\}$-valued on $\mathbb{B}$ for every $i \geq 1$,
(iii) $\sum_{i \in I} r_i = 1$, and
(iv) $\mu = \sum_{i \in I} r_i \mu_i$.

Further, the decomposition in *(iv)* is unique (obviously, up to permutation of the sequence and non-trivially weighted measures (i.e., $r_i > 0$)).

The Sobczyk–Hammer decomposition theorem allows us to decompose any finitely additive probability measure into a single strongly continuous measure and a sum of (at most countably many) $\{0, 1\}$-valued measures. We will show that if $\varphi(x, y)$ is stable in M, then there do not exist any strongly continuous measures on $\mathrm{Def}_\varphi(M)$. Thus every finitely additive probability measure will be the "weighted sum" of at most countably many types.

2.1 Measures are sums of types

Definition 6. Let $\mathbb{B}$ be a Boolean algebra on a set X. We say that $\mathbb{B}$ has a 2*-tree* if there exists $T \in \mathcal{P}(\mathbb{B})$ such that $(T, \supsetneq)$ is an infinite, complete, binary tree, and if $A, C \in T$, $A \not\supset C$, and $C \not\supset A$, then $A \cap C = \emptyset$.

Fact 2. Let $\mathbb{B}$ be a Boolean algebra on a set X and assume that $\mathbb{B}$ has a 2-tree. Then $|\operatorname{Ult}(\mathbb{B})| \geq 2^{\aleph_0}$ where $\operatorname{Ult}(\mathbb{B})$ is the set of ultrafilters on $\mathbb{B}$.

Proof. Let $A_\gamma = \{B \in T : B \in \gamma\}$ for a given path γ in T. Clearly, A_γ has the finite intersection property (since if $B, C \in A_\gamma$, then either $B \subset C$ or $C \subset B$) and so A_γ can be extended to an ultrafilter over $\mathbb{B}$. This construction gives an injective map from paths in T into ultrafilters on $\mathbb{B}$, proving the claim. □

Lemma 1. Let $\mathbb{B}$ be a Boolean algebra on a set X. Assume that there exists a strongly continuous measure μ over $\mathbb{B}$. Then $\mathbb{B}$ has a 2-tree.

Proof. Using μ, we will build a 2-tree in steps.
Stage 0*:* Let $T_0 = \{X\}$.
Stage $n + 1$*:* We construct a tree of height $n + 1$. Assume that T_n is a (complete) binary tree of height n such that $\mu(A) > 0$, for each $A \in T_n$. Assume furthermore that if $A, B \in T$ and $A \not\supset B$ and $B \not\supset A$, then $A \cap B = \emptyset$. We will construct T_{n+1} by adding two children to each leaf. Let $\mathbb{L}_n$

be the collection of leaves on T_n. Let $\epsilon = \frac{\min\{\mu(L):L\in\mathbb{L}_n\}}{2}$. Since μ is strongly continuous, there exist $H_1,...,H_m \in \mathbb{B}$ such that $\mathbb{H} = \{H_1,...,H_m\}$ partitions X and for each $j \leq m$, $\mu(H_j) < \epsilon$. Now fix a leaf L_i. Consider $L_i \cap \mathbb{H} = \{L_i \cap H : H \in \mathbb{H}\}$. We notice that $L_i \cap \mathbb{H}$ forms a partition of L_i. Therefore,

$$0 < \mu(L_i) = \mu\Big(\bigcup_{K\in L_i\cap\mathbb{H}} K\Big) = \sum_{K\in L_i\cap\mathbb{H}} \mu(K).$$

Hence, there exists $K_r = L_i \cap H_r \in L_i \cap \mathbb{H}$ such that $\mu(K_r) > 0$. Furthermore,

$$\mu(K_r) = \mu(L_i \cap H_r) \leq \mu(H_r) < \epsilon \leq \frac{L_i}{2}.$$

Therefore there must exist some $K_l \in L_i \cap \mathbb{H}$ such that $K_l \neq K_r$ and $\mu(K_l) > 0$. We now add K_r, K_l as children to the leaf L_i. Let T_{n+1} be the tree constructed after repeating this process for each $L \in \mathbb{L}_n$. Clearly, T_{n+1} is a binary tree of height $n+1$ such that $\mu(A) > 0$ for each $A \in T_{n+1}$.

Now let $T = \bigcup_{n\geq 0} T_n$. T is clearly a 2-tree by construction. □

Definition 7. Let M_φ be the reduct of M to the language $\mathcal{L}_\varphi = \{\varphi(x,y)\}$. A subset N of M is a φ-substructure of M, written $N \prec_\varphi M$, if the induced structure on N (in the language $\mathcal{L}_\varphi$) is an elementary substructure of M_φ.

Lemma 2. Assume that $\varphi(x,y)$ is stable in M. Then there are no strongly continuous measures on $\mathrm{Def}_\varphi(M)$.

Proof. Assume that there exists a strongly continuous measure over $\mathrm{Def}_\varphi(M)$. By Lemma 1, there exists a 2-tree. Let $\mathbb{B}_0$ be the Boolean algebra generated by this 2-tree. By Fact 2, $\mathbb{B}_0$ is a countable subalgebra of $\mathrm{Def}_\varphi(M)$ such that $|\operatorname{Ult}(\mathbb{B}_0)| \geq 2^{\aleph_0}$. Choose $C \subset M$ such that for each $B \in \mathbb{B}_0$, there exists $b_1,...,b_n$ in C such that B is an element of the boolean algebra generated by $\{\varphi(x,b_i) : i \leq n\}$. Notice that since $\mathbb{B}$ is countable, we may choose C to be countable. By the Downward Löwenheim–Skolem theorem, there exists an $\mathcal{L}_\varphi$-structure N such that $C \subseteq N$, $N \prec_\varphi M$, and $|N| = \aleph_0$. Then,

$$2^{\aleph_0} \leq |\operatorname{Ult}(\mathbb{B}_0)| \leq |\operatorname{Ult}(\mathrm{Def}_\varphi(C))| \leq |\operatorname{Ult}(\mathrm{Def}_\varphi(N))| = |S_\varphi(N)|.$$

However, since $\varphi(x,y)$ is stable in M, it is also stable in N. By Theorem 2, every φ-type over N is definable by a φ^{opp}-formula with parameters from N. Since $|N| = \aleph_0$, there are only countably many φ^{opp}-formulas. Therefore, not every φ-type over N is definable — a contradiction. □

Theorem 4. Let $\varphi(x,y)$ be stable in M and let $\mu \in \mathfrak{M}_\varphi(M)$. Then there exists an initial segment I of $\mathbb{N}$ such that $\mu = \sum_{i\in I} r_i \delta_{p_i}$ where $p_i \in S_\varphi(M)$, $\sum_{i\in I} r_i = 1$, and each $r_i > 0$. Obviously, the statement also holds when $\varphi(x,y)$ is *stable* (i.e., does not admit the k-order property for some k).

Proof. Direct from the Sobczyk–Hammer Decomposition Theorem and Lemma 2. □

2.2 The Morley product is commutative

We now aim to show that the Morley product commutes on appropriate pairs of measures. First, we need to appropriately define what we mean by the *Morley product* in this context. To define it, we make some quick observations.

Fact 3. Suppose that X is a topological space and Y is a dense subset of X. Let $f : Y \to Z$ be a map. If there exists some $\tilde{f} : X \to Z$ such that $\tilde{f}$ is continuous and $\tilde{f}|_Y = f$, then $\tilde{f}$ is the unique function with such property.

Proof. Clear via the net definition of continuity. □

Proposition 1. Suppose that $\varphi(x,y)$ is stable in M and $\mu \in \mathfrak{M}_{\varphi}(M)$. Consider the map $f_{\mu}^{\varphi} : \{\mathrm{tp}_{\varphi^{\mathrm{opp}}}(b/M) : b \in M^{y}\} \to [0,1]$ via $f_{\mu}^{\varphi}(\mathrm{tp}_{\varphi^{\mathrm{opp}}}(b/M)) = \mu(\varphi(x,b))$. This map is well-defined and there exists a unique continuous function $F_{\mu}^{\varphi} : S_{\varphi^{\mathrm{opp}}}(M) \to [0,1]$ such that $F_{\mu}^{\varphi}|_{\{\mathrm{tp}_{\varphi^{\mathrm{opp}}}(b/M): b\in M^{y}\}} = f_{\mu}^{\varphi}$.

Proof. By Theorem 4, $\mu = \sum_{i\in I} r_i \delta_{p_i}$ where each $p_i \in S_{\varphi}(M)$. We argue that the map f_{μ}^{φ} is well-defined. Notice that if $b \in M^{y}$, then

$$\mu(\varphi(x,b)) = \sum_{i\in I} r_i[\delta_{p_i}(\varphi(x,b))] = \sum_{\substack{i\in I \\ M\models d_{p_i}^{\varphi}(b)}} r_i.$$

By Theorem 2, each formula $d_{p_i}^{\varphi}(y)$ is a φ^{opp}-formula which implies that the value above only depends on the φ^{opp}-type of b, hence f_{μ}^{φ} is indeed well-defined.

Since $\{\mathrm{tp}_{\varphi^{\mathrm{opp}}}(b/M) : b \in M^{y}\}$ is a dense subset of $S_{\varphi^{\mathrm{opp}}}(M)$, by Fact 3 it suffices to prove that there exists a continuous map from $S_{\varphi^{\mathrm{opp}}}(M)$ to $[0,1]$ which restricts to f_{μ}^{φ}. We claim that $\sum_{i\in I} r_i \mathbf{1}_{[d_{p_i}^{\varphi}(y)]}$ is the appropriate map. We remark that we may view $\sum_{i\in I} r_i \mathbf{1}_{[d_{p_i}^{\varphi}(y)]}$ as a map from $S_{\varphi^{\mathrm{opp}}}(M)$ to $[0,1]$ since stability in M implies that every formula of the form $d_{p_i}^{\varphi}(y)$ is a φ^{opp}-formula (Theorem 2). $\square$

We may now define the *Morley product* in this setting.

Definition 8. Suppose that $\varphi(x,y)$ is stable in M. Let $\mu \in \mathfrak{M}_{\varphi}(M)$ and $\nu \in \mathfrak{M}_{\varphi^{\mathrm{opp}}}(M)$. We define the Morley product of μ with ν, denoted $\mu_x \otimes \nu_y$, as follows:

$$(\mu \otimes \nu)(\varphi(x,y)) = \int_{S_{\varphi^{\mathrm{opp}}}(M)} F_{\mu}^{\varphi} d\tilde{\nu},$$

where F_{μ}^{φ} is the function from Proposition 1 and $\tilde{\nu}$ is the regular Borel probability measures corresponding to ν. Likewise, since $\varphi(x,y)$ is stable in M if and only if $\varphi^{\mathrm{opp}}(x,y)$ is stable in M, we may also define

$$(\nu \otimes \mu)(\varphi(x,y)) = \int_{S_{\varphi}(M)} F_{\nu}^{\varphi^{\mathrm{opp}}} d\tilde{\mu},$$

with the obvious analogous definitions.

Remark 1. Since our definition of the Morley product is slightly non-standard, we are careful to make sure it resembles the normal Morley product on types. Suppose that $\varphi(x,y)$ is stable in M, let $p \in S_{\varphi}(M)$, $q \in S_{\varphi^{\mathrm{opp}}}(M)$, and fix $\mathcal{U}$ such that $M \prec \mathcal{U}$. Let $\hat{p} \in S_{\varphi}(\mathcal{U})$ be the unique M-definable extension of p to $\mathcal{U}$. Then $(\delta_p \otimes \delta_q)(\varphi(x,y)) = 1$ if and only if $\mathcal{U} \models \varphi(a,b)$, where $b \models q$ and $a \models \hat{p}|_{Mb}$. Indeed, consider the following sequence of bi-implications:

$$\begin{aligned}(\delta_p \otimes \delta_q)(\varphi(x,y)) = 1 &\iff \int_{S_{\varphi^{\mathrm{opp}}}(M)} \chi_{[d_p^{\varphi}(y)]} d\delta_q = 1 \iff \delta_q(d_p^{\varphi}(y)) = 1 \\ &\iff d_p^{\varphi}(y) \in q \iff \mathcal{U} \models d_p^{\varphi}(b) \iff \varphi(x,b) \in \hat{p} \iff \mathcal{U} \models \varphi(a,b).\end{aligned}$$

Theorem 5. Suppose that $\varphi(x,y)$ is stable in M. Then

$$(\mu \otimes \nu)(\varphi(x,y)) = (\nu \otimes \mu)(\varphi(x,y)).$$

Proof. Consider the following sequence of equations:

$$\begin{aligned}(\mu \otimes \nu)(\varphi(x,y)) &= \int_{S_{\varphi^{\mathrm{opp}}}(M)} F_{\mu}^{\varphi} d\tilde{\nu} = \int_{S_{\varphi^{\mathrm{opp}}}(M)} \sum_{i\in I} r_i \mathbf{1}_{[d_{p_i}^{\varphi}(y)]} d\tilde{\nu} \\ &= \sum_{i\in I} r_i \nu(d_{p_i}^{\varphi}(y)) = \sum_{i\in I} r_i \sum_{j\in J} s_j \delta_{q_j}(d_{p_i}^{\varphi}(y)) \overset{(*)}{=} \sum_{i\in I}\sum_{j\in J} r_i s_j \delta_{p_i}(d_{q_j}^{\varphi^{\mathrm{opp}}}(x)).\end{aligned}$$

Equation $(*)$ is justified by Theorem 2. A symmetric computation shows

$$(\nu \otimes \mu)(\varphi(x,y)) = \sum_{i\in I}\sum_{j\in J} r_i s_j \delta_{p_i}(d_{q_j}^{\varphi^{\mathrm{opp}}}(x)),$$

completing the proof. $\square$

2.3 Some functional analysis and double limits

By Theorem 5, we may define the following evaluation map, E_φ, on appropriate pairs of Keisler measures. We prove that if $\varphi(x,y)$ is stable in M, then E_φ also has the double limit property. Our proof follows directly from classical results in functional analysis, namely Grothendieck's double limit theorem and the Krein-Smulian theorem. For other applications of functional analysis in this area, we refer the reader to [11] and [12]. We first recall the definition of the evaluation map from the introduction.

Definition 9. Suppose that $\varphi(x,y)$ is stable in M. Then we define the map $E_\varphi : \mathfrak{M}_\varphi(M) \times \mathfrak{M}_{\varphi^{\mathrm{opp}}}(M) \to [0,1]$ via

$$E_\varphi(\mu,\nu) = (\mu \otimes \nu)(\varphi(x,y)).$$

By Theorem 5, $E_\varphi(\mu,\nu)$ is also equal to $(\nu \otimes \mu)(\varphi(x,y))$.

A proof of the following theorem can be found in most graduate textbooks on functional analysis.

Theorem 6 (Krein-Smulian Theorem). If Y is a Banach space and K is weakly compact subset of Y, then the closed convex hull of K, denoted $\overline{\mathrm{co}}(K)$, is weakly compact. The closed convex hull of K is the intersection of all norm closed, convex subsets of Y containing K.

Corollary 1. If Y is a Banach space and Z is a relatively weakly compact subset of Y, then $\overline{\mathrm{co}}(Z)$ is a weakly compact subset of Y.

Proof. Let Z^w denote the weak closure of Z. Then Z^w is weakly compact and so by the Krein–Smulian theorem, $\overline{\mathrm{co}}(Z^w)$ is weakly compact. Note that $\overline{\mathrm{co}}(Z) \subseteq \overline{\mathrm{co}}(Z^w)$ and since $\overline{\mathrm{co}}(Z)$ is a closed subset of a compact set, it is also compact. $\square$

Definition 10. Suppose that X is a set. If we endow X with the discrete topology and let $\mathcal{M}(X)$ be the collection of regular Borel probability measures on X, then we can consider the set of finitely supported probability measures (the convex hull of the Dirac measures):

$$\mathrm{conv}_\delta(X) \colon = \left\{ \sum_{i\in I} r_i \delta_{x_i} : I \subseteq \mathbb{N}, x_i \in X, r_i \in \mathbb{R}_{\geq 0}, \sum_{i \in I} r_i = 1 \right\}.$$

For simplicity of notation, we write $\sum_{i\in I} r_i\delta_{x_i}$ simply as $\sum_{i\in I} r_i x_i$.

Definition 11. Suppose that X and Y are sets and $f : X \times Y \to [0,1]$. Then we define $f_c : \mathrm{conv}_\delta(X) \times \mathrm{conv}_\delta(Y) \to [0,1]$ via

$$f_c\left(\sum_{i\in I} r_i x_i, \sum_{j\in J} s_j y_j\right) = \sum_{i\in I}\sum_{j\in J} r_i s_j f(x_i, y_j).$$

We now prove the key combinatorial proposition via functional analysis.

Proposition 2. Suppose that X and Y are sets and $f : X \times Y \to [0,1]$. Then f has the double limit property if and only if $f_c : \mathrm{conv}_\delta(X) \times \mathrm{conv}_\delta(Y) \to [0,1]$ has the double limit property.

Proof. If f_c has the double limit property, then clearly f has the double limit property. Therefore, we only need to prove the other direction. Endow Y with the discrete topology. Let $\mathbf{X} = \{f(a,y) : a \in X\}$. It is obvious that $\mathbf{X} \subset C_b(Y)$ and $\mathbf{X}$ is $||\cdot||_\infty$-bounded. By assumption, f has the double limit property and by Theorem 1, $\mathbf{X}$ is relatively weakly compact in $C_b(Y)$. By Corollary 6, we have that $\overline{\mathrm{co}}(\mathbf{X})$ is a weakly compact subset of $C_b(Y)$. Since $\overline{\mathrm{co}}(\mathbf{X})$ is weakly compact in $C_b(Y)$, it is also relatively weakly compact, and so we can apply Theorem 1. So, for any infinite sequences $g_i \in \overline{\mathrm{co}}(\mathbf{X})$ and $b_j \in Y$

$$\lim_i \lim_j g_i(b_j) = \lim_j \lim_i g_i(b_j),$$

provided both limits exist. In particular, this implies that for $g_i := \sum_{\ell_i \in L_i} r_{\ell_i} f(a_{\ell_i}, y)$,

$$\lim_i \lim_j f_c\left(\sum_{\ell_i \in L_i} r_{\ell_i} a_{\ell_i}, b_j\right) = \lim_j \lim_i f_c\left(\sum_{\ell_i \in L_i} r_{\ell_i} a_{\ell_i}, b_j\right),$$

provided both limits exist.

Notice that the computation above demonstrates that the map $f_c|_{\mathrm{conv}_\delta(X)\times Y}$ has the double limit property. Now consider $\mathrm{conv}_\delta(X)$ endowed with the discrete topology and let $\mathbf{Y} = \{f_c(x,b) : b \in Y\}$. It is clear that $\mathbf{Y} \subset C_b(\mathrm{conv}_\delta(X))$ and that $\mathbf{Y}$ is $||\cdot||_\infty$-bounded since each function is bounded by 1. Since $f_c|_{\mathrm{conv}_\delta(X)\times Y}$ has the double limit property, we can again apply Theorem 1 and so $\mathbf{Y}$ is relatively weakly compact in $C_b(\mathrm{conv}_\delta(X))$. By Corollary 6, $\overline{\mathrm{co}}(\mathbf{Y})$ is weakly compact in $C_b(\mathrm{conv}_\delta(X))$. By Theorem 1, f_c has the double limit property. □

Corollary 2. If $\varphi(x,y)$ is stable in M, then the map $E_\varphi : \mathfrak{M}_\varphi(M) \times \mathfrak{M}_{\varphi^{\mathrm{opp}}}(M) \to [0,1]$ has the double limit property.

Proof. By Fact 1, the map $\chi_\varphi : S_\varphi(M) \times S_{\varphi^{\mathrm{opp}}}(M) \to \{0,1\}$ has the double limit property. By Theorem 4, we have that $\mathfrak{M}_\varphi(M) = \mathrm{conv}_\delta(S_\varphi(M))$ and since φ^{opp} is also stable in M, $\mathfrak{M}_{\varphi^{\mathrm{opp}}}(M) = \mathrm{conv}_\delta(S_{\varphi^{\mathrm{opp}}})$. The computation in Theorem 5 demonstrates that $E_\varphi = (\chi_\varphi)_c$. By Proposition 2, E_φ has the double limit property. □

3 *Proper stability and the order property*

In this section, we work with *honest-to-goodness* stable formulas and give a proof of an implicit theorem of Ben Yaacov and Keisler. Another proof of this theorem is given by Khanaki and Pourmahdian using indiscernible arrays (see [13, Theorem 3.11]). We show that if $\varphi(x,y)$ is stable, then the evaluation map E_φ does not witness the continuous logic analogue of the order property. Throughout this section, we fix $\mathcal{L}$-structures M and $\mathcal{U}$ such that $M \prec \mathcal{U}$ and $\mathcal{U}$ is a monster model. We let T be the theory of M in the language $\mathcal{L}$. We first show how to use the randomization to derive a proof. We then give another proof using the VC theorem. Given a theory T, we denote by T^R its randomization. We refer the reader to Section 3.2 of [14] for background and notation regarding the randomization.

The following fact is due to Ben Yaacov and Keisler [6, Theorem 5.14].

Fact 4. Suppose that $\varphi(x,y)$ is a stable formula with respect to T. Then the randomized formula $\mathbb{E}[\varphi(x,y)]$ is stable (in the sense of continuous logic) with respect to T^R. In other words, if N is a model of T^R then for every $r \in (0,1)$ and $\epsilon > 0$, there exists some integer $n = n(\epsilon, r)$ such that there does not exist an array of elements $(\mathbf{a}_i, \mathbf{b}_j)_{(i,j)\in[n]\times[n]}$ from $N^x \times N^y$ such that

$$\mathbb{E}[\varphi(\mathbf{a}_i, \mathbf{b}_j)] \geq r + \epsilon \quad \text{whenever} \quad i \geq j$$

and

$$\mathbb{E}[\varphi(\mathbf{a}_i, \mathbf{b}_j)] \leq r \quad \text{whenever} \quad i < j.$$

Note that the integer n does not depend on the choice of the model N.

Using the above, it is easy to see that E_φ also does not witness the continuous version of the order property. This follows from the observation that the randomization encodes the computations of the Morley product.

Proposition 3. Suppose that $\varphi(x, y)$ is stable. For every $r \in (0, 1)$ and $\epsilon > 0$, there exists some integer $n = n(\epsilon, r)$ such that there does not exist an array of Keisler measures $(\mu_i, \nu_j)_{(i,j)\in[n]\times[n]}$, where $\mu_i \in \mathfrak{M}_\varphi(M)$ and $\nu_j \in \mathfrak{M}_{\varphi^{\mathrm{opp}}}(M)$ such that

$$(\mu_i \otimes \nu_j)(\varphi(x, y)) \geq r + \epsilon, \quad \text{whenever} \quad i \geq j$$

and

$$(\mu_i \otimes \nu_j)(\varphi(x, y)) \leq r, \quad \text{whenever} \quad i < j.$$

Proof. Consider $[0, 1]^2$ with the corresponding Lebesgue measure L^2 and the simple models of the randomization of T relative to $[0, 1]^2$, namely $M^{[0,1]^2}$ and $\mathcal{U}^{[0,1]^2}$. More explicitly, if N is a model of T then $N^{[0,1]}$ is the collection of measurable maps from $[0, 1]^2$ to N with finite image. It follows from quantifier elimination of T^R that $M^{[0,1]^2} \prec \mathcal{U}^{[0,1]^2}$. If $\mu \in \mathfrak{M}_\varphi(M)$ and $\nu \in \mathfrak{M}_{\varphi^{\mathrm{opp}}}(M)$, then Theorem 4 implies that $\mu = \sum_{k\in K} r_k \delta_{p_k}$ and $\nu = \sum_{w\in W} d_w \delta_{q_w}$ where K and W are initial segments of $\mathbb{N}$ and

1) for each $k \in K$, p_k is in $S_\varphi(M)$,
2) for each $w \in W$, q_w is in $S_{\varphi^{\mathrm{opp}}}(M)$,
3) for each $k \in K$ and $w \in W$, r_k and d_w are positive real numbers,
4) $\sum_{k\in K} r_k = \sum_{w\in W} d_w = 1$.

For each q_w, choose some b_w in $\mathcal{U}^y$ such that $b_w \models q_w$. Let $\mathbf{b}_\nu : [0, 1]^2 \to \mathcal{U}$ via $\mathbf{b}_\nu((s, t)) = b_w$ whenever $s \in [\sum_{\ell=0}^{w-1} d_\ell, \sum_{\ell=0}^{w} d_\ell)$ with the convention that $\sum_{\ell=0}^{-1} d_\ell = 0$. For each p_k, choose some a_k in $\mathcal{U}^x$ such that $a_k \models \hat{p}_k|_{M(b_w)_{w\in W}}$, where $\hat{p}_k$ is the unique M-definable extension of p in $S_\varphi(\mathcal{U})$. Let $\mathbf{a}_\mu : [0, 1]^2 \to \mathcal{U}$ via $\mathbf{a}_\mu((s, t)) = a_k$, when $t \in [\sum_{\ell=0}^{k-1} r_\ell, \sum_{\ell=0}^{k} r_\ell)$ again with the convention that $\sum_{\ell=0}^{-1} r_\ell = 0$. In the following computations, if $(a, b) \in \mathcal{U}^x \times \mathcal{U}^y$, we let $\varphi(a, b) = 1$ if $\mathcal{U} \models \varphi(a, b)$ and 0 otherwise. Then

$$\begin{aligned}(\mu \otimes \nu)(\varphi(x, y)) &\overset{(a)}{=} \sum_{k\in K} \sum_{w\in W} r_k d_w (\delta_{q_w}(d^\varphi_{p_k}(y))) \overset{(b)}{=} \sum_{k\in K} \sum_{w\in W} r_k d_w \varphi(a_k, b_w) \\ &= \int_{(s,t)\in[0,1]^2} \varphi(\mathbf{a}_\mu(s, t)), \mathbf{b}_\nu(s, t)) dL^2 = \mathbb{E}[\varphi(\mathbf{a}_\mu, \mathbf{b}_\nu)].\end{aligned}$$

Equation (a) is derived in the proof of Theorem 4. Equation (b) follows from Remark 1. Thus, if the statement is false, then $\mathbb{E}[\varphi(x, y)]$ witnesses the continuous logic version of the order property. This contradicts Fact 4. □

We now work to give a second proof of Proposition 3 via the VC theorem. The statement of the VC theorem given below is much weaker than the general statement, but it is all that we need.

Theorem 7 (VC–theorem). Suppose that X is a set and $\mathcal{F}$ is a collection of subsets of X. Suppose that the VC-dimension of the class $\mathcal{F}$ is bounded by d. Then for every $\epsilon > 0$, there exists an integer $n = n(\epsilon, d)$ such that for every atomic measure μ on X (i.e., $\mu = \sum_{i\in I} r_i \delta_{x_i}$, where $I \subseteq \mathbb{N}$), there exists $a_1, ..., a_n \in X$ such that for any $F \in \mathcal{F}$,

$$\sup_{F\in\mathcal{F}} |\mu(F) - \mathrm{Av}(a_1, ..., a_n)(F)| < \epsilon.$$

We remark that n does not depend on the choice of measure.

Lemma 3. Suppose that $\varphi(x,y)$ is stable. For every $\epsilon > 0$ there exists some natural number $N = N(\epsilon)$ such that for every $\mu \in \mathfrak{M}_{\varphi}(M)$ there exists $a_1, ..., a_N \in M$ such that for any $b \in M^y$,

$$|\mu(\varphi(x,b)) - \mathrm{Av}(a_1, ..., a_N)(\varphi(x,b))| < \epsilon.$$

Proof. There are several ways to see this. By Corollary 4, we may write $\mu = \sum_{i=0}^{\omega} r_i \delta_{p_i}$, where each p_i is in $S_{\varphi}(M)$. Let $\mathcal{U}$ be a monster model such that $M \prec \mathcal{U}$ and consider the measure $\hat{\mu} \in \mathfrak{M}_{\varphi}(\mathcal{U})$ given by $\sum_{i=0}^{\omega} r_i \delta_{\hat{p}_i}$, where $\hat{p}_i$ is the unique global M-definable extension of p to $\mathcal{U}$. For each $i \in \omega$, we have that $\hat{p}_i$ is both definable over M and finitely satisfiable in M [2, Proposition 2.3]. As a consequence, the measure $\hat{\mu}$ is finitely satisfiable and φ-definable over M (for appropriate definitions in this context, see [15, Section 6]). If $\varphi(x,y)$ is stable, it is NIP and so an application of [15, Theorem 6.4] implies that for every $\epsilon > 0$, there exists $a_1, ..., a_n \in M^x$ such that

$$\sup_{b \in \mathcal{U}^y} |\mu(\varphi(x,b)) - \mathrm{Av}(\bar{a})(\varphi(x,b))| < \epsilon.$$

An application of the VC theorem gives uniform bounds. □

Alternative Proof of Proposition 3. Suppose not. Then there exist $r \in (0,1)$, $\epsilon > 0$ and sequences $(\mu_i, \nu_j)_{(i,j) \in [k] \times [k]}$ for arbitrarily larger k which witnesses the (r,ϵ)-order property. Fix k arbitrarily large. We now construct a discrete formula (which is a Boolean combination of $\varphi(x,y)$) which witnesses the order property — and since stable formulas are closed under Boolean combinations, we obtain a contradiction. By Proposition 3, there exists a natural number N such that

1. For every $\mu \in \mathfrak{M}_{\varphi}(M)$, there exists $a_1, \ldots, a_N \in M^x$ such that
$$\sup_{b \in M^y} |\mu(\varphi(x,b)) - \mathrm{Av}(\bar{a})(\varphi(x,b))| < \frac{\epsilon}{16}.$$
For each $i \leq k$, let $\bar{a}_i = a_1^i, \ldots, a_N^i$ witness the above equation for μ_i.
2. For every $\nu \in \mathfrak{M}_{\varphi^{\mathrm{opp}}}(M)$, there exists $b_1, \ldots, b_N \in M^y$ such that
$$\sup_{a \in M^x} |\nu(\varphi(a,y)) - \mathrm{Av}(\bar{b})(\varphi(a,y))| < \frac{\epsilon}{16}.$$
For each $j \leq k$, let $\bar{b}_j = b_1^j, \ldots, b_N^j$ witness the above equation for ν_j.

Consider the formula given by,

$$\theta(x_1, ..., x_N, y_1, ..., y_N) := \bigvee_{\substack{A \times B \subseteq [N] \times [N] \\ \frac{|A \times B|}{N^2} > r + \frac{\epsilon}{2}}} \left(\bigwedge_{(i,j) \in A \times B} \varphi(x_i, y_j) \right).$$

We claim that $\theta(\bar{x}, \bar{y})$ is unstable. Notice that

$$M \models \theta(\bar{a}_i, \bar{b}_j) \Longrightarrow M \models \bigvee_{\substack{A \times B \subseteq [N] \times [N] \\ \frac{|A \times B|}{N^2} > r + \frac{\epsilon}{2}}} \left(\bigwedge_{(\ell,k) \in A \times B} \varphi(a_{\ell}^i, b_k^j) \right) \Longrightarrow (\mathrm{Av}(\bar{a}_i) \otimes \mathrm{Av}(\bar{b}_j))(\varphi(x,y)) > r + \frac{\epsilon}{2},$$

and likewise,

$$M \models \neg\theta(\bar{a}_i, \bar{b}_j) \Longrightarrow (\mathrm{Av}(\bar{a}_i) \otimes \mathrm{Av}(\bar{b}_j))(\varphi(x,y)) \leq r + \frac{\epsilon}{2}.$$

Moreover, if $i < j$, then

$$\begin{aligned} r \geq (\mu_i \otimes \nu_j)(\varphi(x,y)) &\approx_{\epsilon/16} (\mathrm{Av}(\bar{a}_i) \otimes \nu_j)(\varphi(x,y)) = (\nu_j \otimes \mathrm{Av}(\bar{a}_i))(\varphi(x,y)) \\ &\approx_{\epsilon/16} (\mathrm{Av}(\bar{b}_j) \otimes \mathrm{Av}(\bar{a}_i))(\varphi(x,y)) = (\mathrm{Av}(\bar{a}_i) \otimes \mathrm{Av}(\bar{b}_j))(\varphi(x,y)), \end{aligned}$$

and likewise, if $i \geq j$,

$$\begin{aligned} r + \epsilon \leq (\mu_i \otimes \nu_j)(\varphi(x,y)) &\approx_{\epsilon/16} (\mathrm{Av}(\bar{a}_i) \otimes \nu_j)(\varphi(x,y)) = (\nu_j \otimes \mathrm{Av}(\bar{a}_i))(\varphi(x,y)) \\ &\approx_{\epsilon/16} (\mathrm{Av}(\bar{b}_j) \otimes \mathrm{Av}(\bar{a}_i))(\varphi(x,y)) = (\mathrm{Av}(\bar{a}_i) \otimes \mathrm{Av}(\bar{b}_j))(\varphi(x,y)). \end{aligned}$$

Hence, if $i < j$ and $\models \theta(\bar{a}_i, \bar{b}_j)$, then $(\mathrm{Av}(\bar{a}_i) \otimes \mathrm{Av}(\bar{b}_j))(\varphi(\bar{x},\bar{y}))$ is greater than $r + \frac{\epsilon}{2}$ (by witnessing θ) and less than $r + \frac{\epsilon}{8}$ (by the above implication) — a contradiction. Hence, if $i < j$, then $\models \neg\theta(\bar{a}_i, \bar{b}_j)$. A similar argument shows that if $i \geq j$ then $\theta(\bar{a}_i, \bar{b}_j)$ must hold. Thus $\theta(\bar{x}, \bar{y})$, a Boolean combination of $\varphi(x,y)$, is unstable — a contradiction. $\square$

Conclusion

We prove that if a formula $\varphi(x,y)$ is stable in a model, then all local Keisler measures with respect to this formula decompose into countable sums of types. We also give a functional-analytic proof of the fact that if a formula $\varphi(x,y)$ is stable, then local Keisler measures with respect to this formula do not satisfy the order property. This research demonstrates how tools and techniques from functional analysis can be used to understand the combinatorial properties of measures in model-theoretic settings.

Much is already known about Keisler measures — and in particular local Keisler measures — in the NIP and stable settings. Future research directions include understanding Keisler measures in general contexts with the independence property, such as simple, NTP2, and NSOP1 settings.

Acknowledgments

We thank Sergei Starchenko and Gabriel Conant for helpful discussions during the writing of the initial research note. We thank our Kazakh friends for their warm welcome and hospitality.

Author Contributions

All authors contributed equally to this work.

Conflict of Interest

The authors declare no conflict of interest.

References

1 Ben Yaacov, I. (2014). Model theoretic stability and definability of types, after A. Grothendieck. *Bulletin of Symbolic Logic, 20*(4), 491–496. https://doi.org/10.1017/bsl.2014.33

2 Pillay, A. (2016). Generic stability and Grothendieck. *South American Journal of Logic, 2*(2), 1–6.

3 Conant, G. (2021). Stability in a group. *Groups, Geometry, and Dynamics, 15*(4), 1297–1330. https://doi.org/10.4171/GGD/631

4 Khanaki, K., & Pillay, A. (2018). Remarks on the NIP in a model. *Mathematical Logic Quarterly, 64*(6), 429–434. https://doi.org/10.1002/malq.201700070

5 Khanaki, K. (2024). Grothendieck's Double Limit Theorem and Model Theory. *Mathematical Culture and Thought, 43*(2), 247–259. https://doi.org/10.30504/mct.2023.1376.1967

6 Ben Yaacov, I., & Keisler, H.J. (2009). Randomizations of models as metric structures. *Confluentes Mathematicae, 1*(2), 197–223. https://doi.org/10.1142/S1793744209000080

7 Gannon, K. (2019). Local Keisler measures and NIP formulas. *The Journal of Symbolic Logic, 84*(3), 1279–1292. https://doi.org/10.1017/jsl.2019.34

8 Pillay, A. (2024). *Topics in Model Theory.* Singapore: World Scientific. https://doi.org/10.1142/12455

9 Grothendieck, A. (1952). Criteres de compacite dans les espaces fonctionnels generaux [Criteria for compactness in general functional spaces]. *American Journal of Mathematics, 74*(1), 168–186 [in French]. https://doi.org/10.2307/2372076

10 Bhaskara Rao, K.P.S., & Bhaskara Rao, M. (1983). *Theory of charges: a study of finitely additive measures.* (Vol. 109). London: Academic Press.

11 Khanaki, K. (2020). Stability, the NIP, and the NSOP: model theoretic properties of formulas via topological properties of function spaces. *Mathematical Logic Quarterly, 66*(2), 136–149. https://doi.org/10.1002/malq.201500059

12 Chavarria, N., Conant, G., & Pillay, A. (2024). Continuous stable regularity. *Journal of the London Mathematical Society, 109*(1), Article e12822. https://doi.org/10.1112/jlms.12822

13 Khanaki, K., & Pourmahdian, M. (2024). Simple Models of Randomization and Preservation Theorems. *arXiv preprint,* arXiv:2408.15014. https://arxiv.org/abs/2408.15014

14 Conant, G., Gannon, K., & Hanson, J.E. (2025). Generic stability, randomizations and NIP formulas. *Journal of Mathematical Logic,* Article 2550016. https://doi.org/10.1142/S0219061325500163

15 Gannon, K. (2022). Sequential approximations for types and Keisler measures. *Fundamenta Mathematicae, 257*, 305–336. https://doi.org/10.4064/fm133-12-2021

*Author Information**

Christian Marie Benoît Jean D'Elbée — Doctor of Mathematics, Assistant Professor, University of the Basque Country, Department of Mathematics, Barrio Sarriena s/n, Leioa, 48940, Biscay, Spain; Institute for Logic, Cognition, Language and Information, UPV/EHU, Carlos Santamaria Zentroa, Elhuyar Plaza 2, Donostia-San Sebastian, 20018, Spain; e-mail: *christian.delbee@ehu.eus*; https://orcid.org/0000-0002-0268-1802

Kyle Andrew Cox Gannon (*corresponding author*) — Doctor of Mathematics, Assistant Professor, Peking University, Beijing International Center for Mathematical Research (BICMR), No. 5, Yiheyuan Road, Haidian District, Beijing, China; e-mail: *kgannon@bicmr.pku.edu.cn*; https://orcid.org/0000-0001-5951-5269

*Authors' names are presented in the following order: first name, middle name (if any), last name.

https://doi.org/10.31489/2026M1/122-131 *Research article*

Algebras of binary isolating formulas for theories of modular products of graphs

D.Y. Emelyanov

Novosibirsk State Technical University, Novosibirsk, Russian Federation
(E-mail: dima-pavlyk@mail.ru)

This article discusses the problems of constructing and classifying algebras generated by modular products of cycles. We demonstrate that algebras of binary isolating, used to analyze relationships between binary formulas of a theory, can be naturally interpreted in terms of metric properties of graphs. A characteristic feature of the modular product is that with sufficiently large cycle parameters $(m, n > 4)$, the diameter of such a graph does not exceed three. This makes it possible to define an algebra of binary formulas using only four labels. For small cycle parameters, the presence of simplices is identified and justified. Based on the analysis, we propose a generalized scheme combining modular products of cycles and their extended versions. It is proved that for $m, n > 4$, the algebra of binary isolating formulas for the theory of $C_m \nabla C_n$ is isomorphic to the algebra of simplices of corresponding diameter. Explicit Cayley tables are constructed for products involving small cycles (C_3–C_6), leading to general descriptions of algebras $\mathfrak{M}_o$ (odd) and $\mathfrak{M}_e$ (even). The proposed approach provides new opportunities for classifying theories and establishing correspondences between algebras and graphs, underlining its relevance for modern model theory and structural combinatorics.

Keywords: algebra of binary isolating formulas, modular product, model theory, Cayley tables, classification of theories, simplicial algebras, cycle graphs, graph diameter.

2020 Mathematics Subject Classification: 03C65, 03C25.

Introduction

In modern model theory, a section devoted to the algebras of binary isolating and weakly isolating formulas is actively developing. These algebraic structures, constructed according to an arbitrary complete theory, serve as a powerful tool for analyzing the structure of binary types and the complex system of relations between their implementations. Such algebraic invariants turn out to be a powerful tool for classifying theories, allowing one to establish correspondences between theories and the algebraic structures associated with them. Contributions to the development of this field have been made by works devoted to the study of the general properties of such algebras [1–3], as well as their calculation for specific classes of theories, such as the theories of groups [4], ordered structures [5–7], weakly o-minimal theories [8–10], Cartesian products of graphs and simplices [11].

Modular products of graphs are of particular interest in this context. They are an important construction of structural combinatorics, which makes it possible to build complex graphs from simpler ones. This operation, based on the Cartesian product of sets of vertices with the definition of edges through synchronization of adjacency relations in the original graphs, accumulates both local and global properties of factors. This makes the modular product a promising object for model-theoretical analysis, where vertices and edges are interpreted as carriers of predicates, and the operation itself is interpreted as a composition of logical constructions. This approach makes it possible to study the

The research was supported by the grant from Russian Science Foundation №24-21-00096, https://rscf.ru/en/project/24-21-00096/.

Received: 16 August 2025; *Accepted:* 30 October 2025.

transfer of structural properties, such as connectivity, diameter, or the presence of certain subgraphs, and to study their influence on the expressive power of the corresponding theory.

Throughout this paper, we consider graphs as first-order structures in the language $\mathcal{L} = \{R\}$ with a single binary predicate interpreted as the edge relation. For a graph G, its *theory* $\mathrm{Th}(G)$ is the set of all $\mathcal{L}$-sentences true in G. We study the *algebra of binary isolating formulas* $\mathfrak{M}_T$ associated with a complete theory $T = \mathrm{Th}(G)$. The elements (labels) of this algebra correspond to the isomorphism types of principal formulas $\theta(x, a)$ in two variables that isolate complete types over finite sets. The algebraic operation $\cdot$ on these labels is induced by the superposition of such formulas and yields a set of possible labels for the resulting formula. For a detailed construction, see [1, 3].

The relevance of this study is due to the need for a systematic study of binary formula algebras for theories generated by combinatorial constructions, in particular, modular products. Such an analysis makes it possible not only to classify theories by their derived algebraic invariants, but also to better understand the relationship between the combinatorial structure of the model and the logical properties of its theory.

The novelty of the work lies in the fact that it is the first to study the distribution algebras of binary isolating formulas for the theories of modular products of graphs, primarily regular polygons (cycles). For these products, Cayley tables of the corresponding algebras are constructed and analyzed, which makes it possible to give their explicit description.

The following main results were achieved in the course of the study. First of all, Cayley tables were constructed for the algebras of binary isolating formulas of the theories of modular products of an edge graph on cycles of small length, namely C_3, C_4, C_5 and C_6. Based on the analysis of these specific cases, a general description of the $\mathfrak{M}_\mathfrak{o}$ and $\mathfrak{M}_\mathfrak{e}$ algebras for modular products of an edge graph on cycles of arbitrary odd and even length, respectively, was given. An important result of the work was the formulation and proof of a theorem that gives a classification of binary formula algebras for theories of modular products of graph edges on polygons (Theorem 1). Further, the case of the modular product of two cycles $C_m \nabla C_n$ for $m, n > 4$ was investigated; for such graphs, it was proved that their diameter does not exceed 3, and they always contain triangles (Theorems 3). Finally, it is established that in the case of $m, n > 4$, the algebra of binary isolating formulas of the theory of $C_m \nabla C_n$ turns out to be isomorphic to the algebra of simplices of the corresponding diameter (Theorem 4).

The results obtained demonstrate that the class of algebras of binary isolating formulas for modular products of cycles has a rich but classifiable structure, and reveal clear connections between the combinatorial properties of the products and the algebraic properties of the corresponding theories.

1 *Algebras of binary isolating formulas for theories of modular products of graphs*

Definition 1. [12] Let G and H be graphs. The *modular product* $M = G \nabla H$ is a graph defined as follows. The vertex set of M is the Cartesian product:

$$V(M) = V(G) \times V(H).$$

Two distinct vertices $(u, v), (x, y) \in V(M)$ are joined by an edge if and only if

$$u \neq x, \quad v \neq y, \quad \text{and} \quad \big(u \sim_G x\big) \iff \big(v \sim_H y\big).$$

Let $T = \mathrm{Th}(G \nabla H)$. The algebra $\mathfrak{M}_T$ of binary isolating formulas for T has a finite set of *labels* (denoted by integers) corresponding to the possible distances or isomorphism types of pairs of vertices in the graph. The multiplication $i \cdot j = K$, where K is a set of labels, encodes the possible labels for a pair (a, c) given that (a, b) has label i and (b, c) has label j for some vertex b. The tables below are computed by analyzing the graph structure of $G \nabla H$.

We further consider algebras generated by the operation of modular multiplication of edges in graphs of regular polygons. Graphs of regular polygons can be regarded as cycles, denoted C_n, where n is the length of the cycle.

In the case of a modular product of an edge graph on itself, $H\nabla H$, two identical algebraic structures arise. Their signatures, defined by the set $\rho_{\nu(p)} = \{0,1\}$ of labels, are set by the following multiplication rules:

$$\begin{aligned} &0\cdot 0 = \{0\},\\ &0\cdot 1 = \{1\},\\ &1\cdot 1 = \{0,1\},\\ &1\cdot 0 = \{1\}. \end{aligned}$$

The algebra of the modular product of an edge with a triangle $H\nabla C_3$ with the set $\rho_{\nu(p)} = \{0,1,2,3\}$ of labels is defined by the following label products:

$$\begin{aligned} &0\cdot 0 = \{0\}, && 0\cdot 1 = \{1\},\\ &1\cdot 0 = \{1\}, && 1\cdot 1 = \{0,2\},\\ &1\cdot 2 = \{0,1\}, && 1\cdot 3 = \{0,2\},\\ &2\cdot 0 = \{2\}, && 2\cdot 1 = \{1,2\},\\ &2\cdot 2 = \{0,1\}, && 2\cdot 3 = \{1,3\},\\ &3\cdot 0 = \{3\}, && 3\cdot 1 = \{0,2\},\\ &3\cdot 2 = \{1,3\}, && 3\cdot 3 = \{0,2\}. \end{aligned}$$

The modular product $H\nabla C_4$ (an edge on a quarter-cycle) generates two isomorphic algebras. The set of their labels is $\rho_{\nu(p)} = \{0,1,2\}$, and the multiplicative structure is completely determined by the following rules:

$$\begin{aligned} &0\cdot 0 = \{0\}, && 0\cdot 1 = \{1\},\\ &1\cdot 0 = \{1\}, && 1\cdot 1 = \{0,2\},\\ &1\cdot 2 = \{1,2\}, && 2\cdot 0 = \{2\},\\ &2\cdot 1 = \{1,2\}, && 2\cdot 2 = \{0,2\}. \end{aligned}$$

The algebra for the modular product of an edge with a pentagon $H\nabla C_5$ with label set $\rho_{\nu(p)} = \{0,1,2,3,4,5\}$ is defined by the following label products:

$$\begin{aligned} &0\cdot 0 = \{0\},\\ &0\cdot 1 = 1\cdot 0 = \{1\},\\ &0\cdot 2 = 2\cdot 0 = \{2\},\\ &0\cdot 3 = 3\cdot 0 = \{3\},\\ &0\cdot 4 = 4\cdot 0 = \{4\},\\ &0\cdot 5 = 5\cdot 0 = \{5\},\\ &1\cdot 1 = \{0,2\},\\ &1\cdot 2 = 2\cdot 1 = \{1,3\},\\ &1\cdot 3 = 3\cdot 1 = 1\cdot 5 = 5\cdot 1 = 3\cdot 3 = 3\cdot 5 = 5\cdot 3 = 5\cdot 5 = \{0,2,4\},\\ &2\cdot 2 = 2\cdot 4 = 4\cdot 2 = 4\cdot 4 = \{0,2,4\},\\ &1\cdot 4 = 4\cdot 1 = 2\cdot 3 = 3\cdot 2 = 2\cdot 5 = 5\cdot 2 = \{1,3,5\},\\ &3\cdot 4 = 4\cdot 3 = 4\cdot 5 = 5\cdot 4 = \{1,3,5\}. \end{aligned}$$

Consider the modular product of an edge graph on a hexagon, $H\nabla C_6$. This construction generates two isomorphic algebras supported by the set $\rho_{\nu(p)} = \{0,1,2,3\}$ of labels. The multiplication operation in these algebras is given by the following table:

$$
\begin{array}{ll}
0 \cdot 0 = \{0\}, & 0 \cdot 1 = \{1\}, \\
1 \cdot 0 = \{1\}, & 1 \cdot 1 = \{0,2\}, \\
1 \cdot 2 = \{0,1\}, & 1 \cdot 3 = \{0,2\}, \\
2 \cdot 0 = \{2\}, & 2 \cdot 1 = \{1,2\}, \\
2 \cdot 2 = \{0,1\}, & 2 \cdot 3 = \{1,3\}, \\
3 \cdot 0 = \{3\}, & 3 \cdot 1 = \{0,2\}, \\
3 \cdot 2 = \{1,3\}, & 3 \cdot 3 = \{0,2\}.
\end{array}
$$

For the modular product $H\nabla C_k$, depending on the parity of the diameter $\mathrm{diam}(H\nabla C_k)$, we obtain two possible isomorphism types of algebras of binary isolating formulas. The odd-diameter case is denoted by $\mathfrak{M}_o$, and the even-diameter case is denoted by $\mathfrak{M}_e$. Below we give explicit descriptions of these algebras via Cayley tables.

Definition of $\mathfrak{M}_\mathfrak{o}$. Its signature includes labels $\{0,1,2,3,\ldots,n\}$, where n is an odd number equals to the diameter of the graph resulting from the product. The structure of this algebra is determined by the following multiplication table:

$$
\begin{array}{l}
0 \cdot 0 = \{0\}, \\
0 \cdot 1 = 1 \cdot 0 = \{1\}, \\
0 \cdot 2 = 2 \cdot 0 = \{2\}, \\
0 \cdot 3 = 3 \cdot 0 = \{3\}, \\
0 \cdot 4 = 4 \cdot 0 = \{4\}, \\
\vdots \\
0 \cdot n = n \cdot 0 = \{n\}, \\
1 \cdot 1 = \{0,2\}, \\
1 \cdot 2 = 2 \cdot 1 = \{1,3\}, \\
1 \cdot 3 = 3 \cdot 1 = \{0,2,4\}, \\
2 \cdot 2 = \{0,2,4\}, \\
2 \cdot 3 = 3 \cdot 2 = \{1,3,5\}, \\
3 \cdot 3 = \{0,2,4,6\}, \\
4 \cdot 1 = 1 \cdot 4 = \{1,3,5\}, \\
4 \cdot 2 = 2 \cdot 4 = \{0,2,4,6\}, \\
4 \cdot 3 = 3 \cdot 4 = \{1,3,5,\ldots,n\}, \\
4 \cdot 4 = \{0,2,4,\ldots,n-1\}, \\
\vdots \\
n \cdot 1 = 1 \cdot n = \{1,3,5,\ldots,n\}, \\
n \cdot 2 = 2 \cdot n = \{0,2,4,\ldots,n-1\}, \\
n \cdot 3 = 3 \cdot n = \{1,3,5,\ldots,n\}, \\
n \cdot 4 = 4 \cdot n = \{0,2,4,\ldots,n-1\}, \\
\vdots \\
n \cdot n = \{0,2,4,\ldots,n-1\}.
\end{array}
$$

Definition of $\mathfrak{M}_\mathfrak{e}$. Its carrier is a set of labels $\{0,1,2,3,\ldots,n\}$, where n is an even number equal to the diameter of the graph resulting from the product. The structure of the algebra is determined by the following multiplication table:

$$
\begin{array}{ll}
0 \cdot 0 = \{0\} & 0 \cdot 1 = 1 \cdot 0 = \{1\} \\
0 \cdot 2 = 2 \cdot 0 = \{2\} & 0 \cdot 3 = 3 \cdot 0 = \{3\} \\
0 \cdot 4 = 4 \cdot 0 = \{4\} & \vdots \\
0 \cdot n = n \cdot 0 = \{n\} & 1 \cdot 1 = \{0,2\} \\
1 \cdot 2 = 2 \cdot 1 = \{1,3\} & 1 \cdot 3 = 3 \cdot 1 = \{0,2\} \\
2 \cdot 2 = \{0,2,4\} & 2 \cdot 3 = 3 \cdot 2 = \{1,3,5\} \\
3 \cdot 3 = \{0,2,4,6\} & 4 \cdot 1 = 1 \cdot 4 = \{1,3,5\} \\
4 \cdot 2 = 2 \cdot 4 = \{0,2,4,6\} & 4 \cdot 3 = 3 \cdot 4 = \{1,3,5,\ldots,n-1\} \\
4 \cdot 4 = \{0,2,4,\ldots,n\} & \vdots \\
n \cdot 1 = 1 \cdot n = \{1,3,5,\ldots,n-1\} & n \cdot 2 = 2 \cdot n = \{0,2,4,\ldots,n\} \\
n \cdot 3 = 3 \cdot n = \{1,3,5,\ldots,n-1\} & n \cdot 4 = 4 \cdot n = \{0,2,4,\ldots,n\} \\
\vdots & n \cdot n = \{0,2,4,\ldots,n\}
\end{array}
$$

Remark 1. Let G and H be non-empty graphs. Then for the modular product $G\nabla H$ the following holds:

1. If both graphs G and H are bipartite, then $G\nabla H$ consists of exactly two connected components, isomorphic to each other.
2. If at least one of the graphs G or H is not bipartite, then $G\nabla H$ is connected.

If G and H are bipartite with bipartitions (U_1, U_2) and (V_1, V_2) respectively, then the vertex set of $M = G\nabla H$ splits into two disjoint subsets: $(U_1 \times V_1) \cup (U_2 \times V_2)$ and $(U_1 \times V_2) \cup (U_2 \times V_1)$. By the definition of adjacency in the modular product, no edge connects these two subsets, and each induces a connected component isomorphic to the *modular product with a fixed parity constraint*. This structural dichotomy is reflected in the algebra $\mathfrak{M}_T$: the labels and multiplication become symmetric with respect to these two components, effectively yielding two isomorphic copies of the same algebraic structure.

Reflecting Remark 1 on algebras, we obtain that in the first case we get two identical algebras, in the second one, depending on the diameter of the resulting graph, they will be either $\mathfrak{M}_{\mathfrak{e}}$ or $\mathfrak{M}_{\mathfrak{o}}$.

Theorem 1. If T is the theory of the modular product of an edge with polygons, and $\mathfrak{M}$ is the algebra of binary isolating formulas of the theory T, then the algebra $\mathfrak{M}$ is isomorphic to the algebra $\mathfrak{M}_{\mathfrak{o}}$ or $\mathfrak{M}_{\mathfrak{e}}$.

Proof. Let H be the edge graph with two vertices joined by an edge, and let $k \geq 3$. Put $G = H\nabla C_k$ and $T = \mathrm{Th}(G)$.

We first describe the structure of the graph G. The vertex set of G is $\{0,1\} \times V(C_k)$. Two distinct vertices (i,a) and (j,b) are adjacent in G if and only if $i \neq j$, $a \neq b$, and $(i \sim_H j) \iff (a \sim_{C_k} b)$. Since H is an edge graph, the condition $i \neq j$ already implies $i \sim_H j$. Hence adjacency in G is equivalent to the condition that $i \neq j$ and $a \sim_{C_k} b$.

Therefore, each vertex $(0,a)$ is adjacent exactly to the vertices $(1, a-1)$ and $(1, a+1)$ (indices taken modulo k), and similarly for vertices of the form $(1,a)$. This shows that each connected component of G is a cycle.

If k is odd, then C_k is not bipartite, and by Remark 1 the graph G is connected. Hence G is a single cycle of length $2k$, and $\mathrm{diam}(G) = k$, which is an odd number.

If k is even, then both H and C_k are bipartite, and again by Remark 1 the graph G consists of two isomorphic connected components. Each component is a cycle of length k, and therefore has diameter $k/2$, which is an even number.

We now describe the algebra $\mathfrak{M}_T$. Since G is finite and vertex-transitive on each connected component, the complete 2-types over the empty set are uniquely determined by the graph distance between two vertices lying in the same component. Hence the set of labels of $\mathfrak{M}_T$ can be identified with

$\{0, 1, 2, \ldots, n\}$, where $n = \mathrm{diam}(G)$, and the label t corresponds to the formula expressing that the distance between two vertices equals t.

Let $i, j \in \{0, 1, \ldots, n\}$ and let u, v, w be vertices of G such that $\mathrm{dist}(u, v) = i$ and $\mathrm{dist}(v, w) = j$. Since each connected component of G is a cycle, all possible values of $\mathrm{dist}(u, w)$ are exactly the integers t satisfying

$$|i - j| \leq t \leq \min(n, i + j)$$

and

$$t \equiv i + j \pmod 2.$$

Conversely, for every such t there exist vertices u, v, w with $\mathrm{dist}(u, v) = i$, $\mathrm{dist}(v, w) = j$, and $\mathrm{dist}(u, w) = t$. Therefore, the product of labels $i \cdot j$ in $\mathfrak{M}_T$ is precisely the set of all labels t satisfying the above conditions.

If n is odd, then the maximal admissible element of this set is n when $i + j$ is odd and $n - 1$ when $i + j$ is even, which yields exactly the multiplication rules defining the algebra $\mathfrak{M}_o$. If n is even, then n belongs to the even parity class, and the multiplication rules coincide with those defining the algebra $\mathfrak{M}_e$.

Hence the algebra $\mathfrak{M}_T$ is isomorphic to $\mathfrak{M}_o$ when the diameter of G is odd, and to $\mathfrak{M}_e$ when the diameter of G is even. $\square$

Remark 2. If the graph contains at least one simplex, then the algebra for this graph will be isomorphic to the algebra of simplices [11].

For $M = C_3 \nabla C_3$ and $diam(M) = 2$ the algebra will have the set $\{0, 1, 2\}$ of labels and the following multiplication rules:

$$\begin{array}{ll}
0 \cdot 0 = \{0\}, & 0 \cdot 1 = \{1\}, \\
1 \cdot 0 = \{1\}, & 1 \cdot 1 = \{0, 2\}, \\
1 \cdot 2 = \{0, 1, 2\}, & 2 \cdot 0 = \{2\}, \\
2 \cdot 1 = \{0, 1, 2\}, & 2 \cdot 2 = \{0, 1, 2\}.
\end{array}$$

For $M = C_m \nabla C_n$, where $2 < m < 5$, the graph M may contain simplices, therefore Theorem 1 extends to include also the algebra for simplices [11].

If $diam(C_n \nabla C_m) = n$, the algebra of simplices $\mathfrak{M}_{\mathfrak{s}}$ has the set $\rho_{\nu(p)} = \{0, 1, 2, \ldots, n\}$ of labels, and is defined by the following label products:

$$\begin{array}{c}
0 \cdot k = \{k\}, \quad k \cdot 0 = \{k\}, \quad \text{for all } k \in \rho, \\
a \cdot b = \{0, 1, 2, \ldots, \min(n, a + b)\}, \qquad a, b \in \rho.
\end{array}$$

Theorem 2. If T is the theory of the modular product of the graph C_m with $2 < m < 5$ and polygons, and $\mathfrak{M}$ is the algebra of binary isolating formulas of the theory T, then the algebra $\mathfrak{M}$ is isomorphic to the algebra $\mathfrak{M}_{\mathfrak{o}}$, or $\mathfrak{M}_{\mathfrak{e}}$, or $\mathfrak{M}_{\mathfrak{s}}$.

Proof. Let $2 < m < 5$, so $m \in \{3, 4\}$, and let $n \geq 3$. Put $G = C_m \nabla C_n$ and $T = \mathrm{Th}(G)$.

If the graph G contains a simplex, then by Remark 2 the algebra $\mathfrak{M}_T$ is isomorphic to the algebra of simplices $\mathfrak{M}_s$. Thus, in this case the statement of the theorem holds.

Assume now that G contains no simplex. Then G has no triangles, and the complete 2-types over the empty set are determined by the distances between vertices in connected components of G. Consequently, the labels of $\mathfrak{M}_T$ can again be identified with the possible distances between vertices, and the product of labels is determined by the possible distances obtained by composition through an intermediate vertex.

If at least one of the cycles C_m or C_n is not bipartite, then by Remark 1 the graph G is connected. If both cycles are bipartite, then G consists of two isomorphic connected components. In either case, each connected component of G is a graph of bounded diameter, and the label multiplication is completely determined by the metric structure inside a component.

As in the proof of Theorem 1, the parity of paths in a bipartite component implies that only distances of the same parity as the sum of the factors may occur in a product. Therefore, depending on whether the diameter of G is odd or even, the multiplication table of $\mathfrak{M}_T$ coincides with that of $\mathfrak{M}_o$ or $\mathfrak{M}_e$, respectively.

Thus, if G contains no simplex, the algebra $\mathfrak{M}_T$ is isomorphic to $\mathfrak{M}_o$ or to $\mathfrak{M}_e$. Together with the simplex case considered above, this completes the proof. □

Theorem 3. Let $n, m > 4$. Then the diameter of the modular product $M = C_n \nabla C_m$ satisfies the inequality

$$\operatorname{diam}(M) \leq 3.$$

Proof. Let C_n and C_m be cycles with $n, m > 4$, and let $M = C_n \nabla C_m$ be their modular product. Let $A = (u, v)$ and $B = (x, y)$ be two distinct vertices of M. Denote by

$$a = d_{C_n}(u, x), \qquad b = d_{C_m}(v, y)$$

the distances in the corresponding cycles. Since $A \neq B$, we cannot have $a = b = 0$.

We consider several cases.

Case 1: $u \neq x$ and $v \neq y$ (i.e., $a \geq 1$ and $b \geq 1$).

Subcase 1.1: $a = 1$ and $b = 1$. Then $u \sim x$ and $v \sim y$, hence

$$(u \sim x) \iff (v \sim y),$$

and therefore $A \sim B$. Thus $\operatorname{dist}_M(A, B) = 1$.

Subcase 1.2: $a \geq 2$ and $b \geq 2$. Then $u \nsim x$ and $v \nsim y$, so again

$$(u \sim x) \iff (v \sim y)$$

holds. Hence $A \sim B$ and $\operatorname{dist}_M(A, B) = 1$.

Subcase 1.3: $a = 1$ and $b \geq 2$ (the case $a \geq 2$ and $b = 1$ is symmetric). Then $u \sim x$ and $v \nsim y$, so A and B are not adjacent.

Let u' be the neighbor of x distinct from u. Since $n > 4$, the two neighbors of any vertex in C_n are not adjacent, hence $u' \nsim u$. Similarly, since $b \geq 2$, the vertex v is not adjacent to y. As y has exactly two neighbors in C_m, and v can be adjacent to at most one of them, there exists a neighbor v' of y such that $v' \nsim v$.

Set $C = (u', v')$. Then $A \sim C$, because $u \neq u'$, $v \neq v'$, and $u \nsim u'$, $v \nsim v'$. Moreover, $C \sim B$, since $u' \neq x$, $v' \neq y$, and $u' \sim x$, $v' \sim y$. Thus $A \sim C \sim B$ is a path of length 2, and $\operatorname{dist}_M(A, B) \leq 2$.

Case 2: $u = x$ and $v \neq y$ (i.e., $a = 0$ and $b \geq 1$). (The case $v = y$ and $u \neq x$ is symmetric.)

Since $n > 4$, there exists a vertex $u' \in V(C_n) \setminus \{u\}$ such that $u' \nsim u$. Similarly, since $m > 4$, there exists a vertex $v' \in V(C_m) \setminus \{v, y\}$ such that $v' \nsim v$. Set $C = (u', v')$. Then $A \sim C$ because $u \neq u'$, $v \neq v'$, $u \nsim u'$, and $v \nsim v'$.

Now we consider two subcases.

Subcase 2.1: $v' \nsim y$. Then $C \sim B$ because $u' \neq u$, $v' \neq y$, $u' \nsim u$, and $v' \nsim y$. Hence $A \sim C \sim B$ and $\operatorname{dist}_M(A, B) = 2$.

Subcase 2.2: $v' \sim y$. (Note that $u' \nsim u$ by our choice.) Choose a neighbor u'' of u such that $u'' \nsim u'$. This is possible because in a cycle of length greater than 4, for any vertex u and any vertex u'

with $u' \nsim u$, at least one of the two neighbors of u is not adjacent to u'. Indeed, if both neighbors of u were adjacent to u', then u' would be adjacent to two neighbors of u, which in a cycle of length greater than 4 forces $u' = u$, a contradiction.

Choose a neighbor v'' of y such that $v'' \nsim v'$. This is possible because v' is adjacent to y and, in C_m with $m > 4$, the two neighbors of y are not adjacent; hence the neighbor of y different from v' is not adjacent to v'.

Set $D = (u'', v'')$. Then:
- $C \sim D$, because $u' \neq u''$, $v' \neq v''$, and $u' \nsim u''$, $v' \nsim v''$.
- $D \sim B$, because $u'' \neq u$, $v'' \neq y$, and $u'' \sim u$, $v'' \sim y$.

Thus $A \sim C \sim D \sim B$ is a path of length 3, and $\mathrm{dist}_M(A, B) \leq 3$.

Case 3: $u = x$ and $v = y$ is impossible, since $A \neq B$.

Having considered all cases, we conclude that $\mathrm{dist}_M(A, B) \leq 3$ for any vertices $A, B \in V(M)$. Therefore, $\mathrm{diam}(M) \leq 3$.

The bound is sharp. For example, in $C_5 \nabla C_6$ one has $\mathrm{dist}((0,0),(0,3)) = 3$. □

Theorem 4. If $n, m > 4$, then $C_n \nabla C_m$ contains a triangle.

