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PREFACE

MATHEMATICS

PREFACE

https://doi.org/10.31489/2025M4/4 *Editorial*

Functional analysis in interdisciplinary applications

Guest-Editors: Allaberen Ashyralyev[1,2,3,*], Charyyar Ashyralyyev[1,4,5], Makhmud Sadybekov[2]

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Keywords: boundary value problems, delay differential operators with involution, difference schemes, Dirichlet problem, identification problems, involution, integro-differential equations, numerical methods and solutions, partial differential equations, regular solutions, stability, well-posedness.

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This issue presents a collection of 15 carefully selected papers authored by both international and national researchers. Each paper has undergone rigorous peer review and introduces novel findings in the fields of analysis and applied mathematics, with particular emphasis on their application to the construction and investigation of solutions to well-posed and ill-posed boundary value problems for partial differential equations.

The contributing authors represent a diverse range of countries, including Turkey, Kazakhstan, Sweden, the Russian Federation, Azerbaijan, Uzbekistan, Turkmenistan, Iraq and Cyprus. We are especially pleased to note that many of these articles are co-authored by researchers from different universities across the globe, reflecting the collaborative spirit and international scope of contemporary mathematical research.

Guest-Editors: *A. Ashyralyev, C. Ashyralyyev* and *M. Sadybekov*

September 15, 2025

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MATHEMATICS

https://doi.org/10.31489/2025M4/5-20 *Research article*

The quadratic $\mathcal{B}$-spline method for approximating systems of Volterra integro fractional-differential equations involving both classical and fractional derivatives

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The quadratic $\mathcal{B}$-spline method is a widely recognized numerical technique for solving systems of Volterra integro-differential equations that involve both classical and fractional derivatives (SVIDE's-CF). This study presents an improved application of the quadratic $\mathcal{B}$-spline approach to achieve highly accurate and computationally efficient solutions. In the method developed in this paper, control points are treated as unknown variables within the framework of the approximate solution. The fractional derivative ${}_{a}^{C}\mathcal{D}_{x}^{\sigma}$ is considered in the Caputo sense. First, we divide the domain into subintervals, then construct quadratic $\mathcal{B}$-spline basis functions over each subinterval. The approximate solution is presented as a quadratic combination of these $\mathcal{B}$-spline functions over each subinterval, where the control points act as variables. To simplify the system of (VIDE's-CF) into a solvable set of algebraic equations, the collocation method is applied by discretizing the equations at chosen points within each subinterval. The Jacobian matrix method is employed to perform computations efficiently. In addition, a careful, step-by-step algorithm for employing the proposed method is presented to simplify its use, we implemented the method in a Python program and optimized it for efficiency. Experimental example demonstrates effectiveness and accuracy of the proposed technique and its comparison with present techniques in terms of accuracy and computational efficiency.

Keywords: system of Volterra integro-fractional differential equation (SVIDE's), quadratic $\mathcal{B}$-spline functions, Caputo fractional derivative, collocation method, Jacobian matrix algorithm, Clenshaw-Curtis quadrature rule.

2020 Mathematics Subject Classification: 34K33, 45D05, 45J05.

Introduction

Mathematicians have extended the classical concepts of differentiation and integration to fractional (non-integer) orders over the centuries [1]. This kind of generalization, which is referred to as fractional calculus (FC), is a more general mathematical framework for investigating complex systems [2]. Compared with the ordinary calculus that deals with essentially local and instantaneous changes, the fractional calculus incorporates memory and hereditary properties and therefore is particularly suitable to model those processes where the present state depends not only on the present status but also on the past history [3, 4]. Several real-life phenomena demonstrate such dynamics [5]. For example, diffusion

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in porous media, viscoelasticity, and biological systems with memory function regularly display dynamics that are not possible to describe through classical models. In all these cases, fractional derivatives give more accurate and flexible descriptions, accounting for long-range temporal and spatial dependencies [6–8]. After that, researchers developed integro-fractional differential equations that combine fractional derivatives with integral terms. Such equations extend traditional differential and integral equations to include both instantaneous rates of change and accumulative past effects at the same time. This makes them powerful tools for modelling dynamic processes where past states exert strong impacts on current and future dynamics [8,9]. Furthermore, fractional integro-differential and integro-differential equations of fractional order have garnered significant interest in the literature, leading to the development of several unique methodologies. Benzahi et al. demonstrated a least squares method [10]. Ghosh et al. presented an iterative difference scheme for solving an arbitrary order nonlinear Volterra integro-differential population growth model [11]. Rahimkhani et al. illustrated nonlinear fractional integro-differential equations using fractional alternative Legendre functions [12]. Akbar et al. presented an analysis of delay [13]. Miran et al. presented Laplace transform multi-time delay [14]. Akhlaghi et al. addressed fractional order integro-differential equations via Muntz orthogonal functions [15]. Yuzbai et al. presented a fractional Bell collocation method [16]. In practice, most linear Volterra integro-fractional differential systems with variable coefficients are too complex to solve exactly using analytical methods. Because of this, researchers often turn to approximation techniques and numerical methods. One of the most common and effective tools for this purpose is the use of spline and $\mathcal{B}$-spline functions [17, 18]. These functions play a crucial role in solving both linear and nonlinear functional equations. Many researchers use $\mathcal{B}$-spline functions to solve various mathematical problems because of their flexibility and accuracy [19–21].

This study presents an approximate method for solving the linear system associated with Volterra integro-differential equations, encompassing classical and fractional orders (LSVIDE's-CF). For the derivation, it deals with quadratic $\mathcal{B}$-spline interpolation functions. Which takes the following general forms:

$$\mathcal{P}_i(x)\mathcal{U}_i''(x) + a_{i0}(x)\,{}_a^C\mathcal{D}_x^{\sigma_{i0}}\mathcal{U}_i(x) + a_{i1}(x)\,{}_a^C\mathcal{D}_x^{\sigma_{i1}}\mathcal{U}_i(x) + a_{i2}(x)\mathcal{U}_i(x) = \mathcal{F}_i(x) + \sum_{j=0}^{m} \omega_{ij} \int_a^x \mathcal{K}_{ij}(x,s)\,{}_a^C\mathcal{D}_s^{\beta_{ij}}\mathcal{U}_j(s)\,ds. \quad (1)$$

Under the following conditions:

$$\left[\mathcal{D}_x^{k_i}\mathcal{U}_i(x)\right]_{x=a} = \vartheta_{ik_i}, \forall k_i = 0,1,\ldots,\mu_i - 1, \text{ and } i = 0,1,\ldots,m. \quad (2)$$

The variable coefficients $\mathcal{P}_i(x)(\not\equiv 0), a_{i0}(x),\ a_{i1}(x) \in C([a,b],\mathbb{R})$ and $\mathcal{K}_{ij} \in C(\Theta,\mathbb{R}),\ \Theta = \{(x,s) : a \leq s < x \leq b\}$, with fractional orders: $\sigma_{i1} > \sigma_{i0} > 0$ and $\beta_{im} > \beta_{i(m-1)} > \cdots > \beta_{i1} > \beta_{i0} = 0$. Furthermore, the $\mu_i = \max\left\{2, m_{im}^{\beta}\right\}$, where $m_{ij}^{\beta} - 1 < \beta_{ij} \leq m_{ij}^{\beta}, m_{ij}^{\beta} = \lceil \beta_{ij} \rceil$, $\omega_{ij} \in \mathbb{R}$, for all $i,j = 0,1,\ldots,m$. In the manuscript, we examined and assessed the systems of Volterra integro-differential equations for classical and fractional orders (SVIDE's-CF); according to the conditions, fractional orders between 0 and 1. We approximate these integrals using the Clenshaw-Curtis quadrature rule [17, 22] in conjunction with quadratic $\mathcal{B}$-spline functions. Four algorithms summarized the information, and we later produced Python software to implement each algorithm. These algorithms resolved a few test instances. The paper is structured as follows: in Section 1, we introduce some notions of fractional calculus necessary for the description of our model, and then we define the fundamental concepts of $\mathcal{B}$-spline functions. In Section 2, we introduce the numerical approximation that we use throughout our work. The experimental outcomes are presented in Section 3. Lastly, the concluding remarks on the proposed method are presented in Section 4.

1 Basic definitions and notation

In this section, we will introduce and study the concepts.

Definition 1. [1,2] Let $n-1<\alpha\le n$ $(\in\mathbb{Z}^+)$, $\alpha\in\mathbb{R}^+$. The operators $({}^R_a\mathcal{D}^\alpha_x\mathcal{V}(x))$ and $({}^C_a\mathcal{D}^\alpha_x\mathcal{V}(x))$ of fractional order α are defined as:

$$ {}^R_a\mathcal{D}^\alpha_x\mathcal{V}(x)=\mathcal{D}^n_x\left({}_a\mathcal{J}^{n-\alpha}_x\mathcal{V}(x)\right)=\frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dx^n}\int_a^x\frac{\mathcal{V}(s)}{(x-s)^{\alpha+1-n}}ds,\quad x>a, \tag{3}$$

$$ {}^C_a\mathcal{D}^\alpha_x\mathcal{V}(x)={}_a\mathcal{J}^{n-\alpha}_x\mathcal{D}^n_x\mathcal{V}(x)=\frac{1}{\Gamma(n-\alpha)}\int_a^x\frac{\mathcal{V}^{(n)}(s)}{(x-s)^{\alpha+1-n}}ds,\quad x>a. \tag{4}$$

Equation (3) represents the *Riemann–Liouville fractional differential operator.* Additionally, the operator ${}_a\mathcal{J}^\alpha_x$, known as the *Riemann–Liouville fractional integral* of order α, is defined as

$$ {}_a\mathcal{J}^\alpha_x\mathcal{V}(x)=\frac{1}{\Gamma(\alpha)}\int_a^x(x-s)^{\alpha-1}\mathcal{V}(s)ds,\quad {}_a\mathcal{J}^0_x\mathcal{V}(x)=\mathcal{V}(x),\quad x>a.$$

Equation (4) defines the *Caputo fractional differential operator.* Similar to integer-order differentiation, the Caputo fractional differentiation is a linear operation:

$$ {}^C_a\mathcal{D}^\alpha_x\left[\rho_1\mathcal{V}_1(x)+\rho_2\mathcal{V}_2(x)\right]=\rho_1{}^C_a\mathcal{D}^\alpha_x\mathcal{V}_1(x)+\rho_2{}^C_a\mathcal{D}^\alpha_x\mathcal{V}_2(x).$$

Furthermore, the Caputo derivative of any constant function (say $\mathcal{A}\in\mathbb{R}$) vanishes: ${}^C_a\mathcal{D}^\alpha_x\mathcal{A}=0$.

Lemma 1. [1,9] The function $\mathcal{V}(x)=(x-a)^n$ for $n\ge 0$ has a β-Caputo derivative $(\beta\ge 0)$, which is given as follows: for $n\in\{0,1,2,\dots,\lceil\beta\rceil-1\}$, the β-Caputo derivative vanishes, i.e., ${}^C_a\mathcal{D}^\beta_x\mathcal{V}(x)=0$. While for $n\in\mathbb{N}$ and $n\ge\lceil\beta\rceil$ or $n\notin\mathbb{N}$ and $n>\lceil\beta\rceil-1$, where $\lceil\beta\rceil$ represents the least integer that is not less than β, it is given by:

$$ {}^C_a\mathcal{D}^\beta_x\mathcal{V}(x)=\frac{\Gamma(n+1)}{\Gamma(n-\beta+1)}(x-a)^{n-\beta}.$$

Definition 2. [17,22] In 1960, Clenshaw and Curtis established a method for evaluating a definite integral by representing the integrand through a finite Chebyshev series. This involves sequentially summing the individual terms the series. This approach proves to be highly effective, especially when applied to integral equations.

$$\int_{-1}^1\mathcal{V}(x)\,dx\approx\sum_{\substack{r=0\\ \text{even}}}^{\mathcal{N}}\left(\frac{2}{\mathcal{N}}\sum_{K=0}^{\mathcal{N}}\cos\left(\frac{rK\pi}{\mathcal{N}}\right)\mathcal{V}\left(\cos\left(\frac{K\pi}{\mathcal{N}}\right)\right)\right),\quad K=0,1,\dots,\mathcal{N}.$$

Remark:

(I) The symbol $\sum$ indicates that the initial and final terms should be divided by two before summation.

(II) The transformation $x=\frac{b-a}{2}t+\frac{b+a}{2}$, or $t=2\left(\frac{x-a}{b-a}\right)-1$, where $x\in[a,b]$ and $t\in[-1,1]$.

Definition 3. [23] Let $\mathcal{T}_{\mathcal{N}}=\{x_0,x_1,\dots,x_{\mathcal{N}}\}$ be a uniform or non-uniform partition of the interval $[a,b]$. The $\mathcal{K}$-degree $\mathcal{B}$-spline basis function $\mathcal{B}^{\mathcal{K}}_r(x)$, $r\ge 0$, is defined as follows:

$$\mathcal{B}^{\mathcal{K}}_r(x)=\frac{x-x_r}{x_{r+\mathcal{K}}-x_r}\mathcal{B}^{\mathcal{K}-1}_r(x)+\frac{x_{r+\mathcal{K}+1}-x}{x_{r+\mathcal{K}+1}-x_{r+1}}\mathcal{B}^{\mathcal{K}-1}_{r+1}(x),$$

$$\mathcal{B}_r^0(x) = \begin{cases} 1, & x \in [x_r, x_{r+1}), \\ 0, & \text{otherwise.} \end{cases} \tag{5}$$

$$\mathcal{B}_r^1(x) = \begin{cases} \frac{x-x_r}{x_{r+1}-x_r}, & \text{if } x \in [x_r, x_{r+1}), \\ \frac{x_{r+2}-x}{x_{r+2}-x_{r+1}}, & \text{if } x \in [x_{r+1}, x_{r+2}), \\ 0, & \text{otherwise.} \end{cases} \tag{6}$$

Equations (5) and (6) represent zero-degree and one-degree $\mathcal{B}$-splines, respectively [24, 25].

Note that the local support property is $\mathcal{B}_r^{\mathcal{K}}(x) = 0$ for all $x \notin [x_r, x_{r+\mathcal{K}+1})$ and the nonnegativity property is $\mathcal{B}_r^{\mathcal{K}}(x) \geq 0$ for all $x \in [x_r, x_{r+\mathcal{K}+1})$.

2 Methods analysis

In this section, we utilize the quadratic $\mathcal{B}$-spline collocation method to compute the approximate solution (SVIDE's-CF) of equation (1) subject to equation (2), where the mesh points $a = x_0 < x_1 < x_2 < \cdots < x_{\mathcal{N}-1} < x_{\mathcal{N}} = b$ form a uniform partition of the solution domain $[a, b]$ defined by the knots x_r with $h = \frac{x_{r+1}-x_r}{\mathcal{N}} = \frac{b-a}{\mathcal{N}}$, $r = 0, 1, \ldots, \mathcal{N} - 1$. The numerical solution for treating equations (1) and (2) for all $x \in [a, b]$, $\mathcal{U}_i(x) \approx \mathbb{P}_i^{\mathcal{Q},2}(x)$ for each $i = 0, 1, \ldots, m$, using collocation techniques with quadratic $\mathcal{B}$-spline to find an approximate solution $\mathbb{P}_i^{\mathcal{Q},2}(x)$ given by:

$$\mathbb{P}_i^{\mathcal{Q},2}(x) \approx \sum_{l=0}^{n} \mathcal{C}_i^l \mathcal{B}_{il}^k(x), \quad i = 0, 1, \ldots, m. \tag{7}$$

Here, the general expression of a quadratic $\mathcal{B}$-spline curve defined on the interval $[a, b]$ is,

$$\mathbb{P}_i^{\mathcal{Q},2}(x) = \mathcal{C}^0 \left(\frac{b-x}{b-a}\right)^2 + 2\mathcal{C}^1 \left(\frac{x-a}{b-a}\right)\left(\frac{b-x}{b-a}\right) + \mathcal{C}^2 \left(\frac{x-a}{b-a}\right)^2, \quad x \in [a, b]. \tag{8}$$

Now, the Caputo fractional derivative of order $\alpha \in (0, 1]$, and the recursive derivative formula for quadratic $\mathcal{B}$-spline curves for equation (8) are given, respectively:

$$\begin{aligned} &{}_a^C\mathcal{D}_x^{\alpha} \mathbb{P}_i^{\mathcal{Q},2}(x) \\ &= \frac{2(x-a)^{1-\alpha}}{(\mathcal{N}h)^2 \Gamma(3-\alpha)} \left\{ \mathcal{C}^0[(x-a) - (\mathcal{N}h)(2-\alpha)] + \mathcal{C}^1[(\mathcal{N}h)(2-\alpha) - (x-a)] + \mathcal{C}^2(x-a) \right\}, \end{aligned} \tag{9}$$

$$\frac{d^2}{dx^2}\mathbb{P}_i^{\mathcal{Q},2}(x) = \frac{2\left(\mathcal{C}^0 - 2\mathcal{C}^1 + \mathcal{C}^2\right)}{(\mathcal{N}h)^2}, \tag{10}$$

where h is a step size and $\mathcal{N}$ is the number of iterations. For the numerical approximate solutions of (SVIDE's-CF) based on equation (1) the control points $\mathcal{C}_i^l$ are unknowns. Also, for all $i = 0, 1, \ldots, m$, the control points $\mathcal{C}_i^0$ are determined by initial conditions specified in equation (1), and the control points $\mathcal{C}_i^1 = \frac{(\mathcal{N}h)}{2}\frac{(d\mathcal{B}_i^2(a))}{dt} + \mathcal{C}_i^0$ for the quadratic $\mathcal{B}$-spline curve can be determined for each $i = 0, 1, \ldots, m$, to find $\mathcal{C}_i^2$, from the (VIDE's-CF) in equation (1). The unknown function $\mathbb{P}_i^{\mathcal{Q},2}(x)$ is approximated by $\mathcal{B}$-spline interpolation of degree 2 as in equation (7), so the equation (1) becomes:

$$\begin{aligned} &\mathcal{P}_i(x)\frac{d^2}{dx^2}\mathbb{P}_i^{\mathcal{Q},2}(x) + a_{i0}(x)\,{}_a^C\mathcal{D}_x^{\sigma_{i0}}[\mathbb{P}_i^{\mathcal{Q},2}(x)] + a_{i1}(x)\,{}_a^C\mathcal{D}_x^{\sigma_{i1}}[\mathbb{P}_i^{\mathcal{Q},2}(x)] + a_{i2}(x)\mathbb{P}_i^{\mathcal{Q},2}(x) \\ &\qquad = \mathcal{F}_i(x) + \sum_{j=0}^{m} \omega_{ij} \int_a^x K_{ij}(x, s)\,{}_a^C\mathcal{D}_s^{\beta_{ij}}[\mathbb{P}_j^{\mathcal{Q},2}(s)]\,ds. \end{aligned} \tag{11}$$

The fractional orders $\sigma_{i0,i1}, \beta_{ij} \in (0,1]$, $\forall i,j = 0,1\ldots,m$. Consequently, using equation (11), applying the linearity property of fractional Caputo derivatives, and using equation (8), by defining $[x_r, x_{r+1}]$ and analyzing the collocation points, we use a quadratic $\mathcal{B}$-spline function ($\mathcal{K}=2$, $n=2$) in the interval $[x_r, x_{r+1}]$ as established, from equations (9), and (10), yielding results for each $r = 0,1,\ldots,\mathcal{N}-1$ and $i = 0,1,\ldots,m$, derive the following:

$$\mathcal{P}_i(x_{r+1})\left[\tfrac{2\mathcal{C}_i^0}{(\mathcal{N}h)^2} - \tfrac{4\mathcal{C}_i^1}{(\mathcal{N}h)^2} - \tfrac{2\mathcal{C}_i^2}{(\mathcal{N}h)^2}\right] + a_{i0}(x_{r+1})\frac{(x_{r+1}-a)^{1-\sigma_{i0}}}{(\mathcal{N}h)^2\Gamma(3-\sigma_{i0})}\begin{bmatrix}\mathcal{C}_i^0(-2(\mathcal{N}h)(2-\sigma_{i0})+(x_{r+1}-a))\\ +2\mathcal{C}_i^1((\mathcal{N}h)(2-\sigma_{i0})-2(x_{r+1}-a))\\ +2\mathcal{C}_i^2(x_{r+1}-a)\end{bmatrix}$$
$$+ a_{i1}(x_{r+1})\frac{(x_{r+1}-a)^{1-\sigma_{i1}}}{(\mathcal{N}h)^2\Gamma(3-\sigma_{i1})}\begin{bmatrix}\mathcal{C}_i^0(-2(\mathcal{N}h)(2-\sigma_{i1})+(x_{r+1}-a))\\ +2\mathcal{C}_i^1((\mathcal{N}h)(2-\sigma_{i1})-2(x_{r+1}-a))\\ +2\mathcal{C}_i^2(x_{r+1}-a)\end{bmatrix} + a_{i2}(x_{r+1})\begin{bmatrix}\frac{\mathcal{C}_i^0(b-x_{r+1})^2}{(\mathcal{N}h)^2}+\\ \frac{2\mathcal{C}_i^1(x_{r+1}-a)(b-x_{r+1})}{(\mathcal{N}h)^2}\\ +\frac{\mathcal{C}_i^2(x_{r+1}-a)^2}{(\mathcal{N}h)^2}\end{bmatrix}$$
$$= \mathcal{F}_i(x_{r+1}) + \sum_{j=1}^{m}\omega_{ij}\left\{\sum_{q=0}^{r-1}\int_{x_q}^{x_{q+1}}\mathcal{K}_{ij}(x_{r+1},s)\frac{(s-a)^{1-\beta_{ij}}}{(\mathcal{N}h)^2\Gamma(3-\beta_{ij})}\begin{bmatrix}\mathcal{C}_j^0(-2(\mathcal{N}h)(2-\beta_{ij})+2(s-a))\\ +2\mathcal{C}_j^1((\mathcal{N}h)(2-\beta_{ij})-2(s-a))\\ +2\mathcal{C}_j^2(s-a)\end{bmatrix}ds\right\}$$
$$+\int_{x_r}^{x_{r+1}}\mathcal{K}_{ij}(x_{r+1},s)\frac{(s-a)^{1-\beta_{ij}}}{(\mathcal{N}h)^2\Gamma(3-\beta_{ij})}\begin{bmatrix}\mathcal{C}_j^0(-2(\mathcal{N}h)(2-\beta_{ij})+2(s-a))+2\mathcal{C}_j^1\\ ((\mathcal{N}h)(2-\beta_{ij})-2(s-a))+2\mathcal{C}_j^2(s-a)\end{bmatrix}ds$$
$$+\omega_{i0}\left\{\sum_{q=0}^{r-1}\int_{x_q}^{x_{q+1}}\mathcal{K}_{i0}(x_{r+1},s)\left[\tfrac{\mathcal{C}_0^0}{(\mathcal{N}h)^2}(b-s)+\tfrac{2\mathcal{C}_0^1}{(\mathcal{N}h)^2}(s-a)(b-s)+\tfrac{\mathcal{C}_0^2}{(\mathcal{N}h)^2}(s-a)^2\right]ds\right.$$
$$\left.+\int_{x_r}^{x_{r+1}}\mathcal{K}_{i0}(x_{r+1},s)\left[\tfrac{\mathcal{C}_0^0}{(\mathcal{N}h)^2}(b-s)+\tfrac{2\mathcal{C}_0^1}{(\mathcal{N}h)^2}(s-a)(b-s)+\tfrac{\mathcal{C}_0^2}{(\mathcal{N}h)^2}(s-a)^2\right]ds\right\}. \tag{12}$$

The quadratic $\mathcal{B}$-spline function throughout the interval $[x_r, x_{r+1}]$ is optimized to simplify its representation and promote efficient solution methods; it is also determined, and in practice, these integrals must be approximated using the Clenshaw-Curtis quadrature rule; hence, the equation (12) is applicable for $r = 0,1,\ldots,\mathcal{N}-1$, $i = 0,1,\ldots,m$.

$$\mathcal{H}_i^r(\sigma_{i0})\mathcal{C}_i^2 + \mathcal{O}_i^r(\sigma_{i0})\mathcal{C}_i^1 + \mathcal{T}_i^r(\sigma_{i0})\mathcal{C}_i^0 = \mathcal{F}_i(x_{r+1}) + \sum_{j=1}^{m}\omega_{ij}\left(\mathcal{A}_{K_{ij}}^{\beta_{ij}}\right)\mathcal{C}_j^2 + \sum_{j=1}^{m}\omega_{ij}\left(S_{K_{ij}}^{\beta_{ij}}\right)\mathcal{C}_j^1$$
$$+\sum_{j=1}^{m}\omega_{ij}\left(\mathcal{X}_{K_{ij}}^{\beta_{ij}}\right)\mathcal{C}_j^0 + \left(\omega_{i0}^{(r)}\mathcal{Y}_{K_{i0}}\right)\mathcal{C}_0^0 + \left(\omega_{i0}^{(r)}\bar{\mathcal{Y}}_{K_{i0}}\right)\mathcal{C}_0^1 + \left(\omega_{i0}^{(r)}\check{\mathcal{Y}}_{K_{i0}}\right)\mathcal{C}_0^2,$$

where

$$\mathcal{H}_i^r(\sigma_{i0}) = \frac{2\mathcal{P}_i(x_{r+1})}{(\mathcal{N}h)^2} + \frac{2a_{i0}(x_{r+1})\left((r+1)h\right)^{1-\sigma_{i0}}(x_{r+1}-a)}{(\mathcal{N}h)^2\Gamma(3-\sigma_{i0})}$$
$$+\frac{2a_{i1}(x_{r+1})\left((r+1)h\right)^{1-\sigma_{i1}}(x_{r+1}-a)}{(\mathcal{N}h)^2\Gamma(3-\sigma_{i1})} + \frac{a_{i2}(x_{r+1})(x_{r+1}-a)^2}{(\mathcal{N}h)^2},$$

$$\mathcal{O}_i^r = \frac{-4\mathcal{P}_i(x_{r+1})}{(\mathcal{N}h)^2} + \frac{2a_{i0}(x_{r+1})\left((r+1)h\right)^{1-\sigma_{i0}}}{(\mathcal{N}h)^2\Gamma(3-\sigma_{i0})}\left(-2(\mathcal{N}h)^{2-\sigma_{i0}} + 2(x_{r+1}-a)\right)$$
$$+ \frac{2a_{i1}(x_{r+1})\left((r+1)h\right)^{1-\sigma_{i1}}}{(\mathcal{N}h)^2\Gamma(3-\sigma_{i1})}\left(-2(\mathcal{N}h)^{2-\sigma_{i1}} + 2(x_{r+1}-a)\right) + \frac{2a_{i2}(x_{r+1})(x_{r+1}-a)}{(\mathcal{N}h)^2(b-x_{r+1})},$$

$$\mathcal{T}_i^r = \frac{2\mathcal{P}_i\left(x_{r+1}\right)}{(\mathcal{N}h)^2} + \frac{a_{i0}\left(t_{r+1}\right)\left((r+1)h\right)^{1-\sigma_{i0}}}{(\mathcal{N}h)^2\Gamma(3-\sigma_{i0})}\left\{-2(\mathcal{N}h)\left(2-\sigma_{i0}\right) + 2\left(x_{r+1}-a\right)\right\}$$
$$+ \frac{a_{i1}\left(x_{r+1}\right)\left((r+1)h\right)^{1-\sigma_{i1}}}{(\mathcal{N}h)^2\Gamma(3-\sigma_{i1})}\left\{-2(\mathcal{N}h)\left(2-\sigma_{i1}\right) + 2\left(x_{r+1}-a\right)\right\} + \frac{a_{i2}\left(x_{r+1}\right)}{(\mathcal{N}h)^2}\left(b-x_{r+1}\right)^2,$$

$$ {}_r\mathcal{A}_{\mathcal{K}_{ij}}^{\beta_{ij}} = \sum_{q=0}^{r-1}\sum_{k=0}^{\mathcal{N}}\left[\frac{x_{q+1}-x_q}{2}\omega_k\mathcal{K}_{ij}(x_{r+1},S_k)\frac{\left(2(S_k-a)\right)^{1-\beta_{ij}}(S_k-a)}{(\mathcal{N}h)^2\Gamma(3-\beta_{ij})}\right]$$
$$+\sum_{k=0}^{\mathcal{N}}\left[\frac{x_{r+1}-x_r}{2}\omega_k\mathcal{K}_{ij}(x_{r+1},S_k)\frac{\left(2(S_k-a)\right)^{1-\beta_{ij}}(S_k-a)}{(\mathcal{N}h)^2\Gamma(3-\beta_{ij})}\right], \quad (13)$$

$$ {}_r\mathcal{S}_{\mathcal{K}_{i0}}^{\beta_{ij}} = \sum_{q=0}^{r-1}\sum_{k=0}^{\mathcal{N}}\left[\frac{x_{q+1}-x_q}{2}\omega_k\mathcal{K}_{ij}(x_{r+1},S_k)\frac{2(S_k-a)^{1-\beta_{ij}}}{(\mathcal{N}h)^2\Gamma(3-\beta_{ij})}\right]\left[(\mathcal{N}h)(2-\beta_{ij})-2(S_k-a)\right]$$
$$+\sum_{k=0}^{\mathcal{N}}\left[\frac{x_{r+1}-x_r}{2}\omega_k\mathcal{K}_{ij}(x_{r+1},S_k)\frac{2(S_k-a)^{1-\beta_{ij}}}{(\mathcal{N}h)^2\Gamma(3-\beta_{ij})}\right]\left[(\mathcal{N}h)(2-\beta_{ij})-2(S_k-a)\right], \quad (14)$$

$$ {}_r\mathcal{X}_{\mathcal{K}_{ij}}^{\beta_{ij}} = \sum_{q=0}^{r-1}\sum_{k=0}^{\mathcal{N}}\left[\frac{x_{q+1}-x_q}{2}\omega_k\mathcal{K}_{ij}(x_{r+1},S_k)\frac{(S_k-a)^{1-\beta_{ij}}}{(\mathcal{N}h)^2\Gamma(3-\beta_{ij})}\right]\left[-2(\mathcal{N}h)(2-\beta_{ij})+2(S_k-a)\right]$$
$$+\sum_{k=0}^{\mathcal{N}}\left[\frac{x_{r+1}-x_r}{2}\omega_kK_{ij}(x_{r+1},S_k)\frac{(S_k-a)^{1-\beta_{ij}}}{(\mathcal{N}h)^2\Gamma(3-\beta_{ij})}\right]\left[-2(\mathcal{N}h)(2-\beta_{ij})\right], \quad (15)$$

$$ {}_r^{\omega_{i0}}\mathcal{Y}_{\mathcal{K}_{i0}} = \omega_{i0}\left\{\sum_{q=0}^{r-1}\sum_{k=0}^{\mathcal{N}}\left[\frac{x_{q+1}-x_q}{2}\omega_k\mathcal{K}_{i0}(x_{r+1},S_k)\frac{(b-S_k)^2}{(\mathcal{N}h)^2}\right]\right.$$
$$\left.+\sum_{k=0}^{\mathcal{N}}\left[\frac{x_{r+1}-x_r}{2}\omega_k\mathcal{K}_{i0}(x_{r+1},S_k)\frac{(b-S_k)^2}{(\mathcal{N}h)^2}\right]\right\}, \quad (16)$$

$$ {}_r^{\omega_{i0}}\overline{\mathcal{Y}}_{\mathcal{K}_{i0}} = \omega_{i0}\left\{\sum_{q=0}^{r-1}\sum_{k=0}^{\mathcal{N}}\left[\frac{x_{q+1}-x_q}{2}\omega_k\mathcal{K}_{i0}(x_{r+1},S_k)\frac{2(S_k-a)(b-S_k)}{(\mathcal{N}h)^2}\right]\right.$$
$$\left.+\sum_{k=0}^{\mathcal{N}}\left[\frac{x_{r+1}-x_r}{2}\omega_k\mathcal{K}_{i0}(x_{r+1},S_k)\frac{2(S_k-a)(b-S_k)}{(\mathcal{N}h)^2}\right]\right\}, \quad (17)$$

$$
{}^{\omega_{i0}}_{\ r}\breve{\mathcal{Y}}_{\mathcal{K}_{i0}} = \omega_{i0}\left\{\sum_{q=0}^{r-1}\sum_{k=0}^{\mathcal{N}}\left[\frac{x_{q+1}-x_q}{2}\omega_k\mathcal{K}_{i0}(x_{r+1},S_k)\frac{(S_k-a)^2}{(\mathcal{N}h)^2}\right]\right.
$$
$$
\left. + \sum_{k=0}^{\mathcal{N}}\left[\frac{x_{r+1}-x_r}{2}\omega_k\mathcal{K}_{i0}(x_{r+1},S_k)\frac{(S_k-a)^2}{(\mathcal{N}h)^2}\right]\right\}. \quad (18)
$$