Proof. Number the vertices of the cycles modulo n and modulo m. Take

$$u_1 = 0,\ u_2 = 1,\ u_3 = 3 \in V(C_n), \qquad v_1 = 0,\ v_2 = 1,\ v_3 = 3 \in V(C_m).$$

For $n, m \geq 5$ these vertices are pairwise distinct in each cycle. In each of the cycles,

$$u_1 \sim u_2, \quad u_1 \not\sim u_3, \quad u_2 \not\sim u_3$$

and similarly for v_i. Two vertices a and b are adjacent in C_k if and only if $|a - b| = 1$ or $|a - b| = k - 1$. Since $|0 - 1| = 1$, while $|0 - 3| = 3$, $|1 - 3| = 2$, and for $k \geq 5$ the numbers $2, 3$ are not equal to 1 or $k - 1$. Therefore, for pairs $i \neq j$ it holds that $u_i \sim u_j \iff v_i \sim v_j$, and also $u_i \neq u_j$, $v_i \neq v_j$. By the definition of the modular product, this means that the vertices $P_i = (u_i, v_i)$ are pairwise adjacent in $C_n \nabla C_m$, i.e. they form K_3. □

Theorem 5. Let T be the theory of the modular product of graphs $C_m \nabla C_n$, where $m, n > 5$, and let $\mathfrak{M}$ be the algebra of binary isolating formulas of the theory T. Then the algebra will have, depending on the diameter, two sets of labels: $\{0, 1, 2\}$ or $\{0, 1, 2, 3\}$.

From Theorems 3 and 4 it follows that if $\mathfrak{M}$ has $diam(C_m \nabla C_n) = 2$, the algebra will have the set $\rho_{\nu(p)} = \{0, 1, 2\}$ of labels, and is defined by the following label products:

$$\begin{array}{ll}
0 \cdot 0 = \{0\}, & 0 \cdot 1 = \{1\}, \\
1 \cdot 0 = \{1\}, & 1 \cdot 1 = \{0, 2\}, \\
1 \cdot 2 = \{0, 1, 2\}, & 2 \cdot 0 = \{2\}, \\
2 \cdot 1 = \{0, 1, 2\}, & 2 \cdot 2 = \{0, 1, 2\}.
\end{array}$$

If $diam(C_m \nabla C_n) = 3$, the algebra $\mathfrak{M}$ will have the set $\rho_{\nu(p)} = \{0, 1, 2, 3\}$ of labels, and is defined by the following label products:

$$\begin{array}{ll}
0 \cdot 0 = \{0\}, & 0 \cdot 1 = \{1\}, \\
1 \cdot 0 = \{1\}, & 1 \cdot 1 = \{0, 1, 2\}, \\
1 \cdot 2 = \{0, 1, 3\}, & 1 \cdot 3 = \{0, 1, 2, 3\}, \\
2 \cdot 0 = \{2\}, & 2 \cdot 1 = \{0, 1, 2, 3\}, \\
2 \cdot 2 = \{0, 1, 2, 3\}, & 2 \cdot 3 = \{0, 1, 2, 3\}, \\
3 \cdot 0 = \{3\}, & 3 \cdot 1 = \{0, 1, 2, 3\}, \\
3 \cdot 2 = \{0, 1, 2, 3\}, & 3 \cdot 3 = \{0, 1, 2, 3\}.
\end{array}$$

As we can see, these algebras are isomorphic to the algebras of simplices for similar diameters.

Conclusion

In this work, we investigated the algebraic properties of the modular product of cycles $C_m \nabla C_n$. It was established that for small parameters $2 < m < 5$ the resulting structures admit simplices, and therefore the algebra of binary isolating formulas extends naturally to include the algebra $\mathfrak{M}_{\mathfrak{s}}$. For larger parameters $n, m > 4$, we proved that the diameter of the modular product is bounded above by 3, and moreover, such graphs necessarily contain triangles.

As a consequence, the algebra $\mathfrak{M}$ associated with the theory of $C_m \nabla C_n$ can be fully characterized by its set of labels, which is $\{0, 1, 2\}$ when the diameter equals 2, and $\{0, 1, 2, 3\}$ when the diameter equals 3. In both cases, the multiplication tables of $\mathfrak{M}$ are shown to be isomorphic to the corresponding simplicial algebras.

Thus, the modular product of cycles not only demonstrates bounded diameter and the presence of complete subgraphs, but also provides a natural algebraic structure that generalizes and unifies previously studied cases. This highlights the deep interplay between graph-theoretic properties of modular products and the algebraic theory of binary isolating formulas.

Examples of derived graphs in modular and other products with their Cayley tables can be viewed on the website https://graph-product.ru

Conflict of Interest

The author declares, no conflict of interest.

References

1 Shulepov, I.V., & Sudoplatov, S.V. (2014). Algebras of distributions for isolating formulas of a complete theory. *Siberian Electronic Mathematical Reports*, *11*, 380–407.

2 Sudoplatov, S.V. (2010). Hypergraphs of prime models and distributions of countable models of small theories. *Journal of Mathematical Sciences*, *169*(5), 680–695. https://doi.org/10.1007/s10958-010-0069-9

3 Sudoplatov, S.V. (2018). *Classification of countable models of complete theories. Part 1.* NSTU Publisher.

4 Baikalova, K.A., Emel'yanov, D.Yu., Kulpeshov, B.Sh., Palyutin, E.A., & Sudoplatov, S.V. (2018). On algebras of distributions of binary isolating formulas for theories of abelian groups and their ordered enrichments. *Russian Mathematics, 62*(4), 1–12. https://doi.org/10.3103/S1066369X18040011

5 Kulpeshov, B.Sh., & Sudoplatov, S.V. (2022). Algebras of binary formulas for weakly circularly minimal theories with non-trivial definable closure. *Lobachevskii Journal of Mathematics*, *43*(12), 3532–3540. https://doi.org/10.1134/S199508022215015X

6 Kulpeshov, B.Sh., & Sudoplatov, S.V. (2024). On algebras of binary isolating formulas for weakly circularly minimal theories of convexity rank 2. *Kazakh Mathematical Journal*, *24*(4), 6–21. https://doi.org/10.70474/kmj24-4-01

7 Kulpeshov, B.Sh., & Sudoplatov, S.V. (2025). On algebras of binary formulas for weakly circularly minimal theories of finite convexity rank. *Siberian Electronic Mathematical Reports*, *22*(1), 635–649. https://doi.org/10.33048/semi.2025.22.041

8 Altaeva, A. B., Kulpeshov, B.Sh., & Sudoplatov, S.V. (2025). Nesushchestvennye obogashcheniia slabo o-minimalnykh teorii konechnogo ranga vypuklosti [Inessential expansions of weakly o-minimal theories of finite convexity rank]. *Doklady Rossiiskoi akademii nauk. Matematika,*

informatika, protsessy upravleniia — Reports of the Russian Academy of Sciences. Mathematics, informatics, control processes, 525, 12–23 [in Russian].

9 Kulpeshov, B.Sh. (2021). A criterion for binarity of almost ω-categorical weakly o-minimal theories. *Siberian Mathematical Journal*, *62*(6), 1063–1075. https://doi.org/10.1134/S003744662 1060082

10 Kulpeshov, B.Sh., & Sudoplatov, S.V. (2024). Algebras of binary formulas for $\aleph_0$-categorical weakly circularly minimal theories: monotonic case. *Bulletin of the Karaganda University. Mathematics Series,* 1(113), 112–127. https://doi.org/10.31489/2024M1/112-127

11 Emel'yanov, D. Yu. (2017). *Algebras of binary isolating formulas for simplex theories.* In *Algebra and Model Theory, 11*, 66–74. NSTU Publisher.

12 Barrow, H., & Burstall, R. (1976). Subgraph isomorphism, matching relational structures and maximal cliques. *Information Processing Letters*, *4*(4), 83–84. https://doi.org/10.1016/0020-0190(76)90049-1

*Author Information**

Dmitry Yurevich Emelyanov — Candidate of Physical and Mathematical Sciences, Associate Professor of the Department of Algebra and Mathematical Logic, Novosibirsk State Technical University, 20 Karl Marx Ave., Novosibirsk, 630073, Russian Federation; e-mail: *dima-pavlyk@mail.ru*; https://orcid.org/0000-0003-4005-6060

*Authors' names are presented in the following order: first name, middle name (if any), last name.

Bulletin of the Karaganda University. Mathematics Series, No. 1(121), 2026, pp. 132–139

https://doi.org/10.31489/2026M1/132-139 *Research article*

On the Equivalence of Elementary Surfaces with Respect to the Motion Group of Pseudo-Euclidean Space

R.A. Gafforov[1,*], S. Kerbal[2], S.S. Juraboyev[1]

[1] *Fergana State University, Fergana, Uzbekistan;*
[2] *Sultan Qaboos University, Muscat, Oman*
(E-mail: gafforov.rahmatjon@mail.ru, skerbal@squ.edu.om, saidaxbor.juraboyev@mail.ru)

This paper investigates the conditions for the equivalence of regular surfaces with respect to the action of a certain subgroup of linear transformations. This subgroup is pseudo-orthogonal and preserves a metric structure defined by a matrix with specific sign properties. The study focuses on elementary surfaces, which are considered as mappings from the square of the parameter domain $(0,1)\times(0,1)$ into an n-dimensional real vector space. The regularity of a surface is determined by the non-vanishing determinant of a special matrix composed of its partial derivatives. The paper also introduces the concept of surface equivalence. The main theorem establishes necessary and sufficient conditions for the equivalence of regular surfaces under the action of the pseudo-orthogonal group. These conditions are expressed through equalities between products of matrices constructed from the partial derivatives of the surfaces and the pseudo-orthogonal matrix. The obtained results provide a theoretical foundation for understanding the relationships between regular surfaces under the action of the pseudo-orthogonal group and contribute to the further study of their geometric properties and transformations.

Keywords: pseudo-orthogonal group, G-equivalent, equivalence of the surfaces, semidirect product of the groups, symplectic group, special pseudo-orthogonal group, action of the surfaces, Euclidean space.

2020 Mathematics Subject Classification: 53A05, 53A07, 53A15, 53A55.

Introduction

In differential geometry, one of the central tasks is to establish convenient criteria for determining the equivalence of elementary surfaces. This involves identifying properties that help us assess whether two surfaces are geometrically identical, even if they are positioned differently in space.

A powerful approach to solving this problem relies on the theory of differential invariants. Differential invariants are properties of surfaces that remain unchanged under specific geometric transformations. By utilizing these invariants, we can formulate equivalence criteria, which simplifies the process of classifying surfaces and identifying their similarities, regardless of their position or orientation in space.

Let $\mathbb{R}^n$ denote an n-dimensional linear space over the field of real numbers $\mathbb{R}$, and let $GL(n,\mathbb{R})$ be the group of all invertible linear transformations of the space $\mathbb{R}^n$. Elements of $\mathbb{R}^n$ are represented as n-dimensional column vectors $\overrightarrow{x} = \{\overrightarrow{x}_j\}_{j=1}^n$, while the transformations $g \in GL(n,\mathbb{R})$ are represented as $n \times n$ matrices $(g_{ij})_{i,j=1}^n$, where $x_i, g_{ij} \in \mathbb{R}$ for $i,j = 1,\ldots,n$. The action of $g \in GL(n,\mathbb{R})$ on the vector $\overrightarrow{x} = \{\overrightarrow{x}_j\}_{j=1}^n \in \mathbb{R}^n$ is given by matrix-vector multiplication, denoted as $g\overrightarrow{x}$.

An infinitely differentiable mapping $x : (0,1)\times(0,1) \to \mathbb{R}^n$ is called an elementary surface. If G is a subgroup of $GL(n,\mathbb{R})$, then two elementary surfaces $\overrightarrow{y}(s,t)$ and $\overrightarrow{x}(s,t)$ are said to be G-equivalent if $\overrightarrow{y}(s,t) = g\overrightarrow{x}(s,t)$ for some $g \in G$ and for all $(s,t) \in (0,1)\times(0,1)$.

In this paper, we reformulate the problem of G-equivalence of elementary surfaces for the pseudo-orthogonal group $O(n,p,\mathbb{R})$ using the language of differential algebra. This reformulation facilitates

*Corresponding author. *E-mail: gafforov.rahmatjon@mail.ru*
Received: 1 January 2025; *Accepted:* 1 November 2025.

the application of an algebraic approach to solve the problem. Such an approach has previously been employed to derive the necessary and sufficient conditions for surface equivalence in the context of actions by the general linear, special linear [1], orthogonal, pseudo-orthogonal [2], symplectic [3], and special pseudo-orthogonal groups [4–6].

1 Preliminaries

Let $O(n,p,\mathbb{R})$ be the pseudo-orthogonal subgroup of $GL(n,\mathbb{R})$, i.e.,

$$O(n,p,\mathbb{R}) = \{g \in GL(n,\mathbb{R}) : g^T e_p g = e_p\},$$

where g^T is the transpose of the matrix g, and e_p is an identity matrix in $GL(n,\mathbb{R})$ given by $e_p = (e^p_{ij})^n_{i,j=1}$, where

$$e^p_{ij} = \begin{cases} 1 & \text{for } i = 1, 2, \dots, p, \\ -1 & \text{for } i = p+1,\, p+2, \dots,\, n, \\ 0 & \text{for } i \neq j,\, i,j \in \{1,2,\dots,n\} \end{cases}$$

for some $p \in \{1,\dots,n-1\}$.

For each elementary surface $\overrightarrow{x}(s,t) = (x_j(s,t))^n_{j=1}$, we define $M_s(\overrightarrow{x})$ as the $n \times n$ matrix $(m_{ij}(s,t))^n_{i,j=1}$, where the i-th column has coordinates $m_{ij}(s,t) = \frac{\partial^{i-1} x_j(s,t)}{\partial s^{i-1}}$, with $i,j = 1,\dots,n$, and we set $\frac{\partial^0 x_j(s,t)}{\partial s^0} = x_j(s,t)$ for all $j = 1,\dots,n,\;\; s,t \in (0,1)$.

Let $M_{ss}(\overrightarrow{x})$ be the matrix $\left\{\frac{\partial^i x_j(s,t)}{\partial s^i}\right\}^n_{i,j=1}$, $M_{sss}(\overrightarrow{x})$ be the matrix $\left\{\frac{\partial^{i+1} x_j(s,t)}{\partial s^{i+1}}\right\}^n_{i,j=1}$, and $M_{st}(\overrightarrow{x})$ be the matrix $\left\{\frac{\partial^i x_j(s,t)}{\partial s^{i-1}\partial t}\right\}^n_{i,j=1}$, while $M_{sst}(\overrightarrow{x})$ is the matrix $\left\{\frac{\partial^{i+1} x_j(s,t)}{\partial s^i \partial t}\right\}^n_{i,j=1}$.

Below, we consider only regular surfaces, i.e., elementary surfaces $\overrightarrow{x}(s,t)$ such that the determinant $\det M_s(\overrightarrow{x})(s,t) \neq 0$ for all $s,t \in (0,1)$.

Let $Aff(\mathbb{R}^n)$ be the group of all affine transformations of the n-dimensional linear space $\mathbb{R}^n$. Each affine transformation in $Aff(\mathbb{R}^n)$ is a composition of a non-degenerate linear transformation $g \in GL(n,\mathbb{R})$ and a translation by an element $\overrightarrow{u} = (u_i)^n_{i=1} \in \mathbb{R}^n$, i.e., any affine transformation $(\overrightarrow{u}, g) \in Aff(\mathbb{R}^n)$ acts in $\mathbb{R}^n$ according to the following rule:

$$(\overrightarrow{u}, g)(\overrightarrow{x}) = g\overrightarrow{x} + \overrightarrow{u},$$

where $\overrightarrow{x}, \overrightarrow{u} \in \mathbb{R}^n$ and $g \in GL(n,\mathbb{R})$.

The multiplication operation in $Aff(\mathbb{R}^n)$ is defined by

$$(\overrightarrow{u}, g)(\overrightarrow{v}, h) = (\overrightarrow{u} + g\overrightarrow{v}, gh),$$

where $\overrightarrow{u}, \overrightarrow{v} \in \mathbb{R}^n$ and $g,h \in GL(n,\mathbb{R})$. This implies that $Aff(\mathbb{R}^n)$ is a semidirect product of the groups $\mathbb{R}^n$ and $GL(n,\mathbb{R})$, written as

$$Aff(\mathbb{R}^n) = \mathbb{R}^n \lhd GL(n,\mathbb{R}).$$

If G is a subgroup of $GL(n,\mathbb{R})$, then the set

$$\mathbb{R}^n \lhd G = \{(\overrightarrow{u}, g) \in \mathbb{R}^n \lhd GL(n,\mathbb{R}) : g \in G\}$$

is a subgroup of $\mathbb{R}^n \lhd GL(n,\mathbb{R})$, and is also referred to as the semidirect product of the groups $\mathbb{R}^n$ and G.

It is well-known (see, for example, [7, Chapter XVII, §2]) that the group $\mathbb{R}^n \lhd O(n,\mathbb{R})$ coincides with the group of all motions in Euclidean space $(\mathbb{R}^n, (.,.))$, i.e., the group of all bijections U from $\mathbb{R}^n$ to $\mathbb{R}^n$ such that $(Ux, Uy) = (x, y)$ for all $x, y \in \mathbb{R}^n$. Similarly, the group $\mathbb{R}^n \lhd O(n,p,\mathbb{R})$ is the group of all motions in pseudo-Euclidean space $(\mathbb{R}^n, [.,.]_p)$, i.e., the group of all bijections V from $\mathbb{R}^n$ to $\mathbb{R}^n$ such that

$$[Vx, Vy]_p = [x, y]_p \text{ for all } x, y \in \mathbb{R}^n,$$

(see, for example, [8, Chapter III, §1]).

Let G be a subgroup of $\mathbb{R}^n \lhd GL(n,\mathbb{R})$. Two regular surfaces $\overrightarrow{x}(s,t)$ and $\overrightarrow{y}(s,t)$, defined in $\mathbb{R}^n$, are said to be G-equivalent if there exists $(\overrightarrow{u}, g) \in \mathbb{R}^n$ such that:

$$\overrightarrow{y}(s,t) = g\overrightarrow{x}(s,t) + \overrightarrow{u} \text{ for all } (s,t) \in (0,1).$$

The following statement reduces the problem of $\mathbb{R}^n \lhd G$-equivalence of regular surfaces $\overrightarrow{x}(s,t)$ and $\overrightarrow{y}(s,t)$ to the problem of G-equivalence of regular surfaces $\overrightarrow{x}'_s(s,t)$ and $\overrightarrow{y}'_s(s,t)$.

Statement 1. Two regular surfaces $\overrightarrow{x}(s,t)$ and $\overrightarrow{y}(s,t)$, defined in $\mathbb{R}^n$, are $\mathbb{R}^n \lhd G$-equivalent if and only if the regular surfaces $\overrightarrow{x}'_s(s,t)$ and $\overrightarrow{y}'_s(s,t)$ are G-equivalent.

Proof. If two regular surfaces $\overrightarrow{x}(s,t)$ and $\overrightarrow{y}(s,t)$ are $\mathbb{R}^n \lhd G$-equivalent, then there exist $\overrightarrow{u} = \{u_i\}_{i=1}^n \in \mathbb{R}^n$ and $g = \{g_{ij}\}_{i,j=1}^n \in G$ such that

$$\overrightarrow{y}(s,t) = g\overrightarrow{x}(s,t) + \overrightarrow{u} \text{ for all } (s,t) \in (0,1).$$

For each $i = 1, \ldots, n$, we have

$$\overrightarrow{y}_i(s,t) = \sum_{j=1}^n g_{ij}\overrightarrow{x}_j(s,t) + u_i.$$

Taking the derivatives with respect to s, we get

$$\overrightarrow{y}'_i(s,t) = \sum_{j=1}^n g_{ij}\overrightarrow{x}'_j(s,t). \tag{1}$$

Thus, for all $i = 1, \ldots, n$, we have

$$\overrightarrow{y}'_s(s,t) = g\overrightarrow{x}'_s(s,t), \quad (s,t) \in (0,1),$$

which implies that the regular surfaces $\overrightarrow{x}'_s(s,t)$ and $\overrightarrow{y}'_s(s,t)$ are G-equivalent.

Conversely, suppose that

$$\overrightarrow{y}'_s(s,t) = g\overrightarrow{x}'_s(s,t) \text{ for all } (s,t) \in (0,1),$$

for some $g = \{g_{ij}\}_{i,j=1}^n \in G$. From equation (1), we deduce that

$$\overrightarrow{y}_i(s,t) = \sum_{j=1}^n g_{ij}\overrightarrow{x}_j(s,t) + u_i,$$

where $u_i(s,t)$ is a constant (independent of s and t). To find $u_i(s,t)$, we define $\overrightarrow{u}_i(s,t) = \overrightarrow{y}_i(s,t) - \sum_{j=1}^n g_{ij}\overrightarrow{x}_j(s,t)$, and compute the derivative with respect to s:

$$\overrightarrow{u}'_i(s,t) = 0 \text{ for all } (s,t) \in (0,1).$$

Since $\overrightarrow{u}'_i(s,t) = 0$, we conclude that $\overrightarrow{u}_i(s,t) = \overrightarrow{u}_i(0)$, a constant for each i. Therefore, we define $\overrightarrow{u} = \{u_i(0)\}_{i=1}^n \in \mathbb{R}^n$, and obtain the relation

$$\overrightarrow{y}(s,t) = g\overrightarrow{x}(s,t) + \overrightarrow{u} \text{ for all } (s,t) \in (0,1).$$

This shows that the two regular surfaces $\overrightarrow{x}(s,t)$ and $\overrightarrow{y}(s,t)$ are $\mathbb{R}^n \lhd G$-equivalent.

Thus, we have proven that two regular surfaces $\overrightarrow{x}(s,t)$ and $\overrightarrow{y}(s,t)$ are $\mathbb{R}^n \lhd G$-equivalent if and only if $\overrightarrow{x}'_s(s,t)$ and $\overrightarrow{y}'_s(s,t)$ are G-equivalent. □

2 Equivalence of Elementary Surfaces

The following theorem provides necessary and sufficient conditions for the $\mathbb{R}^n \lhd O(n,p,\mathbb{R})$-equivalence of regular surfaces $\overrightarrow{x}(s,t)$ and $\overrightarrow{y}(s,t)$.

Theorem 1. Two regular surfaces $\overrightarrow{x}(s,t)$ and $\overrightarrow{y}(s,t)$ are $\mathbb{R}^n \lhd O(n,p,\mathbb{R})$-equivalent if and only if for any $(s,t) \in (0,1)$, the following equalities hold:
(a) $M_{ss}^{-1}(\overrightarrow{x}(s,t))M_{sss}(\overrightarrow{x}(s,t)) = M_{ss}^{-1}(\overrightarrow{y}(s,t))M_{sss}(\overrightarrow{y}(s,t))$;
(b) $M_{st}^{-1}(\overrightarrow{x}(s,t))M_{sst}(\overrightarrow{x}(s,t)) = M_{st}^{-1}(\overrightarrow{y}(s,t))M_{sst}(\overrightarrow{y}(s,t))$;
(c) $M_{ss}^{T}(\overrightarrow{x}(s,t))e_pM_{ss}(\overrightarrow{x}(s,t)) = M_{ss}^{T}(\overrightarrow{x}(s,t))e_pM_{ss}(\overrightarrow{y}(s,t))$.

Proof. Let the surfaces $\overrightarrow{x}(s,t)$ and $\overrightarrow{y}(s,t)$ be $\mathbb{R}^n \lhd O(n,p,\mathbb{R})$-equivalent, i.e., there exists an element $g \in \mathbb{R}^n \lhd O(n,p,\mathbb{R})$ such that the following holds:

$$\overrightarrow{y}(s,t) = g\overrightarrow{x}(s,t) + \overrightarrow{u}.$$

Consequently, by the definition of the matrices $M_{ss}(\overrightarrow{x})$, we have

$$M_{ss}(\overrightarrow{y}) = gM_{ss}(\overrightarrow{x}).$$

We will now show that from this equality, the validity of equations (a), (b), and (c) follows. Indeed:

$$M_{ss}^{-1}(\overrightarrow{y}(s,t))M_{sss}(\overrightarrow{y}(s,t)) = (gM_{ss}(\overrightarrow{x}(s,t)))^{-1}(gM_{sss}(\overrightarrow{x}(s,t))).$$

This simplifies to

$$(M_{ss}(\overrightarrow{x}(s,t)))^{-1}(g^{-1}g)(M_{sss}(\overrightarrow{x}(s,t))) = M_{ss}^{-1}(\overrightarrow{x}(s,t))M_{sss}(\overrightarrow{x}(s,t)),$$

which shows that (a) holds.

Similarly,
$$M_{st}^{-1}(\overrightarrow{y}(s,t))M_{sst}(\overrightarrow{y}(s,t)) = (gM_{st}(\overrightarrow{x}(s,t)))^{-1}(gM_{sst}(\overrightarrow{x}(s,t))).$$

This simplifies to

$$(M_{st}(\overrightarrow{x}(s,t)))^{-1}(g^{-1}g)(M_{sst}(\overrightarrow{x}(s,t))) = M_{st}^{-1}(\overrightarrow{x}(s,t))M_{sst}(\overrightarrow{x}(s,t)),$$

which shows that (b) holds.

Finally, for equation (c), we have

$$M_{ss}^{T}(\overrightarrow{y}(s,t))e_pM_{ss}(\overrightarrow{y}(s,t)) = (gM_{ss}(\overrightarrow{x}(s,t)))^{T}e_p(gM_{ss}(\overrightarrow{x}(s,t))).$$

This simplifies to

$$(M_{ss}(\overrightarrow{x}(s,t)))^{T}(g^{T}e_pg)(M_{ss}(\overrightarrow{x}(s,t))) = M_{ss}^{T}(\overrightarrow{x}(s,t))e_pM_{ss}(\overrightarrow{x}(s,t)),$$

which shows that (c) holds.

Conversely, assume that the relations (a), (b), and (c) hold for the surfaces $\overrightarrow{x}(s,t)$ and $\overrightarrow{y}(s,t)$. Since $A(s,t) = A_s$ is invertible, assume that equalities (a) and (b) are valid. Differentiating the equality $A_s^{-1}A_s = A_sA_s^{-1} = E$, we get
$$(A^{-1})_s = -(A_s)^{-1}A_{ss}(A_s)^{-1}.$$

$$M_{ss}(\overrightarrow{x}(s,t)) \cdot M_{ss}^{-1}(\overrightarrow{x}(s,t)) = E,$$
$$(M_{ss}(\overrightarrow{x}(s,t)) \cdot M_{ss}^{-1}(\overrightarrow{x}(s,t)))_s = 0,$$

$$M_{sss}(\overrightarrow{x}(s,t)) \cdot M_{ss}^{-1}(\overrightarrow{x}(s,t)) + M_{ss}(\overrightarrow{x}(s,t)) \cdot M_{sss}^{-1}(\overrightarrow{x}(s,t)) = 0,$$

$$M_{ss}(\overrightarrow{x}(s,t)) \cdot M_{sss}^{-1}(\overrightarrow{x}(s,t)) = -M_{sss}(\overrightarrow{x}(s,t)) \cdot M_{ss}^{-1}(\overrightarrow{x}(s,t)),$$

$$M_{sss}^{-1}(\overrightarrow{x}(s,t)) = -M_{ss}^{-1}(\overrightarrow{x}(s,t)) \cdot M_{sss}(\overrightarrow{x}(s,t)) \cdot M_{ss}^{-1}(\overrightarrow{x}(s,t)),$$

$$M_{ss}(\overrightarrow{y}(s,t)) = gM_{ss}(\overrightarrow{x}(s,t)),$$

$$(M_{ss}(\overrightarrow{y}(s,t))M_{ss}^{-1}(\overrightarrow{x}(s,t)))_s = 0.$$

Therefore, the equalities (a) and (b) can be rewritten as

$$\text{(a')}\ (M_{ss}(\overrightarrow{y}(s,t)) \cdot M_{ss}^{-1}(\overrightarrow{x}(s,t)))_s = 0,$$

$$\text{(b')}\ (M_{ss}(\overrightarrow{y}(s,t)) \cdot M_{ss}^{-1}(\overrightarrow{x}(s,t)))_t = 0.$$

These qualities imply that

$$M_{ss}(\overrightarrow{y}(s,t)) \cdot M_{ss}^{-1}(\overrightarrow{x}(s,t)) = g = (g_{ij})_{i,j=1}^n \in GL(n,\mathbb{R}),$$

where $s,t \in (0,1)$. Consequently, we obtain

$$M_{ss}(\overrightarrow{y}(s,t)) = gM_{ss}(\overrightarrow{x}(s,t)),$$

and in particular,

$$\overrightarrow{y}'_s(s,t) = g\overrightarrow{x}'_s(s,t) \quad \text{for all } (s,t) \in (0,1).$$

Thus, for $g = \{g_{ij}\}_{i,j=1}^n$, $\overrightarrow{y}(s,t) = \{y_i(t)\}_{i=1}^n$, and $\overrightarrow{x}(s,t) = \{x_i(t)\}_{i=1}^n$, we have

$$\overrightarrow{y}'_i(s,t) = \sum_{j=1}^n g_{ij}\overrightarrow{x}'_j(s,t).$$

Let $\overrightarrow{u}_i = \overrightarrow{y}_i(s,t) - \sum_{j=1}^n g_{ij}\overrightarrow{x}_j(s,t)$, which yields

$$\overrightarrow{u} = \{u_i\}_{i=1}^n \in \mathbb{R}^n,$$

and for the element $(\overrightarrow{u}, g) \in \mathbb{R}^n \lhd O(n,p,\mathbb{R})$, we have the equality

$$\overrightarrow{y}(s,t) = g\overrightarrow{x}(s,t) + \overrightarrow{u} \quad \text{for all} \ \ (s,t) \in (0,1).$$

Furthermore, due to equation (c), we have

$$g^T e_p g = (M_{ss}(\overrightarrow{y}(s,t)) \cdot M_{ss}^{-1}(\overrightarrow{x}(s,t)))^T e_p (M_{ss}(\overrightarrow{y}(s,t)) \cdot M_{ss}^{-1}(\overrightarrow{x}(s,t))) = e_p,$$

which implies that

$$g^T e_p g = e_p.$$

Thus, $g \in O(n,p,\mathbb{R})$. Theorem 1 is proved. □

Now consider the problem of the equivalence of k-dimensional surfaces. An infinitely differentiable mapping $x : [0,1]^k \to \mathbb{R}^n$, where $1 \le k < n$, is called a parameterized k-dimensional surface in the finite-dimensional vector space $\mathbb{R}^n$.

Definition 1. [9] Two k-dimensional surfaces $\overrightarrow{x}(t_1,\dots,t_k)$ and $\overrightarrow{y}(t_1,\dots,t_k)$ are said to be G-equivalent if there exists an element $g \in G$ such that

$$\overrightarrow{y}(t_1,\dots,t_k) = g\overrightarrow{x}(t_1,\dots,t_k)$$

for all $(t_1,\dots,t_k) \in (0,1)^k$.

Definition 2. [3] A function f of $\overrightarrow{x}(t_1,\dots,t_k)$ and its finite number of partial derivatives is called G-invariant if its values coincide for G-equivalent surfaces.

Let $\overrightarrow{x}(t_1,\dots,t_k)$ be a k-dimensional surface, and let $s \in \{1,\dots,k\}$ be a fixed index. For each surface $\overrightarrow{x}(t_1,\dots,t_k)$, the $(n \times n)$-matrix $M_{t_s}(\overrightarrow{x})$ has the form

$$M_{t_s}(\overrightarrow{x}) = (\overrightarrow{x}^{\,0}_{t_s},\dots,\overrightarrow{x}^{\,n-1}_{t_s}),$$

where the i-th column consists of the coordinates

$$\frac{\partial^{i-1}x_j(t_1,\dots,t_k)}{\partial t_s^{i-1}}, \quad j=1,\dots,n, \quad i=1,\dots,n.$$

Let $M_{t_st_s}(\overrightarrow{x})$ be the matrix

$$M_{t_st_s}(\overrightarrow{x}) = \left\{\frac{\partial^i x_j(t_1,\dots,t_k)}{\partial t_s^i}\right\}_{i,j=1}^n,$$

and let $M_{t_st_st_l}(\overrightarrow{x})$ denote the matrix

$$M_{t_st_st_l}(\overrightarrow{x}) = \left(\frac{\partial^2 x(t_1,\dots,t_k)}{\partial t_s\partial t_l},\dots,\frac{\partial^{n+1}x(t_1,\dots,t_k)}{\partial t_s^n\partial t_l}\right).$$

Throughout the rest of the text, only regular surfaces are considered, i.e., surfaces $\overrightarrow{x}(t_1,\dots,t_k)$ for which

$$\det M_{t_s}(\overrightarrow{x})(t_1,\dots,t_k) \neq 0$$

for all $(t_1,\dots,t_k) \in (0,1)^k$ and $s=1,\dots,k$.

The following theorem provides the necessary and sufficient conditions for $\mathbb{R}^n \lhd O(n,p,\mathbb{R})$-equivalence of two surfaces.

Theorem 2. Two surfaces $x(t_1,\dots,t_k)$ and $y(t_1,\dots,t_k)$ in $\mathbb{R}^n$ are $\mathbb{R}^n \lhd O(n,p,\mathbb{R})$-equivalent if and only if the following equalities hold
1. $M^{-1}_{t_st_s}(\overrightarrow{x})M_{t_st_st_l}(\overrightarrow{x}) = M^{-1}_{t_st_s}(\overrightarrow{y})M_{t_st_st_l}(\overrightarrow{y})$,
2. $(M_{t_st_s}(\overrightarrow{x}))^T e_p M_{t_st_s}(\overrightarrow{x}) = (M_{t_st_s}(\overrightarrow{y}))^T e_p M_{t_st_s}(\overrightarrow{y})$,

for all $l=1,\dots,k$.

The proof of this theorem follows a similar approach to the previously established theorem.

Remark. For two-dimensional regular surfaces, Theorem 1 under the action of the groups $G(n,\mathbb{C})$, $SL(n,\mathbb{C})$, $O(n,\mathbb{C})$, $SO(n,\mathbb{C})$, and $Sp(2n,\mathbb{C})$ was obtained in [2,3,10], while for m-dimensional regular surfaces $(m<n)$ under the action of the same groups, it was obtained in [5]. For the semidirect product of the same groups, the result was obtained in [4]. For two-dimensional regular surfaces, Theorem 1 and Theorem 2 under the action of the group $SO(n,p,K)$ were obtained in [6], while for k-dimensional regular surfaces $(k<n)$ under the action of $SO(p,q,K)$, they were obtained in [9]. Additionally, the above theorems for paths and curves were studied in [11], while the differential invariants of surfaces were examined in [12].

Conclusion

In the article, the problem of determining the equivalence of elementary and k-dimensional surfaces is considered. As the main result, the necessary and sufficient conditions for the equivalence of surfaces with respect to the action of the pseudo-orthogonal group are identified. The equivalence of surfaces under the action of this pseudo-orthogonal group is proven through the relations between special matrices constructed on the basis of the partial derivatives of the surfaces. The results presented in the article can be applied in the future for the classification and study of surfaces in differential geometry.

Author Contributions

All authors contributed equally to this work.

Conflict of Interest

The authors declare no conflict of interest.

References

1 Bekbaev, U.D. (2005). On differential rational invariants of finite subgroups of affine group. *Bulletin of the Malaysian Mathematical Sciences Society, Series 2, 28*(1), 55–60.

2 Muminov, K.K. (2005). Ekvivalentnost putei i poverkhnostei dlia deistviia psevdoortogonalnoi gruppy [Equivalence of paths and surfaces for the action of a pseudo-orthogonal group]. *Uzbekskii matematicheskii zhurnal — Uzbek Mathematical Journal, 2,* 35–43 [in Russian].

3 Muminov, K.K. (1997). Ekvivalentnost poverkhnostei v kompleksnykh vektornykh prostranstvakh otnositelno $Sp(2,C)$ grupp [Equivalence of surfaces in complex vector spaces with respect to $Sp(2,C)$ groups]. *Uzbekskii matematicheskii zhurnal — Uzbek Mathematical Journal, 2*, 53–57 [in Russian].

4 Bekbaev, U.D., & Muminov, K.K. (2004). On differential rational invariants of the classical groups of motions of a vector space. *Methods of Functional Analysis and Topology, 10*(3), 26–30.

5 Muminov, K.K. (2010). Equivalence of multidimensional surfaces with to the acting of classical groups. *Uzbek Mathematical Journal 1,* 99–107.

6 Muminov, K.K., & Gafforov, R.A. (2024). Systems of matrix differential equations for surfaces. *Journal of Mathematical Sciences, 278*(4), 623–632. https://doi.org/10.1007/s10958-024-06944-1

7 Aleksandrov, P.S. (1988). *Kurs analiticheskoi geometrii i lineinoi algebry [Course of analytical geometry and linear algebra].* Moscow: Nauka [in Russian].

8 Rosenfeld, B.A. (1969). *Neevklidovy prostranstva [Non-Euclidean Spaces].* Moscow: Nauka [in Russian].

9 Gafforov, R.A. (2022). Equivalence of multidimensional surfaces under the action of the special pseudo-orthogonal group. *Bulletin of the Institute of Mathematics, 5*(4), 48–54.

10 Muminov, K.K., & Bekbaev, U.D. (1997). Ob ekvivalentnosti i invariantakh elementarnykh poverkhnostei otnositelno simplekticheskoi gruppy [On equivalence and invariants of elementary surfaces with respect to the symplectic group]. *Uzbekskii matematicheskii zhurnal — Uzbek Mathematical Journal, 4*, 26–30 [in Russian].

11 Muminov, K.K., & Chilin, V.I. (2015). *Ekvivalentnost krivykh v konechnomernykh vektornykh prostranstvakh [Equivalence of curves in finite-dimensional spaces].* Deutschland: LAMBERT Academic Publishing [in Russian].

12 Olver, P.J. (2009). Differential invariants of surfaces. *Differential Geometry and its Applications, 27*(2), 230–239. https://doi.org/10.1016/j.difgeo.2008.06.020

*Author Information**

Rakhmatjon Abdukakhkhorovich Gafforov (*corresponding author*) — Doctor of Philosophy (PhD) in Physical and Mathematical Sciences, Associate Professor, Fergana State University, 19 Murabbiylar St., Fergana, 150100, Uzbekistan; e-mail: *gafforov.rahmatjon@mail.ru*

Sebti Kerbal — Doctor of Physical and Mathematical Sciences, Professor, Department of Mathematics, Sultan Qaboos University, P.O. Box 36, Al-Khoudh 123, Muscat, Oman, FracDiff Research Group DR/RG/03, Sultan Qaboos University, Muscat, Oman; e-mail: *skerbal@squ.edu.om*

Saidaxbor Solijonovich Juraboyev — Doctor of Physical and Mathematical Sciences, Teacher, Fergana State University, 19 Murabbiylar St., Fergana, 150100, Uzbekistan; e-mail: *saidaxbor.juraboyev@mail.ru*

*Authors' names are presented in the following order: first name, middle name (if any), last name.

https://doi.org/10.31489/2026M1/140-155 *Research article*

Integral representations for a class of triple confluent hypergeometric functions and their applications in boundary value problems

A. Hasanov[1,3], T.G. Ergashev[1,2,3], A.R. Ryskan[4,5,*]

[1] *Institute of Mathematics of Uzbekistan Academy of Sciences, Tashkent, Uzbekistan;*
[2] *"TIIAME" National Research University, Tashkent, Uzbekistan;*
[3] *Ghent University, Ghent, Belgium;*
[4] *Narxoz University, Almaty, Kazakhstan;*
[5] *UC Davis, Davis, USA*
(E-mail: anvarhasanov@yahoo.com, ergashev.tukhtasin@gmail.com, ryskan.a727@gmail.com)

Hypergeometric functions are divided into complete and confluent functions. Srivastava and Karlsson were the first to propose a method for constructing the complete set of triple Gaussian hypergeometric series and compiled a table containing definitions and regions of convergence for 205 distinct complete series in three variables. Subsequently, several authors obtained various integral representations and transformation formulas for the functions introduced by Srivastava and Karlsson. More recently, Ergashev identified 395 hypergeometric series of three variables that represent confluent forms of the known 205 complete hypergeometric series. In the present study, new Euler-type integral representations are derived for certain Gaussian hypergeometric functions of three variables. The main results are obtained using properties of the gamma and beta functions. New integral representations are established for 14 functions from the list of confluent hypergeometric functions of three variables. All derived integrals can be regarded as generalized Euler type representations of the classical Gaussian hypergeometric functions of one and two variables. In addition, it is demonstrated how one of these confluent functions, together with its integral representation, can be applied to construct solutions of the three-dimensional singular Helmholtz equation.

Keywords: Gaussian hypergeometric function, Appell functions, Humbert functions, Srivastava–Karlsson hypergeometric functions, triple confluent hypergeometric series, integral representation, singular Helmholtz equation, application of hypergeometric functions.

2020 Mathematics Subject Classification: 33C15, 33C20, 33C65, 35C15, 35J75.

Introduction

The great interest in the theory of hypergeometric functions (including functions of one, two or more variables) is primarily due to the fact that hypergeometric functions allow us to find solutions to various applied problems related to thermal conductivity and dynamic processes, electromagnetic oscillations, aerodynamics, quantum mechanics and potential theory. These functions, which relate to higher and special functions [1–3], are often called special functions of mathematical physics.

It is known that hypergeometric function $F(a, b; c; z)$ was studied by Leonhard Euler, but the first complete and systematic interpretation of it was given by Carl Friedrich Gauss in 1813. In the Gaussian hypergeometric function $F(a, b; c; z)$ there are two numerator parameters a, b, and one denominator parameter c. A natural generalization of this function is to introduce an arbitrary number of parameters for both the numerator and the denominator. The resulting function, denoted as ${}_pF_q$, is called the generalized Gauss function or generalized hypergeometric function (for more information, see [4, p. 19]). In 1880, Appell introduced four series F_1 to F_4, each of which is an analogue of the Gauss function $F(a, b; c; z)$. Horn [5] introduced the following ten hypergeometric functions in two variables, denoting

*Corresponding author. *E-mail: ryskan.a727@gmail.com*
Received: 17 January 2025; *Accepted:* 5 January 2026.

them as G_1, G_2, G_3, H_1, ..., H_7; he thus completed the set of all possible second-order (complete) hypergeometric functions in two variables [4, p. 24]. Humbert defined seven confluent forms for the four Appell functions [6], and he denoted these confluent hypergeometric functions in two variables by Φ_1, Φ_2, Φ_3, Ψ_1, Ψ_2, Ξ_1, Ξ_2. In addition, there exist 13 confluent forms of the Horn functions, which are denoted by Γ_1, Γ_2, H_1, ..., H_{11} [5] (see, also [7]). A significant contribution to the further development of the theory of hypergeometric series in two variables was made by Horn, who proposed a general definition and classification of double hypergeometric series. He studied the convergence properties of hypergeometric series in two variables and identified systems of partial differential equations to which these series correspond. Horn investigated hypergeometric series of the second order. He found that among them there are series that are expressed through one variable, or are products of two hypergeometric series, each of which depends on one variable. In addition, according to his conclusions, there are 14 complete and 20 confluent different convergent series of the second order. Definitions and convergence conditions for all 34 hypergeometric series in two variables are also given in [7].

Lauricella [8, p. 114] further generalized the four Applell series F_1, F_2, F_3, F_4 to series in n variables and defined his multiple hypergeometric series denoted by $F_A^{(n)}$, $F_B^{(n)}$, $F_C^{(n)}$ and $F_D^{(n)}$; in this work he introduced 14 complete hypergeometric series in three variables of the second order . He denoted his triple hypergeometric series by the symbols F_1, F_2, ..., F_{14} of which F_1, F_2, F_5 and F_9 correspond, respectively, to the three-variable Lauricella series $F_A^{(3)}$, $F_B^{(3)}$, $F_C^{(3)}$ and $F_D^{(3)}$. The remaining series F_3, F_4, F_6, F_7,F_8, F_{10},..., F_{14} of Lauricella's set apparently fell into oblivion. Saran [9] initiated a systematic study of these ten triple hypergeometric series of Lauricella's set. Sahai and Verma [10, 11] proved new recursion and infinite summation formulas for the triple Lauricella functions. Currently, Bezrodnykh [12–14] has obtained interesting results in the study of Lauricella functions.

In further study of Lauricella's 14 hypergeometric series in three variables, Srivastava [15, 16] discovered three additional complete triple hypergeometric series of the second order. These series, labeled H_A, H_B, and H_C, were not part of Lauricella's set and had not been previously reported in the literature. At present, the properties of various generalizations of the Srivastava's triple hypergeometric functions are being studied [17], integral representations are established for them, and they are applied to solving fractional differential equations [18]. A (p, q)-extensions of H_A, H_B, and H_C are defined and investigated in [19–21], respectively. Applications of the hypergeometric structures to the theory of Feynman integrals are found recently in [22].

An extended presentation of results on hypergeometric functions of three variables, as well as references to the original sources, is presented in the monograph by Srivastava and Karlsson [4], which is considered a classic work in this area. This monograph contains an extensive bibliography, including all relevant publications up to 1985. In particular, the authors compiled a table of 205 different complete triple hypergeometric Gauss series, accompanied by references to their sources, if known. Realizing the importance of integral representations of multiple hypergeometric functions for solving applied problems, Hasanov and Ruzhansky [23] developed Euler-type integral representations for all 205 complete triple hypergeometric functions. Authors of the paper [24] established several new more interesting integral representations of the Euler type and Laplace type for ten Gauss hypergeometric functions of three variables. Later, a systems of partial differential equations that the indicated 205 functions satisfy are constructed and all their linearly independent solutions near the origin are found, in those cases where such solutions exist [25].

However, comparatively less attention has been paid to the study of confluent hypergeometric functions of three variables. In the work of Jain [26], individual functions representing confluent forms of complete hypergeometric functions of three variables were investigated. In his paper [27] Ergashev identified 395 degenerate hypergeometric functions of three variables, denoting them as E_1, ..., E_{395}. He thus probably completed the classification of all possible second-order confluent hypergeometric functions for three variables. The study also includes an analysis of systems of partial differential equations associated with these 395 functions. In addition, particular solutions of some systems of differential equations near the origin were found, if such solutions exist.

Here we present integral representations for the functions E_1, ..., E_{14} defined in [27]. We also demonstrate the application of the confluent hypergeometric function E_2 by constructing particular solutions of the three-dimensional Helmholtz equation with singular coefficients.

This paper uses standard definitions and notations, including the Pochhammer symbol $(\lambda)_n$, the beta function $\mathrm{B}(x, y)$, the gamma function $\Gamma(z)$, the Gauss hypergeometric function and its generalization ${}_pF_q$ [7], the Appell functions [28], the Humbert functions [6], the Horn functions [5] in two variables, and the complete [4] and confluent [27] hypergeometric functions of three variables.

1 *Preliminaries*

A function

$$F(a,b;c;z) \equiv F\left[\begin{array}{ll} a,b; & \\ c; & z \end{array}\right] := \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k} \frac{z^k}{k!}, |z|<1, \;\; c \neq 0,-1,-2,... \tag{1}$$

is known as the Gaussian hypergeometric function.

The Gaussian hypergeometric series $F(a, b; c; z)$ includes two numerator parameters a and b, and one denominator parameter c. Its natural generalization is the introduction of an arbitrary number of parameters in both the numerator and the denominator. The resulting series

$${}_pF_q\left[\begin{array}{ll} (a_p); & \\ (b_q); & z \end{array}\right] \equiv {}_pF_q\left[(a_p);(b_q);z\right] := \sum_{k=0}^{\infty} \frac{\prod_{j=1}^{p} (a_j)_k}{\prod_{j=1}^{q} (b_j)_k} \frac{z^k}{k!}, \;\; |z|<1,$$

is known as the generalized Gauss series [7, p. 182], or simply, the generalized hypergeometric series. Here p and q are positive integers or zero (interpreting an empty product as 1), and we assume that the variable z, the numerator parameters a_1,..., a_p, and the denominator parameters b_1, ..., b_q take on complex values, provided that $b_j \neq 0, -1, -2, \ldots; \; j = 1, \ldots, q$. In general (that is, except for certain integer values of the parameters for which the series terminates or is undefined) ${}_pF_q$ converges for all finite z if $p \leq q$, converges for $|z| < 1$ if $p = q + 1$, and diverges for all $z \neq 0$ if $p > q + 1$.

Gauss' series (1) in the present notation is ${}_2F_1(a, b; c; z) \equiv F(a, b; c; z)$.

The double Appell hypergeometric functions are defined as following [28]:

$$F_1\left(a,b,b';c;x,y\right) = \sum_{m,n=0}^{\infty} \frac{(a)_{m+n}(b)_m\left(b'\right)_n}{(c)_{m+n} m! n!} x^m y^n, \quad \max\left\{|x|,|y|\right\}<1, \tag{2}$$

$$F_2\left(a,b,b';c,c';x,y\right) = \sum_{m,n=0}^{\infty} \frac{(a)_{m+n}(b)_m\left(b'\right)_n}{(c)_m\left(c'\right)_n m! n!} x^m y^n, \quad |x|+|y|<1, \tag{3}$$

$$F_3\left(a,a',b,b';c;x,y\right) = \sum_{m,n=0}^{\infty} \frac{(a)_m\left(a'\right)_n(b)_m\left(b'\right)_n}{(c)_{m+n} m! n!} x^m y^n, \quad \max\left\{|x|,|y|\right\}<1, \tag{4}$$

$$F_4\left(a,b;c,c';x,y\right) = \sum_{m,n=0}^{\infty} \frac{(a)_{m+n}(b)_{m+n}}{(c)_m\left(c'\right)_n m! n!} x^m y^n, \quad \sqrt{|x|}+\sqrt{|y|}<1, \tag{5}$$

here, in all definitions (2)–(5), as usual, the denominator parameters c and c' are neither zero nor a negative integer.

Seven confluent forms of the four Appell series were introduced by Humbert [6], who denoted these confluent hypergeometric series of two variables as follows:

$$\Phi_1(\alpha,\beta;\gamma;x,y)=\sum_{m,n=0}^{\infty}\frac{(\alpha)_{m+n}(\beta)_m}{(\gamma)_{m+n}m!n!}x^m y^n,\ \ |x|<1, \tag{6}$$

$$\Phi_2\left(\beta,\beta';\gamma;x,y\right)=\sum_{m,n=0}^{\infty}\frac{(\beta)_m(\beta')_n}{(\gamma)_{m+n}m!n!}x^m y^n, \tag{7}$$

$$\Phi_3(\beta;\gamma;x,y)=\sum_{m,n=0}^{\infty}\frac{(\beta)_m}{(\gamma)_{m+n}\,m!n!}x^m y^n, \tag{8}$$

$$\Psi_1\left(\alpha,\beta;\gamma,\gamma';x,y\right)=\sum_{m,n=0}^{\infty}\frac{(\alpha)_{m+n}(\beta)_m}{(\gamma)_m(\gamma')_n m!n!}x^m y^n,\ \ |x|<1, \tag{9}$$

$$\Psi_2\left(\alpha;\gamma,\gamma';x,y\right)=\sum_{m,n=0}^{\infty}\frac{(\alpha)_{m+n}}{(\gamma)_m(\gamma')_n\,m!n!}x^m y^n, \tag{10}$$

$$\Xi_1\left(\alpha,\alpha',\beta;\gamma;x,y\right)=\sum_{m,n=0}^{\infty}\frac{(\alpha)_m(\alpha')_n(\beta)_m}{(\gamma)_{m+n}\,m!n!}x^m y^n,\ \ |x|<1, \tag{11}$$

$$\Xi_2(\alpha,\beta;\gamma;x,y)=\sum_{m,n=0}^{\infty}\frac{(\alpha)_m(\beta)_m}{(\gamma)_{m+n}\,m!n!}x^m y^n,\ \ |x|<1, \tag{12}$$

where the denominator parameters γ and γ' are neither zero nor a negative integer. The hypergeometric functions defined in (6)–(12) are called *Humbert functions*.

In this paper we will establish integral representations for the following functions:

$$\mathrm{E}_1(a_1,a_2,a_3,a_4,a_5;c;x,y,z)=\sum_{m,n,p=0}^{\infty}\frac{(a_1)_m(a_2)_m(a_3)_n(a_4)_n(a_5)_p}{(c)_{m+n+p}}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!},\ \ \frac{1}{|x|}+\frac{1}{|y|}>1, \tag{13}$$

$$\mathrm{E}_2(a_1,a_2,a_3,a_4;c;x,y,z)=\sum_{m,n,p=0}^{\infty}\frac{(a_1)_m(a_2)_m(a_3)_n(a_4)_n}{(c)_{m+n+p}}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!},\ \ \frac{1}{|x|}+\frac{1}{|y|}>1, \tag{14}$$

$$\mathrm{E}_3(a_1,a_2,a_3,a_4;c;x,y,z)=\sum_{m,n,p=0}^{\infty}\frac{(a_1)_m(a_2)_m(a_3)_n(a_4)_p}{(c)_{m+n+p}}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!},\ \ |x|<1, \tag{15}$$

$$\mathrm{E}_4(a_1,a_2,a_3;c;x,y,z)=\sum_{m,n,p=0}^{\infty}\frac{(a_1)_m(a_2)_m(a_3)_n}{(c)_{m+n+p}}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!},\ \ |x|<1, \tag{16}$$

$$\mathrm{E}_5(a_1,a_2,a_3;c;x,y,z)=\sum_{m,n,p=0}^{\infty}\frac{(a_1)_m(a_2)_n(a_3)_p}{(c)_{m+n+p}}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}, \tag{17}$$

$$\mathrm{E}_6(a,b;c;x,y,z)=\sum_{m,n,p=0}^{\infty}\frac{(a)_m(b)_n}{(c)_{m+n+p}}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}, \tag{18}$$

$$\mathrm{E}_7(a_1,a_2,a_3,a_4;c;x,y,z)=\sum_{m,n,p=0}^{\infty}\frac{(a_1)_{m+n}(a_2)_m(a_3)_n(a_4)_p}{(c)_{m+n+p}}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!},\ \ |x|<1,\ |y|<1, \tag{19}$$

$$\mathrm{E}_8\left(a_1, a_2, a_3, a_4; c; x, y, z\right) = \sum_{m,n,p=0}^{\infty} \frac{(a_1)_{m+n}(a_2)_m(a_3)_p(a_4)_p}{(c)_{m+n+p}} \frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}, \quad \frac{1}{|x|} + \frac{1}{|z|} > 1, \tag{20}$$

$$\mathrm{E}_9\left(a_1, a_2, a_3; c; x, y, z\right) = \sum_{m,n,p=0}^{\infty} \frac{(a_1)_{m+n}(a_2)_m(a_3)_n}{(c)_{m+n+p}} \frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}, \quad |x| < 1,\ |y| < 1, \tag{21}$$

$$\mathrm{E}_{10}\left(a_1, a_2, a_3; c; x, y, z\right) = \sum_{m,n,p=0}^{\infty} \frac{(a_1)_{m+n}(a_2)_m(a_3)_p}{(c)_{m+n+p}} \frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}, \quad |x| < 1, \tag{22}$$

$$\mathrm{E}_{11}\left(a, b; c; x, y, z\right) = \sum_{m,n,p=0}^{\infty} \frac{(a)_{m+n}(b)_m}{(c)_{m+n+p}} \frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}, \quad |x| < 1, \tag{23}$$

$$\mathrm{E}_{12}\left(a_1, a_2, a_3; c; x, y, z\right) = \sum_{m,n,p=0}^{\infty} \frac{(a_1)_{m+n}(a_2)_{n+p}(a_3)_m}{(c)_{m+n+p}} \frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}, \quad |x| < 1,\ |y| < 1, \tag{24}$$

$$\mathrm{E}_{13}\left(a, b; c; x, y, z\right) = \sum_{m,n,p=0}^{\infty} \frac{(a)_{m+n}(b)_{n+p}}{(c)_{m+n+p}} \frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}, \quad |y| < 1, \tag{25}$$

$$\mathrm{E}_{14}\left(a_1, a_2, a_3; c; x, y, z\right) = \sum_{m,n,p=0}^{\infty} \frac{(a_1)_{m+n+p}(a_2)_m(a_3)_n}{(c)_{m+n+p}} \frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}, \quad |x| < 1,\ |y| < 1, \tag{26}$$

New integral transforms for the two-variable analogous of the confluent hypergeometric functions (13)–(26) are found in [29].

2 Single integral Representations

Theorem 1. If $\mathrm{Re}(\alpha) > 0$ and $\mathrm{Re}(\beta) > 0$, then each of the following integral representation for E_1–E_8 holds true:

$$\mathrm{E}_1\left(\alpha, a_2, \beta, a_4, a_5; c; x, y, z\right) = k_1 \int_0^1 \xi^{\alpha-1}\left(1-\xi\right)^{\beta-1} \mathrm{E}_7\left(\alpha+\beta, a_2, a_4, a_5; c; x\xi, y - y\xi, z\right) d\xi, \tag{27}$$

$$\mathrm{E}_2\left(\alpha, a_2, \beta, a_4, a_5; c; x, y, z\right) = k_1 \int_0^1 \xi^{\alpha-1}\left(1-\xi\right)^{\beta-1} \mathrm{E}_9\left(\alpha+\beta, a_2, a_4; c; x\xi, y - y\xi, z\right) d\xi, \tag{28}$$

$$\mathrm{E}_3\left(\alpha, a_2, \beta, a_4; c; x, y, z\right) = k_1 \int_0^1 \xi^{\alpha-1}\left(1-\xi\right)^{\beta-1} \mathrm{E}_{10}\left(\alpha+\beta, a_2, a_4; c; x\xi, y - y\xi, z\right) d\xi, \tag{29}$$

$$\mathrm{E}_4\left(a_1, \alpha, \beta; c; x, y, z\right) = k_1 \int_0^1 \xi^{\alpha-1}\left(1-\xi\right)^{\beta-1} \mathrm{E}_{11}\left(\alpha+\beta, a_1; c; x\xi, y - y\xi, z\right) d\xi, \tag{30}$$

$$\mathrm{E}_5\left(\alpha, \beta, a; c; x, y, z\right) = k_1 \int_0^1 \xi^{\alpha-1}\left(1-\xi\right)^{\beta-1} \Xi_2\left(\alpha+\beta, a; c; x\xi + y - y\xi, z\right) d\xi, \tag{31}$$

$$\mathrm{E}_6\left(\alpha, \beta; c; x, y, z\right) = k_1 \int_0^1 \xi^{\alpha-1}\left(1-\xi\right)^{\beta-1} \Phi_3\left(\alpha+\beta; c; x\xi + y - y\xi, z\right) d\xi, \tag{32}$$

$$\mathrm{E}_7\left(a, \alpha, \beta, b; c; x, y, z\right) = k_1 \int_0^1 \xi^{\alpha-1}\left(1-\xi\right)^{\beta-1} \Xi_1\left(a, b, \alpha+\beta; c; x\xi + y - y\xi, z\right) d\xi, \tag{33}$$

$$\mathrm{E}_8\left(a_1,\alpha,\beta,a_4;c;x,y,z\right)=k_1\int_0^1\xi^{\alpha-1}\left(1-\xi\right)^{\beta-1}\mathrm{E}_{12}\left(\alpha+\beta,a_1,a_4;c;z-z\xi,x\xi,y\right)d\xi,\qquad(34)$$

where $k_1=\dfrac{\Gamma\left(\alpha+\beta\right)}{\Gamma\left(\alpha\right)\Gamma\left(\beta\right)}$.

Proof. Using the definition of the Beta function

$$\mathrm{B}(\alpha,\beta)=\int_0^1\xi^{\alpha-1}(1-\xi)^{\beta-1}d\xi,\ \ Re\alpha>0,\ \ Re\beta>0,$$

it is easy to establish the relation

$$\frac{(\alpha)_m(\beta)_n}{(\alpha+\beta)_{m+n}}=\frac{\Gamma\left(\alpha+\beta\right)}{\Gamma\left(\alpha\right)\Gamma\left(\beta\right)}\mathrm{B}(\alpha+m,\beta+n),\ \ m,n=0,1,2,\dots,$$

i.e.,

$$\frac{(\alpha)_m(\beta)_n}{(\alpha+\beta)_{m+n}}=k_1\int_0^1\xi^{\alpha-1+m}(1-\xi)^{\beta-1+n}d\xi,\ \ Re\alpha>0,\ \ Re\beta>0,\ \ m,n=0,1,2,\dots,\qquad(35)$$

where $(\lambda)_\nu=\Gamma(\lambda+\nu)/\Gamma(\lambda)$ is a Pochhammer symbol.

Applying the relation (35), we obtain the integral representations (27)–(30).

The proof of the representations (31)–(34) can be distinguished from the others by using the well-known property of double power series

$$\sum_{m,n=0}^{\infty}f(m+n)\frac{x^m}{m!}\frac{y^n}{n!}=\sum_{k=0}^{\infty}f(k)\frac{(x+y)^k}{k!}.\qquad(36)$$

To give an example, by virtue of the definition of E_5, we have

$$\mathrm{E}_5\left(\alpha,\beta,a;c;x,y,z\right)=\sum_{m,n,p=0}^{\infty}\frac{(\alpha+\beta)_{m+n}(a)_p}{(c)_{m+n+p}}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}\int_0^1\xi^{\alpha-1+m}(1-\xi)^{\beta-1+n}d\xi$$

$$=\int_0^1\xi^{\alpha-1}(1-\xi)^{\beta-1}\sum_{m,n,p=0}^{\infty}\frac{(\alpha+\beta)_{m+n}(a)_p}{(c)_{m+n+p}}\frac{(x\xi)^m}{m!}\frac{(y-y\xi)^n}{n!}\frac{z^p}{p!}d\xi.$$

Then using the property (36), we obtain

$$\mathrm{E}_5\left(\alpha,\beta,a;c;x,y,z\right)=\int_0^1\xi^{\alpha-1}(1-\xi)^{\beta-1}\sum_{k,p=0}^{\infty}\frac{(\alpha+\beta)_k(a)_p}{(c)_{k+p}}\frac{(x\xi+y-y\eta)^k}{k!}\frac{z^p}{p!}d\xi,$$

which asserts the validity of the representation (31). The Theorem 1 is proven. □

Theorem 2. If $\mathrm{Re}\left(\beta\right)>\mathrm{Re}\left(\alpha\right)>0$, then the following integral representations are valid:

$$\mathrm{E}_7\left(\alpha,a_2,a_3,a_4;\beta;x,y,z\right)=k_2\int_0^1\frac{\xi^{\alpha-1}\left(1-\xi\right)^{\beta-\alpha-1}}{\left(1-x\xi\right)^{a_2}\left(1-y\xi\right)^{a_3}}{}_1F_1\left(a_4,\beta-\alpha;z-z\xi\right)d\xi,\qquad(37)$$

$$\mathrm{E}_8\left(\alpha,a_2,a_3,a_4;\beta;x,y,z\right)=k_2\int_0^1\frac{\xi^{\alpha-1}\left(1-\xi\right)^{\beta-\alpha-1}}{\left(1-x\xi\right)^{a_2}}e^{y\xi}{}_2F_1\left(a_3,a_4;\beta-\alpha;z-z\xi\right)d\xi,\qquad(38)$$

$$\mathrm{E}_9\left(\alpha,a_2,a_3;\beta;x,y,z\right)=k_2\int_0^1\frac{\xi^{\alpha-1}\left(1-\xi\right)^{\beta-\alpha-1}}{\left(1-x\xi\right)^{a_2}\left(1-y\xi\right)^{a_3}}{}_0F_1\left(-;\beta-\alpha;z-z\xi\right)d\xi,\qquad(39)$$

$$\mathrm{E}_{10}\left(\alpha, a_2, a_3; \beta; x, y, z\right) = k_2 \int_0^1 \frac{\xi^{\alpha-1}\left(1-\xi\right)^{\beta-\alpha-1}}{\left(1-x\xi\right)^{a_2}} e^{y\xi}{}_1F_1\left(a_3; \beta-\alpha; z-z\xi\right) d\xi, \tag{40}$$

$$\mathrm{E}_{11}\left(\alpha, b; \beta; x, y, z\right) = k_2 \int_0^1 \frac{\xi^{\alpha-1}\left(1-\xi\right)^{\beta-\alpha-1}}{\left(1-x\xi\right)^{b}} e^{y\xi}{}_0F_1\left(-; \beta-\alpha; z-z\xi\right) d\xi, \tag{41}$$

$$\mathrm{E}_{12}\left(\alpha, a_2, a_3; \beta; x, y, z\right) = k_2 \int_0^1 \frac{\xi^{\alpha-1}\left(1-\xi\right)^{\beta-\alpha-1}}{\left(1-x\xi\right)^{a_3}\left(1-y\xi\right)^{a_2}} {}_1F_1\left(a_2; \beta-\alpha; \frac{z\left(1-\xi\right)}{1-y\xi}\right) d\xi, \tag{42}$$

$$\mathrm{E}_{13}\left(\alpha, b; \beta; x, y, z\right) = k_2 \int_0^1 \frac{\xi^{\alpha-1}\left(1-\xi\right)^{\beta-\alpha-1}}{\left(1-y\xi\right)^{b}} e^{x\xi}{}_1F_1\left(b; \beta-\alpha; \frac{z\left(1-\xi\right)}{1-y\xi}\right) d\xi, \tag{43}$$

$$\mathrm{E}_{14}\left(\alpha, a_2, a_3; \beta; x, y, z\right) = k_2 \int_0^1 \frac{\xi^{\alpha-1}\left(1-\xi\right)^{\beta-\alpha-1}}{\left(1-x\xi\right)^{a_2}\left(1-y\xi\right)^{a_3}} e^{z\xi} d\xi, \tag{44}$$

$$\mathrm{E}_{14}\left(a_1, \alpha, a_3; \beta; x, y, z\right) = k_2 \int_0^1 \frac{\xi^{\alpha-1}\left(1-\xi\right)^{\beta-\alpha-1}}{\left(1-x\xi\right)^{a_1}} \Phi_1\left(a_1, a_3; \beta-\alpha; \frac{y\left(1-\xi\right)}{1-x\xi}, \frac{z\left(1-\xi\right)}{1-x\xi}\right) d\xi, \tag{45}$$

where $k_2 = \dfrac{\Gamma\left(\beta\right)}{\Gamma\left(\alpha\right)\Gamma\left(\beta-\alpha\right)}$.

Proof. To prove integral representations (37)–(45), Euler's formula for Gauss function [7, p. 59]

$$F(a, b; c; x) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int_0^1 \frac{\xi^{a-1}(1-\xi)^{c-a-1}}{(1-x\xi)^b} d\xi, \quad \mathrm{Re}(c) > \mathrm{Re}(a) > 0 \tag{46}$$

and the integral representation formula for Appell function

$$F_1(a, b_1, b_2; c; x, y) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int_0^1 \frac{\xi^{a-1}(1-\xi)^{c-a-1}}{(1-x\xi)^{b_1}(1-y\xi)^{b_2}} d\xi, \quad \mathrm{Re}(c) > \mathrm{Re}(a) > 0$$

are used.