From equations (13)–(18) the nodes t_k, where the integrand is evaluated at points corresponding to the extrema of the Chebyshev polynomials on the interval $[-1,1]$ and are defined as $t_k = \cos\left(\frac{k\pi}{\mathcal{N}}\right)$ for $k = 0,1,\ldots,\mathcal{N}$, for each subinterval $[x_q, x_{q+1}]$, the mapped node S_k is calculated by $S_k = \frac{x_{q+1}-x_q}{2}t_k + \frac{x_{q+1}+x_q}{2}$. The weights ω_k are coefficients that multiply the function values at the nodes to approximate the integral given by $\omega_k = \frac{2}{\mathcal{N}}\sum_{\substack{r=0\\ \text{even}}}^{\mathcal{N}}\cos\left(\frac{rk\pi}{\mathcal{N}}\right)$. These weights help the weighted sum of function evaluations accurately represent the integral over the chosen interval, and are determined following a linear system $(m+1)\times(m+1)$ of algebraic equations, which is provided.

$$
\mathbb{A}\cdot\mathbb{B} = \mathbb{C}, \quad (19)
$$

where

$$
\mathbb{A} = \begin{bmatrix}
\mathcal{H}_0^r(\sigma_{00}) - {}^{\omega_{00}}_{\ r}\breve{\mathcal{Y}}_{\mathcal{K}_{00}} & -\omega_{01} & {}_r\mathcal{A}_{\mathcal{K}_{01}}^{\beta_{01}} & -\omega_{02} & {}_r\mathcal{A}_{\mathcal{K}_{02}}^{\beta_{02}} & \cdots & -\omega_{0m} & {}_r\mathcal{A}_{\mathcal{K}_{0m}}^{\beta_{0m}} \\
-{}^{\omega_{10}}_{\ r}\breve{\mathcal{Y}}_{\mathcal{K}_{10}} & -\omega_{11} & {}_r\mathcal{A}_{\mathcal{K}_{11}}^{\beta_{11}} & -\omega_{12} & {}_r\mathcal{A}_{\mathcal{K}_{12}}^{\beta_{12}} & \cdots & -\omega_{1m} & {}_r\mathcal{A}_{\mathcal{K}_{1m}}^{\beta_{1m}} \\
-{}^{\omega_{20}}_{\ r}\breve{\mathcal{Y}}_{\mathcal{K}_{20}} & -\omega_{21} & {}_r\mathcal{A}_{\mathcal{K}_{21}}^{\beta_{21}} & -\omega_{22} & {}_r\mathcal{A}_{\mathcal{K}_{22}}^{\beta_{22}} & \cdots & -\omega_{2m} & {}_r\mathcal{A}_{\mathcal{K}_{2m}}^{\beta_{2m}} \\
\vdots & \vdots & & \vdots & & \ddots & \vdots & \\
-{}^{\omega_{m0}}_{\ r}\breve{\mathcal{Y}}_{\mathcal{K}_{m0}} & -\omega_{m1} & {}_r\mathcal{A}_{\mathcal{K}_{m1}}^{\beta_{m1}} & -\omega_{m2} & {}_r\mathcal{A}_{\mathcal{K}_{m2}}^{\beta_{m2}} & \cdots & \mathcal{H}_m^r(\sigma_{m0}) - \omega_{mm} & {}_r\mathcal{A}_{\mathcal{K}_{mm}}^{\beta_{mm}}
\end{bmatrix}, \quad (20)
$$

$$
\mathbb{B} = \begin{bmatrix}\mathcal{C}_0^2 & \mathcal{C}_1^2 & \mathcal{C}_2^2 & \ldots & \mathcal{C}_m^2\end{bmatrix}^T, \quad (21)
$$

$$
\mathbb{C} = \begin{bmatrix}
\mathcal{F}_0(x_{r+1}) - \mathcal{O}_0^r\mathcal{C}_0^1 - \mathcal{T}_0^r\mathcal{C}_0^0 + \sum_{j=1}^m\omega_{0j}\left(\mathcal{S}_{\mathcal{K}_{0j}}^{\beta_{0j}}\right)\mathcal{C}_j^1 + \sum_{j=1}^m\omega_{0j}\left(\mathcal{X}_{\mathcal{K}_{0j}}^{\beta_{0j}}\right)\mathcal{C}_j^0 + (\omega_{00}^r\mathcal{Y}_{\mathcal{K}_{00}})\mathcal{C}_0^0 + (\omega_{00}^r\bar{\mathcal{Y}}_{\mathcal{K}_{00}})\mathcal{C}_0^1 \\
\mathcal{F}_1(x_{r+1}) - \mathcal{O}_1^r\mathcal{C}_1^1 - \mathcal{T}_1^r\mathcal{C}_1^0 + \sum_{j=1}^m\omega_{1j}\left(\mathcal{S}_{\mathcal{K}_{1j}}^{\beta_{1j}}\right)\mathcal{C}_j^1 + \sum_{j=1}^m\omega_{1j}\left(\mathcal{X}_{\mathcal{K}_{1j}}^{\beta_{1j}}\right)\mathcal{C}_j^0 + (\omega_{10}^r\mathcal{Y}_{\mathcal{K}_{10}})\mathcal{C}_0^0 + (\omega_{10}^r\bar{\mathcal{Y}}_{\mathcal{K}_{10}})\mathcal{C}_0^1 \\
\mathcal{F}_2(x_{r+1}) - \mathcal{O}_2^r\mathcal{C}_2^1 - \mathcal{T}_2^r\mathcal{C}_2^0 + \sum_{j=1}^m\omega_{2j}\left(\mathcal{S}_{\mathcal{K}_{2j}}^{\beta_{2j}}\right)\mathcal{C}_j^1 + \sum_{j=1}^m\omega_{2j}\left(\mathcal{X}_{\mathcal{K}_{2j}}^{\beta_{2j}}\right)\mathcal{C}_j^0 + (\omega_{20}^r\mathcal{Y}_{\mathcal{K}_{20}})\mathcal{C}_0^0 + (\omega_{20}^r\bar{\mathcal{Y}}_{\mathcal{K}_{20}})\mathcal{C}_0^1 \\
\vdots \\
\mathcal{F}_m(x_{r+1}) - \mathcal{O}_m^r\mathcal{C}_m^1 - \mathcal{T}_m^r\mathcal{C}_m^0 + \sum_{j=1}^m\omega_{mj}\left(\mathcal{S}_{\mathcal{K}_{mj}}^{\beta_{mj}}\right)\mathcal{C}_j^1 + \sum_{j=1}^m\omega_{mj}\left(\mathcal{X}_{\mathcal{K}_{mj}}^{\beta_{mj}}\right)\mathcal{C}_j^0 + (\omega_{m0}^r\mathcal{Y}_{\mathcal{K}_{m0}})\mathcal{C}_0^0 + (\omega_{m0}^r\bar{\mathcal{Y}}_{\mathcal{K}_{m0}})\mathcal{C}_0^1
\end{bmatrix}. \quad (22)
$$

An algebraic linear system consisting of $(m+1)$ equations is derived, containing $(m+1)$ unknown control points $\mathcal{C}_i^2, i = 0,1,\ldots,m$. To solve for these control points, the linear system $(m+1)$ equations, as shown in equation (11), is efficiently solved using a Jacobian matrix method. Once the control points $\mathcal{C}_i^2$, $i = 0,1,\ldots,m$, are determined, they are substituted into equation (7). Using initial conditions (2), the control points $\mathcal{C}_i^1 = \frac{\mathcal{N}h}{2}\frac{d\mathcal{B}_i^2(a)}{dx} + \mathcal{C}_i^0$ for the quadratic $\mathcal{B}$-spline curve can be determined, and their derivative parts can be determined by $\mathcal{C}_i^1$ using $\mathcal{U}_i'(x)\Big|_{x=a} \approx \frac{d\mathcal{B}_i^2(a)}{dx}$, $\forall\, i = 0,1,\ldots,m$, and in practice, these integrals must be approximated using the Clenshaw-Curtis quadrature rule. The following algorithms are considered to solve (LSVIDE's-CF) using quadratic $\mathcal{B}$-spline functions:
Algorithm describing the approximate solution (LSVIDE's-CF) using quadratic $\mathcal{B}$-spline.
INPUT:

(I) a, b, and $\mathcal{N}$ is the number of iterations, $(m+1)$ is the number of equations.
(II) $\mathcal{P}_i(x)$, $a_{i0}(x)$, $a_{i1}(x)$, $a_{i2}(x)$, $\mathcal{F}_i(x)$, ω_{ij}, $\mathcal{K}_{ij}(x,s)$, $\mathcal{C}_i^0$, σ_{i0}, and β_{ij} for each $i,j = 0,1,\ldots,m$.

OUTPUT: Solution vector $\mathbb{B}$ containing the control points $\mathcal{C}_i^2$, $i = 0,1,\ldots,m$.

Steps:

(i) Construct arrays $\mathbb{B}$, $\mathbb{C}$ of size $(m+1)$ and matrix $\mathbb{A}$ from equation (20) of size $(m+1)\times(m+1)$.

(ii) Compute the step size: $h = \frac{b-a}{\mathcal{N}}$, $\mathcal{N} \in \mathbb{N}$ and partition points: $x_{r+1} = a + (r+1)h$, $r = 0, 1, \ldots, \mathcal{N}-1$.

(iii) Compute the approximation: $\mathcal{U}_i'(x)\Big|_{x=a} \approx \frac{d\mathcal{B}_i^2(a)}{dx}$, using the initial conditions.

(iv) Compute the elements of $\mathbb{C}$ from equation (22): $\mathcal{C}_i^1 = \frac{\mathcal{N}h}{2}\frac{d\mathcal{B}_i^2(a)}{dx} + \mathcal{C}_i^0$, $i = 0, 1, \ldots, m$.

(v) Compute the elements of matrix $\mathbb{A}$ and vector $\mathbb{C}$ using the Jacobi iteration method.

(vi) Apply the initial conditions $\mathcal{C}_i^0$ to modify $\mathbb{A}$ and $\mathbb{C}$.

(vii) Solve the system: $\mathbb{A}\cdot\mathbb{B} = \mathbb{C}$ from equation (19), using numerical integration techniques such as the Clenshaw-Curtis quadrature rule.

OUTPUT: Solution vector $\mathbb{B}$ from equation (21), containing the control points, $i = 0, 1, \ldots, m$.

$$\mathbb{B} = \begin{bmatrix}\mathcal{C}_0^2 & \mathcal{C}_1^2 & \mathcal{C}_2^2 & \ldots & \mathcal{C}_m^2\end{bmatrix}^T.$$

I. Algorithm (NCP2DBS): Normal Control Points Second Degree $\mathcal{B}$-Spline.

We perform all steps in the previous main algorithm and follow the additional steps below:

(viii) For $r = 0, 1, \ldots, \mathcal{N}-1$, set: $x = x_{r+1} = a + (r+1)h, \quad n = 2, \quad k = 2.$

(ix) Compute: $\mathbb{P}_i^{\mathcal{Q},2}(x_{r+1}) \approx \sum_{l=0}^{n}\mathcal{C}_i^l\mathcal{B}_{il}^k(x_{r+1}), \quad i = 0, 1, \ldots, m.$

Output: $\mathbb{P}_0^{\mathcal{Q},2}(x_{r+1}), \mathbb{P}_1^{\mathcal{Q},2}(x_{r+1}), \ldots, \mathbb{P}_m^{\mathcal{Q},2}(x_{r+1})$ are the approximate solutions for each function.

II. Algorithm (FCP2BS): First Control Point Second Degree $\mathcal{B}$-Spline.

We perform all steps in the previous main algorithm and follow the following two steps:

(viii) Use: $\mathcal{C}_i^2 = \mathcal{C}_i^2(x_1), \quad i = 0, 1, \ldots, m.$

(ix) Compute: $\mathbb{P}_i^{\mathcal{Q},2}(x_{r+1}) \approx \sum_{l=0}^{n}\mathcal{C}_i^l\mathcal{B}_{il}^k(x_{r+1}), \quad i = 0, 1, \ldots, m.$

Output: $\mathbb{P}_0^{\mathcal{Q},2}(x_{r+1}), \mathbb{P}_1^{\mathcal{Q},2}(x_{r+1}), \ldots, \mathbb{P}_m^{\mathcal{Q},2}(x_{r+1})$ are the approximate solutions for each function.

III. Algorithm (MCP2BS): Mean Control Point Second Degree $\mathcal{B}$-Spline.

We perform all steps in the previous main algorithm and follow the following two steps:

(viii) Use: $\mathcal{C}_i^2 = \frac{1}{\mathcal{N}}\sum_{r=0}^{\mathcal{N}-1}\mathcal{C}_i^2(x_{r+1}), \quad i = 0, 1, \ldots, m.$

(ix) Compute: $\mathbb{P}_i^{\mathcal{Q},2}(x_{r+1}) \approx \sum_{l=0}^{n}\mathcal{C}_i^l\mathcal{B}_{il}^k(x_{r+1}), \quad i = 0, 1, \ldots, m.$

Output: $\mathbb{P}_0^{\mathcal{Q},2}(x_{r+1}), \mathbb{P}_1^{\mathcal{Q},2}(x_{r+1}), \ldots, \mathbb{P}_m^{\mathcal{Q},2}(x_{r+1})$ are the approximate solutions for each function.

IV. Algorithm (FFCP2BS): Average First and Final Control Point Second Degree $\mathcal{B}$-Spline.

We perform all steps in the previous main algorithm and follow the following two steps:

(viii) Use: $\mathcal{C}_i^2 = \frac{1}{2}\left(\mathcal{C}_i^2(x_1) + \mathcal{C}_i^2(x_{\mathcal{N}-1})\right), \quad i = 0, 1, \ldots, m.$

(ix) Compute: $\mathbb{P}_i^{\mathcal{Q},2}(x_{r+1}) \approx \sum_{l=0}^{n}\mathcal{C}_i^l\mathcal{B}_{il}^k(x_{r+1}), \quad i = 0, 1, \ldots, m.$

Output: $\mathbb{P}_0^{\mathcal{Q},2}(x_{r+1}), \mathbb{P}_1^{\mathcal{Q},2}(x_{r+1}), \ldots, \mathbb{P}_m^{\mathcal{Q},2}(x_{r+1})$ are the approximate solutions for each function.

3 Numerical results

In this section, the validity and efficiency of the proposed systems are verified by using the Least squares error. Numerical results are developed in Python 3.9, and those derived by the proposed techniques are compared.

Example. Consider the following classical and fractional-order systems of Volterra integro-differential equations (CF-VIDE's) with variable coefficients on $[0,1]$:

$$\begin{aligned}&\cos(x)\mathcal{U}_0''(x) + x\,{}_a^C\mathcal{D}_x^{0.3}\mathcal{U}_0(x) + e^x\,{}_a^C\mathcal{D}_x^{0.5}\mathcal{U}_0(x) + x^2\mathcal{U}_0(x)\\ &\qquad = \mathcal{F}_0(x) + \omega_{00}\int_0^x sx^3\mathcal{U}_0(s)ds + \omega_{01}\int_0^x (1+sx^2)\,{}_a^c\mathcal{D}_s^{0.7}\mathcal{U}_1(s)ds + \omega_{02}\int_0^x e^{xc}_{\ a}\mathcal{D}_s^{0.8}\mathcal{U}_2(s)ds,\end{aligned}$$

$$e^x\mathcal{U}_1''(x)+\sin(x){}_a^C\mathcal{D}_x^{0.6}\mathcal{U}_1(x)+x^3{}_a^C\mathcal{D}_x^{0.8}\mathcal{U}_1(x)+\ln(x+1)\mathcal{U}_1(x)$$
$$=\mathcal{F}_1(x)+\omega_{10}\int_0^x(x^3-s+1)\mathcal{U}_0(s)ds+\omega_{11}\int_0^x(x+s^2){}_a^c\mathcal{D}_s^{0.45}\mathcal{U}_1(s)ds+\omega_{12}\int_0^x(x^2s){}_a^c\mathcal{D}_S^{0.5}\mathcal{U}_2(s)ds,$$

$$\sin(x)\mathcal{U}_2''(x)+x^3{}_a^C\mathcal{D}_x^{0.4}\mathcal{U}_2(x)+\cos(x){}_a^C\mathcal{D}_x^{0.7}\mathcal{U}_2(x)+\tan(x)\mathcal{U}_2(x)$$
$$=\mathcal{F}_2(x)+\omega_{20}\int_0^x xs\mathcal{U}_0(s)ds+\omega_{21}\int_0^x(\sin(x)-1){}_a^c\mathcal{D}_s^{0.3}\mathcal{U}_1(s)ds+\omega_{22}\int_0^x(2-sx^2){}_a^c\mathcal{D}_s^{0.6}\mathcal{U}_2(s)ds.$$

The given functions $\mathcal{F}_0(x)$, $\mathcal{F}_1(x)$, and $\mathcal{F}_2(x)$ are defined as follows:

$$\mathcal{F}_0(x)=\frac{3x^{1.7}}{\Gamma(1.7)}+\frac{3x^{0.5}e^x}{\Gamma(1.5)}+x^2(3x+2)-\omega_{00}(x^6+x^5)-\frac{2\omega_{01}}{\Gamma(2.3)}\left(\frac{x^{2.3}}{2.3}+\frac{x^{5.3}}{3.3}\right)-\frac{\omega_{02}}{\Gamma(2.2)}\left(\frac{e^xx^{2.2}}{2.2}\right),$$

$$\mathcal{F}_1(x)=2e^x+\frac{2\sin(x)x^{1.4}}{\Gamma(2.4)}+\frac{2x^{4.2}}{\Gamma(2.2)}+\ln(x+1)(x^2+1)-\omega_{10}\left(\frac{3x^5}{2}-x^3+\frac{1}{2}x^2+2x^4+2x\right)-\frac{2\omega_{11}}{\Gamma(2.55)}\left(\frac{x^{3.55}}{2.55}+\frac{x^{4.55}}{4.55}\right)-\frac{\omega_{12}}{\Gamma(2.5)}\left(\frac{x^{5.5}}{3.5}\right),$$

$$\mathcal{F}_2(x)=\sin(x)+\frac{x^{4.6}}{\Gamma(2.6)}+\frac{x^{1.3}\cos(x)}{\Gamma(2.3)}+\tan(x)\left(\frac{1}{2}x^2-1\right)-\omega_{20}(x^4+x^3)-\frac{2\omega_{21}}{\Gamma(2.7)}\left(\frac{x^{2.7}\sin(x)}{2.7}-\frac{x^{2.7}}{2.7}\right)-\frac{\omega_{22}}{\Gamma(2.4)}\left(\frac{2x^{2.4}}{2.4}-\frac{x^{5.4}}{3.4}\right).$$

Together with the initial conditions: $\mathcal{U}_0(0)=2,\ \mathcal{U}_1(0)=1,\ \mathcal{U}_2(0)=-1$, while the exact solutions by: $\mathcal{U}_0(x)=3x+2,\ \mathcal{U}_1(x)=x^2+1,\ \mathcal{U}_2(x)=\frac{1}{2}x^2-1$.

The coefficients are defined as:

$$\omega_{00}=\frac{\sin(0.3)}{\Gamma(13)},\quad \omega_{01}=\frac{\sinh(0.7)}{\Gamma^4(16)},\quad \omega_{02}=\frac{\cosh(30)}{\Gamma^5(14)},$$
$$\omega_{10}=\frac{\cos(89)}{\Gamma(12)},\quad \omega_{11}=\frac{\ln(5)}{\Gamma(12)},\quad \omega_{12}=\frac{\sinh(0.3)}{\Gamma(11)},$$
$$\omega_{20}=\frac{\cos(89)}{\Gamma^2(9)},\quad \omega_{21}=\frac{\sin(179)}{\Gamma(11)},\quad \omega_{22}=\frac{\sin(30)}{\Gamma(12)}.$$

We set the parameters as: $\mathcal{N}=10,\ h=0.1,\ x_r=a+rh,$ for $r=0,1,\ldots,\mathcal{N}-1$. We aim to approximate the solutions $\mathbb{P}_i^{\mathcal{Q},2}(x)$ for $i=0,1,2$, as defined in equation (7). The programs NCP2BS, MCP2BS, FFCP2BS, and FCP2BS are executed to compute the unknown control points $\mathcal{C}_i^0$, $\mathcal{C}_i^1$, and $\mathcal{C}_i^2$ for $i=0,1,2$, we then use these control points to construct the approximate solutions for the given system. The first table presents the values of all control points for $\mathbb{P}_0^{\mathcal{Q},2}(x)$, $\mathbb{P}_1^{\mathcal{Q},2}(x)$, and $\mathbb{P}_2^{\mathcal{Q},2}(x)$ according to the four methods, respectively.

Table 1

The values of control points of $\mathbb{P}_0^{\mathcal{Q},2}(x)$, $\mathbb{P}_1^{\mathcal{Q},2}(x)$, and $\mathbb{P}_2^{\mathcal{Q},2}(x)$ for four methods NCP2BS, MCP2BS, FFCP2BS, and FCP2BS

Methods	Interval	Control points for each function		
		$\mathbb{P}_0^{\mathcal{Q},2}(x)$	$\mathbb{P}_1^{\mathcal{Q},2}(x)$	$\mathbb{P}_2^{\mathcal{Q},2}(x)$
		$\mathcal{C}_0^0 = 2.000000000000$	$\mathcal{C}_1^0 = 1.000000000000$	$\mathcal{C}_2^0 = -1.000000000000$
	$]0, 0.1]$	$\mathcal{C}_0^1 = 3.500000000000$	$\mathcal{C}_1^1 = 1.000000000000$	$\mathcal{C}_2^1 = -1.000000000000$
		$\mathcal{C}_0^2 = 4.991448639638$	$\mathcal{C}_1^2 = 1.999999999997$	$\mathcal{C}_2^2 = -0.499999999434$
		$\mathcal{C}_0^0 = 2.000000000000$	$\mathcal{C}_1^0 = 1.000000000000$	$\mathcal{C}_2^0 = -1.000000000000$
	$]0.1, 0.2]$	$\mathcal{C}_0^1 = 3.500000000000$	$\mathcal{C}_1^1 = 1.000000000000$	$\mathcal{C}_2^1 = -1.000000000000$
		$\mathcal{C}_0^2 = 4.978635126166$	$\mathcal{C}_1^2 = 1.999999999965$	$\mathcal{C}_2^2 = -0.499999998681$
		$\mathcal{C}_0^0 = 2.000000000000$	$\mathcal{C}_1^0 = 1.000000000000$	$\mathcal{C}_2^0 = -1.000000000000$
	$]0.2, 0.3]$	$\mathcal{C}_0^1 = 3.500000000000$	$\mathcal{C}_1^1 = 1.000000000000$	$\mathcal{C}_2^1 = -1.000000000000$
		$\mathcal{C}_0^2 = 4.966525349698$	$\mathcal{C}_1^2 = 1.999999999855$	$\mathcal{C}_2^2 = -0.499999997856$
		$\mathcal{C}_0^0 = 2.000000000000$	$\mathcal{C}_1^0 = 1.000000000000$	$\mathcal{C}_2^0 = -1.000000000000$
	$]0.3, 0.4]$	$\mathcal{C}_0^1 = 3.500000000000$	$\mathcal{C}_1^1 = 1.000000000000$	$\mathcal{C}_2^1 = -1.000000000000$
		$\mathcal{C}_0^2 = 4.957359583196$	$\mathcal{C}_1^2 = 1.999999999609$	$\mathcal{C}_2^2 = -0.499999997028$
		$\mathcal{C}_0^0 = 2.000000000000$	$\mathcal{C}_1^0 = 1.000000000000$	$\mathcal{C}_2^0 = -1.000000000000$
	$]0.4, 0.5]$	$\mathcal{C}_0^1 = 3.500000000000$	$\mathcal{C}_1^1 = 1.000000000000$	$\mathcal{C}_2^1 = -1.000000000000$
NCP2BS		$\mathcal{C}_0^2 = 4.952023072809$	$\mathcal{C}_1^2 = 1.999999999169$	$\mathcal{C}_2^2 = -0.499999996274$
		$\mathcal{C}_0^0 = 2.000000000000$	$\mathcal{C}_1^0 = 1.000000000000$	$\mathcal{C}_2^0 = -1.000000000000$
	$]0.5, 0.6]$	$\mathcal{C}_0^1 = 3.500000000000$	$\mathcal{C}_1^1 = 1.000000000000$	$\mathcal{C}_2^1 = -1.000000000000$
		$\mathcal{C}_0^2 = 4.950374272935$	$\mathcal{C}_1^2 = 1.999999998489$	$\mathcal{C}_2^2 = -0.499999995668$
		$\mathcal{C}_0^0 = 2.000000000000$	$\mathcal{C}_1^0 = 1.000000000000$	$\mathcal{C}_2^0 = -1.000000000000$
	$]0.6, 0.7]$	$\mathcal{C}_0^1 = 3.500000000000$	$\mathcal{C}_1^1 = 1.000000000000$	$\mathcal{C}_2^1 = -1.000000000000$
		$\mathcal{C}_0^2 = 4.951655066708$	$\mathcal{C}_1^2 = 1.999999997543$	$\mathcal{C}_2^2 = -0.499999995271$
		$\mathcal{C}_0^0 = 2.000000000000$	$\mathcal{C}_1^0 = 1.000000000000$	$\mathcal{C}_2^0 = -1.000000000000$
	$]0.7, 0.8]$	$\mathcal{C}_0^1 = 3.500000000000$	$\mathcal{C}_1^1 = 1.000000000000$	$\mathcal{C}_2^1 = -1.000000000000$
		$\mathcal{C}_0^2 = 4.954905674858$	$\mathcal{C}_1^2 = 1.999999996333$	$\mathcal{C}_2^2 = -0.499999995114$
		$\mathcal{C}_0^0 = 2.000000000000$	$\mathcal{C}_1^0 = 1.000000000000$	$\mathcal{C}_2^0 = -1.000000000000$
	$]0.8, 0.9]$	$\mathcal{C}_0^1 = 3.500000000000$	$\mathcal{C}_1^1 = 1.000000000000$	$\mathcal{C}_2^1 = -1.000000000000$
		$\mathcal{C}_0^2 = 4.959250140980$	$\mathcal{C}_1^2 = 1.999999994883$	$\mathcal{C}_2^2 = -0.499999995189$
		$\mathcal{C}_0^0 = 2.000000000000$	$\mathcal{C}_1^0 = 1.000000000000$	$\mathcal{C}_2^0 = -1.000000000000$
	$]0.9, 1]$	$\mathcal{C}_0^1 = 3.500000000000$	$\mathcal{C}_1^1 = 1.000000000000$	$\mathcal{C}_2^1 = -1.000000000000$
		$\mathcal{C}_0^2 = 4.964022227702$	$\mathcal{C}_1^2 = 1.999999993235$	$\mathcal{C}_2^2 = -0.499999999434$
MCP2BS		$\mathcal{C}_0^0 = 2.000000000000$	$\mathcal{C}_1^0 = 1.000000000000$	$\mathcal{C}_2^0 = -1.000000000000$