Let's consider the last equality (45) in Theorem 2. One can represent the confluent hypergeometric function E_{14} in the form

$$\mathrm{E}_{14}\left(a_1, \alpha, a_3; \beta; x, y, z\right) = \sum_{n,p=0}^{\infty} \frac{\left(a_1\right)_{n+p}\left(a_3\right)_n}{\left(\beta\right)_{n+p}} F\left(\alpha, a_1+n+p; \beta+n+p; x\right) \frac{y^n}{n!} \frac{z^p}{p!}.$$

Then, using the Euler's formula (46) and the definition of the Humbert function Φ_1 we obtain the integral representation (45). The remaining integral representations in the Theorem 2 are proved similarly. The Theorem 2 is proven. □

3 Double integral Representations

Theorem 3. If $\mathrm{Re}\left(\alpha\right) > 0$, $\mathrm{Re}\left(\beta\right) > 0$ and $\mathrm{Re}\left(\gamma\right) > 0$, then the following integral representations are valid:

$$\begin{aligned}\mathrm{E}_1\left(a_1, a_2, a_3, a_4, a_5; \alpha+\beta+\gamma; x, y, z\right) =& k_3 \int_0^1 \int_0^1 \xi^{\alpha-1}\left(1-\xi\right)^{\beta+\gamma-1}\eta^{\beta-1}\left(1-\eta\right)^{\gamma-1} \times \\ &\times {}_2F_1\left(a_1, a_2; \alpha; x\xi\right){}_2F_1\left(a_3, a_4; \beta; y\eta\left(1-\xi\right)\right){}_1F_1\left(a_5; \gamma; z\left(1-\xi\right)\left(1-\eta\right)\right) d\xi d\eta, \end{aligned} \tag{47}$$

$$\mathrm{E}_1\left(\alpha, a_2, \beta, a_4, \gamma; \alpha+\beta+\gamma; x, y, z\right) = k_3 \int_0^1 \int_0^1 \xi^{\alpha-1} \left(1-\xi\right)^{\beta+\gamma-1} \eta^{\beta-1} \left(1-\eta\right)^{\gamma-1} \times$$
$$\times \left(1-x\xi\right)^{-a_2} \left(1-y\eta+y\xi\eta\right)^{-a_4} e^{z(1-\xi)(1-\eta)} d\xi d\eta, \tag{48}$$

$$\mathrm{E}_1\left(\alpha, \beta, \gamma, a_4, a_5; c; x, y, z\right) = k_3 \int_0^1 \int_0^1 \xi^{\alpha-1} \left(1-\xi\right)^{\beta+\gamma-1} \eta^{\beta-1} \left(1-\eta\right)^{\gamma-1} \times$$
$$\times \mathrm{E}_{15}\left(\alpha+\beta+\gamma, a_4, a_5; c; x\xi\left(1-\xi\right)\eta, y\left(1-\xi\right)\left(1-\eta\right), z\right) d\xi d\eta, \tag{49}$$

$$\mathrm{E}_1\left(\alpha, a_2, \beta, a_4, \gamma; c; x, y, z\right) = k_3 \int_0^1 \int_0^1 \xi^{\alpha-1} \left(1-\xi\right)^{\beta-1} \eta^{\alpha+\beta-1} \left(1-\eta\right)^{\gamma-1} \times$$
$$\times \mathrm{E}_{14}\left(a_2+\gamma, a_2, a_4; c; x\xi\eta, y\left(1-\xi\right)\eta, z\left(1-\eta\right)\right) d\xi d\eta, \tag{50}$$

$$\mathrm{E}_1\left(\alpha, a_2, \beta, a_4, \gamma; a_2+\gamma; x, y, z\right) = k_3 \int_0^1 \int_0^1 \xi^{\alpha-1} \left(1-\xi\right)^{\beta-1} \eta^{\alpha+\beta-1} \left(1-\eta\right)^{\gamma-1} \times$$
$$\times \left(1-x\xi\eta\right)^{-a_2} \left(1-y\eta+y\xi\eta\right)^{-a_4} e^{z(1-\eta)} d\xi d\eta, \tag{51}$$

$$\mathrm{E}_2\left(a_1, a_2, a_3, a_4, a_5; \alpha+\beta+\gamma; x, y, z\right) = k_3 \int_0^1 \int_0^1 \xi^{\alpha-1} \left(1-\xi\right)^{\beta+\gamma-1} \eta^{\beta-1} \left(1-\eta\right)^{\gamma-1} \times$$
$$\times {}_2F_1\left(a_1, a_2; \alpha; x\xi\right) {}_2F_1\left(a_3, a_4; \beta; y\eta\left(1-\xi\right)\right) {}_0F_1\left(-; \gamma; z\left(1-\xi\right)\left(1-\eta\right)\right) d\xi d\eta, \tag{52}$$

$$\mathrm{E}_2\left(\alpha, a_2, c_2, a_4, a_5; \alpha+\beta+\gamma; x, y, z\right) = k_3 \int_0^1 \int_0^1 \xi^{\alpha-1} \left(1-\xi\right)^{\beta+\gamma-1} \eta^{\beta-1} \left(1-\eta\right)^{\gamma-1} \times$$
$$\times \left(1-x\xi\right)^{-a_2} \left(1-y\eta+y\xi\eta\right)^{-a_4} {}_0F_1\left(-; \gamma; z\left(1-\xi\right)\left(1-\eta\right)\right) d\xi d\eta, \tag{53}$$

$$\mathrm{E}_2\left(\alpha, \beta, \gamma, a_4, a_5; c; x, y, z\right) = k_3 \int_0^1 \int_0^1 \xi^{\alpha-1} \left(1-\xi\right)^{\beta+\gamma-1} \eta^{\beta-1} \left(1-\eta\right)^{\gamma-1} \times$$
$$\times \mathrm{E}_{17}\left(\alpha+\beta+\gamma, a_4; c; x\xi\left(1-\xi\right)\eta, y\left(1-\xi\right)\left(1-\eta\right), z\right) d\xi d\eta, \tag{54}$$

$$\mathrm{E}_3\left(a_1, a_2, a_3, a_4; \alpha+\beta+\gamma; x, y, z\right) = k_3 \int_0^1 \int_0^1 \xi^{\alpha-1} \left(1-\xi\right)^{\beta+\gamma-1} \eta^{\beta-1} \left(1-\eta\right)^{\gamma-1} \times$$
$$\times {}_2F_1\left(a_1, a_2; \alpha; x\xi\right) {}_1F_1\left(a_3; \beta; y\eta\left(1-\xi\right)\right) {}_1F_1\left(a_4; \gamma; z\left(1-\xi\right)\left(1-\eta\right)\right) d\xi d\eta, \tag{55}$$

$$\mathrm{E}_3\left(\alpha, a_2, \beta, a_4; \alpha+\beta+\gamma; x, y, z\right) =$$
$$= k_3 \int_0^1 \int_0^1 \xi^{\alpha-1} \left(1-\xi\right)^{\beta+\gamma-1} \eta^{\beta-1} \left(1-\eta\right)^{\gamma-1} \left(1-x\xi\right)^{-a_2} e^{(1-\xi)(y\eta+z-z\eta)} d\xi d\eta, \tag{56}$$

$$\mathrm{E}_3\left(\alpha, a_2, a_3, a_4; c; x, y, z\right) = k_3 \int_0^1 \int_0^1 \xi^{\alpha-1} \left(1-\xi\right)^{\beta-1} \eta^{\alpha+\beta-1} \left(1-\eta\right)^{\gamma-1} \times$$
$$\times \Phi_1\left(\alpha+\beta+\gamma, a_2; c; y\eta\left(1-\xi\right)+z\left(1-\eta\right), x\xi\eta\right) d\eta d\xi, \tag{57}$$

$$\mathrm{E}_3\left(\alpha, \beta, \gamma, a_4; c; x, y, z\right) = k_3 \int_0^1 \int_0^1 \xi^{\alpha-1} \left(1-\xi\right)^{\beta+\gamma-1} \eta^{\beta-1} \left(1-\eta\right)^{\gamma-1} \times$$
$$\times \mathrm{E}_{18}\left(\alpha+\beta+\gamma, b; c; x\xi\left(1-\xi\right)\eta, y\left(1-\xi\right)\left(1-\eta\right), z\right) d\xi d\eta, \tag{58}$$

$$\mathrm{E}_4\left(a_1,a_2,a_3;\alpha+\beta+\gamma;x,y,z\right)=k_3\int_0^1\int_0^1\xi^{\alpha-1}\left(1-\xi\right)^{\beta+\gamma-1}\eta^{\beta-1}\left(1-\eta\right)^{\gamma-1}\times$$
$$\times{}_2F_1\left(a_1,a_2;\alpha;x\xi\right){}_1F_1\left(a_3;\beta;y\eta\left(1-\xi\right)\right){}_0F_1\left(-;\gamma;z\left(1-\xi\right)\left(1-\eta\right)\right)d\xi d\eta, \tag{59}$$

$$\mathrm{E}_4\left(\alpha,\beta,\gamma;c;x,y,z\right)=k_3\int_0^1\int_0^1\xi^{\alpha-1}\left(1-\xi\right)^{\beta+\gamma-1}\eta^{\beta-1}\left(1-\eta\right)^{\gamma-1}\times$$
$$\times\mathrm{E}_{19}\left(\alpha+\beta+\gamma;c;x\xi\left(1-\xi\right)\eta,y\left(1-\xi\right)\left(1-\eta\right),z\right)d\xi d\eta, \tag{60}$$

$$\mathrm{E}_5\left(a_1,a_2,a_3;\alpha+\beta+\gamma;x,y,z\right)=k_3\int_0^1\int_0^1\xi^{\alpha-1}\left(1-\xi\right)^{\beta+\gamma-1}\eta^{\beta-1}\left(1-\eta\right)^{\gamma-1}\times$$
$$\times{}_1F_1\left(a_1;\alpha;x\xi\right){}_1F_1\left(a_2;\beta;y\eta\left(1-\xi\right)\right){}_1F_1\left(a_3;\gamma;z\left(1-\xi\right)\left(1-\eta\right)\right)d\xi d\eta, \tag{61}$$

$$\mathrm{E}_6\left(a,b;\alpha+\beta+\gamma;x,y,z\right)=k_3\int_0^1\int_0^1\xi^{\alpha-1}\left(1-\xi\right)^{\beta+\gamma-1}\eta^{\beta-1}\left(1-\eta\right)^{\gamma-1}\times$$
$$\times{}_1F_1\left(a;\alpha;x\xi\right){}_1F_1\left(b;\beta;y\eta\left(1-\xi\right)\right){}_0F_1\left(-;\gamma;z\left(1-\xi\right)\left(1-\eta\right)\right)d\xi d\eta, \tag{62}$$

$$\mathrm{E}_7\left(a_1,a_2,a_3,a_4;\alpha+\beta+\gamma;x,y,z\right)=k_3\int_0^1\int_0^1\xi^{\alpha-1}\left(1-\xi\right)^{\beta+\gamma-1}\eta^{\beta-1}\left(1-\eta\right)^{\gamma-1}\times$$
$$\times F_2\left(a_1;a_2,a_3;\alpha,\beta;x\xi,y\eta\left(1-\xi\right)\right){}_1F_1\left(a_4;\gamma;z\left(1-\xi\right)\left(1-\eta\right)\right)d\xi d\eta, \tag{63}$$

$$\mathrm{E}_7\left(\beta,a_2,a_3,a_4;\alpha+\beta+\gamma;x,y,z\right)=k_3\int_0^1\int_0^1\frac{\xi^{\alpha-1}\left(1-\xi\right)^{\beta+\gamma-1}\eta^{\beta-1}\left(1-\eta\right)^{\gamma-1}}{\left(1-y\eta+y\xi\eta\right)^{a_3}}\times$$
$$\times F_1\left(a_2;\beta-\gamma,a_3;\alpha;x\xi,\frac{x\xi}{1-y\eta+y\xi\eta}\right){}_1F_1\left(a_4;\gamma;z\left(1-\xi\right)\left(1-\eta\right)\right)d\xi d\eta, \tag{64}$$

$$\mathrm{E}_7\left(\alpha,a_2,a_3,a_4;2\alpha+\beta;x,y,z\right)=k_4\int_0^1\int_0^1\frac{\xi^{\alpha-1}\left(1-\xi\right)^{\alpha+\beta-1}\eta^{\alpha-1}\left(1-\eta\right)^{\beta-1}}{\left(1-x\xi\right)^{a_2}\left(1-y\eta+y\xi\eta\right)^{a_3}}\times$$
$$\times{}_2F_1\left(a_2,a_3;\alpha;\frac{xy\xi\eta\left(1-\xi\right)}{\left(1-x\xi\right)\left(1-y\eta+y\xi\eta\right)}\right){}_1F_1\left(a_4;\beta;z\left(1-\xi\right)\left(1-\eta\right)\right)d\xi d\eta, \tag{65}$$

$$\mathrm{E}_7\left(a_1,a_2,a_3,a_4;\alpha+\beta+\gamma;x,y,z\right)=$$
$$=k_3\int_0^1\int_0^1\frac{\xi^{\alpha-1}\left(1-\xi\right)^{\beta+\gamma-1}\eta^{\beta-1}\left(1-\eta\right)^{\gamma-1}}{\left(1-x\xi-y\eta+y\xi\eta\right)^{a_1}}{}_1F_1\left(a_4;\gamma;z\left(1-\xi\right)\left(1-\eta\right)\right)\times$$
$$\times F_2\left(a_1,\alpha-a_2,\beta-a_3;\alpha,\beta;\frac{x\xi}{x\xi+y\eta-y\xi\eta-1},\frac{y}{x\xi+y\eta-y\xi\eta-1}\right)d\xi d\eta, \tag{66}$$

$$\mathrm{E}_7\left(a_1,\alpha,\beta,\gamma;c;x,y,z\right)=k_3\int_0^1\int_0^1\xi^{\alpha-1}\left(1-\xi\right)^{\beta-1}\eta^{\alpha+\beta-1}\left(1-\eta\right)^{\gamma-1}\times$$
$$\times\Phi_1\left(\alpha+\beta+\gamma,a_1;c;z-z\eta,x\xi\eta+y\eta-y\xi\eta\right)d\xi d\eta, \tag{67}$$

$$\mathrm{E}_7\left(a_1,\alpha,\beta,\gamma;c;x,y,z\right)=k_3\int_0^1\int_0^1\frac{\xi^{\alpha-1}\eta^{\alpha+\beta-1}\left(1-\xi\right)^{\beta-1}\left(1-\eta\right)^{\gamma-1}}{\left(1-z+z\eta\right)^{a_1}}\times$$
$$\times e^{x\xi\eta+y\eta-y\xi\eta}\Phi_1\left(c-\alpha-\beta-\gamma,a_1;c;\frac{z-z\eta}{z-z\eta-1},-x\xi\eta-y\eta+y\xi\eta\right)d\xi d\eta, \tag{68}$$

$$\mathrm{E}_8\left(a_1,a_2,a_3,a_4;\alpha+\beta+\gamma;x,y,z\right)=k_3\int_0^1\int_0^1\xi^{\alpha-1}\left(1-\xi\right)^{\beta+\gamma-1}\eta^{\beta-1}\left(1-\eta\right)^{\gamma-1}\times$$
$$\times\Psi_1\left(a_1,a_2;\beta,\alpha;y\eta-y\xi\eta,x\xi\right){}_2F_1\left(a_3,a_4;\gamma;z\left(1-\xi\right)\left(1-\eta\right)\right)d\xi d\eta, \tag{69}$$

$$\mathrm{E}_8\left(\alpha,a_2,a_3,a_4;c;x,y,z\right)=k_3\int_0^1\int_0^1\xi^{\alpha-1}\left(1-\xi\right)^{\beta-1}\eta^{\alpha+\beta-1}\left(1-\eta\right)^{\gamma-1}\times$$
$$\times\mathrm{E}_{21}\left(\alpha+\beta+\gamma,a_4;c;x\xi\eta^2,z,y\xi\eta\right)d\xi d\eta, \tag{70}$$

$$\mathrm{E}_9\left(a_1,a_2,a_3;\alpha+\beta+\gamma;x,y,z\right)=k_3\int_0^1\int_0^1\xi^{\alpha-1}\left(1-\xi\right)^{\beta+\gamma-1}\eta^{\beta-1}\left(1-\eta\right)^{\gamma-1}\times$$
$$\times F_2\left(a_1,a_2,a_3;\alpha,\beta;x\xi,y\eta-y\xi\eta\right){}_0F_1\left(-;\gamma;z\left(1-\xi\right)\left(1-\eta\right)\right)d\xi d\eta, \tag{71}$$

$$\mathrm{E}_9\left(a_1,a_2,a_3;\alpha+\beta+\gamma;x,y,z\right)=k_3\int_0^1\int_0^1\frac{\xi^{\alpha-1}\left(1-\xi\right)^{\beta+\gamma-1}\eta^{\beta-1}\left(1-\eta\right)^{\gamma-1}}{\left(1-y\eta+y\xi\eta\right)^{a_3}}\times$$
$$\times F_1\left(a_2;\beta-a_3,a_3;\alpha;x\xi,\frac{x\xi}{1-y\eta+y\xi\eta}\right){}_0F_1\left(-;\gamma;z\left(1-\xi\right)\left(1-\eta\right)\right)d\xi d\eta, \tag{72}$$

$$\mathrm{E}_{10}\left(a_1,a_2,a_3;\alpha+\beta+\gamma;x,y,z\right)=k_3\int_0^1\int_0^1\xi^{\alpha-1}\left(1-\xi\right)^{\beta+\gamma-1}\eta^{\beta-1}\left(1-\eta\right)^{\gamma-1}\times$$
$$\times\Psi_1\left(a_1,a_2;\alpha,\beta;x\xi,y\eta\left(1-\xi\right)\right){}_1F_1\left(a_3;\gamma;z\left(1-\xi\right)\left(1-\eta\right)\right)d\xi d\eta, \tag{73}$$

$$\mathrm{E}_{10}\left(a_1,a_2,a_3;\alpha+\beta+\gamma;x,y,z\right)=k_3\int_0^1\int_0^1\xi^{\alpha-1}\left(1-\xi\right)^{\beta+\gamma-1}\eta^{\beta-1}\left(1-\eta\right)^{\gamma-1}\times$$
$$\times\Psi_1\left(a_1,\alpha-a_2;\alpha,\beta;\frac{x\xi}{x\xi-1},\frac{y\eta\left(1-\xi\right)}{1-x\xi}\right){}_1F_1\left(a_3;\gamma;z\left(1-\xi\right)\left(1-\eta\right)\right)d\xi d\eta, \tag{74}$$

$$\mathrm{E}_{11}\left(a,b;\alpha+\beta+\gamma;x,y,z\right)=k_3\int_0^1\int_0^1\xi^{\alpha-1}\left(1-\xi\right)^{\beta+\gamma-1}\eta^{\beta-1}\left(1-\eta\right)^{\gamma-1}\times$$
$$\times\Psi_1\left(a,b;\alpha,\beta;x\xi,y\eta\left(1-\xi\right)\right){}_0F_1\left(-;\gamma;z\left(1-\xi\right)\left(1-\eta\right)\right)d\xi d\eta, \tag{75}$$

$$\mathrm{E}_{12}\left(a_1,a_2,a_3;\alpha+\beta+\gamma;x,y,z\right)=k_3\int_0^1\int_0^1\xi^{\alpha-1}\left(1-\xi\right)^{\beta+\gamma-1}\eta^{\beta-1}\left(1-\eta\right)^{\gamma-1}\times$$
$$\times\mathrm{E}_{62}\left(a_1,a_2,a_3;\alpha,\beta,\gamma;x\xi,y\eta\left(1-\xi\right),z\left(1-\xi\right)\left(1-\eta\right)\right)d\xi d\eta, \tag{76}$$

$$\mathrm{E}_{12}\left(\alpha,\beta,\gamma;c;x,y,z\right)=k_3\int\limits_0^1\int\limits_0^1\xi^{\alpha+\gamma-1}\left(1-\xi\right)^{\beta-1}\eta^{\alpha-1}\left(1-\eta\right)^{\gamma-1}\times$$
$$\times\mathrm{H}_6\left(\alpha+\beta+\gamma;c;x\xi^2\eta\left(1-\eta\right)+y\xi\left(1-\xi\right)\eta,z\left(1-\xi\right)\right)d\xi d\eta, \tag{77}$$

$$\mathrm{E}_{13}\left(a,b;\alpha+\beta+\gamma;x,y,z\right)=k_3\int_0^1\int_0^1\xi^{\alpha-1}\left(1-\xi\right)^{\beta+\gamma-1}\eta^{\beta-1}\left(1-\eta\right)^{\gamma-1}\times$$
$$\times\mathrm{E}_{63}\left(a,b;\alpha,\beta,\gamma;x\xi,y\eta\left(1-\xi\right),z\left(1-\xi\right)\left(1-\eta\right)\right)d\xi d\eta, \tag{78}$$

$$\mathrm{E}_{14}\left(a_1,a_2,a_3;\alpha+\beta+\gamma;x,y,z\right)=k_3\int_0^1\int_0^1 \xi^{\alpha-1}\left(1-\xi\right)^{\beta+\gamma-1}\eta^{\beta-1}\left(1-\eta\right)^{\gamma-1}\times$$
$$\times\mathrm{E}_{64}\left(a_1,a_2,a_3;\alpha,\beta,\gamma;x\xi,y\eta\left(1-\xi\right),z\left(1-\xi\right)\left(1-\eta\right)\right)d\xi d\eta, \tag{79}$$

$$\mathrm{E}_{1}\left(\alpha,a_2,a_3,a_4,a_5;\alpha+\beta+\gamma+\delta;x,y,z\right)=k_5\int_0^1\int_0^1 \xi^{\alpha-1}\left(1-\xi\right)^{\beta-1}\eta^{\gamma-1}\left(1-\eta\right)^{\delta-1}\times$$
$$\times\,\Xi_1\left(\alpha+\beta,a_5,\gamma+\delta;\alpha+\beta+\gamma+\delta;x\xi\eta+y\left(1-\xi\right)\left(1-\eta\right),z\right)d\xi d\eta, \tag{80}$$

where

$$k_3=\frac{\Gamma\left(\alpha+\beta+\gamma\right)}{\Gamma\left(\alpha\right)\Gamma\left(\beta\right)\Gamma\left(\gamma\right)},\quad k_4=\frac{\Gamma\left(2\alpha+\beta\right)}{\Gamma^2\left(\alpha\right)\Gamma\left(\beta\right)},\quad k_5=\frac{\Gamma\left(\alpha+\beta\right)\Gamma\left(\gamma+\delta\right)}{\Gamma\left(\alpha\right)\Gamma\left(\beta\right)\Gamma\left(\gamma\right)\Gamma\left(\delta\right)},$$

confluent hypergeometric functions E_{15}, E_{17}, E_{18}, E_{19}, E_{21}, E_{62}, E_{63} and E_{64} are defined in [27].

Proof. The proofs of the representations (47)–(80) are similar to the proofs of the previous theorems. □

Theorem 4. If $\mathrm{Re}\left(\alpha\right)>0$, $\mathrm{Re}\left(\beta\right)>0$ and $\mathrm{Re}\left(\gamma\right)>\mathrm{Re}\left(\alpha\right)+\mathrm{Re}\left(\beta\right)$, then the following integral representations are valid:

$$\mathrm{E}_{5}\left(\alpha,\beta,a;\gamma;x,y,z\right)=k_6\int_0^1\int_0^1 \frac{\xi^{\alpha-1}\left(1-\xi\right)^{\beta-1}\eta^{\alpha+\beta-1}\left(1-\eta\right)^{\gamma-\alpha-\beta-1}}{\left(1-y\eta-x\xi\eta+y\xi\eta\right)^{a}}\times$$
$$\times\,{}_0F_1\left(-;\gamma-\alpha-\beta;z-z\eta\right)d\eta d\xi, \tag{81}$$

$$\mathrm{E}_{6}\left(\alpha,\beta;\gamma;x,y,z\right)=k_6\int_0^1\int_0^1 \xi^{\alpha-1}\left(1-\xi\right)^{\beta-1}\eta^{\alpha+\beta-1}\left(1-\eta\right)^{\gamma-\alpha-\beta-1}\times$$
$$\times\,e^{x\xi\eta+y\eta-y\xi\eta}{}_0F_1\left(-;\gamma-\alpha-\beta;z-z\eta\right)d\xi d\eta, \tag{82}$$

where

$$k_6=\frac{\Gamma\left(\gamma\right)}{\Gamma\left(\alpha\right)\Gamma\left(\beta\right)\Gamma\left(\gamma-\alpha-\beta\right)}.$$

Proof. The proofs of the representations (81) and (82) are similar to the proofs of the previous theorems. □

4 Triple integral Representations

Theorem 5. The following integral representations are valid under certain restrictions on the numerical parameters:

$$\mathrm{E}_{2}\left(a_1,a_2,a_3,a_4,a_5;c;x,y,z\right)=\frac{\Gamma\left(a\right)}{\Gamma\left(a_1\right)\Gamma\left(a_2\right)\Gamma\left(a_3\right)\Gamma\left(a_4\right)}\times$$
$$\times\int_0^1\int_0^1\int_0^1 \xi^{a_1-1}\left(1-\xi\right)^{a_2+a_3-1}\eta^{a_2-1}\left(1-\eta\right)^{a_3-1}\zeta^{a_1+a_2+a_3-1}\left(1-\zeta\right)^{a_4-1}\times$$
$$\times\Xi_2\left(\frac{a}{2},\frac{a+1}{2};c;\;4x\xi\eta\left(1-\xi\right)\zeta^2+4y\left(1-\xi\right)\left(1-\eta\right)\zeta\left(1-\zeta\right),z\right)d\xi d\eta d\zeta, \tag{83}$$

$$a:=a_1+a_2+a_3+a_4,\;\;\mathrm{Re}\left(a_k\right)>0,\;\;k=1,2,3,4;$$

$$\mathrm{E}_3\left(a_1,a_2,a_3,a_4;c;x,y,z\right)=\frac{\Gamma\left(a\right)}{\Gamma\left(a_1\right)\Gamma\left(a_2\right)\Gamma\left(a_3\right)\Gamma\left(a_4\right)}\times$$
$$\times\int\limits_0^1\int\limits_0^1\int\limits_0^1\xi^{a_1-1}\left(1-\xi\right)^{a_2+a_3-1}\eta^{a_2-1}\left(1-\eta\right)^{a_3-1}\zeta^{a_1+a_2+a_3-1}\left(1-\zeta\right)^{a_4-1}\times$$
$$\times\mathrm{H}_6\left(a;c;x\xi\left(1-\xi\right)\eta\zeta^2,y\left(1-\xi\right)\left(1-\eta\right)\zeta+z\left(1-\zeta\right)\right)d\xi d\eta d\zeta,\tag{84}$$

$$a:=a_1+a_2+a_3+a_4,\ \ \mathrm{Re}\left(a_k\right)>0,\ \ k=1,2,3,4;$$

$$\mathrm{E}_7\left(a_1,a_2,a_3,a_4;c_1+c_2+c_3;x,y,z\right)=\frac{\Gamma\left(a_2+a_3\right)}{\Gamma\left(a_2\right)\Gamma\left(a_3\right)}\frac{\Gamma\left(c_1+c_2+c_3\right)}{\Gamma\left(c_1\right)\Gamma\left(c_2\right)\Gamma\left(c_3\right)}\times$$
$$\times\int\limits_0^1\int\limits_0^1\int\limits_0^1\xi^{c_1-1}\left(1-\xi\right)^{c_2+c_3-1}\eta^{c_2-1}\left(1-\eta\right)^{c_3-1}\zeta^{a_2-1}\left(1-\zeta\right)^{a_3-1}\times$$
$$\times F_4\left(a_1,a_2+a_3;c_1,c_2;x\xi\zeta,y\eta\left(1-\xi\right)\left(1-\zeta\right)\right){}_1F_1\left(a_4;c_3;z\left(1-\xi\right)\left(1-\eta\right)\right)d\xi d\eta d\zeta,\tag{85}$$

$$\mathrm{Re}\left(a_2\right)>0,\ \ \mathrm{Re}\left(a_3\right)>0,\ \ \mathrm{Re}\left(c_k\right)>0,\ \ k=1,2,3;$$

$$\mathrm{E}_7\left(a_1,a_2,a_3,a_4;c;x,y,z\right)=\frac{\Gamma\left(c\right)}{\Gamma\left(a_2\right)\Gamma\left(a_3\right)\Gamma\left(a_4\right)\Gamma\left(c-a_2-a_3-a_4\right)}\times$$
$$\times\int\limits_0^1\int\limits_0^1\int\limits_0^1\xi^{a_2-1}\left(1-\xi\right)^{a_3-1}\eta^{a_4-1}(1-\eta)^{a_2+a_3-1}\zeta^{a_2+a_3+a_4-1}\left(1-\zeta\right)^{c-a_2-a_3-a_4-1}\times$$
$$\times\left(1-z\eta\zeta\right)^{-a_4}e^{x\xi\eta\zeta+y(1-\xi)(1-\eta)\zeta}d\zeta d\xi d\eta,\tag{86}$$

$$\mathrm{Re}\left(a_k\right)>0,\ \ k=2,3,4,\ \ \mathrm{Re}\left(c-a_2-a_3-a_4\right)>0;$$

$$\mathrm{E}_9\left(a_1,a_2,a_3;c_1+c_2+c_3;x,y,z\right)=\frac{\Gamma\left(a_2+a_3\right)}{\Gamma\left(a_2\right)\Gamma\left(a_3\right)}\frac{\Gamma\left(c_1+c_2+c_3\right)}{\Gamma\left(c_1\right)\Gamma\left(c_2\right)\Gamma\left(c_3\right)}\times$$
$$\times\int\limits_0^1\int\limits_0^1\int\limits_0^1\xi^{c_1-1}\left(1-\xi\right)^{c_2+c_3-1}\eta^{c_2-1}\left(1-\eta\right)^{c_3-1}\zeta^{a_2-1}\left(1-\zeta\right)^{a_3-1}\times$$
$$\times F_4\left(a_1,a_2+a_3;c_1,c_2;x\xi\zeta,y\eta\left(1-\xi\right)\left(1-\zeta\right)\right){}_0F_1\left(-;c_3;z\left(1-\xi\right)\left(1-\eta\right)\right)d\xi d\eta d\zeta,\tag{87}$$

$$\mathrm{Re}\left(a_2\right)>0,\ \ \mathrm{Re}\left(a_3\right)>0,\ \ \mathrm{Re}\left(c_k\right)>0,\ \ k=1,2,3;$$

$$\mathrm{E}_9\left(a_1,a_2,a_3;c_1+c_2+c_3;x,y,z\right)=\frac{\Gamma\left(a_1+a_2\right)}{\Gamma\left(a_1\right)\Gamma\left(a_2\right)}\frac{\Gamma\left(c_1+c_2+c_3\right)}{\Gamma\left(c_1\right)\Gamma\left(c_2\right)\Gamma\left(c_3\right)}\times$$
$$\times\int\limits_0^1\int\limits_0^1\int\limits_0^1\xi^{c_1-1}\left(1-\xi\right)^{c_2+c_3-1}\eta^{c_2-1}\left(1-\eta\right)^{c_3-1}\zeta^{a_1-1}\left(1-\zeta\right)^{a_2-1}\times$$
$$\times H_4\left(a_1+a_2,a_3;c_1,c_2;x\xi\zeta(1-\zeta),y\eta\zeta(1-\xi)\right){}_0F_1\left(-;c_3;z\left(1-\xi\right)\left(1-\eta\right)\right)d\xi d\eta d\zeta,\tag{88}$$

$$\mathrm{Re}\left(a_2\right)>0,\ \ \mathrm{Re}\left(a_3\right)>0,\ \ \mathrm{Re}\left(c_k\right)>0,\ \ k=1,2,3;$$

$$\mathrm{E}_{10}\left(a_1, a_2, a_3; c_1+c_2+c_3; x, y, z\right) = \frac{\Gamma\left(c_1+c_2+c_3\right)}{\Gamma\left(a_2\right)\Gamma\left(c_1-a_2\right)\Gamma\left(c_2\right)\Gamma\left(c_3\right)} \times$$
$$\times \int\limits_0^1 \int\limits_0^1 \int\limits_0^1 \frac{\xi^{c_1-1}\left(1-\xi\right)^{c_2+c_3-1}\eta^{c_2-1}\left(1-\eta\right)^{c_3-1}\zeta^{a_2-1}\left(1-\zeta\right)^{c_1-a_2-1}}{\left(1-x\xi\zeta\right)^{a_1}} \times$$
$$\times {}_1F_1\left(a_1; c_2; \frac{y\eta\left(1-\xi\right)}{1-x\xi\zeta}\right) {}_1F_1\left(a_3; c_3; z\left(1-\xi\right)\left(1-\eta\right)\right) d\xi d\eta d\zeta, \tag{89}$$

where

$$\mathrm{Re}\left(c_1-a_2\right)>0, \ \ \mathrm{Re}\left(c_k\right)>0, \ \ k=1,2,3.$$

Proof. The proofs of the representations (83)–(89) are similar to the proofs of the previous theorems. □

Similar integral formulas involving confluent hypergeometric functions of three variables are discussed in [30].

5 Application

The confluent hypergeometric function E_2 has important applications. In the recent paper [31] particular solutions, including the solution of the Dirichlet problem for the three-dimensional singular Helmholtz equation

$$u_{xx}+u_{yy}+u_{zz}+\frac{2\alpha}{x}u_x+\frac{2\beta}{y}u_y+\frac{2\gamma}{z}u_z-\lambda^2 u=0, \quad 0<2\alpha,\, 2\beta,\, 2\gamma<1$$

in the infinite first octant $\Omega \equiv \{(x,y,z): x>0, \ y>0, \ z>0\}$ are expressed through a function E_2 (in the work [31] the function E_2 is denoted as A_2).

Another recent paper [32] derives important relations linking function E_1 with Appell function F_3, Humbert functions Φ_1, Φ_2, Φ_3, Ξ_1, Ξ_2, generalized hypergeometric function ${}_pF_q$ and Kampé de Fériet function $F^{p:q;k}_{l:m;n}$ (for details on Kampé de Fériet function see [33–35]).

Conclusion

As is known [27], the list of confluent hypergeometric functions of three variables was compiled recently, however, the confluent functions E_1–E_{14} investigated in this paper were first introduced by Jain [26] in 1966, who limited himself to composing systems of partial differential equations corresponding to these functions. Until now, the scientific community has not known any applications of the confluent hypergeometric functions E_1–E_{14}, except for the function E_2 discussed in the Application section.

Author Contributions

A. Hasanov served as the principal investigator of the research and supervised the research process. T.G. Ergashev collected and analyzed data and led manuscript preparation. A.R. Ryskan assisted in data collection and analysis. All authors participated in the revision of the manuscript and approved the final submission. All authors contributed equally to this work.

Conflict of Interest

The authors declare no conflict of interest.

References

1 Brychkov, Y.A., & Savischenko, N.V. (2019). Application of hypergeometric functions of two variables in Wireless communication theory. *Lobachevskii Journal of Mathematics, 40*(7), 938–953. https://doi.org/10.1134/S1995080219070096

2 Karlsson, P.W., & Ernst, T. (2024). Applications of quadratic and cubic hypergeometric transformations. *Annales Universitatis Mariae Curie Sklodowska, Sectio A Mathematica, 78*(1), 37–73. https://doi.org/10.17951/a.2024.78.1.37-73

3 Irgashev, B.Y. (2024). Application of hypergeometric functions to the construction of particular solutions. *Complex Variables and Elliptic Equations, 69*(12), 2025–2047. https://doi.org/10.1080/17476933.2023.2270910

4 Srivastava, H.M., & Karlsson, P.W. (1985). Multiple Gaussian Hypergeometric Series. *Ellis Horwood Series in Mathematics and Its Applications.* Chichester, West Sussex, England: E. Horwood; New York: Halsted Press.

5 Horn, J. (1931). Hypergeometrische Funktionen zweier Veränderlichen. *Mathematische Annalen, 105*, 381–407. https://doi.org/10.1007/BF01455825

6 Humbert, P. (1922). The confluent hypergeometric functions of two variables. *Proceedings of the Royal Society of Edinburgh, 41*(1), 73–96.

7 Erdelyi, A., Magnus, W., Oberhettinger, F., & Tricomi, F.G. (Eds.). (1953). *Higher Transcendental Functions, Volume I.* New York, Toronto, London: McGraw-Hill Book Company, Inc.

8 Lauricella, G. (1893). Sulle funzione ipergeometriche a più variabili. *Rendiconti del Circolo Matematico di Palermo, 7*, 111–158. https://doi.org/10.1007/BF03012437

9 Saran, S. (1954). Hypergeometric functions of three variables. *Ganita, 5*, 77–91.

10 Sahai, V., & Verma, A. (2015). Recursion formulas for multivariable hypergeometric functions. *Asian-European Journal of Mathematics, 8*(4), Article 1550082. https://doi.org/10.1142/S1793557115500825

11 Sahai, V., & Verma, A. (2022). Infinite summation formulas of Srivastava's general triple hypergeometric function. *Mathematics in Engineering, Science and Aerospace (MESA), 13*(4), 212–221.

12 Bezrodnykh, S.I. (2018). The Lauricella hypergeometric function $F_D^{(N)}$, the Riemann–Hilbert problem, and some applications. *Russian Mathematical Surveys, 73*(6), 941–1031. https://doi.org/10.1070/RM9841

13 Bezrodnykh, S.I. (2018). Analytic continuation of the Lauricella function $F_D^{(N)}$ with arbitrary number of variables. *Integral Transforms and Special Functions, 29*(1), 21–42. https://doi.org/10.1080/10652469.2017.1402017

14 Bezrodnykh, S.I. (2022). Formulas for computing the Lauricella function in the case of crowding of variables. *Computational Mathematics and Mathematical Physics, 62*(12), 2069–2090. https://doi.org/10.1134/S0965542522120041

15 Srivastava, H.M. (1964). Hypergeometric functions of three variables. *Ganita, 15*, 97–108.

16 Srivastava, H.M. (1967). Some integrals representing triple hypergeometric functions. *Rendiconti del Circolo Matematico di Palermo, 16*, 99–115.

17 Choi, J., & Parmar, R.K. (2017). Generalized Srivastava's triple hypergeometric functions and their associated properties. *Journal of Nonlinear Sciences and Applications, 10*(2), 817–827. https://doi.org/10.22436/jnsa.010.02.41

18 Ata, E. (2024). M-Srivastava hypergeometric functions: integral representations and solution of the fractional differential equations. *Miskolc Mathematical Notes, 25*(1), 93–108. https://doi.org/10.18514/MMN.2024.4326

19 Parmar, R.K., & Pogány, T.K. (2017). Extended Srivastava's triple hypergeometric $H_{A,p,q}$ function and related bounding inequalities. *Journal of Contemporary Mathematical Analysis, 52*(6), 276–287. https://doi.org/10.3103/S1068362317060036

20 Dar, S.A., & Paris, R.B. (2019). A (p, q)-extension of Srivastava's triple hypergeometric function H_B and its properties. *Journal of Computational and Applied Mathematics, 348*, 237–245. https://doi.org/10.1016/j.cam.2018.08.045

21 Dar, S.A., & Paris, R.B. (2020). A (p, q)-extension of Srivastava's triple hypergeometric function H_C. *Publications De L Institut Mathematique, 108*(122), 33–45. https://doi.org/10.2298/PIM2022033D

22 Blümlein, J., & Schneider, C. (2023). Hypergeometric structures in Feynman integrals. *Annals of Mathematics and Artificial Intelligence, 91*(5), 591–649. https://doi.org/10.1007/s10472-023-09831-8

23 Hasanov, A., & Ruzhansky, M. (2019). Euler-type integral representations for the hypergeometric functions in three variables of second order. *Bulletin of the Institute of Mathematics, 2*(6), 73–223.

24 Bin-Saad, M.G., Shahwan, M.J.S., Younis, J.A., Aydi, H., & Abd El Salam, M.A. (2022). On Gaussian hypergeometric functions of three variables: Some new integral representations. *Journal of Mathematics, 2022*, Article 1914498. https://doi.org/10.1155/2022/1914498

25 Hasanov A., & Ruzhansky M. (2022). Systems of differential equations of Gaussian hypergeometric functions in three variables and their linearly-independent solutions. *Bulletin of the Institute of Mathematics, 5*(3), 50–142.

26 Jain, R.N. (1966). The confluent hypergeometric functions of three variables. *Proceedings of the National Academy of Sciences, India, 36*(2), 395–408.

27 Ergashev, T.G. (2024). Confluent hypergeometric functions in three variables and associated systems of partial differential equations. *Bulletin of the Institute of Mathematics, 7*(4), 121–215.

28 Srivastava, H.M., & Shpot, M.A. (2017). Reduction and transformation formulas for the Appell and related functions in two variables. *Mathematical Methods in the Applied Sciences, 40*(11), 4102–4108. https://doi.org/10.1002/mma.4289

29 Belafhal, A., Nossir, N., Dalil-Essakali, L., & Usman, T. (2020). Integral transforms involving the product of Humbert and Bessel functions and its application. *AIMS Mathematics, 5*(2), 1260–1274. https://doi.org/10.3934/math.2020086

30 Atash, A.A., & Bellahaj, H.S. (2018). Integrals formulas involving confluent hypergeometric functions of three variables $\Phi_2^{(3)}$ and $\Psi_2^{(3)}$. *University of Aden Journal of Natural and Applied Sciences, 22*(1), 143–149. https://doi.org/10.47372/uajnas.2018.n1.a11

31 Arzikulov, Z.O., Hasanov, A., & Ergashev, T.G. (2025). Konfliuentnye gipergeometricheskie funktsii i ikh primenenie k resheniiu zadachi Dirikhle dlia uravneniia Gelmgoltsa s tremia singuliarnymi koeffitsientami [Confluent hypergeometric functions and their application to the solution of Dirichlet problem for the Helmholtz equation with three singular coefficients.] *Vestnik Samarskogo gosudarstvennogo tekhnicheskogo universiteta. Seriia «Fiziko-matematicheskie nauki» — Journal of Samara State Technical University, Ser. Physical and Mathematical*

Sciences, 29(3), 407–429. https://doi.org/10.14498/vsgtu2156

32 Hasanov, A., & Yuldashova, H. (2025). Some functional identities of confluent hypergeometric function E1 of three variables. *Bulletin of the Institute of Mathematics, 8*(5), 18–29.

33 Kim, I., Paris, R.B., & Rathie, A.K. (2022). Some new results for the Kampé de Fériet function with an application. *Symmetry, 14*(12), Article 2588. https://doi.org/10.3390/sym14122588

34 Choi, J., & Rathie, A.K. (2019). General summation formulas for the Kampé de Fériet function. *Montes Taurus Journal of Pure and Applied Mathematics, 1*(1), 107–128.

35 Choi, J.J., Milovanović, C.V., & Rathie, A.K. (2021). Generalized summation formulas for the Kampé de Fériet functions. *Axioms, 10*(4), Article 318. https://doi.org/10.3390/axioms10040318

Author Information[*]

Anvar Hasanov — Doctor of Physical and Mathematical Sciences, Professor, Institute of Mathematics, Uzbekistan Academy of Sciences, 4 University St., Tashkent, 700174, Uzbekistan; e-mail: *anvarhasanov@yahoo.com*; https://orcid.org/0000-0002-9849-4103

Tukhtasin Gulamzhanovich Ergashev — Doctor of Physical and Mathematical Sciences, Professor, Department of Higher Mathematics, "Tashkent Institute of Irrigation and Agricultural Mechanization Engineer" National Research University, 39 Kari-Niyazi St., Tashkent, 100000, Uzbekistan; e-mail: *ergashev.tukhtasin@gmail.com*; https://orcid.org/0000-0003-3542-8309

Ainur Ryskankyzy Ryskan (*corresponding author*) — PhD, Associate Professor, School of Digital Technologies, Narxoz University, 55 Zhandosov St., Almaty, 050035, Kazakhstan; e-mail: *ryskan.a727@gmail.com*; https://orcid.org/0000-0002-8764-4751

[*]Authors' names are presented in the following order: first name, middle name (if any), last name.

https://doi.org/10.31489/2026M1/156-168 *Research article*

On estimates for norms of matrix operators: the case $q < p$

K.D. Kuliev[1,*], G.T. Kulieva[1], T.D. Turakulov[2]

[1]*Samarkand State University named after Sh. Rashidov, Samarkand, Uzbekistan;*
[2]*Samarkand Branch of Oriental University, Samarkand, Uzbekistan*
(E-mail: komilkuliev@gmail.com, kulievagt@gmail.com, tohirbekt92@gmail.com)

The study of matrix operators acting between weighted sequence spaces $l_{p,v}$ and $l_{q,u}$ has become an important direction in functional analysis, particularly due to its close connection with Hardy-type inequalities and the general theory of linear operators on discrete structures. A key problem in this framework is determining when such operators are bounded and obtaining precise value or sharp estimates for their operator norms. Although considerable attention has been devoted to matrix operators whose entries satisfy the so-called *Oinarov conditions*, including several extensions to broader classes of kernels, the literature still lacks comprehensive norm estimates, especially in the case $1 < q < p < \infty$. In this paper, we establish necessary and sufficient criteria for the boundedness of matrix operators with entries satisfying the Oinarov conditions. Furthermore, we provide both lower and upper estimates for their norms. These results not only refine previously known inequalities but also provide new tools for analyzing the structure and behavior of weighted sequence spaces. Applications of our findings include spectral characterization of matrix operators, investigation of oscillatory and non-oscillatory properties of solutions to higher-order difference equations, and the evaluation of sequences via their discrete derivatives.

Keywords: operator, matrix, sequence, norm, discrete analysis, Hardy-type inequality, weight function, kernel, Lebesgue sequence space, best constant.

2020 Mathematics Subject Classification: Primary 47B37, 47A30; Secondary 26D10, 39A12.

Introduction

Let $1 < p,\, q < \infty$ and let $u = \{u_n\}_{n=1}^{\infty}$ and $v = \{v_n\}_{n=1}^{\infty}$ be positive sequences of real numbers, which we call weights. Denote by $l_{p,v}$ the space of all sequences $f = \{f_n\}_{n=1}^{\infty}$ of real numbers for which the norm

$$\|f\|_{l_{p,v}} = \left(\sum_{n=1}^{\infty} |f_n|^p v_n\right)^{\frac{1}{p}}$$

is finite. Similarly, the space $l_{q,u}$ is defined. Let us consider the following triangular matrix operator

$$A : l_{p,v} \to l_{q,u}, \quad (Af)_n := \sum_{k=1}^{n} a_{n,k} f_k \tag{1}$$

acting between weighted sequence spaces $l_{p,v}$ and $l_{q,u}$. The entries $a := \{a_{n,k}\}_{n,k=1}^{\infty}$, $n \geq k \geq 1$ are nonnegative $a_{n,k} \geq 0$.

Since the 1980s, conditions ensuring the boundedness of the operator, that is, the existence of a constant $C > 0$ such that the inequality

$$\|Af\|_{l_{q,u}} \leq C\|f\|_{l_{p,v}},$$

*Corresponding author. *E-mail: komilkuliev@gmail.com*
Received: 28 May 2025; *Accepted:* 25 December 2025.

i.e., discrete weighted Hardy type inequality (HTI)

$$\left(\sum_{n=1}^{\infty}\left|\sum_{k=1}^{n}a_{n,k}f_k\right|^q u_n\right)^{\frac{1}{q}} \leq C\left(\sum_{n=1}^{\infty}|f_n|^p v_n\right)^{\frac{1}{p}} \tag{2}$$

holds for all $f \in l_{p,v}$ — for various forms of $a_{n,k}$ have been established.

If $a_{n,k} \equiv 1$, then the inequality takes the form

$$\left(\sum_{n=1}^{\infty}\left|\sum_{k=1}^{n}f_k\right|^q u_n\right)^{\frac{1}{q}} \leq C\left(\sum_{n=1}^{\infty}|f_n|^p v_n\right)^{\frac{1}{p}}, \tag{3}$$

which is called general weighted inequality (GWI). In 1983, K.F. Andersen and H.P. Heinig provided a characterization of the weight conditions ensuring the validity of inequality (3) in the case $1 < p \leq q < \infty$. Subsequently, in 1985, H.P. Heinig established a sufficient condition for the same inequality and derived estimates for its best constant $C = \|A\|_{l_{p,v}\to l_{q,u}}$ in the case $1 < q < p < \infty$, see e.g. [1]. We now present Heinig's result in a form adapted to the GWI

$$A = \left(\sum_{n=1}^{\infty}\left(\sum_{k=n}^{\infty}u_k\right)^{\frac{r}{q}}\left(\sum_{k=1}^{n}v_k^{1-p'}\right)^{\frac{r}{q'}} v_n^{1-p'}\right)^{\frac{1}{r}} < \infty \tag{4}$$

and the corresponding upper estimate

$$C \leq q^{\frac{1}{q}}(p')^{\frac{1}{q'}}A, \tag{5}$$

which will be used in the proof of the main theorem of this work, where $\frac{1}{p}+\frac{1}{p'}=1$ and $\frac{1}{q}+\frac{1}{q'}=1$.

If $\{a_{n,k}\}$ is different from constant sequence then the analysis of HTI becomes considerable more complicated. The first result was obtained by K.F. Andersen and H.P. Heinig, who established sufficient conditions for the validity of the inequality in the case $1 < p \leq q < \infty$ under a specific choice of the kernel $a_{n,k}$, which was assumed to be non-increasing in k and non-decreasing in n. For further details, see [1] and the references therein.

Later, R. Oinarov, C.A. Okpoti, and L.-E. Persson [2], as well as R. Oinarov, S. Shalginbayeva, T. Temirkhanova and others [3–5] obtained necessary and sufficient conditions for the boundedness of the matrix operator in the case $1 < q < p < \infty$. Their results hold under hypotheses on the entries $a_{n,k}$ that are weaker than those originally imposed by Oinarov. Estimates of the best constant in the discrete Hardy-type inequality for matrix operators satisfying Oinarov condition are given in [6].

Most of the works cited above focus on characterizing boundedness; by contrast, exact values or even two-sided estimates for the operator norm are rarely treated. Recently, several papers have appeared that provide lower and upper bounds for the norm of the integral analogue of this matrix operator (see, for example, [7, 8]).

There are also a number of related contributions concerning iterated discrete Hardy inequalities [9, 10], three-weight inequalities [11, 12], and the boundedness of iterated matrix operators [13, 14], which may be consulted for further reference.

In this paper, we derive lower and upper estimates for the norm of the matrix operator, as well as necessary and sufficient conditions for its boundedness, in the case $1 < q < p < \infty$. We consider the operators whose entries $\{a_{n,k}\}$ are non-decreasing in n, non-increasing in k, and satisfy the condition

$$a_{n,k} \leq h(a_{n,l} + a_{l,k}) \quad \text{for all} \quad n \geq l \geq k \geq 1 \tag{6}$$

for some $h \geq 1$. These conditions are known in the theory of Hardy-type inequalities as *Oinarov conditions.*

1 Main results

In this section, we present the main result of the paper along with the auxiliary lemmas needed for its proof. We begin by introducing the following notations:

$$B_1 := \left(\sum_{n=1}^{\infty} \left(\sum_{k=n}^{\infty} a_{k,n}^q u_k \right)^{\frac{r}{q}} \left(\sum_{k=1}^{n} v_k^{1-p'} \right)^{\frac{r}{q'}} v_n^{1-p'} \right)^{\frac{1}{r}}$$

and

$$B_2 := \left(\sum_{n=1}^{\infty} \left(\sum_{k=n}^{\infty} u_k \right)^{\frac{r}{p}} \left(\sum_{k=1}^{n} a_{n,k}^{p'} v_k^{1-p'} \right)^{\frac{r}{p'}} u_n \right)^{\frac{1}{r}},$$

where $\frac{1}{r} = \frac{1}{q} - \frac{1}{p}$.

Let us present the auxiliary lemmas, with particular emphasis on the constants appearing in the estimates, as they play a crucial role in determining sharp estimates for the norm of the operator.

Lemma 1. Let $1 < q < \infty$ and $\{u_n\} \in \ell_1$ be a nonnegative sequence. Then for any $m \in \mathbb{N}$ the following estimate holds

$$\sum_{n=m}^{\infty} \left(\sum_{k=n}^{\infty} u_k \right)^{-\frac{1}{q}} u_n \leq q' \left(\sum_{k=m}^{\infty} u_k \right)^{\frac{1}{q'}}, \tag{7}$$

where $q' = \frac{q}{q-1}$.

Proof. Let denote

$$x_n = \sum_{k=n}^{\infty} u_k,$$

so that $x_m \geq ... \geq x_n \geq x_{n+1} \geq ... \geq 0$. Then inequality (7) can be written as

$$\sum_{n=m}^{\infty} x_n^{-\frac{1}{q}} (x_n - x_{n+1}) \leq q' x_m^{\frac{1}{q'}},$$

where $q' = \frac{q}{q-1}$. To estimate the left hand side, observe that

$$\sum_{n=m}^{\infty} \frac{x_n - x_{n+1}}{x_n^{\frac{1}{q}}} = \sum_{n=m}^{\infty} \left(\int_{x_{n+1}}^{x_n} \frac{1}{x_n^{\frac{1}{q}}} dx \right) \leq \sum_{n=m}^{\infty} \left(\int_{x_{n+1}}^{x_n} \frac{1}{x^{\frac{1}{q}}} dx \right) = \int_0^{x_m} \frac{1}{x^{\frac{1}{q}}} dx = q' x_m^{\frac{1}{q'}}.$$

The proof is complete. □

Lemma 2. Let $1 < q < \infty$, and $\{u_n\} \in \ell_1$ be a nonnegative sequence. Suppose the entries $\{a_{n,k}\}$ of the matrix operator satisfy Oinarov conditions. Then, for every nonnegative sequence $\{f_n\} \in l_{p,v}$, the following estimate holds:

$$\sum_{n=1}^{\infty} \left(\sum_{k=1}^{n} a_{n,k} f_k \right)^q u_n \leq (2h)^{q-1} q [S_1 + S_2], \tag{8}$$

where

$$S_1 = \sum_{m=1}^{\infty} f_m \left(\sum_{n=m}^{\infty} a_{n,m}^q u_n \right) \left(\sum_{k=1}^{m} f_k \right)^{q-1}$$

and

$$S_2 = \sum_{m=1}^{\infty} f_m \left(\sum_{n=m}^{\infty} a_{n,m} u_n \right) \left(\sum_{k=1}^{m} a_{m,k} f_k \right)^{q-1}.$$

Proof. Using Lagrange's mean value theorem, Fubini's theorem, Oinarov condition (6) and the inequality

$$(a+b)^{q-1} \le 2^{q-1}(a^{q-1}+b^{q-1}) \quad \text{for all} \quad a,b \ge 0,$$

we estimate the left hand side of (8) in the form

$$\begin{aligned}
&\sum_{n=1}^{\infty} \left(\sum_{k=1}^{n} a_{n,k} f_k \right)^q u_n = \sum_{n=1}^{\infty} \left(\sum_{m=1}^{n} \left[\left(\sum_{k=1}^{m} a_{n,k} f_k \right)^q - \left(\sum_{k=1}^{m-1} a_{n,k} f_k \right)^q \right] \right) u_n \\
&= q \sum_{n=1}^{\infty} \left(\sum_{m=1}^{n} a_{n,m} f_m \left(\sum_{k=1}^{m-1} a_{n,k} f_k + \xi_{n,m} a_{n,m} f_m \right)^{q-1} \right) u_n \\
&\le q \sum_{n=1}^{\infty} \left(\sum_{m=1}^{n} a_{n,m} f_m \left(\sum_{k=1}^{m} a_{n,k} f_k \right)^{q-1} \right) u_n \\
&= q \sum_{m=1}^{\infty} f_m \left(\sum_{n=m}^{\infty} a_{n,m} u_n \left(\sum_{k=1}^{m} a_{n,k} f_k \right)^{q-1} \right) \\
&\le q h^{q-1} \sum_{m=1}^{\infty} f_m \left(\sum_{n=m}^{\infty} a_{n,m} u_n \left(a_{n,m} \sum_{k=1}^{m} f_k + \sum_{k=1}^{m} a_{m,k} f_k \right)^{q-1} \right) \\
&\le (2h)^{q-1} q \sum_{m=1}^{\infty} f_m \left(\sum_{n=m}^{\infty} a_{n,m}^q u_n \left(\sum_{k=1}^{m} f_k \right)^{q-1} + \sum_{n=m}^{\infty} a_{n,m} u_n \left(\sum_{k=1}^{m} a_{m,k} f_k \right)^{q-1} \right) \\
&= (2h)^{q-1} q \left[\sum_{m=1}^{\infty} f_m \left(\sum_{n=m}^{\infty} a_{n,m}^q u_n \right) \left(\sum_{k=1}^{m} f_k \right)^{q-1} + \sum_{m=1}^{\infty} f_m \left(\sum_{n=m}^{\infty} a_{n,m} u_n \right) \left(\sum_{k=1}^{m} a_{m,k} f_k \right)^{q-1} \right] \\
&= (2h)^{q-1} q [S_1 + S_2],
\end{aligned}$$

where $\xi_{n,m} \in (0,1)$, $m = 1, 2, ..., n$.

The proof is complete. □

Theorem 1. Let $1 < q < p < \infty$ and the entries $\{a_{n,k}\}$ of matrix operator (1) satisfy Oinarov conditions. Then the operator acting between weighted sequence spaces $l_{p,v}$ and $l_{q,u}$ is bounded, i.e., inequality (2) holds if and only if

$$B = \max\{B_1, B_2\} < \infty.$$

Moreover, the following estimates are valid for its norm

$$\max\left\{ \left(\frac{p'q}{r} \right)^{\frac{1}{q'}} B_1, \left(\frac{p'q}{r} \right)^{\frac{1}{p}} B_2 \right\} \le \|A\|_{l_p \to l_q} \le \left((2h)^{q-1} q^3 (p')^{q-1} + (2h)^{q(q-1)} q^q \left(q'\right)^{\frac{q^2}{p}} \right)^{\frac{1}{q}} B,$$

where $p' = \frac{p}{p-1}$, $q' = \frac{q}{q-1}$.

Proof. Let us recall that, for inequality (2) to hold, it is necessary and sufficient that the inequality

$$\left(\sum_{n=1}^{\infty}\left(\sum_{k=1}^{n}a_{n,k}f_k\right)^q u_n\right)^{\frac{1}{q}} \leq C\left(\sum_{n=1}^{\infty}f_n^p v_n\right)^{\frac{1}{p}} \tag{9}$$

holds for all nonnegative sequences $\{f_n\}\in l_{p,v}$.

(*Necessity and lower estimate.*) To investigate the necessity of the conditions and to derive a lower estimate for the best constant C, we consider the dual form of inequality (9):

$$\left(\sum_{k=1}^{\infty}v_k^{1-p'}\left(\sum_{n=k}^{\infty}a_{n,k}g_n\right)^{p'}\right)^{\frac{1}{p'}} \leq C\left(\sum_{k=1}^{\infty}u_k^{1-q'}g_k^{q'}\right)^{\frac{1}{q'}}, \tag{10}$$

where $\{g_k\}\in l_{q',u^{1-q'}}$. It is well known that these inequalities are equivalent in the sense that the validity of the dual inequality is both necessary and sufficient for the validity of (9). Moreover, their best constants coincide (see, for example, [2]).

Let us assume that (9) holds for a finite constant C and, for fixed $n\in Z^+$, apply the following test sequence to (9):

$$(f_N)_k := \left(\sum_{n=k}^{N}a_{n,k}^q u_n\right)^{\frac{1}{p-q}}\left(\sum_{n=1}^{k}v_n^{1-p'}\right)^{\frac{q-1}{p-q}}v_k^{1-p'}\chi_{[1,N]}(k), \tag{11}$$

where $\chi_{[1,N]}(k)$ is the characteristic sequence. Putting (11) to the right hand side of (9), we have

$$\begin{aligned}\left(\sum_{k=1}^{\infty}(f_N)_k^p v_k\right)^{\frac{1}{p}} &= \left(\sum_{k=1}^{N}\left(\sum_{n=k}^{N}a_{n,k}^q u_n\right)^{\frac{p}{p-q}}\left(\sum_{n=1}^{k}v_n^{1-p'}\right)^{\frac{p(q-1)}{p-q}}v_k^{1-p'}\right)^{\frac{1}{p}}\\ &= \left(\sum_{k=1}^{N}\left(\sum_{n=k}^{N}a_{n,k}^q u_n\right)^{\frac{r}{q}}\left(\sum_{n=1}^{k}v_n^{1-p'}\right)^{\frac{r}{q'}}v_k^{1-p'}\right)^{\frac{1}{p}}.\end{aligned} \tag{12}$$

For the left hand side of (9), apply Fubini's theorem and we find that

$$\begin{aligned}\sum_{n=1}^{\infty}\left(\sum_{k=1}^{n}a_{n,k}(f_N)_k\right)^q u_n &\geq \sum_{n=1}^{N}\left(\sum_{k=1}^{n}a_{n,k}(f_N)_k\right)^q u_n\\ &= \sum_{n=1}^{N}\left(\sum_{k=1}^{n}a_{n,k}(f_N)_k\left(\sum_{k=1}^{n}a_{n,k}(f_N)_k\right)^{q-1}\right)u_n\\ &\geq \sum_{n=1}^{N}\left(\sum_{k=1}^{n}a_{n,k}(f_N)_k\left(\sum_{l=1}^{k}a_{n,l}(f_N)_l\right)^{q-1}\right)u_n\\ &= \sum_{k=1}^{N}(f_N)_k\left[\sum_{n=k}^{N}a_{n,k}\left(\sum_{l=1}^{k}a_{n,l}(f_N)_l\right)^{q-1}u_n\right]\end{aligned}$$

[using $a_{n,k} \le a_{n,l}$ for $n \ge k \ge l \ge 1$]

$$\ge \sum_{k=1}^{N} (f_N)_k \sum_{n=k}^{N} a_{n,k}^q u_n \left(\sum_{l=1}^{k} (f_N)_l \right)^{q-1}$$

$$= \sum_{k=1}^{N} (f_N)_k \sum_{n=k}^{N} a_{n,k}^q u_n \left(\sum_{l=1}^{k} v_l^{1-p'} \left(\sum_{n=l}^{N} a_{n,l}^q u_n \right)^{\frac{1}{p-q}} \left(\sum_{n=1}^{l} v_n^{1-p'} \right)^{\frac{q-1}{p-q}} \right)^{q-1}$$

[using now $a_{n,k} \le a_{n,l}$ for $n \ge k \ge l \ge 1$ again and then Lemma 1]

$$\ge \sum_{k=1}^{N} (f_N)_k \left(\sum_{n=k}^{N} a_{n,k}^q u_n \right)^{\frac{p-1}{p-q}} \left(\sum_{l=1}^{k} v_l^{1-p'} \left(\sum_{n=1}^{l} v_n^{1-p'} \right)^{\frac{q-1}{p-q}} \right)^{q-1}$$

$$\ge \left(\frac{p-q}{p-1} \right)^{q-1} \sum_{k=1}^{N} (f_N)_k \left(\sum_{n=k}^{N} a_{n,k}^q u_n \right)^{\frac{p-1}{p-q}} \left(\sum_{l=1}^{k} v_l^{1-p'} \right)^{\frac{(p-1)(q-1)}{p-q}}$$

$$= \left(\frac{p'q}{r} \right)^{q-1} \sum_{k=1}^{N} \left(\sum_{n=k}^{N} a_{n,k}^q u_n \right)^{\frac{p}{p-q}} \left(\sum_{l=1}^{k} v_l^{1-p'} \right)^{\frac{p(q-1)}{p-q}} v_k^{1-p'},$$

i.e.,

$$\left(\sum_{n=1}^{\infty} \left(\sum_{k=1}^{n} a_{n,k} (f_N)_k \right)^q u_n \right)^{\frac{1}{q}} \ge \left(\frac{p'q}{r} \right)^{\frac{1}{q'}} \left(\sum_{k=1}^{N} \left(\sum_{n=k}^{N} a_{n,k}^q u_n \right)^{\frac{r}{q}} \left(\sum_{l=1}^{k} v_l^{1-p'} \right)^{\frac{r}{q'}} v_k^{1-p'} \right)^{\frac{1}{q}}. \tag{13}$$

Using (12) and (13) estimates in inequality (9), we obtain

$$\left(\frac{p'q}{r} \right)^{\frac{1}{q'}} \left(\sum_{k=1}^{N} \left(\sum_{n=k}^{N} a_{n,k}^q u_n \right)^{\frac{r}{q}} \left(\sum_{l=1}^{k} v_l^{1-p'} \right)^{\frac{r}{q'}} v_k^{1-p'} \right)^{\frac{1}{r}} \le C$$

and from this with a constant independent of N and hence, letting $N \to \infty$, we obtain

$$\left(\frac{p'q}{r} \right)^{\frac{1}{q'}} B_1 \le C. \tag{14}$$

Now for fixed $N \in Z^+$ apply the following test sequence to (10) $g_N = \{(g_N)_k\}_{k=1}^{\infty}$, where

$$(g_N)_k := u_k \left(\sum_{n=k}^{N} u_n \right)^{\frac{q-1}{p-q}} \left(\sum_{l=1}^{k} a_{k,l}^{p'} v_l^{1-p'} \right)^{\frac{(p-1)(q-1)}{p-q}} \chi_{[1,N]}(k). \tag{15}$$

Applying (15) to the right hand side of (10), we have

$$\left(\sum_{k=1}^{\infty} u^{1-q'} {(g_N)_k}^{q'} \right)^{\frac{1}{q'}} = \left(\sum_{k=1}^{N} u_k \left(\sum_{n=k}^{N} u_n \right)^{\frac{q}{p-q}} \left(\sum_{l=1}^{k} a_{k,l}^{p'} v_l^{1-p'} \right)^{\frac{q(p-1)}{p-q}} \right)^{\frac{1}{q'}}$$

$$= \left(\sum_{k=1}^{N} u_k \left(\sum_{n=k}^{N} u_n \right)^{\frac{r}{p}} \left(\sum_{l=1}^{k} a_{k,l}^{p'} v_l^{1-p'} \right)^{\frac{r}{p'}} \right)^{\frac{1}{q'}}. \tag{16}$$

Applying Fubini's theorem for the left-hand side of (10), we obtain

$$\sum_{k=1}^{\infty} v_k^{1-p'} \left(\sum_{n=k}^{\infty} a_{n,k}(g_N)_n \right)^{p'} = \sum_{k=1}^{N} v_k^{1-p'} \left(\sum_{n=k}^{N} a_{n,k}(g_N)_n \right)^{p'}$$
$$= \sum_{k=1}^{N} v_k^{1-p'} \left(\sum_{n=k}^{N} a_{n,k}(g_N)_n \left(\sum_{n=k}^{N} a_{n,k}(g_N)_n \right)^{p'-1} \right)$$
$$\geq \sum_{k=1}^{N} v_k^{1-p'} \left(\sum_{n=k}^{N} a_{n,k}(g_N)_n \left(\sum_{l=n}^{N} a_{l,k}(g_N)_l \right)^{p'-1} \right)$$
$$= \sum_{n=1}^{N} (g_N)_n \left(\sum_{k=1}^{n} a_{n,k} v_k^{1-p'} \left(\sum_{l=n}^{N} a_{l,k}(g_N)_l \right)^{p'-1} \right)$$
$$\geq \sum_{n=1}^{N} (g_N)_n \left(\sum_{l=n}^{N} (g_N)_l \right)^{p'-1} \left(\sum_{k=1}^{n} a_{n,k}^{p'} v_k^{1-p'} \right)$$
$$\geq \sum_{n=1}^{N} (g_N)_n \left(\sum_{l=n}^{N} u_l \left(\sum_{k=1}^{l} a_{l,k}^{p'} v_k^{1-p'} \right)^{\frac{(p-1)(q-1)}{p-q}} \left(\sum_{k=l}^{N} u_k \right)^{\frac{q-1}{p-q}} \right)^{p'-1} \left(\sum_{k=1}^{n} a_{n,k}^{p'} v_k^{1-p'} \right)$$
$$\geq \sum_{n=1}^{N} (g_N)_n \left(\sum_{k=1}^{n} a_{n,k}^{p'} v_k^{1-p'} \right)^{\frac{p-1}{p-q}} \left(\sum_{l=n}^{N} u_l \left(\sum_{k=l}^{N} u_k \right)^{\frac{q-1}{p-q}} \right)^{p'-1}$$
$$\geq \left(\frac{p-q}{p-1} \right)^{p'-1} \sum_{n=1}^{N} (g_N)_n \left(\sum_{k=1}^{n} a_{n,k}^{p'} v_k^{1-p'} \right)^{\frac{p-1}{p-q}} \left(\sum_{l=n}^{N} u_l \right)^{\frac{1}{p-q}}$$
$$= \left(\frac{p-q}{p-1} \right)^{p'-1} \sum_{n=1}^{N} u_n \left(\sum_{k=1}^{n} a_{n,k}^{p'} v_k^{1-p'} \right)^{\frac{(p-1)(q-1)}{p-q}} \left(\sum_{l=n}^{N} u_l \right)^{\frac{q-1}{p-q}} \left(\sum_{k=1}^{n} a_{n,k}^{p'} v_k^{1-p'} \right)^{\frac{p-1}{p-q}} \left(\sum_{l=n}^{N} u_l \right)^{\frac{1}{p-q}}$$
$$= \left(\frac{p-q}{p-1} \right)^{p'-1} \sum_{n=1}^{N} u_n \left(\sum_{k=1}^{n} a_{n,k}^{p'} v_k^{1-p'} \right)^{\frac{q(p-1)}{p-q}} \left(\sum_{l=n}^{N} u_l \right)^{\frac{q}{p-q}}$$
$$= \left(\frac{p'q}{r} \right)^{p'-1} \sum_{n=1}^{N} u_n \left(\sum_{k=1}^{n} a_{n,k}^{p'} v_k^{1-p'} \right)^{\frac{r}{p'}} \left(\sum_{l=n}^{N} u_l \right)^{\frac{r}{p}},$$

i.e.,

$$\left(\sum_{k=1}^{\infty} v_k^{1-p'} \left(\sum_{n=k}^{\infty} a_{n,k}(g_N)_n \right)^{p'} \right)^{\frac{1}{p'}} \geq \left(\frac{p'q}{r} \right)^{\frac{1}{p}} \left(\sum_{n=1}^{N} u_n \left(\sum_{l=n}^{N} u_l \right)^{\frac{r}{p}} \left(\sum_{k=1}^{n} a_{n,k}^{p'} v_k^{1-p'} \right)^{\frac{r}{p'}} \right)^{\frac{1}{p'}}. \tag{17}$$

Using (16) and (17) in inequality (10), we obtain

$$\left(\frac{p'q}{r} \right)^{\frac{1}{p}} \left(\sum_{n=1}^{N} u_n \left(\sum_{l=n}^{N} u_l \right)^{\frac{r}{p}} \left(\sum_{k=1}^{n} a_{n,k}^{p'} v_k^{1-p'} \right)^{\frac{r}{p'}} \right)^{\frac{1}{r}} \leq C$$

with a constant independent of N and hence, letting $N \to \infty$, we obtain

$$\left(\frac{p'q}{r}\right)^{\frac{1}{p}} B_2 \leq C. \tag{18}$$

Thus, in view of (14), (18) and our assumption, we have

$$\max\left\{\left(\frac{p'q}{r}\right)^{\frac{1}{q'}} B_1, \left(\frac{p'q}{r}\right)^{\frac{1}{p}} B_2\right\} \leq C.$$

The necessity part is proved.

(*Sufficiency and upper estimate.*) Now, we proceed to estimate the right-hand side of (9). To this end, taking into account Lemma 2, we estimate the terms S_1 and S_2 separately:

$$\begin{aligned}
S_1 &= \sum_{m=1}^{\infty} f_m \left(\sum_{n=m}^{\infty} a_{n,m}^q u_n\right) \left(\sum_{k=1}^{m} f_k\right)^{q-1} \\
&= \sum_{m=1}^{\infty} \left(\sum_{n=m}^{\infty} a_{n,m}^q u_n\right) \left(\sum_{k=1}^{m} f_k - \sum_{k=1}^{m-1} f_k\right) \left(\sum_{k=1}^{m} f_k\right)^{q-1} \\
&\leq \sum_{m=1}^{\infty} \left(\sum_{n=m}^{\infty} a_{n,m}^q u_n\right) \left[\left(\sum_{k=1}^{m} f_k\right)^q - \left(\sum_{k=1}^{m-1} f_k\right)^q\right] \\
&= \sum_{m=1}^{\infty} \left(\sum_{n=m}^{\infty} a_{n,m}^q u_n\right) \left(\sum_{k=1}^{m} f_k\right)^q - \sum_{m=1}^{\infty} \left(\sum_{n=m}^{\infty} a_{n,m}^q u_n\right) \left(\sum_{k=1}^{m-1} f_k\right)^q \\
&= \sum_{m=1}^{\infty} \left(\sum_{n=m}^{\infty} a_{n,m}^q u_n\right) \left(\sum_{k=1}^{m} f_k\right)^q - \sum_{m=1}^{\infty} \left(\sum_{n=m+1}^{\infty} a_{n,m+1}^q u_n\right) \left(\sum_{k=1}^{m} f_k\right)^q \\
&= \sum_{m=1}^{\infty} \left[\left(\sum_{n=m}^{\infty} a_{n,m}^q u_n\right) - \left(\sum_{n=m+1}^{\infty} a_{n,m+1}^q u_n\right)\right] \left(\sum_{k=1}^{m} f_k\right)^q \\
&= \sum_{m=1}^{\infty} \bar{u}_m \left(\sum_{k=1}^{m} f_k\right)^q \\
&\leq C_{p,q}^q \left(\sum_{m=1}^{\infty} f_m^p v_m\right)^{\frac{q}{p}}.
\end{aligned}$$

To get the last estimate we used inequality (3) with the weight sequence $\bar{u}_m := \left(\sum_{n=m}^{\infty} a_{n,m}^q u_n\right) - \left(\sum_{n=m+1}^{\infty} a_{n,m+1}^q u_n\right)$, since the satisfying of which follows from condition (4), i.e.,

$$\begin{aligned}
&\sum_{n=1}^{\infty} \left(\sum_{k=n}^{\infty} \bar{u}_k\right)^{\frac{r}{q}} \left(\sum_{k=1}^{n} v_k^{1-p'}\right)^{\frac{r}{q'}} v_n^{1-p'} = \\
&= \sum_{n=1}^{\infty} \left(\sum_{k=n}^{\infty} \left[\left(\sum_{m=k}^{\infty} a_{m,k}^q u_m\right) - \left(\sum_{m=k+1}^{\infty} a_{m,k+1}^q u_m\right)\right]\right)^{\frac{r}{q}} \left(\sum_{k=1}^{n} v_k^{1-p'}\right)^{\frac{r}{q'}} v_n^{1-p'} \\
&= \sum_{n=1}^{\infty} \left(\sum_{m=n}^{\infty} a_{m,n}^q u_m\right)^{\frac{r}{q}} \left(\sum_{k=1}^{n} v_k^{1-p'}\right)^{\frac{r}{q'}} v_n^{1-p'} = B_1^r < \infty.
\end{aligned}$$

Moreover, using (5), we obtain the estimate for the best constant $C_{p,q}$, i.e.,

$$C_{p,q} \leq q^{\frac{1}{q}} (p')^{\frac{1}{q'}} B_1.$$

Therefore,

$$S_1^{\frac{1}{q}} \leq q^{\frac{1}{q}} (p')^{\frac{1}{q'}} B_1 \|f\|_{p,v},$$

i.e.,

$$S_1 \leq q(p')^{q-1} B_1^q \|f\|_{p,v}^q. \tag{19}$$

Now we estimate S_2. Applying Hölder inequality, we get

$$\begin{aligned}
S_2 &= \sum_{m=1}^{\infty} f_m \left(\sum_{n=m}^{\infty} a_{n,m} u_n \right) \left(\sum_{k=1}^{m} a_{m,k} f_k \right)^{q-1} \\
&\leq \left(\sum_{m=1}^{\infty} f_m^p v_m \right)^{\frac{1}{p}} \left(\sum_{m=1}^{\infty} v_m^{1-p'} \left(\sum_{n=m}^{\infty} a_{n,m} u_n \right)^{p'} \left(\sum_{k=1}^{m} a_{m,k} f_k \right)^{(q-1)p'} \right)^{\frac{1}{p'}} \\
&= \|f\|_{p,v} \bar{S}_2^{\frac{1}{p'}},
\end{aligned}$$

where

$$\bar{S}_2 = \sum_{m=1}^{\infty} \left(\sum_{n=m}^{\infty} a_{n,m} u_n \right)^{p'} \left(\sum_{k=1}^{m} a_{m,k} f_k \right)^{(q-1)p'} v_m^{1-p'}.$$

To estimate $\bar{S}_2$, we proceed as follows:

$$\begin{aligned}
\bar{S}_2 &= \sum_{m=1}^{\infty} v_m^{1-p'} \left(\sum_{n=m}^{\infty} a_{n,m} u_n \right)^{p'} \sum_{l=1}^{m} \left[\left(\sum_{k=1}^{l} a_{l,k} f_k \right)^{(q-1)p'} - \left(\sum_{k=1}^{l-1} a_{l,k} f_k \right)^{(q-1)p'} \right] \\
&= \sum_{l=1}^{\infty} \left[\sum_{m=l}^{\infty} v_m^{1-p'} \left(\sum_{n=m}^{\infty} a_{n,m} u_n \right)^{p'} \right] \left[\left(\sum_{k=1}^{l} a_{l,k} f_k \right)^{(q-1)p'} - \left(\sum_{k=1}^{l-1} a_{l,k} f_k \right)^{(q-1)p'} \right]
\end{aligned}$$

[we apply the generalized Minkowskii inequality to the second bracket]

$$\begin{aligned}
&\leq \sum_{l=1}^{\infty} \left[\left(\sum_{k=1}^{l} a_{l,k} f_k \right)^{(q-1)p'} - \left(\sum_{k=1}^{l-1} a_{l-1,k} f_k \right)^{(q-1)p'} \right] \left(\sum_{n=l}^{\infty} u_n \left(\sum_{m=1}^{n} a_{n,m}^{p'} v_m^{1-p'} \right)^{\frac{1}{p'}} \right)^{p'} \\
&= \sum_{l=1}^{\infty} \left[\sum_{n=l}^{\infty} \left(\sum_{m=n}^{\infty} u_m \right)^{\frac{1}{p}} \left(\sum_{m=1}^{n} a_{n,m}^{p'} v_m^{1-p'} \right)^{\frac{1}{p'}} u_n^{1-\frac{q}{p}} \left(\sum_{m=n}^{\infty} u_m \right)^{-\frac{1}{p}} u_n^{\frac{q}{p}} \right]^{p'} \\
&\times \left[\left(\sum_{k=1}^{l} a_{l,k} f_k \right)^{(q-1)p'} - \left(\sum_{k=1}^{l-1} a_{l-1,k} f_k \right)^{(q-1)p'} \right]
\end{aligned}$$

[applying Hölder inequality with the degrees $\frac{p}{p-q}$ and $\frac{p}{q}$ and according to (7), we have]

$$\leq \sum_{l=1}^{\infty}\left(\sum_{n=l}^{\infty}\left(\sum_{m=n}^{\infty}u_m\right)^{\frac{1}{p-q}}\left(\sum_{m=1}^{n}a_{n,m}^{p'}v_m^{1-p'}\right)^{\frac{p-1}{p-q}}u_n\right)^{\frac{p-q}{p-1}}\left(\sum_{n=l}^{\infty}\left(\sum_{m=n}^{\infty}u_m\right)^{-\frac{1}{q}}u_n\right)^{\frac{q}{p-1}}$$
$$\times\left[\left(\sum_{k=1}^{l}a_{l,k}f_k\right)^{(q-1)p'}-\left(\sum_{k=1}^{l-1}a_{l-1,k}f_k\right)^{(q-1)p'}\right]$$
$$\leq (q')^{\frac{q}{p-1}}\sum_{l=1}^{\infty}\left(\sum_{n=l}^{\infty}\left(\sum_{m=n}^{\infty}u_m\right)^{\frac{1}{p-q}}\left(\sum_{m=1}^{n}a_{n,m}^{p'}v_m^{1-p'}\right)^{\frac{p-1}{p-q}}u_n\right)^{\frac{p-q}{p-1}}$$
$$\times\left(\sum_{m=l}^{\infty}u_m\right)^{\frac{q-1}{p-1}}\left[\left(\sum_{k=1}^{l}a_{l,k}f_k\right)^{(q-1)p'}-\left(\sum_{k=1}^{l-1}a_{l-1,k}f_k\right)^{(q-1)p'}\right]$$

[again applying Hölder inequality with the degrees $\frac{p-1}{p-q}$ and $\frac{p-1}{q-1}$, we get]

$$\leq (q')^{\frac{q}{p-1}}\left[\sum_{l=1}^{\infty}\left(\sum_{n=l}^{\infty}\left(\sum_{m=n}^{\infty}u_m\right)^{\frac{1}{p-q}}\left(\sum_{m=1}^{n}a_{n,m}^{p'}v_m^{1-p'}\right)^{\frac{p-1}{p-q}}u_n\right)\right.$$
$$\left.\times\left[\left(\sum_{k=1}^{l}a_{l,k}f_k\right)^{(q-1)p'}-\left(\sum_{k=1}^{l-1}a_{l-1,k}f_k\right)^{(q-1)p'}\right]^{\frac{1}{p'}}\right]^{\frac{p-q}{p-1}}$$
$$\times\left[\sum_{l=1}^{\infty}\left(\sum_{m=l}^{\infty}u_m\right)\left[\left(\sum_{k=1}^{l}a_{l,k}f_k\right)^{(q-1)p'}-\left(\sum_{k=1}^{l-1}a_{l-1,k}f_k\right)^{(q-1)p'}\right]^{\frac{q'}{p'}}\right]^{\frac{q-1}{p-1}}$$

[applying the inequality $(x-y)^{\alpha}\leq x^{\alpha}-y^{\alpha}$ $(0<\alpha<1)$, followed by Fubini's theorem on the first and second products, we get]

$$\leq (q')^{\frac{q}{p-1}}\left[\sum_{l=1}^{\infty}\left(\sum_{n=l}^{\infty}\left(\sum_{m=n}^{\infty}u_m\right)^{\frac{1}{p-q}}\left(\sum_{m=1}^{n}a_{n,m}^{p'}v_m^{1-p'}\right)^{\frac{p-1}{p-q}}u_n\right)\right.$$
$$\left.\times\left[\left(\sum_{k=1}^{l}a_{l,k}f_k\right)^{q-1}-\left(\sum_{k=1}^{l-1}a_{l-1,k}f_k\right)^{q-1}\right]\right]^{\frac{p-q}{p-1}}$$
$$\times\left[\sum_{l=1}^{\infty}\left(\sum_{m=l}^{\infty}u_m\right)\left(\left(\sum_{k=1}^{l}a_{l,k}f_k\right)^{q}-\left(\sum_{k=1}^{l-1}a_{l-1,k}f_k\right)^{q}\right)\right]^{\frac{q-1}{p-1}}$$

$$= \left(q'\right)^{\frac{q}{p-1}} \left[\sum_{n=1}^{\infty}\left(\sum_{m=n}^{\infty} u_m\right)^{\frac{1}{p-q}} \left(\sum_{m=1}^{n} a_{n,m}^{p'} v_m^{1-p'}\right)^{\frac{p-1}{p-q}} u_n\right.$$

$$\left.\times \left(\sum_{l=1}^{n}\left[\left(\sum_{k=1}^{l} a_{l,k} f_k\right)^{q-1} - \left(\sum_{k=1}^{l-1} a_{l-1,k} f_k\right)^{q-1}\right]\right)\right]^{\frac{p-q}{p-1}}$$

$$\times \left[\sum_{m=1}^{\infty} u_m \left(\sum_{l=1}^{m}\left(\sum_{k=1}^{l} a_{l,k} f_k\right)^{q} - \left(\sum_{k=1}^{l-1} a_{l-1,k} f_k\right)^{q}\right)\right]^{\frac{q-1}{p-1}}$$

$$= \left(q'\right)^{\frac{q}{p-1}} \left[\sum_{n=1}^{\infty}\left(\sum_{m=n}^{\infty} u_m\right)^{\frac{1}{p-q}} \left(\sum_{m=1}^{n} a_{n,m}^{p'} v_m^{1-p'}\right)^{\frac{p-1}{p-q}} u_n^{\frac{1}{q}} u_n^{\frac{1}{q'}} \left(\sum_{k=1}^{n} a_{n,k} f_k\right)^{q-1}\right]^{\frac{p-q}{p-1}}$$

$$\times \left[\sum_{m=1}^{\infty}\left(\sum_{k=1}^{m} a_{m,k} f_k\right)^{q} u_m\right]^{\frac{q-1}{p-1}}$$

[applying Hölder inequality with exponents q and q' to the first product, we obtain]

$$= \left(q'\right)^{\frac{q}{p-1}} \left(\left[\sum_{n=1}^{\infty}\left(\sum_{m=n}^{\infty} u_m\right)^{\frac{q}{p-q}} \left(\sum_{m=1}^{n} a_{n,m}^{p'} v_m^{1-p'}\right)^{\frac{q(p-1)}{p-q}} u_n\right]^{\frac{1}{q}} \left[\sum_{n=1}^{\infty}\left(\sum_{k=1}^{n} a_{n,k} f_k\right)^{q} u_n\right]^{\frac{1}{q'}}\right)^{\frac{p-q}{p-1}} S^{\frac{q-1}{p-1}}$$

$$= \left(q'\right)^{\frac{q}{p-1}} \left[\sum_{n=1}^{\infty}\left(\sum_{m=n}^{\infty} u_m\right)^{\frac{r}{p}} \left(\sum_{m=1}^{n} a_{n,m}^{p'} v_m^{1-p'}\right)^{\frac{r}{p'}} u_n\right]^{\frac{p-q}{q(p-1)}} \left[\sum_{n=1}^{\infty}\left(\sum_{k=1}^{n} a_{n,k} f_k\right)^{q} u_n\right]^{\frac{p-q}{q'(p-1)}} S^{\frac{q-1}{p-1}}$$

$$= \left(q'\right)^{\frac{q}{p-1}} \left[\sum_{n=1}^{\infty}\left(\sum_{m=n}^{\infty} u_m\right)^{\frac{r}{p}} \left(\sum_{m=1}^{n} a_{n,m}^{p'} v_m^{1-p'}\right)^{\frac{r}{p'}} u_n\right]^{\frac{p'}{r}} S^{\frac{p-q}{q'(p-1)}} S^{\frac{q-1}{p-1}}$$

$$= \left(q'\right)^{\frac{q}{p-1}} B_2^{p'} S^{\frac{p'}{q'}}.$$

Thus, we have

$$\bar{S}_2 \le \left(q'\right)^{\frac{q}{p-1}} B_2^{p'} S^{\frac{p'}{q'}}$$

and then

$$S_2 \le \|f\|_{p,v} \bar{S}_2^{\frac{1}{p'}} \le \|f\|_{p,v} \left(q'\right)^{\frac{q}{p}} B_2 S^{\frac{1}{q'}}. \tag{20}$$

By summing estimates (19) and (20), we obtain

$$\begin{aligned} S &\le (2h)^{q-1} q [S_1 + S_2] \\ &\le (2h)^{q-1} q \left[q(p')^{q-1} B_1^q \|f\|_{p,v}^q + \|f\|_{p,v} (q')^{\frac{q}{p}} B_2 S^{\frac{1}{q'}}\right] \\ &= (2h)^{q-1} q^2 (p')^{q-1} B_1^q \|f\|_{p,v}^q + (2h)^{q-1} q \|f\|_{p,v} \left(q'\right)^{\frac{q}{p}} B_2 S^{\frac{1}{q'}}. \end{aligned}$$

Using Young's inequality on the second term yields

$$S \le (2h)^{q-1} q^2 (p')^{q-1} B_1^q \|f\|_{p,v}^q + \frac{\left((2h)^{q-1} q \|f\|_{p,v} (q')^{\frac{q}{p}} B_2\right)^q}{q} + \frac{S}{q'}$$

and then

$$S \le (2h)^{q-1} q^3 (p')^{q-1} B_1^q \|f\|_{p,v}^q + \left((2h)^{q-1} q \|f\|_{p,v} \left(q'\right)^{\frac{q}{p}} B_2\right)^q,$$

i.e.,

$$\left(\sum_{n=1}^{\infty}\left(\sum_{k=1}^{n} a_{n,k} f_k\right)^q u_n\right)^{\frac{1}{q}} \le \left((2h)^{q-1} q^3 (p')^{q-1} B_1^q + (2h)^{q(q-1)} q^q \left(q'\right)^{\frac{q^2}{p}} B_2^q\right)^{\frac{1}{q}} \left(\sum_{n=1}^{\infty} f_n^p v_n\right)^{\frac{1}{p}}.$$

Finally, we arrive at the following estimate

$$\left(\sum_{n=1}^{\infty}\left(\sum_{k=1}^{n} a_{n,k} f_k\right)^q u_n\right)^{\frac{1}{q}} \le \left((2h)^{q-1} q^3 (p')^{q-1} + (2h)^{q(q-1)} q^q \left(q'\right)^{\frac{q^2}{p}}\right)^{\frac{1}{q}} B\|f\|_{p,v}.$$

Hence, the sufficiency of the condition and the upper for the norm are established.

The proof is complete. □

Conclusion

The necessary and sufficient conditions for the boundedness of the matrix operators whose entries satisfy the so-called *Oinarov conditions* are established. In addition, both lower and upper estimates for the norms of such operators are derived. The obtained results can be used in the spectral analysis of matrix operators and in determining the oscillatory or non-oscillatory behavior of solutions to certain difference equations. Furthermore, the derived inequalities can be employed to evaluate sequences via their higher-order differences.

Acknowledgments

The research of the first author was supported by the project AL-9224104685 of the Republic of Uzbekistan.

Author Contributions

All authors contributed equally to this work.

Conflict of Interest

The authors declare no conflict of interest.

References

1 Kufner, A., Maligranda, L., & Persson, L.-E. (2007). *The Hardy inequality. About its history and some related results.* Pilsen.

2 Oinarov, R., Okpoti, C.A., & Persson, L.-E. (2007). Weighted inequalities of Hardy type for matrix operators: the case $q < p$. *Mathematic Inequality and Applications, 10*(4), 843–861. https://doi.org/10.7153/mia-10-77

3 Taspaganbetova, Zh., & Temirkhanova, T. (2011). Criteria on boundedness of matrix operators in weighted spaces of sequences and their applications. *Annals of Functional Analysis, 2*(1), 114–127. https://doi.org/10.15352/afa/1399900267

4 Oinarov, R., & Shalginbayeva, S.Kh. (2004). Vesovaia additivnaia otsenka odnogo klassa matrichnykh operatorov [Weighted additive estimate of one class of matrix operators]. *Izvestiia NAN RK. Seriia fiziko-matematicheskaia — Bulletin of the National Academy of Sciences of the Republic of Kazakhstan. Physics and Mathematics Series,* (1), 39–49 [in Russian].

5 Shaimardan, S., & Shalgynbaeva, S. (2017). Hardy-type inequalities for matrix operators. *Bulletin of the Karaganda University. Mathematics Series,* 4(88), 63–72. https://doi.org/10.31489/2017m4/63-72

6 Kalybay, A., & Shalginbayeva, S. (2024). Estimate of the best constant of discrete Hardy-type inequality with matrix operator satisfying the Oinarov condition. *Eurasian Mathematical Journal, 15*(2), 42–47. https://doi.org/10.32523/2077-9879-2024-15-2-42-47

7 Kufner, A., Kuliev, K., Kulieva, G., & Eshimova, M. (2024). New equivalent conditions for Hardy-type inequalities. *Mathematica Bohemica, 149*(1), 57–73. https://doi.org/10.21136/MB.2023.0088-22

8 Kuliev, K., Kulieva, G., & Eshimova, M. (2023). On estimates for norm of some integral operators in weighted Lebesgue spaces. *Mathematic Inequality and Applications, 26*(1), 27–37. http://dx.doi.org/10.7153/mia-2023-26-03

9 Omarbayeva, B.K., Persson, L.-E., & Temirkhanova, A.M. (2020). Weighted iterated discrete Hardy-type inequalities. *Mathematical Inequalities and Applications, 23*(3), 943–959. https://doi.org/10.7153/mia-2020-23-73

10 Zhangabergenova, N., & Temirkhanova, A. (2023). Iterated discrete Hardy-type inequalities. *Eurasian Mathematical Journal, 14*(1), 81–95. https://doi.org/10.32523/2077-9879-2023-14-1-81-95

11 Oinarov, R., Omarbayeva, B.K., & Temirkhanova, A.M. (2020). Discrete iterated Hardy-type inequalities with three weights. *Journal of Mathematics, Mechanics, Computer Science, 105*(1), 19–29. https://doi.org/10.26577/JMMCS.2020.v105.i1.03

12 Zhangabergenova, N.S., & Temirkhanova, A.M. (2023). Iterated discrete Hardy-type inequalities with three weights for a class of matrix operators. *Bulletin of the Karaganda University. Mathematics Series,* 4(112), 163–172. https://doi.org/10.31489/2023M4/163-172

13 Gogatishvili, A., Křepela, M., Ol'hava, R., & Pick, L. (2020). Weighted inequalities for discrete iterated Hardy operators. *Mediterranean Journal of Mathematics, 17*, Article 132, 132–148. https://doi.org/10.1007/s00009-020-01526-2

14 Stepanov, V.D., & Shambilova, G.E. (2019). On iterated and bilinear integral Hardy-type operators. *Mathematical Inequalities and Applications, 22*(4), 1505–1533. https://doi.org/10.7153/mia-2019-22-105

*Author Information**

Komil Danaboyevich Kuliev (*corresponding author*) — Doctor of Physical and Mathematical Sciences, Professor, Department of Mathematics, Samarkand State University named after Sh. Rashidov, 15 University Blvd., Samarkand, 140104, Uzbekistan; e-mail: *komilkuliev@gmail.com*; https://orcid.org/0000-0002-1111-456X

Gulchehra Taxirovna Kulieva — Doctor of Phylosophy (PhD), Department of Mathematics, Samarkand State University named after Sh. Rashidov, 15 University Blvd., Samarkand, 140104, Uzbekistan; e-mail: *kulievagt@gmail.com*; https://orcid.org/0009-0007-3866-8507

Tohirbek Davronovich Turakulov — Assistent, Department of Mathematics, Samarkand Branch of Oriental University, 139 Spitamen St., Samarkand, 140104, Uzbekistan; e-mail: *tohirbekt92@gmail.com*; https://orcid.org/0009-0005-1783-9325

*Authors' names are presented in the following order: first name, middle name (if any), last name.

Bulletin of the Karaganda University. Mathematics Series, No. 1(121), 2026, pp. 169–178

https://doi.org/10.31489/2026M1/169-178 *Research article*

Heritability of types of a pregeometry relative to a family of relational structures

S.B. Malyshev[1], S.V. Sudoplatov[1,2,*]

[1] *Novosibirsk State Technical University, Novosibirsk, Russian Federation;*
[2] *Sobolev Institute of Mathematics, Novosibirsk, Russian Federation*
(E-mail: sergei2-mal1@yandex.ru, sudoplat@math.nsc.ru)

A series of geometrical and topological properties induced by structures, including degeneration, modularity, local modularity, projectivity, local finiteness, etc., play an important role in clarifying structural relationships and in classifying basic and derivative semantical and syntactical objects related to a given class of structures and their valuable characteristics. It is natural to turn to the family of all structures on a given finite or infinite universe, which allows us to represent all possible structures of a given cardinality up to the definability and to describe relationships, possibilities of preserving and violating structural properties during enrichments and restrictions of structures within the framework of the chosen family. This paper studies the behavior of pregeometry types (degenerate, locally finite, modular) within the Boolean algebra $\mathcal{B}(M)$ of regular expansions and reducts of a relational structure M. We establish criteria for the inheritance of pregeometry properties under Boolean operations, proving that degeneracy and local finiteness are preserved under intersections. In contrast, we show through counterexamples that modularity generally fails to be preserved, as does local finiteness under unions. We formulate a sufficient condition of linear growth of the closure operator under which the union of locally finite structures remains locally finite. These results reveal a fundamental asymmetry between intersection and union operations, contributing to geometric stability theory by delineating the preservation boundaries of pregeometries in Boolean families of structures.

Keywords: pregeometry, Boolean algebra, degeneracy, modularity, local finiteness, algebraic closure, relational structure, intersection of structures, union of structures.

2020 Mathematics Subject Classification: 03C30, 03C52.

Introduction

Pregeometry and the geometric structure of models remain among the central objects of study in Mathematical Logic and Model Theory. Since the 1970s, approaches to the description and classification of pregeometries arising in various theories, including o-minimal, ω-stable, and strongly minimal theories, have been actively developed. Substantial contributions to this area were made by B.I. Zilber for strongly minimal [1–3] and uncountably categorical theories [4], A. Pillay [5], E. Hrushovski [6], as well as works related to ω-categorical structures [7] and systematic presentations of Model Theory [8].

By now, the literature contains a wide range of results closely related to the topic of this paper. For instance, the works of A. Bernstein and E. Vasiliev [9, 10] are devoted to the study of geometric structures and their extensions, including cases where dense independent sets and homogeneous matroids are present. Closures of special atomic subsets of semantic model were described by A.R. Yeshkeyev, A.K. Issaeva, N.V. Popova [11]. A.R. Yeshkeyev, M.T. Kassymetova, O.I. Ulbrikht [12] studied independence and simplicity in Jonsson theories with abstract geometry. Research by B.Sh. Kulpeshov,

*Corresponding author. *E-mail: sergei2-mal1@yandex.ru*

The research was supported by the grant from Russian Science Foundation, No. 24-21-00096, https://rscf.ru/en/project/24-21-00096/.

Received: 25 August 2025; *Accepted:* 1 November 2025.

S.V. Sudoplatov, D.Yu. Emelyanov, and In.I. Pavlyuk concerns various aspects of closures [13] including algebraic and definable closures for theories of abelian groups [14], combinations of structures [15], algebras of binary formulas for a series of structural operations: compositions [16], tensor products [17], strong products [18], homomorphic products [19], ordered structures [20–22], etc., which are closely connected to the preservation of pregeometry properties under transitions between various structures.

An important line of research connected to this topic is the study of *fusions* and *combinations* of structures. In [23], Hasson and Hils introduced the notion of *fusion over sublanguages*, which allows for the construction of composite structures that accumulate properties of the original ones. This operator was further generalized in [15], where Sudoplatov developed a unified framework for combinations of structures and theories, involving both unary predicates and equivalence classes as carriers of data. These works describe structural properties of fusions and combinations, providing a broader perspective on how structural and geometric characteristics behave under composition.

One approach to studying these families is to consider the Boolean algebra of relational structures [24]. This algebra is formed on the set of regular enrichments and restrictions of a fixed structure, that is, structures obtained by adding or removing predicate symbols from the signature while keeping the underlying domain fixed. The Boolean algebra $\mathcal{B}(M)$ of a relational structure M is naturally equipped with the operations of intersection, union, and complement of structures, allowing one to formally consider transitions between different signature representations of the same underlying set.

A natural question arises: which properties of pregeometries are preserved under these transformations? In particular, is the type of pregeometry (e.g., degeneracy, local finiteness, modularity) preserved under the intersection or union of two structures in the Boolean algebra?

In this paper, we study structures in the Boolean algebra $\mathcal{B}(M)$ endowed with an algebraic closure operator. We focus on the inheritance of pregeometry types under intersection and union of such structures.

The main result is as follows: if at least one of the structures $M_1, M_2 \in \mathcal{B}(M)$ has a pregeometry of degenerate or locally finite type, then the pregeometry of the intersection $M_1 \cap M_2$ inherits the same type. However, modularity is not, in general, preserved under intersection.

For unions, the situation is different: even if both structures have a locally finite pregeometry, their union may result in the loss of local finiteness.

Thus, the obtained results refine the boundaries of inheritance of pregeometry properties under composition of structures in $\mathcal{B}(M)$. We show that degeneracy and local finiteness are preserved under intersection, while modularity and local finiteness may fail to hold in the case of unions. These observations highlight an asymmetry between the operations of intersection and union and indicate directions for further investigation of the preservation of pregeometries in more general structural compositions.

The results obtained develop the ideas presented in [25], where pregeometries arising from structural compositions were studied, and complement the existing theory by describing the behavior of pregeometries in the broader context of Boolean families of structures.

1 *Pregeometries and their types*

We recall the necessary definitions from [5, 7, 8].

Definition 1. A *pregeometry* is a set S together with a closure operation $\mathrm{cl} : P(S) \to P(S)$ satisfying the following conditions:

1) for any $X \subseteq S$ we have $X \subseteq \mathrm{cl}(X)$;
2) for any $X \subseteq S$ we have $\mathrm{cl}(\mathrm{cl}(X)) = \mathrm{cl}(X)$;
3) for any $X \subseteq S$ and any $a, b \in S$, if $a \in \mathrm{cl}(X \cup \{b\}) \setminus \mathrm{cl}(X)$, then $b \in \mathrm{cl}(X \cup \{a\})$;
4) for any $X \subseteq S$, if $a \in \mathrm{cl}(X)$, then $a \in \mathrm{cl}(Y)$ for some finite $Y \subseteq X$.

Given a pregeometry $\langle S, \mathrm{cl}\rangle$, every subset $X \subseteq S$ has a minimal subset $X' \subseteq X$ such that $\mathrm{cl}(X) = \mathrm{cl}(X')$. This minimal subset X' is called a *basis* of X. Moreover, all bases have the same cardinality, which is called the *dimension* of X in the pregeometry $\langle S, \mathrm{cl}\rangle$ and is denoted by $\dim(X)$.

By definition, $\dim(X) = \dim(\mathrm{cl}(X))$, i.e., the dimension is preserved under taking the closure of a set X in the pregeometry $\langle S, \mathrm{cl}\rangle$.

If $\dim(X) \in \omega$, then the set X is called *finite-dimensional*.

Definition 2. A subset $X \subseteq S$ is called *closed* if $X = \mathrm{cl}(X)$.

Definition 3. A pregeometry $\langle S, \mathrm{cl}\rangle$ is called *trivial* or *degenerate* if for every $X \subseteq S$ we have

$$\mathrm{cl}(X) = \bigcup\{\mathrm{cl}(\{a\}) \mid a \in X\}.$$

A pregeometry $\langle S, \mathrm{cl}\rangle$ is called *modular* if for any closed sets $X_0, Y_0 \subseteq S$, the set X_0 is independent from Y_0 over $X_0 \cap Y_0$, i.e., for any finite-dimensional closed sets $X \subseteq X_0$, $Y \subseteq Y_0$ we have

$$\dim(X) + \dim(Y) - \dim(X \cap Y) = \dim(X \cup Y).$$

A pregeometry $\langle S, \mathrm{cl}\rangle$ is called *locally modular* if for every $a \in S$, the pregeometry $\langle S, \mathrm{cl}_{\{a\}}\rangle$ is modular, where $\mathrm{cl}_{\{a\}}(X) = \mathrm{cl}(X \cup \{a\})$.

A pregeometry $\langle S, \mathrm{cl}\rangle$ is called *projective* if it is modular and non-trivial, and *locally projective* if it is locally modular and non-trivial.

A pregeometry $\langle S, \mathrm{cl}\rangle$ is called *locally finite* if for every finite subset $A \subseteq S$ we have that $\mathrm{cl}(A)$ is finite.

Definition 4. Let S be a model of a theory T. The *algebraic closure* operator for the model M is defined as the operator $\mathrm{acl} : P(M) \to P(M)$ such that for any subset $X \subseteq S$ we have

$$\mathrm{acl}(X) = \{a \in S \mid \text{ for some formula } \phi(x, \bar{y}) \text{ and } \bar{b} \in X,\ S \models \exists^{<\omega} x\, \phi(x, \bar{b}) \wedge \phi(a, \bar{b})\}.$$

In what follows, we will consider pregeometries of the form $\langle S, \mathrm{acl}\rangle$.

2 Families of Relational Structures

We consider regular enrichments and reducts of relational structures, which form a natural Boolean algebra. We provide a description of the types of pregeometries [5] with the algebraic closure operator for the family of structures in this Boolean algebra.

Definition 5. Let M be a relational structure with a signature Σ.

A *reduct* of the structure M is a structure obtained by removing some predicates from the signature Σ.

An *enrichment* of the structure M is a structure obtained by adding new predicates to the signature Σ and assigning their interpretations on the same universe.

Definition 6. [24] A structure is called *regular* if it is a relational structure without repetitions of interpretations of the signature symbols.

The procedure transforming an arbitrary structure M into a regular structure N is called *regularization*, and the structure N is called *regularized* with respect to M. The inverse procedure, transforming N back into the original structure M, is called *deregularization*, and M is called *deregularized* with respect to N.

Definition 7. [24] Let M be a fixed relational structure with signature Σ. The *Boolean algebra* $\mathcal{B}(M)$ is the set of all structures obtained from M by adding and removing predicates in the signature, while all structures are defined on the same universe $|M|$.

The operations in the Boolean algebra $\mathcal{B}(M)$ are defined as follows:

- *Intersection of structures.* For $M_1, M_2 \in \mathcal{B}(M)$, the intersection $M_1 \cap M_2$ is the structure with signature $\Sigma_1 \cap \Sigma_2$, where Σ_i is the signature of M_i. A predicate in the intersection is preserved if it is present and identically interpreted in both structures.
- *Union of structures.* For $M_1, M_2 \in \mathcal{B}(M)$, the union $M_1 \cup M_2$ is the structure with signature $\Sigma_1 \cup \Sigma_2$, where each predicate is taken from the structure in which it is defined. If a predicate occurs in both structures, it is assumed to be identically interpreted.
- *Complement.* If $\Sigma_0 \subseteq \Sigma$, then the complement of a structure with respect to Σ_0 means removing these predicates from the signature.

3 Inheritance of Pregeometry Properties under Boolean Algebra Operations

Before formulating the main results on the preservation of pregeometry types, we prove several auxiliary statements that clarify the behavior of the acl operator when passing to substructures and superstructures in $\mathcal{B}(M)$.

Lemma 1 (Monotonicity of Algebraic Closure). Let $M_1 = \langle M, \Sigma_1 \rangle$ and $M_2 = \langle M, \Sigma_2 \rangle$ be structures from $\mathcal{B}(M)$, and let $\Sigma_1 \subseteq \Sigma_2$. Denote by acl_1 and acl_2 the algebraic closure operators in M_1 and M_2, respectively. Then for any set $X \subseteq M$, the following holds:

$$\mathrm{acl}_1(X) \subseteq \mathrm{acl}_2(X).$$

In other words, enriching the signature can only expand the algebraic closure.

Proof. If an element $a \in \mathrm{acl}_1(X)$, then there exists a formula $\phi(x, \bar{y})$ in the signature Σ_1 and a tuple $\bar{b} \in X$ such that $M_1 \models \phi(a, \bar{b})$ and the set $\{x \in M : M_1 \models \phi(x, \bar{b})\}$ is finite. Since $\Sigma_1 \subseteq \Sigma_2$, the formula ϕ is also a formula in the signature Σ_2, and its solution set in M_2 coincides with its solution set in M_1. Consequently, $a \in \mathrm{acl}_2(X)$. $\square$

Corollary 1. For any $M_1, M_2 \in \mathcal{B}(M)$ and any $X \subseteq M$, the following holds:

$$\mathrm{acl}_{M_1 \cap M_2}(X) \subseteq \mathrm{acl}_{M_1}(X) \cap \mathrm{acl}_{M_2}(X).$$

In particular, if M_1 or M_2 are locally finite, then $M_1 \cap M_2$ is also locally finite.

Proof. The signature of the intersection $M_1 \cap M_2$ is contained in both Σ_1 and Σ_2. By Lemma 1, $\mathrm{acl}_{M_1 \cap M_2}(X) \subseteq \mathrm{acl}_{M_1}(X)$ and $\mathrm{acl}_{M_1 \cap M_2}(X) \subseteq \mathrm{acl}_{M_2}(X)$, whence the inclusion follows. If M_1 is locally finite, then $\mathrm{acl}_{M_1}(X)$ is finite for finite X, and hence its subset $\mathrm{acl}_{M_1 \cap M_2}(X)$ is also finite. $\square$

Theorem 1. Let $\mathcal{B}(M)$ be the Boolean algebra of regular expansions and reducts of a relational structure M. Suppose the structures in $\mathcal{B}(M)$ are endowed with a pregeometry given by an algebraic closure operator. Then, if at least one of the structures $M_1, M_2 \in \mathcal{B}(M)$ has a pregeometry of degenerate or locally finite type, the pregeometry of the intersection $M_1 \cap M_2$ inherits the same type.

Proof. The intersection of two structures $M_1 = \langle M, \Sigma_1 \rangle$, $M_2 = \langle M, \Sigma_2 \rangle \in \mathcal{B}(M)$ is the structure $M' = \langle M, \Sigma_1 \cap \Sigma_2 \rangle \in \mathcal{B}(M)$.

For the degenerate or locally finite type of pregeometry, we prove that if $\langle M_1, \mathrm{acl} \rangle$ and $\langle M_2, \mathrm{acl} \rangle$ share the same pregeometry type, then $\langle M', \mathrm{acl} \rangle$ inherits this type.

Degeneracy. By definition, a pregeometry $\langle M', \mathrm{cl} \rangle$ is called *trivial* or *degenerate* if for any $X \subseteq M$, $\mathrm{cl}(X) = \bigcup \{\mathrm{cl}(\{a\}) \mid a \in X\}$.

When taking the intersection of two structures, the set of predicates in the new signature may either decrease or remain the same. Thus, we prove that for any reduct of the structure, the equality $\mathrm{acl}(X) = \bigcup\{\mathrm{acl}(\{a\}) \mid a \in X\}$ holds for all $X \subseteq M$.

By definition, the algebraic closure $\mathrm{acl}(X)$ of a set X is the union of the finite solution sets of all possible formulas in one variable with parameters from X. Therefore, for pregeometries with the algebraic closure operator $\langle M', \mathrm{acl}\rangle$, we have

$$\forall a \in X \subseteq M \quad \mathrm{acl}(a) \subseteq \mathrm{acl}(X).$$

Hence, one inclusion of the equality is preserved: $\mathrm{acl}(X) \supseteq \bigcup\{\mathrm{acl}(\{a\}) \mid a \in X\}$.

The inclusion $\mathrm{acl}(X) \subseteq \bigcup\{\mathrm{acl}(\{a\}) \mid a \in X\}$ fails precisely when there exist formulas with two or more distinct parameters that have finitely many solutions, and these solutions are not captured by formulas using a single parameter. Note that, by definition, passing to a reduct only removes predicates. This means the number of formulas whose solutions contribute to the closure can only decrease or remain unchanged. However, if the original structures were degenerate, they initially lacked such formulas violating degeneracy. Therefore, such formulas cannot appear in the reduct structure. We conclude that $\mathrm{acl}(X) = \bigcup\{\mathrm{acl}(\{a\}) \mid a \in X\}$.

Local Finiteness. By definition, a pregeometry $\langle M', \mathrm{acl}\rangle$ is called *locally finite* if for any finite subset $A \subseteq M$, the set $\mathrm{acl}(A)$ is finite.

If $\mathrm{acl}(A)$ was finite in the original structure (before taking the reduct), this means all algebraic formulas with parameters from A had finite solution sets. Passing to a reduct removes predicates from the signature, so the set of formulas can only shrink. This can only reduce the size of the solution sets, and therefore the intersection structure inherits local finiteness of the pregeometry. □

The following lemma follows from the proof of Theorem 1 in the part concerning the inheritance of degeneracy.

Lemma 2 (Preservation of Degeneracy under Reducts). Let $M_1 = \langle M, \Sigma_1\rangle$ have a degenerate pregeometry. Then for any structure $M_2 = \langle M, \Sigma_2\rangle \in \mathcal{B}(M)$ such that $\Sigma_2 \subseteq \Sigma_1$, the pregeometry of M_2 is also degenerate.

Remark 1. The statement of Theorem 1 cannot be strengthened to the condition "at least one of the structures has the property of modularity". As shown in Example 1, modularity is not preserved even when passing to a subsignature, as it critically depends on the presence or absence of specific predicates.

Theorem 2 (Non-Preservation of Properties under Union). The properties of local finiteness and modularity are not generally preserved under the union of structures in $\mathcal{B}(M)$.

1. There exist locally finite structures $M_1, M_2 \in \mathcal{B}(M)$ such that $M_1 \cup M_2$ is not locally finite (see Example 2).
2. There exist modular structures $M_1, M_2 \in \mathcal{B}(M)$ such that $M_1 \cup M_2$ is not modular.

Proof. Part 1 is proved in Example 2.

For Part 2, one can modify Example 1. Consider modular structures $M_1 = \langle M, \{R\}\rangle$ and $M_2 = \langle M, \{P\}\rangle$, where R and P are independent modular pregeometries (e.g., projective planes over different prime fields). Their union $M_1 \cup M_2 = \langle M, \{R, P\}\rangle$ may yield a non-modular pregeometry if the relations R and P are "intertwined" in a specific way, creating dependencies that violate the modular law (for instance, analogous to Hrushovski's construction). □

Example 1 (Failure of Modularity under Intersection). Consider pregeometries $M_1 = \langle M, \Sigma_1\rangle$ and $M_2 = \langle M, \Sigma_2\rangle \in \mathcal{B}(M)$, where $\Sigma_1 = \{R, Q\}$ and $\Sigma_2 = \{Q, P\}$.

Let R and P be infinite trees connecting all elements of the set M, where each vertex in each tree has a unique degree (distinct from all others). Then the closure of the empty set in the pregeometries M_1 and M_2 coincides with M, i.e., $\mathrm{acl}(\emptyset) = M$.

In this case, the dimension of any set is zero,

$$\forall A \subseteq M \quad \dim(A) = 0,$$

and consequently, for any finite-dimensional subsets $X, Y \subseteq M$, the identity holds

$$\dim(X) + \dim(Y) - \dim(X \cap Y) = \dim(X \cup Y).$$

Thus, the pregeometries M_1 and M_2 are modular, and their modularity does not depend on the relation Q. However, the modularity of the intersection pregeometry is determined exclusively by the relation Q. Therefore, if Q is not modular, then the pregeometry $\langle M, \Sigma_1 \cap \Sigma_2 \rangle$ will not be modular.

Note that the corresponding statements for unions are false in general.

Example 2 (Local Finiteness Fails under Union). Consider two acyclic graph structures $M_1 = \langle M, \Sigma_1 \rangle$ and $M_2 = \langle M, \Sigma_2 \rangle \in \mathcal{B}(M)$, where $\Sigma_1 = \{R_1\}$, $\Sigma_2 = \{R_2\}$. Let M_1 and M_2 share the domain M, and let each be an infinite tree in which every vertex has countable (infinite) degree.

Definition 8. We define the *n-neighborhood* of a vertex a as the set of vertices connected to it by a path of n edges. This set is denoted by $N_n(a)$.

We construct R_2 from R_1 as follows: for each vertex $a \in M$, we reassign the edges incident to vertices in $N_1(a)$ (i.e., the immediate neighbors) to vertices in $N_{i(b)}(a)$, where $i(b) \in \mathbb{N}$ is chosen individually for each vertex $b \in N_1(a)$ such that for distinct $b \in N_1(a)$ the values $i(b)$ are distinct. Acyclicity and the infinite degree of each vertex are preserved.

Thus, each of the structures M_1 and M_2 individually possesses local finiteness: the algebraic closure of any set is finite. However, in the union of the signatures, where edges present in either R_1 or R_2 are included, the closure of a previously chosen vertex a becomes infinite due to the interaction of edges from $N_1(a)$ and $N_{i(b)}(a)$, leading to the loss of the local finiteness property for the entire structure.

This demonstrates that the union of two locally finite structures can violate the local finiteness of the pregeometry.

Theorem 3 (Sufficient Condition for Local Finiteness of the Union). Let M_1, M_2 be locally finite structures from $\mathcal{B}(M)$. If for every finite $A \subseteq M$ the following holds:

$$\mathrm{acl}_{M_1}(\mathrm{acl}_{M_2}(A)) = \mathrm{acl}_{M_2}(\mathrm{acl}_{M_1}(A)),$$

and the set $\mathrm{acl}_{M_1 \cup M_2}(A)$ minus the iterations of $\mathrm{acl}_{M_1}(A)$ and $\mathrm{acl}_{M_2}(A)$ is finite, then the union $M_1 \cup M_2$ is locally finite.

Proof. Take an arbitrary finite set $A \subseteq M$ and prove that $\mathrm{acl}_{M_1 \cup M_2}(A)$ is finite. Since M_1 and M_2 are locally finite, the sets

$$\mathrm{acl}_{M_1}(A) \quad \text{and} \quad \mathrm{acl}_{M_2}(A)$$

are finite.

Consider the alternating sequence of sets defined recursively:

$$S_0 := A, \quad S_{2k+1} := \mathrm{acl}_{M_1}(S_{2k}), \quad S_{2k+2} := \mathrm{acl}_{M_2}(S_{2k+1}) \quad (k \geq 0).$$

Clearly, each S_n is finite (since M_1, M_2 are locally finite) and the chain is monotone, i.e., $S_n \subseteq S_{n+1}$ for all n.

Denote by

$$S := \bigcup_{n=0}^{\infty} S_n$$

the union of all iterations. We show that under the condition of commutativity of algebraic closures (i.e., $\mathrm{acl}_{M_1} \circ \mathrm{acl}_{M_2} = \mathrm{acl}_{M_2} \circ \mathrm{acl}_{M_1}$ on finite sets) the set S is finite.

Indeed, let w be an arbitrary composition of operators acl_{M_1} and acl_{M_2} applied to A. Since the closure operations are idempotent (for any structure N we have $\mathrm{acl}_N(\mathrm{acl}_N(X)) = \mathrm{acl}_N(X)$), any such composition can be reduced to one of the following forms:

- $\mathrm{acl}_{M_1}(A)$,
- $\mathrm{acl}_{M_2}(A)$,
- $\mathrm{acl}_{M_1}(\mathrm{acl}_{M_2}(A))$ (or, equivalently by the condition, $\mathrm{acl}_{M_2}(\mathrm{acl}_{M_1}(A))$).

This follows because due to idempotence and the commutativity condition, any sequence of operators becomes equivalent to one operator of each type or their composition of two different operators, and repeated application of the same operator does not yield new elements.

Hence,

$$S = A \cup \mathrm{acl}_{M_1}(A) \cup \mathrm{acl}_{M_2}(A) \cup \mathrm{acl}_{M_1}(\mathrm{acl}_{M_2}(A)).$$

Since all the listed sets are finite, S is finite.

By the definition of algebraic closure in the expanded signature,

$$\mathrm{acl}_{M_1 \cup M_2}(A)$$

consists of elements that appear either in one of the iterations of acl_{M_1} and acl_{M_2}, or are "new" elements not captured by these iterations. By the theorem's condition, the set of such "new" elements (i.e., the difference $\mathrm{acl}_{M_1 \cup M_2}(A) \setminus S$) is finite. Since S is finite, we conclude that $\mathrm{acl}_{M_1 \cup M_2}(A)$ is the union of two finite sets and therefore finite.

Since A was an arbitrary finite subset, we obtain that for every finite A, $\mathrm{acl}_{M_1 \cup M_2}(A)$ is finite, meaning that the pregeometry given by the operator $\mathrm{acl}_{M_1 \cup M_2}$ is locally finite. This implies that the structure $M_1 \cup M_2$ is locally finite. $\square$

Remark 2. The conditions of the theorem should be viewed as two independent constraints that together prevent the "mutual amplification" of new algebraic points when merging signatures:

- The commutativity condition $\mathrm{acl}_{M_1} \circ \mathrm{acl}_{M_2} = \mathrm{acl}_{M_2} \circ \mathrm{acl}_{M_1}$ on finite sets ensures that alternating closure operations does not produce "new type" of dependencies regardless of the order of operator application: any composition reduces to a simple combination of finitely many applications of the operators.
- The additional condition of finiteness of the set $\mathrm{acl}_{M_1 \cup M_2}(A)$ outside of the iterations of acl_{M_1} and acl_{M_2} from A. prevents the emergence of an infinite set of elements that cannot be obtained from iterations of the operators acl_{M_1} and acl_{M_2} alone; otherwise, the interaction of predicates from different signatures could generate an infinite closure, as demonstrated in the provided example 2.

Note also that these conditions are sufficient but not necessarily necessary: more subtle criteria for the local finiteness of the union may exist which relax one condition while strengthening the other. Furthermore, example 2 shows that without at least one of these constraints, the union can indeed lose local finiteness.

Conclusion

This work presents a systematic analysis of the preservation of pregeometry types under intersection and union operations in the Boolean algebra of structures $\mathcal{B}(M)$. The main results are summarized as follows.

1. *Stability under Intersection:* It is proved that important properties such as degeneracy and local finiteness remain preserved under the intersection operation. If at least one of the structures M_1 or M_2 possesses a pregeometry of one of these types, then their intersection $M_1 \cap M_2$ inherits this type. In contrast, modularity is not preserved under intersection.
2. *Non-Preservation under Union:* It is shown that the union operation is significantly less preserving. Even the union of two locally finite structures can yield a pregeometry that fails to be locally finite. This highlights a fundamental asymmetry between the operations in the Boolean algebra $\mathcal{B}(M)$.
3. *Sufficient Condition:* To address the instability of the union operation, a sufficient condition ensuring the preservation of local finiteness in the union is established. Specifically, if the algebraic closures in M_1 and M_2 commute on all finite sets, and the union does not introduce infinitely many new algebraic dependencies beyond those generated by iterating $\mathrm{acl}M_1$ and $\mathrm{acl}M_2$, then the combined structure $M_1 \cup M_2$ retains local finiteness.

The established asymmetry between intersection and union opens several directions for further research. These include a systematic study of other geometric properties (such as local projectivity), the search for necessary and sufficient conditions for preserving modularity, and the analysis of pregeometry interactions in more complex lattices of structures generated by Boolean operations.

Acknowledgments

The research was supported by the grant from Russian Science Foundation No. 24-21-00096, https://rscf.ru/en/project/24-21-00096/.

Author Contributions

The motivational idea, approach to studying Boolean algebras $\mathcal{B}(M)$, and commentaries in the exposition are due to S.V. Sudoplatov. Main assertions, including Theorems 1, 2, 3, and their Corollaries are due to S.B. Malyshev.

Conflict of Interest

The authors declare no conflict of interest.

References

1 Zilber, B.I. (1980). Strongly minimal countably categorical theories. *Siberian Mathematical Journal, 21*(2), 219–230. https://doi.org/10.1007/BF00968268

2 Zilber, B.I. (1984). Strongly minimal countably categorical theories II. *Siberian Mathematical Journal, 25*(3), 396–412. https://doi.org/10.1007/BF00968979

3 Zilber, B.I. (1984). Strongly minimal countably categorical theories III. *Siberian Mathematical Journal, 25*(4), 559–571. https://doi.org/10.1007/BF00968893

4 Zilber, B. (1986). Structural properties of models of ω_1-categorical theories. *Logic, Methodology and Philosophy of Science VII.* Amsterdam: North-Holland, 115–128.

5 Pillay, A. (1996). *Geometric Stability Theory.* Oxford: Clarendon Press.

6 Hrushovski, E. (1993) A new strongly minimal set. *Annals of Pure and Applied Logic, 62*(2), 147–166. https://doi.org/10.1016/0168-0072(93)90171-9

7 Cherlin, G., Harrington, L., & Lachlan, A.H. (1986). $\aleph_0$-categorical, $\aleph_0$-stable structures. *Annals of Pure and Applied Logic, 28*(2), 103–135. https://doi.org/10.1016/0168-0072(85)90023-5

8 Hodges, W. (1994). *Model theory. Encyclopedia of Mathematics and its Applications* (Vol. 42). Cambridge: Cambridge University Press.

9 Berenstein, A., & Vassiliev, E. (2016). Geometric structures with a dense independent subset. *Selecta Mathematica, 22*(1), 191–225. https://doi.org/10.1007/s00029-015-0190-1

10 Mukhopadhyay, M.M., & Vassiliev, E. (2019). On the Vamos matroid, homogeneous pregeometries and dense pairs. *Australian Journal of Combinatorics, 75*(1), 158–170.

11 Yeshkeyev, A.R., Issaeva, A.K., & Popova N.V. (2020). Closure of special atomic subsets of semantic model. *Bulletin of the Karaganda University. Mathematics Series,* 1(97), 97–103. https://doi.org/10.31489/2020M1/97-103

12 Yeshkeyev, A.R., Kassymetova, M.T., & Ulbrikht, O.I. (2021). Independence and simplicity in Jonsson theories with abstract geometry. *Siberian Electronic Mathematical Reports, 18*(1), 433–455. https://doi.org/10.33048/semi.2021.18.030

13 Sudoplatov, S.V. (2016). Closures and generating sets related to combinations of structures. *Bulletin of Irkutsk State University. Series Mathematics, 16*, 131–144.

14 Pavlyuk, In.I. (2024). On algebraic and definable closures for theories of abelian groups. *Bulletin of Irkutsk State University. Series Mathematics, 47*, 107–118. https://doi.org/10.26516/1997-7670.2024.47.107

15 Sudoplatov, S.V. (2018). Combinations of structures. *Bulletin of Irkutsk State University. Series Mathematics, 24*, 82–101. https://doi.org/10.26516/1997-7670.2018.24.82

16 Emel'yanov, D.Y., Kulpeshov, B.Sh., & Sudoplatov, S.V. (2020). Algebras of binary formulas for compositions of theories. *Algebra and Logic, 59*(4), 295–312. https://doi.org/10.1007/s10469-020-09602-y

17 Emelyanov, D.Yu. (2022). Algebras of binary isolating formulas for tensor product theories. *Bulletin of Irkutsk State University. Series Mathematics, 41*, 131–139. https://doi.org/10.26516/1997-7670.2022.41.131

18 Emelyanov, D.Yu. (2023). Algebras of binary isolating formulas for strong product theories. *Bulletin of Irkutsk State University. Series Mathematics, 45*, 138–144. https://doi.org/10.26516/1997-7670.2023.45.138

19 Emelyanov, D.Yu. (2025). Algebras of binary isolating formulas for homomorphic product theories. *Bulletin of Irkutsk State University. Series Mathematics, 52*, 153–161. https://doi.org/10.26516/1997-7670.2025.52.153

20 Kulpeshov, B.Sh. (2023). Algebras of binary formulas for $\aleph_0$-categorical weakly circularly minimal theories: piecewise monotonic case. *Siberian Electronic Mathematical Reports, 20*(2), 824–832. https://doi.org/10.33048/semi.2023.20.049

21 Kulpeshov, B.Sh. (2024). Algebras of binary formulas for weakly circularly minimal theories with trivial definable closure. *Siberian Mathematical Journal, 65*(2), 316–327. https://doi.org/10.1134/S0037446624020071

22 Kulpeshov, B.Sh. (2025). Algebras of binary formulas for weakly circularly minimal theories with equivalence relations. *Eurasian Mathematical Journal, 16*(3), 42–56. https://doi.org/10.32523/2077-9879-2025-16-3-42-56

23 Hasson, A., & Hils, M. (2006). Fusion over sublanguages. *The Journal of Symbolic Logic, 71*(2),

361–398. https://doi.org/10.2178/jsl/1146620149

24 Sudoplatov, S.V. (2025). Expansions and restrictions of structures and theories, their hierarchies. *Bulletin of Irkutsk State University. Series Mathematics, 54*, 143–159. https://doi.org/10.26516/1997-7670.2025.54.143

25 Malyshev, S.B. (2025). Nasleduemost tipov predgeometrii otnositelno kompozitsii struktur [Heritability of Types of Pregeometry with Respect to Compositions of Structures]. *Izvestiia Irkutskogo gosudarstvennogo universiteta. Seriia «Matematika» — The Bulletin of Irkutsk State University. Series «Mathematics», 52*, 162–174 [in Russian]. https://doi.org/10.26516/1997-7670.2025.52.162

*Author Information**

Sergey Borisovich Malyshev (*corresponding author*) — Assistant of the Department of Algebra and Mathematical Logic, Novosibirsk State Technical University, 20 Karl Marx Ave., Novosibirsk, 630073, Russian Federation; e-mail: *sergei2-mal1@yandex.ru*; https://orcid.org/0000-0002-6868-7545

Sergey Vladimirovich Sudoplatov — Professor, Doctor of Physical and Mathematical Sciences, Head of Chair of Algebra and Mathematical Logic, Novosibirsk State Technical University, 20 Karl Marx Ave., Novosibirsk, 630073, Russian Federation; Principal Researcher, Sobolev Institute of Mathematics; 4 Academician Koptyug Ave., Novosibirsk, 630090, Russian Federation; e-mail: *sudoplat@math.nsc.ru*; https://orcid.org/0000-0002-3268-9389

*Authors' names are presented in the following order: first name, middle name (if any), last name.

Bulletin of the Karaganda University. Mathematics Series, No. 1(121), 2026, pp. 179–193

https://doi.org/10.31489/2026M1/179-193 *Research article*

Automorphisms of free braided nonassociative algebras of rank 2

R. Mutalip, B.A. Duisengaliyeva, A.S. Naurazbekova*

L.N. Gumilyov Eurasian National University, Astana, Kazakhstan
(E-mail: mutalipriza@yahoo.com, bibinur959@gmail.com, naurazbekova82@gmail.com)

We prove the elementary reducibility of any nonaffine automorphism of a free nonassociative algebra of rank two over an arbitrary field. Using this result we establish a property of automorphisms of this algebra that will be needed later. We then derive a necessary and sufficient condition for the isomorphism of two free braided nonassociative algebras of rank two over a field with diagonal braidings. We describe the automorphism groups of two generated free braided nonassociative algebras with involutive diagonal braidings over an arbitrary field of characteristic not equal to two. Depending on the form of the diagonal involutive braiding, five different automorphism groups of a two-generated free nonassociative algebra arise in this case: 1) the group of all automorphisms, 2) the group of all odd automorphisms, 3) the subgroup of the group of triangular automorphisms, 4) the toric automorphism group, 5) the semidirect product of the toric automorphism group with the subgroup generated by an automorphism that permutes two variables.

Keywords: Yang–Baxter equation, braided space, braiding, diagonal braiding, involutive braiding, braided algebra, free nonassociative algebra, automorphism, odd automorphism, toric automorphism group.

2020 Mathematics Subject Classification: 17A36, 17B37, 16T25.

Introduction

A *braided space* (see, for example, [1,2]) is a linear space V over an arbitrary field K with a linear map $\beta : V \otimes V \to V \otimes V$, called a *braiding*, that satisfies the Yang–Baxter equation

$$(\beta \otimes \mathrm{id})(\mathrm{id} \otimes \beta)(\beta \otimes \mathrm{id}) = (\mathrm{id} \otimes \beta)(\beta \otimes \mathrm{id})(\mathrm{id} \otimes \beta). \tag{1}$$

Let $X = \{x_1, x_2, \ldots, x_n\}$ be a basis of V. The linear map

$$\beta : x_i \otimes x_s \mapsto \beta_{is} x_s \otimes x_i, \text{ where } \beta_{is} \in K,\ 1 \le i, s \le n, \tag{2}$$

is a braiding and it is called a *diagonal braiding*. In the case $\beta^2 = \mathrm{id}$ the braiding β is called *involutive*. The diagonal braiding β is involutive if and only if

$$\beta_{ij}\beta_{ji} = 1, \quad \text{for all } 1 \le i, j \le n. \tag{3}$$

In particular, $\beta_{ii} = \pm 1$ for all i.