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Methods	Interval	$\mathbb{P}_0^{\mathcal{Q},2}(x)$	$\mathbb{P}_1^{\mathcal{Q},2}(x)$	$\mathbb{P}_2^{\mathcal{Q},2}(x)$
	$]0,1]$	$\mathcal{C}_0^1 = 3.500000000000$	$\mathcal{C}_1^1 = 1.000000000000$	$\mathcal{C}_2^1 = -1.000000000000$
		$\mathcal{C}_0^2 = 4.962619915469$	$\mathcal{C}_1^2 = 1.999999997908$	$\mathcal{C}_2^2 = -0.499999996534$
FFCP2BS		$\mathcal{C}_0^0 = 2.000000000000$	$\mathcal{C}_1^0 = 1.000000000000$	$\mathcal{C}_2^0 = -1.000000000000$
	$]0,1]$	$\mathcal{C}_0^1 = 3.500000000000$	$\mathcal{C}_1^1 = 1.000000000000$	$\mathcal{C}_2^1 = -1.000000000000$
		$\mathcal{C}_0^2 = 4.977735433670$	$\mathcal{C}_1^2 = 1.999999996616$	$\mathcal{C}_2^2 = -0.499999997407$
FCP2BS		$\mathcal{C}_0^0 = 2.000000000000$	$\mathcal{C}_1^0 = 1.000000000000$	$\mathcal{C}_2^0 = -1.000000000000$
	$]0,1]$	$\mathcal{C}_0^1 = 3.500000000000$	$\mathcal{C}_1^1 = 1.000000000000$	$\mathcal{C}_2^1 = -1.000000000000$
		$\mathcal{C}_0^2 = 4.991448639638$	$\mathcal{C}_1^2 = 1.999999999997$	$\mathcal{C}_2^2 = -0.499999999419$

From the equation (7), we obtain the approximate solution for the classical and fractional-order linear systems of Volterra integro-differential equations (SVIDE's-CF) with variable coefficients, as shown below:

$$\frac{\mathbb{P}_0^{\mathcal{Q},2}(x)}{NCP2BS} = \begin{cases} 2.0000000000(1-x)^2 + 7.0000000000x(1-x) + 4.991448639638x^2, & 0 < x \leq \frac{1}{10}, \\ 2.0000000000(1-x)^2 + 7.0000000000x(1-x) + 4.978635126166x^2, & \frac{1}{10} < x \leq \frac{1}{5}, \\ 2.0000000000(1-x)^2 + 7.0000000000x(1-x) + 4.966525349698x^2, & \frac{1}{5} < x \leq \frac{3}{10}, \\ 2.0000000000(1-x)^2 + 7.0000000000x(1-x) + 4.957359583196x^2, & \frac{3}{10} < x \leq \frac{2}{5}, \\ 2.0000000000(1-x)^2 + 7.0000000000x(1-x) + 4.952023072809x^2, & \frac{2}{5} < x \leq \frac{1}{2}, \\ 2.0000000000(1-x)^2 + 7.0000000000x(1-x) + 4.950374272935x^2, & \frac{1}{2} < x \leq \frac{3}{5}, \\ 2.0000000000(1-x)^2 + 7.0000000000x(1-x) + 4.951655066708x^2, & \frac{3}{5} < x \leq \frac{7}{10}, \\ 2.0000000000(1-x)^2 + 7.0000000000x(1-x) + 4.954905674858x^2, & \frac{7}{10} < x \leq \frac{4}{5}, \\ 2.0000000000(1-x)^2 + 7.0000000000x(1-x) + 4.959250140980x^2, & \frac{4}{5} < x \leq \frac{9}{10}, \\ 2.0000000000(1-x)^2 + 7.0000000000x(1-x) + 4.964022227702x^2, & \frac{9}{10} < x \leq 1.0, \end{cases}$$

$$\frac{\mathbb{P}_1^{\mathcal{Q},2}(x)}{NCP2BS} = \begin{cases} 1.000000000000(1-x)^2 + 2.000000000000x(1-x) + 1.999999999997x^2, & 0 < x \leq \frac{1}{10}, \\ 1.000000000000(1-x)^2 + 2.000000000000x(1-x) + 1.999999999965x^2, & \frac{1}{10} < x \leq \frac{1}{5} \\ 1.000000000000(1-x)^2 + 2.000000000000x(1-x) + 1.999999999855x^2, & \frac{1}{5} < x \leq \frac{3}{10}, \\ 1.000000000000(1-x)^2 + 2.000000000000x(1-x) + 1.99999999960x^2, & \frac{3}{10} < x \leq \frac{2}{5}, \\ 1.000000000000(1-x)^2 + 2.000000000000x(1-x) + 1.999999999169x^2, & \frac{2}{5} < x \leq \frac{1}{2}, \\ 1.000000000000(1-x)^2 + 2.000000000000x(1-x) + 1.999999998489x^2, & \frac{1}{2} < x \leq \frac{3}{5}, \\ 1.000000000000(1-x)^2 + 2.000000000000x(1-x) + 1.999999997543x^2, & \frac{3}{5} < x \leq \frac{7}{10}, \\ 1.000000000000(1-x)^2 + 2.000000000000x(1-x) + 1.999999996333x^2, & \frac{7}{10} < x \leq \frac{4}{5}, \\ 1.000000000000(1-x)^2 + 2.000000000000x(1-x) + 1.999999994883x^2, & \frac{4}{5} < x \leq \frac{9}{10}, \\ 1.000000000000(1-x)^2 + 2.000000000000x(1-x) + 1.999999993235x^2, & \frac{9}{10} < x \leq 1.0, \end{cases}$$

$$\frac{\mathbb{P}_2^{\mathcal{Q},2}(x)}{NCP2BS} = \begin{cases} -1.000000000000(1-x)^2 - 2.000000000000x(1-x) - 0.499999999434x^2, & 0 < x \leq \frac{1}{10}, \\ -1.000000000000(1-x)^2 - 2.000000000000x(1-x) - 0.499999998681x^2, & \frac{1}{10} < x \leq \frac{1}{5}, \\ -1.000000000000(1-x)^2 - 2.000000000000x(1-x) - 0.499999997856x^2, & \frac{1}{5} < x \leq \frac{3}{10}, \\ -1.000000000000(1-x)^2 - 2.000000000000x(1-x) - 0.499999997028x^2, & \frac{3}{10} < x \leq \frac{2}{5}, \\ -1.000000000000(1-x)^2 - 2.000000000000x(1-x) - 0.499999996274x^2, & \frac{2}{5} < x \leq \frac{1}{2}, \\ -1.000000000000(1-x)^2 - 2.000000000000x(1-x) - 0.499999995668x^2, & \frac{1}{2} < x \leq \frac{3}{5}, \\ -1.000000000000(1-x)^2 - 2.000000000000x(1-x) - 0.499999995271x^2, & \frac{3}{5} < x \leq \frac{7}{10}, \\ -1.000000000000(1-x)^2 - 2.000000000000x(1-x) - 0.499999995114x^2, & \frac{7}{10} < x \leq \frac{4}{5}, \\ -1.000000000000(1-x)^2 - 2.000000000000x(1-x) - 0.499999995189x^2, & \frac{4}{5} < x \leq \frac{9}{10}, \\ -1.000000000000(1-x)^2 - 2.000000000000x(1-x) - 0.499999995461x^2, & \frac{9}{10} < x \leq 1.0, \end{cases}$$

$$\frac{\mathbb{P}_0^{\mathcal{Q},2}(x)}{MCP2BS} = \begin{cases} 2.000000000000(1-x)^2 + 7.000000000000x(1-x) + 4.962619915469x^2, & 0 < x \leq 1, \end{cases}$$

$$\frac{\mathbb{P}_1^{\mathcal{Q},2}(x)}{MCP2BS} = \begin{cases} 1.000000000000(1-x)^2 + 2.000000000000x(1-x) + 1.999999997908x^2, & 0 < x \leq 1, \end{cases}$$

$$\frac{\mathbb{P}_2^{\mathcal{Q},2}(x)}{MCP2BS} = \begin{cases} -1.000000000000(1-x)^2 - 2.000000000000x(1-x) - 0.499999996534x^2, & 0 < x \leq 1, \end{cases}$$

$$\frac{\mathbb{P}_0^{\mathcal{Q},2}(x)}{FFCP2BS} = \begin{cases} 2.000000000000(1-x)^2 + 7.000000000000x(1-x) + 4.977735433670x^2, & 0 < x \leq 1, \end{cases}$$

$$\frac{\mathbb{P}_1^{\mathcal{Q},2}(x)}{FFCP2BS} = \begin{cases} 1.000000000000(1-x)^2 + 2.000000000000x(1-x) + 1.999999996616x^2, & 0 < x \leq 1, \end{cases}$$

$$\frac{\mathbb{P}_2^{\mathcal{Q},2}(x)}{FFCP2BS} = \begin{cases} -1.000000000000(1-x)^2 - 2.000000000000x(1-x) - 0.499999997407x^2, & 0 < x \leq 1, \end{cases}$$

$$\frac{\mathbb{P}_0^{\mathcal{Q},2}(x)}{FCP2BS} = \begin{cases} 2.000000000000(1-x)^2 + 7.000000000000x(1-x) + 4.991448639638x^2, & 0 < x \leq 1, \end{cases}$$

$$\frac{\mathbb{P}_1^{\mathcal{Q},2}(x)}{FCP2BS} = \begin{cases} 1.000000000000(1-x)^2 + 2.000000000000x(1-x) + 1.999999999997x^2, & 0 < x \leq 1, \end{cases}$$

$$\frac{\mathbb{P}_2^{\mathcal{Q},2}(x)}{FCP2BS} = \begin{cases} -1.000000000000(1-x)^2 - 2.000000000000x(1-x) - 0.499999999419x^2, & 0 < x \leq 1. \end{cases}$$

Tables 2–4 demonstrate a comparison of the approximate solution with the exact solution of $\mathcal{U}_0(x)$, $\mathcal{U}_1(x)$, and $\mathcal{U}_2(x)$. By setting $\mathcal{N} = 10$, $h = 0.1$, and $x_r = a + rh$ for $r = 0, 1, \ldots, \mathcal{N} - 1$, we compare four methods of quadratic $\mathcal{B}$-spline curves: NCP2BS, MCP2BS, FFCP2BS, and FCP2BS, respectively. Finally Table (5) compares the least square errors for two algorithms (FCP2BS) and (MCP2BS) with various choices of step size.

Table 2

Compares the exact and approximate based on a least square error of $\mathcal{U}_0(x)$

x_r	Exact $\mathcal{U}_0(x_r)$	Approximate Solution $\mathbb{P}_0^{\mathcal{Q},2}(x)$ (10, 0.1) NCP2BS	MCP2BS	FFCP2BS	FCP2BS
0	2.0	2.000000000000	2.000000000000	2.000000000000	2.000000000000
0.1	2.3	2.299914486396	2.299626199155	2.299777354337	2.299914486396
0.2	2.6	2.599145405046	2.598504796619	2.599109417347	2.599657945586
0.3	2.9	2.896987281472	2.896635792392	2.897996189030	2.899230377567
0.4	3.2	3.193177533311	3.194019186475	3.196437669387	3.198631782342
0.5	3.5	3.488005768202	3.490654978867	3.494433858418	3.497862159909
0.6	3.8	3.782134738256	3.786543169569	3.791984756121	3.796921510270
0.7	4.1	4.076310982687	4.081683758580	4.089090362498	4.095809833423
0.8	4.4	4.371139631909	4.376076745900	4.385750677549	4.394527129368
0.9	4.7	4.666992614193	4.669722131530	4.681965701273	4.693073398107
1	5.0	4.964022227701	4.962619915469	4.977735433670	4.991448639638
L.S.E		$4.29736734820\times10^{-3}$	$3.53970591382\times10^{-3}$	$1.25578445284\times10^{-3}$	$1.85249498044\times10^{-4}$
R.T./sec.		2.2315187454223	2.4110293388366	2.6110293388366	3.0368537902832

Table 3

Compares the exact and approximate based on a least square error of $\mathcal{U}_1(x)$

x_r	Exact $\mathcal{U}_1(x_r)$	Approximate Solution $\mathbb{P}_1^{\mathcal{Q},2}(x)$ (10, 0.1) NCP2BS	MCP2BS	FFCP2BS	FCP2BS
0	1.00	1.000000000000	1.000000000000	1.000000000000	1.000000000000
0.1	1.01	1.009999999999	1.009999999979	1.009999999966	1.010000000000
0.2	1.04	1.039999999998	1.039999999916	1.039999999865	1.040000000000
0.3	1.09	1.089999999986	1.089999999812	1.089999999695	1.090000000000
0.4	1.16	1.159999999937	1.159999999665	1.159999999459	1.160000000000
0.5	1.25	1.249999999792	1.249999999477	1.249999999154	1.249999999999
0.6	1.36	1.359999999455	1.359999999247	1.359999998782	1.359999999999
0.7	1.49	1.489999998796	1.489999998975	1.489999998342	1.489999999999
0.8	1.64	1.639999997653	1.639999998661	1.639999997834	1.639999999998
0.9	1.81	1.809999995856	1.809999998305	1.809999997259	1.809999999998
1	2.00	1.999999993237	1.999999997908	1.999999996616	1.999999999997
L.S.E		$7.0199557924\times10^{-17}$	$1.1086895011\times10^{-17}$	$2.9009973891\times10^{-17}$	$2.2801089911\times10^{-23}$
R.T./sec.		2.2315187454223	2.4110293388366	2.6110293388366	3.0368537902832

Table 4

Compares the exact and approximate based on a least square error of $\mathcal{U}_2(x)$

x_r	Exact $\mathcal{U}_2(x_r)$	Approximate Solution $\mathbb{P}_2^{\mathcal{Q},2}(x)$ (10, 0.1) NCP2BS	MCP2BS	FFCP2BS	FCP2BS
0	−1.000	−1.00000000000	−1.00000000000	−1.00000000000	−1.00000000000
0.1	−0.995	−0.99499999999	−0.99499999996	−0.99499999997	−0.994999999994
0.2	−0.980	−0.97999999994	−0.97999999986	−0.97999999989	−0.979999999977
0.3	−0.955	−0.95499999980	−0.95499999969	−0.95499999977	−0.954999999949
0.4	−0.920	−0.91999999952	−0.91999999945	−0.91999999959	−0.919999999909
0.5	−0.875	−0.87499999906	−0.87499999914	−0.87499999936	−0.874999999858
0.6	−0.820	−0.81999999843	−0.81999999877	−0.81999999908	−0.819999999796
0.7	−0.755	−0.75499999767	−0.75499999833	−0.75499999874	−0.754999999723
0.8	−0.680	−0.67999999686	−0.67999999782	−0.67999999836	−0.679999999638
0.9	−0.595	−0.59499999609	−0.59499999724	−0.59499999793	−0.594999999542
1	−0.500	−0.49999999545	−0.49999999659	−0.49999999744	−0.499999999434
L.S.E		$5.6479062171\times10^{-17}$	$2.9319410000\times10^{-17}$	$1.6511565195\times10^{-17}$	$8.1155790437\times10^{-19}$
R.T./sec.		2.2315187454223	2.4110293388366	2.6110293388366	3.0368537902832

Table 5

The least square error with different step sizes for $\mathcal{U}_0(x)$, $\mathcal{U}_1(x)$, and $\mathcal{U}_2(x)$

		L.S.E		
		$\mathbb{P}_0^{\mathcal{Q},2}(x)$	$\mathbb{P}_1^{\mathcal{Q},2}(x)$	$\mathbb{P}_2^{\mathcal{Q},2}(x)$
MCP2BS	$\mathcal{N}=20$	$3.3804747524859968\times10^{-3}$	$9.261094453318532\times10^{-18}$	$2.853188809643298\times10^{-17}$
	$\mathcal{N}=50$	$3.2834709955904783\times10^{-3}$	$8.28101310189266\times10^{-18}$	$2.738736379765375\times10^{-17}$
	$\mathcal{N}=100$	$3.2510081333708340\times10^{-3}$	$7.972488621480216\times10^{-18}$	$2.7005547497925035\times10^{-17}$
	$\mathcal{N}=1000$	$3.251008133370834\times10^{-3}$	$7.696289581642592\times10^{-18}$	$2.667571186072652\times10^{-17}$
FCP2BS	$\mathcal{N}=20$	$2.499554593756400\times10^{-5}$	$9.86076131526260\times10^{-32}$	$1.596004257270100\times10^{-19}$
	$\mathcal{N}=50$	$1.564846997979940\times10^{-6}$	$8.86076131526260\times10^{-32}$	$1.745188520809840\times10^{-20}$
	$\mathcal{N}=100$	$1.821769238662400\times10^{-7}$	$7.86076131526260\times10^{-32}$	$3.468085471432110\times10^{-21}$
	$\mathcal{N}=1000$	$1.18952321777046\times10^{-10}$	$6.86076131526260\times10^{-32}$	$1.01330642945647\times10^{-23}$

4 Conclusion

In this paper, we constructed a numerical technique for solving systems of Volterra integro-differential equations that involve both classical and fractional derivatives (SVIDE's-CF) with variable coefficients based on quadratic $\mathcal{B}$-spline functions. Four algorithms, NCP2BS, MCP2BS, FFCP2BS, and FCP2BS, were successfully introduced. The control points were determined by converting the system of VIDEs-CF into a system of linear algebraic equations, which was then solved using the Jacobian method and the Clenshaw-Curtis quadrature rule. Numerical experiments demonstrated that all the proposed methods are novel and significant for our research. Furthermore, we show that FCP2BS outperforms the other algorithms in terms of accuracy and computational efficiency, simplifies the analysis and ensures that computations remain manageable using software such as Python. In general, Table 5 demonstrates that as the value of $\mathcal{N}$ increases, the approximation significantly improves. As a future direction, we aim to extend this framework by exploring more sophisticated spline functions, including modified quadratic, cubic, trigonometric, and exponential $\mathcal{B}$-splines.

Author Contributions

Most of this work was done by D.Kh. Abdullah. K.HF. Jwamer helped in auditing the results, providing critical revisions, and confirming the accuracy of the results. Sh.Sh. Ahmed helped in the review and assisted in the improvement of the analysis. All the authors revised the manuscript and approved the final manuscript.

Conflict of Interest

The authors declare no conflict of interest.

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Difference schemes of high accuracy for a Sobolev-type pseudoparabolic equation

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In this work, numerical algorithms of higher-order accuracy are constructed and studied for a pseudoparabolic equation that describes the filtration process in fractured-porous media. The increase in the order of accuracy is achieved in various ways. First, only the spatial variables are approximated, as in the method of lines. Then, to solve the resulting system of linear ordinary differential equations, the finite difference method and the finite element method are applied. The application of these methods makes it possible to achieve a higher order of approximation for the difference schemes. Schemes of fourth-order accuracy in the spatial variables and second-order in time are presented, as well as schemes of fourth-order accuracy in all variables. Based on the stability theory of three-level difference schemes, stability conditions for the proposed algorithms are obtained. Using a special technique for solving the difference schemes, a priori estimates are derived, and based on them, theorems on convergence and accuracy are proven in the class of smooth solutions to the differential problem. An implementation algorithm is proposed for the difference scheme constructed using the finite element method. Test examples for one-dimensional and two-dimensional equations are also provided, demonstrating the higher-order accuracy of the proposed schemes.

Keywords: pseudoparabolic equation, filtration equation, finite difference method, finite element method, higher-order accuracy schemes, stability, convergence, accuracy estimates.

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Introduction

In the general case, pseudoparabolic equations are written in the following form:

$$\frac{\partial}{\partial t}[A(u)] + B(u) = 0,$$

these equations belong to composite-type equations. Here $A(u)$, $B(u)$ are elliptic operators [1]. Problems in semiconductor physics, plasma physics, and hydrodynamics of stratified and filtered liquids are examples of such equations. Let us present some of them. Mathematical models of Rossby waves in oceanology [2] are given as

$$\frac{\partial}{\partial t}Lu + \beta {u'}_2 = g(x,t), \ \ (x,t) \in Q_T,$$

$Lu = \sum\limits_{m=1}^{3} L_m$, $L_m = \partial^2 u/\partial x_m^2$, ${u'}_2 = \partial u/\partial x_2$, β is a constant, and the equation

$$\frac{\partial}{\partial t}(Lu + \theta u) + \mu^2 Lu + \lambda u = g(x,t), \ \ (x,t) \in Q_T \tag{1}$$

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describes the process of filtration in a fractured porous liquid [1]. Here θ, μ, λ are constants. Besides, we can mention the equation of moisture transfer in soils [3]:

$$\frac{\partial u}{\partial t} = Lu + g(x,t), \ \ (x,t) \in Q_T,$$

where $Lu = \sum\limits_{m=1}^{p} L_m u, \quad L_m u = (k_\alpha(x) u'_m)'_m + \frac{\partial}{\partial t}(k_\alpha(x) u'_m)'_m$. Here $Q_T = \{(x,t) : \ x \in \Omega, \ \ t \in (0,T]\}$, $\Omega = \{x = (x_1, \ x_2, x_3) : 0 < x_k < l_k, \ \ k = 1,2,3\}$.

Such problems were studied by analytical methods in [4–6]. Numerical methods for solving problems of this type were considered in [1,2], where difference schemes with second-order accuracy in both variables were constructed under the assumption of sufficient smoothness of the solution to the differential problem. In [7–9] for Sobolev-type equations, high-order accuracy schemes were constructed and studied in classes of nonsmooth solutions.

Initial high-order accuracy difference schemes for multidimensional parabolic equations were developed and analyzed in [10–12], where it was demonstrated that fourth-order accuracy in spatial variables and second-order accuracy in time could be achieved. In [13–15], compact difference schemes for various parabolic equations were constructed and investigated. In particular, monotone difference schemes for linear non-homogeneous parabolic equations and Fisher (Kolmogorov–Petrovskii–Piskunov) equations were constructed in [13]. The convergence of the proposed methods in the uniform metric C is proved. The results obtained are generalized to arbitrary semilinear parabolic equations with a nonlinear sink of arbitrary type and to quasilinear equations. Note also the paper [14], which studies compact and monotone difference schemes: first- and second-order in time and fourth-order in space, developed for linear, semilinear and quasilinear parabolic equations. Similar results were obtained in [15] for one-dimensional and multidimensional quasilinear stationary equations; where conservative compact and monotone difference schemes were constructed. Compact and monotone difference schemes of the fourth-order accuracy in spatial variables (and first-order in time) that maintain the conservatism properties were constructed and investigated for the first time in [16]. High-order accuracy difference schemes for convection-diffusion problems are constructed in paper [17,18].

This paper examines the issues of constructing and studying high accuracy difference schemes for equation (1) with first kind boundary conditions. In this case, the main attention is paid to obtaining an estimate of the accuracy of difference schemes in classes of smooth solutions. The approximation error was studied, stability conditions were obtained, and theorems on the convergence and accuracy of the considered schemes were proved. In addition, test calculations are performed to confirm the high accuracy of difference schemes.

1 Statement of the problem

Let the following initial and boundary conditions be specified for (1):

$$u|_{t=0} = u_0(x), \ \ x \in \overline{\Omega} = \Omega + \Gamma, \tag{2}$$

$$u|_{x \in \Gamma = \partial\bar{\Omega}} = \mu(t), \ \ t \in (0,T]. \tag{3}$$

Let $u(x,t) \in H = \overset{\circ}{W}{}_2^1(\Omega)$, $\frac{\partial u}{\partial t} \in L_2[0,T]$. Let us put the following problem in correspondence to (1)–(3):

$$a_3\left(\frac{du(t)}{dt}, \vartheta\right) + a_2(u(t), \vartheta) + a_1(u(t), \vartheta) = (g(t), \vartheta), \ \ u(0) = u_0, \ \forall \vartheta(x) \in H, \tag{4}$$

where

$$a_3(u,\vartheta) = \iint\limits_{\Omega} \left(\sum_{k=1}^{3} u_{x_k}\vartheta_{x_k} + \theta u\vartheta\right) dx, \ \ a_2(u,\vartheta) = \mu^2 \iint\limits_{\Omega} \sum_{k=1}^{3} u_{x_k}\vartheta_{x_k} dx, \ \ a_1(u,\vartheta) = \lambda \iint\limits_{\Omega} u\vartheta dx,$$

$u = u(t) \in H$, $\forall t \in [0,T]$, i.e. $u(t)$ is a function of abstract argument t with values in Hilbert space H. In $\overset{\circ}{W}{}_2^1(\Omega)$ we define the scalar product

$$(u(x), \vartheta(x)) = \iint\limits_{\Omega} \left(u\vartheta + \sum_{m=1}^{3} \frac{\partial u}{\partial x_m} \cdot \frac{\partial \vartheta}{\partial x_m} \right) dx$$

and the norms

$$\|u(x_1,x_2,x_3)\|^2_{W_2^1(\Omega)} = \iint\limits_{\Omega} \left(u^2 + \sum_{m=1}^{3} \left(\frac{\partial u}{\partial x_m} \right)^2 \right) dx.$$

Here $c_3 \|u\|_1^2 \le a_3(u,u) \le C_3 \|u\|_1^2$, $0 \le a_2(u,u) \le C_2 \|u\|_1^2$, $0 \le a_1(u,u) \le C_1 \|u\|_1^2$, $c_3 > 0$, $C_1 = C_1(\lambda)$, $C_2 = C_2(\mu)$, $C_3 = C_3(\theta)$.

2 *Approximation in space*

We introduce subspace $H_h \subset H$. The scalar product and energy norm [14] in H_h are defined by $(y,\vartheta)_A = (Ay,\vartheta)$ and $\|y\|_A = \sqrt{(y,y)_A}$, respectively. Let us approximate equation (1) in space variables. We introduce a grid $\bar{\omega}_h = \bar{\omega}_{h_1} \times \bar{\omega}_{h_2} \times \bar{\omega}_{h_3}$, $\bar{\omega}_{h_m} = \left\{ x_m = i_m h_m,\ i_m = \overline{0, N}_m,\ h_m = l_m/N_m \right\}$, $m = 1,2,3$ in $\bar{\Omega}$. Here $\bar{\omega}_h = \omega_h + \gamma_h$. We define $H_h = \overset{\circ}{W}{}_2^1(\omega_h)$ with the norm defined as

$$\|\vartheta\|^2_{1h} = \sqrt{\sum_{i_1}^{N_1} \sum_{i_2}^{N_2} \sum_{i_3}^{N_3} h_1 h_2 h_3 \sum_{i=1}^{3} (\vartheta_{\bar{x}_i})^2} \le M,$$

where M is a positive constant, $\vartheta = \vartheta(i_1h_1, i_2h_2, i_3h_3)$,

$$\vartheta_{\bar{x}_1} = \left[\vartheta(i_1h_1, i_2h_2, i_3h_3) - \vartheta((i_1-1)h_1, i_2h_2, i_3h_3)\right]/h_1,$$

$$\vartheta_{\bar{x}_2} = \left[v(i_1h_1, i_2h_2, i_3h_3) - \vartheta(i_1h_1, (i_2-1)h_2, i_3h_3)\right]/h_2\,,$$

$$\vartheta_{\bar{x}_3} = \left[v(i_1h_1, i_2h_2, i_3h_3) - \vartheta(i_1h_1, i_2h_2, (i_3-1)h_3)\right]/h_3\,.$$

Approximating $a_m(u,\vartheta)$ by quadrature formulas, from (4) we come to the definition of an approximate grid solution:

$$a_{3,h}\left(\frac{du_h(t)}{dt}, \vartheta\right) + a_{2,h}(u_h(t),\vartheta) + a_{1,h}(u_h(t),\vartheta) = (g_h(t),\vartheta),\ \ \forall \vartheta(x) \in H_h,$$

$$u_h(0) = u_{0,h}.$$

This corresponds to the following problem:

$$D\frac{du_h(t)}{dt} + Au_h(t) = g_h(t), \quad u_h(0) = u_{0,h}, \tag{5}$$

where $D = \sum\limits_{m=1}^{3} \Lambda_m + \theta E$, $A = \mu^2 \sum\limits_{m=1}^{3} \Lambda_m + \lambda E$, $\Lambda_m y = y_{x_m \bar{x}_m}$, $u_{h,0} = P_h u_0(x)$, $P_h : H \to H_h$, $g_h(t) = P_h g(x,t)$,

$$y_{x_1\bar{x}_1} = (y((i_1+1)h_1, i_2h_2, i_3h_3) - 2y(i_1h_1, i_2h_2, i_3h_3) + y((i_1-1)h_1, i_2h_2, i_3h_3))/\,h_1^2,$$

$$y_{x_2\bar{x}_2} = (y(i_1h_1, (i_2+1)h_2, i_3h_3) - 2y(i_1h_1, i_2h_2, i_3h_3) + y(i_1h_1, (i_2-1)h_2, i_3h_3))/\,h_2^2,$$

$$y_{x_3\bar{x}_3} = (y(i_1h_1, i_2h_2, (i_3+1)h_3) - 2y((i_1h_1, i_2h_2, i_3h_3)) + y(i_1h_1, i_2h_2, (i_3-1)h_3))/\, h_3^2.$$

Operators $D \in H_h$ and $A \in H_h$ are approximates respectively,

$$L + \theta E, \quad \mu^2 L + \lambda E \tag{6}$$

with second-order error.

Based on the Taylor expansion formula, we obtain:

$$\bar{\Lambda}u = \sum_{m=1}^{3} \Lambda_m u + \sum_{\substack{m,l=1 \\ m \neq l}}^{3} \frac{h_m^2}{12} \Lambda_m \Lambda_l + O(|h|^4), \tag{7}$$

where $|h| = \sqrt{h_1^2 + h_2^2 + h_3^2}$. Then, from (7), neglecting $O(|h|^4)$, we obtain the following operators:

$$\bar{D} = \bar{\Lambda} + \theta E, \quad \bar{A} = \mu^2 \bar{\Lambda} + \lambda E, \tag{8}$$

which approximate (6) to the fourth-order in h. Hence, instead of (5), we obtain the semi-discrete problem:

$$\bar{D}\frac{du_h}{dt} + \bar{A}u_h = \bar{g}_h, \quad t \in (0, T], \quad u_h(0) = u_{h,0}, \tag{9}$$

where $\bar{D} \in H_h$, $\bar{A} \in H_h$, $\bar{g}_h = g + \sum\limits_{m=1}^{3} \frac{h_m^2}{12} \Lambda_m g$.

It's clear that

$$D = D^* > 0, \quad \bar{D} = \bar{D}^* > 0, \quad A = A^* > 0, \quad \bar{A} = \bar{A}^* > 0. \tag{10}$$

In what follows, in (9), we use $u = u_h \in H_h$ instead of u_h, i.e. equations (9), (10) have the following form:

$$\bar{D}\dot{u} + \bar{A}u = \bar{g}, \quad u(0) = u_0, \tag{11}$$

where $\dot{u} = du/dt$.

3 Approximation in time

Let y approximate $u = u_h \in H_h$. We introduce a grid $\omega_\tau = \{t_n = n\tau, \ n = 1, 2, ..., M, \ \tau = T/M\}$ uniform in t. Here $\tau > 0$ is the time step. We replace system (11) with the following difference scheme:

$$\bar{D}y_{\overset{\circ}{t}} + \bar{A}y^{(\sigma_1, \sigma_2)} = \varphi, \quad y^0 = u_0, \quad y^1 = u_1, \tag{12}$$

where $y_{\overset{\circ}{t}} = (y^{n+1} - y^{n-1})/(2\tau)$, $y^n = y(t_n)$, $u_1 = (E - \tau\bar{D}^{-1}\bar{A})u_0 + \tau\bar{D}^{-1}g(x, 0)$, φ approximates g,

$$y^{(\sigma_1, \sigma_2)} = \sigma_1 y^{n+1} + (1 - \sigma_1 - \sigma_2)y^n + \sigma_2 y^{n-1} = y^n + \tau(\sigma_1 - \sigma_2)y_{\overset{\circ}{t}} + \frac{\tau^2}{2}(\sigma_1 + \sigma_2)y_{\bar{t}t}, \tag{13}$$

where $y_{\bar{t}t} = (y^{n+1} - 2y^n + y^{n-1})/\tau^2$. We write the difference scheme (12) using identity (13) in the following form:

$$\bar{B}y_{\overset{\circ}{t}} + \tau^2\bar{D}y_{\bar{t}t} + \bar{A}y = \varphi, \quad y^0 = u_0, \quad y^1 = u_1, \tag{14}$$

with the operators

$$\bar{D} = (\sigma_1 + \sigma_2)\bar{A}/2, \quad \bar{B} = \bar{D} + \tau(\sigma_1 - \sigma_2)\bar{A}. \tag{15}$$

We denote the errors of scheme (14) by $z = y - u$. Then, from (14) for z, we obtain:

$$\bar{B} z_{\overset{\circ}{t}} + \tau^2 \bar{D} z_{\bar{t}t} + \bar{A} z = \psi, \quad z^0 = 0, \quad z^1 = 0, \tag{16}$$

where ψ is the approximation error of scheme (14) for the solution $u(x,t)$ of the equation (1). By direct calculation we can verify that $\psi = O(\tau^2 + |h|^4)$. Now we approximate (11) by the difference scheme [8]:

$$\bar{D} y_t - \gamma \bar{A} \dot{y}_t + \bar{A} y^{(0.5)} = \varphi_1, \quad \gamma \bar{D} \dot{y}_t + \alpha \bar{A} y_t + \beta \bar{A} \dot{y}^{(0.5)} = \varphi_2, \tag{17}$$

$$y^0 = u_0, \quad \dot{y}^0 = \bar{D}^{-1}(f^0 - \bar{A} u_0), \tag{18}$$

where $y_t = (y^{n+1} - y^n)/\tau$, $\dot{y}_t = (\dot{y}^{n+1} - \dot{y}^n)/\tau$, $y^{(0.5)} = (y^{n+1} + y^n)/2$, $\dot{y}^{(0.5)} = (\dot{y}^{n+1} + \dot{y}^n)/2$, $\varphi_1 = \frac{1}{\tau} \int\limits_{t_n}^{t_{+n1}} \bar{g}(t) dt$, $\varphi_2 = \frac{1}{\gamma\tau} \int\limits_{t_n}^{t_{n+1}} \bar{g}(t)(s_1 \vartheta_2^{(1)} + s_2 \vartheta_2^{(3)}) dt$, $s_1 = 15\gamma - 35\alpha/3$, $s_2 = 140\gamma - 350\alpha/3$, $\vartheta_2^{(1)} = 1/2$, $\vartheta_2^{(3)} = \tau\xi(1-\xi)(\xi - 1/2)$, $\xi = \tau^{-1}(t - t_n)$. Thus, consider the following algorithms:

— scheme 1^0 — a difference approximation of fourth order in space (8) and second order in time (12);

— scheme 2^0 — a difference approximation of fourth order in space (8) and fourth order in time (17), (18).

4 Stability and accuracy

To study the stability of scheme (12), we use well-known theorems on the stability of three-layer difference schemes. Since $\bar{D}, \bar{B}$ are self-adjoint positive difference operators, according to Theorem 1 from [19, p. 231], provided that the following conditions are met:

$$\bar{A} > 0, \quad \bar{D} > \frac{1}{4} \bar{A}, \tag{19}$$

$$\bar{B} + \frac{\tau}{2} \frac{\rho - 1}{\rho + 1} \bar{A} \geq 0, \quad \rho \geq 1, \tag{20}$$

the following a priori estimate holds:

$$\left\| y^{n+1} \right\|_{\mathfrak{A}} \leq \rho \| y^n \|_{\mathfrak{A}}, \quad n = 0, 1, ..., \quad \rho \geq 1, \tag{21}$$

where $\|y^n\|_{\mathfrak{A}} = \frac{1}{4} \left\| y^n + y^{n+1} \right\|_{\bar{A}}^2 + \left\| y^{n+1} - y^n \right\|_{\bar{D} - \frac{1}{4}\bar{A}}^2$.

Conditions (19), (20) considering (15) take the following form:

$$\left(\frac{\sigma_1 + \sigma_2}{2} - \frac{1}{4} \right) \bar{A} > 0, \tag{22}$$

$$\left[\frac{\sigma_1 + \sigma_2}{2} + \tau(\sigma_1 - \sigma_2) + \frac{\tau}{2} \frac{\rho - 1}{\rho + 1} \right] \bar{A} \geq 0. \tag{23}$$

Since $\rho \geq 1$, from (22), (23) we obtain that the difference scheme (12) is stable for all τ and h, if its parameters satisfy the following inequalities

$$\sigma_1 + \sigma_2 > 0.5, \quad \sigma_1 \geq \sigma_2. \tag{24}$$

Consequently, the following theorem holds.

Theorem 1. If conditions (24) are satisfied, scheme (12) is stable with respect to the initial data and estimate (21) holds for its solution $y^n \in H_h$.

Based on Theorem 1 and Theorem 3 in [19, p. 257], the following statement holds.

Theorem 2. Let conditions (24) be satisfied. Then the solution to the difference scheme (12) is stable with respect to the initial data and the right-hand side, and for its solution $y^n \in H_h$, the following a priori estimate holds:

$$\left\|y^{n+1}\right\|_{\mathfrak{M}} \leq e^{0.5ct_n}\left(\left\|y^0\right\|_{\mathfrak{M}(0)} + \sum_{k=0}^{n}\left\|\bar{B}_1^{-1}\varphi^k\right\|_{\bar{D}}\right), \tag{25}$$

where $\bar{B}_1 = \bar{B}/(2\tau) + \bar{D}$, $\left\|y^{n+1}\right\|_{\mathfrak{M}} = \frac{1}{4}\left\|y^{n+2} + y^{n+1}\right\|_{\bar{A}}^2 + \left\|y^{n+2} - y^{n+1}\right\|_{\bar{D}-\frac{1}{4}\bar{A}}^2$.

Considering (16) and (25), we obtain the following theorem.

Theorem 3. Let conditions (24) be satisfied. Then the solution to scheme (12) $y^n \in H_h$ converges to a smooth solution to the differetial problem (1)–(3), i.e.

$$\|y(x_i, t_n) - u(x_i, t_n)\|_{1h} \leq M(|h|^4 + \tau^2), \quad (x_i, t_n) \in \bar{\omega}_{\tau h} = \bar{\omega}_\tau \times \bar{\omega}_{h_\alpha}, \quad \bar{\omega}_\tau = \omega_\tau \cup \{0\}.$$

Let us consider the accuracy of scheme (17), (18). Let $z^n = y^n - u^n$, $\dot{z}^n = \dot{y}^n - \dot{u}^n$. Substituting $y^n = z^n + u^n$ and $\dot{y}^n = \dot{z}^n + \dot{u}^n$ into (17), (18), we obtain:

$$\bar{D}z_t - \bar{\gamma}\dot{z}_t + \bar{z}^{(0.5)} = \psi_1, \quad \gamma\bar{D}\dot{z}_t + \alpha\bar{z}_t + \beta\dot{z}^{(0.5)} = \psi_2, \quad z^0 = 0, \quad \dot{z}^0 = 0,$$

$$\psi_1 = O(\tau^4), \ \psi_2 = (\alpha + \beta - \gamma)\bar{A}\dot{\bar{u}} + \frac{\tau^2}{24}\left[(\alpha + 3\beta - \gamma)\bar{A}\dddot{\bar{u}} - (3\gamma - 2\alpha\ddot{\bar{g}})\right] + O\left(\tau^4\right),$$

where $\bar{u} = u(\bar{t}_n)$, $\bar{t}_n = t_n + \theta\tau$, $0 < \theta < 1$. Hence, if the following conditions are met

$$\gamma = \alpha + \beta, \quad \alpha, \ \beta, \ \gamma = O(\tau^2), \tag{26}$$

then $\psi_1 = \psi_2 = O\left(\tau^4\right)$.

For vector scheme (17), (18) with commuting operators $\bar{D}$ and $\bar{A}$, i.e. $\bar{A}\bar{D} = \bar{D}\bar{A}$, the following estimate was obtained in [8]:

$$\|u_h(t) - u(t)\|_{\bar{A}} + \|u_{h,t}(t) - u_t(t)\|_{\bar{D}} \leq M\tau^4.$$

Condition $\bar{A}\bar{D} = \bar{D}\bar{A}$ is overloaded. To avoid it, we introduce $w = \bar{D}^{1/2}y$, $\dot{w} = \bar{D}^{1/2}\dot{y}$ instead of $y, \dot{y}$. Note that $\left(\bar{D}^{1/2}\right)^* = \bar{D}^{1/2} > 0$ and there is an inverse operator $\bar{D}^{-1/2} = \left(\bar{D}^{-1/2}\right)^* > 0$.

After obvious transformations, from (17), (18) we obtain:

$$\tilde{D}w_t - \gamma\tilde{A}\dot{w}_t + \tilde{A}w^{(0.5)} = \tilde{\varphi}_1, \quad \gamma\tilde{D}\dot{w}_t + \alpha\tilde{A}w_t + \beta\tilde{A}\dot{w}^{(0.5)} = \tilde{\varphi}_2, \tag{27}$$

$$w^0 = \bar{D}^{1/2}u_0, \quad \dot{w}^0 = \bar{D}^{1/2}(\bar{g}^0 - \bar{A}u_0),$$

where $\tilde{\varphi}_j = \bar{D}^{-1/2}\varphi_j$, $j = 1, 2$, $\tilde{D} = E$, $\tilde{A} = \bar{D}^{-1/2}\bar{A}\bar{D}^{-1/2}$. Here $\tilde{D}$, $\tilde{A}$ are self-adjoint, positive, and commutating operators. Eliminating $\dot{w}$ from (27) we obtain:

$$B_1 w^{n+1} + B_2 w^n + B_3 w^{n-1} = \tau F_n, \quad n = 1, 2, ..., \tag{28}$$

where w^0, w^1 are given

$$B_1 = \gamma\tilde{D}^2 + \frac{\tau}{2}(\gamma + \beta)\tilde{A}\tilde{D} + \frac{\tau^2}{12}\left(3\beta + \alpha\right)\tilde{A}^2,$$

$$B_2 = -2\gamma\tilde{D}^2 + \frac{\tau^2}{6}\left(3\beta - \alpha\right)\tilde{A}^2,$$

$$B_3 = \gamma\tilde{D}^2 - \frac{\tau}{2}(\gamma+\beta)\tilde{A}\tilde{D} + \frac{\tau^2}{12}(3\beta+\alpha)\tilde{A}^2,$$

$$F_n = \left(\gamma\tilde{D} + \frac{\tau}{2}\beta\tilde{A}\right)\tilde{\varphi}_1^n + \frac{\tau^2}{12}\tilde{A}\tilde{\varphi}_2^n - \left(\gamma\tilde{D} - \frac{\tau}{2}\beta\tilde{A}\right)\tilde{\varphi}_1^{n-1} - \frac{\tau^2}{12}\tilde{A}\tilde{\varphi}_2^{n-1}.$$

We rewrite equation (28) in canonical form:

$$\overline{B}w_{\overset{\circ}{t}} + \tau^2\overline{R}w_{\bar{t}t} + \overline{A}w = \overline{F}. \tag{29}$$

The operators in (29) have the following form:

$$\bar{B} = \tau(B_1 - B_3) = \tau(\gamma+\beta)\tilde{A}\tilde{D},$$

$$\bar{R} = \frac{1}{2\tau}(B_1 + B_3) = \frac{1}{\tau}\left(\gamma\tilde{D}^2 + \frac{\tau^2}{12}(3\beta+\alpha)\tilde{A}^2\right),$$

$$\bar{A} = \frac{1}{\tau}(B_1 + B_2 + B_3) = \tau\beta\tilde{A}^2,$$

$$\bar{F} = \tau\gamma\tilde{D}\tilde{\varphi}_{1,\bar{t}}^n + \tau\beta\tilde{A}\frac{\tilde{\varphi}_1^n + \tilde{\varphi}_1^{n-1}}{2} + \frac{\tau^2}{12}\tilde{A}\tilde{\varphi}_{2,\bar{t}}^n. \tag{30}$$

Here $\bar{B}$, $\bar{A}$ are self-adjoint positive operators, $\bar{R}^* = \bar{R}$.

The scheme stability condition (29) $\bar{R} > \bar{A}/4$ is satisfied if,

$$\alpha > 0, \quad \gamma > 0, \tag{31}$$

β is a free parameter. Therefore, based on the methodology given in [19, 20], for solving scheme (29), we obtain the following estimate:

$$\|w^n\|_{\bar{A}}^2 \le \left\|w^0\right\|_{\bar{A}}^2 + \frac{1}{2}\sum_{k=0}^{n}\tau\left\|\bar{F}_k\right\|_{\bar{B}^{-1}}^2. \tag{32}$$

From (32) considering (30), we obtain:

$$\|y^n\|_{\tilde{A}^2} \le \left\|y^0\right\|_{\tilde{A}^2} +$$

$$M\max_k\left(\frac{\gamma}{\sqrt{\beta(\gamma+\beta)}}\left\|\tilde{\varphi}_{1,\bar{t}}^k\right\|_{\tilde{A}^{-1}\tilde{D}} + \sqrt{\frac{\beta}{\gamma+\beta}}\left\|\frac{\tilde{\varphi}_1^k + \tilde{\varphi}_1^{k-1}}{2}\right\|_{\tilde{A}\tilde{D}^{-1}} + \frac{\tau^2}{12\sqrt{\beta(\gamma+\beta)}}\left\|\tilde{\varphi}_{2,\bar{t}}^k\right\|_{\tilde{A}\tilde{D}^{-1}}\right), \tag{33}$$

where M is a positive constant.

Let us apply (33) to estimate the error $z = y - u$ of scheme (29), which satisfies equation $\bar{B}z_{\overset{\circ}{t}} + \tau^2\bar{R}z_{\bar{t}t} + \bar{A}z = \psi$, where $\psi = \bar{F} - (\bar{B}u_{\overset{\circ}{t}} + \tau^2\bar{R}u_{\bar{t}t} + \bar{A}u)$. Hance, we get the following estimate for z:

$$\|z^n\|_{\tilde{A}^2} \le$$

$$M\max_k\left(\frac{\gamma}{\sqrt{\beta(\gamma+\beta)}}\left\|\psi_{1,\bar{t}}^k\right\|_{\tilde{A}^{-1}\tilde{D}} + \sqrt{\frac{\beta}{\gamma+\beta}}\left\|\frac{\psi_1^k + \psi_1^{k-1}}{2}\right\|_{\tilde{A}\tilde{D}^{-1}} + \frac{\tau^2}{12\sqrt{\beta(\gamma+\beta)}}\left\|\psi_{2,\bar{t}}^k\right\|_{\tilde{A}\tilde{D}^{-1}}\right).$$

Here $\psi_1,\ \psi_2$ are the approximation errors of the vector scheme (17).

Similarly we obtain results for $\dot{z} = \dot{y} - \dot{u}(t_n)$. Therefore, $\|z^n\|_{\tilde{A}^2} = \|u^n - y^n\|_{\tilde{A}^2} = O(\tau^4)$ and $\|\dot{z}^n\|_{\tilde{A}^2} = \|\dot{u}^n - \dot{y}^n\|_{\tilde{A}^2} = O(\tau^4)$ at time point t_n, $n = 1, 2, \ldots$ Based on (26), (31), (33), we obtain the following result.

Theorem 4. Let conditions (26), (31) be satisfied. Then, for $u(x,t) \in C^6[0,T]$, scheme (17), (18) converges to the solution to problem (11), i.e. the following accuracy estimates hold:

$$\|z(t)\|_{\tilde{A}^2} \leq M\tau^4, \quad \|\dot{z}(t))\|_{\tilde{A}^2} \leq M\tau^4, \quad \forall t \in [0,T].$$

Similarly, we obtain accuracy estimates for scheme 2^0.

Theorem 5. Let the approximation conditions (26) be satisfied. Then, if condition (31) is satisfied, the solution to scheme 2^0 converges to a sufficiently smooth solution to problem (1)–(3), i.e.

$$\|z(t)\|_{1h} + \|\dot{z}(t)\|_{1h} \leq M(|h|^4 + \tau^4), \quad z, \dot{z} \in H_h.$$

5 *Algorithm for implementing the scheme*

To implement (27) we rewrite it in the following form:

$$m_{11}w^{n+1} + m_{12}\dot{w}^{n+1} = \phi_1, \quad m_{21}w^{n+1} + m_{22}\dot{w}^{n+1} = \phi_2, \tag{34}$$

where

$$m_{11} = \tilde{D} + \tau\tilde{A}/2, \ \ m_{12} = -\tau^2\tilde{A}/12, \ \ m_{21} = \alpha\tilde{A}, \ \ m_{22} = \gamma\tilde{D} + \tau\beta\tilde{A}/2,$$

$$\phi_1 = \tau\tilde{\varphi}_1 + \left(\tilde{D} + \tau\tilde{A}/2\right)w^n - \tau^2\tilde{A}\dot{w}^n/12, \quad \phi_2 = \tau\tilde{\varphi}_2 + \alpha\tilde{A}w^n + \left(\gamma\tilde{D} + \tau\beta\tilde{A}/2\right)\dot{w}^n.$$

The integrals $\tilde{\varphi}_1$, $\tilde{\varphi}_2$ can be calculated, for example, using Simpson's formula.

Considering the commutability of $\tilde{A}$, $\tilde{D}$, we eliminate $\dot{w}^{n+1}$ from (34):

$$Cy^{n+1} = F, \tag{35}$$

where $C = \gamma\tilde{D}^2 + \tau(\beta+\gamma)\tilde{A}\tilde{D}/2 + \tau^2(3\beta+\alpha)\tilde{A}^2/12$, $F = m_{22}\phi_1 - m_{12}\phi_2$.

To solve (35), we factorize the operator C:

$$C = \gamma\, C_1C_2 = \gamma[\tilde{D}^2 - (x_1+x_2)\tau\tilde{A}\tilde{D} + x_1x_2\tau^2\tilde{A}^2], \ C_k = (\tilde{D} - x_k\tau\tilde{A}), \ \ k = 1,2.$$

Therefore, the algorithm for solving (35) has the following form:

$$\gamma C_1\bar{w} = F, \quad C_2w^{n+1} = \bar{w}.$$

The value of $\dot{w}^{n+1}$ is determined from

$$\left(\gamma\tilde{D} + \tau\beta\tilde{A}/2\right)\dot{w}^{n+1} = \phi_2 - \alpha\tilde{A}w^{n+1}.$$

The implementation of scheme (12) is not difficult, for example, for $\sigma_1 = \sigma_2 = \sigma$, it is implemented as follows:

$$(\bar{D} - \sigma\tau\bar{A})y^{n+1} = (1-2\sigma)\tau\bar{A}y^n + (\bar{D} + \sigma\tau\bar{A})y^{n-1} + \tau\phi, \quad n = 1,2,...,$$

$$y^0 = u_{h,0}, \quad y^1 = u_{h,1}.$$

6 Numerical experiments

6.1 One-dimensional case

Let us choose the parameters of problem (1)–(3): $l = \pi$, $T = 1$, $\mu = \theta = 1$, $\lambda = -1$. Then, instead of (1)–(3), we obtain:

$$\frac{\partial}{\partial t}\left(\frac{\partial^2 u}{\partial x^2} - u\right) + \frac{\partial^2 u}{\partial x^2} - u = 0,$$

$$(x,t) \in Q_T = \{x:\ \ 0 < x < \pi,\ \ t \in (0,1]\},$$

$$u(0,t) = u(\pi,t) = 0, \quad t \in (0,1],$$

$$u(x,0) = \sin x, \quad x \in [0,\pi].$$

The exact solution is $u(x,t) = e^{-t}\sin x$. The parameters of scheme (17), (18) are given by the values of $\gamma = \tau^2$, $\alpha = 9\tau^2/7$, $\beta = -2\tau^2/7$.

The order of the convergence rate is determined by the following formulas: $p^h = \log_2(\|z\| \,/\, \|z_{1/2}\|)$, $p^\tau = \log_2(\|z\| \,/\, \|z_{1/2}\|)$, where $z_{1/2} = y_{h/2,\tau/2} - u_{h/2,\tau/2}$.

Table 1

Convergence rates in spatial and temporal variables

h	τ	Error	Order
0.01	0.01	0.00038	–
0.005	0.005	$1.93E-05$	4.26
0.0025	0.0025	$1.27E-06$	3.93
0.00125	0.00125	$8.09E-08$	3.98

6.2 Two-dimensional case

We choose the parameters of problem (1)–(3) in the following form: $l_1 = l_2 = \pi$, $T = 1$, $\mu = \theta = 1$, $\lambda = -1$. Then, instead of (1)–(3), we obtain:

$$\frac{\partial}{\partial t}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} - u\right) + \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} - u = 0,$$

$$(x,y,t) \in Q_T = \{(x,y):\ \ 0 < x < \pi,\ \ 0 < y < \pi,\ \ t \in (0,1]\},$$

$$u(x,y,t) = 0, \quad (x,y) \in \partial\bar{\Omega}, \quad t \in (0,1],$$

$$u(x,y,0) = \sin x \sin y, \quad x \in [0,\pi], \quad y \in [0,\pi].$$

The exact solutions is $u(x,y,t) = e^{-t}\sin x \sin y$. The parameters of scheme (17), (18) are given by the values of $\gamma = \tau^2$, $\alpha = 9\tau^2/7$, $\beta = -2\tau^2/7$.

Table 2

Convergence rates in spatial and temporal variables

h_1	h_2	τ	Error	Order
1/10	1/10	0.05	$3.78E-02$	–
1/20	1/20	0.05	$2.49E-03$	3.97
1/40	1/40	0.05	$1.61E-04$	3.98
1/80	1/80	0.05	$1.01E-05$	3.97

Tables 1 and 2 show the rate of convergence of the approximate solution to the exact solution when conditions (26), (31) are satisfied.

Conclusion

A high-accuracy numerical method was developed and investigated for solving the first boundary value problem for a pseudoparabolic equation. Based on the stability theory results for difference schemes, it was possible to obtain a priori estimates and, on their basis, prove the convergence of the constructed algorithm with a fourth-order rate in both variables. An algorithm for implementing the methods is given. Based on a computational experiment, test calculations were verified to confirm the theoretical data.

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Author Contributions

All authors contributed equally to this work.

Conflict of Interest

The authors declare no conflict of interest.

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The intrinsic geometry of a convex surface in Galilean space

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This paper investigates the intrinsic geometry of a convex surface in the Galilean space R_3^1. The Galilean space, as a special case of a pseudo-Euclidean space, possesses a degenerate metric. The angle between two directions is defined using a parabolic method, which is itself degenerate. The three-dimensional Galilean space, similar to the Euclidean space, is based on a three-dimensional affine space. While the fundamental geometric objects in these spaces coincide structurally, the geometric quantities associated with them differ significantly from those in Euclidean geometry. It becomes necessary to introduce and rigorously define various geometric characteristics of objects in Galilean space. Therefore, special attention in this work is given to the total angle around the vertex of a cone, the angle between curves on a convex surface, and the variation of curve turning on a convex surface. A geodesic on a convex surface is defined as a curve with bounded variation of turning. A triangle is defined as a curve homeomorphic to a circle, bounded by three geodesics. Using the concept of the total angle around the vertex of a cone, we define the intrinsic curvature of convex surfaces in Galilean space and obtain an analogue of the Gauss–Bonnet theorem for convex surfaces in Galilean geometry. The results obtained extend classical notions of intrinsic geometry under a degenerate metric.

Keywords: Galilean space, convex surface, intrinsic geometry, intrinsic curvature, Gauss–Bonnet theorem, degenerate metric, tangent cone, geodesic, curves with bounded variation of turning.

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Introduction

Modern differential geometry successfully applies methods of both intrinsic and extrinsic geometry to the study of curves and surfaces in various spaces. One such space is the Galilean space R_3^1, where a degenerate metric coexists with the affine structure. This metric does not depend on all coordinates, leading to fundamental differences in the definitions of distances, angles, and curvature, as compared to the Euclidean case.

It is well known that the study of surface geometry is traditionally divided into "intrinsic" and "extrinsic" components. In Euclidean space, the first fundamental form plays a central role in intrinsic geometry. However, in Galilean space, the first fundamental form of a surface is degenerate, and Gauss's theorem, stating that the Gaussian curvature of a surface can be expressed entirely in terms of the coefficients of the first fundamental form and their derivatives-does not hold. Therefore, it becomes necessary to redefine intrinsic curvature, highlighting specific geometric characteristics that arise due to the degeneracy of the metric.

The aim of this paper is to define the fundamental geometric characteristics of convex surfaces and to construct an analogue to the intrinsic geometry of a surface within the Galilean space. Due to the degenerate nature of the metric, it is not possible to directly apply classical Euclidean definitions such as geodesics, arc length, or intrinsic curvature. Consequently, this paper introduces new approaches: using angles between generators of tangent cones, curves with bounded variation of turning, and cylindrical mappings.

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This work builds upon the results of previous studies by A.D. Alexandrov and O. Roschel, and contributes to the further developing of the theory of convex surfaces in pseudo-Euclidean spaces [1,2].

Since the 2000s, there has been an increasing interest in the geometry of Galilean space. In this context, special attention should be given to the works [3–5].

1 *Galilean space and fundamental concepts*

The Galilean space R_3^1 is an affine space A_3 equipped with two scalar products defined for vectors $X = \{x_1, y_1, z_1\}$ and $Y = \{x_2, y_2, z_2\}$:
1. $(X, Y) = (X, Y)_1 = x_1 \cdot x_2$,
2. $(X, Y) = (X, Y)_2 = y_1 \cdot y_2 + z_1 \cdot z_2$, when $(X, Y)_1 = 0$.

The norm of a vector is defined as the square root of its scalar square, and the distance between two points equals the norm of the vector connecting them [6].

The motions of Galilean space, i.e., linear transformations preserving distances between corresponding points, are described by the system [7]:

$$\begin{aligned} x' &= x + a, \\ y' &= \alpha x + y\cos\phi + z\sin\phi + b, \\ z' &= \beta x - y\sin\phi + z\cos\phi + c. \end{aligned}$$

Here a, b, c are translation parameters, α, β correspond to a Galilean shear (related to the parabolic angle h), and φ denotes the Euclidean rotation angle in the (y, z)-plane.

Let $e_1 = (1, 0, 0)$, $e_2 = (0, 1, 0)$, $e_3 = (0, 0, 1)$ be an orthonormal basis of the space. Then it is easy to establish that a motion maps a plane parallel to the vectors e_2, e_3 into another parallel plane. These planes are Euclidean and are called special planes. Planes not parallel to e_2 and e_3 are called planes in general position. Vectors parallel to special planes are also called special vectors.