Every braiding β of V can be uniquely extended to the free associative algebra $K\langle X\rangle$ [3] and to the free nonassociative algebra $K\{X\}$ [4] freely generated by set X over a field K so that $K\langle X\rangle$ and $K\{X\}$ are braided algebras. Moreover, they also proved that the free braided algebra $K\{X\}$ has a natural structure of a braided nonassociative Hopf algebra [4]. This algebra plays an important role in quantum Lie theory (see, for example, [3,5]). R. Mutalip, A. Naurazbekova and U. Umirbaev [6] described all automorphism groups of free braided associative algebras of rank 2 with diagonal involutive braidings

*Corresponding author. *E-mail: naurazbekova82@gmail.com*

This research was funded by the grant of the Ministry of Science and Higher Education of the Republic of Kazakhstan (project AP23487886).

Received: 15 July 2025; *Accepted:* 12 December 2025.

over a field of characteristic $\neq 2$. Many papers are devoted to the investigation of structures of braided algebras (see, for example, [7–9]).

Consider the following grading

$$K\{x_1, x_2\} = C_0 \oplus C_1 \tag{4}$$

of the free nonassociative algebra $K\{x_1, x_2\}$ in two variables x_1, x_2 over K, where C_0 and C_1 are the linear spans of all even length monomials and a linear span of all odd length monomials, respectively.

Denote by $\mathrm{Aut}A$ the automorphism group of an algebra A. Let us call an automorphism $\varphi \in \mathrm{Aut}K\{x_1, x_2\}$ *odd* if $\varphi(x_1), \varphi(x_2) \in C_1$. The set of all odd automorphisms of $K\{x_1, x_2\}$ denote by G_{odd}. It is clear that G_{odd} is subgroup of $\mathrm{Aut}K\{x_1, x_2\}$.

Let $(K\{x_1, x_2\}, \beta)$ be a free braided nonassociative algebra in two variables x_1, x_2 with a diagonal involutive braiding $\beta = (\beta_{11}, \beta_{12}, \beta_{21}, \beta_{22})$ over a field K of characteristic $\neq 2$. In this paper we show that

(1) if $\beta_{ij} = 1$ for all i, j then $\mathrm{Aut}(K\{x_1, x_2\}, \beta) = \mathrm{Aut}K\{x_1, x_2\}$;

(2) if $\beta_{ij} = -1$ for all i, j then $\mathrm{Aut}(K\{x_1, x_2\}, \beta) = G_{\mathrm{odd}}$;

(3) if $\beta_{11} = \beta_{22}$, $\beta_{12} = \beta_{21}$, and $\beta_{11}\beta_{12} = -1$ then $\mathrm{Aut}(K\{x_1, x_2\}, \beta) \cong (K^* \times K^*) \rtimes \mathbb{Z}_2$, where $\mathbb{Z}_2$ is the subgroup of $\mathrm{Aut}K\{x_1, x_2\}$ generated by (x_2, x_1);

(4) if $\beta_{12} = 1$ and $\beta_{11}\beta_{22} = -1$ then $\mathrm{Aut}(K\{x_1, x_2\}, \beta) \cong \{\varphi \in \mathrm{Aut}K\{x_1, x_2\} \mid \varphi(x_1) = ax_1 + g(x_2^2),$ $\varphi(x_2) = bx_2,\ a, b \in K^*,\ g(x) \in K\{x\}\}$;

(5) if $\beta_{12} \neq \pm 1$ or $\beta_{12} = -1,\ \beta_{11}\beta_{22} = -1$ then $\mathrm{Aut}(K\{x_1, x_2\}, \beta) \cong K^* \times K^*$.

The paper is organized as follows. In Section 1, we give some definitions and facts on free braided nonassociative algebras. In Section 2, we prove elementary reducibility of any nonaffine automorphism of a free nonassociative algebra of rank two over an arbitrary field. Using this result we prove the property of automorphisms of this algebra that we need in the future. In Section 3, we derive a necessary and sufficient condition for the isomorphism of two free braided nonassociative algebras of rank two over a field with diagonal braidings. In Section 4, we describe all automorphism groups of free braided nonassociative algebras of rank 2 equipped with diagonal involutive braidings over a field of characteristic $\neq 2$.

1 Free braided nonassociative algebra

The *braid monoid* B_n [3] is an associative monoid generated by elements $b_1, b_2, \ldots, b_{n-1}$, called braids, subject to the following relations:

$$b_t b_{t+1} b_t = b_{t+1} b_t b_{t+1}, \quad b_i b_j = b_j b_i, \quad 1 \leq t < n-1, \quad |i - j| > 1. \tag{5}$$

Let V be a linear space over a field K equipped with a braiding $\beta : V \otimes V \to V \otimes V$. Using (1), it is easy to see that the linear maps

$$\beta_i = \mathrm{id}^{\otimes(i-1)} \otimes \beta \otimes \mathrm{id}^{\otimes(n-i-1)} : V^{\otimes n} \to V^{\otimes n}, \quad 1 \leq i < n,$$

satisfy all relations (5).

Introduce the following notation:

$$[t;t] = 1, \quad [m;t] = \beta_{t-1}\beta_{t-2}\cdots\beta_{m+1}\beta_m, \quad [t;m] = \beta_m\beta_{m+1}\cdots\beta_{t-2}\beta_{t-1}, \quad m < t.$$

Define the map $\nu_r^{t,n} : V^{\otimes n} \to V^{\otimes n}$, $t \leq r < n$ as a superposition of maps β_i:

$$\nu_r^{t,n} = [t; r+1][t+1; r+2]\cdots[t+n-r-1; n].$$

V. Kharchenko proved [3], that

$$\nu_r^{t,n} = [n; r][n-1; r-1]\cdots[n-r+t; t].$$

An algebra A with a multiplication $\mathrm{m} : A \otimes A \to A$ is called a *braided algebra* [1] if A is a braided space and

$$(\mathrm{m} \otimes \mathrm{id})\beta = \beta_2\beta_1(\mathrm{id} \otimes \mathrm{m}), \quad (\mathrm{id} \otimes \mathrm{m})\beta = \beta_1\beta_2(\mathrm{m} \otimes \mathrm{id})$$

(the operators in the superposition act from left to right).

A linear map $\varphi : V \to V'$ of linear spaces V and V' equipped with braidings β and β', respectively, is called a *homomorphism of braided spaces* if

$$\beta(\varphi \otimes \varphi) = (\varphi \otimes \varphi)\beta'. \tag{6}$$

A *homomorphism of braided algebras* is a linear map that is simultaneously a homomorphism of algebras and braided spaces.

Let $X = \{x_1, x_2, \ldots, x_n\}$ be a basis of V. Denote by $K\langle X\rangle$ the free associative algebra generated by the set X over a field K. The set of all associative words X' in the alphabet X forms a linear basis of $K\langle X\rangle$. Set $\mathrm{mdeg}(x_i) = e_i$, where $e_1, \ldots, e_n$ is the standard basis for $\mathbb{Z}^n$. If $v = x_{i_1}x_{i_2}\ldots x_{i_k} \in X'$, then we put

$$d(v) = k \ \text{ and } \ \mathrm{mdeg}(v) = \sum_{j=1}^{k} \mathrm{mdeg}(x_{i_j}).$$

Consider the tensor algebra $T(V) = \bigoplus_{i=0}^{\infty} V^{\otimes i}$ of a linear space V. It is clear that $T(V) = K\langle X\rangle$. We will write $v_1v_2\ldots v_k$ instead of $v_1 \otimes v_2 \otimes \ldots \otimes v_k \in V^{\otimes k}$. Denote by $\mathrm{m}' : T(V) \oplus' T(V) \to T(V)$ the product in $T(V)$, where $\otimes'$ is the tensor product $\otimes$ with the separation function of a pair of tensors. So, $(u \otimes' v)\mathrm{m}' = u \otimes v$, where $u, v \in T(V)$.

Consider the linear map

$$\theta_t : V^{\otimes n} \to V^{\otimes t} \otimes' V^{\otimes(n-t)}, \ \ 0 \le t \le n,$$

defined by

$$(x_{i_1}x_{i_2}\cdots x_{i_n})\theta_t = x_{i_1}x_{i_2}\cdots x_{i_t} \otimes' x_{i_{t+1}}\cdots x_{i_n}.$$

V. Kharchenko [3] proved that every braiding β of V has a unique extension β' on $K\langle X\rangle$ so that $K\langle X\rangle$ is a braided algebra. The braiding β' is defined in [3] by

$$(u \otimes' v)\beta' = (u \otimes v)\nu_t^{1,n}\theta_{n-t}, \quad u \in V^{\otimes t}, \quad v \in V^{\otimes(n-t)}. \tag{7}$$

We set $(1 \otimes' v)\beta' = v \otimes' 1$ and $(u \otimes' 1)\beta' = 1 \otimes' u$.

Denote by $K\{X\}$ and $K\{y\}$ free nonassociative algebras over a field K freely generated by the set X and one variable y, respectively. The set of all nonassociative words X^* in the alphabet X and the set of all nonassociative words Y^* in the alphabet y form linear basis for $K\{X\}$ and $K\{y\}$, respectively.

Every nonassociative word $vf \in X^*$ of length t has a unique representation $vf = v \cdot f$, where $v \in X'$, $f \in Y^*$, $d(v) = d(f) = t$. We can consider f as a linear map

$$f : V^{\otimes t} \to K\{X\}.$$

We can linearly extend the action f on $K\langle X\rangle$ by $V^{\otimes n}f = 0$ if $n \neq t$. We also can extend this action on $K\langle X\rangle$ to an action of the algebra $K\{y\}$ by linearity. The linear map [4]

$$K\langle X\rangle \otimes K\{y\} \to K\{X\} \ \text{ defined by } \ (a \otimes g) \mapsto a \cdot g$$

is an isomorphism of linear spaces.

Let $u \in X'$, $f \in Y^*$, and $d(u) = d(f) = t$. Then $uf \in X^*$. Using this, we have

$$(uf)(vg) = (uv)(fg), \quad u, v \in X', \quad f, g \in Y^*.$$

U. Umirbaev and V. Kharchenko [4] proved that the every braiding β of V has a unique extension β^* on $K\{X\}$ so that $K\{X\}$ is a braided algebra. The braiding β^* is defined in [4] by

$$(uf \otimes vg)\beta^* = (u \otimes' v)\beta'(g \otimes f), \tag{8}$$

where $u, v \in X'$, $f, g \in Y^*$, and β' is the braiding of $K\langle X\rangle$ defined by (7). It is clear that

$$(1 \otimes vf)\beta^* = vf \otimes 1 \quad \text{and} \quad (ug \otimes 1)\beta^* = 1 \otimes ug. \tag{9}$$

For convenience, denote the extensions β' and β^* by the same symbol β.

Let $w \in X^*$ and let $w = w'f$, where $w' \in X'$ and $f \in Y^*$. We put

$$\text{mdeg}(w) = \text{mdeg}(w').$$

Lemma 1. Let V be a linear space over a field K with a linear basis $X = \{x_1, x_2, ..., x_n\}$ and a diagonal braiding $\beta : x_i \otimes x_j \mapsto \beta_{ij} \cdot x_j \otimes x_i$. If $u, v \in X^*$, $\text{mdeg}(u) = (k_1, k_2, \ldots, k_n)$ and $\text{mdeg}(v) = (l_1, l_2, \ldots, l_n)$, then

$$(u \otimes v)\beta^* = \prod_{ij} \beta_{ij}^{k_i l_j}(v \otimes u).$$

Proof. Let $u, v \in X^*$. Denote by u' and v' the associative words obtained from the nonassociative words u and v, respectively, by removing all brackets. Then $u = u'f$ and $v = v'g$ for some $f, g \in Y^*$. It is proven in [6] that

$$(u' \otimes' v')\beta' = \prod_{i,j} \beta_{ij}^{k_i l_j}(v' \otimes' u').$$

By this and (8), we have

$$(u \otimes v)\beta = (u'f \otimes v'g)\beta^* = (u' \otimes' v')\beta(g \otimes f) = \prod_{i,j} \beta_{ij}^{k_i l_j}(v' \otimes' u')(g \otimes f)$$

$$= \prod_{i,j} \beta_{ij}^{k_i l_j}(v'g \otimes u'f) = \prod_{i,j} \beta_{ij}^{k_i l_j}(v \otimes u). \qquad \square$$

2 Properties of automorphisms of $K\{x_1, x_2\}$

Denote by $\deg(u)$ the degree function on X^* such that $\deg(x_i) = 1$ for all i. Every nonassociative word u of degree ≥ 2 is uniquely represented as $u = u_1 \cdot u_2$, where $\deg(u_1),\ \deg(u_2) < \deg(u)$.

Set $x_1 < x_2 < \cdots < x_n$. Let u, v be arbitrary elements of X^*. We say that $u < v$, if $\deg(u) < \deg(v)$. If $\deg(u) = \deg(v) \geq 2$, where $u = u_1 \cdot u_2$, $v = v_1 \cdot v_2$, then we say that $u < v$ if $u_1 < v_1$ or if $u_1 = v_1$ and $u_2 < v_2$.

It is not difficult to see that the statement of next lemma is true.

Lemma 2. Let $u, v, w \in X^*$. If $u > v$ then $wu > wv$ and $uw > vw$.

Arbitrary element $f \in K\{X\}$ can be uniquely written as

$$f = \alpha_1 f_1 + \alpha_2 f_2 + \cdots + \alpha_k f_k,$$

where $f_i \in X^*$, $0 \neq \alpha_i \in K$ for all i and $f_1 > f_2 > ... > f_k$. Let us call f_1 *the leader* of f and denote it by $\widehat{f}$.

Corollary 1. Let $0 \neq f, g \in K\{X\}$. Then

$$\widehat{f \cdot g} = \widehat{f} \cdot \widehat{g} \quad \text{and} \quad \deg(f \cdot g) = \deg(f) + \deg(g).$$

Lemma 3. Let u, v be nonassociative words of $K\{y\}$ and let $f \in X^*$. If $u > v$, then $u(f) > v(f)$.

Proof. It is clear that $u(f), v(f) \in X^*$. Let $\deg(v) = k$. Establish the lemma's statement by induction on k. If $\deg(v) = 1$ then $v = y$ and the inequality $u > v$ holds if and only if $\deg(u) > 1$. Therefore, $u(f) > v(f)$.

Let $\deg(v) \geq 2$. We can represent u and v as $u = u_1 \cdot u_2$ and $v = v_1 \cdot v_2$, respectively. $u > v$ implies that one of the following conditions is satisfied: 1) $\deg(u) > \deg(v)$; 2) $\deg(u) = \deg(v)$ and $u_1 > v_1$; 3) $\deg(u) = \deg(v)$, $u_1 = v_1$, and $u_2 > v_2$.

If $\deg(u) > \deg(v)$ then $\deg(u(f)) > \deg(v(f))$ and $u(f) > v(f)$.

If $\deg(u) = \deg(v)$ and $u_1 > v_1$ then, by induction proposition, $u_1(f) > v_1(f)$. Therefore, $u(f) > v(f)$.

If $\deg(u) = \deg(v)$, $u_1 = v_1$, and $u_2 > v_2$ then, by induction proposition, $u_1(f) = v_1(f)$ and $u_2(f) > v_2(f)$. Therefore, $u(f) > v(f)$. □

Denote by $K\{x_1, x_2\}$ the free nonassociative algebra in two variables x_1, x_2 over a field K. The next corollary follows immediately from Corollary 1 and Lemma 2.

Corollary 2. Let $u \in K\{y\}$ and $0 \neq f \in K\{x_1, x_2\}$. Then

$$\widehat{u(f)} = \widehat{u}(\widehat{f}) \text{ and } \deg(u(f)) = \deg(u) \cdot \deg(f).$$

Denote by $\varphi = (f_1, f_2)$ the automorphism of $K\{x_1, x_2\}$ such that $\varphi(x_1) = f_1, \varphi(x_2) = f_2$. If $f_i = x_i$, $f_j = \alpha x_j + g(x_i)$, $i \neq j$, $0 \neq \alpha \in K$, $g \in K\{x_i\}$, then the automorphism φ is called *elementary.* The automorphism generated by elementary automorphisms is called *tame.*

P. Cohn [10] proved that all automorphisms of free Lie algebras of a finite rank over a field are tame. J. Lewin [11] extended this result to free algebras in Nielsen–Schreier varieties. It is well known that the variety of all nonassociative algebras [12] over a field is Nielsen–Schreier (see also [13]). As a consequence, every automorphism of $K\{x_1, x_2\}$ is tame.

The degree of $\varphi = (f_1, f_2)$ is defined by

$$\deg(\varphi) = \deg(f_1) + \deg(f_2).$$

An automorphism φ is called *elementary reducible* if there exists an elementary automorphism λ such that $\deg(\varphi \circ \lambda) < \deg(\varphi)$.

Every nonzero element $g \in K\{x_1, x_2\}$ can be uniquely represented as

$$g = g_0 + g_1 + \cdots + g_{m-1} + g_m,$$

where g_i is homogenous element of degree i, $g_m \neq 0$. Let us call g_m *the highest homogeneous part* of g and denote it by $\overline{g}$.

Let $\mathrm{Af}K\{x_1, x_2\}$ be the affine automorphism group of $K\{x_1, x_2\}$, i.e., the group of automorphisms of the form

$$(a_1x_1 + b_1x_2 + c_1, a_2x_1 + b_2x_2 + c_2),$$

where $a_i, b_i, c_i \in K$, $a_1b_2 \neq a_2b_1$, let $\mathrm{Tr}K\{x_1, x_2\}$ be the triangular automorphism group of $K\{x_1, x_2\}$, i.e., the group of automorphisms of the form

$$(ax_1 + f(x_2), bx_2 + c),$$

where $0 \neq a, b \in K$, $c \in K$, $f(x_2) \in K\{x_2\}$, and let $C = \mathrm{Af}K\{x_1, x_2\} \cap \mathrm{Tr}K\{x_1, x_2\}$.

The notation $h_i(x_2)$ means that $h_i(x_2) \in K\{x_2\}$ is a homogeneous element of degree i with respect to the degree function deg in one variable x_2. It is clear that $h_0(x_2) \in K$.

A. Alimbaev, A. Naurazbekova, and D. Kozybaev [14] showed that the system of elements

$$A_0 = \{\mathrm{id} = (x_1, x_2), \gamma = (x_2, x_1 + ax_2) | a \in K\}$$

is a left coset representative system for $\mathrm{Af}K\{x_1, x_2\}$ modulo C, and the system of elements

$$B_0 = \{\delta = (x_1 + q(x_2), x_2) | q(x_2) = h_2(x_2) + \ldots + h_n(x_2)\}$$

is a left coset representative system for $\mathrm{Tr}K\{x_1, x_2\}$ modulo C. They also proved the following two lemmas.

Lemma 4. [14] Any automorphism φ of $K\{x_1, x_2\}$ can be representation as

$$\varphi = \gamma_1 \circ \delta_1 \circ \gamma_2 \circ \delta_2 \circ \ldots \circ \gamma_k \circ \delta_k \circ \lambda,$$

where $\gamma_i \in A_0$, $\gamma_2, \ldots, \gamma_k \neq \mathrm{id}$, $\delta_i \in B_0$, $\delta_1, \ldots, \delta_k \neq \mathrm{id}$, and $\lambda \in \mathrm{Af}K\{x_1, x_2\}$. Moreover, this representation of φ is unique.

Lemma 5. [14] Let

$$\varphi_k = \delta_1 \circ \gamma_2 \circ \delta_2 \circ \ldots \circ \gamma_k \circ \delta_k,$$

where $\mathrm{id} \neq \gamma_i \in A_0$, $\mathrm{id} \neq \delta_i \in B_0$ for all i. If $\delta_i = (x_1 + q_i(x_2), x_2)$ and $\deg(q_i(x_2)) = n_i$ for all $1 \leq i \leq k$ then

$$\deg(\varphi_k(x_1)) = n_1 n_2 \ldots n_{k-1} n_k,$$

$$deg(\varphi_k(x_2)) = \begin{cases} n_1 n_2 \ldots n_{k-1}, & \text{if } k > 1, \\ 1, & \text{if } k = 1. \end{cases}$$

Proposition 1. Nonaffine automorphisms of a free nonassociative algebra $K\{x_1, x_2\}$ of rank 2 over an arbitrary field K are elementary reducible.

Proof. Let φ be a nonaffine automorphism of $K\{x_1, x_2\}$. It is not difficult to see that if $\gamma^{-1} \circ \varphi$, where $\gamma \in A_0$, is elementary reducible then φ is also elementary reducible. By Lemma 4, φ can be uniquely represented as the product

$$\varphi = \gamma_1 \circ \delta_1 \circ \gamma_2 \circ \delta_2 \circ \cdots \circ \gamma_k \circ \delta_k \circ \lambda,$$

where $\gamma_i \in A_0$, $\gamma_2, \ldots, \gamma_k \neq \mathrm{id}$, $\tau_i \in B_0$, $\delta_1, \ldots, \delta_k \neq \mathrm{id}$, and $\lambda \in \mathrm{Af}K\{x_1, x_2\}$.

Let $\lambda = \mathrm{id}$. Then as in Lemma 5

$$\gamma_1^{-1} \circ \varphi = \varphi_k = \delta_1 \circ \gamma_2 \circ \delta_2 \circ \cdots \circ \gamma_k \circ \delta_k = \phi_{k-1} \circ \gamma_k \circ \delta_k.$$

Put $\varphi_{k-1} = (u_1, u_2)$. By Lemma 5, $\deg(u_1) > \deg(u_2)$. Since $\gamma_k = (x_2, x_1 + a_k x_2)$, $a_k \in K$, and $\delta_k = (x_1 + q_k(x_2), x_2)$ it follows that

$$\varphi_{k-1} \circ \gamma_k = (u_2, u_1 + a_k u_2)$$

and

$$\varphi_k = \varphi_{k-1} \circ \gamma_k \circ \delta_k = (u_2 + q_k(u_1 + a_k u_2), u_1 + a_k u_2) = (w_1, w_2).$$

By Lemma 5,

$$\deg w_1 > \deg w_2 = \deg(u_1 + a_k u_2) > \deg(u_2). \tag{10}$$

Consequently,

$$\deg(\varphi_{k-1} \circ \gamma_k) < \deg \varphi_k.$$

Since δ_k is the elementary automorphism, it follows that the automorphism $\gamma_1^{-1} \circ \varphi = \varphi_k$ is elementary reducible. This means that φ is also elementary reducible.

Let now

$$\lambda = (a_1 x_1 + b_1 x_2 + c_1, a_2 x_1 + b_2 x_2 + c_2) \neq \mathrm{id},$$

where $a_1 b_2 - a_2 b_1 \in K^*$. We have

$$\gamma_1^{-1} \circ \varphi = \varphi_k \circ \lambda = (a_1 w_1 + b_1 w_2 + c_1, a_2 w_1 + b_2 w_2 + c_2) = (f_1, f_2).$$

Let a_1, a_2 be non-zero. Consider the automorphism

$$\gamma_1^{-1} \circ \varphi \circ (x_1 - \frac{a_1}{a_2} x_2, x_2) = (f_1, f_2) \circ (x_1 - \frac{a_1}{a_2} x_2, x_2) = (f_1 - \frac{a_1}{a_2} f_2, f_2).$$

By Lemma 5, $\deg(f_1 - \frac{a_1}{a_2} f_2) < \deg f_1$. Consequently, the automorphism φ is elementary reducible.

Let now one of two coefficients a_1, a_2 be zero. Without loss of generality, we may assume that $a_1 = 0$ and $a_2 \neq 0$. Then $b_1 \in K^*$.

Since $K\{w_2\} = K\{b_1 w_2 + c_1\}$, it follows that there exists $v_k(y) \in K\{y\}$ such that

$$v_k(b_1 w_2 + c_1) = q_k(w_2).$$

Consider the automorphism

$$\psi = \gamma_1^{-1} \circ \varphi \circ (x_1, a_2^{-1} x_2 - v_k(x_1)) = (f_1, a_2^{-1} f_2 - v_k(f_1))$$
$$= (b_1 w_2 + c_1, a_2^{-1}(a_2 w_1 + b_2 w_2 + c_2) - v_k(b_1 w_2 + c_1)),$$

where

$$a_2^{-1}(a_2 w_1 + b_2 w_2 + c_2) - v_k(b_1 w_2 + c_1) = w_1 + a_2^{-1}(b_2 w_2 + c_2) - q_k(w_2)$$
$$= u_2 + q_k(w_2) + a_2^{-1}(b_2 w_2 + c_2) - q_k(w_2) = u_2 + a_2^{-1}(b_2 w_2 + c_2).$$

By(10), $\deg \psi < \deg \gamma_1^{-1} \circ \phi$. Since $(x_1, a_2^{-1} x_2 - v_k(x_1))$ is the elementary automorphism, it follows that the automorphism φ is elementary reducible. □

Proposition 2. If $\varphi = (f_1, f_2)$ is an automorphism of a free nonassociative algebra $K\{x_1, x_2\}$ in two variables x_1, x_2 over a field K and $\deg(\varphi) \geq 3$, then $\overline{f_1}, \overline{f_2}$ are homogeneous elements of a free nonassociative algebra $K\{ax_1 + bx_2\}$ in one variable $ax_1 + bx_2$, $a, b \in K$, $(a, b) \neq (0, 0)$, and $\deg(f_1) \mid \deg(f_2)$ or $\deg(f_2) \mid \deg(f_1)$.

Proof. Let $\varphi = (f_1, f_2)$ be an automorphism of $K\{x_1, x_2\}$ with $\deg(\varphi) \geq 3$. Without loss of generality, we can assume that $\deg(f_1) \leq \deg(f_2)$. Establish the proposition's statement by induction on $\deg(\varphi) = \deg(f_1) + \deg(f_2)$.

By Proposition 1, φ is elementary reducible. Therefore, there exists an elementary automorphism $\epsilon = (x_1, cx_2 - g(x_1))$, where $c \neq 0$, $g(x_1) \in K\{x_1\}$, such that $\deg(\varphi \circ \epsilon) < \deg(\varphi)$. Since $\varphi \circ \epsilon = (f_1, cf_2 - g(f_1))$, it follows that $\deg(f_2) = \deg(g(f_1))$. By Corollary 2,

$$\deg(g(f_1)) = \deg(g) \cdot \deg(f_1).$$

It means that $\deg(f_1) \mid \deg(f_2)$. If $\deg(\varphi \circ \epsilon) \geq 3$ then, by the induction proposition, $\overline{f_1}$ is homogeneous element of $K\{ax_1 + bx_2\}$. Notice that this is true even if $\deg(\varphi \circ \epsilon) = 2$. Hence, by Corollary 2, $c\overline{f_2} = \overline{g(f_1)} = \overline{g}(\overline{f_1})$. Consequently, $\overline{f_2}$ is the homogeneous element of $K\{ax_1 + bx_2\}$. □

3 Diagonal braidings on $K\{x_1,x_2\}$

Let V be a linear space over a field K with a linear basis x_1,x_2 and with a diagonal braiding

$$\beta=(\beta_{11},\beta_{12},\beta_{21},\beta_{22})$$

defined by (2). Denoted by

$$\overline{\beta}=(\beta_{22},\beta_{21},\beta_{12},\beta_{11})$$

the diagonal braiding obtained from β by exchanging the variables x_1 and x_2.

Proposition 3. Let $(K\{x_1,x_2\},\beta)$ and $(K\{x_1,x_2\},\gamma)$ be free braided nonassociative algebras in two variables x_1,x_2 over a field K equipped with diagonal braidings β and γ, respectively. Then $(K\{x_1,x_2\},\beta)$ is isomorphic to $(K\{x_1,x_2\},\gamma)$ if and only if $\gamma=\beta$ or $\gamma=\overline{\beta}$.

Proof. Let $\varphi:(K\{x_1,\,x_2\},\beta)\to(K\{x_1,\,x_2\},\gamma)$ is an isomorphism and $\varphi(x_1)=g_1$, $\varphi(x_2)=g_2$. Denote by $Lin(g)$ the linear part of g. Let $Lin(g_1)=a_1x_1+b_1x_2$ and $Lin(g_2)=a_2x_1+b_2x_2$. Since φ is an isomorphism, it follows that

$$a_1b_2-a_2b_1\neq 0 \tag{11}$$

and

$$((x_i\otimes x_j)\beta)(\varphi\otimes\varphi)=((x_i\otimes x_j)(\varphi\otimes\varphi))\gamma,\ \text{where } 1\leq i,j\leq 2.$$

Denote by $Qu(g)$ the quadratic part of g. Let $\beta=(\beta_{11},\beta_{12},\beta_{21},\beta_{22})$ and $\gamma=(\gamma_{11},\gamma_{12},\gamma_{21},\gamma_{22})$. Then

$$Qu(\beta_{ij}(\varphi(x_j)\otimes\varphi(x_i)))=Qu\left((\varphi(x_i)\otimes\varphi(x_j))\gamma\right),\ 1\leq i,j\leq 2.$$

It follows that

$$\begin{aligned}&\beta_{ij}(a_ja_ix_1\otimes x_1+b_ja_ix_2\otimes x_1+a_jb_ix_1\otimes x_2+b_jb_ix_2\otimes x_2)\\ &=\gamma_{11}a_ia_jx_1\otimes x_1+\gamma_{21}b_ia_jx_1\otimes x_2+\gamma_{12}a_ib_jx_2\otimes x_1+\gamma_{22}b_ib_jx_2\otimes x_2\end{aligned} \tag{12}$$

since, by (9), for any $w\in X^*$ and any $c\in K$

$$((w\otimes c)\beta)(\varphi\otimes\varphi)=(c\otimes w)(\varphi\otimes\varphi)=(c\otimes\varphi(w))=(\varphi(w)\otimes c)\gamma=((w\otimes c)(\varphi\otimes\varphi))\gamma.$$

Comparing the coefficients of the terms $x_i\otimes x_j$ in (12), we obtain

$$(\gamma_{11}-\beta_{ij})a_ia_j=(\gamma_{21}-\beta_{ij})a_jb_i=(\gamma_{12}-\beta_{ij})a_ib_j=(\gamma_{22}-\beta_{ij})b_ib_j=0,\ 1\leq i,j\leq 2. \tag{13}$$

By (11), $a_1b_2\neq 0$ or $a_2b_1\neq 0$. If $a_1b_2\neq 0$ then (13) implies that

$$(\gamma_{11},\gamma_{12},\gamma_{21},\gamma_{22})=(\beta_{11},\beta_{12},\beta_{21},\beta_{22}).$$

If $a_2b_1\neq 0$ then (13) implies that

$$(\gamma_{11},\gamma_{12},\gamma_{21},\gamma_{22})=(\beta_{22},\beta_{21},\beta_{12},\beta_{11}).$$

Hence, if $(K\{x_1,x_2\},\beta)$ is isomorphic to $(K\{x_1,x_2\},\gamma)$ then $\gamma=\beta$ or $\gamma=\overline{\beta}$.

Let φ be an automorphism of $K\{x_1,\,x_2\}$ such that $\varphi(x_1)=x_2$ and $\varphi(x_2)=x_1$. Let $u,v\in X^*$, $\mathrm{mdeg}(u)=(k_1,k_2)$ and $\mathrm{mdeg}(v)=(l_1,l_2)$. Then $\mathrm{mdeg}(\varphi(u))=(k_2,k_1)$ and $\mathrm{mdeg}(\varphi(v))=(l_2,l_1)$. By Lemma 1, we have

$$((u\otimes v)\beta)(\varphi\otimes\varphi)=\beta_{11}^{k_1l_1}\beta_{12}^{k_1l_2}\beta_{21}^{k_2l_1}\beta_{22}^{k_2l_2}(\varphi(v)\otimes\varphi(u))$$

and

$$((u\otimes v)(\varphi\otimes\varphi))\overline{\beta}=\beta_{11}^{k_1l_1}\beta_{12}^{k_1l_2}\beta_{21}^{k_2l_1}\beta_{22}^{k_2l_2}(\varphi(v)\otimes\varphi(u)).$$

Hence,

$$((u\otimes v)\beta)(\varphi\otimes\varphi)=((u\otimes v)(\varphi\otimes\varphi))\overline{\beta},$$

and $(K\{x_1,x_2\},\beta)$ is isomorphic to $(K\{x_1,x_2\},\overline{\beta})$. □

If $n=2$ then, by (3), the braiding β is involutive if and only if

$$\beta_{11}=\pm 1,\ \beta_{22}=\pm 1,\ \beta_{12}\beta_{21}=1.$$

Note that if β is involutive then β' is also involutive.

4 Automorphisms of $(K\{x_1,\, x_2\},\beta)$

Introduce the following notations:

(1) $G_1=\{\varphi\in \mathrm{Aut}K\{x_1,x_2\}\mid \varphi=(a_1x_1,b_2x_2)\,\mathrm{or}\,\varphi=(b_1x_2,\ a_2x_1),\ a_1,b_2,a_2,b_1\in K^*\}$;

(2) $G_2=\{\varphi\in \mathrm{Aut}K\{x_1,x_2\}\mid \varphi=(a_1x_1+g(x_2^2),b_2x_2),\ a_1,\, b_2\in K^*, g(x)\in K\,\{x\}\}$;

(3) $G_{tor}=\{\varphi\in \mathrm{Aut}K\{x_1,x_2\}\mid \varphi=(a_1x_1,b_2x_2),\ a_1,\ b_2\in K^*\}$ is the group of all *toric* automorphisms of $K\{x_1,x_2\}$;

(4) $\mathbb{Z}_2$ is the subgroup of $\mathrm{Aut}K\{x_1,x_2\}$ generated by (x_2,x_1).

Note that, by (6), if $\varphi\in \mathrm{Aut}K\{x_1,x_2\}$ then $\varphi\in \mathrm{Aut}(K\{x_1,x_2\},\beta)$ if and only if

$$\beta(\varphi\otimes\varphi)=(\varphi\otimes\varphi)\beta. \tag{14}$$

The main result of this section is the following theorem.

Theorem 1. Let $(K\{x_1,x_2\},\beta)$ be a free braided nonassociative algebra in two generators x_1, x_2 over a field K of arbitrary characteristic $\neq 2$ equipped with an involutive diagonal braiding $\beta=(\beta_{11},\beta_{12},\beta_{21},\beta_{22})$. Then

(1) $\mathrm{Aut}(K\{x_1,x_2\},\beta)=\mathrm{Aut}K\{x_1,x_2\}$ if $\beta_{ij}=1$ for all i,j;

(2) $\mathrm{Aut}(K\{x_1,x_2\},\beta)\cong(K^*\times K^*)\rtimes\mathbb{Z}_2$ if $\beta_{11}=\beta_{22}$, $\beta_{12}=\beta_{21}$, and $\beta_{11}\beta_{12}=-1$;

(3) $\mathrm{Aut}(K\{x_1,x_2\},\beta)=G_{\mathrm{odd}}$ if $\beta_{ij}=-1$ for all i,j;

(4) $\mathrm{Aut}(K\{x_1,x_2\},\beta)\cong G_2$ if $\beta_{12}=1$ and $\beta_{11}\beta_{22}=-1$;

(5) $\mathrm{Aut}(K\{x_1,x_2\},\beta)\cong K^*\times K^*$ if $\beta_{12}\neq\pm1$ or $\beta_{12}=-1,\beta_{11}\beta_{22}=-1$.

Following Lemmas 6, 7, 8, 9, 10, and 11 immediately imply the statement of this theorem.

Lemma 6. If $\beta_{ij}=1$ for all i,j then $\mathrm{Aut}(K\{x_1,x_2\},\beta)=\mathrm{Aut}K\{x_1,x_2\}$.

Proof. Let $\varphi\in \mathrm{Aut}K\{x_1,x_2\}$ and let u, $v\in X^*$. By Lemma 1, we have

$$((u\otimes v)\beta)(\varphi\otimes\varphi)=(v\otimes u)(\varphi\otimes\varphi)=\varphi(v)\otimes\varphi(u)=(\varphi(u)\otimes\varphi(v))\beta=(u\otimes v)(\varphi\otimes\varphi)\beta.$$

By (14), $\varphi\in \mathrm{Aut}(K\{x_1,x_2\},\beta)$. Consequently, $\mathrm{Aut}(K\{x_1,x_2\},\beta)=\mathrm{Aut}K\{x_1,x_2\}$. □

Lemma 7. If $\beta_{11}=\beta_{22}$, $\beta_{12}=\beta_{21}$, and $\beta_{11}\beta_{12}=-1$ then $\mathrm{Aut}(K\{x_1,x_2\},\beta)=G_1\cong(K^*\times K^*)\rtimes\mathbb{Z}_2$.

Proof. By Proposition 3, $\beta=(-1,1,1,-1)$ or $\beta=(1,-1,-1,1)$. Let $\varphi=(f_1,f_2)$ is an automorphism of $(K\{x_1,x_2\},\beta)$ and $\deg(\varphi)\geq 3$. Since $\mathrm{Aut}(K\{x_1,x_2\},\beta)\subseteq \mathrm{Aut}K\{x_1,x_2\}$, it follows that, by Proposition 2,

$$\overline{f_1}=h_1(ax_1+bx_2),\overline{f_2}=h_2(ax_1+bx_2), \tag{15}$$

where h_1,h_2 are homogeneous elements of $K\{y\}$, $\deg(h_1)=m_1$, $\deg(h_2)=m_2$, $m_1\mid m_2$ or $m_2\mid m_1$, $a,b\in K$, and $(a,b)\neq(0,0)$. By (14), we get

$$\overline{((x_i\otimes x_j)\beta)(\varphi\otimes\varphi)}=\overline{((x_i\otimes x_j)(\varphi\otimes\varphi))\beta},\quad 1\leq i,j\leq 2.$$

Hence,

$$\overline{\beta_{ij}\varphi(x_j)\otimes\varphi(x_i)}=\overline{(\varphi(x_i)\otimes\varphi(x_j))\beta},\quad 1\leq i,j\leq 2.$$

Using (15), we get

$$\beta_{ij}h_j(ax_1+bx_2)\otimes h_i(ax_1+bx_2)=\left(h_i(ax_1+bx_2)\otimes h_j(ax_1+bx_2)\right)\beta,\quad 1\leq i,j\leq 2. \tag{16}$$

We can write $h_k(ax_1+bx_2)$ as

$$h_k(ax_1+bx_2)=a^{m_k}h_k(x_1)+b^{m_k}h_k(x_2)+w_k,\ 1\le k\le 2,$$

where each term of w_k contains both x_1 and x_2.

Using Lemma 1 and comparing coefficients of the terms $h_j(x_1)\otimes h_i(x_1)$, $h_j(x_2)\otimes h_i(x_1)$, $h_j(x_1)\otimes h_i(x_2)$, $h_j(x_2)\otimes h_i(x_2)$ in (16), we obtain

$$\begin{aligned}(\beta_{ij}-\beta_{11}^{m_im_j})a^{m_i+m_j}=(\beta_{ij}-\beta_{12}^{m_im_j})a^{m_i}b^{m_j}=(\beta_{ij}-\beta_{21}^{m_im_j})a^{m_j}b^{m_i}\\ =(\beta_{ij}-\beta_{22}^{m_im_j})b^{m_i+m_j}=0,\quad 1\le i,j\le 2.\end{aligned}\tag{17}$$

If $\beta=(-1,1,1,-1)$ then, by (17),

$$(\beta_{ij}-(-1)^{m_im_j})a^{m_i+m_j}=(\beta_{ij}-(-1)^{m_im_j})b^{m_i+m_j}=0,\quad 1\le i,j\le 2.$$

This implies that

$$(-1-(-1)^{m_1^2})a^{2m_1}=(-1-(-1)^{m_1^2})b^{2m_1}=(1-(-1)^{m_1m_2})a^{m_1+m_2}$$

$$=(1-(-1)^{m_1m_2})b^{m_1+m_2}=(-1-(-1)^{m_2^2})a^{2m_2}=(-1-(-1)^{m_2^2})b^{2m_2}=0.$$

It easily follows from this that $a=b=0$ over a field of characteristic $\neq 2$.

If $\beta=(1,-1,-1,1)$ then, by (17),

$$(\beta_{ij}-1)a^{m_i+m_j}=(\beta_{ij}-1)b^{m_i+m_j}=0,\quad 1\le i,j\le 2.$$

This implies that

$$-2a^{m_1+m_2}=-2b^{m_1+m_2}=0.$$

Hence, $a=b=0$ over a field of characteristic $\neq 2$.

Consequently, if $\beta=(-1,1,1,-1)$ or $\beta=(1,-1,-1,1)$ then $(K\{x_1,x_2\},\beta)$ has only automorphisms of degree 2. Therefore,

$$\varphi=(a_1x_1+b_1x_2+c_1,\ a_2x_1+b_2x_2+c_2),\ a_i,\ b_i,\ c_i\in K.$$

Using (14), we get

$$((x_i\otimes x_j)\beta)\varphi\otimes\varphi=((x_i\otimes x_j)\varphi\otimes\varphi)\beta,\quad 1\le i,j\le 2.$$

Hence,

$$\beta_{ij}(a_jx_1+b_jx_2+c_j)\otimes(a_ix_1+b_ix_2+c_i)=((a_ix_1+b_ix_2+c_i)\otimes(a_jx_1+b_jx_2+c_j))\beta,\quad 1\le i,j\le 2.$$

By comparing the coefficients of the terms $x_i\otimes x_j$, x_i, $1\le i,j\le 2$, and the term 1 on both sides of the equality, we obtain the following relations:

$$(\beta_{ij}-\beta_{11})a_ia_j=(\beta_{ij}-\beta_{12})a_ib_j=(\beta_{ij}-\beta_{21})a_jb_i=(\beta_{ij}-\beta_{22})b_ib_j$$

$$=(\beta_{ij}-1)(a_ic_j+a_jc_i)=(\beta_{ij}-1)(b_ic_j+b_jc_i)=(\beta_{ij}-1)c_ic_j=0,\quad 1\le i,j\le 2.$$

Varying the values of $1\le i,j\le 2$, we get

$$\begin{aligned}(\beta_{11}-\beta_{12})a_1b_1=(\beta_{11}-\beta_{21})a_1b_1=(\beta_{11}-\beta_{22})b_1^2=(\beta_{11}-1)a_1c_1\\ =(\beta_{11}-1)b_1c_1=(\beta_{11}-1)c_1^2=0,\end{aligned}\tag{18}$$

$$\begin{aligned}(\beta_{12}-\beta_{11})a_1a_2 &= (\beta_{12}-\beta_{21})a_2b_1 = (\beta_{12}-\beta_{22})b_1b_2 \\ = (\beta_{12}-1)(a_1c_2+a_2c_1) &= (\beta_{12}-1)(b_1c_2+b_2c_1) = (\beta_{12}-1)c_1c_2 = 0,\end{aligned} \tag{19}$$

$$\begin{aligned}(\beta_{21}-\beta_{11})a_1a_2 &= (\beta_{21}-\beta_{12})a_2b_1 = (\beta_{21}-\beta_{22})b_1b_2 \\ = (\beta_{21}-1)(a_2c_1+a_1c_2) &= (\beta_{21}-1)(b_2c_1+b_1c_2) = (\beta_{21}-1)c_2c_1 = 0,\end{aligned} \tag{20}$$

$$\begin{aligned}(\beta_{22}-\beta_{11})a_2^2 = (\beta_{22}-\beta_{12})a_2b_2 = (\beta_{22}-\beta_{21})a_2b_2 = (\beta_{22}-1)a_2c_2 \\ = (\beta_{22}-1)b_2c_2 = (\beta_{22}-1)c_2^2 = 0.\end{aligned} \tag{21}$$

If $\beta=(-1,1,1,-1)$ or $\beta=(1,-1,-1,1)$, then it follows from (18), (19), (20), and (21) that

$$a_1b_1 = a_1a_2 = b_1b_2 = a_2b_2 = c_1^2 = c_2^2 = 0$$

or

$$a_1b_1 = a_1a_2 = b_1b_2 = a_2b_2 = a_1c_2+a_2c_1 = b_1c_2+b_2c_1 = c_1c_2 = 0,$$

respectively. Using (11), this implies

$$a_1\neq 0,\ b_2\neq 0,\ b_1=a_2=c_1=c_2=0$$

or

$$b_1\neq 0,\ a_2\neq 0,\ a_1=b_2=c_1=c_2=0.$$

So we have

$$\varphi=(a_1x_1,\ b_2x_2) \ \text{ or } \ \varphi=(b_1x_2,\ a_2x_1).$$

Using Lemma 1, it is easy to see that $\varphi\in \mathrm{Aut}(K\{x_1,x_2\},\beta)$. Consequently, $\mathrm{Aut}(K\{x_1,x_2\},\beta)=G_1$. □

Lemma 8. If $\beta_{ij}=-1$ for all i,j then $\mathrm{Aut}(K\{x_1,x_2\},\beta)=G_{\mathrm{odd}}$.

Proof. Let $u,v\in X^*$ with $\mathrm{mdeg}(u)=(m_1,m_2)$ and $\mathrm{mdeg}(v)=(t_1,t_2)$. If $\beta_{ij}=-1$ then, by Lemma 1,

$$(u\otimes v)\beta=(-1)^{(m_1+m_2)(t_1+t_2)}(v\otimes u).$$

It is clear that $(-1)^{(m_1+m_2)(t_1+t_2)}=-1$ is equivalent to both m_1+m_2 and t_1+t_2 being odd, and hence we conclude

$$(f\otimes g)\beta=(-1)^{ij}(g\otimes f), \tag{22}$$

where $f\in C_i$ and $g\in C_j$ with respect to the grading (4).

Let $\varphi\in G_{\mathrm{odd}}$. Therefore $\varphi(C_i)\subseteq C_i$. Using (22), we obtain

$$(f\otimes g)\beta(\varphi\otimes\varphi)=(-1)^{ij}(g\otimes f)(\varphi\otimes\varphi)$$

$$=(-1)^{ij}(\varphi(g)\otimes\varphi(f))=(\varphi(f)\otimes\varphi(g))\beta=(f\otimes g)(\varphi\otimes\varphi)\beta.$$

By (14), $\varphi\in \mathrm{Aut}(K\{x_1,x_2\},\beta)$ and $G_{\mathrm{odd}}\subseteq \mathrm{Aut}(K\{x_1,x_2\},\beta)$.

Let $\varphi=(f_1,\ f_2)\in \mathrm{Aut}(K\{x_1,x_2\},\beta)$, $\deg(f_1)=m_1$ and $\deg(f_2)=m_2$. We prove that $\varphi\in G_{\mathrm{odd}}$ by induction on $\deg(\varphi)=m_1+m_2$. Let $\deg(\varphi)=2$, i.e.,

$$\varphi=(a_1x_1+b_1x_2+c_1, a_2x_1+b_2x_2+c_2), \quad a_i,b_i,c_i\in K.$$

It follows from (18), (19), (20), and (21) that $c_1 = c_2 = 0$. So we have

$$\varphi = (a_1x_1 + b_1x_2, a_2x_1 + b_2x_2) \in G_{\text{odd}}.$$

Assume that $\deg(\varphi) = m_1 + m_2 \geq 3$. Then (17) implies that

$$\begin{aligned}(-1-(-1)^{m_im_j})a^{m_i+m_j} = (-1-(-1)^{m_im_j})a^{m_i}b^{m_j} = (-1-(-1)^{m_im_j})a^{m_j}b^{m_i} \\ = (-1-(-1)^{m_im_j})b^{m_i+m_j} = 0, \quad i,\, j \in \{1,\, 2\}.\end{aligned} \tag{23}$$

Assume that m_1m_2 is even. It follows from (23) that $a = b = 0$ over a field of characteristic $\neq 2$. Consequently, m_1m_2 is odd. This means that m_1 and m_2 are odd. Without loss of generality, we can assume that $m_1 \leq m_2$. By Proposition 2, $m_1|m_2$. Then there exists the odd automorphism $\lambda = (x_1, x_2 + dx_1^{m_2/m_1})$, where $d \in K^*$, such that $\deg(\varphi \circ \lambda) < \deg(\varphi)$. Since $\lambda \in G_{\text{odd}} \subseteq \text{Aut}(K\{x_1, x_2\}, \beta)$ it follows that $\varphi\circ\lambda \in \text{Aut}(K\{x_1, x_2\}, \beta)$. By the induction proposition, $\varphi\circ\lambda \in G_{\text{odd}}$. Consequently, $\varphi \in G_{\text{odd}}$. □

Lemma 9. If $\beta_{12} = 1$ and $\beta_{11}\beta_{22} = -1$ then $\text{Aut}(K\{x_1, x_2\}, \beta) \cong G_2$.

Proof. By Proposition 3, $\beta = (1, 1, 1, -1)$. Consider the grading

$$K\{x_1, x_2\} = D_0 \oplus D_1$$

of $K\{x_1, x_2\}$, where D_0 and D_1 are linear spans of all monomials of even degree and all monomials of odd degree in variable x_2, respectively.

Let $u, v \in X^*$, $\text{mdeg}(u) = (s, s')$ and $\text{mdeg}(v) = (t, t')$. By Lemma 1,

$$(u \otimes v)\beta = (-1)^{s't'}(v \otimes u).$$

$(-1)^{s't'} = -1$ is equivalent to both s' and t' being odd. Therefore, for any homogeneous elements $f \in D_i$ and $g \in D_j$,

$$(f \otimes g)\beta = (-1)^{ij}(g \otimes f). \tag{24}$$

Let $\varphi \in G_2$. Then $\varphi(D_i) \subseteq D_i$. Using (24), we obtain

$$(f \otimes g)\beta(\varphi \otimes \varphi) = (-1)^{ij}(g \otimes f)(\varphi \otimes \varphi)$$

$$= (-1)^{ij}(\varphi(g) \otimes \varphi(f)) = (\varphi(f) \otimes \varphi(g))\beta = (f \otimes g)(\varphi \otimes \varphi)\beta.$$

Consequently, $\varphi \in \text{Aut}(K\{x_1, x_2\}, \beta)$ and $G_2 \subseteq \text{Aut}(K\{x_1, x_2\}, \beta)$.

Let $\varphi = (f_1, f_2) \in \text{Aut}(K\{x_1, x_2\}, \beta)$ with $\deg(f_1) = m_1$ and $\deg(f_2) = m_2$. We prove that $\varphi \in G_2$ by induction on $\deg(\varphi) = m_1 + m_2$. Let

$$\varphi = (a_1x_1 + b_1x_2 + c_1, a_2x_1 + b_2x_2 + c_2), \quad a_i, b_i, c_i \in K.$$

It follows from (18), (19), (20), and (21) that

$$b_1^2 = b_1b_2 = a_2^2 = a_2b_2 = a_2c_2 = b_2c_2 = c_2^2 = 0.$$

Using this and (11), we get

$$a_1 \neq 0,\ b_2 \neq 0,\ b_1 = a_2 = c_2 = 0.$$

So

$$\varphi = (a_1x_1 + c_1,\ b_2x_2) \in G_2.$$

Assume that $\deg(\varphi) = m_1 + m_2 \geq 3$. Then (17) implies that

$$(-1-1^{m_2^2})a^{2m_2} = (1-(-1)^{m_1^2})b^{2m_1} = (1-(-1)^{m_1m_2})b^{m_1+m_2} = (-1-(-1)^{m_2^2})b^{2m_2} = 0. \qquad (25)$$

It follows from this $a = 0$ over a field of characteristic $\neq 2$. By (25), $b \neq 0$ in the case m_1 is even and m_2 is odd. Therefore, by Proposition 2, $m_2|m_1$ and m_1/m_2 is even. Then there exists the automorphism $\lambda = (x_1 + dx_2^{m_1/m_2}, x_2) \in G_2$ such that $\deg(\varphi \circ \lambda) < \deg(\varphi)$. Since $\lambda \in G_2 \subseteq (K\{x_1, x_2\}, \beta)$ it follows that $\varphi \circ \lambda \in (K\{x_1, x_2\}, \beta)$. By the induction proposition, $\varphi \circ \lambda \in G_2$. Consequently, $\varphi \in G_2$. □

Lemma 10. If $\beta_{12} = -1,\ \beta_{11}\beta_{22} = -1$, then $\mathrm{Aut}(K\{x_1, x_2\}, \beta) \cong G_{\mathrm{tor}} \cong K^* \times K^*$.

Proof. By Proposition 3, $\beta = (1, -1, -1, -1)$. Let $\varphi = (f_1,\ f_2) \in \mathrm{Aut}(K\{x_1, x_2\}, \beta)$, $\deg(f_1) = m_1$ and $\deg(f_2) = m_2$. If $\deg(\varphi) = m_1 + m_2 \geq 3$, then (17) implies that

$$(-1-1^{m_1m_2})a^{m_1+m_2} = (1-(-1)^{m_1^2})b^{2m_1}$$

$$= (-1-(-1)^{m_1m_2})b^{m_1+m_2} = (-1-(-1)^{m_2^2})b^{2m_2} = 0.$$

It follows from this that $a = b = 0$ over a field of characteristic $\neq 2$. Thus, the algebra $(K\{x_1, x_2\}, \beta)$ has only automorphisms of degree 2.

Let

$$\varphi = (a_1x_1 + b_1x_2 + c_1,\ a_2x_1 + b_2x_2 + c_2),\ a_i,\ b_i,\ c_i \in K.$$

It follows from (18), (19), (20), and (21) that

$$a_1b_1 = b_1^2 = a_1a_2 = a_1c_2 + a_2c_1 = b_1c_2 + b_2c_1 = c_1c_2 = a_2^2 = a_2c_2 = b_2c_2 = c_2^2 = 0.$$

By this and (11), we get

$$a_1 \neq 0, b_2 \neq 0,\ b_1 = a_2 = c_1 = c_2 = 0.$$

So

$$\varphi = (a_1x_1,\ b_2x_2) \in G_{\mathrm{tor}}.$$

Using Lemma 1, it is not difficult to show that $G_{\mathrm{tor}} \in \mathrm{Aut}(K\{x_1, x_2\}, \beta)$. Consequently,

$$\mathrm{Aut}(K\{x_1, x_2\}, \beta) = G_{\mathrm{tor}}$$

□

Lemma 11. If $\beta_{12} \neq \pm 1$ then $\mathrm{Aut}(K\{x_1, x_2\}, \beta) = G_{\mathrm{tor}} \cong K^* \times K^*$.

Proof. By (3), $\beta_{11} = \pm 1$, $\beta_{22} = \pm 1$, $\beta_{21} \neq \beta_{12}$, $\beta_{12} \neq \pm 1$. Let $\varphi = (f_1,\ f_2) \in \mathrm{Aut}(K\{x_1, x_2\}, \beta)$, $\deg(f_1) = m_1$ and $\deg(f_2) = m_2$. If $\deg(\varphi) = m_1 + m_2 \geq 3$, then it follows from (17) that $a = b = 0$ over a field of characteristic $\neq 2$. Thus, the algebra $(K\{x_1, x_2\}, \beta)$ has only automorphisms of degree 2.

Let

$$\varphi = (a_1x_1 + b_1x_2 + c_1,\ a_2x_1 + b_2x_2 + c_2),\ a_i,\ b_i,\ c_i \in K.$$

It follows from (18), (19), (20), and (21) that

$$a_1b_1 = a_1a_2 = a_2b_1 = b_1b_2 = a_2b_2 = a_2c_1 + a_1c_2 = b_2c_1 + b_1c_2 = c_1c_2 = 0.$$

Using this and (11), we get

$$a_1 \neq 0,\ b_2 \neq 0,\ b_1 = a_2 = c_1 = c_2 = 0.$$

So

$$\varphi = (a_1x_1,\ b_2x_2) \in G_{\mathrm{tor}}.$$

Using Lemma 1, it is easy to see that $G_{\mathrm{tor}} \in \mathrm{Aut}(K\{x_1, x_2\}, \beta)$. Consequently, $\mathrm{Aut}(K\{x_1, x_2\}, \beta) = G_{\mathrm{tor}}$. □

Conclusion

Using properties of automorphisms of a two generated free nonassociative algebra we describe the automorphism groups of two generated free braided nonassociative algebras with involutive diagonal braidings over a field of characteristic not equal to two. The obtained results can be used to study automorphisms of other free nonassociative braided algebras.

Acknowledgments

This work was supported by the grant of the Ministry of Science and Higher Education of the Republic of Kazakhstan (project AP23487886).

Author Contributions

All authors contributed equally to this work.

Conflict of Interest

The authors declare no conflict of interest.

References

1 Guo, L., & Li, Y. (2021). Braided dendriform and tridendriform algebras, and braided Hopf algebras of rooted trees. *Journal of Algebraic Combinatorics, 53*, 1147–1185. https://doi.org/10.1007/s10801-020-00957-0

2 Ion, B. (2023). Braided shuffle algebras and Lyndon words. *Bulletin of the Belgian Mathematical Society - Simon Stevin, 30*(5), 601–617. https://doi.org/10.36045/j.bbms.230206

3 Kharchenko, V. (2016). *Quantum Lie Theory: A Multilinear Approach.* Springer International Publishing. https://doi.org/10.1007/978-3-319-22704-7

4 Umirbaev, U., & Kharchenko, V. (2017). Free braided nonassociative Hopf algebras and Sabinin τ-algebras. *Journal of Algebra, 492*, 130–156. https://doi.org/10.1016/j.jalgebra.2017.08.019

5 Takeuchi, M. (2000). Survey of braided Hopf algebras. In *New Trends in Hopf Algebra Theory, Contemporary Mathematics, 267*, 301–324. Providence, RI: American Mathematical Society. https://doi.org/10.1090/conm/267/04277

6 Mutalip, R., Naurazbekova, A., & Umirbaev, U. (2022). Automorphisms of free braided associative algebras in two variables. *Communications in Algebra, 50*(2), 848–860. https://doi.org/10.1080/00927872.2021.1974027

7 Grewcoe, C.J., Jonke, L., Kodzoman, T., & Manolakos, G. (2022). From Hopf Algebra to Braided L_∞-Algebra. *Universe, 8*(4), 222. https://doi.org/10.3390/universe8040222

8 Bochniak, A., & Sitarz, A. (2019). Braided Hopf algebras from twisting. *Journal of Algebra and Its Applications, 18*(09), 1950178. https://doi.org/10.1142/S0219498819501780

9 Ciric, M.D., Giotopoulos, G., Radovanovic, V., & Szabo, R.J. (2021). Braided L_∞-algebras, braided field theory and noncommutative gravity. *Letters in Mathematical Physics, 111*, Article 148. https://doi.org/10.1007/s11005-021-01487-x

10 Cohn, P.M. (1964). Subalgebras of free associative algebras. *Proceedings of the London Mathematical Society, Third Series, 14*(4), 618–632. https://doi.org/10.1112/plms/s3-14.4.618

11 Lewin, J. (1968). On Schreier varieties of linear algebras. *Transactions of the American Mathematical Society, 132*(2), 553–562. https://doi.org/10.1090/S0002-9947-1968-0224663-5

12 Kurosh, A.G. (1947). Neassotsiativnye svobodnye algebry i svobodnye proizvedeniia algebr [Non-associative free algebras and free products of algebras]. *Matematicheskii Sbornik — Mathematical collection, 20*(62), 239–262 [in Russian].

13 Dotsenko, V., & Umirbaev, U. (2023). An Effective Criterion for Nielsen–Schreier Varieties. *International Mathematics Research Notices, 2023*(23), 20385–20432. https://doi.org/10.1093/imrn/rnad092

14 Alimbaev, A., Naurazbekova, A., & Kozybaev, D. (2019). Linearizatsiia avtomorfizmov i trianguliatsiia differentsirovanii svobodnykh algebr ranga 2 [Linearization of automorphisms and triangulation of derivations of free algebras of rank 2]. *Sibirskie elektronnye matematicheskie izvestiia — Siberian electronic mathematical reports, 16*, 1133–1146 [in Russian]. https://doi.org/10.33048/semi.2019.16.077

*Author Information**

Riza Mutalip — Master of Mathematics, Senior Lecturer, L.N. Gumilyov Eurasian National University, 2 Satbaev St., Astana, 010008, Kazakhstan; e-mail: *mutalipriza@yahoo.com*; https://orcid.org/ 0009-0009-9393-7794

Bibinur Amyrkanovna Duisengaliyeva — PhD, Senior Lecturer, L.N. Gumilyov Eurasian National University, 2 Satbaev St., Astana, 010008, Kazakhstan; e-mail: *bibinur959@gmail.com*; https://orcid.org/0000-0003-3087-0683

Altyngul Serikovna Naurazbekova (*corresponding author*) — PhD, Associate Professor, L.N. Gumilyov Eurasian National University, 2 Satbaev St., Astana, 010008, Kazakhstan; e-mail: *naurazbekova82@gmail.com*; https://orcid.org/0000-0002-8609-8904

*Authors' names are presented in the following order: first name, middle name (if any), last name.

Bulletin of the Karaganda University. Mathematics Series, No. 1(121), 2026, pp. 194–201

https://doi.org/10.31489/2026M1/194-201 *Research article*

On a solution of the periodic boundary value problem for a hyperbolic equation with a fractional derivative

N.T. Orumbayeva, B.B. Zhantassova*

Buketov Karaganda National Research University, Karaganda, Kazakhstan
(E-mail: orumbayevanurgul@gmail.com, zh.botagoz.b@gmail.com)

The article investigates a boundary value problem for a hyperbolic equation with the Riemann–Liouville fractional derivative, which is periodic in one variable. Such equations are widely used in modeling complex physical processes with memory effects, including viscoelasticity, anomalous diffusion, and thermoviscoelasticity phenomena, where classical integer-order models fail to adequately describe the hereditary properties of materials and transport processes. To solve this problem, an iterative algorithm is proposed based on domain decomposition and the reduction of the original problem to a system of integro-differential equations. A theorem on the existence and uniqueness of the solution is proved, and an estimate of the convergence rate of the method is obtained using matrix analysis and a strengthened Gronwall–Bellman inequality. It is established that the choice of the decomposition step plays a key role in ensuring the stability of the algorithm. The conducted analysis extends the class of problems for which efficient computational algorithms can be constructed and may serve as a foundation for studying more complex nonlinear cases and problems in irregular domains.

Keywords: hyperbolic equation, Riemann–Liouville fractional derivative, periodic boundary value problem, integro-differential equations, algorithm, Gronwall–Bellman inequality, variable coefficients, nonlinear terms.

2020 Mathematics Subject Classification: 34B05, 35A05, 35A20, 35A22.

Introduction

In modern mathematical physics, considerable attention is devoted to the study of differential equations with fractional derivatives, which are applied to modeling memory processes such as viscoelasticity, anomalous diffusion, and thermo-viscoelasticity [1–3]. Hyperbolic equations containing fractional derivatives are of particular interest [4–6], as they describe wave phenomena in media with hereditary properties. This work considers a periodic boundary value problem for a hyperbolic equation with the Riemann–Liouville fractional derivative, which arises in the study of processes with periodic boundary conditions. An iterative algorithm is proposed, based on decomposing the spatial domain and reducing the original problem to a system of integro-differential equations. The paper proves a theorem on the existence and uniqueness of the solution, as well as derives estimates of the convergence rate of the proposed method. The use of the parametrization method [7] enabled establishing sufficient convergence conditions for the algorithm. The results of this work extend the class of problems for which efficient computational algorithms can be constructed and can be applied to the numerical simulation of memory processes [8–10]. The conducted research broadens the range of problems for which effective computational algorithms can be developed [11, 12]. The obtained results are consistent with known approaches described in and complement them with new estimates [13, 14].

*Corresponding author. *E-mail: zh.botagoz.b@gmail.com*

This research was funded by the Science Committee of the Ministry of Science and Higher Education of the Republic of Kazakhstan (Grant No. AP23488729, 2024-2026).

Received: 22 April 2025; *Accepted:* 12 December 2025.

1 Problem statement

Consider a periodic boundary value problem for a hyperbolic equation with fractional derivative

$$\frac{\partial^2 w(t,x)}{\partial t \partial x} = A(t,x)\frac{\partial w(t,x)}{\partial t} + B(t,x)D_t^{\mu}w(t,x) + f(t,x), \quad (t,x) \in \Omega = [0,T]\times[0,X], \quad w \in R^n, \tag{1}$$

$$w(0,x) = 0, \qquad x \in [0,X], \tag{2}$$

$$w(t,0) = w(t,X), \qquad t \in [0,T], \tag{3}$$

where the $(n\times n)$ matrices $A(t,x), B(t,x)$, and the n-vector function $f(x,t)$ are continuous on Ω,

$$\|w\| = \max_{i=\overline{1,n}} |w_i|, \quad \|A(t,x)\| = \max_{i=\overline{1,n}} \sum_{j=1}^{n} |a_{ij}(t,x)| \le \alpha, \quad \|B(t,x)\| = \max_{i=\overline{1,n}} \sum_{j=1}^{n} |b_{ij}(t,x)| \le \beta,$$

$\alpha, \beta = \text{const}$, $0<\mu<1$, $D_t^{\mu}w(t,x)$ is the Riemann–Liouville fractional derivative of order μ, defined by the formula

$$D_t^{\mu}w(t,x) = \frac{1}{\Gamma(1-\mu)}\frac{d}{dt}\int_0^t \frac{w(\tau,x)}{(t-\tau)^{\mu}}d\tau, \quad t\in[0,T],$$

where $\Gamma(z)$ is gamma function: $\Gamma(z) = \int\limits_0^{\infty} x^{z-1}e^{-x}dx$.

Let $C(\Omega,\mathbb{R}^n)$ be the space of continuous functions $w:\Omega\to\mathbb{R}^n$ on Ω with the norm

$$\|w(t,x)\|_1 = \max_{(t,x)\in\Omega}\|w(t,x)\|.$$

The function $w(t,x)\in C(\Omega,\mathbb{R}^n)$, having partial derivatives $\frac{\partial w(t,x)}{\partial t}\in C(\Omega,\mathbb{R}^n)$, $\frac{\partial^2 w(t,x)}{\partial t\partial x}\in C(\Omega,\mathbb{R}^n)$, provided that $D_t^{\mu}w(t,x)$ exists in the Riemann–Liouville sense, is called a solution of the problem (1)–(3), if it satisfies the system (1) for all $(t,x)\in\Omega$, has zero value on the characteristic $t=0$, and has equal values on the characteristics $x=0$, $x=X$ for all $t\in[0,T]$.

To solve this problem, we introduce a new function $u(t,x) = \frac{\partial w(t,x)}{\partial t}$, then $w(t,x) = \int\limits_0^t u(\tau,x)d\tau$. Substituting $u(t,x)$ into equation (1) gives

$$\frac{\partial u(t,x)}{\partial x} = A(t,x)u(t,x) + B(t,x)D_t^{\mu}\left(\int_0^t u(\tau,x)d\tau\right) + f(t,x), \quad (t,x)\in\Omega=[0,T]\times[0,X],$$

with the boundary condition:

$$u(t,0) = u(t,X), \quad t\in[0,T].$$

Let us partition the interval $[0,X]$ into N parts with a step size $h = X/N$. We introduce the partition points: $x_r = (r-1)h$, $r = 1,2,\ldots,N$. We define the restrictions of the function $u(t,x)$ on each interval $u_r(t,x) = u(t,x)$, $x\in[(r-1)h, rh)$.

The original problem is equivalent to the problem

$$\frac{\partial u_r(t,x)}{\partial x} = A(t,x)u_r(t,x) + B(t,x)D_t^{\mu}\left(\int_0^t u_r(\tau,x)d\tau\right) + f(t,x), \tag{4}$$

$$u_1(t,0) = \lim_{x\to Nh-0} u_N(t,x), \quad t\in[0,T], \tag{5}$$

$$\lim_{x\to rh-0} u_r(t,x) = u_{r+1}(t,rh), \quad r = 1,2,...,N-1, \tag{6}$$

where $(t,x) \in \Omega_r = [0,T] \times [(r-1)h, rh)$, (6) is the matching condition of the solution on the internal partition lines,

$$D_t^{\mu}\left(\int\limits_0^t u_r(\tau,x)d\tau\right) = \frac{1}{\Gamma(1-\mu)}\frac{\partial}{\partial t}\int\limits_0^t \frac{\int\limits_0^{\tau} u_r(\tau_1,x)d\tau_1}{(t-\tau)^{\mu}}d\tau = \frac{1}{\Gamma(1-\mu)}\int\limits_0^t \frac{u_r(\tau,x)}{(t-\tau)^{\mu}}d\tau.$$

2 *Main results*

We introduce the substitution $v_r(t,x) = u_r(t,x) - \lambda_r(t)$, where $\lambda_r(t) = u_r(t,(r-1)h)$. Then $u_r(t,x) = v_r(t,x) + \lambda_r(t)$. Substituting into equations (4)–(6), we obtain

$$\frac{\partial v_r(t,x)}{\partial x} = A(t,x)v_r(t,x) + A(t,x)\lambda_r(t) +$$

$$+\frac{B(t,x)}{\Gamma(1-\mu)}\int\limits_0^t \frac{v_r(\tau,x)}{(t-\tau)^{\mu}}d\tau + \frac{B(t,x)}{\Gamma(1-\mu)}\int\limits_0^t \frac{\lambda_r(\tau)}{(t-\tau)^{\mu}}d\tau + f(t,x), \tag{7}$$

$$v_r(t,(r-1)h) = 0, \quad t \in [0,T], \quad r = \overline{1,N}, \tag{8}$$

$$\lambda_1(t) = \lambda_N(t) + \lim_{x\to X-0} v_N(t,x), \quad t \in [0,T], \tag{9}$$

$$\lambda_r(t) + \lim_{x\to rh-0} v_r(t,x) = \lambda_{r+1}(t), \quad r = 1,2,...,N-1. \tag{10}$$

The problem (7), (8) with fixed $\lambda_r(t)$ is a one-parameter family of Cauchy problems for systems of integro-differential equations, where $t \in [0,T]$, and is equivalent to the integral equation

$$v_r(t,x) = \int\limits_{(r-1)h}^{x} A(t,\xi)v_r(t,\xi)d\xi + \lambda_r(t)\int\limits_{(r-1)h}^{x} A(t,\xi)d\xi +$$

$$+\int\limits_{(r-1)h}^{x} \frac{B(t,\xi)}{\Gamma(1-\mu)}\int\limits_0^t \frac{v_r(\tau,\xi)}{(t-\tau)^{\mu}}d\tau d\xi + \frac{1}{\Gamma(1-\mu)}\int\limits_0^t \frac{\lambda_r(\tau)}{(t-\tau)^{\mu}}d\tau \int\limits_{(r-1)h}^{x} B(t,\xi)d\xi + \int\limits_{(r-1)h}^{x} f(t,\xi)d\xi. \tag{11}$$

Passing to the limit as $x \to rh-0$ in (11) and substituting in (9), (10) instead of $\lim\limits_{x\to rh-0} v_r(t,x)$, $r = \overline{1,N}$, the corresponding right-hand sides for the unknown functions $\lambda_r(t)$, $r = \overline{1,N}$ we obtain a system of equations

$$h\lambda_1(t) = h\lambda_N(t) + h\int\limits_{(N-1)h}^{Nh} A(t,\xi)v_N(t,\xi)d\xi + h\lambda_N(t)\int\limits_{(N-1)h}^{Nh} A(t,\xi)d\xi +$$

$$+h\int\limits_{(N-1)h}^{Nh} \frac{B(t,\xi)}{\Gamma(1-\mu)}\int\limits_0^t \frac{v_N(\tau,\xi)}{(t-\tau)^{\mu}}d\tau d\xi + \frac{h}{\Gamma(1-\mu)}\int\limits_0^t \frac{\lambda_N(\tau)}{(t-\tau)^{\mu}}d\tau \int\limits_{(N-1)h}^{Nh} B(t,\xi)d\xi + h\int\limits_{(N-1)h}^{Nh} f(t,\xi)d\xi,$$

$$\lambda_r(t)+\int\limits_{(r-1)h}^{rh} A(t,\xi)v_r(t,\xi)d\xi+\lambda_r(t)\int\limits_{(r-1)h}^{rh} A(t,\xi)d\xi+\int\limits_{(r-1)h}^{rh}\frac{B(t,\xi)}{\Gamma(1-\mu)}\int\limits_0^t\frac{v_r(\tau,\xi)}{(t-\tau)^\mu}d\tau d\xi+$$

$$+\frac{1}{\Gamma(1-\mu)}\int\limits_0^t\frac{\lambda_r(\tau)}{(t-\tau)^\mu}d\tau\int\limits_{(r-1)h}^{rh}B(t,\xi)d\xi+\int\limits_{(r-1)h}^{rh}f(t,\xi)d\xi=\lambda_{r+1}(t),\quad r=\overline{1,N-1}.$$

This system can be written in matrix form as

$$Q(h)\lambda(t)=-\Lambda(t,h,\lambda)-V_1(t,h,v)-V_2(t,h,v)-F(t,h), \tag{12}$$

where $\lambda(t)=\big(\lambda_1(t),\lambda_2(t),\dots,\lambda_N(t)\big)'$,

$$Q(h)=\begin{pmatrix} h & 0 & \dots & 0 & -h\Big(1+\int\limits_{(N-1)h}^{Nh}A(t,\xi)d\xi\Big)\\ 1+\int\limits_0^h A(t,\xi)d\xi & -1 & \dots & 0 & 0\\ 0 & 1+\int\limits_h^{2h}A(t,\xi)d\xi & \dots & 0 & 0\\ \dots & \dots & \dots & \dots & \dots\\ 0 & 0 & \dots & 1+\int\limits_{(N-2)h}^{(N-1)h}A(t,\xi)d\xi & -1 \end{pmatrix},$$

$$\Lambda(t,h,\lambda)=\Bigg(h\int\limits_{(N-1)h}^{Nh}B(t,\xi)d\xi\cdot\frac{1}{\Gamma(1-\mu)}\int\limits_0^t\frac{\lambda_N(\tau)}{(t-\tau)^\mu}d\tau,$$

$$\int\limits_0^h B(t,\xi)d\xi\cdot\frac{1}{\Gamma(1-\mu)}\int\limits_0^t\frac{\lambda_1(\tau)}{(t-\tau)^\mu}d\tau,\ \dots,\int\limits_{(N-2)h}^{(N-1)h}B(t,\xi)d\xi\cdot\frac{1}{\Gamma(1-\mu)}\int\limits_0^t\frac{\lambda_{N-1}(\tau)}{(t-\tau)^\mu}d\tau\Bigg)',$$

$$V_1(t,h,v)=\Bigg(h\int\limits_{(N-1)h}^{Nh}A(t,\xi)v_N(t,\xi)d\xi,\int\limits_0^h A(t,\xi)v_1(t,\xi)d\xi,\ \dots,\int\limits_{(N-2)h}^{(N-1)h}A(t,\xi)v_{N-1}(t,\xi)d\xi\Bigg)',$$

$$V_2(t,h,v)=\Bigg(h\int\limits_{(N-1)h}^{Nh}\frac{B(t,\xi)}{\Gamma(1-\mu)}\int\limits_0^t\frac{v_N(\tau,\xi)}{(t-\tau)^\mu}d\tau d\xi,$$

$$\int\limits_0^h\frac{B(t,\xi)}{\Gamma(1-\mu)}\int\limits_0^t\frac{v_1(\tau,\xi)}{(t-\tau)^\mu}d\tau d\xi,\ \dots,\int\limits_{(N-2)h}^{(N-1)h}\frac{B(t,\xi)}{\Gamma(1-\mu)}\int\limits_0^t\frac{v_{N-1}(\tau,\xi)}{(t-\tau)^\mu}d\tau d\xi\Bigg)',$$

$$F(t,h)=\Bigg(h\int\limits_{(N-1)h}^{Nh}f(t,\xi)d\xi,\int\limits_0^h f(t,\xi)d\xi,\ \dots,\int\limits_{(N-2)h}^{(N-1)h}f(t,\xi)d\xi\Bigg)'.$$

To find the system consisting of the functions $\{\lambda_r(t),v_r(t,x)\}$, $r=\overline{1,N}$, we have a closed system consisting of equations (11) and (12).

Suppose the matrix $Q(h)$ is invertible for all $t \in [0,T]$ [15]. Taking as the initial approximation $v_r(t,x) = 0$, from system (12), we find $\lambda_r^{(0)}(t)$, $r = \overline{1,N}$. From equation (11) with $\lambda_r(t) = \lambda_r^{(0)}(t)$, we find $v_r^{(0)}(t,x)$, $r = \overline{1,N}$.

Step 1. Using equation (12) with $v_r(t,x) = v_r^{(0)}(t,x)$, we find $\lambda_r^{(1)}(t)$, $r = \overline{1,N}$. From equation (11) with $\lambda_r(t) = \lambda_r^{(1)}(t)$, we find $v_r^{(1)}(t,x)$, $r = \overline{1,N}$.

Continuing the process, at the k-th step, we obtain a system of pairs $\{\lambda_r^{(k)}(t), v_r^{(k)}(t,x)\}$.

Sufficient conditions for the feasibility and convergence of the proposed algorithm, as well as an estimate of the difference between the exact and approximate solutions, are established by

Theorem 1. Let for some $h > 0 : Nh = X, N = 1,2,...$, $(nN \times nN)$ matrix $Q(h)$ is invertible and the inequalities hold

1) $\|[Q(h)]^{-1}\| < \gamma(h)$,

2) $q(h) = h^2 \max\{1,h\}\Big(\alpha + \frac{\beta T^{1-\mu}}{\Gamma(2-\mu)}\Big)^2 E_{1-\mu}\big(h\gamma(h)\max\{1,h\}\cdot\beta\cdot T^{1-\mu}\big) e^{h\left(\alpha + \frac{\beta T^{1-\mu}}{\Gamma(2-\mu)}\right)} < 1$,

where $E_{1-\mu}\left(h\gamma(h)\max\{1,h\}\cdot\beta\cdot T^{1-\mu}\right) = \sum_{k=0}^{\infty} \frac{\left(h\gamma(h)\max\{1,h\}\cdot\beta\cdot T^{1-\mu}\right)^k}{\Gamma((1-\mu)k+1)}$, $0 < \mu < 1$, $\|A(t,x)\| \le \alpha$, $\|B(t,x)\| \le \beta$. Then the boundary value problem (1)–(3) has a unique solution $w^*(t,x)$ and the estimate holds

$$\|w^*(t,x) - w^{(k)}(t,x)\| \le$$

$$\le T\bigg(1 + h\max\{1,h\}\Big(\alpha + \frac{\beta T^{1-\mu}}{\Gamma(2-\mu)}\Big) E_{1-\mu}\big(h\gamma(h)\max\{1,h\}\cdot\beta\cdot T^{1-\mu}\big)\bigg)\frac{[q(h)]^k}{1-q(h)} M(h)\|f(t,x)\|_1.$$

Proof. The following inequalities hold

$$\|\Lambda(t,h,\lambda)\| \le \frac{\max\{1,h\}h\beta}{\Gamma(1-\mu)}\left\|\int_0^t \frac{\lambda_N(\tau)}{(t-\tau)^\mu}d\tau\right\| \le \frac{\max\{1,h\}h\beta}{\Gamma(1-\mu)}\int_0^t \frac{1}{(t-\tau)^\mu}\max_{r=\overline{1,N}}\|\lambda_r(\tau)\|d\tau,$$

$$\|V_1(t,h,v)\| \le \max\{1,h\}\alpha h \max_{r=\overline{1,N}} \sup_{x\in[(r-1)h,rh)} \|v_r(t,x)\|,$$

$$\|V_2(t,h,v)\| \le \frac{\max\{1,h\}h\beta}{\Gamma(1-\mu)}\int_0^t \frac{1}{(t-\tau)^\mu}\max_{r=\overline{1,N}} \sup_{x\in[(r-1)h,rh)} \|v_r(\tau,x)\|d\tau,$$

$$\|F(t,h)\| \le h\max\{1,h\}\|f(t,x)\|_1.$$

Assume that the matrix $Q(h)$ is invertible. Taking as the initial approximation $v_r(t,x) = 0$, from system (12), we find $\lambda_r^{(0)}(t)$.

Let us use the generalized Gronwall–Bellman inequality for equations with a fractional integral [16]

$$\|\lambda^{(0)}(t)\| = \max_{r=\overline{1,N}}\|\lambda_r^{(0)}(t)\| \le h\gamma(h)\max\{1,h\}\cdot\|f(t,x)\|_1 \cdot E_{1-\mu}\big(h\gamma(h)\max\{1,h\}\cdot\beta\cdot t^{1-\mu}\big),$$

where $E_{1-\mu}\big(h\gamma(h)\max\{1,h\}\cdot\beta\cdot t^{1-\mu}\big)$ is the two-parameter Mittag-Leffler function.

So, we have

$$\|\lambda^{(0)}(t)\|_2 = \max_{t\in[0,T]}\|\lambda^{(0)}(t)\| \le h\gamma(h)\max\{1,h\}E_{1-\mu}\big(h\gamma(h)\max\{1,h\}\cdot\beta\cdot T^{1-\mu}\big)\|f(t,x)\|_1.$$

Under our assumptions, the Cauchy problem (7), (8) with $\lambda_r(t) = \lambda_r^{(0)}(t)$, has a unique solution $v_r^{(0)}(t,x)$. By the Gronwall–Bellman inequality,

$$\|v^{(0)}(t,x)\|_3 = \max_{r=\overline{1,N}} \max_{(t,x)\in\Omega_r} \|v_r^{(0)}(t,x)\| \le$$

$$\le h\left[\left(\alpha + \frac{\beta T^{1-\mu}}{\Gamma(2-\mu)}\right)\|\lambda^{(0)}(t)\|_2 + \|f(t,x)\|_1\right] e^{h\left(\alpha+\frac{\beta T^{1-\mu}}{\Gamma(2-\mu)}\right)} \le M(h)\|f(t,x)\|_1,$$

$$M(h) = h\left[\left(\alpha + \tfrac{\beta T^{1-\mu}}{\Gamma(2-\mu)}\right) h\gamma(h)\max\{1,h\} E_{1-\mu}\big(h\gamma(h)\max\{1,h\}\cdot\beta\cdot T^{1-\mu}\big) + 1\right] e^{h\left(\alpha+\frac{\beta T^{1-\mu}}{\Gamma(2-\mu)}\right)}.$$

According to the algorithm, we determine $\lambda^{(1)}(t)$ and estimate $\lambda_r^{(1)}(t) - \lambda_r^{(0)}(t)$. Let us use the generalized Gronwall–Bellman inequality for equations with a fractional integral [16]

$$\|\lambda^{(1)}(t) - \lambda^{(0)}(t)\|_2 \le$$

$$\le h\max\{1,h\}\left(\alpha + \frac{\beta T^{1-\mu}}{\Gamma(2-\mu)}\right) E_{1-\mu}\big(h\gamma(h)\max\{1,h\}\cdot\beta\cdot T^{1-\mu}\big)\|v^{(0)}(t,x)\|_3.$$

Then

$$\left\|v^{(1)}(t,x) - v^{(0)}(t,x)\right\|_3 \le q(h)\|v^{(0)}(t,x)\|_3.$$

Continuing the iterative process, we obtain a sequence of systems of pairs $\{\lambda_r^{(k)}(t), v_r^{(k)}(t,x)\}$, $r = 1,2,...,N$, $k = 1,2,...$:

$$\|\lambda^{(k+1)}(t) - \lambda^{(k)}(t)\|_2 \le$$

$$\le h\max\{1,h\}\left(\alpha + \frac{\beta T^{1-\mu}}{\Gamma(2-\mu)}\right) E_{1-\mu}\big(h\gamma(h)\max\{1,h\}\cdot\beta\cdot T^{1-\mu}\big)\|v^{(k)}(t,x) - v^{(k-1)}(t,x)\|_3, \quad (13)$$

$$\|v^{(k+1)}(t,x) - v^{(k)}(t,x)\|_3 \le q(h)\|v^{(k)}(t,x) - v^{(k-1)}(t,x)\|_3, \quad k = 1,2,\dots \quad (14)$$

By virtue of condition 2) of Theorem 1 and inequalities (13), (14), the sequence $\{\lambda_r^{(k)}(t), v_r^{(k)}(t,x)\}$ converges to $\{\lambda_r^*(t), v_r^*(t,x)\}$ as $k\to\infty$ and the following estimates hold:

$$\|\lambda^*(t) - \lambda^{(k)}(t)\|_2 \le$$

$$\le h\max\{1,h\}\left(\alpha + \frac{\beta T^{1-\mu}}{\Gamma(2-\mu)}\right) E_{1-\mu}\big(h\gamma(h)\max\{1,h\}\cdot\beta\cdot T^{1-\mu}\big)\frac{[q(h)]^k}{1-q(h)} M(h)\|f(t,x)\|_1,$$

$$\|v^*(t,x) - v^{(k)}(t,x)\|_3 \le \frac{[q(h)]^k}{1-q(h)} M(h)\|f(t,x)\|_1.$$

Since $\{\lambda_r^*(t), v_r^*(t,x)\}$ is a solution of the problem (7)–(10), the function $w^*(t,x)$, obtained by joining together the systems of functions $\int\limits_0^t (\lambda_r^*(\tau) + v_r^*(\tau,x))d\tau$, will be a solution of the original problem (1)–(3), and the estimate of the Theorem 1 holds.

Let us prove uniqueness. Suppose that $w^{**}(t,x), w^*(t,x)$ are two solutions of the problem (1)–(3). Then the corresponding systems of pairs $\{\lambda_r^{**}(t), v_r^{**}(t,x)\}$, $\{\lambda_r^*(t), v_r^*(t,x)\}$, $r = \overline{1,N}$, will be solutions of the boundary value problem (7)–(10), and similarly to (13), (14):

$$\|\lambda^{**}(t) - \lambda^*(t)\|_2 \le h\max\{1,h\}\left(\alpha + \frac{\beta T^{1-\mu}}{\Gamma(2-\mu)}\right) E_{1-\mu}\big(h\gamma(h)\max\{1,h\}\cdot\beta\cdot T^{1-\mu}\big)\|v^*(t,x) - v^{**}(t,x)\|_3,$$

$$\|v^{**}(t,x) - v^*(t,x)\|_3 \le q(h)\|v^{**}(t,x) - v^*(t,x)\|_3, \quad q(h) < 1.$$

It follows from this that $\lambda_r^*(t) = \lambda_r^{**}(t), v_r^*(t,x) = v_r^{**}(t,x)$, i.e. $w^{**}(t,x) = w^*(t,x)$, for $(t,x)\in\Omega$. The theorem is proved. □

Conclusion

An iterative algorithm has been proposed in this work for solving the periodic boundary value problem of a hyperbolic equation with the Riemann–Liouville fractional derivative. A theorem on the existence and uniqueness of the solution has been proved, and estimates of its approximate values have been established. The convergence conditions of the algorithm have been derived using matrix analysis and the Gronwall–Bellman inequality. It has been shown that the choice of the partition step of the spatial interval plays a key role in ensuring the stability of the method. The results of this work can be applied to the modeling of a wide class of physical processes with memory and periodic boundary conditions. Further research may be aimed at solving nonlocal boundary value problems for nonlinear hyperbolic equations with a fractional derivative.

Acknowledgments

The authors thank the reviewers for their valuable comments and suggestions.

Author Contributions

All authors contributed equally to this work.

Conflict of Interest

The authors declare no conflict of interest.

References

1. Samko, S.G., Kilbas, A.A., & Marichev, O.I. (1993). *Fractional integrals and derivatives: Theory and applications.* Yverdon: Gordon and Breach.
2. Nakhushev, A.M. (2003). *Drobnye ischisleniia i ikh primeneniia [Fractional calculus and its applications]*. Moscow: FIZMATLIT [in Russian].
3. Leonenko, N., & Podlubny, I. (2022). Monte Carlo method for fractional-order differentiation extended to higher orders. *Fractional Calculus and Applied Analysis, 25*(3), 841–857. https://doi.org/10.1007/s13540-022-00048-w
4. Abdimanapova, P.B., & Temesheva, S.M. (2023). On a Solution of a Nonlinear Nonlocal Boundary Value Problem for one Class of Hyperbolic Equation. *Lobachevskii Journal of Mathematics Open source preview, 44*(7), 2529–2541. https://doi.org/10.1134/S1995080223070028
5. Pskhu, A.V. (2022). Green Function of the First Boundary-Value Problem for the Fractional Diffusion-Wave Equation in a Multidimensional Rectangular Domain. *Journal of Mathematical Sciences, 260*(3), 325–334. https://doi.org/10.1007/s10958-022-05697-z
6. Khubiev, K.U. (2018). Zadachi so smeshcheniem dlia nagruzhennogo uravneniia giperbolo-parabolicheskogo tipa s operatorom drobnoi diffuzii [Boundary value problem with shift for loaded hyperbolic-parabolic type equation involving fractional diffusion operator]. *Vestnik Udmurtskogo universiteta. Matematika. Mekhanika. Kompiuternye nauki — Bulletin of Udmurt University. Mathematics. Mechanics. Computer Science, 28*(1), 82–90 [in Russian]. https://doi.org/10.20537/vm180108
7. Dzhumabayev, D.S. (1989). Criteria for the unique solvability of a linear boundary-value problem for an ordinary differential equation. *USSR Computational Mathematics and Mathematical Physics, 29*(1), 34–46. https://doi.org/10.1016/0041-5553(89)90038-4

8. Assanova, A. (2025). Novel Approach to Solving Boundary Value Problems for Systems of Hyperbolic Second Order Differential Equations. *Journal of Mathematical Sciences United States Open source preview, 291*(2), 190–201. https://doi.org/10.1007/s10958-025-07800-6
9. Iskakova, N.B., Bakirova, E.A., Khanzharova, B.S., & Kadirbayeva, Z.M. (2024). An Algorithm for Solving Boundary Value Problems for Delay Differential Equations with Loadings. *Lobachevskii Journal of Mathematics Open source preview, 45*(10), 5032–5042. https://doi.org/10.1134/S1995080224606064
10. Pskhu, A.V., Kosmakova, M.T., & Izhanova, K.A. (2024). Cauchy Problem for a Loaded Fractional Diffusion Equation. *Lobachevskii Journal of Mathematics Open source preview, 45*(9), 4574–4581. https://doi.org/10.1134/S1995080224605228
11. Ramazanov, M.I., Gulmanov, N.K., Kopbalina, S.S., & Omarov, M.T. (2024). Solution of a Singular Integral Equation of Volterra Type of the Second Kind. *Lobachevskii Journal of Mathematics, 45*(11), 5898–5906. https://doi.org/10.1134/S1995080224606830
12. Pskhu, A.V., Ramazanov, M.I., & Kosmakova, M.T. (2023). Boundary value problem for a loaded fractional diffusion equation. *Turkish Journal of Mathematics, 47*(5), 1585–1594. https://doi.org/10.55730/1300-0098.3450
13. Pskhu, A.V. (2025). On overdetermination conditions in order determination inverse problems for fractional diffusion equations. *Uzbek Mathematical Journal Open source preview, 69*(2), 222–231. https://doi.org/10.29229/uzmj.2025-2-21
14. Orumbayeva, N.T., Kosmakova, M.T., Tokmagambetova, T.D., & Manat, A.M. (2025). Solutions of boundary value problems for loaded hyperbolic type equations. *Bulletin of the Karaganda University. Mathematics Series,* 2(118), 177–188. https://doi.org/10.31489/2025m2/177-188
15. Orumbayeva, N.T., Assanova, A.T., & Keldibekova, A.B. (2022). On an algorithm of finding an approximate solution of a periodic problem for a third-order differential equation. *Eurasian Mathematical Journal, 13*(1), 69–85. https://doi.org/10.32523/2077-9879-2022-13-1-69-85
16. Ye, H., Gao, J., & Ding, Y. (2007). A generalized Gronwall inequality and its application to a fractional differential equation. *Journal of Mathematical Analysis and Applications, 328*, 1075–1081. https://doi.org/10.1016/j.jmaa.2006.05.061

*Author Information**

Nurgul Tumarbekovna Orumbayeva — Candidate in Physical and Mathematical Sciences, Associate Professor, Buketov Karaganda National Research University, Institute of Applied Mathematics, 28 Universitetskaya St., Karaganda, 100028, Kazakhstan; e-mail: *orumbayevanurgul@gmail.com*; https://orcid.org/0000-0003-1714-6850

Botagoz Beketovna Zhantassova (corresponding author) — PhD doctoral student, Senior lecturer, Buketov Karaganda National Research University, Institute of Applied Mathematics, 28 Universitetskaya St., Karaganda, 100028, Kazakhstan; e-mail: *zh.botagoz.b@gmail.com*; https://orcid.org/0009-0008-6712-4928

*Authors' names are presented in the following order: first name, middle name (if any), last name.

https://doi.org/10.31489/2026M1/202-216 *Research article*

Higher-order balancing numbers: new sequences, recurrence relations, generating functions and identities

K. Prasad[1], Indubala[2], M. Kumari[1,*]

[1] *Government Engineering College Bhojpur, Bihar, India;*
[2] *Indian Institute of Science, Bengaluru, India*
(E-mail: klkaprsd@gmail.com, indubala@iisc.ac.in, muneshnasir94@gmail.com)

In this article, we study a novel extension of the classic balancing numbers, referred to as the higher-order balancing numbers and denoted by $B_n^{(k)}$. This sequence is analogous to the higher-order Fibonacci numbers and follows the same recurrence relation as the balancing sequence itself. The case $k = 1$ gives the classic balancing numbers (A001109) and for $k = 2$ gives the sequence A029547, thus establishing a direct link to existing number sequences. Here, we first establish the Binet-like formula and then, with its help, present various algebraic properties of this newly introduced sequence, such as recurrence relations, generating functions (both ordinary and exponential), partial sums, binomial sums, combined identities, and more. We also obtain the limiting ratio and establish several well-known identities, including Catalan's identity, d'Ocagane's identity, Vajda's identity, Honsberger's identity, using the Binet-like formula. Finally, we give some mixed identity and series sum formulae. In this study, the obtained identities and algebraic properties are expressed in terms of the existing balancing and Lucas-balancing numbers.

Keywords: balancing numbers, Binet's formula, partial sums, binomial sums, Vajda's identity, binomial transform, generating functions, recurrence relations.

2020 Mathematics Subject Classification: 11B37, 11B39, 11B83.

Introduction

In number theory, the Fibonacci numbers emerges as celebrity kind numbers. Beside Fibonacci numbers, in number theory Lucas, Pell, Jacobsthal, Mersenne, Leonardo, Perrin, Padovan, etc. also follows the same pattern of study and opens an area of research for further investigation. One of such fascinating number sequences is the balancing sequence. The concept of balancing numbers (and balancers) was originally introduced in 1999 by Behera et al. [1]. Let us recall some important properties of it. A natural number n is said to be balancing number with balancer r if it satisfy the Diophantine equation

$$1 + 2 + 3 + ... + (n - 1) = (n + 1) + (n + 2) + ... + (n + r).$$

Thus the balancing numbers $\{B_n\}_{n\geq 0}$ are defined recursively as

$$B_{n+2} = 6B_{n+1} - B_n \quad \text{with} \quad B_0 = 0,\ B_1 = 1.$$

Another sequence close to balancing numbers is "Lucas-balancing numbers $\{C_n\}$" which is defined as $C_n = \sqrt{8B_n^2 + 1}$ and that satisfy the same recurrence relation but with initial condition $C_0 = 1,\ C_1 = 3$. The first few values of these sequence are:

n	**0**	**1**	2	3	4	5	6	7	8	...
B_n	**0**	**1**	6	35	204	1189	6930	40391	235416	...
C_n	**1**	**3**	17	99	577	3363	19601	114243	665857	...

*Corresponding author. *E-mail: muneshnasir94@gmail.com*
Received: 3 February 2025; *Accepted:* 11 December 2025.

The closed form formula for the above sequences are given as

$$B_n = \frac{\lambda_1^n - \lambda_2^n}{4\sqrt{2}} \quad \text{and} \quad C_n = \frac{\lambda_1^n + \lambda_2^n}{2}, \tag{1}$$

where $\lambda_1 = 3 + \sqrt{8}$ and $\lambda_2 = 3 - \sqrt{8}$ are the roots of the characteristic equation $x^2 - 6x + 1 = 0$ and hold the following relations:

$$\lambda_1 + \lambda_2 = 6, \quad \lambda_1^2 + \lambda_2^2 = 34, \quad \lambda_1 - \lambda_2 = 2\sqrt{8} \quad \text{and} \quad \lambda_1\lambda_2 = 1.$$

Some identities involving the balancing and Lucas-balancing numbers which we need later are given in the following lemma.

Lemma 1. For integers m and n, we have
1. $B_{n-m}B_{n+m} - B_n^2 = -B_m^2$.
2. $C_{n-m}C_{n+m} - C_n^2 = \frac{1}{2}(C_{2m} - 1)$.
3. $C_{2n} = 2C_n^2 - 1$ and $C_n^2 = 8B_n^2 + 1$.

Theorem 1. For positive integer k and $n \geq 0$, we have

$$B_{k(n+1)} = \lambda_1^k B_{kn} + \lambda_2^{kn} B_k. \tag{2}$$

Proof. From Binet-like formula (1), the LHS can be written as

$$\begin{aligned}
B_{k(n+1)} &= \frac{\lambda_1^{k(n+1)} - \lambda_2^{k(n+1)}}{4\sqrt{2}} \\
&= \frac{\lambda_1^{k(n+1)} - \lambda_2^{k(n+1)} + \lambda_1^k\lambda_2^{kn} - \lambda_1^k\lambda_2^{kn}}{4\sqrt{2}} \\
&= \frac{\lambda_1^k(\lambda_1^{kn} - \lambda_2^{kn}) + \lambda_2^{kn}(\lambda_1^k - \lambda_2^k)}{4\sqrt{2}} \\
&= \lambda_1^k B_{kn} + \lambda_2^{kn} B_k.
\end{aligned}$$

□

In the next section, we first define the higher-order balancing numbers, and then we study their algebraic properties. In recent years, several articles have appeared on higher-order sequences associated with a famous number sequence, where the authors presented study on their algebraic properties. The study begins with the earlier work of Randić et al. (1996) [2], where the authors proposed higher-order Fibonacci numbers and investigated their properties in the context of applications in chemistry. Later, Ozvatan [3] studied a generalization of these higher-order Fibonacci numbers and obtained several algebraic properties.

Recently, Uysal and Özkan [4, 5] studied the quaternion algebra of higher-order Jacobsthal–Lucas numbers. Kızılateş [6] studied the hypercomplex numbers whose components are higher-order Fibonacci numbers while in Özimamoğlu [7] they considered higher-order Pell numbers. Kızılateş in their recent study [8], presented a comprehensive survey on the generalization of hybrid numbers (polynomial) with higher-order generalized Fibonacci polynomials. Kumari et al. [9] studied the algebra of quaternions and octonions of the higher-order Mersenne numbers. A similar concept was applied in [10] to study new sequences of balancing numbers. A polynomial version of the balancing numbers and their algebraic properties were studied by Ray [11] and Frontczak [12]. Some recent developments and applications of balancing numbers are due to Ray [13, 14], Frontczak [12, 15, 16], Liptai [17], Özkoc [18], Panda [19], Prasad et al. [10, 20], etc. Similar to the above work, many algebraic properties of our proposed new sequences can be investigated after this study.

1 Higher-order balancing numbers

Throughout the article, we adopt the symbols $\{B_n\}$, $\{C_n\}$ and $\{B_n^{(k)}\}$ for the n-th balancing, Lucas-balancing and higher-order balancing numbers, respectively.

Definition 1. Let $k \in \mathbb{N}$, then the higher-order balancing numbers $\{B_n^{(k)}\}$ is defined as

$$B_n^{(k)} = \frac{B_{kn}}{B_k}, \quad n = 0, 1, 2, ... \tag{3}$$

From Definition 1, note that the following identities hold obviously:

1. $B_0^{(k)} = 0$ and $B_1^{(k)} = 1$.
2. $B_2^{(k)} = \lambda_1^k + \lambda_2^k = 2C_k$.

For different values of k, some higher-order balancing numbers are listed in Table 1:

Table 1

List of some higher-order balancing numbers ($B_n^{(k)}$)

Numbers	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$
$B_0^{(k)}$	0	0	0	0	0
$B_1^{(k)}$	1	1	1	1	1
$B_2^{(k)}$	6	34	198	1154	6726
$B_3^{(k)}$	35	1155	39203	1331715	45239075
$B_4^{(k)}$	204	39236	7761996	1536797956	304278011724
$B_5^{(k)}$	1189	1332869	1536836005	1773463509509	2046573861616547

Note that for $k = 2$, relation (3) gives a new sequence $B_n^{(2)} = (1/6)B_{2n}$ which generates the terms $1, 34, 1155, 39236, 1332869, 45278310, 1538129671, 52251130504, \ldots$, indexed as A029547 on OEIS and satisfy the recurrence relation $a_{n+1} = 34a_n - a_{n-1}$, with $a_{-1} = 0$, $a_0 = 1$.

Theorem 2. The sequence $\{B_n^{(k)}\}$ satisfies the following recurrence relation

$$B_{n+2}^{(k)} - (\lambda_1^k + \lambda_2^k)B_{n+1}^{(k)} + B_n^{(k)} = 0, \quad \text{with } B_0^{(k)} = 0,\ B_1^{(k)} = 1. \tag{4}$$

Proof. Dividing both sides of (2) by B_k and then using Definition 1, we get

$$B_{n+1}^{(k)} = \lambda_1^k B_n^{(k)} + \lambda_2^{kn}. \tag{5}$$

Now replacing n by $n+1$ in (5) yields

$$B_{n+2}^{(k)} = \lambda_1^k B_{n+1}^{(k)} + \lambda_2^{k(n+1)}. \tag{6}$$

Multiplying both sides of (6) by λ_2^{-k}, we get

$$B_{n+2}^{(k)}\lambda_2^{-k} = \lambda_1^k B_{n+1}^{(k)}\lambda_2^{-k} + \lambda_2^{kn}. \tag{7}$$

Thus, subtracting (5) from (7) gives the required result. □

Now we give the Binet's formula for higher-order balancing numbers and using the Binet's formula, we prove some algebraic identities for this sequence.

1.1 Binet's formula and some identities

For a fixed positive integer k, the characteristic equation of recurrences (4) is $x^2-(\lambda_1^k+\lambda_2^k)x+1=0$, whose roots are λ_1^k and λ_2^k. Thus, the Binet's formula for the higher-order balancing numbers $B_n^{(k)}$ is given by

$$B_n^{(k)} = \frac{\lambda_1^{kn} - \lambda_2^{kn}}{\lambda_1^k - \lambda_2^k}. \tag{8}$$

Theorem 3. The limiting ratio of higher-order balancing numbers is λ_1^k, i.e.,

$$\lim_{n\to\infty} \frac{B_{n+1}^{(k)}}{B_n^{(k)}} = \lambda_1^k.$$

Proof. From Binet's formula (8), we can write

$$\begin{aligned}
\lim_{n\to\infty} \frac{B_{n+1}^{(k)}}{B_n^{(k)}} &= \lim_{n\to\infty} \left(\frac{\frac{(\lambda_1^k)^{n+1}-(\lambda_2^k)^{n+1}}{\lambda_1^k-\lambda_2^k}}{\frac{(\lambda_1^k)^{n}-(\lambda_2^k)^{n}}{\lambda_1^k-\lambda_2^k}} \right) \\
&= \lim_{n\to\infty} \left(\frac{(\lambda_1^k)^{n+1} - (\lambda_2^k)^{n+1}}{(\lambda_1^k)^{n} - (\lambda_2^k)^{n}} \right) \\
&= \lim_{n\to\infty} \left(\frac{(\lambda_1^k)^{n}\lambda_1^k - (\lambda_2^k)^{n}\lambda_2^k}{(\lambda_1^k)^{n} - (\lambda_2^k)^{n}} \right) \\
&= \lim_{n\to\infty} \left(\frac{(\lambda_1^k)\left(\frac{(\lambda_2^k)^{n+1}}{(\lambda_1^k)^{n+1}} - 1\right)}{\left(\frac{(\lambda_2^k)^{n}}{(\lambda_1^k)^{n}} - 1\right)} \right).
\end{aligned}$$

Now taking into account that $\lim_{n\to\infty}\left(\frac{(\lambda_2^k)^{n+1}}{(\lambda_1^k)^{n+1}}\right) = 0$ and $\lim_{n\to\infty}\left(\frac{(\lambda_2^k)^{n}}{(\lambda_1^k)^{n}}\right) = 0$ as $|\frac{\lambda_2}{\lambda_1}| < 1$ for $k \in \mathbb{N}$, we have

$$\lim_{n\to\infty} \left(\left(\frac{(\lambda_2^k)^{n+1}}{(\lambda_1^k)^{n+1}} \right) - 1 \right) = -1.$$

So the original limit simplifies to

$$\lim_{n\to\infty} \frac{B_{n+1}^{(k)}}{B_n^{(k)}} = \lambda_1^k. \qquad \square$$

Theorem 4 (Catalan's identity). For $n \geq r$, we have

$$B_{n-r}^{(k)} B_{n+r}^{(k)} - \left(B_n^{(k)}\right)^2 = -\left(B_r^{(k)}\right)^2.$$

Proof. Using Binet's formula (8) in LHS, we have

$$\begin{aligned}
{B_{n-r}}^{(k)} B_{n+r}^{(k)} - \left(B_n^{(k)}\right)^2 &= \left(\frac{(\lambda_1^k)^{n-r} - (\lambda_2^k)^{n-r}}{\lambda_1^k - \lambda_2^k} \right) \left(\frac{(\lambda_1^k)^{n+r} - (\lambda_2^k)^{n+r}}{\lambda_1^k - \lambda_2^k} \right) - \left(\frac{(\lambda_1^k)^{n} - (\lambda_2^k)^{n}}{\lambda_1^k - \lambda_2^k} \right)^2 \\
&= \frac{1}{(\lambda_1^k - \lambda_2^k)^2} \left[(\lambda_1^k)^{n-r}(\lambda_1^k)^{n+r} - (\lambda_1^k)^{n-r}(\lambda_2^k)^{n+r} - (\lambda_2^k)^{n-r}(\lambda_1^k)^{n+r} + (\lambda_2^k)^{n-r}(\lambda_2^k)^{n+r} \right] \\
&\quad - \frac{1}{(\lambda_1^k - \lambda_2^k)^2} \left[(\lambda_1^k)^{2n} - 2(\lambda_1^k)^n(\lambda_2^k)^n + (\lambda_2^k)^{2n} \right] \\
&= \frac{1}{(\lambda_1^k - \lambda_2^k)^2} \left[\lambda_1^{2kn} - (\lambda_1^k)^{n-r}(\lambda_2^k)^{n+r} - (\lambda_2^k)^{n-r}(\lambda_1^k)^{n+r} + (\lambda_2^k)^{2n} - (\lambda_1^k)^{2n} + 2(\lambda_1^k)^n(\lambda_2^k)^n - \lambda_2^{2kn} \right]
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{(\lambda_1^k-\lambda_2^k)^2}\left[2(\lambda_1^k\lambda_2^k)^n-(\lambda_1^k\lambda_2^k)^n\frac{(\lambda_2^k)^r}{(\lambda_1^k)^r}-(\lambda_1^k\lambda_2^k)^n\frac{(\lambda_1^k)^r}{(\lambda_2^k)^r}\right]\\
&= \frac{(1)^{kn}}{(\lambda_1^k-\lambda_2^k)^2}\Big(2-\frac{(\lambda_2^k)^r}{(\lambda_1^k)^r}-\frac{(\lambda_1^k)^r}{(\lambda_2^k)^r}\Big) \qquad \text{(since } \lambda_1\lambda_2=1)\\
&= \frac{1}{(\lambda_1^k-\lambda_2^k)^2(\lambda_1\lambda_2)^{kr}}\Big(2(\lambda_1\lambda_2)^{kr}-\lambda_1^{2kr}-\lambda_2^{2kr}\Big)\\
&= -\frac{1}{(\lambda_1^k-\lambda_2^k)^2}(\lambda_1^{kr}-\lambda_2^{kr})^2=-\big(B_r^{(k)}\big)^2.
\end{aligned}
$$

□

Corollary 1 (Cassini's identity). For positive integer n, we have

$$B_{n-1}^{(k)}B_{n+1}^{(k)}-\Big(B_n^{(k)}\Big)^2=-(B_1^k)^2=-1.$$

Theorem 5 (d'Ocagane identity). For integers m,n with $m>n$, we have

$$B_{n+1}^{(k)}B_m^{(k)}-B_n^{(k)}B_{m+1}^{(k)}=B_{m-n}^{(k)}.$$

Proof. Proceeding similar to the above theorem using Binet's formula (8) proves the given identity.

□

Theorem 6 (Vajda's identity). For every $n,m,r\geq 0$, we have

$$B_{n+m}^{(k)}B_{n+r}^{(k)}-B_n^{(k)}B_{n+m+r}^{(k)}=B_m^{(k)}B_r^{(k)}.$$

Proof. Substituting Binet's formula (8) in the LHS, we get

$$
\begin{aligned}
&B_{n+m}^{(k)}B_{n+r}^{(k)}-B_n^{(k)}B_{n+m+r}^{(k)}\\
&=\left(\frac{\lambda_1^{k(n+m)}-\lambda_2^{k(n+m)}}{\lambda_1^k-\lambda_2^k}\right)\left(\frac{\lambda_1^{k(n+r)}-\lambda_2^{k(n+r)}}{\lambda_1^k-\lambda_2^k}\right)-\left(\frac{\lambda_1^{kn}-\lambda_2^{kn}}{\lambda_1^k-\lambda_2^k}\right)\left(\frac{\lambda_1^{k(n+m+r)}-\lambda_2^{k(n+m+r)}}{\lambda_1^k-\lambda_2^k}\right).
\end{aligned}
$$

Expanding the product of RHS, it becomes

$$
\begin{aligned}
&=\frac{\Big(\lambda_1^{k(n+m)}-\lambda_2^{k(n+m)}\Big)\Big(\lambda_1^{k(n+r)}-\lambda_2^{k(n+r)}\Big)-\Big(\lambda_1^{kn}-\lambda_2^{kn}\Big)\Big(\lambda_1^{k(n+m+r)}-\lambda_2^{k(n+m+r)}\Big)}{\Big(\lambda_1^k-\lambda_2^k\Big)^2}\\
&=\frac{1}{\Big(\lambda_1^k-\lambda_2^k\Big)^2}\Bigg(\Big(\lambda_1^{k(n+m+n+r)}-\lambda_1^{k(n+m)}\lambda_2^{k(n+r)}-\lambda_2^{k(n+m)}\lambda_1^{k(n+r)}+\lambda_2^{k(n+m+n+r)}\Big)\\
&\quad-\Big(\lambda_1^{k(n+n+m+r)}-\lambda_1^{kn}\lambda_2^{k(n+m+r)}-\lambda_2^{kn}\lambda_1^{k(n+m+r)}+\lambda_2^{k(n+n+m+r)}\Big)\Bigg)\\
&=\frac{1}{\Big(\lambda_1^k-\lambda_2^k\Big)^2}\Bigg(\Big(\lambda_1^{k(2n+m+r)}-\lambda_1^{k(n+m)}\lambda_2^{k(n+r)}-\lambda_2^{k(n+m)}\lambda_1^{k(n+r)}+\lambda_2^{k(2n+m+r)}\Big)\\
&\quad-\Big(\lambda_1^{k(2n+m+r)}-\lambda_1^{kn}\lambda_2^{k(n+m+r)}-\lambda_2^{kn}\lambda_1^{k(n+m+r)}+\lambda_2^{k(2n+m+r)}\Big)\Bigg).
\end{aligned}
$$

Since $\lambda_1\lambda_2 = 1$, so the above quantity reduced to

$$\begin{aligned}
&= \frac{1}{\left(\lambda_1^k - \lambda_2^k\right)^2}\left(\left(\lambda_1^{k(m+r)} + \lambda_2^{k(m+r)} - \lambda_1^{km}\lambda_2^{kr} - \lambda_2^{km}\lambda_1^{kr}\right)\right)\\
&= \frac{1}{\left(\lambda_1^k - \lambda_2^k\right)^2}\left(\left(\lambda_1^{km}(\lambda_1^{kr} - \lambda_2^{kr}) - \lambda_2^{km}(\lambda_1^{kr} - \lambda_2^{kr}\right)\right)\\
&= \left(\frac{\lambda_1^{km} - \lambda_2^{km}}{\lambda_1^k - \lambda_2^k}\right)\left(\frac{\lambda_1^{kr} - \lambda_2^{kr}}{\lambda_1^k - \lambda_2^k}\right)\\
&= B_m^{(k)}B_r^{(k)}.
\end{aligned}$$

Thus, we conclude that

$$B_{n+m}^{(k)}B_{n+r}^{(k)} - B_n^{(k)}B_{n+m+r}^{(k)} = B_m^{(k)}B_r^{(k)}.$$

□

Theorem 7 (Honsberger's identity). For any integers $p,\ n > 0$, we have

$$B_{p-1}^{(k)}B_n^{(k)} + B_p^{(k)}B_{n+1}^{(k)} = \frac{C_{kn}C_{k(p+n)} - C_{k(p-n-1)}}{8B_k^2}.$$

Proof. Taking into account $\lambda_1\lambda_2 = 1$, we obtain

$$\begin{aligned}
&B_{p-1}^{(k)}B_n^{(k)} + B_p^{(k)}B_{n+1}^{(k)}\\
&\quad= \left(\frac{\lambda_1^{k(p-1)} - \lambda_2^{k(p-1)}}{\lambda_1^k - \lambda_2^k}\right)\left(\frac{\lambda_1^{kn} - \lambda_2^{kn}}{\lambda_1^k - \lambda_2^k}\right) + \left(\frac{\lambda_1^{kp} - \lambda_2^{kp}}{\lambda_1^k - \lambda_2^k}\right)\left(\frac{\lambda_1^{k(n+1)} - \lambda_2^{k(n+1)}}{\lambda_1^k - \lambda_2^k}\right)\\
&\quad= \frac{1}{\left(\lambda_1^k - \lambda_2^k\right)^2}\left[\left(\lambda_1^{k(p-1+n)} - \lambda_1^{k(p-1)}\lambda_2^{kn} - \lambda_2^{k(p-1)}\lambda_1^{kn} + \lambda_2^{k(p-1+n)}\right)\right]\\
&\qquad+ \frac{1}{\left(\lambda_1^k - \lambda_2^k\right)^2}\left[\left(\lambda_1^{k(p+1+n)} - \lambda_1^{kp}\lambda_2^{k(n+1)} - \lambda_2^{kp}\lambda_1^{k(n+1)} + \lambda_2^{k(p+1+n)}\right)\right]\\
&\quad= \frac{1}{\left(\lambda_1^k - \lambda_2^k\right)^2}\Big[\lambda_1^{k(p+n)}\left[\lambda_1^{-kn} + \lambda_1^{kn}\right] + \lambda_2^{k(p+n)}\left[\lambda_2^{-kn} + \lambda_2^{kn}\right]\\
&\qquad- \lambda_1^{k(p-1-n)} - \lambda_2^{k(p-n-1)} - \lambda_1^{k(p-n-1)} - \lambda_2^{k(p-n-1)}\Big]\\
&\quad= \frac{1}{32B_k^2}\left[\left(\lambda_1^{k(p+n)} + \lambda_2^{k(p+n)}\right)\left(\lambda_1^{kn} + \lambda_2^{kn}\right) - 2\left(\lambda_1^{k(p-1-n)} + \lambda_2^{k(p-n-1)}\right)\right]\\
&\quad= \frac{C_{kn}C_{k(p+n)} - C_{k(p-n-1)}}{8B_k^2} \qquad \text{(using (1))}.
\end{aligned}$$

□

2 *Some identities involving HOBN*

Theorem 8 (Generating function). For higher-order balancing sequence, we have

$$\sum_{n=0}^{\infty} B_n^{(k)}x^n = \frac{x}{x^2 - 2C_kx + 1}.$$

Proof. Let $G(x,k)=\sum_{n=0}^{\infty} B_n^{(k)}x^n$ be the generating function for the higher-order balancing sequence. Now multiplying both sides of (4) by x^{n+2} and then taking the summation over 0 to ∞, we get

$$\sum_{n=0}^{\infty} B_{n+2}^{(k)}x^{n+2}-(\lambda_1^k+\lambda_2^k)\sum_{n=0}^{\infty} B_{n+1}^{(k)}x^{n+2}+\sum_{n=0}^{\infty} B_n^{(k)}x^{n+2}=0.$$

Now substituting

$$\sum_{n=0}^{\infty} B_{n+2}^{(k)}x^{n+2}=G(x,k)-B_1^{(k)}x-B_0^{(k)},\quad \sum_{n=0}^{\infty} B_{n+1}^{(k)}x^{n+2}=x[G(x,k)-B_0^{(k)}]$$
$$\text{and}\quad \sum_{n=0}^{\infty} B_n^{(k)}x^{n+2}=x^2G(x,k),$$

we get

$$\left(G(x,k)-B_1^{(k)}x-B_0^{(k)}\right)-\left(\lambda_1^k+\lambda_2^k\right)x\left[G(x,k)-B_0^{(k)}\right]+x^2G(x,k)=0,$$
$$G(x,k)\left[1-\left(\lambda_1^k+\lambda_2^k\right)x+x^2\right]+\left(\lambda_1^k+\lambda_2^k\right)xB_0^{(k)}-B_1^{(k)}x-B_0^{(k)}=0.$$

Since $B_0^{(k)}=0$ and $B_1^{(k)}=1$, so after simplifications it gives the proposed identity. □

For instance, setting $k=1$ in the above theorem gives the generating function for the balancing sequence, i.e.,

$$G(x,1)=\frac{x}{x^2-6x+1}.$$

Theorem 9. The generating function for odd and even indexed sequences of higher-order balancing numbers $\{B_n^{(k)}\}$ are:

$$\sum_{r=0}^{\infty} B_{2r}^{(k)}x^r=\frac{2xC_k}{x^2-2C_{2k}x+1}\quad\text{and}\quad \sum_{r=0}^{\infty} B_{2r+1}^{(k)}x^r=\frac{x+1}{x^2-2C_{2k}x+1}.$$

Proof. Using Binet's formula of $B_{2r}^{(k)}$, we write

$$\begin{aligned}\sum_{r=0}^{\infty} B_{2r}^{(k)}x^r&=\sum_{r=0}^{\infty}\frac{\lambda_1^{2kr}-\lambda_2^{2kr}}{\lambda_1^k-\lambda_2^k}x^r=\frac{1}{\lambda_1^k-\lambda_2^k}\left(\sum_{r=0}^{\infty}\lambda_1^{2kr}x^r-\lambda_2^{2kr}x^r\right)\\&=\frac{1}{\lambda_1^k-\lambda_2^k}\left[\frac{1}{1-\lambda_1^{2k}x}-\frac{1}{1-\lambda_2^{2k}x}\right]\\&=\frac{1}{\lambda_1^k-\lambda_2^k}\left[\frac{x(\lambda_1^{2k}-\lambda_2^{2k})}{x^2-x(\lambda_1^{2k}+\lambda_2^{2k})+1}\right]\\&=\frac{xB_2^{(k)}}{x^2-2C_{2k}x+1}=\frac{2xC_k}{x^2-2C_{2k}x+1}.\end{aligned}$$

In a similar fashion, the second identity can be also obtained. □

Theorem 10. The exponential generating function $E(x,k)$ for higher-order balancing numbers is

$$E(x,k)=\frac{e^{\lambda_1^k x}-e^{\lambda_2^k x}}{4\sqrt{2}B_k}.$$

Proof. Using Binet's formula (8) in exponential generating function $E(x,k)=\sum_{n=0}^{\infty}\frac{B_n^{(k)}x^n}{n!}$, we get

$$
\begin{aligned}
E(x,k) &= \sum_{n=0}^{\infty}\Big(\frac{\lambda_1^{kn}-\lambda_2^{kn}}{\lambda_1^k-\lambda_2^k}\Big)\frac{x^n}{n!}\\
&= \frac{1}{\lambda_1^k-\lambda_2^k}\Big(\sum_{n=0}^{\infty}\frac{(\lambda_1^k x)^n}{n!}-\sum_{n=0}^{\infty}\frac{(\lambda_2^k x)^n}{n!}\Big)\\
&= \frac{e^{\lambda_1^k x}-e^{\lambda_2^k x}}{\lambda_1^k-\lambda_2^k}.
\end{aligned}
$$

Substituting $\lambda_1^k-\lambda_2^k=4\sqrt{2}B_k$ proves the required identity. □

Theorem 11. The Poisson generating function for the higher-order balancing numbers is

$$
\sum_{r=0}^{\infty}B_r^{(k)}\frac{x^{-r}}{r!}=\frac{e^{\frac{\lambda_1^k}{x}}-e^{\frac{\lambda_2^k}{x}}}{4\sqrt{2}B_k}.
$$

Proof. In the previous theorem replacing x by x^{-1} gives the required identity. □

2.1 Partial sum and Binomial sum

Theorem 12 (Partial sum). For $k,n\in\mathbb{N}$, we have

$$
\sum_{i=1}^{n}B_i^{(k)}=\frac{B_n^{(k)}-B_{n+1}^{(k)}+1}{2(1-C_k)}.
$$

Proof. Using Binet's formula of the higher-order balancing numbers in $\sum_{i=1}^{n}B_i^{(k)}$, we have

$$
\begin{aligned}
&\sum_{i=1}^{n}\frac{1}{\lambda_1^k-\lambda_2^k}(\lambda_1^{ki}-\lambda_2^{ki})\\
&=\frac{1}{\lambda_1^k-\lambda_2^k}\left(\lambda_1^k\frac{(\lambda_1^k)^n-1}{\lambda_1^k-1}-\lambda_2^k\frac{(\lambda_2^k)^n-1}{\lambda_2^k-1}\right)\\
&=\frac{1}{\lambda_1^k-\lambda_2^k}\left[\frac{\big((\lambda_1^k)^{n+1}\lambda_2^k-\lambda_1^k\lambda_2^k-(\lambda_1^k)^{n+1}+\lambda_1^k\big)-\big((\lambda_2^k)^{n+1}\lambda_1^k-\lambda_1^k\lambda_2^k-(\lambda_2^k)^{n+1}+\lambda_2^k\big)}{(\lambda_1^k-1)(\lambda_2^k-1)}\right]\\
&=\frac{1}{\lambda_1^k-\lambda_2^k}\left[\frac{\Big((\lambda_1^k)^n-(\lambda_2^k)^n\Big)-\Big((\lambda_1^k)^{n+1}-(\lambda_2^k)^{n+1}\Big)+\Big(\lambda_1^k-\lambda_2^k\Big)}{(\lambda_1\lambda_2)^k-\lambda_1^k-\lambda_2^k+1}\right]\\
&=\frac{B_n^{(k)}-B_{n+1}^{(k)}+1}{2(1-C_k)}.
\end{aligned}
$$

□

Theorem 13 (Partial sum with even indexes). For $k,n\in\mathbb{N}$, we have

$$
\sum_{i=1}^{n}B_{2i}^{(k)}=\frac{B_2^{(k)}+B_{2n}^{(k)}-B_{2(n+1)}^{(k)}}{2-2C_{2k}}.
$$

Proof. We write

$$\begin{aligned}\sum_{i=1}^{n} B_{2i}^{(k)} &= \sum_{i=1}^{n} \frac{1}{\lambda_1^k - \lambda_2^k}\left(\lambda_1^{2ki} - \lambda_2^{2ki}\right)\\ &= \frac{1}{\lambda_1^k - \lambda_2^k}\left(\lambda_1^{2k}\frac{(\lambda_1^{2k})^n - 1}{\lambda_1^{2k} - 1} - \lambda_2^{2k}\frac{(\lambda_2^{2k})^n - 1}{\lambda_2^{2k} - 1}\right).\end{aligned}$$

After simplifying using Binet's formula, we get

$$\sum_{i=1}^{n} B_{2i}^{(k)} = \frac{B_2^{(k)} + B_{2n}^{(k)} - B_{2(n+1)}^{(k)}}{2 - 2C_{2k}}.$$

□

Theorem 14 (Partial sum with odd indexes). For $k, n \in \mathbb{N}$, we have

$$\sum_{i=0}^{n} B_{2i+1}^{(k)} = \frac{C_{2k(n+1)-1}}{C_{2k} - 1}.$$

Proof. Using the Binet's formula, we have

$$\begin{aligned}\sum_{i=0}^{n} B_{2i+1}^{(k)} &= \sum_{i=0}^{n} \frac{1}{\lambda_1^k - \lambda_2^k}\left(\lambda_1^{k(2i+1)} - \lambda_2^{k(2i+1)}\right)\\ &= \frac{1}{\lambda_1^k - \lambda_2^k}\left(\lambda_1^{k}\frac{(\lambda_1^{2k})^{n+1} - 1}{\lambda_1^{2k} - 1} - \lambda_2^{k}\frac{(\lambda_2^{2k})^{n+1} - 1}{\lambda_2^{2k} - 1}\right)\end{aligned}$$

$$\begin{aligned}&= \frac{1}{(\lambda_1^k - \lambda_2^k)(\lambda_1^k - 1)(\lambda_2^k - 1)}\Big[\Big(\lambda_1^k(\lambda_1^{2k})^{n+1}\lambda_2^{2k} - \lambda_1^k\lambda_2^{2k} - \lambda_1^k(\lambda_1^{2k})^{n+1} + \lambda_1^k\Big)\\ &\quad - \Big(\lambda_2^k(\lambda_2^{2k})^{n+1}\lambda_1^{2k} - \lambda_1^{2k}\lambda_2^k - \lambda_2^k(\lambda_2^{2k})^{n+1} + \lambda_2^k\Big)\Big]\\ &= \frac{1}{\lambda_1^k - \lambda_2^k}\left[\frac{\left(\lambda_1^k(\lambda_1^{2k})^n - \lambda_1^k(\lambda_1^{2k})^{n+1} - \lambda_2^k(\lambda_2^{2k})^n + \lambda_2^k(\lambda_2^{2k})^{n+1} + 2(\lambda_1^k - \lambda_2^k)\right)}{(\lambda_1^k - 1)(\lambda_2^k - 1)}\right] \quad (\text{by } \lambda_1\lambda_2 = 1)\\ &= \frac{1}{\lambda_1^k - \lambda_2^k}\left[\frac{\left((\lambda_1^{2k})^{n+1}(\lambda_2^k - \lambda_1^k) + (\lambda_1^{2k})^{n+1}(\lambda_2^k - \lambda_1^k) + 2(\lambda_1^k - \lambda_2^k)\right)}{(\lambda_1\lambda_2)^k - \lambda_1^k - \lambda_2^k + 1}\right]\\ &= \frac{C_{2k(n+1)-1}}{C_{2k} - 1}.\end{aligned}$$

□

Theorem 15 (Binomial sum). For the higher-order balancing numbers $\{B_n^{(k)}\}$, we have

1. $\sum_{n=0}^{s-1}\binom{s-1}{n}B_n^{(k)} = \dfrac{1}{4\sqrt{2}B_k}\left((1+\lambda_1^k)^{s-1} - (1+\lambda_2^k)^{s-1}\right)$.
2. $\sum_{n=0}^{s-1}(-1)^n\binom{s-1}{n}B_n^{(k)} = \dfrac{1}{4\sqrt{2}B_k}\left((1-\lambda_1^k)^{s-1} - (1-\lambda_2^k)^{s-1}\right)$.

Proof. 1. From Binet's formula (8), we write

$$\begin{aligned}\sum_{n=0}^{s-1}\binom{s-1}{n}B_n^{(k)} &= \sum_{n=0}^{s-1}\binom{s-1}{n}\frac{\lambda_1^{kn} - \lambda_2^{kn}}{\lambda_1^k - \lambda_2^k}\\ &= \frac{1}{\lambda_1^k - \lambda_2^k}\left(\sum_{n=0}^{s-1}\binom{s-1}{n}\lambda_1^{kn} - \sum_{n=0}^{s-1}\binom{s-1}{n}\lambda_2^{kn}\right)\\ &= \frac{1}{4\sqrt{2}B_k}\left((1+\lambda_1^k)^{s-1} - (1+\lambda_2^k)^{s-1}\right) \quad (\text{using the Binomial theorem}).\end{aligned}$$

2. From Binet's formula (8), we write

$$\begin{aligned}\sum_{n=0}^{s-1}(-1)^n\binom{s-1}{n}B_n^{(k)} &= \sum_{n=0}^{s-1}\binom{s-1}{n}\frac{(-\lambda_1^k)^n-(-\lambda_2^k)^n}{\lambda_1^k-\lambda_2^k}\\ &= \frac{1}{\lambda_1^k-\lambda_2^k}\left(\sum_{n=0}^{s-1}\binom{s-1}{n}(-\lambda_1^k)^n-\sum_{n=0}^{s-1}\binom{s-1}{n}(-\lambda_2^k)^n\right)\\ &= \frac{1}{4\sqrt{2}B_k}\left((1-\lambda_1^k)^{s-1}-(1-\lambda_2^k)^{s-1}\right) \quad \text{(using the Binomial theorem).} \qquad \square\end{aligned}$$

Theorem 16. If m and n are positive integers, then

1. $\sum_{r=0}^{2n}\binom{2n}{r}B_{2r}^{(k)} = \dfrac{1}{4\sqrt{2}B_k}\left((1+\lambda_1^{2k})^{2n}-(1+\lambda_2^{2k})^{2n}\right)$.
2. $\sum_{r=0}^{2n+1}\binom{2n+1}{r}B_{2r}^{(k)} = \dfrac{1}{4\sqrt{2}B_k}\left((1+\lambda_1^{2k})^{2n+1}-(1+\lambda_2^{2k})^{2n+1}\right)$.

Proof. 1. From Binet's formula (8), we write

$$\begin{aligned}\sum_{r=0}^{2n}\binom{2n}{r}B_{2r}^{(k)} &= \sum_{r=0}^{2n}\binom{2n}{r}\frac{\lambda_1^{2kr}-\lambda_2^{2kr}}{\lambda_1^k-\lambda_2^k}\\ &= \frac{1}{\lambda_1^k-\lambda_2^k}\left(\sum_{r=0}^{2n}\binom{2n}{r}\lambda_1^{2kr}-\sum_{r=0}^{2n}\binom{2n}{r}\lambda_2^{2kr}\right)\\ &= \frac{1}{\lambda_1^k-\lambda_2^k}\left((1+\lambda_1^{2k})^{2n}-(1+\lambda_2^{2k})^{2n}\right) \quad \text{(using the Binomial theorem)}\\ &= \frac{1}{4\sqrt{2}B_k}\left((1+\lambda_1^{2k})^{2n}-(1+\lambda_2^{2k})^{2n}\right).\end{aligned}$$

2. From Binet's formula (8), we write

$$\begin{aligned}\sum_{r=0}^{2n+1}\binom{2n+1}{r}B_{2r}^{(k)} &= \sum_{r=0}^{2n+1}\binom{2n+1}{r}\frac{\lambda_1^{2kr}-\lambda_2^{2kr}}{\lambda_1^k-\lambda_2^k}\\ &= \frac{1}{\lambda_1^k-\lambda_2^k}\left(\sum_{r=0}^{2n+1}\binom{2n+1}{r}\lambda_1^{2kr}-\sum_{r=0}^{2n+1}\binom{2n+1}{r}\lambda_2^{2kr}\right)\\ &= \frac{1}{\lambda_1^k-\lambda_2^k}\left((1+\lambda_1^{2k})^{2n+1}-(1+\lambda_2^{2k})^{2n+1}\right) \quad \text{(using the Binomial theorem)}\\ &= \frac{1}{4\sqrt{2}B_k}\left((1+\lambda_1^{2k})^{2n+1}-(1+\lambda_2^{2k})^{2n+1}\right). \qquad \square\end{aligned}$$

With the help of Binet's formula, the following identities can be easily established.

Theorem 17. For $n \geq 0$, the following identitites are verified:

1. $B_{-n}^{(k)} = -B_n^{(k)}$.
2. $B_{2n}^{(k)} = 2C_{kn}B_n^{(k)}$.
3. $B_{3n}^{(k)} = 32B_n^{(k)}(B_{kn}^2+3)$.
4. $B_{n\pm r}^{(k)} = B_n^{(k)}C_{kr} \pm B_r^{(k)}C_{kn}$.
5. $B_{n+r}^{(k)} + B_{n-r}^{(k)} = 2C_{kr}B_n^{(k)}$.
6. $B_{n+r}^{(k)} - B_{n-r}^{(k)} = 2C_{kn}B_r^{(k)}$.
7. $(B_{n+r}^{(k)})^2 - (B_{n-r}^{(k)})^2 = 4C_{kn}C_{kr}B_n^{(k)}B_r^{(k)}$.
8. $B_{n+r}^{(k)}B_{n-r}^{(k)} = (B_n^{(k)})^2 - (B_r^{(k)})^2$.