A sphere in Galilean space R_3^1 is a set of all points equidistant from a point X_0 and is defined by the equation:

$$(x - x_0, x - x_0) = r^2.$$

If the center of the sphere is at the origin and the radius is 1, then

$$(x, x)_1 = x^2 = 1.$$

The set of all points whose coordinates satisfy the sphere equation forms a set of parallel special planes located at unit distance from the origin.

Unit vectors in the directions of X and Y have the coordinates:

$$\widetilde{X} = \left\{1, \frac{y_1}{x_1}, \frac{z_1}{x_1}\right\}, \quad \widetilde{Y} = \left\{1, \frac{y_2}{x_2}, \frac{z_2}{x_2}\right\}.$$

These vectors are the radius vectors of points on the unit sphere.

The angle between spatial vectors is defined as the distance between the endpoints of their corresponding unit vectors on the sphere, and is given by:

$$h = \sqrt{\left(\frac{y_1}{x_1} - \frac{y_2}{x_2}\right)^2 + \left(\frac{z_1}{x_1} - \frac{z_2}{x_2}\right)^2}.$$

It is evident that $0 \leq h < \infty$, and $h \to \infty$ if one of the vectors approaches a special direction. When $h = 0$, the vectors are parallel.

The angle between a spatial vector $\widetilde{X} = (x_1, y_1, z_1)$ and a special vector $\widetilde{Y} = (x_2, y_2, z_2)$ is defined as:

$$f = \frac{(\vec{X}, \vec{Y})}{|Y|_2} = \frac{\frac{y_1}{x_1} y_2 + \frac{z_1}{x_1} z_2}{\sqrt{y_2^2 + z_2^2}}.$$

The geometric interpretation of the angle f is the projection length of the unit vector $\widetilde{X}$ onto the direction of $\widetilde{Y}$ in the special plane. The projection is taken along the vector e_1. If $\vec{X}$ is parallel to e_1, then $f = 0$.

The angle between special vectors is given by the standard Euclidean formula:

$$\cos\varphi = \frac{y_1 y_2 + z_1 z_2}{\sqrt{y_1^2 + z_1^2} \cdot \sqrt{y_2^2 + z_2^2}} = \frac{(\vec{X}, \vec{Y})_2}{|\vec{X}|_2 \cdot |\vec{Y}|_2}.$$

Thus, the angle between lines in Galilean space is defined via the angle between their direction vectors.

Let F be a surface in R_3^1 that does not possess special tangent planes. We introduce a special curvilinear coordinate system by considering all intersections of F with special planes $x = \text{const}$. We choose the family of curves formed by these intersections as $u = u_0$ coordinate lines, and arbitrary transverse curves on F as $v = v_0$ lines. Then the surface can be parameterized as:

$$\vec{r}(u, v) = u e_1 + y(u, v) e_2 + z(u, v) e_3.$$

Here, the vectors $\vec{r}_u$ and $\vec{r}_v$ form a basis of the Galilean tangent plane at each point. The direction of $\vec{r}_v$ corresponds to the distinguished direction in the Galilean plane.

Let a curve on F be given by the equation $v = v(u)$. The arc length of the curve between points $A(u_0)$ and $A(u_1)$, where $u_0 \neq u_1$, is computed as follows:

$$ds = |\vec{r}_u du + \vec{r}_v dv| = |du|.$$

Hence, the square of the arc length differential on the surface equals the square of the increment of the coordinate u:

$$ds^2 = du^2.$$

This expression is referred to as the first fundamental form of the surface.

When $du = 0$, i.e., $u = \text{const}$, the corresponding curve lies entirely in a special plane. In this case, the differential of arc length is given by

$$ds^2 = (y_v^2 + z_v^2) dv^2 = G(u, v) dv^2.$$

We refer to this as the first supplementary fundamental form of the surface. Thus, with the chosen curvilinear coordinates, the coefficients of the first fundamental forms are $E_1 = 1$, and $G = y_v^2 + z_v^2$.

Suppose two points emanate from a point $M(u_0, v_0)$ on a surface in general position (i.e., whose tangents are not parallel to a special plane). Let $d\vec{r}$ and $\delta\vec{r}$ be the differentials of the radius vector along these curves. The angle θ between them is defined as the angle between the vectors $d\vec{r}$ and $\delta\vec{r}$.

Hence,

$$\theta = \sqrt{G(u, v)} \left(\frac{dv}{du} - \frac{\delta v}{\delta u} \right).$$

Similar to the Euclidean case, the concept of surface area can be introduced. The area of a domain D on the surface is given by

$$S = \iint\limits_D \sqrt{G(u, v)}\, du\, dv.$$

2 Convergence of convex surfaces in R_3^1

The degeneracy of the scalar product induces a degenerate metric in the Galilean space R_3^1. If two points lie on different planes, then the distance between the special planes to which they belong is defined as the distance between the points. When the points lie on the same special plane, the distance between them is defined as the length of the segment connecting them. Special planes in R_3^1 are Euclidean planes.

Suppose that a sequence of convex polyhedra F_n converges to a convex surface F, and a sequence of points $x_n \in F_n$ converges to a point $x \in F$.

We consider only such approximations for which the points x_n and y_n-converging respectively to x and y-remain at distances of the same order.

Theorem 1. Let a sequence of closed convex polyhedra F_n converge to a closed convex surface F, and let sequences of points $x_n, y_n \in F_n$ converge to points $x, y \in F$, respectively. Then the distances between x_n and y_n, measured on F_n, converge to the distance between x and y, measured on F, i.e.,

$$\rho_F(x, y) = \lim_{n\to\infty} \rho_{F_n}(x_n, y_n).$$

Proof. Suppose the points x_n and y_n lie on different special planes and converge to points x and y lying on corresponding special planes. Then we have:

$$\rho_{E_n}(x_n, y_n) \leq \rho_E(x, y),$$

where ρ_E denotes Euclidean distance. Moreover, in Galilean space, for points lying on different special planes, the distances are equal:

$$\rho_{F_n}(x_n, y_n) = \rho_F(x, y),$$

since in this case the measured distance is formally defined: it does not depend on the surface itself.

If the points x_n and y_n lie on the same special plane, then the metric is considered as a second-order metric, and we have $\rho_2(x_n, y_n) = \rho_E(x_n, y_n)$. Instead of computing the direct distance between the points, we consider the length L_n of a polygonal line on the special plane connecting x_n and y_n. This broken line arises from the intersection of F_n with the special plane. Since the special plane is Euclidean, distances on F_n within it are measured via the polygonal path joining x_n and y_n, and thus

$$\rho_2(x_n, y_n) = L_n(x_n, y_n).$$

When $F_n \to F$, the Euclidean length of the polygonal line $L_n(x_n, y_n)$ converges to the length of the curve $L(x, y)$ on the special plane. Therefore,

$$\lim_{n\to\infty} \rho(x_n, y_n) = \lim_{n\to\infty} L_n(x_n, y_n) = L(x, y) = \rho_2(x, y).$$

The theorem is thus proved.

□

3 The total angle around the vertex of a cone

The definition of the total angle around the vertex of a cone in Galilean space was introduced in the work [8] of A. Artykbayev. The main challenge in this definition lies in the concept of the angle between a vector and a special plane. Therefore, cones are divided into two classes: cones that do not have special supporting planes and cones that do.

In both cases, the total angle around the cone's vertex is defined using a circle of unit radius centered at the vertex of the cone.

When the cone does not possess special supporting planes, the total angle around its vertex is defined via the intersection of the cone with special planes. Since special planes determine a sphere in Galilean space [8], intersecting the cone with these planes yields hyperbolas with asymptotes parallel to lines passing through the cone's vertex.

The sphere in Galilean space consists of two parallel special planes. If one of these sections is reflected symmetrically onto the other, we obtain both branches of the hyperbola on the same special plane.

Let V be a convex cone in R_3^1 that does not have any special supporting plane. Intersect V with a special plane π_0 passing through the cone's vertex. Let μ_1 and μ_2 be the generatrices of the cone lying on this intersection. Let γ_1 and γ_2 be the curves formed by intersecting the cone V with the unit sphere of Galilean space, i.e., with the pair of special planes located at unit distance from the cone's vertex. Clearly, the curves γ_1 and γ_2 have asymptotes parallel to the lines μ_1 and μ_2, respectively.

This configuration, when visualized on a special plane, appears as shown in Figure 1.

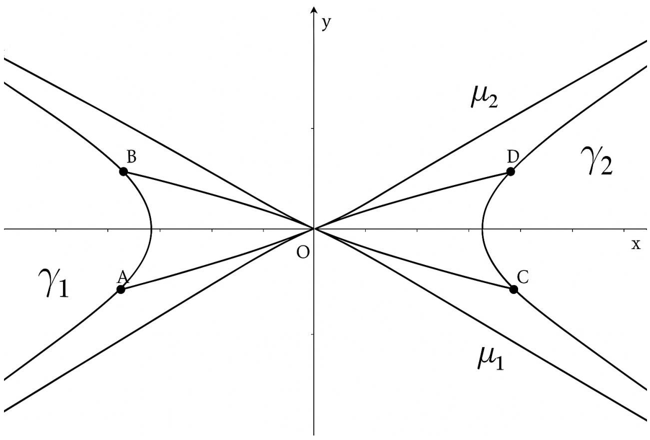

Figure 1. Intersection of a convex cone with special planes in Galilean space R_3^1

Let us denote the angular quantity by

$$\omega = AD + BC - \breve{AB} - \breve{DC} > 0.$$

The total angle around the cone's vertex is defined as the limit of ω as the points A, B, C, D on the respective branches of the curves γ_1 and γ_2 tend to infinity.

In [8], the limit was proven to be bounded. Furthermore, in [9], an analytical formula for this total angle was obtained when the equations of the curves γ_1 and γ_2 are known explicitly.

To define the curvature of fundamental sets on a convex surface, we use of the total angle around the cone's vertex in Galilean space.

When the cone has special supporting planes, its intersection with the unit sphere centered at the vertex is a closed curve. The length of this closed curve is then taken as the total angle around the vertex of the cone possessing a special supporting plane [10, 11].

4 Angle between curves on a convex surface in Galilean space R_3^1

To define the angle between two curves on a convex surface, we use the angle between the generatrices of the tangent cone. At every point on a convex surface in Galilean space, a tangent cone exists. This follows from the fact that Galilean space is affine, and affine structures are preserved in Galilean geometry.

When the convex surface is regular, the tangent cone degenerates into a plane. The geometry on this plane is Galilean.

Let l_1 and l_2 be two generatrices of a convex cone V, directed into the same half-space with respect to a special plane π_0. The generatrices l_1 and l_2 intersect the curve γ_i (for $i = 1$ or 2, depending on the direction of l_1 and l_2). The length of the arc of the curve γ_i enclosed between l_1 and l_2 is taken as the angle φ^+ between them.

We intersect the cone V with a plane passing through the bisector of the angle formed by the generatrices μ_1 and μ_2, and parallel to the Ox-axis. This intersection is referred to as the principal section of the cone V.

Generatrices of the cone directed into opposite half-spaces with respect to both the special plane and the principal section, and forming equal angles with the generatrices lying in the principal section, are called conjugate generatrices. The angle between conjugate generatrices is defined to be half of the total angle around the vertex of the cone.

When $\widetilde{l_1}$ and $\widetilde{l_2}$ are generatrices directed into different half-spaces divided by the special plane π_0, the angle between them is given by

$$\varphi^-\{\widetilde{l_1}, \widetilde{l_2}\} = \frac{\omega}{1} - \beta^\star,$$

where $\beta^\star$ is the angle $\varphi^-\{\widetilde{l_1}, \widetilde{l_2}^\star\}$, and $\widetilde{l_2}^\star$ is the generatrix conjugate to $\widetilde{l_2}$. It is easy to verify that

$$\varphi^+\{\widetilde{l_1}, \widetilde{l_2}^\star\} = \varphi^-\{\widetilde{l_1}^\star, \widetilde{l_2}\}.$$

It can be shown that for three generatrices of the cone not directed into the same half-space and distinct from μ_1 and μ_2, the sum of the angles between them equals the total angle around the vertex of the cone.

If the cone V degenerates into a plane or a dihedral angle, the defined angles φ^+ and φ^- coincide with the angle between rays in the Galilean plane R_2. In such cases, the total angle is zero.

Now consider an arbitrary point M on the surface F, and let V be the tangent cone at this point. Let $\{\gamma\}$ denote the family of curves on the surface F issuing from the point M and having a direction not lying in the special plane π_0. The direction of any curve in $\{\gamma\}$ coincides with a generatrix of the tangent cone V.

The angle between two such curves issuing from the point M on the convex surface F is defined as the angle between their directions — that is, the angle between the corresponding generatrices of the tangent cone.

This notion of angle does not satisfy all the properties of angles between curves on convex surfaces in Euclidean geometry. For instance, in Euclidean space, if L_1, L_2, and L_3 are three curves forming angles α_1, α_2, and α_3, then the sum of any two of these angles is at least as great as the third.

This property holds in Galilean space only for curves directed into the same half-space.

The angle defined in this manner is naturally called the "parabolic angle". It can take any positive value. When the direction of one of the curves tends to lie infinitely close to the special plane, the angle increases without bound.

5 *Curves of bounded variation of turning in R_3^1*

To introduce the analogue of a shortest path in R_3^1, we first define curves of bounded variation of turning. Let γ be a curve in the space R_3^1 connecting points A and B that lie on different special planes. Inscribe a polygonal line L_n into γ, and denote by $\mu(L_n)$ the sum of (parabolic) angles of this polygonal line. The upper limit of the values $\mu(L_n)$ over all such inscribed polygonal lines L_n is called the variation of turning of the curve γ. If $\mu(\gamma)$ is finite, then γ is called a curve of bounded variation of turning.

Lemma 1. If γ is a curve of bounded variation of turning connecting points A and B on different special planes, then it intersects each special plane of R_3^1 in at most one point.

Proof. Suppose γ has two points of intersection with some special plane, or contains a component lying entirely within a special plane. Then one can inscribe a polygonal line L_n such that at least one of its segments lies entirely within the special plane. The angle of the polygon at the ends of such a segment becomes unbounded. This contradicts the boundedness of the variation of turning. □

Lemma 2. If γ is a curve of bounded variation of turning in Galilean space R_3^1, then it also has bounded variation of turning in Euclidean space.

Proof. Let $A_{i-1}A_i$ and A_iA_{i+1} be segments of a polygonal line inscribed in γ. Let h_i be the angle between these segments in R_3^1, and $\varphi_i^\star$ be the Euclidean measure of that angle. Then the following inequality holds:

$$0 \leq \varphi_i^\star \leq \tan \varphi_i^\star \leq h_i.$$

Since γ is of bounded variation in R_3^1, we have $\sum\limits_{i=1}^{n} h_i < \infty$, and thus $\sum\limits_{i=1}^{n} \varphi_i^\star$ is also finite. Therefore, the variation of turning in Euclidean space is bounded. □

Lemma 3. Curves of bounded variation of turning have right and left semi-tangents at every point. These are not parallel to the special plane.

This follows from the properties of Euclidean curves of bounded variation of turning. Since such curves also have bounded variation in Euclidean space, the tangents cannot be parallel to the special plane; otherwise, it contradicts boundedness.

Variation of turning can also be defined equivalently. Let γ be a curve with a right (or left) semi-tangent at each point. Take a finite number of points A_k on γ, and at each point place the right semi-tangent t_k. The supremum of the sum of angles between successive semi-tangents over all such finite systems of points A_k is called the variation of turning of γ. This definition is equivalent to the one given above, as proved analogously in Euclidean geometry [12].

Let A and B be points on different special planes in R_3^1. Consider circular cones S_A and S_B with vertices at A and B, respectively, and with their directrices centered along the segment AB (lying in a special plane). These cones intersect. The class of closed convex surfaces formed by all possible intersections of such cones is denoted by S_{AB}.

Lemma 4. If m_{AB} is a family of curves connecting A and B and having variation of turning not greater than N, then there exists a surface F in the class S_{AB} such that all curves in the family lie within F.

Proof. From the set of surfaces, choose one. For this surface the total angle around the vertices satisfies:

$$\gamma_A = \gamma_B = 2\pi N.$$

Consider a broken line consisting of the generatrices of intersecting cones S_A and S_B, with a vertex at their intersection point. The turn at this vertex is not less than N. This follows from the triangle formed by the broken line and the segment AB, where the base angles are N, and the vertex angle is at least the sum of the base angles. The same argument applies if any vertex of the broken line does not correspond to a generatrix of surface F. In such case, that part cannot lie on the cone, implying the curve cannot lie outside F. □

Theorem 2. Let $\gamma_1, \gamma_2, \ldots, \gamma_n$ be an infinite sequence of curves with bounded turning variation, each with variation no greater than N. If γ_n converges to a curve γ, then γ is also with bounded turning variation and its variation does not exceed N.

Proof. The proof is analogous to the corresponding result in Euclidean space [12].

Let X and Y be arbitrary points on a convex polyhedron Q without special supporting planes. If X and Y lie on the same special plane, they are connected by a planar convex polygon with bounded turning. If they lie on different special planes, they can still be connected by a polygonal line on Q with bounded turning. □

Lemma 5. Any two points X and Y on a convex polyhedron Q without special planes can be connected by a polygonal line L_{XY} on Q with bounded turning.

Proof. Let points X and Y lie on the singular planes π_X and π_Y, respectively. Consider the intersection of the polyhedron Q with the singular planes π_X and π_Y, and denote by Q_{AB} the portion of Q between these two planes. At the points X and Y, the boundary of the polyhedron Q_{AB} has two directions: $l_{1X},\ l_{2X}$ at X and $l_{1Y},\ l_{2Y}$ at Y.

Now consider the spatial segment connecting the points X and Y. We construct through X and Y a general-position plane π_{XY} such that the boundary edge directions of the broken line L_{XY}, formed by the intersection of π_{XY} with the polyhedron Q_{XY}, do not coincide with the directions of the boundary at X and Y. The broken line L_{XY} contains no segments lying on singular planes. Indeed, the extreme segments are chosen in such a way that they do not lie on any singular plane. The intermediate segments cannot lie on singular planes either, since the polyhedron Q contains no such planes.

Since the plane π_{XY} is in general position, it is a Galilean plane. Consider the convex polygon formed by the segment XY and the broken line L_{XY} on this plane. In this polygon, the sum of the angles not supported by singular directions equals the sum of the angles with singular supporting directions. Hence,

$$\mu(L_{XY}) = \alpha_X + \alpha_Y,$$

where α_X and α_Y are the angles at the vertices X and Y supported by singular directions. These values α_X and α_Y are finite because the segment XY and the extreme edges of the broken line L_{XY}, which form these angles, are spatial (i.e., not singular). Therefore, the value $\mu(L_{XY})$ is bounded. This completes the proof of the lemma. □

Consider a sequence of convex polyhedra F_n with common boundary L_n, all lacking special supporting planes. Suppose each L_n is a polygonal line with bounded turning on F_n, and the sequence F_n converges to a convex surface F with boundary L.

Lemma 6. If a sequence of polygonal lines γ_n with bounded turning on F_n converges to a curve γ on F, then γ also has bounded turning variation.

This follows from Theorem 2 and the assumption that F does not have special supporting planes.

6 *An analogue of a geodesic on a surface in R_3^1*

The degeneracy of the metric in Galilean space R_3^1 prevents the definition of a geodesic via standard metric methods. The distance between two points lying on different special planes is equal to the interval between those planes [13].

An interesting phenomenon arises: all curves connecting two given points that do not lie on the same special plane have equal length. This effect can be interpreted within Newtonian mechanics as worldline length invariance connecting given events. In other words, in Newtonian mechanics, time is independent of the velocity of bodies. Hence, the natural question arises — by what criterion can a curve on the surface be distinguished as a substitute for the shortest path, i.e., a curve possessing properties similar to those of a Euclidean geodesic?

Let F be a convex surface in Galilean space. Consider a family $\{\gamma\}$ of curves lying on the surface and connecting two given points on the Galilean surface.

Definition 1. A curve γ from the family $\{\gamma\}$ that has the least turning variation is called the shortest curve between the given points on the surface.

This provides one possible definition of a geodesic in R^1_3.

Accordingly, a geodesic is defined as a continuous curve that is the shortest (in the sense of minimal turning variation) over each of its sufficiently small subarcs. A triangle on the surface is defined as a figure homeomorphic to a circle and bounded by three such shortest curves. A geodesic triangle is defined as a figure bounded by three geodesics.

7 Intrinsic curvature of a convex surface

The degeneracy of the metric in Galilean space necessitates a new approach to defining the intrinsic geometry of a surface. Intrinsic geometry includes those properties of a surface that depend on the properties of its metric. In Euclidean space, the intrinsic curvature of a convex surface is fully determined by the internal metric of the surface. A similar approach in Galilean space does not yield satisfactory results. Therefore, we attempt to study the intrinsic geometry of a convex surface using its extrinsic geometry in Galilean space. We define the intrinsic curvature of a set on a convex surface in Galilean space by analogy with the Euclidean case, initially for three types of "elementary" sets: open triangles, open geodesics, and points. An open triangle excludes its vertices and sides; its sides do not lie on special planes. An open geodesic is a geodesic excluding its endpoints.

For an open triangle T on F, the curvature is defined as

$$\omega(T) = \alpha - \beta + \gamma.$$

Here α, β, γ are the triangle's angles, and each side lies in a different half-space determined by a special plane through the vertex.

The curvature of an open geodesic is taken to be zero.

A point's curvature on a convex surface is defined as the total angle around the vertex of the tangent cone at that point.

We consider sets on a convex surface that do not share common points pairwise. Such sets are called "elementary". Based on the definition of elementary set curvature, we define a bounded set's curvature on a convex surface.

Definition 2. The intrinsic curvature of a bounded closed set on a convex surface is defined as the infimum of the curvatures of all elementary sets containing it.

We define the intrinsic curvature of Borel sets on a convex surface as the least upper bound (supremum) of the curvatures of all bounded closed subsets contained in it.

The definitions of intrinsic curvature of a set on a convex surface in Galilean space given above are analogous to those in Euclidean geometry. The main difference lies in how the curvature of the "elementary" (or "basic") sets is defined.

Let M be an "elementary" set on a convex surface F. Suppose it can be represented as a disjoint union of basic sets

$$M = \sum_{i=1}^{n} B_i.$$

Then, the intrinsic curvature of the set M is defined as the sum of the curvatures of its basic components:

$$\omega(M) = \sum_{i=1}^{n} \omega(B_i).$$

The intrinsic curvature of a Borel set on a convex surface is defined as the exact least upper bound of the curvatures of all bounded closed subsets contained in it.

The curvature value of a set on a convex surface does not depend on the particular way it is decomposed into basic sets.

This fact, along with the non-negativity and complete additivity of the intrinsic curvature of a convex surface for elementary sets, is proved in the same way as in Euclidean geometry. This is justified by the observation that the cylindrical mapping of a convex surface can be regarded as the projection of its spherical mapping onto a cylinder. The generating curve of the cylinder corresponds to a great circle on the unit sphere. As a result, the cylindrical mapping of a convex surface in Galilean space inherits all the essential properties of the spherical mapping. These properties ensure the correctness of the intrinsic curvature's stated properties.

Theorem 3. The intrinsic curvature of a Borel set on a convex surface is equal to its extrinsic curvature.

Proof. The concept of extrinsic curvature is defined in [8]. The authors show the cylindrical mapping is a projection of the Euclidean spherical mapping onto the sphere in the isotropic space R_3^1. The isotropic sphere is interpreted as the co-Euclidean plane S_2^1. To prove the theorem, it suffices to show that the curvature of basic sets equals the area of their cylindrical image. Indeed, the spherical image of an open triangle maps to a triangle on the co-Euclidean plane. The quantity defining the curvature of the open triangle on F equals the area of the triangle on S_3^1. The intrinsic curvature of an open geodesic equals the area of its cylindrical image, which is a curve on the plane.

The total angle around the vertex of a cone is taken to be equal to the area of its cylindrical image.

The theorem for any Borel set on a convex surface follows from the fact that the cylindrical mapping of a convex surface in Galilean space is a central projection of the Euclidean spherical mapping. □

Theorem 4. Intrinsic curvature is a non-negative and fully additive function on Borel sets of a convex surface.

Proof. The extrinsic curvature of convex surfaces in Galilean space is a non-negative and fully additive function on Borel sets of the surface. Therefore, intrinsic curvature, being equal to extrinsic curvature, also possesses these properties. □

8 Gauss–Bonnet formula in Galilean space

The results obtained in the previous sections allow us to approach a generalization of the Gauss–Bonnet formula for an arbitrary domain on a convex surface in Galilean space. However, a completely new difficulty arises here, related to the discontinuity of the angle between vectors when a vector traverses a closed region. In particular, when one of the vectors is parallel to a singular plane, the angle between vectors becomes unbounded. To eliminate this peculiarity, the domain must satisfy certain conditions.

Let Q be a convex domain on a convex surface F, that has no singular supporting planes, and is bounded by smooth curves

$$\alpha_1, \alpha_2, \ldots, \alpha_k, \quad \beta_1, \beta_2, \ldots, \beta_p.$$

Assume that the curves α_1 and β_p, as well as α_k and β_1, share common endpoints A and B, respectively. The points A and B lie on the singular planes that bound the domain. The directions of the curves α_1, β_p at point A, and α_k, β_1 at point B, are not parallel to the singular planes.

Let φ_i and ψ_j denote the angles between the curves (α_i, α_{i+1}) and (β_j, β_{j+1}), respectively. Let φ and ψ denote the angles at the points A and B, respectively.

A domain Q satisfying the above conditions is called admissible.

Then, the following theorem holds.

Theorem 5. Let $D \subset F$ be an admissible domain on a convex surface F in Galilean space. Then the generalized Gauss–Bonnet formula holds:

$$\iint\limits_{D} K\, d\sigma = \varphi + \psi - \sum_{i=1}^{k} \left[\varphi_i + \int\limits_{\alpha_i} k(\alpha_i)\, ds \right] - \sum_{j=1}^{n} \left[\psi_j + \int\limits_{\beta_j} k(\beta_j)\, ds \right],$$

where:

- K is the *Gaussian curvature* on the surface F,
- $d\sigma$ is the *surface area element*,
- $k(\alpha_i)$, $k(\beta_j)$ are the *geodesic curvatures* of the boundary curves,
- ds is the *arc length element*,
- φ_i, ψ_j are the *turning angles* between boundary curve segments,
- φ, ψ are the *interior angles* at the corner points A and B.

Proof. We begin by computing the intrinsic curvature of a convex geodesic polygon on a convex polyhedral surface. Let F_n be a sequence of convex polyhedral surfaces converging to a convex surface F that has no singular supporting planes.

Let Q_n be a geodesic polygon on F_n, bounded by geodesic arcs α_{in} and β_{jm}, such that Q_n consists of a collection of non-overlapping geodesic triangles. These triangles are chosen in such a way that none of their sides lie on singular planes. Furthermore, the vertices of the polyhedron F_n contained in Q_n are the vertices of these triangles.

By definition, the intrinsic curvature $\omega(Q_n)$ of the polygon Q_n is equal to the sum of the intrinsic curvatures of the sets contained within it:

$$\omega(Q_n) = \sum \omega(T_e) + \sum \omega(X_m) + \sum \omega(L_n),$$

where:

- T_e are the open triangles in the triangulation,
- X_m are the vertices of the triangles T_e contained in Q_n,
- L_n are the sides of the triangles (excluding endpoints).

The intrinsic curvature $\omega(L_n) = 0$ for all segments L_n, since geodesic arcs have zero intrinsic curvature except at their endpoints.

The vertices of triangles T_e in Q_n are of two types:

1. vertices located *inside* the polygon Q_n,
2. vertices lying *on the boundary* of the polygon.

The boundary vertices are further subdivided into:

- points lying on A_n or B_n,
- points lying on the geodesic arcs α_{in} or β_{jm}.

The sum of all angles around an interior vertex of Q_n is equal to the negative of the intrinsic curvature at that vertex. The angle at a boundary vertex equals the turning angle of the boundary at that point.

Thus, we obtain:

$$\sum \omega(T_e) = \varphi_n + \psi_n - \sum_{i=1}^{k} (\varphi_{in} + \Delta\alpha_{in}) - \sum_{j=1}^{p} (\psi_{jn} + \Delta\beta_{jn}) - \sum \omega(X_m),$$

where $\Delta\alpha_{in}$, $\Delta\beta_{jn}$ denote the total turning (geodesic curvature integrals) along the respective arcs α_{in}, β_{jn}.

Hence, the total intrinsic curvature of the polygon Q_n is

$$\omega(Q_n) = \varphi_n + \psi_n - \sum_{i=1}^{k} (\varphi_{in} + \Delta\alpha_{in}) - \sum_{j=1}^{p} (\psi_{jn} + \Delta\beta_{jn}) - \sum \omega(X_m).$$

Finally, passing to the limit and applying arguments analogous to those used in Euclidean geometry, we obtain the required formula. □

In Galilean space, consider a closed surface F possessing two conical points A and B, each admitting a singular supporting plane. Assume that S_A and S_B are the tangent cones at points A and B, respectively. Let the total angles around the vertices of these cones be γ_A and γ_B.

Then the Gauss–Bonnet formula for the closed surface F takes the form

$$\int\limits_{\Phi} K\, d\sigma = \gamma_A + \gamma_B,$$

where:

- K is the *Gaussian curvature*,
- $d\sigma$ is the *surface area element*,
- γ_A, γ_B are the *total cone angles* at the conical points A and B.

This formula reflects the concentration of curvature at the conical points on the surface and generalizes the classical result to surfaces with isolated singularities in Galilean geometry.

Conclusion

This work presents a systematic exposition of the intrinsic geometry of convex surfaces in Galilean space. It is shown that, despite the degeneracy of the metric, it is possible to construct a consistent theory that incorporates the notions of length, angle, geodesics, and curvature. One of the key results is the formulation and proof of an analogue of the Gauss–Bonnet theorem, valid for convex surfaces without special supporting planes. It is also demonstrated that the intrinsic curvature coincides with the extrinsic curvature defined via cylindrical mapping, highlighting the deep connection between the intrinsic and extrinsic properties of convex geometry in Galilean space. The results obtained may serve as a foundation for further investigations of geometric structures in non-smooth spaces and have potential applications in mechanics, optics, and relativity theory, where space-time models may admit degenerate metrics. These results can be applied in classical mechanics, where Galilean space models Newtonian spacetime. They may also be useful in optics and relativity theory for studying degenerate metrics.

Author Contributions

All authors contributed equally to this work.

Conflict of Interest

The authors declare no conflict of interest.

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Forward and inverse problems for a mixed-type equation with the Caputo fractional derivative and Dezin-type non-local condition

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This paper investigates a mixed-type partial differential equation involving the Caputo fractional derivative of order $\rho \in (0,1)$ for $t > 0$, and a classical parabolic equation for $t < 0$. The problem is studied in an arbitrary N-dimensional domain Ω with smooth boundary, subject to Dezin-type non-local boundary and gluing conditions. For the forward problem, existence and uniqueness of the classical solution are established under suitable assumptions on the data, employing the Fourier method. The influence of the parameter λ in the non-local boundary condition on solvability is analyzed. Additionally, an inverse problem is considered, where the source term is separable as $F(x,t) = f(x)g(t)$, with known $g(t)$ and unknown spatial function $f(x)$. Under certain conditions on $g(t)$, the uniqueness and existence of the solution are proven. This work extends previous results on mixed-type equations, highlighting the role of fractional derivatives and non-local conditions in both forward and inverse settings. The findings contribute to the theory of mixed-type and fractional differential equations, with potential applications in subdiffusion and related processes.

Keywords: mixed type equation, the Caputo derivative, forward problem, inverse problem, Fourier method, Dezin-type non-local condition, existence and uniqueness, gluing conditions.

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Introduction and formulation of problems

Numerous researchers have investigated boundary value problems for differential equations of mixed type. These problems first attracted attention through the work of S. Chaplygin, who applied mixed-type partial differential equations to model gas dynamics. Later, A. Bitsadze [1] demonstrated the ill-posedness of the Dirichlet problem for the equation $u_{xx} + \mathrm{sgn}(y)u_{yy} = 0$.

Let $0 < \rho < 1$. The Caputo fractional derivative of order ρ of a function f is given by [2, p. 336]

$$D_t^\rho f(t) = \frac{1}{\Gamma(1-\rho)} \int\limits_0^t \frac{f'(\tau)}{(t-\tau)^{1-\rho}} d\tau, \quad t > 0,$$

provided the right-hand side exists. Here $\Gamma(\cdot)$ denotes the well-known gamma function.

Let Ω be an arbitrary N-dimensional domain with a sufficiently smooth boundary $\partial\Omega$. Consider the following mixed-type equation:

$$\begin{cases} D_t^\rho u - \Delta u = F(x,t), & x \in \Omega, \quad 0 < t \le \beta, \\ u_t + \Delta u = F(x,t), & x \in \Omega, \quad -\alpha < t < 0, \end{cases} \tag{1}$$

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where $F(x,t)$ is a continuous function and $\alpha > 0$, $\beta > 0$ are given real numbers and Δ is the Laplace operator.

The Dezin problem. Find a function $u(x,t)$ satisfying equation (1) and the boundary condition

$$u(x,t)|_{\partial\Omega} = 0, \quad t \in [\alpha, \beta], \tag{2}$$

and the gluing condition

$$\lim_{t\to+0} u(x,t) = \lim_{t\to-0} u(x,t), \quad x \in \Omega, \tag{3}$$

and also the non-local condition

$$u(x,-\alpha) = \lambda u(x,0), \quad x \in \Omega, \tag{4}$$

where $\lambda = const$, $\lambda \neq 0$.

This problem is called the Dezin problem due to condition (4). Note that if $\lambda = 0$ then we arrive at the backward problem for the subdiffusion equation.

Definition 1. A function $u(t,x) \in AC([0,\beta]; C(\overline{\Omega}))$ with the properties

1. $u(x,t) \in C(\overline{\Omega} \times [-\alpha, 0])$,
2. $\Delta u(x,t) \in C(\overline{\Omega} \times (-\alpha, 0) \cup (0, \beta])$,
3. $D_t^\rho u(x,t) \in C(\overline{\Omega} \times (0, \beta])$,
4. $u_t(x,t) \in C(\overline{\Omega} \times (-\alpha, 0))$,

and satisfying conditions (1)–(4) is called a (classical) solution of the problem (1)–(4).

In equation (1), the derivatives of the function $u(x,t)$ are considered in the open domain. The condition of continuity for these derivatives in the closed domain $\overline{\Omega}$ is imposed to facilitate a straightforward proof of the solution's uniqueness. The requirement of absolute continuity of the solution for $t \geq 0$ is necessary to exclude singular functions from consideration, as their inclusion would violate the uniqueness of the solution. Notably, the solution derived via the Fourier method inherently satisfies these continuity and absolute continuity requirements.

Inverse problem. Let $F(x,t) = f(x)g(t)$, and let the function $g(t)$ be known. Find functions $f(x)$ and $u(x,t)$, such that $f(x) \in C(\overline{\Omega})$ and the function $u(x,t)$ satisfies conditions (1)–(4) and conditions of Definition 1, also an additional condition

$$u(x,t_0) = \varphi_0(x), \quad x \in \Omega, \tag{5}$$

here $\varphi_0(x)$ is a given sufficiently smooth function and t_0 is a given point in $(0,\beta)$.

In 1963 A.A. Dezin [3] (see the condition (Γ_1)) studied solvable extensions of mixed-type differential equations. He formulated a boundary value problem characterized by 2π-periodicity and non-local conditions, where the value of the unknown function within a rectangular domain is related to the value of its derivative on the boundary. This formulation involves the Lavrentiev–Bitsadze operator and reflects a significant development in the theory of mixed-type equations.

In works [4–7] non-local boundary value problems of Dezin's type for mixed-type differential equations have been investigated. Let us dwell in more detail on these works.

In [4], the following degenerating mixed type equation is considered:

$$Lu \equiv K(t)u_{xx} + u_{tt} - bK(t)u = F(x,t), \tag{6}$$

in the rectangular domain $D = \{(x,t) : 0 < x < l, \ -\alpha < t < \beta\}$, where $K(t) = (\operatorname{sgn} t)|t|^m$, and $m, b, l > 0$ are given real constants. The study addresses an inhomogeneous Dezin-type non-local boundary condition of the form $u_t(x,-\alpha) - \lambda u(x,0) = \psi(x)$. In [5], a similar problem is examined under the assumptions $m = b = 0$, $\alpha = l$, $\psi(x) = 0$, and $F(x,t) = f(x,t)H(t)$ ($H(t)$ is the Heaviside function), with $\lambda \geq 0$. It is also shown that in the case $\lambda < 0$, the homogeneous problem admits a nontrivial solution. In [6], equation (6) is investigated under the same conditions as in [4], except for the homogeneous case where $F(x,t) \equiv 0$. It should be emphasized that all the abovementioned works focus on forward problems. In the work [8], the forward and inverse problems for equation (1) were studied. In solving the forward problem, instead of the non-local condition (4), the gluing condition $D_t^\rho u(x,+0) = u_t(x,-0)$ was used. The inverse problem of determining the unknown function $f(x)$ was investigated for the case where $g(t) \equiv 1$. In [9], the inverse problem is also considered, where the equation involves for $t > 0$ a Caputo fractional derivative of order ρ, and for $t < 0$ the equation is of hyperbolic type. Furthermore, in [10–12], similar inverse problems are studied for the subdiffusion and mixed-type equations.

In this paper, we consider the forward problem (1)–(4) and the inverse problem (1)–(5) of determining the right-hand side.

1 *Preliminaries*

Let us denote by $\{v_k\}$ the complete orthonormal eigenfunctions in $L_2(\Omega)$ and by λ_k (where the values λ_k are a sequence of non-negative integers that do not decrease with increasing index k: $0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq ...$) the set of positive eigenvalues of the following spectral problem

$$\begin{cases} -\Delta v(x) = \lambda v(x), \ x \in \Omega, \\ v(x)\,|_{\partial\Omega} = 0. \end{cases} \tag{7}$$

Let σ be an arbitrary real number. In the space $L_2(\Omega)$, we introduce the operator $\hat{A}^\sigma$, which operates according to the rule

$$\hat{A}^\sigma g(x) = \sum_{k=1}^{\infty} \lambda_k^\sigma g_k v_k(x).$$

Here $g_k = (g, v_k)$ are the Fourier coefficients of an element $g \in L_2(\Omega)$. Obviously, this operator $\hat{A}^\sigma$ with the domain $D\left(\hat{A}^\sigma\right) = \left\{g \in L_2(\Omega) : \sum_{k=1}^{\infty} \lambda_k^{2\sigma} |g_k|^2 < \infty\right\}$ is selfadjoint. If we denote the operator by A in $L_2(\Omega)$ acting according to the rule $Ag(x) = -\Delta g(x)$ and with the domain of definition $D(A) = \left\{g \in C^2(\overline{\Omega}) : g(x) = 0, \ x \in \partial\Omega\right\}$, then the operator $\hat{A} = \hat{A}^1$ is the selfadjoint extension in $L_2(\Omega)$ of the operator A [13, p. 139].

Our reasoning will largely rely on the methodology developed in the monograph [14].

Lemma 1. [14, p. 453] Let $\sigma > \frac{N}{4}$. Then the following estimate $||\hat{A}^{-\sigma} g||_{C(\Omega)} \leq C||g||_{L_2(\Omega)}$ holds.

In order to prove the existence of a solution to the forward and inverse problems, it is necessary to study the convergence of the following series:

$$\sum_{k=1}^{\infty} \lambda_k^\tau |h_k|^2, \qquad \tau > \frac{N}{2}, \tag{8}$$

here h_k are the Fourier coefficients of the function $h(x) \in L_2(\Omega)$. In the case of integer τ, in the paper by Il'in [13] we obtain the conditions for convergence of such series in terms of function $h(x)$ belonging to the classical Sobolev space. In order to formulate this condition, let us introduce the class $\hat{W}_2^1(\Omega)$ as a closure in the $W_2^1(\Omega)$ norm of the set of functions from $C_0^\infty(\Omega)$ that vanish on the boundary of the domain Ω. Il'in's lemma states that if the function $h(x)$ satisfies the following conditions (we can take $\tau = \frac{N}{2} + 1$, if N is even and $\tau = \frac{N+1}{2}$, if N is odd):

$$h(x) \in W_2^{\left[\frac{N}{2}\right]+1}(\Omega), \quad h(x), \, \Delta h(x), ..., \Delta^{\left[\frac{N}{4}\right]} h(x) \in \hat{W}_2^1(\Omega), \tag{9}$$

then the series (8) converges. Similarly, if in (8) τ is replaced by $\tau+1$, then the convergence conditions are

$$h(x) \in W_2^{\left[\frac{N}{2}\right]+2}(\Omega), \quad h(x),\ \Delta h(x), ...,\ \Delta^{\left[\frac{N}{4}\right]} h(x) \in \hat{W}_2^1(\Omega). \tag{10}$$

Next we recall some properties of the Mittag-Leffler function.

Let μ be an arbitrary complex number. The function defined by the following infinite series

$$E_{\rho,\mu}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\rho k + \mu)}$$

is called a Mittag-Leffler function with two parameters [2, p. 56]. If the parameter $\mu = 1$, then we have the classical Mittag-Leffler function: $E_\rho(z) = E_{\rho,1}(z)$.

Lemma 2. [2, p. 61, Eq. (4.4.5)] For any $t \geq 0$ one has

$$0 < E_{\rho,\mu}(-t) \leq \frac{C_0}{1+t},$$

where the constant C_0 does not depend on t and μ.

Lemma 3. [2, p. 47]) The classical Mittag-Leffler function of the negative argument $E_\rho(-t)$ is a monotonically decreasing function for all $0 < \rho < 1$ and

$$0 < E_\rho(-t) < 1, \quad E_\rho(0) = 1.$$

Lemma 4. [2, p. 61, Eq. (4.4.5)] Let $\rho > 0$, $\mu > 0$ and $\lambda \in C$. Then for all positive t one has

$$\int_0^t \eta^{\rho-1} E_{\rho,\rho}(\lambda \eta^\rho) d\eta = t^\rho E_{\rho,\rho+1}(\lambda t^\rho).$$

Lemma 5. [2, p. 57, Eq. (4.2.3)] For all $\alpha > 0$, $\mu \in \mathbb{C}$, the following recurrence relation holds:

$$E_{\rho,\mu}(-t) = \frac{1}{\Gamma(\mu)} - t E_{\rho,\mu+\rho}(-t).$$

Lemma 6. [15] Let $\lambda > 0$, $0 < \varepsilon < \rho$. Then, for all $t > 0$, the following coarser estimate holds:

$$\left| t^{\rho-1} E_{\rho,\rho}\left(-\lambda t^\rho\right) \right| \leq C \lambda^{\varepsilon-1} t^{\varepsilon\rho-1},$$

where $C > 0$ is a constant independent of λ and t.

2 Constructing the solution of the forward problem (1)–(4)

We seek the unknown function $u(x,t)$, which is a solution to the problem (1)–(4), in the form

$$u(x,t) = \sum_{k=1}^{\infty} T_k(t) v_k(x).$$

It is easy to see that the unknown coefficients $T_k(t)$ have the form [2, p. 174]

$$T_k(t) = \begin{cases} a_k E_{\rho,1}(-\lambda_k t^\rho) + \int\limits_0^t s^{\rho-1} E_{\rho,\rho}\left(-\lambda_k s^\rho\right) F_k(t-s)\, ds, & t > 0, \\ b_k e^{\lambda_k t} - \int\limits_t^0 F_k(s) e^{\lambda_k (t-s)} ds, & t < 0, \end{cases}$$

where a_k, b_k are arbitrary constants, and $F_k(t)$ are the Fourier coefficients of the function $F(x,t)$.

By the gluing condition (3) one has $a_k = b_k$. The non-local condition (4) implies:

$$a_k\delta_k = F_k^*, \tag{11}$$

where $F_k^* = \int\limits_{-\alpha}^{0} F_k(s)e^{\lambda_k(-\alpha-s)}ds$ and $\delta_k := \delta_k(\lambda) = e^{-\lambda_k\alpha} - \lambda,\ k \geq 1$.

If for some k, we have $\delta_k = 0$, then equation (11) has a solution only if the free term is zero, i.e., $F_k^* = 0$. In this case, the coefficients a_k remain arbitrary, and problem (1)–(4) does not have a unique solution.

Thus, if $\delta_k \neq 0$ for all k, then the unknown coefficients a_k are uniquely determined, and problem (1)–(4) has a unique solution. Indeed let $u \equiv u_1 - u_2$. We have the following problem for $u(x,t)$:

$$\begin{cases} D_t^\rho u(x,t) - \Delta u(x,t) = 0, & 0 < t < \beta,\ \ x \in \Omega, \\ u_t(x,t) + \Delta u(x,t) = 0, & -\alpha < t < 0,\ \ x \in \Omega, \end{cases} \tag{12}$$

and the conditions (2), (3) and (4).

Assume that $u(x,t)$ satisfies all the conditions of the homogeneous problem, and let v_k be an arbitrary eigenfunction of the spectral problem (7) corresponding to the eigenvalue λ_k. Let

$$T_k(t) = \int\limits_\Omega u(x,t)v_k(x)\,dx, \quad k = 1, 2, \ldots$$

Differentiating under the integral sign with respect to t, which is allowed by the definition of the solution, and using equation (12), we obtain

$$D_t^\rho T_k(t) = \int\limits_\Omega D_t^\rho u(x,t)v_k(x)\,dx = \int\limits_\Omega \Delta u(x,t)v_k(x)\,dx, \quad t > 0,$$

$$\frac{dT_k(t)}{dt} = \int\limits_\Omega \frac{\partial u(x,t)}{\partial t}v_k(x)\,dx = -\int\limits_\Omega \Delta u(x,t)v_k(x)\,dx, \quad t < 0.$$

Integrating by parts and using condition (2), we get:

$$D_t^\alpha T_k(t) = -\lambda_k T_k(t), \quad t > 0, \quad \frac{dT_k(t)}{dt} = \lambda_k T_k(t), \quad t < 0.$$

The solutions to these equations are given by [2, p. 175]:

$$T_k(t) = a_k E_{\rho,1}(-\lambda_k t^\rho), \quad t > 0, \quad T_k(t) = b_k e^{\lambda_k t}, \quad k = 1, 2, \ldots, \quad t < 0. \tag{13}$$

Gluing condition (3) translates into: $T_k(+0) = T_k(-0)$. Using this condition, we find $a_k = b_k$. Applying the non-local condition (4) to get: $a_k\delta_k = 0$. Since $\delta_k \neq 0$ for all $k \in \mathbb{N}$, $a_k = b_k = 0$. Therefore, from (13), we can see that the right-hand sides must be identically zero, which implies that $u(x,t)$ is orthogonal to the complete system $\{v_k(x)\}$. As a result, we conclude that $u(x,t) \equiv 0$ in $\overline{\Omega}$.

Thus, we arrive at the criterion for the uniqueness of the solution to the forward problem (1)–(4):

Theorem 1. If there is a solution to the forward problem (1)–(4), then this solution is unique if and only if the condition $\delta_k \neq 0$ is satisfied for all $k \in \mathbb{N}$.

So we obtain a formal solution to problem (1)–(4) represented in the form

$$u(x,t)=\begin{cases}\sum\limits_{k=1}^{\infty}\left(\frac{F_k^*}{\delta_k}E_{\rho,1}(-\lambda_k t^{\rho})+\int\limits_0^t s^{\rho-1}E_{\rho,\rho}\left(-\lambda_k s^{\rho}\right)F_k(t-s)\,ds\right)v_k(x), & 0\le t\le\beta,\\ \sum\limits_{k=1}^{\infty}\left(\frac{F_k^*}{\delta_k}e^{\lambda_k t}-\int\limits_t^0 F_k(s)e^{\lambda_k(t-s)}ds\right)v_k(x), & -\alpha\le t\le 0.\end{cases}\tag{14}$$

To show that these series satisfy the conditions of Definition 1, we need to estimate the denominator δ_k from below.

3 Lower estimates for the denominator of the solution to the forward problem (1)–(4)

In this section, we investigate the conditions under which δ_k may be equal to zero, and for those cases where $\delta_k \neq 0$, we derive lower bounds for δ_k. It is not hard to see that the following lemma is true:

Lemma 7. Let $\lambda \notin [0,1)$. Then there exists a constant $\delta_0>0$ such that, for all $k\in\mathbb{N}$, the following estimate holds:

$$|\delta_k|>\delta_0, \quad \delta_0=\begin{cases}|\lambda|, & \lambda<0,\\ \lambda-e^{-\lambda_1\alpha}, & \lambda\ge 1.\end{cases}$$

Proof. We consider two separate cases based on the value of the parameter λ.
Case 1. $\lambda<0$. In this case, since $e^{-\lambda_k\alpha}>0$, we have:

$$|\delta_k|=|e^{-\lambda_k\alpha}-\lambda|=|\lambda|+e^{-\lambda_k\alpha}\ge|\lambda|=\delta_0>0.$$

Case 2. $\lambda\ge 1$. In this case, we observe that $e^{-\lambda_k\alpha}\in(0,1)$ for all $k\in\mathbb{N}$, and therefore:

$$|\delta_k|=\left|e^{-\lambda_k\alpha}-\lambda\right|=\lambda-e^{-\lambda_k\alpha}\ge\lambda-e^{-\lambda_1\alpha}=\delta_0>0.$$

This completes the proof. □

Theorem 2. Let $\lambda\notin[0,1)$. Let the function $F(x,t)$ be continuous for all $t\in[-\alpha,\beta]$ and satisfy condition (10) uniformly with respect to t. Then there exists a unique solution of the forward problem (1)–(4), determined by the series (14).

Proof. Now we will show the existence of a solution. The formal solution of problem (1)–(4) has the form

$$u(x,t)=\begin{cases}\sum\limits_{k=1}^{\infty}\left(\frac{F_k^*}{\delta_k}E_{\rho,1}(-\lambda_k t^{\rho})+\int\limits_0^t s^{\rho-1}E_{\rho,\rho}\left(-\lambda_k s^{\rho}\right)F_k(t-s)\,ds\right)v_k(x), & t>0,\\ \sum\limits_{k=1}^{\infty}\left(\frac{F_k^*}{\delta_k}e^{\lambda_k t}-\int\limits_t^0 F_k(s)e^{\lambda_k(t-s)}ds\right)v_k(x), & t<0.\end{cases}\tag{15}$$

Let us now show that the sum of series (15) is indeed a solution to the forward problem. Consider the case for $t>0$, and in the case $t<0$ the absolute convergence of the solution (15) is proved in a similar way. This series is the sum of two series. We denote the first sum by $-\Delta S_1(x,t)$, and the second by $-\Delta S_2(x,t)$. Let the partial sums of the first and second terms have the following forms, respectively:

$$-\Delta S_1^j(x,t) = \sum_{k=1}^{j} \frac{\lambda_k \left(\int\limits_{-\alpha}^{0} F_k(s) e^{\lambda_k(-\alpha-s)} ds \right) E_{\rho,1}(-\lambda_k t^\rho)}{\delta_k} v_k(x), \tag{16}$$

$$-\Delta S_2^j(x,t) = \sum_{k=1}^{j} \lambda_k \left(\int\limits_{0}^{t} \eta^{\rho-1} E_{\rho,\rho}\left(-\lambda_k \eta^\rho\right) F_k(t-s)\, ds \right) v_k(x). \tag{17}$$

In what follows, the symbol C will denote a positive constant, not necessarily the same one.

Let $\sigma > \frac{N}{4}$. Since $\hat{A}^{-\sigma} v_k(x) = \lambda_k^{-\sigma} v_k(x)$, we have by (16)

$$-\Delta S_1^j(x,t) = \hat{A}^{-\sigma} \sum_{k=1}^{j} \frac{\lambda_k^{\sigma+1} \left(\int\limits_{-\alpha}^{0} F_k(s) e^{\lambda_k(-\alpha-s)} ds \right) E_{\rho,1}(-\lambda_k t^\rho)}{\delta_k} v_k(x).$$

By virtue of Lemma 1 we obtain

$$\left\| -\Delta S_1^j(x,t) \right\|_{C(\Omega)}^2 \le C \left\| \sum_{k=1}^{j} \frac{\lambda_k^{\sigma+1} \left(\int\limits_{-\alpha}^{0} F_k(s) e^{\lambda_k(-\alpha-s)} ds \right) E_{\rho,1}(-\lambda_k t^\rho)}{\delta_k} v_k(x) \right\|_{L_2(\Omega)}^2.$$

Since the system $\{v_k\}$ is orthonormal, by applying Parseval's equality and using Lemma 2 we have

$$\left\| -\Delta S_1^j(x,t) \right\|_{C(\Omega)}^2 \le C t^{-2\rho} \sum_{k=1}^{j} \lambda_k^{2\sigma} \left| \int\limits_{-\alpha}^{0} F_k(s) e^{\lambda_k(-\alpha-s)} ds \right|^2.$$

Applying the Cauchy-Schwarz inequality

$$\left\| -\Delta S_1^j(x,t) \right\|_{C(\Omega)}^2 \le \frac{C t^{-2\rho}}{\lambda_1^2} \int\limits_{-\alpha}^{0} \sum_{k=1}^{j} \lambda_k^{2\sigma} \left| F_k(s) \right|^2 ds, \quad \tau = 2\sigma > \frac{N}{2}.$$

This means that we have the series, similar to the series (8). Thus, if the function $F(x,t)$ satisfies the conditions (10) with $\tau > \frac{N}{2}$, then the series $|-\Delta S_1(x,t)|^2_{C(\overline{\Omega})} \le C$ will converge if $t > 0$.

For the series (17) by virtue of Lemma 6 we get

$$\left\| -\Delta S_2^j(x,t) \right\|_{C(\Omega)}^2 \le C \sum_{k=1}^{j} \left| \int_0^t s^{\varepsilon\rho-1} \lambda_k^{\sigma+\varepsilon} F_k(t-s) ds \right|^2.$$

Further, we will apply the generalized Minkowski inequality. Then

$$\left\| -\Delta S_2^j(x,t) \right\|_{C(\Omega)}^2 \le C \left[\int_0^t s^{\rho\varepsilon-1} \left(\sum_{k=1}^{j} \left| \lambda_k^{2(\sigma+\varepsilon)} \right| \left| F_k(t-s) \right|^2 \right)^{\frac{1}{2}} ds \right]^2, \quad \tau = 2\sigma + 2\varepsilon > \frac{N}{2}. \tag{18}$$

Here we again get a series similar to (8). In this case, $\tau = 2\sigma + 2\varepsilon$. Since ε is an arbitrarily small number, the series (18) converges under the same conditions (10) for the function $F(x,t)$.

Consequently, $|-\Delta S_1(x,t)|^2_{C(\overline{\Omega})} \le C$, $|-\Delta S_2(x,t)|^2_{C(\Omega)} \le C$, $t > 0$. Thus $\Delta u(x,t) \in C(\overline{\Omega} \times (0,\beta])$, in particular $u(x,t) \in C(\overline{\Omega} \times [0,\beta])$. Using completely similar reasoning, it can be shown that sum (15) for $t < 0$ has the same properties as sum (15) for $t > 0$. Hence, $\Delta u(x,t) \in C(\overline{\Omega} \times (-\alpha, 0))$, in particular $u(x,t) \in C(\overline{\Omega} \times [-\alpha, 0])$.

From equation (1), we have $D_t^\rho u(x,t) \in C(\overline{\Omega} \times (0,\beta])$, $u_t(x,t) \in C(\overline{\Omega} \times (-\alpha, 0))$. That $u(x,t)$ is absolutely continuous in a closed region follows from the fact that every function $T_k(t)v_k(x)$ is such. Theorem 2 is proved. □

Lemma 8. Let $0 < \lambda < 1$. Then there exists a number $k_0 \in \mathbb{N}$, such that for all $k > k_0$, the following estimate holds:

$$|\delta_k| \ge \frac{\lambda}{2}.$$

If $0 < \lambda < 1$, then obviously, there is a unique $\lambda_0 > 0$ such that $e^{-\lambda_0 \alpha} = \lambda$. If $\lambda_k \ne \lambda_0$ for all $k \in \mathbb{N}$ then the formal solution of problem (1)–(4) has the form (14).

If $\lambda_k = \lambda_0$ for $k = k_0,\ k_0 + 1,\ \ldots,\ k_0 + p_0 - 1$, where p_0 is the multiplicity of the eigenvalue λ_{k_0}, then for the solvability of problem (1)–(4) it is necessary and sufficient that the following equality holds (see (11)):

$$F_k^* = (F^*, v_k) = 0, \quad k \in K_0, \quad K_0 = \{k_0, k_0+1, \ldots, k_0 + p_0 - 1\}. \tag{19}$$

In this case, the solution of problem (1)–(4) can be written as follows:

$$u(x,t) = \begin{cases} \sum\limits_{k \notin K_0} T_k(t) v_k(x) + \sum\limits_{k \in K_0} a_k E_{\rho,1}(\lambda_k t^\rho) v_k(x), & t > 0, \\ \sum\limits_{k \notin K_0} T_k(t) v_k(x) + \sum\limits_{k \in K_0} a_k e^{\lambda_k t} v_k(x), & t < 0, \end{cases} \tag{20}$$

here, a_k are arbitrary constants.

Thus, we obtain the following statement:

Theorem 3. Let $0 < \lambda < 1$ and let the function $F(x,t)$ be continuous for all $t \in [-\alpha, \beta]$ and satisfy condition (10) uniformly with respect to t.

1) If $\lambda_k \ne \lambda_0$, for all $k \ge 1$, then there exists a unique solution of the problem (1)–(4) and it can be represented in the form (14).

2) If $\lambda_k = \lambda_0$, for some k and the orthogonality condition (19) holds for indices $k \in K_0$, then the problem (1)–(4) has a solution, which is expressed in the form (20) with arbitrary coefficients a_k.