Proof. 1. Since $B_{-nk} = -B_{nk}$, so

$$B_{-n}^{(k)} = \frac{B_{-nk}}{B_k} = -\frac{B_{nk}}{B_k} = -B_n^{(k)}.$$

2. Using $B_{2k} = 2B_kC_k$, we can write $B_{2n}^{(k)} = \dfrac{B_{2kn}}{B_k} = 2C_{kn}\dfrac{B_{kn}}{B_k} = 2C_{kn}B_n^{(k)}$.

3. Taking into account the identity $B_{3k} = 32B_k^3 + 3B_k$, we have

$$B_{3n}^{(k)} = \frac{B_{3nk}}{B_k} = \frac{32B_{kn}^3 + 3B_{kn}}{B_k} = 32B_n^{(k)}(B_{kn}^2 + 3).$$

For identity 4, 5 and 6, along with Definition 1, use $B_{m\pm n} = B_mC_n \pm C_mB_n)$, $B_{a+b} + B_{a-b} = 2B_aC_b$ and $B_{a+b} - B_{a-b} = 2C_aB_b$, respectively.

7. Using $B_{a+b} + B_{a-b} = 2B_aC_b$ and $B_{a+b} - B_{a-b} = 2C_aB_b$, we have

$$\begin{aligned}
(B_{n+r}^{(k)})^2 - (B_{n-r}^{(k)})^2 &= \frac{(B_{k(n+r)})^2 - (B_{k(n-r)})^2}{(B_k)^2} \\
&= \frac{(B_{kn+kr} + B_{kn-kr})(B_{kn+kr} - B_{kn-kr})}{(B_k)^2} \\
&= \frac{4B_{kn}B_{kr}C_{kn}C_{kr}}{(B_k)^2} \\
&= 4C_{kn}C_{kr}B_n^{(k)}B_r^{(k)}.
\end{aligned}$$

8. Since $B_{a+b}B_{a-b} = B_a^2 - B_b^2$, so

$$B_{n+r}^{(k)}B_{n-r}^{(k)} = \frac{B_{kn+kr}B_{kn-kr}}{B_k^2} = \frac{B_{kn}^2 - B_{kr}^2}{B_k^2} = (B_n^{(k)})^2 - (B_r^{(k)})^2. \qquad \square$$

Theorem 18. If m and n are positive integers, then

1. $\sum_{r=1}^{n} B_{2mr}^{(k)} = \dfrac{B_{2mn}^{(k)} - B_{2m(n+1)}^{(k)} + B_{2m}^{(k)}}{2 - 2C_{2mk}}$.
2. $\sum_{r=1}^{n} (-1)^r B_{2mr}^{(k)} = \dfrac{(-1)^n\left(B_{2mn}^{(k)} + B_{2m(n+1)}^{(k)}\right) - B_{2m}^{(k)}}{2 + 2C_{2mk}}$.
3. $\sum_{r=1}^{n} (B_{mr}^{(k)})^2 = \dfrac{1}{C_{2k} - 1}\left(\dfrac{C_{2mk} + C_{2mkn} - C_{2mk(1+n)} - 1}{2(1 - C_{2mk})} - n\right)$.
4. $\sum_{r=1}^{n} (-1)^r (B_{mr}^{(k)})^2 = \begin{cases} \dfrac{1}{2(C_{2k} - 1)}\left[\frac{(-1)^n(C_{2mkn} + C_{2mk(n+1)}) - C_{2mk}}{C_{2mk} + 1}\right] & : n = \text{even} \\ \dfrac{1}{2(C_{2k} - 1)}\left[\frac{(-1)^n(C_{2mkn} + C_{2mk(n+1)}) - C_{2mk}}{C_{2mk} + 1} + 2\right] & : n = \text{odd.} \end{cases}$

Proof. 1. Using the Binet's formula (8), we have

$$
\begin{aligned}
\sum_{r=1}^{n} B_{2mr}^{(k)} &= \sum_{r=1}^{n} \frac{1}{\lambda_1^k-\lambda_2^k}(\lambda_1^{2mrk}-\lambda_2^{2mrk}) \\
&= \frac{1}{\lambda_1^k-\lambda_2^k}\left(\lambda_1^{2mk}\frac{(\lambda_1^{2mk})^n-1}{\lambda_1^{2mk}-1}-\lambda_2^{2mk}\frac{(\lambda_2^{2mk})^n-1}{\lambda_2^{2mk}-1}\right) \\
&= \frac{1}{\lambda_1^k-\lambda_2^k}\left[\frac{\left((\lambda_1^{2mk})^{n+1}\lambda_2^{2mk}-(\lambda_1\lambda_2)^{2mk}-(\lambda_1^{2mk})^{n+1}+\lambda_1^{2mk}\right)}{(\lambda_1^{2mk}-1)(\lambda_2^{2mk}-1)}\right] \\
&\quad -\frac{1}{\lambda_1^k-\lambda_2^k}\left[\frac{\left((\lambda_2^{2mk})^{n+1}\lambda_1^{2mk}-(\lambda_1\lambda_2)^{2mk}-(\lambda_2^{2mk})^{n+1}+\lambda_2^{2mk}\right)}{(\lambda_1^{2mk}-1)(\lambda_2^{2mk}-1)}\right] \\
&= \frac{1}{\lambda_1^k-\lambda_2^k}\left[\frac{\left((\lambda_1^{2mk})^n-(\lambda_2^{2mk})^n\right)-\left((\lambda_1^{2mk})^{n+1}-(\lambda_2^{2mk})^{n+1}\right)+\left(\lambda_1^{2mk}-\lambda_2^{2mk}\right)}{(\lambda_1\lambda_2)^{2mk}-\lambda_1^{2mk}-\lambda_2^{2mk}+1}\right] \\
&= \frac{B_{2mn}^{(k)}-B_{2m(n+1)}^{(k)}+B_{2m}^{(k)}}{2-2C_{2mk}}.
\end{aligned}
$$

2. Using the Binet's formula (8), we have

$$
\begin{aligned}
\sum_{r=1}^{n}(-1)^r B_{2mr}^{(k)} &= \sum_{r=1}^{n}\frac{(-1)^r}{\lambda_1^k-\lambda_2^k}(\lambda_1^{2mrk}-\lambda_2^{2mrk}) \\
&= \frac{1}{\lambda_1^k-\lambda_2^k}\left(\sum_{r=1}^{n}(-\lambda_1^{2mk})^r-\sum_{r=1}^{n}(-\lambda_2^{2mk})^r\right) \\
&= \frac{1}{\lambda_1^k-\lambda_2^k}\left(-\lambda_1^{2mk}\frac{1-(-\lambda_1^{2mk})^n}{1-(-\lambda_1^{2mk})}-(-\lambda_2^{2mk})\frac{1-(-\lambda_2^{2mk})^n}{1-(-\lambda_2^{2mk})}\right).
\end{aligned}
$$

After simplifying, we get

$$
\sum_{r=1}^{n}(-1)^r B_{2mr}^{k} = \frac{(-1)^n\left(B_{2mn}^{(k)}+B_{2m(n+1)}^{(k)}\right)-B_{2m}^{(k)}}{2+2C_{2mk}}.
$$

3. Using the Binet's formula (8) in $\sum_{r=1}^{n}(B_{mr}^{(k)})^2$, we have

$$
\begin{aligned}
\sum_{r=1}^{n}\left(\frac{\lambda_1^{mrk}-\lambda_2^{mrk}}{\lambda_1^k-\lambda_2^k}\right)^2 &= \frac{1}{(\lambda_1^k-\lambda_2^k)^2}\sum_{r=1}^{n}\left(\lambda_1^{2mkr}+\lambda_2^{2mkr}-2(\lambda_1\lambda_2)^{mkr}\right) \\
&= \frac{1}{(\lambda_1^k-\lambda_2^k)^2}\left(\lambda_1^{2mk}\frac{(\lambda_1^{2mk})^n-1}{\lambda_1^{2mk}-1}+\lambda_2^{2mk}\frac{(\lambda_2^{2mk})^n-1}{\lambda_2^{2mk}-1}-2n\right).
\end{aligned}
$$

After simplifying and rearrangement, it becomes

$$
\sum_{r=1}^{n}(B_{mr}^{(k)})^2 = \frac{1}{C_{2k}-1}\left(\frac{C_{2mk}+C_{2mkn}-C_{2mk(1+n)}-1}{2(1-C_{2mk})}-n\right).
$$

4. Using the Binet's formula of higher-order balancing number, we have

$$\begin{aligned}
\sum_{r=1}^{n}(-1)^r (B_{mr}^{(k)})^2 &= \sum_{r=1}^{n}(-1)^r \left(\frac{\lambda_1^{mrk}-\lambda_2^{mrk}}{\lambda_1^k-\lambda_2^k}\right)^2 \\
&= \frac{1}{(\lambda_1^k-\lambda_2^k)^2}\sum_{r=1}^{n}(-1)^r \Big(\lambda_1^{2mkr}+\lambda_2^{2mkr}-2(\lambda_1\lambda_2)^{mkr}\Big) \\
&= \frac{1}{(\lambda_1^k-\lambda_2^k)^2}\left[\sum_{r=1}^{n}(-\lambda_1^{2mk})^r+\sum_{r=1}^{n}(-\lambda_2^{2mk})^r-2\sum_{r=1}^{n}(-1)^r\right] \\
&= \frac{1}{(\lambda_1^k-\lambda_2^k)^2}\left[(-\lambda_1^{2mk})\frac{(-\lambda_1^{2mk})^n-1}{(-\lambda_1^{2mk})-1}+(-\lambda_2^{2mk})\frac{(-\lambda_2^{2mk})^n-1}{(-\lambda_2^{2mk})-1}-2\sum_{r=1}^{n}(-1)^r\right] \\
&= \frac{1}{2(C_{2k}-1)}\left[\frac{(-1)^n(C_{2mkn}+C_{2mk(n+1)})-C_{2mk}}{C_{2mk}+1}-2\sum_{r=1}^{n}(-1)^r\right].
\end{aligned}$$

Since $\sum_{r=1}^{n}(-1)^r = 0$ and -1 for $n =$ even and odd, respectively, so

$$\sum_{r=1}^{n}(-1)^r (B_{mr}^{(k)})^2 = \begin{cases} \dfrac{1}{2(C_{2k}-1)}\left[\dfrac{(-1)^n(C_{2mkn}+C_{2mk(n+1)})-C_{2mk}}{C_{2mk}+1}\right] & : \ n = \text{even} \\ \dfrac{1}{2(C_{2k}-1)}\left[\dfrac{(-1)^n(C_{2mkn}+C_{2mk(n+1)})-C_{2mk}}{C_{2mk}+1}+2\right] & : \ n = \text{odd}. \end{cases}$$

□

Conclusion

In this article, we introduce a new extension of the classical balancing numbers, which generalizes the balancing numbers in a different way. For this integer sequence, we present a Binet-like formula and various algebraic properties, such as generating functions (both ordinary and exponential), various partial and binomial sums, etc. In addition, we present several identities in connection with the existing balancing and Lucas-balancing numbers. In particular, for $k = 1$ it gives the classical balancing numbers (A001109) and for $k = 2$ it gives the sequence A029547.

Following this study, this sequence can be extended and applied in many directions, such as hypercomplex numbers, polynomials, quaternions, octonions, matrix theory, transforms, etc. We are currently working on hypercomplex/2^m-ions numbers using this new sequence.

Acknowledgments

The second author acknowledges the DST, Govt. of India, for INSPIRE SHE scholarship.

Author Contributions

K.P.: Conceptualization, Methodology, Formal Analysis, Supervision; **I.:** Investigation, Formal Analysis, Validation; **M.K.:** Conceptualization, Formal Analysis, Validation, Manuscript preparation; All authors contributed equally to this work and approved the final submission.

Conflict of Interest

The authors declare no conflict of interest.

References

1 Behera, A., & Panda, G. (1999). On the square roots of triangular numbers. *Fibonacci Quarterly, 37*(2), 98–105. https://doi.org/10.1080/00150517.1999.12428864

2 Randić, M., Morales, D.A., & Araujo, O. (1996). Higher-order Fibonacci numbers. *Journal of Mathematical Chemistry, 20*, 79–94. https://doi.org/10.1007/BF01165157

3 Ozvatan, M. (2018). Generalized Golden–Fibonacci Calculus and Applications. *Master's thesis.* Izmir: Izmir Institute of Technology.

4 Uysal, M., & Özkan, E. (2022). Higher-order Jacobsthal–Lucas quaternions. *Axioms, 11*(12), Article 671. https://doi.org/10.3390/axioms11120671

5 Özkan, E., & Uysal, M. (2023). On quaternions with higher order Jacobsthal numbers components. *Gazi University Journal of Science, 36*(1), 336–347. https://doi.org/10.35378/gujs.1002454

6 Kızılateş, C., & Kone, T. (2021). On higher order Fibonacci hyper complex numbers. *Chaos, Solitons & Fractals, 148*, Article 111044. https://doi.org/10.1016/j.chaos.2021.111044

7 Özimamoğlu, H. (2023). On hyper complex numbers with higher order Pell numbers components. *The Journal of Analysis, 31*, 2443–2457. https://doi.org/10.1007/s41478-023-00579-2

8 Kızılateş, C., Du, W.S., & Terzioğlu, N. (2024). On Higher-Order Generalized Fibonacci Hybrinomials: New Properties, Recurrence Relations and Matrix Representations. *Mathematics, 12*(8), Article 1156. https://doi.org/10.3390/math12081156

9 Kumari, M., Prasad, K., & Mohanta, R. (2024). Algebra of quaternions and octonions involving higher order Mersenne numbers. *Proceedings of the Indian National Science Academy, 91*, 333–342. https://doi.org/10.1007/s43538-024-00297-8

10 Prasad, K., Kumari, M., & Tanti, J. (2023). Generalized k-balancing and k-Lucas balancing numbers and associated polynomials. *Kyungpook Mathematical Journal, 63*(4), 539–550. https://doi.org/10.5666/KMJ.2023.63.4.539

11 Ray, P. (2017). Balancing polynomials and their derivatives. *Ukrainian mathematical journal, 69*(4), 646–663. https://doi.org/10.1007/s11253-017-1386-7

12 Frontczak, R. (2019). On balancing polynomials. *Applied Mathematical Sciences, 13*(2), 57–66. https://doi.org/10.12988/ams.2019.812183

13 Ray, P.K. (2015). Balancing and Lucas-balancing sums by matrix methods. *Mathematical Reports, 17*(2), 225–233.

14 Ray, P.K. (2017). On the properties of k-balancing and k-Lucas-balancing numbers. *Acta et commentationes universitatis tartuensis de mathematica, 21*(2), 259–274. https://doi.org/10.12697/ACUTM.2017.21.18

15 Frontczak, R., & Goy, T. (2020). Combinatorial sums associated with balancing and Lucas-balancing polynomials. *Annales Mathematicae et Informaticae, 52*, 97–105. https://doi.org/10.33039/ami.2020.10.002

16 Frontczak, R., & Prasad, K. (2023). Balancing polynomials, Fibonacci numbers and some new series for π. *Mediterranean Journal of Mathematics, 20*(4), Article 207. https://doi.org/10.1007/s00009-023-02413-2

17 Liptai, K., Luca, F., Pintér, Á., & Szalay, L. (2009). Generalized balancing numbers. *Indagationes mathematicae, 20*(1), 87–100. https://doi.org/10.1016/S0019-3577(09)80005-0

18 Özkoc, A. (2015). Tridigonal matrices via k-balancing number. *British Journal of Mathematics and Computer Science, 10*(4), 1–11. https://doi.org/10.9734/BJMCS/2015/19014

19 Panda, G.K. (2006). Some fascinating properties of balancing numbers. In *Fibonacci Numbers and Their Applications, 10. Proceedings of the International Conference.*

20 Prasad, K., Kumari, M., Mohanty, R., & Mahato, H. (2026). Spinor algebra of k-balancing and k-Lucas-balancing numbers. *Journal of Algebra and Its Applications, 25*(04), Article 2550367. https://doi.org/10.1142/S0219498825503670

*Author Information**

Kalika Prasad — Doctor of Mathematical Sciences, Assistant Professor, Department of Mathematics, Government Engineering College Bhojpur (Under DSTTE Bihar), Bihar, 802301, India; e-mail: *klkaprsd@gmail.com*; https://orcid.org/0000-0002-3653-5854

Indubala — Bachelor of Science (Research) Student, Indian Institute of Science, Bengaluru, Karnataka, 560012, India; e-mail: *indubala@iisc.ac.in*

Munesh Kumari (*corresponding author*) — Doctor of Mathematical Sciences, Assistant Professor, Department of Mathematics, Government Engineering College Bhojpur (Under DSTTE Bihar), Bihar, 802301, India; e-mail: *muneshnasir94@gmail.com*; https://orcid.org/0000-0002-6541-0284

*Authors' names are presented in the following order: first name, middle name (if any), last name.

https://doi.org/10.31489/2026M1/217-227 *Research article*

Numerical Scheme for Singularly Perturbed Differential Equations with Small Shifts Using Non-Polynomial Quartic Spline

K. Ragula[1,*], G.B.S.L. Soujanya[2]

[1]*Rajiv Gandhi University of Knowledge Technologies, Basar, India;*
[2]*Kakatiya University, Warangal, India*
(E-mail: rksm39@gmail.com, gbslsoujanya@gmail.com)

In this paper, a non-polynomial quartic spline technique with a fitting parameter is applied to solve a second-order singularly perturbed differential-difference equation (SPDDE) having small shifts. Taylor series expansion is employed for the delayed and advanced terms in the considered problem to produce a singularly perturbed differential equation (SPDE), and then a non-polynomial quartic spline technique is applied. To manage the layer structure, a fitting parameter is introduced in the proposed computational method; based on the step size, this parameter is evaluated using the theory of singular perturbation theory. Two model examples with left-end boundary layer behavior are considered to theory validate the theoretical finding. The convergence method is analyzed, and the solutions are reported in terms of maximum absolute error with quadratic convergence rate using the fitting parameter. For comparison, solutions without the fitting parameter are reported for test problems. The graphs depict the layer profile for the values of perturbation and shift parameters using the fitting factor and the oscillations without it. The proposed scheme gives uniformly convergent and valid results.

Keywords: singularly perturbed differential-difference equation, fitting parameter, non-polynomial spline, small shifts, boundary layer, truncation Error, maximum absolute error, convergence.

2020 Mathematics Subject Classification: 65L10, 65L11, 65L12.

Introduction

The differential equation in which the highest derivative is multiplied by a small parameter and having delay/advanced on the terms different from the highest derivative is known as a singularly perturbed differential-difference equation (SPDDE). A challenging and frequent task in the mathematical modeling of many physical, engineering, and biological problems is finding solutions for SPDDEs in the interval of boundary conditions. Such problems include the initial exit time problem in neuronal variability activation models [1] and oscillations of the human pupil light reflex with delayed and mixed responses [2]. A comprehensive overview of SPDDEs is given in [3]. The efficient approximation schemes SCEM and MMAE to solve SPDDE were introduced in [4]. The authors suggested a hybrid technique and a midpoint upwind strategy for inside and outside the boundary layer region on the Shishkin mesh in [5]. In [6], the researchers proposed a numerical method for solving SPDDEs with small and large delays using a non-polynomial spline. In [7], the authors developed two adaptive methods based on the r-refinement strategy to solve SPDDE with mixed parameters. In [8, 9], the authors suggested spline methods with fitting factor to solve the problem of SPDDEs. An exponentially fitted spline method was proposed to solve SPDDE with delay in the convection term in [10]. The authors of [11] came up with a way to solve the singularly perturbed boundary value problems with mixed shifts using non polynomial splines. In [12], the authors developed a uniform convergent computational technique for solving singularly perturbed delay reaction-diffusion equations. The authors

*Corresponding author. *E-mail: rksm39@gmail.com*
Received: 30 August 2024; *Accepted:* 18 December 2025.

of [13, 14] proposed numerical methods for solving SPDDE with mixed shifts using non-polynomial cubic splines. The fourth-order finite difference method (FDM) with a fitting parameter is suggested to solve SPDDE with mixed shifts in [15]. The mixed finite difference method (FDM) was proposed to solve SPDDE with mixed shifts in [16]. The parametric spline approach developed by the authors in [17] is used to solve differential-difference equations (DDEs) with mixed parameters having twin layers. In [18], the authors proposed a second-order computational approach to solving SPDDEs using Stormer's method.

1 Description of the Problem

Consider the following linear SPDDE with mixed shifts

$$\varepsilon u''(\vartheta) + p(\vartheta)\, u'(\vartheta - \delta) + q(\vartheta)\, u(\vartheta + \eta) + r(\vartheta)\, u(\vartheta) = s(\vartheta), \quad 0 < \vartheta < 1 \tag{1}$$

with boundary conditions

$$u(\vartheta) = \varphi(\vartheta), \ -\delta \le \vartheta \le 0; \quad u(\vartheta) = \psi(\vartheta), \ 1 \le \vartheta \le 1 + \eta, \tag{2}$$

where $0 < \varepsilon \ll 1$ and $p(\vartheta),\ q(\vartheta),\ r(\vartheta),\ s(\vartheta),\ \varphi(\vartheta)$ and $\psi(\vartheta)$ are sufficiently smooth functions on $(0,1)$ and $0 < \delta = o(\varepsilon),\ 0 < \eta = o(\varepsilon),\ \delta$ is the delay parameter, η is the advanced parameter.

The Taylor series expansions of $u'(\vartheta - \delta),\ u(\vartheta + \eta)$ about the point ϑ, we have

$$u'(\vartheta - \delta) = u'(\vartheta) - \delta u''(\vartheta) + o\left(\delta^2\right); \quad u(\vartheta + \eta) = u(\vartheta) + \eta u'(\vartheta) + \frac{\eta^2}{2} u''(\vartheta) + o\left(\eta^3\right). \tag{3}$$

Using Eq. (3) in Eq. (1), we get

$$\varepsilon u''(\vartheta) + a(\vartheta)\, u'(\vartheta) + b(\vartheta)\, u(\vartheta) = f(\vartheta), \quad 0 < \vartheta < 1 \tag{4}$$

with the boundary constraints $u(0) = \varphi_0 = \varphi(0)$ and $u(1) = \psi_1 = \psi(1)$,

where $a(\vartheta) = \left(\dfrac{p(\vartheta) + \eta q(\vartheta)}{1 - \frac{p(\vartheta)\delta}{\varepsilon} + \frac{q(\vartheta)\eta^2}{2\varepsilon}}\right)$, $b(\vartheta) = \left(\dfrac{q(\vartheta) + r(\vartheta)}{1 - \frac{p(\vartheta)\delta}{\varepsilon} + \frac{q(\vartheta)\eta^2}{2\varepsilon}}\right)$ and $f(\vartheta) = \left(\dfrac{s(\vartheta)}{1 - \frac{p(\vartheta)\delta}{\varepsilon} + \frac{q(\vartheta)\eta^2}{2\varepsilon}}\right)$.

2 Quartic Non-Polynomial Spline Approach

The interval $[0,1]$ partitioned into $\mathcal{N}$ sub intervals of equal length with constant step size h. Let $0 = \vartheta_0 < \vartheta_1 < \ldots < \vartheta_{\mathcal{N}} = 1$ be the $\mathcal{N}$ grid points. Then we have $h = \frac{1}{\mathcal{N}}$ and $\vartheta_i = ih,\ i = 0, 1, 2, \ldots, \mathcal{N}$.

In [19], the authors explained about the non-polynomial quartic spline and defined in $[\vartheta_i, \vartheta_{i+1}]$, $i = 0, 1, \ldots, \mathcal{N} - 1$ the spline is of the form

$$\mathcal{G}_i(\vartheta) = p_i(\vartheta - \vartheta_i)^2 + q_i(\vartheta - \vartheta_i) + r_i \sin\tau(\vartheta - \vartheta_i) + s_i \cos\tau(\vartheta - \vartheta_i) + t_i, \tag{5}$$

where $p_i,\ q_i,\ r_i,\ s_i$ and t_i are unknown constants and the function $\mathcal{G}_i(\vartheta)$ interpolates $u(\vartheta_i)$ at the points ϑ_i by depending on arbitrary parameter τ and reducing to quartic spline in $[0,1]$ as $\tau \to 0$.

To examine the coefficients $p_i,\ q_i,\ r_i,\ s_i$ and t_i in Eq. (5) in terms of $u_i,\ u_{i+1},\ \mathcal{M}_i,\ \mathcal{M}_{i+1}, \mathcal{F}_i$ and $\mathcal{F}_{i+1}$, we define

$$\mathcal{G}_i(\vartheta_i) = u_i, \ \ \mathcal{G}_i(\vartheta_{i+1}) = u_{i+1}, \ \ \mathcal{G}_i''(\vartheta_i) = \mathcal{M}_i, \ \ \mathcal{G}_i''(\vartheta_{i+1}) = \mathcal{M}_{i+1}, \ \ \mathcal{G}_i^{(4)}(\vartheta_i) = \frac{1}{2}(\mathcal{F}_i + \mathcal{F}_{i+1}).$$

By using the conditions, we calculate the coefficients in Eq.(5) as

$$\begin{cases} p_i = \frac{\mathcal{M}_i}{2} + \frac{\mathcal{F}_i+\mathcal{F}_{i+1}}{4\tau^2}, \\ q_i = \frac{1}{h}(\ u_{i+1} - u_i) - (\frac{1}{h\tau^2} + \frac{h}{2})\mathcal{M}_i + \frac{1}{h\tau^2}\ \mathcal{M}_{i+1} - \frac{h}{4\tau^2}\left(\mathcal{F}_i + \mathcal{F}_{i+1}\right), \\ r_i = \frac{1}{\tau^2 \sin\omega}\left(\mathcal{M}_i - \mathcal{M}_{i+1}\right) + \frac{1-\cos\omega}{2\tau^4 \sin\omega}\left(\mathcal{F}_i + \mathcal{F}_{i+1}\right), \\ s_i = \frac{\mathcal{F}_i\ +\ \mathcal{F}_{i+1}}{2\tau^4}, \\ t_i = u_i - \frac{1}{2\tau^4}\left(\mathcal{F}_i + \mathcal{F}_{i+1}\right), \end{cases}$$

where $\tau h = \omega$.

Using continuity of first and third derivatives, $\mathcal{G}_{i-1}{}^{(m)}\left(\vartheta_i\right) = \mathcal{G}_i{}^{(m)}\left(\vartheta_i\right),\ m = 1, 3$ then, we get relations

$$\frac{4h\tau^3 \sin\omega}{2\left(1-\cos\omega\right) - h\tau\sin\omega}\left[\frac{u_{i+1} - 2u_i + u_{i-1}}{h^2}\right] + \left(2\mathcal{F}_i + \mathcal{F}_{i+1} + \mathcal{F}_{i-1}\right)$$

$$= \frac{2\tau\left(2h\tau\cos\omega + h^2\tau^2\sin\omega - 2\sin\omega\right)}{h\left[2\left(1-\cos\omega\right) - h\tau\sin\omega\right]}\mathcal{M}_{i-1} + \frac{2\tau\left(4\sin\omega + h^2\tau^2\sin\omega - 2h\tau\left(\cos\omega+1\right)\right)}{h\left[2\left(1-\cos\omega\right) - h\tau\sin\omega\right]}\mathcal{M}_i \quad (6)$$

$$+ \frac{4\tau\left(h\tau - \sin\omega\right)}{h\left[2\left(1-\cos\omega\right) - h\tau\sin\omega\right]}\mathcal{M}_{i+1},$$

$$\left(2\mathcal{F}_i + \mathcal{F}_{i+1} + \mathcal{F}_{i-1}\right) = \frac{2\tau^2\cos\omega}{\left(1-\cos\omega\right)}\mathcal{M}_{i-1} + \frac{2\tau^2\left(\cos\omega+1\right)}{\left(1-\cos\omega\right)}\mathcal{M}_i + \frac{2\tau^2}{\left(1-\cos\omega\right)}\mathcal{M}_{i+1}. \quad (7)$$

Substituting Eq. (7) in Eq. (6), we get the consistent relation

$$u_{i-1} - 2u_i + u_{i+1} = h^2\left[\alpha\left(\mathcal{M}_{i-1} + \mathcal{M}_{i+1}\right) + 2\beta\mathcal{M}_i\right], \quad i = 1, 2, \ldots, \mathcal{N}-1, \quad (8)$$

where

$$\alpha = \frac{\omega^2 - 2\left(1-\cos\omega\right)}{2\omega^2\left(1-\cos\omega\right)}, \quad \beta = \frac{4\left(1-\cos\omega\right) + \omega^2\left(1-3\cos\omega\right)}{4\omega^2\left(1-\cos\omega\right)}.$$

If $h \to 0$, then $\omega = h\tau \to 0$. Thus, using L'Hospital's rule, we have $(\beta, \alpha) \to \left(\frac{5}{12}, \frac{1}{12}\right)$.

3 Numerical Algorithm

At each ϑ_i, Eq. (4) can be written as

$$\varepsilon u_i'' = -a\left(\vartheta_i\right)u_i' - b\left(\vartheta_i\right)u_i + f\left(\vartheta_i\right).$$

Using $\mathcal{G}_i''\left(\vartheta_i\right) = \mathcal{M}_i = u_i''$ in above equation, we get

$$\varepsilon\mathcal{M}_j = -a_j\left(\vartheta_i\right)u_i' - b_j\left(\vartheta_i\right)u_i + f_j\left(\vartheta_i\right) \quad \text{for} \quad j = i,\ i \pm 1. \quad (9)$$

Using Eq. (9) in Eq. (8) and then using u_j', for $j = i-1,\ i,\ i+1$

$$u_i' \approx \frac{1}{2h}\left(u_{i+1} - u_{i-1}\right),\ u_{i+1}' \approx \frac{1}{2h}\left(-\ 4u_{\ i} + u_{i-1} + 3u_{\ i+1}\right)$$

and

$$u_{i-1}' \approx \frac{1}{2h}\left(+\ 4u_i - 3\ u_{i-1} - u_{i+1}\right),$$

$$\frac{\varepsilon}{h^2}(u_{i+1} - 2u_i + u_{i-1}) = -\alpha a_{i-1}\frac{\left(-u_{i+1} + 4u_i - 3\ u_{i-1}\right)}{2h} - 2\beta a_i\frac{\left(u_{i+1} - u_{i-1}\right)}{2h}$$

$$-\alpha a_{i+1}\frac{\left(u_{i-1} - 4u_{\ i} + 3u_{i+1}\right)}{2h} - \alpha b_{i-1}\ u_{i-1} - 2\beta b_i u_i - \alpha\ u_{i+1} + \left(\alpha(f_{i-1} + f_{i+1}) + 2\beta f_i\ \right).$$

To control the oscillations and increase the accuracy of the solution, we introduce the fitting parameter $\sigma\ (\rho)$ in the proposed approach then, we have

$$\frac{\varepsilon\sigma\ (\rho)}{h^2}(u_{i+1}-2u_i+u_{i-1})=-\alpha a_{i-1}\frac{(4u_i-u_{i+1}-3u_{i-1})}{2h}-2\beta a_i\frac{(u_{i+1}-u_{i-1})}{2h}$$

$$-\alpha a_{i+1}\frac{(u_{i-1}-4u\ _i+3u_{i+1})}{2h}-\alpha b_{i-1}u_{i-1}-2\beta b_iu_i-\alpha b_{i+1}u_{i+1}+\left(\alpha f_{\,i-1}+2\beta f_i+\ \alpha f_{\,i+1}\right). \quad (10)$$

Eq. (10) can be written as

$$E_iu_{i-1}+F_iu_i+G_iu_{i+1}=H_i,\quad i=1,2,\ldots,\mathcal{N}-1, \quad (11)$$

where

$$E_i=\varepsilon\sigma-h\frac{3\alpha a_{i-1}}{2}-h\beta a_i+h\frac{\alpha a_{i+1}}{2}+h^2\alpha b_{i-1},$$

$$F_i=-2\sigma\varepsilon+2\alpha ha_{i-1}-2\alpha ha_{i+1}+2h^2\beta b_i,$$

$$G_i=\varepsilon\sigma-h\frac{\alpha a_{i-1}}{2}+h\beta a_i+h\frac{3\alpha a_{i+1}}{2}+h^2\alpha b_{i+1},$$

$$H_i=h^2\left(\alpha(f_{i-1}+f_{i+1})\ +2\beta f_i\right).$$

With the help of Thomas algorithm and boundary conditions $u\,(0)=\varphi_0$, $u\,(1)=\psi_1$ Eq. (11) can be solved.

To calculate fitting parameter from singular perturbations theory, an approximation for the solution of Eq. (4)

$$u\,(\vartheta)=u_0\,(\vartheta)+\frac{a\,(0)}{a\,(\vartheta)}\left(\varphi_0-u_0\,(0)\right)\exp^{-\int_0^\vartheta\left(\frac{a(\vartheta)}{\varepsilon}\ \right)d\vartheta}+o\,(\varepsilon)\,, \quad (12)$$

where $u_0\,(\vartheta)$ is the solution of

$$a\,(\vartheta)\,u_0'\,(\vartheta)+b\,(\vartheta)\,u_0\,(\vartheta)=f\,(\vartheta)\,,\quad u_0\,(1)=\psi_1.$$

If we expand $a\,(\vartheta)$ and $b\,(\vartheta)$ about the point zero using Taylor's series, then Eq. (12) becomes

$$u\,(\vartheta)=u_0\,(\vartheta)+\left(\varphi_0-u_0\,(0)\right)\exp^{-\left(\frac{a(\vartheta)}{\varepsilon}\ \right)\vartheta}+o\,(\varepsilon)\,. \quad (13)$$

From Eq. (13), we have

$$\lim_{h\to 0}u\,(ih)=u_0\,(0)+\exp^{-a(\vartheta_i)\rho}\left(\varphi_0-u_0\,(0)\right).$$

These limit values used in Eq. (10), we obtain the $\sigma\,(\rho)$

$$\sigma\,(\rho)=\rho\,(\beta+\alpha)\,a_i\coth\left(\frac{a_i\rho}{2}\right),\ \text{where}\ \ \rho=\frac{h}{\varepsilon}.$$

4 Convergence Analysis

The local truncation error estimate for the computational scheme of Eq. (11) is

$$T\,(h)=h^2\left[1-2\,(\beta+\alpha)\right]\varepsilon u_i''+$$

$$h^4\left[\alpha\left(b_i''u_i+a_i''u_i'+2b_i'u_i'-2a_i'u_i''+b_iu_i''+\frac{a_i}{3}u_i'''-f_i''\right)+\frac{\beta a_i}{3}u_i'''\right]+o\left(h^6\right).$$

Hence, the truncation error of order four as $(\alpha,\beta)\to\left(\frac{1}{12},\frac{5}{12}\right)$.

Using the boundary conditions in Eq. (2), the matrix form of Eq. (11) is

$$(\mathbb{Q}+\mathbb{R})\,U+\widetilde{M}+\mathbb{T}\,(h)=O, \tag{14}$$

where

$$\mathbb{Q}=\begin{bmatrix} -2\sigma\varepsilon & \sigma\varepsilon & 0 & 0 & ... & 0\\ \sigma\varepsilon & -2\sigma\varepsilon & \sigma\varepsilon & .. & ... & 0\\ 0 & .. & .. & .. & ... & 0\\ .. & .. & .. & .. & .. & ..\\ .. & .. & .. & .. & .. & ..\\ 0 & .. & .. & .. & \sigma\varepsilon & -2\sigma\varepsilon \end{bmatrix},$$

$$\mathbb{R}=\begin{bmatrix} v_1 & w_1 & 0 & 0 & .... & 0\\ x_2 & v_2 & w_2 & 0 & .... & 0\\ 0 & x_3 & v_3 & w_3 & .... & 0\\ .. & .. & .. & .. & .. & ..\\ .. & .. & .. & .. & .. & ..\\ 0 & .. & .. & 0 & x_{\mathcal{N}-1} & v_{\mathcal{N}-1} \end{bmatrix},$$

with

$$x_i=-\frac{3\alpha h a_{i-1}}{2}-\beta h a_i+\frac{\alpha h a_{i+1}}{2}+h^2\alpha b_{i-1},\quad v_i=2\alpha h a_{i-1}-2\alpha h a_{i+1}+2h^2\beta b_i,$$

$$w_i=-\frac{\alpha h a_{i-1}}{2}+\beta h a_i+\frac{3\alpha h a_{i+1}}{2}+h^2\alpha b_{i+1},\ \forall\, i=1,2,...,\mathcal{N}-1,$$

and

$$\widetilde{M}\ =\ [m_1+(\sigma\varepsilon+x_1)\,\varphi_0,m_2,m_3\ ,\dots,m_{\mathcal{N}-2},m_{\mathcal{N}-1}+(\varepsilon\sigma+w_{\mathcal{N}-1})\,\psi_1]^T\,,$$

with $m_i=h^2\left(\alpha(f_{i-1}+f_{i+1})+2\beta f_i\right)$ for $i=1,2,\dots,\mathcal{N}-1,\ \mathbb{T}\,(h)=O\left(h^4\right);\,U=[U_1,U_2,...,U_{\mathcal{N}-1}]^T$, $\mathbb{T}\,(h)=[\bar{t}_1,\bar{t}_2,...,\bar{t}_{\mathcal{N}-1}]^T,\ O=[0,0,...,0]^T$are corresponding vectors of Eq. (14).

Let $u=[u_1,u_2,...,u_{\mathcal{N}-1}]^T\cong U$ which satisfies the equation

$$(\mathbb{Q}+\mathbb{R})\,u+\widetilde{M}=0. \tag{15}$$

If $e_i=u_i-U_i$, $i=1,2,\dots,\mathcal{N}-1$ denote discretization error, then $\widetilde{E}=[\bar{e}_1,\bar{e}_2,...,\bar{e}_{\mathcal{N}-1}]^T=u-U$.

Subtracting Eq. (14) from Eq. (15), we obtain

$$(\mathbb{Q}+\mathbb{R})\,E=\mathbb{T}(h). \tag{16}$$

Let $|a_i|\le P_1,\ |b_i|\le P_2$ so that if $R_{i,j}$ is the $(i,j)^{th}$ element of matrix $\mathbb{R}$, then

$$|R_{i,i+1}|=|w_i|\le\varepsilon+(h\alpha+h\beta)\,P_1+h^2\alpha P_2,\quad i=1,2,\dots,\mathcal{N}-2,$$

$$|R_{i,\ i-1}|=|x_i|\le\varepsilon+(h\alpha+h\beta)\,P_1+h^2\alpha P_2,\quad i=2,3\dots,\mathcal{N}-1.$$

Thus, for relatively small h $(h\to 0)$, we observe that

$$|R_{i,i+1}|<\varepsilon,\quad \forall\, i=1,2,\dots,\mathcal{N}-2,$$
$$|R_{i,i+1}|<\varepsilon,\quad \forall\, i=2,3,\dots,\mathcal{N}-1.$$

Hence $(\mathbb{Q}+\mathbb{R})$ is irreducible [20].

Let $\mathbb{S}_i$ be sum of i^{th} row elements of the matrix $(\mathbb{Q}+\mathbb{R})$, then

$$\mathbb{S}_i=-\sigma\varepsilon+h\frac{3\alpha a_{i-1}}{2}+\beta h a_i-h\frac{\alpha a_{i+1}}{2}+h^2\left(\alpha b_{i+1}+2\beta b_i\right)\ \text{for}\ \ i=1,$$

$$\mathbb{S}_i = h^2\left(\alpha(b_{i-1}+b_{i+1})+2\beta b_i\right) \text{ for } \; i=2,3,\ldots,\mathcal{N}-2,$$

$$\mathbb{S}_i = -\sigma\varepsilon + h\frac{\alpha a_{i-1}}{2} - \beta h a_i - h\frac{3\alpha a_{i+1}}{2} + h^2\left(\alpha b_{i-1}+2\beta b_i\right) \text{ for } \; i=\mathcal{N}-1.$$

Let

$$P_{1*} = \min_{1\ \leq i\leq\ \mathcal{N}-1}|a_i|,\quad P_1{}^* = \max_{1\ \leq i\ \leq \mathcal{N}}|a_i|\,,\quad P_{2*} = \min_{1\ \leq i\ \leq \mathcal{N}-1}|b_i|,\quad P_2{}^* = \max_{1\ \leq i\ \leq \mathcal{N}}|b_i|,$$

then $0 \leq P_{1*} \leq P_1 \leq P_1{}^*,\ 0 \leq P_{2*} \leq P_2 \leq P_2{}^*$.

If h sufficiently small h $(h \to 0)$, then $(\mathbb{Q}+\mathbb{R})$ is monotone [20]. Hence, $(\mathbb{Q}+\mathbb{R})^{-1}$ exists and $(\mathbb{Q}+\mathbb{R})^{-1} \geq 0$. Therefore, from Eq. (16), we have

$$||E|| \leq \ ||\mathbb{Q}+\mathbb{R}||^{-1}||\mathbb{T}||. \tag{17}$$

Let $(\mathbb{Q}+\mathbb{R})^{-1}_{i,k}$ be the $(i,k)^{th}$ element of $(\mathbb{Q}+\mathbb{R})^{-1}$ and define

$$||\mathbb{Q}+\mathbb{R}||^{-1} = \max_{1\leq\ i\leq\ \mathcal{N}-1}\sum_{k=1}^{\mathcal{N}-1}(\mathbb{Q}+\mathbb{R})^{-1}_{i,k}\,,\ \text{and}\quad ||\mathbb{T}(h)|| = \max_{1\leq\ i\leq\ \mathcal{N}-1}|\mathbb{T}_i|.$$

Since $(\mathbb{Q}+\mathbb{R})^{-1}_{i,k} \geq 0$ and $\sum_{k=1}^{\mathcal{N}-1}(\mathbb{Q}+\mathbb{R})^{-1}_{i,k}\,,\ \mathbb{S}_k = 1$ for $i=1,2,...,\mathcal{N}-1$, hence,

$$(\mathbb{Q}+\mathbb{R})^{-1}_{i,k} \leq \frac{1}{\mathbb{S}_i} < \frac{1}{h^2P_2},\quad i=1, \tag{18a}$$

$$(\mathbb{Q}+\mathbb{R})^{-1}_{i,k} \leq \frac{1}{\mathbb{S}_i} < \frac{1}{h^2P_2},\quad i=\mathcal{N}-1. \tag{18b}$$

Furthermore,

$$\sum_{k=1}^{\mathcal{N}-1}(\mathbb{Q}+\mathbb{R})^{-1}_{i,k} \leq \frac{1}{\min\limits_{2\leq i\leq\mathcal{N}-2}\mathbb{S}_i} < \frac{1}{h^2P_2}. \tag{18c}$$

From Eq. (17) and Eq. (18a)–(18c), we get

$$||E|| \leq O\ (h^2).$$

This illustrates the quadratic rate of convergence for Eq. (11), as $(\alpha,\beta) \to \left(\frac{1}{12},\frac{5}{12}\right)$.

5 Numerical Illustrations

To examine the quality and robustness of the suggested technique, we solved two different test problems and reported the numerical results in the form of maximum absolute errors (MAEs) with and without fitting parameter and the computed rates of convergence (ROC) in the tables. The MAEs are calculated with the double mesh principle because the exact solutions to test problems are unknown.

$$E_{\mathcal{N}} = \max_{0\leq i\leq\mathcal{N}}\left|u_{2i}{}^{2\mathcal{N}} - u_i{}^{\mathcal{N}}\right|,$$

where $u_i{}^{\mathcal{N}}$ and $u_{2i}{}^{2\mathcal{N}}$ are the numerical solutions of the problem for $\mathcal{N}$ and $2\mathcal{N}$ mesh points respectively. Further, formula was used to determine the numerical rate of convergence (ROC).

$$R_{\mathcal{N}} = \log_2\left|\frac{E_{\mathcal{N}}}{E_{2\mathcal{N}}}\right|.$$

Example 1.

$$\varepsilon u''(\vartheta) + (1+\vartheta)\, u'(\vartheta-\delta) + \exp(-2\vartheta)\, u(\vartheta+\eta) - 2\exp(-\vartheta)\, u(\vartheta) = 0,$$

with boundary constraints $u(\vartheta) = 1;\ -\delta \le \vartheta \le 0,\ u(1) = 0.$

Example 2.

$$\varepsilon u''(\vartheta) + (1+\vartheta)\, u'(\vartheta-\delta) + \sin(2\vartheta)\, u(\vartheta+\eta) - \exp(-\vartheta)\, u(\vartheta) = \sin(2\vartheta) + 3\exp(-\vartheta),$$

with boundary constraints $u(\vartheta) = -1;\ -\delta \le \vartheta \le 0,\ u(1) = 1.$

Table 1

MAEs and ROCs of Example 1 (with fitting factor)

$\varepsilon \downarrow \mathcal{N} \rightarrow$	2^4	2^5	2^6	2^7	2^8	2^9
$\eta = \delta = 0.5\varepsilon$						
2^{-1}	5.6311e-04	1.3401e-04	3.3083e-05	8.2448e-06	2.0596e-06	5.1483e-07
	2.0710	2.1199	2.0045	2.0011	2.0001	
2^{-2}	9.6942e-04	2.2072e-04	5.3696e-05	1.3362e-05	3.3344e-06	8.3321e-07
	2.1349	2.0393	2.0104	2.0066	2.0006	
2^{-3}	2.3099e-03	4.2972e-04	9.8534e-05	2.4310e-05	6.0432e-06	1.5087e-06
	2.4263	2.1247	2.0313	2.0190	2.0020	
2^{-4}	5.9356e-03	1.0579e-03	2.0386e-04	4.7300e-05	1.1613e-05	2.8940e-06
	2.4881	2.3755	2.1076	2.0260	2.0046	
2^{-5}	1.1620e-02	2.8395e-03	5.0813e-04	9.9643e-05	2.3249e-05	5.7093e-06
	2.0329	2.4823	2.3503	2.0995	2.0257	
2^{-6}	1.4231e-02	5.8252e-03	1.3925e-03	2.4936e-04	4.9330e-05	1.1542e-05
	1.2886	2.0646	2.4813	2.3376	2.0955	
2^{-7}	1.4493e-02	7.3470e-03	2.9211e-03	6.9013e-04	1.2358e-04	2.4555e-05
	0.9801	1.3306	2.0815	2.4814	2.3313	
2^{-8}	1.4524e-02	7.5002e-03	3.7407e-03	1.4634e-03	3.4363e-04	6.1524e-05
	0.9534	1.0036	1.3539	2.0903	2.4816	
2^{-9}	1.4538e-02	7.5104e-03	3.8238e-03	1.8884e-03	7.3247e-04	1.7147e-04
	0.9528	0.9738	1.0178	1.36623	2.0948	

Table 2

MAEs of Example 1 (without fitting factor)

$\varepsilon \downarrow \mathcal{N} \rightarrow$	2^4	2^5	2^6	2^7	2^8	2^9
$\eta = \delta = 0.5\varepsilon$						
2^{-1}	2.5261e-03	6.2615e-04	1.5621e-04	3.9031e-05	9.7564e-06	2.4391e-06
2^{-2}	8.1260e-03	1.9974e-03	4.9626e-04	1.2401e-04	3.0985e-05	7.7463e-06
2^{-3}	3.2720e-02	7.0941e-03	1.7178e-03	4.2932e-04	1.0711e-04	2.6766e-05
2^{-4}	1.1969e-01	2.9731e-02	6.5329e-03	1.5862e-03	3.9373e-04	9.8484e-05
2^{-5}	3.3077e-01	1.1540e-01	2.8201e-02	6.2431e-03	1.5181e-03	3.7697e-04
2^{-6}	6.6691e-01	3.3194e-01	1.1324e-01	2.7433e-02	6.0967e-03	1.4837e-03
2^{-7}	1.0199e+00	6.9033e-01	3.3256e-01	1.1217e-01	2.7049e-02	6.0235e-03
2^{-8}	1.2902e+00	1.0854e+00	7.0056e-01	3.3292e-01	1.1164e-01	2.6858e-02
2^{-9}	1.4613e+00	1.3962e+00	1.1125e+00	7.0578e-01	3.3312e-01	1.1137e-01

Table 3

MAEs and ROCs of Example 2 (with fitting factor)

$\varepsilon \downarrow \mathcal{N} \rightarrow$	2^4	2^5	2^6	2^7	2^8	2^9
$\eta = \delta = 0.5\varepsilon$						
2^{-1}	5.8352e-03	1.8204e-03	5.1499e-04	1.3599e-04	3.4639e-05	8.7050e-06
	1.6805	1.8216	1.9210	1.9730	1.9924	
2^{-2}	1.0479e-02	3.6556e-03	1.1655e-03	3.5046e-04	1.0087e-04	2.7900e-05
	1.5193	1.6491	1.7336	1.7967	1.8541	
2^{-3}	1.5717e-02	5.9555e-03	2.0005e-03	6.2793e-04	1.8872e-04	5.5003e-05
	1.4000	1.5742	1.6716	1.7343	1.9153	
2^{-4}	2.0188e-02	8.6036e-03	3.1435e-03	1.0417e-03	3.2435e-04	9.6961e-05
	1.2304	1.4525	1.5934	1.6833	1.7420	
2^{-5}	2.2262e-02	1.0756e-02	4.4448e-03	1.6094e-03	5.3115e-04	1.6496e-04
	1.0494	1.2749	1.4655	1.5993	1.6870	
2^{-6}	2.2569e-02	1.1731e-02	5.5199e-03	2.2582e-03	8.1433e-04	2.6826e-04
	0.9440	1.1740	1.2894	1.4714	1.6019	
2^{-7}	2.2572e-02	1.1895e-02	6.0190e-03	2.7959e-03	1.1381e-03	4.0962e-04
	0.9241	0.9827	1.1062	1.2966	1.4742	
2^{-8}	2.2569e-02	1.1900e-02	6.1045e-03	3.0472e-03	1.4070e-03	5.7135e-04
	0.9233	0.9630	1.0023	1.1148	1.3001	
2^{-9}	2.2568e-02	1.1899e-02	6.1074e-03	3.0909e-03	1.5330e-03	7.0573e-04
	0.9234	0.9622	0.9825	1.0116	1.1191	

Table 4

MAEs of Example 2 (without fitting factor)

$\varepsilon \downarrow \mathcal{N} \rightarrow$	2^4	2^5	2^6	2^7	2^8	2^9
$\eta = \delta = 0.5\varepsilon$						
2^{-1}	4.4262e-03	1.0866e-03	2.7065e-04	6.7609e-05	1.6897e-05	4.2241e-06
2^{-2}	1.3543e-02	3.2492e-03	8.1299e-04	2.0259e-04	5.0602e-05	1.2647e-05
2^{-3}	5.2911e-02	1.1506e-02	2.7860e-03	6.9285e-04	1.7304e-04	4.3238e-05
2^{-4}	1.9685e-01	4.8317e-02	1.0644e-02	2.5854e-03	6.4179e-04	1.6035e-04
2^{-5}	5.4951e-01	1.9032e-01	4.6290e-02	1.0263e-02	2.4963e-03	6.1989e-04
2^{-6}	1.1029e+00	5.5157e-01	1.8761e-01	4.5356e-02	1.0088e-02	2.4554e-03
2^{-7}	1.6521e+00	1.1505e+00	5.5337e-01	1.8642e-01	4.4912e-02	1.0005e-02
2^{-8}	2.0454e+00	1.8037e+00	1.1678e+00	5.5460e-01	1.8587e-01	4.4696e-02
2^{-9}	2.2859e+00	2.2936e+00	1.8556e+00	1.1768e+00	5.5530e-01	1.8561e-01

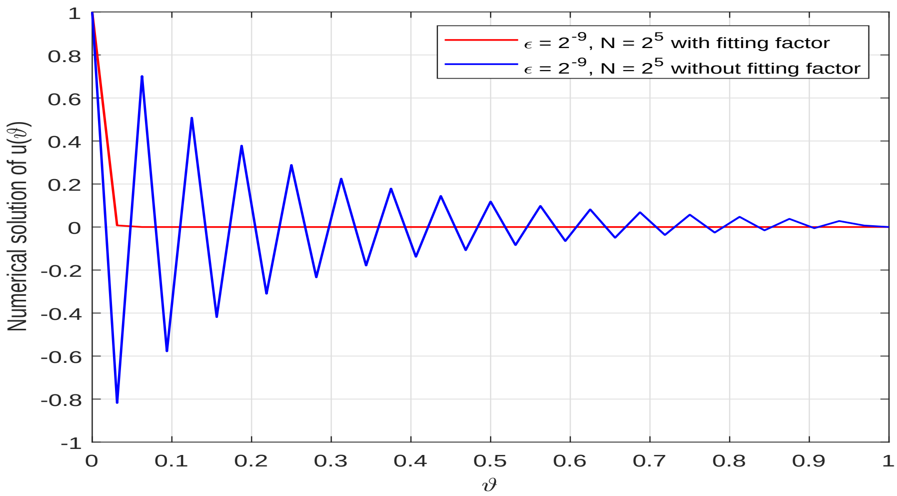


Figure 1. Solution profile of Ex.1 for $\eta = 0.5\varepsilon = \delta$ with and without fitting parameter

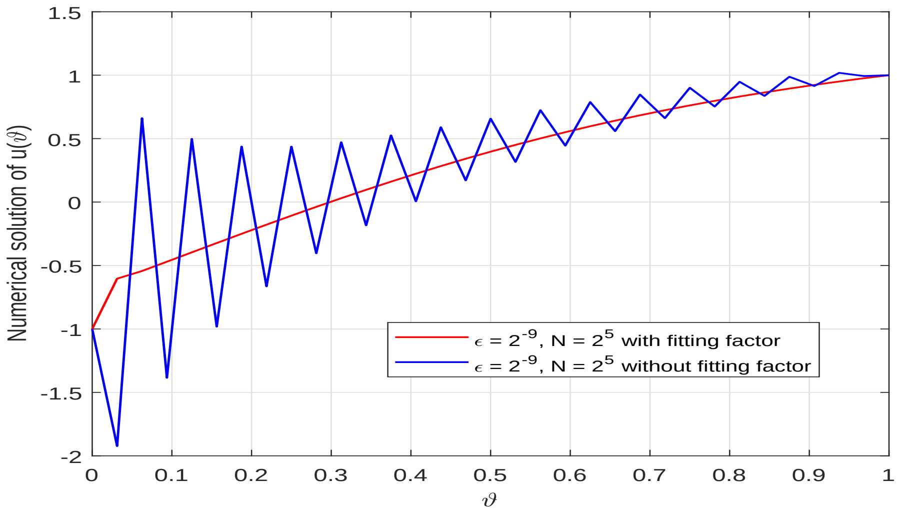


Figure 2. Solution profile of Ex.2 for $\eta = 0.5\varepsilon = \delta$ with and without fitting parameter

Conclusion

A simple and efficient computational non polynomial quartic spline technique is presented for solving second-order SPDDE with mixed parameters on the convection and reaction terms. To illustrate the accuracy and effectiveness of the approach, we solved two example problems for different values of ε, N with $\eta = \delta = 0.5\varepsilon$ and reported the numerical results in terms of maximum absolute errors (MAEs) and rate of convergence (ROC) with and without fitting factor. Using MATLAB, the results of the examples are listed in Tables 1, 2, 3, and 4 in terms of MAEs. Numerical solutions of Example 1 and Example 2 with fitting factors are shown in Table 1 and Table 3 in terms of MAEs and numerical ROCs, and Tables 2 and 4 show the numerical solutions of Examples 1 and 2 without fitting factor. The figures (1-2) depict graphs of the solutions of test problems with and without fitting factor respectively. As ε decreases for different values of $\mathcal{N}$, the solution can exhibit oscillatory behaviour. To manage these drawbacks in solutions of test problems, we introduced a fitting parameter so that we can control the layer structure shown in Figures 1-2. The maximum elapsed time is approximately 0.77 seconds for various values of $\mathcal{N}$ and ε for two test problems.

Author Contributions

All authors contributed equally to this work.

Conflict of Interest

The authors declare that they have no conflict of interest.

References

1 Stein, R.B. (1967). Some models of neuronal variability. *Biophysical Journal, 7*(1), 37–68. https://doi.org/10.1016/S0006-3495(67)86574-3

2 Longtin, A., & Milton, J.G. (1988). Complex oscillations in the human pupil light reflex with mixed and delayed feedback. *Mathematical Biosciences, 90*(1-2), 183–199. https://doi.org/10.1016/0025-5564(88)90064-8

3 Roos, H.-G., Stynes, M., & Tobiska, L. (2008). *Robust Numerical Methods for Singularly Perturbed Differential Equations: Convection-Diffusion-Reaction and Flow Problems.* Springer Berlin Heidelberg. https://doi.org/10.1007/978-3-540-34467-4

4 Priyadarshana, S., Sahu, S.R., & Mohapatra, J. (2022). Asymptotic and numerical methods for solving singularly perturbed differential difference equations with mixed shifts. *Iranian Journal of Numerical Analysis and Optimization, 1*(12), 55–72. https://doi.org/10.22067/ijnao.2021.731.1038

5 Kadalbajoo, M.K., & Ramesh, V.P. (2007). Hybrid method for numerical solution of singularly perturbed delay differential equations. *Applied Mathematics and Computation, 187*(2), 797–814. https://doi.org/10.1016/j.amc.2006.08.159

6 Lalu, M., Phaneendra, K., & Emineni, S.P. (2021). Numerical approach for differential-difference equations having layer behaviour with small or large delay using non-polynomial spline. *Soft Computing, 25*, 13709–13722. https://doi.org/10.1007/s00500-021-06032-5

7 Das, P., Rana, S., & Vigo-Aguiar, J. (2020). Higher order accurate approximations on equidistributed meshes for boundary layer originated mixed type reaction diffusion systems with multiple scale nature. *Applied numerical mathematics, 148*, 79–97. https://doi.org/10.1016/j.apnum.2019.08.028

8 Ragula, K., Soujanya, G.B.S.L., & Swarnakar, D. (2023). Computational Approach for a Singularly Perturbed Differential Equations With Mixed Shifts Using a Non-Polynomial Spline. *International Journal of Analysis and Applications, 21*, Article 5. https://doi.org/10.28924/2291-8639-21-2023-5.

9 Ranjan, R., & Prasad, H.S. (2021). A novel approach for the numerical approximation to the solution of singularly perturbed differential-difference equations with small shift. *Journal of Applied Mathematics and Computing, 65*, 403–427. https://doi.org/10.1007/s12190-020-01397-6

10 Adivi Sri Venkata, R.K., & Palli, M.M.K. (2017). A numerical approach for solving singularly perturbed convection delay problems via exponentially fitted spline method. *Calcolo, 54*, 943–961. https://doi.org/10.1007/s10092-017-0215-6

11 Prathap, T., & Rao, R.N. (2024). Fitted mesh methods based on non-polynomial splines for singularly perturbed boundary value problems with mixed shifts. *AIMS Mathematics, 9*(10), 26403–26434. https://doi.org/10.3934/math.20241285

12 Kiltu, G.G., Duressa, G.F., & Bullo, T.A. (2020). Numerical treatment of singularly perturbed delay reaction-diffusion equations. *International Journal of Engineering, Science and Technology, 12*(1), 15–24. https://doi.org/10.4314/ijest.v12i1.2

13 Debela, H.G., & Duressa, G.F. (2022). Robust Numerical Method for Singularly Perturbed Two-Parameter Differential-Difference Equations. *Applied Mathematics E-Notes, 22*(1), 200–209.

14 Tirfesa, B.B., Duressa, G.F., & Garoma, H. (2022). Non-polynomial cubic spline method for solving singularly perturbed delay reaction-diffusion equations. *Thai Journal of Mathematics, 20*(2), 679–692.

15 Ayalew, M., Kiltu, G.G., & Duressa, G.F. (2021). Fitted Numerical Scheme for Second-Order Singularly Perturbed Differential-Difference Equations with Mixed Shifts. *Abstract and Applied Analysis*, Article 4573847. https://doi.org/10.1155/2021/4573847

16 Sirisha, L., Phaneendra, K., & Reddy, Y.N. (2018). Mixed finite difference method for singularly perturbed differential difference equations with mixed shifts via domain decomposition. *Ain Shams Engineering Journal, 9*(4), 647–654. https://doi.org/10.1016/j.asej.2016.03.009

17 Soujanya, G., Ragula, K., & Phaneendra, K. (2023). A difference scheme using a parametric spline for differential difference equation with twin layers. *International Journal of Nonlinear Analysis and Applications, 14*(1), 2469–2479. https://doi.org/10.22075/ijnaa.2022.28237.3841

18 Joy, D., & Kumar, S.D. (2024). Computational techniques for singularly perturbed reaction-diffusion delay differential equations: a second-order approach. *Journal of mathematics and computer science, 35*, 304–318.

19 Justine, H., Chew, J., & Sulaiman, J. (2017). Quartic non-polynomial spline solution for solving two-point boundary value problems by using Conjugate Gradient Iterative Method. *Journal of Applied Mathematics and Computational Mechanics, 16*(1), 41–50. https://doi.org/10.17512/jamcm.2017.1.04

20 Varga, R.S. (1962). *Matrix iterative analysis.* Englewood Cliffs, N.J.: Prentice-Hall.

*Author Information**

Kumar Ragula (*corresponding author*) — Doctor of Mathematical Sciences, Assistant Professor, Department of Mathematics, Rajiv Gandhi University of Knowledge Technologies, Basar, 504107, India; e-mail: *rksm39@gmail.com*; https://orcid.org/0000-0003-3671-3590

G.B.S.L. Soujanya — Doctor of Mathematical Sciences, Assistant Professor, Department of Mathematics, Kakatiya University, Warangal, 506009, India; e-mail: *gbslsoujanya@gmail.com*; https://orcid.org/0000-0002-4522-7896

*Authors' names are presented in the following order: first name, middle name (if any), last name.

https://doi.org/10.31489/2026M1/228-237 *Research article*

Integration of the negative order Nonlinear Schrödinger Equation with self-consistent source

G.U. Urazboev[1,2], I.I. Baltaeva[1]

[1] *Urgench State University, Urgench, Uzbekistan;*
[2] *V.I. Romanovskiy Institute of Mathematics, Uzbekistan Academy of Sciences, Tashkent, Uzbekistan*
(E-mail: gayrat71@mail.ru, iroda-b@mail.ru)

This paper focuses on the integrability properties of the negative-order nonlinear Schrödinger equation with a source. The source consists of the combination of the eigenfunctions of the corresponding spectral problem for the Dirac system which has not spectral singularities. The connection between the negative-order nonlinear Schrödinger equation with a self-consistent source and the Dirac system of equations is crucial, as it allows the complex dynamics of the original nonlinear model to be interpreted through the spectral theory of the Dirac operator. Building on this relationship, the evolution equations for the scattering data of the Dirac operator are derived, which is the central part in the inverse scattering transform (IST) framework. Due to the IST procedure, the rapidly decaying potential of the Dirac operator can be reconstructed from the derived differential equations for the scattering data, and this potential corresponds precisely to the solution of the problem under consideration. To illustrate the practical value of the theoretical results, the paper presents a detailed example demonstrating each stage of the method, from the formulation of the scattering data to the final reconstruction of the potential. This example clarifies the overall procedure and highlights the effectiveness of the approach in concrete applications.

Keywords: soliton solution, negative-order, nonlinear equations, nonlinear Schrödinger equation, self-consistent source, inverse scattering transform, eigenvalue, eigenfunction.

2020 Mathematics Subject Classification: 35P25, 35P30, 35Q51, 35Q53, 37K15.

Introduction

The nonlinear Schrödinger equation (NLSE) arises in several physical systems characterized by wave-like behavior interacting with nonlinear effects, resulting in distinct phenomena such as optical solitons in fiber optics, Bose–Einstein condensates in ultracold atomic gases, and wave dynamics in plasma physics [1, 2]. The NLSE provides a mathematical foundation to understand nonlinear wave dynamics, emphasizing the interplay between dispersion, which tends to spread wave packets, and nonlinearity, which can counteract dispersion through self-interaction. This framework helps elucidate the emergence of solitons, vortices, and other intricate wave phenomena observed in both natural and engineered systems. Well-known bright, dark, and gray solitons, as well as so-called optical rogue waves, whose experimental observations in optical systems are supported by numerical simulations based on probabilistic supercontinuum generation in highly nonlinear microstructured optical fibers, are modeled using the generalized NLSE [3]. The NLSE can be written in several forms depending on the context, but a common form for the NLSE in one spatial dimension is:

$$iu_t(x,t) = u_{xx}(x,t) \pm 2u^*(x,t)u^2(x,t).$$

The motivation for developing negative-order nonlinear equations originates from Peter J. Olver's [4] on recursion operators for symmetries of evolution equations, which was subsequently extended to derive negative-order analogs of evolution equations [5–7].

*Corresponding author. *E-mail: iroda-b@mail.ru*
Received: 27 November 2024; *Accepted:* 26 December 2025.

Note that in [8], the focusing NLSE was derived as the following

$$\begin{cases} \mu_x = |u|_t^2\,, \\ u_{xt} + 2\mu u + iu_x = 0, \quad x \in \mathbb{R}, \quad t \geq 0 \end{cases}$$

negative-order nonlinear Schrödinger equation (nNLSE) and subsequently solved using the IST.

The study of soliton equations with self-consistent sources has been particularly pursued in the work of Mel'nikov [9]. Subsequently, numerous works [10–12] have explored various methods for solving these types of equations. Mathematically, such systems of equations arise through a multiscaling limit of well-known integrable systems. Several of these systems have also independently appeared in various physical contexts [13, 14]. In this work we concern on applying the inverse scattering transform (IST) for integrating the nNLSE with a self-consistent source in the class of decreasing function.