Proof. We have considered the proof of the first part of the theorem above in Theorem 2. Now, we need to show the convergence of the series (20). If $k \in K_0$, then in the solution (20) additional series are formed as

$$u_0(x,t) = \begin{cases} \sum\limits_{k \in K_0} a_k E_{\rho,1}\left(-\lambda_k t^\rho\right) v_k(x), & t > 0, \\ \sum\limits_{k \in K_0} a_k e^{\lambda_k t} v_k(x), & t < 0. \end{cases}$$

Since K_0 has a finite number of elements, these series consist of finite sum of smooth functions. Therefore, these series satisfy all conditions of Definition 1. □

4 *Existence and uniqueness of the solution of the inverse problem (1)–(5)*

We study the inverse problem for the equation (1) with the right-hand side of the form $F(x,t) = f(x)g(t)$, where $g(t)$ is a given function and $f(x)$ is an unknown function. Furthermore, since we use the solution of the forward problem when solving the inverse problem, in all subsequent sections we assume that $\delta_k \neq 0$ for all k. According to the additional condition (5), it is sufficient to construct the solution of the inverse problem (1)–(5) only for $t > 0$. Using the representation (14), we obtain the following solution to the inverse problem (1)–(5):

$$u(x,t) = \sum_{k=1}^{\infty} \left(a_{1k} E_{\rho,1}(-\lambda_k t^\rho) + f_k \int_0^t s^{\rho-1} E_{\rho,\rho}\left(-\lambda_k s^\rho\right) g(t-s)\, ds \right) v_k(x), \quad 0 \leq t \leq \beta, \tag{21}$$

where

$$a_{1k} = \frac{f_k \int\limits_{-\alpha}^{0} g(s) e^{\lambda_k(-\alpha-s)} ds}{\delta_k}.$$

Substituting the function (21) into the condition (5), we obtain the equation

$$\sum_{k=1}^{\infty} T_k(t_0) v_k(x) = \varphi_0(x) = \sum_{k=1}^{\infty} \varphi_{0k} v_k(x), \tag{22}$$

where

$$T_k(t_0) = \frac{f_k \int\limits_{-\alpha}^{0} g(s) e^{\lambda_k(-\alpha-s)} ds}{\delta_k} E_{\rho,1}(-\lambda_k t_0^\rho) + f_k \int_0^{t_0} \eta^{\rho-1} E_{\rho,\rho}\left(-\lambda_k \eta^\rho\right) g(s)\, ds,$$

and

$$\varphi_{0k} = \int_\Omega \varphi_0(x) v_k(x)\, dx, \quad k = 1, 2, \ldots,$$

the numbers f_k are so far unknown and have to be determined.

From the relation (22), we have

$$f_k \Delta_k(t_0) = \delta_k \varphi_{0k} = (e^{-\lambda_k \alpha} - \lambda)\varphi_{0k}, \tag{23}$$

here

$$\Delta_k(t_0) = E_{\rho,1}(-\lambda_k t_0^\rho) \int_{-\alpha}^{0} g(s) e^{\lambda_k(-\alpha-s)} ds + (e^{-\lambda_k \alpha} - \lambda) \int_0^{t_0} s^{\rho-1} E_{\rho,\rho}\left(-\lambda_k s^\rho\right) g(t_0 - s)\, ds.$$

Let us introduce the following notation:

$$I_k(\alpha) = \int_{-\alpha}^{0} g(s) e^{\lambda_k(-\alpha-s)} ds, \quad I_{k,\rho}(t_0) = \int_0^{t_0} s^{\rho-1} E_{\rho,\rho}\left(-\lambda_k s^\rho\right) g(t_0 - s)\, ds.$$

Again, as we noted above, if $\Delta_k(t_0) \neq 0$ for all k, then the coefficients f_k are found uniquely, otherwise, i.e. if $\Delta_k(t_0) = 0$ for some k, according to the equation (23), the coefficients f_k are chosen arbitrarily. Therefore, we have the following uniqueness criterion for the inverse problem (1)–(5):

Theorem 4. The uniqueness of the solution to the inverse problem (1)–(5) is guaranteed if and only if $\Delta_k(t_0) \neq 0$ for all $k \geq 1$.

The uniqueness of the solution of the inverse problem follows from the completeness of the eigenfunctions (see the proof of Theorem 1).

5 Lower estimates for the denominator of the solution to the inverse problem (1)–(5)

We now provide a lower estimate for $\Delta_k(t_0)$. Let $g \in C[-\alpha, \beta]$ and $g(t) \neq 0$, we define

$$m = \min_{t\in[-\alpha,t_0]} |g(t)| > 0, \quad M = \max_{t\in[-\alpha,t_0]} |g(t)| > 0.$$

Lemma 9. Let $\lambda < 0$, $g(t) \in C[-\alpha, \beta]$ and $g(t) \neq 0$, $t \in [-\alpha, \beta]$. Then, there is a constant $C > 0$, depending on t_0 and α, such that for all k:

$$\Delta_k(t_0) \geq \frac{C}{\lambda_k}.$$

Proof. It is sufficient to consider the case $g(t) > 0$, $t \in [-\alpha, \beta]$. If $t_0 \in (0, \beta]$, then (see Lemma 4)

$$I_{k,\rho}(t_0) \geq m \int\limits_0^{t_0} s^{\rho-1} E_{\rho,\rho}(-\lambda_k s^\rho) ds = m t_0^\rho E_{\rho,\rho+1}(-\lambda_k t_0^\rho).$$

Taking into account (see Lemma 5), we obtain

$$I_{k,\rho}(t_0) \geq \frac{1}{\lambda_k}(1 - E_\rho(-\lambda_k t_0^\rho))m \geq \frac{1}{\lambda_k}(1 - E_\rho(-\lambda_1 t_0^\rho))m \geq \frac{C_{t_0}}{\lambda_k},\ C_{t_0} > 0,$$

$$I_k(\alpha) \geq m \int\limits_{-\alpha}^{0} e^{\lambda_k(-\alpha-s)} ds = m\frac{1 - e^{-\lambda_k \alpha}}{\lambda_k} \geq \frac{C_\alpha}{\lambda_k},\ C_\alpha > 0.$$

Therefore,

$$\Delta_k(t_0) \geq E_{\rho,1}(-\lambda_k t_0^\rho)\frac{C_\alpha}{\lambda_k} + (e^{-\lambda_k \alpha} - \lambda)\frac{C_{t_0}}{\lambda_k} \geq (e^{-\lambda_k \alpha} - \lambda)\frac{C_{t_0}}{\lambda_k},$$

which implies the desired assertion because $\lambda < 0$. Lemma 9 is proved. □

Lemma 10. Let $\lambda \geq 1$, $g(t) \in C[-\alpha, \beta]$ and $g(t) \neq 0$, $t \in [-\alpha, \beta]$.
If the number t_0 satisfies the following condition

$$t_0^\rho > \frac{C_0}{\lambda_1}\left(1 + \frac{M}{m}\right), \tag{24}$$

where C_0 is the number in Lemma 2 then, there is a constant $C > 0$ depending on t_0, ρ and α, such that for all k:

$$|\Delta_k(t_0)| \geq \frac{C}{\lambda_k}. \tag{25}$$

If the number t_0 does not satisfy condition (24), then there exists a number k_l, $l \in \mathbb{N}$, such that the estimate (25) holds for all $k > k_l$.

Proof. We begin by estimating $\Delta_k(t_0)$ from below. From its definition, it consists of a sum of two integrals. For the first and second integrals, using Lemma 4 and Lemma 5, we get:

$$\int_{-\alpha}^{0} g(s) e^{\lambda_k(-\alpha-s)}\, ds \geq m\frac{1 - e^{-\lambda_k \alpha}}{\lambda_k}, \quad \int_0^{t_0} s^{\rho-1} E_{\rho,\rho}(-\lambda_k s^\rho) g(t_0 - s)\, ds \leq M\frac{1 - E_{\rho,1}(-\lambda_k t_0^\rho)}{\lambda_k}.$$

Hence,

$$\Delta_k(t_0) \geq \frac{E_{\rho,1}(-\lambda_k t_0^\rho)}{\lambda_k}\left[m(1 - e^{-\lambda_k \alpha}) + (\lambda - e^{-\lambda_k \alpha})M\right] - \frac{M}{\lambda_k},$$

which implies

$$\Delta_k(t_0) \geq -\frac{M}{\lambda_k},$$

where $C_1 = M$.

Next, to estimate $\Delta_k(t_0)$ from above using Lemma 2, we obtain:

$$\Delta_k(t_0) \leq \left(\frac{1 - e^{-\lambda_k \alpha}}{\lambda_k}\right)\left(\frac{C_0(M+m)}{\lambda_k t_0^\rho} - m\right). \tag{26}$$

Note that the expression in parentheses becomes negative under the assumption:

$$t_0^\rho > \frac{C_0}{\lambda_1}\left(1 + \frac{M}{m}\right).$$

Thus, for all $k \in \mathbb{N}$, we have:

$$\Delta_k(t_0) \leq -\frac{C_2}{\lambda_k},$$

where $C_2 = \left(\frac{M+m}{\lambda_1 t_0^\rho} - m\right) > 0$.

Hence, there exists a constant $C = \min\{C_1, C_2\}$ such that the required lower bound holds.

Now let $\lambda \geq 1$ and assume that, condition (24) not be satisfied for the given values of the parameter. However, there exists an index k_l, such that for all $k > k_l$ the condition $t_0^\rho > \frac{C_0}{\lambda_k}\left(1 + \frac{M}{m}\right)$ is satisfied, since $\frac{C_0}{\lambda_k}\left(\frac{M+m}{m}\right) \to 0$ as $k \to \infty$, (see (26)). Therefore, for all $k > k_l$ the estimate (25) holds. Lemma 10 is proved. □

Lemma 11. Let $0 < \lambda < 1$, $g(t) \in C[-\alpha, \beta]$ and $g(t) \neq 0$, $t \in [-\alpha, \beta]$. Then for all $k > k_r$, $r \in \mathbb{N}$ the following estimate

$$|\Delta_{k,\rho}(t_0)| \geq \frac{C}{\lambda_k} \tag{27}$$

is valid, where a constant $C > 0$ depends on ρ, t_0 and α.

Proof. Since $\delta_k \neq 0$, it follows that $\lambda_k \neq \lambda_0$ for all k. Therefore, we consider only the following two cases.

Case 1. Let $\lambda_k < \lambda_0$. In this case, based on the proof of Lemma 9, it is not difficult to see that for all $k < k_0$, the following estimate holds:

$$\Delta_{k,\rho}(t_0) > c_0,$$

where $c_0 > 0$ is a constant depending on α, t_0, and ρ.

Case 2. Let $\lambda_k > \lambda_0$. We prove this case of the lemma similarly to the proofs of the previous lemmas. The lower bound of $\Delta_k(t_0)$ has the form (see Lemma 10)

$$\Delta_k(t_0) \geq -\frac{C_1}{\lambda_k}.$$

Now, we establish an upper bound for $\Delta_k(t_0)$. To this end, using Lemma 4, Lemma 5, and Lemma 2, we obtain:

$$\Delta_k(t_0) \leq \frac{C_0}{\lambda_k^2 t_0^\rho}\Big(M(1 - e^{-\lambda_k \alpha}) + (\lambda - e^{-\lambda_k \alpha})m\Big) - \frac{(\lambda - e^{-\lambda_k \alpha})m}{\lambda_k}.$$

Thus, for all $k > k_r$, we have

$$\Delta_k(t_0) \leq -\frac{C_3}{\lambda_k},$$

where $C_3 = \left(\lambda - e^{-\lambda_k \alpha}\right) m > 0$.

Therefore, there exists a constant $C = \min\{c_0, C_1, C_3\}$ such that for all $k > k_r$ the required lower bound holds. Lemma 11 is proved. □

The above estimates (25) and (27) allows to determine explicitly the index from which they hold. For example, according to the proof of the second condition of Lemma 10, the index k_l is given by

$$k_l = \min\left\{k : t_0^\rho > \frac{1}{\lambda_k}\left(1+\frac{M}{m}\right)\right\}.$$

Similarly, for estimate (27), the index k_r can be determined in the same way.

Hence, we introduce the set:

$$\mathbb{K}_0 = \{k \in \mathbb{N} : \Delta_k(t_0) = 0\}.$$

Remark 1. Note that if $k \in \mathbb{K}_0$, then obviously $\delta_k \neq 0$.

Lemma 12. The set $\mathbb{K}_0$ is either empty or contains only finitely many elements.

Proof. From the proof of Lemma 10, it follows that if there exists an index $k \in \mathbb{K}_0$, then necessarily $k \leq k_l$. Therefore, $\mathbb{K}_0$ is a finite set. Moreover, as mentioned in Section 1, the sequence $\{\lambda_k\}$ consists of discrete values. Hence, $\Delta_k(t_0)$ can vanish only at isolated indices, and it is possible that no such index exists. In this case, the set $\mathbb{K}_0$ is empty. A similar argument is valid for the elements of the set $\mathbb{K}_0$ when $k \leq k_r$. This completes the proof of Lemma 12. □

Theorem 5. Let $g(t) \in C[-\alpha, \beta]$ and $g(t) \neq 0$, $t \in [-\alpha, \beta]$. Let $\lambda < 0$ and the function $\varphi_0(x)$ satisfies the conditions (9). Then there exists a unique solution of the inverse problem (1)–(5) and it can be represented as:

$$\begin{gathered} u(x,t) = \sum_{k=1}^{\infty}\left(\frac{\varphi_{0k}}{\Delta_k(t_0)}E_{\rho,1}(\lambda_k t^\rho)\int\limits_{-\alpha}^{0} g(s)e^{\lambda_k(-\alpha-s)}ds\right)v_k(x) \\ +\sum_{k=1}^{\infty}\left(\frac{\varphi_{0k}(e^{-\lambda_k\alpha}-\lambda)}{\Delta_k(t_0)}\int\limits_{0}^{t} s^{\rho-1}E_{\rho,\rho}\left(-\lambda_k s^\rho\right)g(t-s)\,ds\right)v_k(x), \quad t>0, \end{gathered} \tag{28}$$

$$u(x,t) = \sum_{k=1}^{\infty}\left(\frac{\varphi_{0k}}{\Delta_k(t_0)}e^{\lambda_k t}\int\limits_{-\alpha}^{0} g(s)e^{\lambda_k(-\alpha-s)}ds - \frac{\varphi_{0k}(e^{-\lambda_k\alpha}-\lambda)}{\Delta_k(t_0)}\int\limits_{t}^{0} g(s)e^{\lambda_k(t-s)}ds\right)v_k(x), \quad t<0.$$

$$f(x) = \sum_{k=1}^{\infty}\frac{\varphi_{0k}(e^{-\lambda_k\alpha}-\lambda)}{\Delta_k(t_0)}v_k(x). \tag{29}$$

Proof. We write the series (28) as sums of two series: $I_1(x,t)$ and $I_2(x,t)$. If $I_1^j(x,t)$ and $I_2^j(x,t)$ are the corresponding partial sums, then we have:

$$-\Delta I_1^j(x,t) = \sum_{k=1}^{j}\left(\frac{\lambda_k\varphi_{0k}}{\Delta_k(t_0)}E_{\rho,1}(\lambda_k t^\rho)\int\limits_{-\alpha}^{0} g(s)e^{\lambda_k(-\alpha-s)}ds\right)v_k(x),$$

$$-\Delta I_2^j(x,t) = \sum_{k=1}^{j}\left(\frac{\lambda_k\varphi_{0k}(e^{-\lambda_k\alpha}-\lambda)}{\Delta_k(t_0)}\int\limits_{0}^{t} s^{\rho-1}E_{\rho,\rho}\left(-\lambda_k s^\rho\right)g(t-s)\,ds\right)v_k(x).$$

Next, applying the identity $\hat{A}^{-\sigma}v_k(x) = \lambda_k^{-\sigma}v_k(x)$ and using Lemma 1, Lemma 2, and by applying Parseval's equality, we obtain:

$$\left\|-\Delta I_1^j(x,t)\right\|_{C(\Omega)}^2 \leq \frac{MCt^{-2\rho}}{\lambda_1}\sum_{k=1}^{j}|\varphi_{0k}|^2\lambda_k^{2\sigma}, \quad \tau = 2\sigma > \frac{N}{2}.$$

By the Lemma 4 and Lemma 2 we have

$$\left\|-\Delta I_2^j(x,t)\right\|^2_{C(\Omega)} \leq \frac{MC}{\lambda_1}\left(\sum_{k=1}^{j}\lambda_k^{2\sigma}\left|\varphi_{0k}\right|^2\right), \quad \tau = 2\sigma > \frac{N}{2}.$$

It is easy to see that

$$\|f(x)\|^2_{C(\Omega)} \leq C\sum_{k=1}^{j}\lambda^{2\sigma}\left|\varphi_{0k}\right|^2, \quad \tau = 2\sigma > \frac{N}{2}.$$

Therefore, if the function $\varphi_0(x)$ satisfies the conditions (10), then the following estimates hold:

$$\left\|-\Delta I_1^j(x,t)\right\|^2_{C(\Omega)} \leq C, \quad \left\|-\Delta I_2^j(x,t)\right\|^2_{C(\Omega)} \leq C, \quad \|f(x)\|^2_{C(\Omega)} \leq C, \quad t > 0.$$

Thus, we conclude that $\Delta u(x,t) \in C(\overline{\Omega} \times (0,\beta])$. In particular, $u(x,t) \in C(\overline{\Omega} \times [0,\beta])$, and $f(x) \in C(\overline{\Omega})$. Theorem 5 is proved. □

Theorem 6. Let $\varphi_0(x)$ satisfy the conditions (9) and $g(t) \in C[-\alpha,\beta]$, $g(t) \neq 0$, $t \in [-\alpha,\beta]$ and let $\delta_k \neq 0$ for all k. Moreover, let the assumptions of Lemma 10 or Lemma 11 hold.

1) If the set $\mathbb{K}_0$ is empty, then there exists a unique solution of the inverse problem (1)–(5) and it can be represented as the series in Theorem 5.

2) If the set $\mathbb{K}_0$ is not empty, then for the existence of a solution to the inverse problem (1)–(5), it is necessary and sufficient that the following conditions

$$\varphi_{0k} = (\varphi_0, v_k) = 0, \ k \in \mathbb{K}_0$$

be satisfied. In this case, the solution to inverse problem (1)–(5) exists, but is not unique:

$$f(x) = \sum_{k\notin\mathbb{K}_0}\frac{\delta_k\varphi_{0k}}{\Delta_k(t_0)}v_k(x) + \sum_{k\in\mathbb{K}_0}f_k v_k(x), \tag{30}$$

$$u(x,t) = \sum_{k=1}^{\infty} f_k\left(\frac{\int\limits_{-\alpha}^{0} g(s)e^{\lambda_k(-\alpha-s)}ds}{\delta_k}E_{\rho,1}(-\lambda_k t^\rho) + \int\limits_0^t s^{\rho-1}E_{\rho,\rho}\left(-\lambda_k s^\rho\right)g(t-s)\,ds\right)v_k(x), \quad t>0, \tag{31}$$

$$u(x,t) = \sum_{k=1}^{\infty} f_k\left(\frac{\int\limits_{-\alpha}^{0} g(s)e^{\lambda_k(-\alpha-s)}ds}{\delta_k}e^{\lambda_k t} - \int\limits_t^0 g(s)e^{\lambda_k(t-s)}ds\right)v_k(x), \quad t<0, \tag{32}$$

where if $k \notin \mathbb{K}_0$ then f_k has the form (23) and if $k \in \mathbb{K}_0$, then f_k are arbitrary real numbers.

Proof. To prove the theorem we need to show that the series (30), (31) and (32) satisfy all the conditions of Definition 1. This follows directly from the proof of Theorem 5, and the proof is almost the same when any of the conditions of Lemma 10 or Lemma 11 hold. For clarity, let us suppose that the assumptions of Lemma 10 are satisfied. Series (31) and (32) are divided into two parts, following the structure given in (30). The second part of both these series, as stated in Lemma 12, is a finite sum of smooth functions. In the first part, the satisfaction of the series of the conditions of Definition 1 can be proved in the same way as for the series (28). Here we use the lower bound (25) for $\Delta_k(t_0)$. The convergence of the first part of (30) is shown similarly to that of the series (29), while the second part is a sum of finitely many smooth functions. □

Conclusion

In this work, a subdifusion equation with the Caputo fractional derivative of order $\rho \in (0, 1)$ is studied for $t > 0$, while a classical parabolic equation is considered for $t < 0$. Following the work [3], forward and inverse problems ($f(x)$ is unknown) are considered with a non-local Dezin type condition. The solutions are constructed using the classical Fourier method. The main contribution of the authors is that such non-local direct and inverse problems for mixed-type equations with a fractional order have not been previously studied. In the process of studying these problems, we investigate the effect of the parameter λ in Dezin's condition, on the existence and uniqueness of the solution. As proved, it is shown that for certain values of λ, the uniqueness of the solution may fail, and in order to recover the solution, orthogonality conditions on the given functions $\varphi_0(x)$ and $F(x, t)$ are required.

In the future, it would be of interest to consider other types of fractional derivatives instead of the Caputo derivative, in order to investigate whether similar effects occur. Another promising direction is the study of inverse problems aimed at determining fractional orders in mixed-type equations for such nonlocal problems.

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Author Contributions

All authors contributed equally to this work.

Conflict of Interest

The authors declare no conflict of interest.

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Numerical solutions of source identification problems for telegraph-parabolic equations

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This paper presents a numerical study of source identification problems for one-dimensional telegraph-parabolic equations subject to Dirichlet and Neumann boundary conditions. In these inverse problems, the unknown source terms are assumed to be space-dependent, which introduces both analytical and computational challenges. The study begins by discretizing the considered problems using the finite difference method – first in space and subsequently in time – resulting in a system of discrete equations. Stability results for the solutions of the resulting finite difference schemes are established to ensure the reliability of the numerical approach. A numerical algorithm is proposed for solving the discrete inverse problems. The algorithm begins by eliminating the unknown source terms, which transforms the original discretized problem into a new nonlocal problem with unknown initial data. To approximate this initial data, an iterative procedure based on fixed-point iterations is constructed. Once the transformed nonlocal problem is solved, the solution of the main finite difference scheme and approximations of the unknown source term are recovered. Numerical results for two test problems are presented to illustrate the proposed method in practice. The findings confirm the accuracy of the approach in solving space-dependent inverse source problems.

Keywords: source identification problem, inverse problem, mixed-type differential equation, telegraph-parabolic equation, finite difference scheme, numerical algorithm, nonlocal problems, fixed-point iterations.

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Introduction

Partial differential equations with unknown source terms are widely used in the mathematical modelling of real-world phenomena in various applied fields (see, e.g., [1] and the references therein). A problem involving a differential equation with a time- and/or space-dependent source term is referred to as a source identification problem (SIP). These types of inverse problems have been extensively studied in the literature (see, e.g., [2–4] and the references therein).

In recent years, the analysis of SIPs for mixed-type differential equations, as well as the development and investigation of numerical methods for their solution, has attracted significant attention (see, e.g., [5, 6] for parabolic-elliptic, [7–9] for elliptic-hyperbolic, and [10, 11] for telegraph-parabolic SIPs). By mixed-type, we mean that the differential equation is of one type in one part of the domain and of a different type in another part. For instance, consider a physical system initially modelled by the heat equation. At a certain moment in time, due to an instantaneous change in the system, the governing model transitions to the wave equation with a damping term. In such cases, the resulting differential equations are referred to as telegraph-parabolic equations.

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Consider the following abstract formulation for telegraph-parabolic equations with an unknown space-dependent source term p:

$$\begin{cases} w''(t) + \alpha w'(t) + Aw(t) = p + f(t), & t \in (0,1), \\ w'(t) + Aw(t) = p + g(t), & t \in (-1,0), \\ w(0^+) = w(0^-),\ w'(0^+) = w'(0^-), \\ w(-1) = \varphi,\ w(\lambda) = \psi,\ \lambda \in (-1,1], \end{cases} \tag{1}$$

where the problem is posed in a Hilbert space H with a self-adjoint positive definite (SAPD) operator A satisfying $A \geq \delta I$, for some $\delta > \frac{\alpha^2}{4}$ and $\alpha > 0$. Here, φ, $\psi \in D(A)$ and the functions $f(t)$ and $g(t)$ are assumed to be continuously differentiable on $[0,1]$ and $[-1,0]$, respectively. The existence, uniqueness, and stability of solutions of the problem (1) in the space $C(H)$ of continuous H-valued functions $w(t)$ defined on the interval $[-1,1]$, equipped with the norm

$$\|w\|_{C(H)} = \max_{t\in[-1,1]} \|w(t)\|_H$$

are established in [10].

For the approximate solution of the abstract problem (1), the following stable difference scheme (DS) of first-order accuracy is constructed in [11]:

$$\begin{cases} \frac{w_{k+1}-2w_k+w_{k-1}}{\tau^2} + \alpha\frac{w_{k+1}-w_k}{\tau} + Aw_{k+1} = p + f_k, & 1 \leq k \leq N-1, \\ \frac{w_k-w_{k-1}}{\tau} + Aw_k = p + g_k, & -N+1 \leq k \leq 0, \\ \frac{w_1-w_0}{\tau} = p - Aw_0 + g_0, \\ w_{-N} = \varphi, \quad w_\ell = \psi, \end{cases} \tag{2}$$

where $\tau = \frac{1}{N}$ is sufficiently small positive number, $t_k = k\tau$, $-N \leq k \leq N$, $\ell = \lceil \frac{\lambda}{\tau} \rceil$, $f_k = f(t_k)$, $1 \leq k \leq N-1$ and $g_k = g(t_k)$, $-N+1 \leq k \leq 0$.

The unique solvability of the DS (2) and the stability estimates for its solution were established in [11]. However, the abstract results for the DS (2), presented in [11], require further investigation from an implementation perspective. In the present paper, we consider the application of the aforementioned abstract results to two SIPs for one-dimensional telegraph-parabolic equations with Dirichlet and Neumann boundary conditions. We provide a complete discretization of the considered problems and propose a numerical algorithm for solving the resulting DSs. Numerical examples are presented to illustrate the proposed numerical procedure.

1 *SIPs for one-dimensional telegraph-parabolic equations*

In this section, we consider two SIPs for one-dimensional telegraph-parabolic equations: one with Dirichlet boundary conditions and the other with Neumann boundary conditions. Since the discretization procedures for the considered problems are very similar, we describe the approach for both SIPs simultaneously.

First, consider the following SIP for one-dimensional telegraph-parabolic equations

$$\begin{cases} w_{tt}(t,x) + \alpha\ w_t(t,x) - \big(a(x)\ w_x(t,x)\big)_x = p(x) + f(t,x), & x \in (0,1),\ t \in (0,1), \\ w_t(t,x) - \big(a(x)\ w_x(t,x)\big)_x = p(x) + g(t,x), & x \in (0,1),\ t \in (-1,0), \\ w(0^+,x) = w(0^-,x),\ w_t(0^+,x) = w_t(0^-,x), & x \in [0,1], \\ w(-1,x) = \varphi(x),\ w(1,x) = \psi(x), & x \in [0,1], \\ w(t,0) = w(t,1) = 0, & t \in [-1,1] \end{cases} \tag{3}$$

with homogeneous Dirichlet boundary conditions. Here and throughout the paper, $p(x)$ denotes the unknown source term, $a(x) \geq a > 0$, $\varphi(x)$, $\psi(x)$, $f(t,x)$, and $g(t,x)$ are given sufficiently smooth functions, and α is a positive constant. SIP (3) can be reduced to the abstract problem (1) in a Hilbert space $H = L_2(0,1)$ with a SAPD operator $A = A^x$ defined by the formula

$$A^x w(x) = -\big(a(x)w_x(x)\big)_x \tag{4}$$

with domain $D(A^x) = \big\{w(x) \ : \ w(x),\ w_x(x),\ \big(a(x)w_x\big)_x \in L_2[0,1],\ w(0) = w(1) = 0\big\}$.

Second, consider the SIP for one-dimensional telegraph-parabolic equations with Neumann boundary conditions

$$\begin{cases} w_{tt}(t,x) + \alpha w_t(t,x) - \big(a(x)w_x(t,x)\big)_x + \delta w(t,x) = p(x) + f(t,x), & x \in (0,1),\ t \in (0,1), \\ w_t(t,x) - \big(a(x)w_x(t,x)\big)_x + \delta w(t,x) = p(x) + g(t,x), & x \in (0,1),\ t \in (-1,0), \\ w(0^+,x) = w(0^-,x),\ w_t(0^+,x) = w_t(0^-,x), & x \in [0,1], \\ w(-1,x) = \varphi(x),\ w(1,x) = \psi(x), & x \in [0,1], \\ w_x(t,0) = w_x(t,1) = 0, & t \in [-1,1], \end{cases} \tag{5}$$

where δ is a positive constant. SIP (5) can be reduced to the abstract problem (1) in a Hilbert space $H = L_2(0,1)$ with a SAPD operator $A = A^x$ defined by the formula

$$A^x w(x) = -\big(a(x)w_x(x)\big)_x + \delta w \tag{6}$$

with domain $D(A^x) = \big\{w(x) \ : \ w(x),\ w_x(x),\ \big(a(x)w_x\big)_x \in L_2(0,1),\ w_x(0) = w_x(1) = 0\big\}$.

By means of the abstract result from [10], both problems (3) and (5) have a unique smooth solution $\big\{w(t,x), p(x)\big\}$ for given smooth data satisfying all compatibility conditions.

We start the discretization of SIPs (3) and (5) by defining the grid space $[0,1]_h = \big\{x \ \big| \ x_m = m\,h,\ 0 \leq m \leq M,\ \ M\,h = 1\big\}$.

Let us introduce the Hilbert space $L_{2h} = L_2\left([0,1]_h\right)$ of grid functions $\varphi^h(x) = \{\varphi^m\}_0^M$ defined on $[0,1]_h$ and equipped with the norm $\left\|\varphi^h\right\|_{L_{2h}} = \left(\sum\limits_{x \in [0,1]_h} \left|\varphi^h(x)\right|^2 h\right)^{1/2}$. To the differential operator A^x, defined by formula (4), we associate the difference operator A_h^x, given by the formula

$$A_h^x \varphi^h(x) = \Big\{ -\big(a(x)\varphi_{\bar{x}}^m(x)\big)_x \Big\}_1^{M-1},$$

which acts in the space of grid functions $\varphi^h(x) = \{\varphi^m\}_0^M$ satisfying boundary conditions $\varphi_0 = \varphi_M = 0$. Here and throughout the paper,

$$\varphi_{\bar{x}}^m = \left\{\frac{\varphi^m - \varphi^{m-1}}{h}\right\}_1^M \quad \text{and} \quad \varphi_x^m = \left\{\frac{\varphi^{m+1} - \varphi^m}{h}\right\}_0^{M-1}.$$

Similarly, to the differential operator A^x, defined by formula (6), we assign the corresponding difference operator A_h^x, given by the formula

$$A_h^x \varphi^h(x) = \Big\{ -\big(a(x)\varphi_{\bar{x}}^m(x)\big)_x + \delta\varphi^m(x) \Big\}_1^{M-1},$$

acting in the space of grid functions $\varphi^h(x) = \{\varphi^m\}_0^M$, subject to the boundary conditions $\varphi_0 = \varphi_1$ and $\varphi_M = \varphi_{M-1}$.