1 *Statement of the problem*

This paper is concerned with the following system of equations:

$$\begin{cases} \mu_x = |u|_t^2\,, \\ iu_{xt} + 2\mu u + iu_x = i\sum\limits_{k=1}^{2N} (\Phi_{k_1}^2 - \Phi^{*2}_{k_2}), \\ L\Phi_k = \xi_k \Phi_k,\ k = 1, 2, ..., 2N,\ x \in R, \end{cases} \tag{1}$$

for the complex-valued continuous function $u(x,t)$ and $\Phi_k = (\Phi_{k1}(x,t),\ \Phi_{k2}(x,t))^T,\ \ k = 1, 2, ..., N$ is the eigenvector functions of the operator $L(t) = i\begin{pmatrix} \frac{d}{dx} & -u(x,t) \\ -u^*(x,t) & -\frac{d}{dx} \end{pmatrix}$, corresponding to the eigenvalue ξ_k. (*) means the complex conjugate of the function. For definiteness, we will assume that the sum involved in the right-hand side of (1) first includes terms with $Im\xi_k > 0,\ \ k = 1, 2, ..., N$. It is also assumed that

$$\int_{-\infty}^{\infty} \Phi_{k1}\Phi_{k2}dx = A_k(t), \qquad k = 1, 2, ..., 2N, \tag{2}$$

with given continuous, non-zero functions $A_k(t)$, that satisfy the conditions $A_k(t) = A_n(t)$ for $\xi_k = -\xi_n$, under initial condition

$$u(x,0) = u_0(x), \quad x \in R, \tag{3}$$

where, the initial function $u_0(x)$ $(-\infty < x < \infty)$ has the following properties:

1) $\int_{-\infty}^{\infty} (1 + |x|)\ |u_0(x)|\, dx < \infty$,

2) the operator $L\,(0) = i\begin{pmatrix} \frac{\partial}{\partial x} & -u_0(x) \\ -u_0^*(x) & -\frac{\partial}{\partial x} \end{pmatrix}$ has no spectral singularities and has exactly $2N$ simple eigenvalues $\xi_1(0),\ \xi_2(0),\ \ldots,\ \xi_{2N}(0)$. Here the function $u_0^*(x)$ is the complex conjugate of the function $u_0(x)$.

Let the function $u = u(x,t)$ be a complex valued and sufficiently smooth function of x and t, $\mu = \mu(x,t)$ is the sufficiently smooth real function of x and t, for all $t \geq 0$ satisfies the requirement

$$\begin{gathered} \int\limits_{-\infty}^{\infty} \left((1 + |x|)(|u(x,t)| + |u_x(x,t)|)\right) dx < \infty, \\ \mu(x,t) \to c^2 \text{ as } x \to \pm\infty, \end{gathered} \tag{4}$$

respectively.

In this work, we obtain the evolution of scattering data for the system with a potential that is a solution to the considered Cauchy problem (1)–(4) for the nNLSE with a simple self-consistent source.

2 *Scattering problem*

In this section, we provide known brief facts from the Dirac spectral problem on the real axis [15].

$$\begin{cases} y_{1x} = -i\xi y_1 + u(x)\, y_2, \\ y_{2x} = i\xi y_2 - u^*(x)\, y_1, \end{cases} \tag{5}$$

where $u(x)$ is complex-valued potential satisfies the condition

$$\int_{-\infty}^{\infty} (1+|x|)\ |u(x)|\, dx < \infty \tag{6}$$

and $\xi \in C$ is spectral parameter.

The system (5) under the condition (6) posses "Jost solutions" $\varphi(x,\xi)$ and $\psi(x,\xi)$, for real ξ Jost solutions have the following asymptotes:

$$\left.\begin{array}{l} \varphi(x,\xi)e^{i\xi x} \to \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ \bar{\varphi}(x,\xi)e^{-i\xi x} \to \begin{pmatrix} 0 \\ -1 \end{pmatrix} \end{array}\right\}, \quad x \to -\infty, \qquad \left.\begin{array}{l} \psi(x,\xi)e^{-i\xi x} \to \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ \bar{\psi}(x,\xi)e^{i\xi x} \to \begin{pmatrix} 1 \\ 0 \end{pmatrix} \end{array}\right\}, \quad x \to +\infty.$$

Note that here and below the function $\bar{\varphi}$ is not complex conjugate to φ. On the continuous spectrum of the problem (5)-(6) pairs of vector functions $\{\psi, \bar{\psi}\}$ are linearly independent, and for them holds the following relation

$$\varphi(x,\xi) = a(\xi)\bar{\psi}(x,\xi) + b(\xi)\psi(x,\xi), \tag{7}$$

where

$$a(\xi) = W\{\varphi,\psi\} \equiv \varphi_1(x,\xi)\psi_2(x,\xi) - \varphi_2(x,\xi)\psi_1(x,\xi) \tag{8}$$

$$\bar{\psi}(x,\xi) = \begin{pmatrix} \psi_2^*(x,\xi^*) \\ -\psi_1^*(x,\xi^*) \end{pmatrix}, \qquad \bar{\varphi}(x,\xi) = \begin{pmatrix} \varphi_2^*(x,\xi^*) \\ -\varphi_1^*(x,\xi^*) \end{pmatrix}. \tag{9}$$

The quantity $r^+(\xi) = \frac{b(\xi)}{a(\xi)}$ is known as reflection coefficient. Furthermore [4], $r^+(\xi)$ uniquely determines $a(\xi)$. The function $a(\xi)$ admits analytic continuation into the upper half-plane $Im\,\xi > 0$. The function $a(\xi)$ can only have a finite number of zeros $\xi_k,\ k = 1,\ 2,\ ...,\ N$ in the half-plane $Im\,\xi > 0$. The zeros $\xi_k,\ k = 1,\ 2,\ ...,\ N$, of the functions $a(\xi)$ correspond to the eigenvalues $\xi_k,\ k = 1,\ 2,\ ...,\ N$, of the operator L in the upper half-plane.

In general, the operator L may have multiple eigenvalues and spectral singularities that lie on the continuous spectrum. The continuous spectrum of the operator fills the real axis.

In this paper, we assume that the operator L has no spectral singularities and all its eigenvalues ξ_k are simple, so that since the quantities ξ_k are the zeros of $a(\xi)$, it follows from relation (8) that

$$\varphi(x,\xi_k) = C_k\psi(x,\xi_k), \quad k = 1, 2, ..., N,$$

where C_k do not depend on x.

Definition. The set $\{r^+(\xi),\ C_k,\ \xi_k,\ k = 1, 2, ..., N\}$ is called scattering data for the system of equations (5). The set of scattering data uniquely determines the potential $u(x)$.

The potential function $u(x)$ is determined by the equality

$$u(x) = -2K(x,x).$$

Here $K(x,y)$ is a solution of the integral equation Gelfand–Levitan–Marchenko

$$K(x,y) - F^*(x+y) + \int_x^{+\infty}\int_x^{+\infty} K(x,y)\, F(z+s)\, F^*(s+y)\, ds dz = 0,$$

where $F(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \frac{b(\xi)}{a(\xi)} e^{i\xi x} d\xi - i\sum_{j=1}^{N} C_j e^{i\xi_j x}$.

3 Evolution of the scattering data

For further calculations and to obtain the main result, we provide the necessary notes.

Lemma 1. If the vector functions $Y(x,\zeta) = (y_1, y_2)^T$ and $Z(x,\eta) = (z_1, z_2)^T$ are solutions of the equations $LY = \zeta Y$ and $LZ = \eta Z$, respectively, then their components satisfy the equalities

$$\frac{d}{dx}(y_1 z_2 - y_2 z_1) = -i(\zeta - \eta)(y_1 z_2 + y_2 z_1),$$

$$\frac{d}{dx}(y_1 z_1^* + y_2 z_2^*) = -i(\zeta - \eta^*)(y_1 z_1^* - y_2 z_2^*).$$

This Lemma 1 can be proved by direct verification.

It easy to show that the vector functions

$$h_n(x) = \frac{\left.\frac{d}{d\xi}(\varphi - C_n\psi)\right|_{\xi=\xi_n}}{\dot{a}(\xi_n)}, \quad n = 1, 2, ..., N, \tag{10}$$

are solutions to the system of equations $Lh_n = \xi_n h_n$. According to the equality (8), which can be rewritten in the $Im\,\xi > 0$, and using the equality (10), we obtain the following asymptotes

$$\begin{gathered} h_n \sim -C_n \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{i\xi_n x} \quad at \quad x \to -\infty, \\ h_n \sim \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-i\xi_n x} \quad at \quad x \to \infty. \end{gathered} \tag{11}$$

In particular,

$$W\{\varphi_n, h_n\} \equiv \varphi_{n1} h_{n2} - \varphi_{n2} h_{n1} = -C_n, \tag{12}$$

where $\varphi_n \equiv \varphi(x, \xi_n)$, $n = 1, 2, ..., N$.

Now, let's consider the nNLSE with the source

$$\begin{cases} \mu_x = |u|_t^2, \\ iu_{xt} + 2\mu u + iu_x = iG, \end{cases} \tag{13}$$

where $G = G(x,t)$ is a sufficiently smooth function and for any nonnegative value t, satisfies the condition

$$G(x,t) = o(1) \text{ at } x \to \pm\infty.$$

Equation (13) is considered under the initial condition (3).

Lemma 2. If the potential $u(x,t)$, in the system of equations (5), is a solution to equation (13) in the class of functions (4), then the scattering data of the system of equations (5) varies as follows:

$$\frac{\partial r^+}{\partial t} = \frac{i}{\xi}\left(c^2 + \xi\right) r^+ + \frac{1}{2\xi a^2(\xi)} \int_{-\infty}^{\infty} \left(G\varphi_2^2 + G^*\varphi_1^2\right) dx, \quad (Im\xi = 0),$$

$$\frac{dC_n}{dt} = \left(i\left(1 + \frac{c^2}{\xi_n}\right) + \frac{1}{2\xi_n}\int_{-\infty}^{\infty} (G^* h_{n1}\psi_{n1} + G h_{n2}\psi_{n2})\, dx\right) C_n,$$

$$\frac{d\xi_n}{dt} = \frac{i}{4\xi_n} \frac{\int_{-\infty}^{\infty}(G^*\varphi_{n1}^2 + G\varphi_{n2}^2)dx}{\int_{-\infty}^{\infty} \varphi_{n1}\varphi_{n2} dx}, \quad n = 1, 2, ..., N.$$

Proof. The system of equation (13) can be represented as a Lax equation:

$$L_t + [L, A] = R, \tag{14}$$

where $[L,\ A]=\ LA-AL$ and

$$L(t)=i\begin{pmatrix}\frac{d}{dx} & -u(x,t)\\ -u^*(x,t) & -\frac{d}{dx}\end{pmatrix},$$

$$A=\frac{i}{2\xi}\begin{pmatrix}-\mu-\xi & -iu-u_t\\ iu^*-u_t^* & \mu+\xi\end{pmatrix},$$

$$R=\begin{pmatrix}0 & -\frac{G}{2\xi}\\ \frac{G^*}{2\xi} & 0\end{pmatrix}.$$

Let $\varphi\left(x,\xi,t\right)$ is the Jost solution of the equation $L\varphi=\xi\varphi$. Differentiating this equality with respect to t, we obtain

$$L_t\varphi+L\varphi_t=\xi\varphi_t,$$

substituting L_t from (14), we get

$$\left(L-\xi\right)\left(\varphi_t-A\varphi\right)=R\varphi. \tag{15}$$

We will look for a solution (15) in the form

$$\varphi_t-A\varphi=\alpha\left(x\right)\psi+\beta\left(x\right)\varphi.$$

Using the approach described in [5], we obtain the following equality

$$\varphi_t-A\varphi=-\frac{1}{a\left(\xi\right)}\int_{-\infty}^{x}\hat{\varphi}^TR\varphi dx\cdot\psi+\left(\frac{1}{a\left(\xi\right)}\int_{-\infty}^{x}\hat{\psi}^TR\varphi dx+\frac{i\left(c^2+\xi\right)}{2\xi}\right)\varphi. \tag{16}$$

Using (7) and passing to the limit in (16) at $x\to\infty$, using the definition of the reflection coefficient for the scattering problem (5)-(6), we obtain

$$\frac{\partial r^+}{\partial t}=\frac{i}{\xi}\left(c^2+\xi\right)r^++\frac{1}{2\xi a^2\left(\xi\right)}\int_{-\infty}^{\infty}\left(G\varphi_2^2+G^*\varphi_1^2\right)dx,\quad\left(Im\xi=0\right).$$

Differentiating the equality $\varphi_n=C_n\psi_n$ by t, we receive

$$\left.\frac{\partial\varphi}{\partial t}\right|_{\xi=\xi_n}+\left.\frac{\partial\varphi}{\partial\xi}\right|_{\xi=\xi_n}\frac{d\xi_n}{dt}=\frac{dC_n}{dt}\psi_n+C_n\left.\frac{\partial\psi}{\partial t}\right|_{\xi=\xi_n}+C_n\left.\frac{\partial\psi}{\partial\xi}\right|_{\xi=\xi_n}\frac{d\xi_n}{dt}.$$

Substituting, instead $\left.\frac{d}{d\xi}\left(\varphi-C_n\psi\right)\right|_{\xi=\xi_n}$, the expression $h_n(x)$ from (10), we obtain

$$\frac{\partial\varphi_n}{\partial t}=\frac{dC_n}{dt}\psi_n+C_n\frac{\partial\psi_n}{\partial t}-\dot{a}\left(\xi_n\right)h_n\frac{d\xi_n}{dt}, \tag{17}$$

where $\frac{\partial\varphi_n}{\partial t}\equiv\left.\frac{\partial\varphi}{\partial t}\right|_{\xi=\xi_n}$.

Similarly to the continuous spectrum, for the discrete spectrum, we obtain the equality

$$\frac{\partial\varphi_n}{\partial t}-A\varphi_n=\left(\frac{1}{C_n}\int\limits_{-\infty}^{x}\hat{\varphi}_n^TR\varphi_n dx\right)h_n+\left(-\frac{1}{C_n}\int\limits_{-\infty}^{x}\hat{h}_n^TR\varphi_n\,dx+\frac{i}{2\xi_n}\left(c^2+\xi_n\right)\right)\varphi_n,$$

which is an analog of equality (16) for the continuous spectrum. According to (17), the last equality can be rewritten as

$$\frac{dC_n}{dt}\psi_n+C_n\frac{\partial\psi_n}{\partial t}-\dot{a}\left(\xi_n\right)\frac{d\xi_n}{dt}h_n-C_nA\psi_n=$$

$$= \left(\frac{1}{C_n} \int\limits_{-\infty}^{x} \hat{\varphi}_n^T R \varphi_n dx \right) h_n + \left(-\frac{1}{C_n} \int\limits_{-\infty}^{x} \hat{h}_n^T R \varphi_n \, dx + \frac{i}{2\xi_n} \left(c^2 + \xi_n \right) \right) C_n \psi_n. \quad (18)$$

Using asymptotics (11) and passing to the limit as $x \to \infty$ in (18), we obtain

$$\frac{dC_n}{dt} = \frac{i}{\xi_n} \left(c^2 + \xi_n \right) C_n - \int_{-\infty}^{\infty} \hat{h}_n R \varphi_n dx,$$

$$\frac{d\xi_n}{dt} = -\frac{\int_{-\infty}^{\infty} \hat{\varphi}_n R \varphi_n \, dx}{C_n \dot{a} \left(\xi_n \right)},$$

where $\dot{a}\left(\xi_n\right) = -\frac{2i}{\xi_n} \int_{-\infty}^{\infty} \varphi_{n1} \, \varphi_{n2} dx$.

As a result, we have

$$\frac{dC_n}{dt} = \left(i \left(1 + \frac{c^2}{\xi_n} \right) + \frac{1}{2\xi_n} \int_{-\infty}^{\infty} \left(G^* h_{n1} \psi_{n1} + G h_{n2} \psi_{n2} \right) dx \right) C_{n,},$$

$$\frac{d\xi_n}{dt} = \frac{i}{4\xi_n} \frac{\int_{-\infty}^{\infty} (G^* \varphi_{n1}^2 + G \varphi_{n2}^2) dx}{\int_{-\infty}^{\infty} \varphi_{n1} \varphi_{n2} dx}, \quad n = 1, 2, ..., N.$$

Lemma 2 is proved.

Based on the conditions given for the function $A_k(t)$ in formula (2) and equalities (9), the right-hand side in equation (1) can be rewritten as

$$\sum_{k=1}^{2N} \left(\Phi_{k1}^2 - \Phi_{k2}^{*2} \right) = 2 \sum_{\substack{k=1 \\ Im\xi_k > 0}}^{N} \left(\Phi_{k1}^2 - \Phi_{k2}^{*2} \right).$$

Let us apply the results of Lemma 2 to the system of equations (1) assuming

$$G = 2 \sum_{\substack{k=1, \\ Im\xi_k > 0}}^{N} \left(\Phi_{k1}^2 - \Phi_{k2}^{*2} \right).$$

For ξ_k, $k \neq n$, according to Lemma 1, we have the following equality:

$$\left(\Phi_{k1}^2 - \Phi_{k2}^{*2} \right) \left(h_{n1} \psi_{n1} + h_{n2} \psi_{n2} \right) = -\frac{1}{2i} (\frac{1}{\xi_k + \xi_n} \frac{d}{dx} ((\Phi_{k1} h_{n1} + \Phi_{k2}^* h_{n2}) \times$$

$$\times (\Phi_{k1} \psi_{n1} + \Phi_{k2}^* \psi_{n2})) + \frac{1}{\xi_k - \xi_n} \frac{d}{dx} ((\Phi_{k1} h_{n2} - \Phi_{k2}^* h_{n1})(\Phi_{k1} \psi_{n2} - \Phi_{k2}^* \psi_{n1}))),$$

hence

$$\int_{-\infty}^{\infty} \left(\Phi_{k1}^2 - \Phi_{k2}^{*2} \right) \left(h_{n1} \psi_{n1} + h_{n2} \psi_{n2} \right) dx = 0.$$

If $\xi_k = \xi_n$, then

$$\left(\Phi_{n1}^2 - \Phi_{n2}^{*2} \right) \left(h_{n1} \psi_{n1} + h_{n2} \psi_{n2} \right) = -\frac{1}{4i\xi_n} \frac{d}{dx} ((\Phi_{n1} h_{n1} + \Phi_{n2}^* h_{n2}) \times$$

$$\times (\Phi_{n1} \psi_{n1} + \Phi_{n2}^* \psi_{n2})) + \Phi_{n1} \Phi_{n2}^* \left(\psi_{n1} h_{n2} - \psi_{n2} h_{n1} \right),$$

therefore, taking into account (2) and (12), we obtain

$$\int_{-\infty}^{\infty}\left(\Phi_{n1}^2-\Phi_{n2}^{*2}\right)\left(h_{n1}\psi_{n1}+h_{n2}\psi_{n2}\right)dx=\int_{-\infty}^{\infty}\Phi_{n1}\Phi_{n2}^{*}W\left\{\psi_n,h_n\right\}dx=$$

$$=\frac{1}{C_n}\int_{-\infty}^{\infty}\Phi_{n1}\Phi_{n2}^{*}W\left\{\varphi_n,h_n\right\}dx=-A_n\left(t\right).$$

Additionally, according to Lemma 2, we have

$$\frac{dC_n}{dt}=i\left(1+\frac{c^2}{\xi_n}\right)C_n-\frac{A_n}{\xi_n}C_n.$$

Similarly, it can be shown that

$$\int\limits_{-\infty}^{\infty}(G\varphi_2^2+G^*\varphi_1^2)dx=0,$$

$$\int\limits_{-\infty}^{\infty}(G\varphi_{n2}^2+G^*\varphi_{n1}^2)dx=0,$$

thus, we find

$$\frac{\partial r^+}{\partial t}=\frac{i}{\xi}\left(c^2+\xi\right)r^+,\quad\left(Im\xi=0\right),$$

$$\frac{d\xi_n}{dt}=0.$$

Accordingly, we have proved the following theorem:

Theorem 1. If the set of functions $\{u\left(x,t\right),\mu\left(x,t\right),\Phi_k\left(x,t\right),\,k=1,2,\ldots,N\}$ is a solution to the problem (1)–(3) in the class of functions (4), then the scattering data of the system of equations (5) with potential $u\left(x,t\right)$ changes on t as follows:

$$\frac{\partial r^+}{\partial t}=\frac{i}{\xi}\left(c^2+\xi\right)r^+,\quad\left(Im\xi=0\right),$$

$$\frac{dC_n}{dt}=i\left(1+\frac{c^2}{\xi_n}\right)C_n-\frac{A_n}{\xi_n}C_n,$$

$$\frac{d\xi_n}{dt}=0,n=1,2,...,N.$$

The results allows us to apply the inverse scattering problem method to solve the Cauchy problem of the system of equations (1).

Consider the following example. Let the system of equations (1) be considered under the initial condition

$$u|_{t=0}=-\frac{2}{\mathrm{ch}2x},$$

which implies that the initial scattering data will be as follows

$$r^+\left(\xi,0\right)=0,\quad\xi_1\left(0\right)=i,\quad C_1\left(0\right)=2i.$$

According to the main theorem, we have

$$r^+\left(\xi,t\right)=0,\;\xi_1\left(t\right)=i,\quad C_1\left(t\right)=2i\exp\left\{\left(i+c^2\right)t-\int_0^t A_1\left(\tau\right)d\tau\right\},$$

from which, solving the inverse scattering problem, we obtain

$$K(x,y,t) = \frac{2\exp\{-x-y-it+\gamma(t)\}}{1+\exp\{-4x+2\gamma(t)\}}.$$

Applying the relation between potential and kernel $u(x) = -2K(x,x)$ yields that

$$u(x,t) = -\frac{2\exp\{-it\}}{\operatorname{ch}(2x-\gamma(t))},$$

$$\mu(x,t) = \frac{2\left(A_1(t)-c^2\right)}{\operatorname{ch}^2(2x-\gamma(t))} + c^2,$$

and consequently, using the integral representation of Levin for the vector function $\psi(x,\xi)$, we find

$$\psi_{11}(x,t) = \frac{\exp\{-it\}}{2\operatorname{ch}(2x-\gamma(t))},$$

$$\psi_{12}(x,t) = \frac{\exp\{-x\}}{2} + \exp\left\{\frac{-x}{2}\right\}\operatorname{th}(2x-\gamma(t)).$$

Since, $\Phi_{11}(x) = d\psi_{11}$, $\Phi_{12}(x) = d\psi_{12}$, $d^2 = 4iA_1\exp\{it+\gamma(t)\}$ and from the normalization condition (2), we get

$$\Phi_{11} = \frac{\sqrt{iA_1(t)}}{\operatorname{ch}(2x-\gamma(t))}\exp\left\{-x-\frac{it-\gamma(t)}{2}\right\},$$

$$\Phi_{12} = 2\sqrt{iA_1(t)}\exp\left\{-x+\frac{it+\gamma(t)}{2}\right\}(1+\operatorname{th}(2x-\gamma(t))),$$

where $\gamma(t) = c^2 t - \int_0^t A_1(\tau)d\tau$.

Conclusion

This paper studies the integrability of the nNLSE with a self-consistent source using the inverse scattering transform (IST). The problem statement is presented in Section 1. Section 2 reviews the scattering theory for the Dirac system, including the definition of the scattering data and the IST technique. In Section 3, we obtain the evolution of scattering data for the system with a potential that is a solution to the considered Cauchy problem for the nNLSE with a simple self-consistent source.

Author Contributions

All authors contributed equally to this work.

Conflict of Interest

The authors declare no conflict of interest.

References

1 Sulem, C., & Sulem, P.-L. (2007). The nonlinear Schrödinger equation: self-focusing and wave collapse. *Springer Science and Business Media,* 139.

2 Sheveleva, A., Andral, U., Kibler, B., Colman, P., & Dudley, J.M. (2022). Ideal Four Wave Mixing Dynamics in a Nonlinear Schrödinger Equation Fibre System. *Optica, 9*(6), 656–662. https://doi.org/10.1364/OPTICA.445172

3 Bonatto, C., Feyereisen, M., Barland, S., Giudici, M., Masoller, C., Rios Leite, J.R., & Tredicce, J.R. (2011). Deterministic optical rogue waves. *Physical Review Letters, 107*, Article 053901. https://doi.org/10.1103/PhysRevLett.107.053901

4 Olver, P.J. (1977). Evolution equations possessing infinitely many symmetries. *Journal of Mathematical Physics, 18*(6), 1212–1215. https://doi.org/10.1063/1.523393

5 Urazboev, G.U., Baltaeva, I.I., & Atanazarova, Sh.E. (2025). Analysis of the solitary wave solutions of the negative order modified Korteweg-de Vries equation with a self-consistent source. *Partial Differential Equations in Applied Mathematics, 13*, Article 101108. https://doi.org/10.1016/j.padiff.2025.101108

6 Khasanov, M.M., & Rakhimov, I.D. (2023). Integrirovanie uravneniia KdF otritsatelnogo poriadka so svobodnym chlenom v klasse periodicheskikh funktsii [Integration of the KdV equation of negative order with a free term in the class of periodic functions]. *Chebyshevskii Sbornik — Chebyshev Proceedings, 24*(2), 266–275 [in Russian]. https://doi.org/10.22405/2226-8383-2023-24-2-266-275

7 Ismailov, M.I., & Sabaz, C. (2025). Inverse scattering method via the Gelfand–Levitan–Marchenko equation for some negative-order nonlinear wave equations. *Theoretical and Mathematical Physics, 222*, 20–33. https://doi.org/10.1134/S0040577925010039

8 Urazboev, G.U., Baltaeva, I.I., & Babadjanova, A.K. (2024). Soliton solutions of the negative-order nonlinear Schrödinger equation. *Theoretical and Mathematical Physics, 219*, 761–769. https://doi.org/10.1134/S0040577924050052

9 Mel'nikov, V. (1992). Integration of the nonlinear Schrödinger equation with a source. *Inverse Problem, 8*(1), 133–147. http://dx.doi.org/10.1088/0266-5611/8/1/009

10 Hasanov, A.B., & Hasanov, M.M. (2019). Integration of the Nonlinear Schrödinger Equation with an Additional Term in the Class of Periodic Functions. *Theoretical and Mathematical Physics, 199*, 525–532. https://doi.org/10.1134/S0040577919040044

11 Reyimberganov, A.A., & Rakhimov, I.D. (2021). The soliton solutions for the nonlinear Schrödinger equation with self-consistent source. *Bulletin of Irkutsk State University. Series Mathematics, 36*, 84–94. https://doi.org/10.26516/1997-7670.2021.36.84

12 Khasanov, A.B., & Reyimberganov, A.A. (2025). On the Complex Modified Korteweg-de Vries Equation with a Self-Consistent Source and Nonzero Boundary Conditions. *Ukrainian Mathematical Journal, 77*, 149–164. https://doi.org/10.1007/s11253-025-02459-3

13 Liu, Y., & Zhou, J. (2022). The envelope soliton for the nonlinear interaction of Langmuir waves with electron acoustic waves in the Earth's inner magnetosphere. *Physics of Plasmas, 29*, Article 092302. https://doi.org/10.1063/5.0096999

14 Karimov, A.R, & Buyanov, G.O. (2025). Nonlinear Dynamics of Cylindrical Waves in Isentropic Plasma. *Physics, 7*(4), Article 54. https://doi.org/10.3390/physics7040054

15 Zakharov, V.E., & Shabat, A.B. (1972). Exact theory of two-dimensional self-focusing and one-dimensional self-modulation of waves in nonlinear media. *Soviet Physics JETP, 34*(1), 62–69.

*Author Information**

Gayrat Urazalievich Urazboev — Doctor of Physical and Mathematical Sciences, Vice-rector for International Cooperation, Urgench State University; Leading researcher, V.I. Romanovskiy Institute of Mathematics, Uzbekistan Academy of Sciences, 14 H. Alimdjan St., Urgench, 220100, Uzbekistan; e-mail: *gayrat71@mail.ru*; https://orcid.org/0000-0002-7420-2516

Iroda Ismailovna Baltaeva (*corresponding author*) — Candidate of Physical and Mathematical Sciences, Associate Professor, Urgench State University, 14 H. Alimdjan St., Urgench, 220100, Uzbekistan; e-mail: *iroda-b@mail.ru*; https://orcid.org/0000-0001-5624-7529

*Authors' names are presented in the following order: first name, middle name (if any), last name.

https://doi.org/10.31489/2026M1/238-247 *Research article*

Model-theoretic properties of J-superstable Jonsson theories in classes defined by cosemanticness

A.R. Yeshkeyev, G.E. Zhumabekova*, A.K. Issayeva

Buketov Karaganda National Research University, Karaganda, Kazakhstan
(E-mail: aibat.kz@gmail.com, galkatai@mail.ru, isaevaaiga@gmail.com)

This article deals with the problems of model-theoretic characterization of J-superstable Jonsson theories. The characteristic features of such theories are analyzed in terms of J-stability, J-P-superstability, and J-nonmultidimensionality. The need to generalize classical stability notions to the framework of Jonsson theories is identified and justified. The concepts of J-stationary and J-orthogonal types are introduced, and their role in describing the dimensional structure of existentially closed models is examined. It is shown that every JF^{α}_{λ}-saturated model embeds as a submodel of the semantic model of a Jonsson theory T. Based on the notion of J-stationarity, a theory of independence for existential types is developed, and the notions of J-basis and J-dimension are defined. The equivalence between J-nonmultidimensionality, J-P-superstability, and J-P-stability is established, providing precise criteria for the model-theoretic classification of Jonsson theories. The results contribute to the refinement of model-theoretic tools for analyzing stability and dimensionality within the framework of Jonsson theories and place these findings in the broader context of modern classification theory, highlighting Jonsson theories as a natural generalization of elementary theories. These results clarify the interaction between saturation, independence, and dimensionality in Jonsson theories and provide a unified framework for further developments in their model-theoretic classification.

Keywords: Jonsson theory, semantic model, perfect Jonsson theory, hereditary Jonsson theory, permissible enrichment, J-λ-stable theory, J-superstable theory, J-P-superstable theory, J-orthogonal type, J-nonmultidimensional theory.

2020 Mathematics Subject Classification: 03C45, 03C52, 03C65.

Introduction

Model-theoretic stability is a central topic in contemporary mathematical logic and model theory. In particular, the notions of stability and superstability play a crucial role in the classification of theories and the analysis of their model structures. Among the various developments in this field, Jonsson theories, which are characterized by the existence of infinite models preserving homogeneity with respect to subsets of a fixed cardinality, have attracted significant attention.

The relationship between the properties of a complete theory T and those of the theory of elementary pairs T_P was first studied by B. Poizat [1], who formulated the problem of determining under what conditions the theory of elementary pairs is complete. This line of research was later extended by E. Bouscaren [2] and others. Bouscaren demonstrated that the completeness of the theory of elementary pairs depends on the nature of the underlying stable theory, providing necessary and sufficient conditions for both the stable and the superstable cases.

In 1990, in the Proceedings of the Soviet–French Colloquium on Model Theory, A. Nurtazin solved this problem for uncountably categorical theories, and T. Mustafin introduced the concept of

*Corresponding author. *E-mail: galkatai@mail.ru*

This research was funded by the Science Committee of the Ministry of Science and Higher Education of the Republic of Kazakhstan (Grant No. AP22686827 "The central types of hereditary Jonsson theories").

Received: 20 April 2025; *Accepted:* 29 December 2025.

T^*-λ-stability, studying its properties, including P-stability. E. Palyutin [3] proposed the notion of E^*-stability and proved the definability of types for such theories. In particular, in [4], a classification of ω-stable theories by P-stability was obtained. Further, A.R. Yeshkeyev showed the equivalence of the conditions of P-stability and J-P-stability for $\exists$-complete perfect Jonsson theories. In the paper [4] authors investigated the concept of P-superstability with respect to the nonmultidimensionality property. The study of J-P-stable theories was also continued in [5] and [6].

In recent years, the toolkit for studying Jonsson theories has expanded markedly. This development is reflected in the diverse approaches of works [7–9], which emphasize foundational aspects; studies [10–12], which examine generalizations and structural properties; and research [13], which explores new applications and classification problems.

This article focuses on J-superstable Jonsson theories, in which the notions of J-stability and its variants, such as J-P-superstability and J-nonmultidimensionality, play roles analogous to the classical stability properties. The relevance of this study lies in the need to extend classical stability methods to broader classes of theories, including Jonsson structures, and to clarify the relationships between different forms of J-stability and the inheritance of structural properties within a theory.

The main objective of this work is to analyze the model-theoretic properties of J-superstable theories, formulate classification criteria, and investigate the structural and dimensional behavior of their models. Special attention is given to the introduction of J-stationary and J-orthogonal types, which provide a refined understanding of model structure and allow the identification of essential patterns of property inheritance.

Through this study, we establish the equivalence of J-nonmultidimensionality, J-P-superstability, and J-P-stability, and show that every J-P-superstable perfect Jonsson theory preserves its fundamental structural properties under admissible extensions. These results enhance the model-theoretic tools available for analyzing stability and dimensionality in the context of Jonsson theories and lay the groundwork for further research in generalized stability theory.

1 *Preliminaries on Jonsson Theory*

The first section discusses the basic concepts and fundamental results related to Jonsson theories, providing an overview of their model-theoretic foundations, key definitions, and the main developments that have shaped their study within the broader context of stability theory.

Definition 1. [14, p. 80] A theory T is called a *Jonsson theory* if the following conditions are satisfied:
1. The theory T has at least one infinite model.
2. T is *inductive*, that is, it is equivalent to a set of sentences of the form $\forall\exists$.
3. T has the *joint embedding property (JEP)*, meaning that for any two models $A, B \models T$, there exists a model $C \models T$ into which both A and B can be embedded isomorphically.
4. T has the *amalgamation property (AP)*: for any models $A, B_1, B_2 \models T$ and isomorphic embeddings $f_1 \colon A \to B_1$ and $f_2 \colon A \to B_2$, there exists a model $C \models T$ together with isomorphic embeddings $g_1 \colon B_1 \to C$ and $g_2 \colon B_2 \to C$ such that $g_1 f_1 = g_2 f_2$.

It is worth noting that Jonsson theories naturally appear in various classical areas of algebra and model theory. In particular, typical examples of Jonsson theories include:
- the theory of groups;
- the theory of abelian groups;
- the theory of fields of fixed characteristic;
- the theory of Boolean algebras;
- the theory of polygons over a fixed monoid;
- the theory of modules over a fixed ring;
- the theory of linearly ordered sets.

Hence, Jonsson theories encompass a broad class of algebraic and logical structures characterized by strong embedding and amalgamation properties, making them a significant object of study in modern model theory.

Further, let us consider in more detail the semantic and syntactic invariants associated with Jonsson theory. These invariants play a fundamental role in understanding the structural properties and model-theoretic behavior of Jonsson theories.

Definition 2. [15] A model C_T of a Jonsson theory T such that $|C_T| = 2^\omega$ is called a semantic model, if it is ω^+-homogeneous-universal.

Definition 3. [15] The *center* of a Jonsson theory T is the elementary theory of its semantic model C_T, denoted by T^*, where $T^* = \mathrm{Th}(C_T)$ is a complete theory.

Further, we note that a Jonsson theory may exhibit an additional property called perfection, which is related to the saturation of its semantic model.

Definition 4. [15] A Jonsson theory T is said to be perfect, if C_T is ω^+-saturated.

In Jonsson theory, the notion of perfection is closely tied to the class of existentially closed models. Such models play a key role in understanding the realizability of existential statements within larger models of the theory, providing a natural link between semantic completeness and structural homogeneity.

Definition 5. [14, p. 97] A model A of a theory T is said to be *existentially closed in* T if for every model B and every existential formula $\varphi(x)$ with parameters from A, whenever $A \subseteq B$ and $B \vDash \exists x\, \varphi(x)$, we also have $A \vDash \exists x\, \varphi(x)$.

Denote by E_T the class of all existentially closed models of the Jonsson theory T.

Theorem 1. If T is a perfect Jonsson theory, then $E_T = \operatorname{Mod} T^*$.

In the study of Jonsson theories, the behavior of types over small subsets of existentially closed models plays a crucial role in understanding the theory stability properties. To capture this behavior, the notions of J-λ-stability, J-stability, and J-superstability are introduced, which formalize the control over the number of J-types realized over subsets of a given size.

Let T be a Jonsson theory, $S^J(X)$ be a set of all existential complete n-types over X, consistent with T, for each finite n.

Definition 6. (A.R. Yeshkeyev) A Jonsson theory T is called J-λ-stable if for any existentially closed model A and any its subset X: $|X| \leq \lambda \Rightarrow |S^J(X)| \leq \lambda$.

We say that a Jonsson theory T is J-stable if it is J-λ-stable for some cardinal λ.

Definition 7. We say that a Jonsson theory T is J-superstable if it is J-λ-stable for some cardinal λ and $\lambda \geq 2^\omega$.

Let us consider the theorem that connects the Jonsson stability and the classical stability, which is an important property for Jonsson theories.

Theorem 2. (A.R. Yeshkeyev) Let T be a perfect $\exists$-complete (complete for existential sentences) Jonsson theory, $\lambda \geq \omega$. Then the following conditions are equivalent:

1) T is J-λ-stable;

2) T^* is λ-stable (in classical sense), where T^* is the center of T.

Let T be an arbitrary Jonsson theory of a signature σ, and let C_T be its semantic model. Let $A \subseteq C_T$, and let P be a new unary predicate symbol.

Consider the (generally incomplete) theory in the signature $\sigma_P(A) = \sigma_A \cup \{P\}$, defined as follows:

$$T_P^J(A) = \mathrm{Th}_{\forall\exists}(C_T, a)_{a \in A} \cup \{P(c_a) \mid a \in A\} \cup \{\text{“}P \subseteq\text{”}\},$$

where

- $\mathrm{Th}_{\forall\exists}(C_T, a)_{a\in A}$ is the set of all universal-existential sentences true in the structure $(C, a)_{a\in A}$;
- $\{P(c_a) \mid a \in A\}$ asserts that all elements of A are included in the interpretation of the predicate P;
- $\{$"$P \subseteq$"$\}$ is an infinite set of sentences expressing that the interpretation of P is an existentially closed submodel in the signature σ.

Remark. The requirement of existential closedness of the submodel is essential in the sense that the submodel must *not* be finite.

Let $S_p^J(A)$ denote the set of all $\exists$-completions of the theory $T_P^J(A)$.

Let λ be an arbitrary cardinal.

Definition 8. (A.R. Yeshkeyev) A Jonsson theory T is said to be *Jonsson P-λ-stable* (or, briefly, *J-P-λ-stable*) if for every set A of cardinality at most λ, the following holds:

$$|S_P^J(A)| \leq \lambda.$$

Definition 9. (A.R. Yeshkeyev) A Jonsson theory T is said to be *Jonsson P-stable* (or simply *J-P-stable*) if T is J-P-λ-stable for some cardinal λ.

Definition 10. A Jonsson theory T is said to be *Jonsson P-superstable* (or simply *J-P-superstable*) if there exists a cardinal λ_0 such that for all $\lambda \geq \lambda_0$, the theory T is J-P-λ-stable.

That is, for all $\lambda \geq \lambda_0$ and for every set $A \subseteq C_T$ with $|A| \leq \lambda$, the number of existential types with predicate P over A satisfies $|S_P^J(A)| \leq \lambda$.

Definition 11. A Jonsson theory T is called *Jonsson P-unstable* (or simply *J-P-unstable*) if it is not J-P-λ-stable for any cardinal λ.

The following theorem establishes a connection between the P-λ-stability studied in [4] and J-P-λ-stability in the context of a perfect $\exists$-complete Jonsson theory. It is fair to say that, thanks to this theorem, attributed to the first author, a number of significant results have been obtained within the framework of Jonsson theories.

Theorem 3. (A.R. Yeshkeyev) Let T be a perfect $\exists$-complete Jonsson theory. Then the following conditions are equivalent:

1) the center of the theory T is P-λ-stable (in the sense of [4]),

2) the theory T is J-P-λ-stable.

Definition 12. (A.R. Yeshkeyev) Let K be a class of L-structures. The following set $JSp(K)$ of Jonsson theories is called a Jonsson spectrum of the class K:

$$JSp(K) = \{T \mid T \text{ is a Jonsson theory and } A \models T \text{ for any } A \in K\}.$$

Definition 13. (T.G. Mustafin) Let T_1 and T_2 be Jonsson theories, C_{T_1} and C_{T_2} be their semantic models, correspondingly. Then T_1 and T_2 are called cosemantic ($T_1 \bowtie T_2$) Jonsson theories, if $C_{T_1} = C_{T_2}$.

It is easy to see that the cosemanticness is an equivalence relation.

Let L be a first-order language, and let T be a Jonsson theory formulated in L. Assume that $K \subseteq E_T$. We now consider the *Jonsson spectrum* $JSp(K)$ and define on it a relation of *cosemantic equivalence*, denoted by $\bowtie$. Factoring $JSp(K)$ by this relation yields the quotient set $JSp(K)_{/\bowtie}$.

Denote by $[T]$ the *cosemantic class* of a theory T, where $T \in JSp(K)$. The class $[T]$ consists of all Jonsson theories that are cosemantically equivalent to T. For every theory $\Delta \in [T]$, we associate a corresponding semantic model C_Δ. By the definition of the cosemantic class, these models coincide with the semantic model C_T of some fixed representative $T \in [T]$. Thus, the entire class $[T]$ shares a single common semantic model, which we denote by $C_{[T]} = C_T$.

Consider now the elementary theory of this model. We refer to it as the *center* of the Jonsson class $[T]$ and denote it by

$$[T]^* = Th(C_{[T]}).$$

This center is identical to the centers of all individual theories $\Delta \in [T]$, that is,

$$[T]^* = Th(C_\Delta) \quad \text{for all } \Delta \in [T].$$

Furthermore, let

$$E_{[T]} = \bigcup_{\Delta \in [T]} E_\Delta$$

be the collection of all existentially closed models corresponding to the theories in $[T]$. Observe that the intersection

$$\bigcap_{\Delta \in [T]} E_\Delta \neq \emptyset,$$

since at least the model $C_{[T]}$ is contained in every E_Δ.

Consequently, all theories belonging to the class $[T]$ possess a shared semantic model and the same center. This provides a coherent structural framework for analyzing Jonsson theories and their models within a unified semantic context.

2 On the classification and structural properties of J-superstable theories

In the study of the model-theoretic properties of Jonsson theories, it is essential to introduce analogues of classical notions such as stationarity, independence, and orthogonality, but formulated in terms of existential types. These concepts play a central role in the classification theory of Jonsson structures and in the analysis of their dimensions.

Let us look at the following definitions, where "J-forks over A" is used in the meaning of Theorem 2.1.2 from [15].

Definition 14. [6] Let p be a complete $\exists$-type over A, where A is a Jonsson subset of C_T. Then p is called *J-stationary* over A if:

1) p does not J-fork over A;
2) p has a unique consistent extension that does not J-fork over A.

Definition 15. Let $p(\bar{x})$ and $q(\bar{y})$ be $\exists$-types over M, where M is a Jonsson subset of C_T. p and q are said to be J-perpendicular ($p \perp_J q$) if $p(\bar{x}) \cup q(\bar{y})$ determines a complete $\exists$-type over M.

Definition 16. 1. A non-algebraic type $p \in S^J(A)$ is *J-regular* if for every subset $B \supseteq A$ of C_T and every $r \in S^J(B)$ with $p \subseteq r$,

$$r \text{ } J\text{-forks over } A \implies p \perp_J r.$$

2. A J-stationary type p is *J-regular* if its stationarization over $\mathrm{Dom}(p)$ is J-regular.

Before we formally define JF^α_λ-saturation, let us clarify the intuition behind it. In stable theories, not all types need to be realized; it suffices to realize those types that are almost over small sets.

In this context, "almost over A" means that there exists $A_0 \subseteq A$ with $|A_0| < \lambda$ such that p does not J-fork over A_0.

Definition 17. An existensional closed model M is called *JF^α_λ-saturated* if for every subset $A \subseteq M$ with $|A| < \lambda$ and every $\exists$-type $p(x) \in S^J_\alpha(A)$ that is *almost over* A, there exists an element $a \in M$ that realizes the $\exists$-type p, i.e., a satisfies all formulas in p.

JF^α_λ-saturated models are submodels of the semantic model of the Jonsson theory T.

The notion of J-stationarity allows us to define a structural measure of independence for existential types in Jonsson theories. This leads naturally to the concept of a *J-basis* and the corresponding notion of dimension.

Definition 18. Let I be a set of tuples and A a set of parameters. We say that I is *J-independent over A* if for every $b \in I$,

$$\mathrm{tp}(b/A \cup (I \setminus \{b\})) \text{ does not } J\text{-fork over } A.$$

Definition 19. Let T be a Jonsson theory, M be an existentially closed model of T, $A \subseteq M$, and let $p \in S^J(A)$. A set $I \subseteq M$ is called a *J-basis* for p in M if
1) every element of I realizes p in M;
2) I is J-independent over A;
3) I is maximal with these properties.

The *dimension* of p in M is defined by

$$\dim_J(p, M) = |I|,$$

where I is any J-basis for p in M.

Definition 20. [6] 1. Let $p(\bar{x}_1)$, $q(\bar{x}_2)$ be complete $\exists$-types over A, where A is a Jonsson subset of C_T. Then p is said to be *J-weakly orthogonal* to q if and only if $p(\bar{x}_1) \cup q(\bar{x}_2)$ is a complete $\exists$-type over A.

2. Let p_1, p_2 be either $\exists$-complete or J-stationary types. Then p_1 is *J-orthogonal* to p_2 if for any set A such that $\mathrm{Dom}(p_1) \cup \mathrm{Dom}(p_2) \subseteq A$, where A is the universe of an $\exists_1$-saturated model, and for any J-non-forking extensions q_1, q_2 of p_1, p_2 over A respectively, the types q_1 and q_2 are weakly J-orthogonal.

The orthogonality of existential types allows us to describe the multidimensional structure of Jonsson theories. The following definition formalizes the notion of multidimensionality within the framework of Jonsson theories.

Definition 21. [6] Let A be a Jonsson subset of the semantic model C_T, where T is a Jonsson theory. An $\exists$-complete type p is said to be *J-multidimensional* if it is J-orthogonal to any complete $\exists$-type over A. If T has a J-multidimensional type, then T is called a *J-multidimensional theory*. Otherwise, T is called a *J-nonmultidimensional theory*, or a theory of *J-restricted dimension*.

The following theorem was obtained in [6]:

Theorem 4. Let T be a perfect, J-λ-stable, $\exists$-complete Jonsson theory:
1) the theory T^* is nonmultidimensional (in the classical sense);
2) the theory T is J-nonmultidimensional.

Since we rely on Lemmas 2 and 3 from [4] and for them the lemmas that we will consider to prove our main theorem are satisfied.

Lemma 1. Let T be a perfect, hereditary $\exists$-complete Jonsson theory, $K \subseteq E_T$, $[\Delta] \in JSp(K)/_{\bowtie}$, $[\Delta]^* = Th(C_{[\Delta]})$ be its center. Let $M, N_1, N_2 \in \mathrm{Mod}([\Delta]^*)$ with $A \subseteq C_{[\Delta]}$ and $N_1, N_2 \prec_{\exists_1} C_{[\Delta]}$, $M \prec_{\exists_1} N_1$, $M \prec_{\exists_1} N_2$, where the models M, N_1, N_2 are JF^{α}_{ω}-saturated. Let $\{p_i \mid i \in I\}$ be a maximal set of pairwise orthogonal J-regular types from $S^J(M)$. Define $J_S \subseteq I$ by

$$J_S = \{i \in I \mid \dim_J(p_i, N_S) < \omega\}, \quad S = 1, 2,$$

and assume $J_1 = J_2$. If for every $i \in J = J_1 = J_2$ we have $\dim_J(p_i, N_1) = \dim_J(p_i, N_2)$, then the pairs of models

$$(N_1, M) \equiv_{\exists_1} (N_2, M)$$

in the signature $\sigma_P(A)$.

Proof. Since $\{p_i \mid i \in I\}$ is a maximal set of orthogonal J-regular types over M and N_1, N_2 are JF^{α}_{ω}-saturated, we can write

$$N_S = M \oplus \bigoplus_{i \in I} X_i^{(S)}, \quad S = 1, 2,$$

where $X_i^{(S)}$ denotes the set of realizations of p_i in N_S, and $\dim_J(p_i, N_S) = |X_i^{(S)}|$.

For $i \in J$, the sets $X_i^{(1)}$ and $X_i^{(2)}$ have the same finite cardinality. For $i \in I \setminus J$, both $X_i^{(S)}$ are infinite by saturation, so their finite partial substructures can always be matched.

Define $f : M \to M$ as the identity. For each $i \in J$, choose a bijection

$$f_i : X_i^{(1)} \to X_i^{(2)}.$$

For $i \in I \setminus J$, by JF^{α}_{ω}-saturation, any finite subset of $X_i^{(1)}$ can be matched with a corresponding finite subset of $X_i^{(2)}$.

Since the types p_i are pairwise orthogonal, the realizations of different p_i are independent. Therefore, the partial isomorphisms on each $X_i^{(S)}$ can be combined with the identity on M to form a partial $\exists$-isomorphism from N_1 to N_2 over M.

Hence, for any $\exists$-formula $\exists \bar{x}\, \varphi(\bar{x}, \bar{m})$ with parameters $\bar{m} \subseteq M$, we have

$$N_1 \models \exists \bar{x}\, \varphi(\bar{x}, \bar{m}) \iff N_2 \models \exists \bar{x}\, \varphi(\bar{x}, \bar{m}),$$

which proves $(N_1, M) \equiv_{\exists_1} (N_2, M)$ in $\sigma_P(A)$. □

Lemma 2. Let T be a perfect, hereditary $\exists$-complete Jonsson theory, $K \subseteq E_T$, $[\Delta] \in JSp(K)/_{\bowtie}$, $[\Delta]^* = Th(C_{[\Delta]})$ be its center. Let $M_1, M_2, N_1, N_2 \in \mathrm{Mod}([\Delta]^*)$ with $A \subseteq M_1 \cap M_2$ and $N_1, N_2 \prec_{\exists_1} C_{[\Delta]}$, $M_1 \prec_{\exists_1} N_1$, $M_2 \prec_{\exists_1} N_2$. Then there exist JF^{α}_{ω}-saturated models M, N_1', N_2' satisfying:
1) $A \subseteq M$;
2) $M \prec_{\exists_1} N_1'$ and $M \prec_{\exists_1} N_2'$;
3) $(N_1, M_1) \equiv_{\exists} (N_1', M)$ and $(N_2, M_2) \equiv_{\exists_1} (N_2', M)$ in the signature $\sigma_P(A)$.

Proof. Let M be any JF^{α}_{ω}-saturated model containing A and existentially embedding both M_1 and M_2 in the sense that

$$M_1 \prec_{\exists_1} M, \quad M_2 \prec_{\exists_1} M.$$

Such a model exists by standard chain arguments for Jonsson theories and saturation: take an increasing chain of countable extensions starting from A that embeds M_1 and M_2, and then take a JF^{α}_{ω}-saturated elementary extension.

Consider N_1' to be a JF^{α}_{ω}-saturated model such that

$$N_1 \prec_{\exists_1} N_1' \quad \text{and} \quad M \subseteq N_1'.$$

Similarly, let N_2' be a JF^{α}_{ω}-saturated model extending N_2 with $M \subseteq N_2'$. By the properties of Jonsson theories and JF^{α}_{ω}-saturation, these extensions can be chosen so that

$$(N_1, M_1) \equiv_{\exists_1} (N_1', M), \quad (N_2, M_2) \equiv_{\exists_1} (N_2', M)$$

in the signature $\sigma_P(A)$.

1. By construction, $A \subseteq M$.
2. $M \prec_{\exists_1} N_1'$ and $M \prec_{\exists_1} N_2'$ hold since N_1' and N_2' are built as saturated extensions containing M.
3. The existential equivalences follow from the JF^{α}_{ω}-saturation and the fact that $M_1 \prec_{\exists_1} N_1$, $M_2 \prec_{\exists_1} N_2$. Any existential statement over M true in N_1' or N_2' can be mirrored in N_1 or N_2 via embeddings extending M_1 and M_2.

Hence, the models M, N_1', N_2' satisfy all required properties. □

We now turn to our main result. This result generalizes Theorem 1 of [4] in the context of the Jonsson spectrum of class K.

Since K is a subclass of E_T, the Kaiser hull of any model in K is a Jonsson theory. Let Δ be one of them.

Theorem 5. Let T be a perfect, hereditary $\exists$-complete, J-superstable Jonsson theory, $K \subseteq E_T$, $[\Delta] \in JSp(K)/_{\bowtie}$, $[\Delta]^*$ be its center. Then the following conditions are equivalent:
(a) $[\Delta]$ is J-nonmultidimensional;
(b) $[\Delta]$ is J-P-superstable;
(c) $[\Delta]$ is J-P-stable.

Proof. It is obvious that (b) implies (c). We shall show that (a) implies (b), and that (c) implies (a).

In the proof of $(a) \Rightarrow (b)$:

A class $[\Delta]$ of Jonsson theories is J-nonmultidimensional if every theory in the class is J-nonmultidimensional. Let Δ a be J-nonmultidimensional theory. By Theorem 4, it follows that Δ^* is a nonmultidimensional theory.

Nonmultidimensionality implies that all regular types are related, meaning that the behavior of models can be described in terms of only one dimension.

From nonmultidimensionality it follows that all regular types are connected, that is, the behavior of models can be described via a single dimension.

Lemma 1 states: if the dimensions coincide, then the models are logically indistinguishable in the signature σ_P.

Therefore, the set of possible pairs (N, M) is bounded, and hence the Δ^* is P superstable.

Before doing so, let us prove the following auxiliary statement.

Lemma 2 is needed to estimate the number of extensions or models, that is, to turn the qualitative result of Lemma 1 (extensions with the same types are equivalent) into a quantitative estimate of the number of such extensions. It is used to control cardinalities and for the subsequent derivation of superstability or bounds on the number of types.

Here we have shown that if Δ^* is nonmultidimensional, then Δ^* is P-superstable. We now need to show that Δ is J-P-superstable.

By Theorem 3 stability properties of Δ^* and Δ are preserved under this correspondence.

Assume that Δ^* is P-superstable. Then there exists a cardinal λ_0 such that Δ^* is P-λ-stable for all $\lambda \geq \lambda_0$. By the equivalence above, Δ is J-P-λ-stable for all such λ. Hence, every Δ is J-P-superstable. Then $[\Delta]$ is J-P-superstable.

Combining Lemmas 1 and 2, we complete the proof of Theorem 5 in the direction $(a) \Rightarrow (b)$.

(c) $\Rightarrow$ (a) Assume $[\Delta]$ is J-P-stable.

Suppose, towards a contradiction, that $[\Delta]$ is multidimensional. Then there exist two J-orthogonal regular types p and q.

Orthogonal types generate uncontrolled growth in the number of possible types over substructures (any combination of realizations of p and q is possible).

This contradicts J-P-stability, which requires the number of J-types to be bounded.

Hence, $[\Delta]$ must be J-nonmultidimensional.

We have established the chain of implications:

$$(a) \Rightarrow (b) \Rightarrow (c) \Rightarrow (a),$$

and therefore all three properties are equivalent. □

Conclusion

In this article, we studied the properties of Jonsson theories with additional predicates, focusing on J-stability and J-superstability. We established conditions under which models exhibit these properties and provided a systematic construction of $T_P^J(A)$, which preserves existential completeness while extending models with new predicates. The results generalize earlier work on superstable theories and offer a broader perspective on stability in models with added structure. The introduction of J-P-stability and the analysis of types in Jonsson structures with predicates represent a novel contribution to model theory and provide a foundation for further research in combinatorial and algebraic model theory. Future work may explore multidimensional types, their interactions with additional predicates, and extensions to more general classes of theories. Overall, this study deepens the understanding of the structure and classification of models in Jonsson theories, highlighting both theoretical significance and potential applications.

Author Contributions

All authors contributed equally to the development of the study, including the formulation of main results, analysis, and writing of the manuscript. All authors have read and approved the final version.

Conflict of Interest

The authors declare no conflict of interest.

References

1 Poizat, B. (1983). Paires de structure stables. *The Journal of Symbolic Logic, 48*(2), 239–249. https://doi.org/10.2307/2273543

2 Bouscaren, E. (1989). Elementary pairs of models. *Annals of Pure and Appl. Logic, 45*(2), 129–137. https://doi.org/10.1016/0168-0072(89)90057-2

3 Palyutin, E.A. (2003). E^*-stabilnye teorii [E^*-stable theories]. *Algebra i logika — Algebra and logic, 42*(2), 194–210 [in Russian].

4 Mustafin, T.G., & Nurmagambetov, T.A. (1990). O P-stabilnosti polnykh teorii [On P-stability of complete theories]. *Strukturnye svoistva algebraicheskikh sistem — Structural properties of algebraic systems*, 88–100 [in Russian].

5 Yeshkeyev, A.R., Zhumabekova, G.E., & Tungushbayeva, I.O. (2023). The central type of a semantic pair. *Logic Colloquium European Summer Meeting of the Association for Symbolic Logic*: Collection of Abstracts, 184. Italy: University of Milan.

6 Kassymetova, M.T., & Zhumabekova, G.E. (2024). Svoistva nemultirazmernosti dlia klassa semanticheskikh par [Nonmultidimensional properties for the class of semantic pairs]. *Traditsionnaia mezhdunarodnaia aprelskaia matematicheskaia konferentsiia v chest Dnia uchenogo Respubliki Kazakhstan — Traditional International April Mathematical Conference in honor of the Day of Scientists of the Republic of Kazakhstan*: Collection of Abstracts, 222. Almaty: Institute of Mathematics and Mathematical Modeling [in Russian].

7 Amanbekov, S.M., Onerkhaan A., & Tungushbayeva, I.O. (2025). The properties of amalgamation and joint embedding in the meaning of positive Jonsson theories. *Kazakh Mathematical Journal, 25*(2), 6–18. https://doi.org/10.70474/18bp7004

8 Yarullina, A.R., & Kaliolla, D.Ye. (2022). The perfect Jonsson S-acts. *Proceedings of the International Scientific Conference "Actual Problems of Mathematics, Mechanics and Informatics" dedicated to the 80th anniversary of Professor T.G. Mustafin,* 40–42. Karaganda: Karaganda Buketov University.

9 Omarova, M.T., & Amanbekov, S.M. (2024). 1-perfectness of fragments of normal Jonsson theory. *International Conference Mal'tsev Meeting*: Collection of Abstracts, 243. Novosibirsk: Sobolev Institute of Mathematics, Novosibirsk State University.

10 Popova, N.V., & Amanbekov, S.M. (2024). The properties of 1-model companions of Jonsson sets in normal Jonsson theory. *International Conference Mal'tsev Meeting*: Collection of Abstracts, 244. Novosibirsk: Sobolev Institute of Mathematics, Novosibirsk State University.

11 Kassymetova, M.T., Tungushbayeva, I.O., & Yarullina, A.R. (2025). Syntactic similarity of Hybrids of classes of normal Jonsson theories. *International Conference Actual problems of analysis, differential equations and algebra (EMJ–2025) dedicated to the 15th anniversary of the Eurasian Mathematical Journal*: Collection of Abstracts, 162. Astana: L.N. Gumilyov Eurasian National University.

12 Kassymetova, M.T., Mussina, N.M., & Amanbekov, S.M. (2021). Companions of hybrids of fragments of theoretical subsets. *International Conference Mal'tsev Meeting*: Collection of Abstracts, 167. Novosibirsk: Sobolev Institute of Mathematics, Novosibirsk State University.

13 Tungushbayeva, I.O., & Ulbrikht, O.I. (2024). On theories with amalgamation and joint embedding properties. *International Conference Mal'tsev Meeting*: Collection of Abstracts, 171. Novosibirsk: Sobolev Institute of Mathematics, Novosibirsk State University.

14 Barwise, J. (1977). *Handbook of Mathematical Logic.* Studies in Logic and the Foundations of Mathematics (Vol. 90). Amsterdam: North-Holland.

15 Yeshkeyev, A.R. (2024). *Teorii i ikh modeli [Theories and their models]: monograph* (Vols. 1–2). Karaganda: Izdatelstvo KarU [in Russian].

*Author Information**

Aibat Rafhatuly Yeshkeyev — Doctor of Physical and Mathematical Sciences, Professor, Professor researcher of the Department of Algebra, Mathematical Logic and Geometry named after Professor T.G. Mustafin, Buketov Karaganda National Research University, 28 Universitetskaya St., Karaganda, 100028, Kazakhstan; e-mail: *aibat.kz@gmail.com*; https://orcid.org/0000-0003-0149-6143

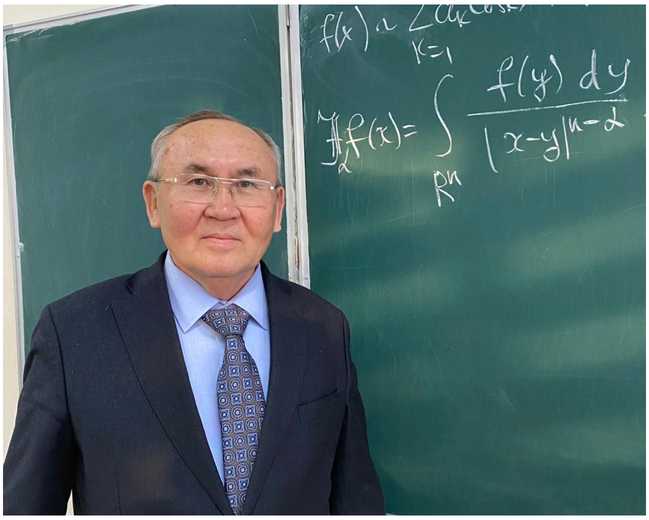
Galiya Erkinovna Zhumabekova (*corresponding author*) — PhD, Associate Professor of the Department of Algebra, Mathematical Logic and Geometry named after Professor T.G. Mustafin, Buketov Karaganda National Research University, 28 Universitetskaya St., Karaganda, 100028, Kazakhstan; e-mail: *galkatai@mail.ru*; https://orcid.org/0009-0002-2785-1756

Aigul Koishibaevna Issayeva — PhD, Assistant Professor of the Department Methods of Teaching Mathematics and Computer Science, Buketov Karaganda National Research University, 28 Universitetskaya St., Karaganda, 100028, Kazakhstan; e-mail: *isaevaaiga@gmail.com*; https://orcid.org/0009-0006-5697-0710

*Authors' names are presented in the following order: first name, middle name (if any), last name.

ANNIVERSARIES

Professor Bokayev's Mathematical Olymp: Celebrating the Scholar's 70th Anniversary



The academic community of Kazakhstan honors Nurzhan Adilkhanovich Bokayev, Doctor of Physical and Mathematical Sciences and Professor of the Department of Fundamental Mathematics at L.N. Gumilyov Eurasian National University for his distinguished contributions to the field. His 70th anniversary is a celebration not only for his colleagues and students, but also for the entire mathematical community of our country.

From an early age, having demonstrated remarkable abilities in the exact sciences, Nurzhan Adilkhanovich entered Karaganda State University named after academician Ye.A. Buketov. His years of study (1972–1977) became a time of rapid intellectual growth. For exceptional achievements and devotion to science, Nurzhan Bokayev was awarded the highest distinction of that era – the Lenin Scholarship. This prestigious status was a natural result of his diligence and opened the way to the principal scientific center of a vast country – Lomonosov Moscow State University.

Moscow State University and Menshov school provided Nurzhan Adilkhanovich with a world-class academic foundation. His initial inspiration, passion for function theory, and his introduction to research came from his Karaganda mentor – Doctor of Physical and Mathematical Sciences, Professor Esmukhanbet Saidakhametovich Smailov. It was under Professor Smailov's attentive guidance that student Bokayev began his first serious research. Esmukhanbet Saidakhametovich, a renowned specialist in the theory of embeddings of functional spaces and approximation theory, recognized in the young man a rare analytical talent. The "Teacher–Student" bond between Smailov and Bokayev grew into many years of fruitful collaboration. Their joint works on functional spaces and multiple Fourier series laid the foundation for what is now known as the Karaganda school of mathematical analysis. It is widely believed that Esmukhanbet Saidakhametovich played a decisive role in directing Bokayev – an outstanding graduate and Lenin Scholarship recipient – toward postgraduate study at Moscow State University under V.A. Skvortsov. He understood that talent of such magnitude must be shaped by a world-class school, in order to return and advance science in Kazakhstan.

Bokayev's formation as a researcher of international stature is inseparably linked with the Faculty of Mechanics and Mathematics of Lomonosov Moscow State University. There he became a direct successor to the great Moscow school of function theory. His academic advisor was the eminent mathematician, Professor Valentin Anatolyevich Skvortsov. Through his mentor, Nurzhan Adilkhanovich became the scholarly "grandson" of the legendary Dmitry Yevgenyevich Menshov, one of the founders of the Soviet school of function theory. This unique continuity of generations enabled the honoree to master the most complex areas of analysis – fields he continues to develop today.

Professor Bokayev's research is a synthesis of classical rigor and modern perspective. He has made a significant contribution to the study of the uniqueness of function expansions in multiple orthogonal series. Nurzhan Adilkhanovich masterfully adapted classical methods of function theory to contemporary problems of signal processing in functional spaces.

After Moscow State University, his professional journey continued within the walls of his alma mater – Karaganda State University named after academician Ye.A. Buketov. Returning to KarSU named after academician Ye.A. Buketov, he advanced from assistant lecturer to associate professor and then full professor. It was there that he began to introduce the high standards of the Moscow mathematical school, teaching courses in mathematical analysis and specialized courses in function theory. In different years, Nurzhan Adilkhanovich held important administrative positions, including Dean of the Faculty of Mathematics at Karaganda State University named after academician Ye.A. Buketov and editor of the journal "Bulletin of the Karaganda University. Mathematics Series". Under his leadership, the faculty strengthened its standing as one of the leading centers for training mathematicians in the region. During this period, he actively supervised his first postgraduate students, laying the groundwork for the constellation of disciples who today represent Kazakhstan's mathematical science. At present, Nurzhan Adilkhanovich is a member of the editorial board of the journal "Bulletin of the Karaganda University. Mathematics Series" and continues to take an active part in the scientific life of his home university.

His work in Karaganda became a period of building the powerful scientific potential that later led to the defense of his doctoral dissertation and his move to L.N. Gumilyov Eurasian National University in the capital.

Today, working at ENU, Professor Bokayev remains a benchmark of academic ethics. A two-time recipient of the state grant "Best University Teacher," he has educated dozens of students who now work successfully in science and education.

Colleagues value him for his encyclopedic knowledge, and students – for clarity of thought and the ability to inspire a love for the beauty of mathematical logic. For him, mathematics has never been merely a profession, but a calling to which he has remained faithful for half a century.

We congratulate Nurzhan Adilkhanovich on his anniversary! We wish him robust health, inexhaustible inspiration for new achievements, and many years of fruitful work for the benefit of Kazakhstan's science.

The staff of the Faculty of Mathematics and Information Technologies
of the Karaganda National Research University Academician named after Ye.A. Buketov
and the editorial board of the journal
"Bulletin of the Karaganda University. Mathematics Series"