Note that in both cases, A_h^x corresponds to the second-order accuracy centered difference approximation of the respective differential operator A^x, incorporating Dirichlet and Neumann boundary conditions, respectively. Moreover, A_h^x is a SAPD operator in L_{2h} in both cases.

With the help of the corresponding operator A_h^x, the first step of the discretization of both SIPs (3) and (5) leads to the following problem:

$$\begin{cases} \frac{d^2w^h(t,x)}{dt^2} + \alpha\frac{dw^h(t,x)}{dt} + A_h^x w^h(t,x) = p^h(x) + f^h(t,x), & t \in (0,1), \\ \frac{dw^h(t,x)}{dt} + A_h^x w^h(t,x) = p^h(x) + g^h(t,x), & t \in (-1,0), \\ w^h(0^+,x) = w^h(0^-,x), \quad \frac{dw^h(0^+,x)}{dt} = \frac{dw^h(0^-,x)}{dt}, \\ w^h(-1,x) = \varphi^h(x), \quad w^h(1,x) = \psi^h(x), \end{cases} \tag{7}$$

where $x \in [0,1]_h$.

Now, in the second step of the discretization process, we define $\tau = \frac{1}{N},\ t_k = k\tau,\ -N \le k \le N$ and replace problem (7) with DS (2)

$$\begin{cases} \frac{w_{k+1}^h(x)-2w_k^h(x)+w_{k-1}^h(x)}{\tau^2} + \alpha\frac{w_{k+1}^h(x)-w_k^h(x)}{\tau} + A_h^x w_{k+1}^h(x) = p^h(x) + f_k^h(x), & 1 \le k \le N-1, \\ \frac{w_k^h(x)-w_{k-1}^h(x)}{\tau} + A_h^x w_k^h(x) = p^h(x) + g_k^h(x), & -N+1 \le k \le 0, \\ \frac{w_1^h(x)-w_0^h(x)}{\tau} = p^h(x) - A_h^x w_0^h(x) + g_0^h(x), \\ w_{-N}^h(x) = \varphi^h(x),\ w_N^h(x) = \psi^h(x), \end{cases} \tag{8}$$

where $x \in [0,1]_h$, $f_k^h(x) = f^h(t_k,x)$, $1 \le k \le N-1$ and $g_k^h(x) = g^h(t_k,x)$, $-N+1 \le k \le 0$. Then, the following theorem follows readily from the abstract result stated in Theorem 1.

Theorem 1. The solution of DS (8) satisfies the following stability estimate

$$\max_{-N\le k\le N} \left\|w_k^h\right\|_{L_{2h}} + \left\|\left(A_h^x\right)^{-1} p^h\right\|_{L_{2h}}$$
$$\le M(\delta,\alpha)\left[\max_{1\le k\le N-1}\left\|f_k^h\right\|_{L_{2h}} + \max_{-N+1\le k\le 0}\left\|g_k^h\right\|_{L_{2h}} + \left\|\varphi^h\right\|_{L_{2h}} + \left\|\psi^h\right\|_{L_{2h}}\right].$$

Here, $M(\delta,\alpha)$ is independent of τ, h, $\varphi^h(x)$, $\psi^h(x)$, $f_k^h(x)$ and $g_k^h(x)$.

2 Numerical algorithm

In this section, we propose a numerical algorithm to solve the difference scheme (8). The approach relies on a suitable substitution that eliminates the unknown source term p^h. Let us denote

$$w_k^h(x) = v_k^h(x) + \left(A_h^x\right)^{-1} p^h(x),\ x \in [0,1]_h,\ -N \le k \le N.$$

Then, the scheme (8) results in the following auxiliary DS

$$\begin{cases} \frac{v_{k+1}^h(x)-2v_k^h(x)+v_{k-1}^h(x)}{\tau^2} + \alpha\frac{v_{k+1}^h(x)-v_k^h(x)}{\tau} + A_h^x v_{k+1}^h(x) = f_k^h(x), & 1 \le k \le N-1, \\ \frac{v_k^h(x)-v_{k-1}^h(x)}{\tau} + A_h^x v_k^h(x) = g_k^h(x), & -N+1 \le k \le 0, \\ \frac{v_1^h(x)-v_0^h(x)}{\tau} = -A_h^x v_0^h(x) + g_0^h(x), \\ v_{-N}^h(x) = v_N^h(x) + \varphi^h(x) - \psi^h(x), \end{cases} \tag{9}$$

where $x \in [0,1]_h$. Note that the scheme (9) no longer involves the unknown source p^h. However, it exhibits a non-local nature due to the coupling between v^h_{-N} and v^h_N, and therefore it cannot be solved using a standard time-marching approach.

We attempt to solve the non-local difference problem (9) iteratively. Let $\{v^h_k(x;\theta)\}$ be the solution of the following scheme

$$\begin{cases} \frac{v^h_{k+1}(x;\theta)-2v^h_k(x;\theta)+v^h_{k-1}(x;\theta)}{\tau^2} + \alpha\frac{v^h_{k+1}(x;\theta)-v^h_k(x;\theta)}{\tau} + A^x_h v^h_{k+1}(x;\theta) = f^h_k(x), & 1 \le k \le N-1, \\ \frac{v^h_k(x;\theta)-v^h_{k-1}(x;\theta)}{\tau} + A^x_h v^h_k(x;\theta) = g^h_k(x), & -N+1 \le k \le 0, \\ \frac{v^h_1(x;\theta)-v^h_0(x;\theta)}{\tau} = -A^x_h v^h_0(x;\theta) + g^h_0(x), \\ v^h_{-N}(x) = \theta^h(x), \end{cases} \tag{10}$$

where $x \in [0,1]_h$. For $\{v^h_k(x;\theta)\}$ to be a solution of the scheme (9), the initial vector $\theta = \theta^h(x)$, where $x \in [0,1]_h$, must satisfy the following condition

$$\theta = v^h_N(x;\theta) + \varphi^h(x) - \psi^h(x), \quad x \in [0,1]_h.$$

We can then construct an iterative procedure, such as fixed point iterations, to approximate the initial vector θ. Taking all of the above into account, the following algorithm can be used to solve the difference scheme (8).

1. To approximate the initial vector θ iteratively, we use the following formula:

$$\theta^{m+1} = v^h_N(x;\theta^m) + \varphi^h(x) - \psi^h(x),\ x \in [0,1]_h,\ m = 0,1,2,\dots$$

   At each iteration step, the difference scheme (10) must be solved to compute $v^h_N(x;\theta^m)$.
2. Next, we approximate the source $p(x)$ using the formula

$$p^h(x) = A^x_h\left(\varphi^h(x) - \theta\right),\ x \in [0,1]_h,$$

   where θ is the initial vector, approximated in the first step.
3. Finally, we obtain the solution of the difference scheme (8) using the formula:

$$w^h_k(x) = v^h_k(x) + \varphi^h(x) - \theta, \quad x \in [0,1]_h, \quad -N+1 \le k \le N-1.$$

   Here, $v^h_k(x)$ is the solution of the difference scheme (10) with the initial vector θ obtained from the iterative procedure.

3 Numerical example

First, we consider the following initial-boundary value problem

$$\begin{cases} w_{tt} + 2w_t - w_{xx} = p(x) + f(t,x), & x \in (0,1),\ t \in (0,1), \\ w_t - w_{xx} = p(x) + g(t,x), & x \in (0,1),\ t \in (-1,0), \\ w(0^+,x) = w(0^-,x),\ w_t(0^+,x) = w_t(0^-,x), & x \in [0,1], \\ w(-1,x) = e^1 \sin\pi x,\ w(1,x) = e^{-1}\sin\pi x, & x \in [0,1], \\ w(t,0) = 0,\ w(t,1) = 0, & t \in [-1,1], \end{cases} \tag{11}$$

where $f(t,x) = g(t,x) = \left((\pi^2-1)e^{-t} - 1\right)\sin\pi x$. The analytical solution of problem (11) is

$$w(t,x) = e^{-t}\sin\pi x, \quad x \in [0,1],\ t \in [-1,1]$$

with the source term $p(x) = \sin \pi x,\ x \in (0,1)$.

Second, we consider the initial-boundary value problem

$$\begin{cases} w_{tt} + 2w_t - w_{xx} + 3w = p(x) + f(t,x), & x \in (0,1),\ t \in (0,1), \\ w_t - w_{xx} + 3w = p(x) + g(t,x), & x \in (0,1),\ t \in (-1,0), \\ w(0^+,x) = w(0^-,x),\ w_t(0^+,\ x) = w_t(0^-,\ x), & x \in [0,1], \\ w(-1,x) = e^1 \cos \pi x,\ w(1,x) = e^{-1} \cos \pi x, & x \in [0,1], \\ w_x(t,0) = w_x(t,1) = 0, & t \in [-1,1], \end{cases} \tag{12}$$

where $f(t,x) = g(t,x) = \left((\pi^2+2)e^{-t} - 1\right) \cos \pi x$. The analytical solution of problem (12) is

$$w(t,x) = e^{-t} \cos \pi x, \quad x \in [0,1],\ t \in [-1,1]$$

with the source term $p(x) = \cos \pi x,\ x \in (0,1)$.

The numerical solutions for SIPs (11) and (12) are computed using the first-order accuracy DS and the aforementioned numerical procedure for various values of $M = N$. To evaluate the accuracy of the method, we compute the error between the analytical and numerical solutions using the following formulas:

$$E_p = \max_n |p(x_n) - p_n|, \quad E_w = \max_{k,n} \left| w(t_k, x_n) - w_n^k \right|.$$

Here, w_n^k and p_n denote the corresponding numerical approximations of the exact solution $\{w(t,x), p(x)\}$ at the grid points $t = t_k$ and $x = x_n$. Figure 1 shows the errors between the exact and numerical solutions of problems (11) and (12) for different values of τ, confirming the first-order convergence of the proposed method. Since we have taken $M = N$, which implies $h = \tau$, and the error of the method is $O(\tau + h^2)$, observe only the temporal errors here, i.e., the first-order convergence of the method.

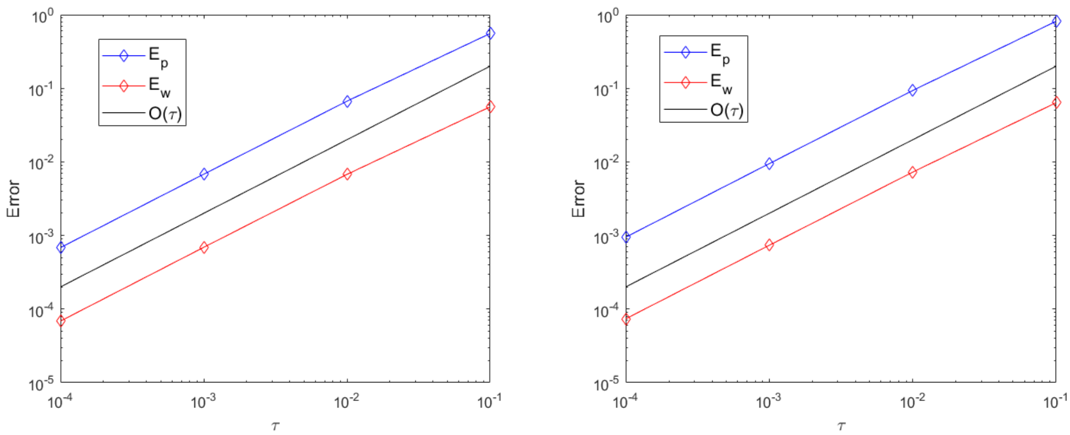

Figure 1. The errors between the analytical solutions of problems (11) (on the left) and (12) (on the right) and their numerical solutions, computed using the first-order DS for various values of the time step τ

Conclusion

In this work, we developed an algorithm for the numerical solution of one-dimensional telegraph-parabolic equations with an unknown source term dependent on a spatial variable. The local inverse problems considered are transformed into corresponding nonlocal direct problems, which are then solved using an iterative technique similar to the shooting method. Numerical experiments are provided to illustrate the procedure in practice.

Our results demonstrate first-order convergence of the proposed numerical method. It is of practical importance to develop higher order accuracy stable DSs so that more accurate results can be obtained in less computational time. Future work will also focus on the investigation of SIPs for telegraph-parabolic equations with time-dependent sources.

Author Contributions

All authors contributed equally to this work.

Conflict of Interest

The authors declare no conflict of interest.

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Bounded solutions in epidemic models governed by semilinear parabolic equations with general semilinear incidence rates

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The transmission mechanisms of most infectious diseases are generally well understood from an epidemiological standpoint. To mathematically and quantitatively characterize the spread of these diseases, various classical epidemic models-such as the SIR, SIS, SEIR, and SIRS frameworks-have been formulated and thoroughly investigated. In the present paper, the initial value problem for the system of semilinear parabolic differential equations arising in epidemic models with a general semilinear incidence rate in a Hilbert space with a self-adjoint positive definite operator is investigated. The main theorem on the existence and uniqueness of bounded solutions for this system is established. In applications, theorems on the existence and uniqueness of bounded solutions for two types of systems of semilinear partial differential equations arising in epidemic models are proved. A first-order accurate finite difference scheme is developed to construct approximate solutions for this system. We further prove a theorem that guarantees the existence and uniqueness of bounded solutions for the discrete problem, independently of the time step. The theoretical results are supported by applications, where bounded solutions of the continuous system and their corresponding discrete approximations are demonstrated. Finally, numerical results are presented to illustrate the effectiveness and accuracy of the proposed scheme.

Keywords: system of semilinear partial differential equations(SPDEs), EM, bounded solution(BS), numerical results, Hilbert space, self-adjoint positive definite operator,existence and uniqueness (EU), difference scheme(DS).

2020 Mathematics Subject Classification: 58J35, 58D25, 65M12, 92B05, 35K61, 35K58, 35K90, 91B76.

Introduction

The mechanism of disease transmission is typically well understood from an epidemiological perspective for most infectious diseases. To describe mathematically and quantitatively the spread of such diseases, numerous classical EMs have been developed and extensively studied, including the SIR, SIS, SEIR, and SIRS models [1–3].

In particular, the studies presented in [1] focus on the numerical solution of systems of linear parabolic equations modeling the transmission of HIV from mother to child. Numerical simulations were provided to support the theoretical results.

In the papers [4–6], the authors study a diffusive SIR epidemic model with nonlinear incidence in a heterogeneous environment. They establish the boundedness and uniform persistence of solutions to the system, as well as the global stability of the constant endemic equilibrium in the case of a homogeneous environment.

The papers [7, 8] study the dynamical behavior of a diffusive epidemic SIRI system with distinct dispersal rates. The overall solution of the system is derived using L_p theory and Young's inequality.

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The uniform boundedness of the solution is established, and the asymptotic smoothness of the semi-flow, as well as the existence of a global attractor, is discussed.

Finally, the papers [9,10] focus on a reaction-diffusion SEIR (susceptible-exposed-infected-recovered) epidemic model with a mass-action infection mechanism. The basic reproduction number of the SEIR model is defined, and its properties are studied under conditions of low mobility of the susceptible and exposed/infected populations, respectively. In a homogeneous environment, the global asymptotic stability of both the disease-free equilibrium and the endemic equilibrium is determined by the basic reproduction number. Furthermore, the asymptotic behavior of the endemic equilibrium (when it exists) is analyzed in a spatially heterogeneous environment with low migration rates of the susceptible, exposed, or infected populations.

Various classes of mixed boundary value problems for systems of partial differential equations can be transformed into initial value problems for systems of semilinear ordinary differential equations [11,12]

$$\begin{cases} \frac{dw^1(t)}{dt} + \mu w^1(t) + Aw^1(t) = -F(t, w^1(t), w^2(t)), \\ \frac{dw^2(t)}{dt} + (\xi + \mu) w^2(t) + Aw^2(t) = F(t, w^1(t), w^2(t)) - \Upsilon(t, w^2(t)), \\ \frac{dw^3(t)}{dt} + \mu w^3(t) + Aw^3(t) = \Upsilon(t, w^2(t)), \ t \in (0, b), \\ w^n(0) = \psi^n, \ n = \overline{1,3} \end{cases} \tag{1}$$

in a Hilbert space $\aleph$ with an unbounded elliptic operator A.

Throughout this paper, a theorem on the EU of BSs to the abstract problem (1) is proved. The results are illustrated by their application to a system of semilinear parabolic equations, demonstrating their effectiveness in both one- and multi-dimensional settings with appropriate boundary conditions. Furthermore, a discrete analogue of the theoretical results is developed for a first-order accurate time-difference scheme. Numerical simulations are included to illustrate and validate the theoretical results.

1 *BS of the differential problem (1)*

Let $\aleph$ be a Hilbert space, and let A be a positive definite self-adjoint operator such that $A \geq \delta I$ for some $\delta > 0$. Throughout this paper, the family $\{\exp(-tA), \ t \geq 0\}$ denotes the strongly continuous exponential operator-function.

By applying the spectral representation of a self-adjoint positive definite operator in a Hilbert space, we obtain the following estimate:

$$\|\exp(-tA)\|_{\aleph \to \aleph} \leq e^{-\delta t}, \quad t \geq 0. \tag{2}$$

A vector-valued function $w(t) = \left(w^1(t), w^2(t), w^3(t)\right)^T$ is said to be a solution of problem (1) if the following conditions are satisfied:

(i) For each $m \in \{1, 2, 3\}$, $w^m(t)$ is a continuously differentiable function on the interval $(0, b)$.

(ii) For all $t \in [0, b]$ and each $m = \overline{1,3}$, the element $w^m(t)$ belongs to the domain $D(A)$ of the operator A, and the function $Aw^m(t)$ is continuous on $[0, b]$.

(iii) The functions $F(t, w^1(t), w^2(t))$ and $\Upsilon(t, w^2(t))$ are continuous for all $t \in [0, b]$.

(iv) The function $w(t)$ satisfies the system of equations and initial conditions given in (1).

The proof of the main theorem regarding the EU of a BS of problem (1) is based on reducing the problem to an equivalent system of integral equations

$$\begin{cases} w^1(t) = e^{-\mu t}e^{-At}\psi^1 - \int\limits_0^t e^{-\mu(t-\lambda)}e^{-A(t-\lambda)}F\left(\lambda, w^1(\lambda), w^2(\lambda)\right)d\lambda, \\ w^2(t) = e^{-(\mu+\xi)t}e^{-At}\psi^2 + \int\limits_0^t e^{-(\mu+\xi)(t-\lambda)}e^{-A(t-\lambda)}F\left(\lambda, w^1(\lambda), w^2(\lambda)\right)d\lambda \\ -\int\limits_0^t e^{-(\mu+\xi)(t-\lambda)}e^{-A(t-\lambda)}\Upsilon\left(\lambda, w^2(\lambda)\right)d\lambda, \\ w^3(t) = e^{-\mu t}e^{-At}\psi^3 + \int\limits_0^t e^{-\mu(t-\lambda)}e^{-A(t-\lambda)}\Upsilon\left(\lambda, w^2(\lambda)\right)d\lambda \end{cases} \tag{3}$$

in $C(\aleph)$ and the use of successive approximations. Here, $C(\aleph)$ stands for the Banach space of the continuous functions $z(t)$ defined on $[0,b]$ with values in $\aleph$, equipped with the norm

$$\|z\|_C = \max_{t\in[0,b]} \|z(t)\|_\aleph .$$

We introduce the equivalent norm

$$\| z \|_{C_L} = \max_{t\in[0,b]} e^{-Lt}\|z(t)\|_\aleph , \quad L > 0.$$

The recursive formula for the solution of problem (3) is

$$\begin{cases} nw^1(t) = e^{-\mu t}e^{-At}\psi^1 - \int\limits_0^t e^{-\mu(t-\lambda)}e^{-A(t-\lambda)}F\left(\lambda, (n-1)w^1(\lambda), (n-1)w^2(\lambda)\right)d\lambda, \\ nw^2(t) = e^{-(\mu+\xi)t}e^{-At}\psi^2 \\ +\int\limits_0^t e^{-(\mu+\xi)(t-\lambda)}e^{-A(t-\lambda)}F\left(\lambda, (n-1)w^1(\lambda), (n-1)w^2(\lambda)\right)d\lambda \\ -\int\limits_0^t e^{-(\mu+\xi)(t-\lambda)}e^{-A(t-\lambda)}\Upsilon\left(\lambda, (n-1)u^2(\lambda)\right)d\lambda, \\ nw^3(t) = e^{-\mu t}e^{-At}\psi^3 + \int\limits_0^t e^{-\mu(t-\lambda)}e^{-A(t-\lambda)}\Upsilon\left(\lambda, (n-1)w^2(\lambda)\right)d\lambda, \ n = 1, 2, ..., \\ 0w^m(t), \ m = \overline{1,3} \quad \text{are given.} \end{cases} \tag{4}$$

Theorem 1. Assume the following conditions are satisfied:

1. For each $m = \overline{1,3}$, the initial function ψ^m belong to the domain $D(A)$ of the operator A, and
$$\|\psi^m\|_{D(A)} = M_1. \tag{5}$$
2. The function $F : [0,b] \times \aleph \times \aleph \to \aleph$ is continuous and satisfies the uniform bound:
$$\|F(t, w(t), u(t))\|_\aleph \le M_2, \tag{6}$$
for all $(t, w, u) \in [0,b] \times \aleph \times \aleph$. In addition, the mapping F fulfills a Lipschitz condition that holds uniformly in t:
$$\|F(t, w, u) - F(t, z, v)\|_\aleph \le L_1 \left(\|w - z\|_\aleph + \|u - v\|_\aleph\right). \tag{7}$$
3. The function $\Upsilon : [0,b] \times \aleph \to \aleph$ is uniformly Lipschitz continuous w.r.t. the variable t:
$$\|\Upsilon(t, w(t))\|_\aleph \le M_3 \tag{8}$$
for all $(t, w) \in [0,b] \times \aleph$. In addition, Υ satisfies a Lipschitz condition uniformly w.r.t. t:
$$\|\Upsilon(t, w) - \Upsilon(t, z)\|_\aleph \le L_2\|w - z\|_\aleph. \tag{9}$$

Here, L_r for $r = 1, 2$ and M_r for $r = 1, 2, 3$ are positive constants. Then, under these assumptions, there exists a unique solution $w(t) = \left(w^1(t), w^2(t), w^3(t)\right)^T$ of the problem (1), which is bounded in the product space $\mathbb{C}^3(\aleph) = C(\aleph) \times C(\aleph) \times C(\aleph)$.

Proof. Since $w^3(t)$ does not appear in equations for $\frac{dw^1(t)}{dt}$ and $\frac{dw^2(t)}{dt}$, it is sufficient to analyze the behaviors of solutions $w^1(t)$ and $w^2(t)$ of (1) in the norm of the space $C_L(\aleph)$.

According to the method of recursive approximation (4), we get

$$w^m(t) = 0w^m(t) + \sum_{i=0}^{\infty} \left[(i+1)w^m(t) - iw^m(t)\right], \quad m = 1, 2, \tag{10}$$

where

$$0w^m(t) = \begin{cases} e^{-\mu t} e^{-At} \psi^1, & m = 1, 3, \\ e^{-(\mu+\xi)t} e^{-At} \psi^2, & m = 2. \end{cases}$$

Using formula (4) and estimates (2), (5), (6) and (8), we obtain

$$e^{-Lt} \| 1w^1(t) - 0u^1(t) \|_{\aleph}$$

$$\leq \int_0^t \exp\left(-(\mu + L)(t - \lambda)\right) \| e^{-A(t-\lambda)} \| \| F\left(\lambda, 0w^1(\lambda), 0w^2(\lambda)\right) \|_{\aleph} d\lambda \leq \frac{M_2}{\mu + L},$$

$$e^{-Lt} \| 1w^2(t) - 0w^2(t) \|_{\aleph}$$

$$\leq \int_0^t \exp\left(-(\mu + \xi + L)(t - \lambda)\right) \| e^{-A(t-\lambda)} \| \left[\| F\left(\lambda, 0w^1(\lambda), 0w^2(\lambda)\right) \|_{\aleph} + \| \Upsilon\left(\lambda, 0w^2(\lambda)\right) \|_{\aleph} \right] d\lambda$$

$$\leq \frac{M_2 + M_3}{\mu + \xi + L}$$

for any $t \in [0, b]$. Using the triangle inequality, we get

$$e^{-Lt} \| 1w^1(t) \|_{\aleph} \leq M_1 + \frac{M_2 + M_3}{\mu + L}, \quad e^{-Lt} \| 1w^2(t) \|_{\aleph} \leq M_1 + \frac{M_2 + M_3}{\mu + L}$$

for any $t \in [0, b]$. Using formula (4) and estimates (2), (6), (7) and (9), we obtain

$$e^{-Lt} \| 2w^1(t) - 1w^1(t) \|_{\aleph}$$

$$\leq \int_0^t \exp\left(-(\mu + L)(t - \lambda)\right) e^{-L\lambda} \| e^{-A(t-\lambda)} \| \| F\left(\lambda, 1w^1(\lambda), 1w^2(\lambda)\right) - F\left(\lambda, 0w^1(\lambda), 0w^2(\lambda)\right) \|_{\aleph} d\lambda$$

$$\leq \frac{2L_1 (M_2 + M_3)}{(\mu + L)^2} \leq \frac{2(L_1 + L_2)(M_2 + M_3)}{(\mu + L)^2},$$

$$e^{-Lt} \| 2w^2(t) - 1w^2(t) \|_{\aleph}$$

$$\leq \int_0^t \exp\left(-(\mu + \xi + L)(t - \lambda)\right) e^{-L\lambda} \| e^{-A(t-\lambda)} \| \| F\left(\lambda, 1w^1(\lambda), 1w^2(\lambda)\right) - F\left(\lambda, 0w^1(\lambda), 0w^2(\lambda)\right) \|_{\aleph} d\lambda$$

$$+ \int_0^t \exp\left(-(\mu + \xi + L)(t - \lambda)\right) e^{-L\lambda} \| e^{-A(t-\lambda)} \| \| \Upsilon\left(\lambda, 1w^2(\lambda)\right) - \Upsilon\left(\lambda, 0w^2(\lambda)\right) \|_{\aleph} d\lambda$$

$$\leq \frac{(2L_1 + L_2)(M_2 + M_3)}{(\mu + L)^2} \leq \frac{2(L_1 + L_2)(M_2 + M_3)}{(\mu + L)^2}$$

for any $t \in [0,b]$. Then,

$$e^{-Lt}\|2w^1(t)\|_{\aleph} \le M_1 + \frac{M_2+M_3}{\mu+L} + \frac{2(L_1+L_2)(M_2+M_3)}{(\mu+L)^2},$$

$$e^{-Lt}\|2w^2(t)\|_{\aleph} \le M_1 + \frac{M_2+M_3}{\mu+L} + \frac{2(L_1+L_2)(M_2+M_3)}{(\mu+L)^2}$$

for any $t \in [0,b]$. Let

$$e^{-Lt}\|nw^m(t)-(n-1)w^m(t)\|_{\aleph} \le \frac{2^{n-1}(L_1+L_2)^{n-1}(M_2+M_3)}{(\mu+L)^n}, \quad m=1,2.$$

Thus, we arrive at

$$e^{-Lt}\|(n+1)w^1(t)-nw^1(t)\|_{\aleph}$$

$$\le \int_0^t e^{-(\mu+L)(t-\lambda)}e^{-L\lambda}\|e^{-A(t-\lambda)}\|\|F\left(\lambda, nw^1(\lambda), nu^2(\lambda)\right) - F\left(\lambda,(n-1)w^1(\lambda),(n-1)w^2(\lambda)\right)\|_{\aleph}d\lambda$$

$$\le \frac{2L_1 2^{n-1}(L_1+L_2)^{n-1}(M_2+M_3)}{(\mu+L)^{n+1}} \le \frac{2^n(L_1+L_2)^n(M_2+M_3)}{(\mu+L)^{n+1}},$$

$$e^{-Lt}\|(n+1)w^2(t)-nw^2(t)\|_{\aleph}$$

$$\le \int_0^t e^{-(\mu+\xi+L)(t-\lambda)}e^{-L\lambda}\|e^{-A(t-\lambda)}\|\|F\left(\lambda, 1w^1(\lambda), 1w^2(\lambda)\right) - F\left(\lambda,0w^1(\lambda),0w^2(\lambda)\right)\|_{\aleph}d\lambda$$

$$\le \frac{(2L_1+L_2)2^{n-1}(L_1+L_2)^{n-1}(M_2+M_3)}{(\mu+L)^{n+1}} \le \frac{2^n(L_1+L_2)^n(M_2+M_3)}{(\mu+L)^{n+1}}$$

for any $t \in [0,b]$. Then,

$$e^{-Lt}\|(n+1)w^m(t)\|_{\aleph}$$

$$\le M_1 + \frac{M_2+M_3}{\mu+L} + \frac{2(L_1+L_2)(M_2+M_3)}{(\mu+L)^2} + \cdots + \frac{2^n(L_1+L_2)^n(M_2+M_3)}{(\mu+L)^{n+1}}, \quad m=1,2$$

for each $t \in [0,b]$. Then, for any n, $n \ge 1$, we have

$$e^{-Lt}\|(n+1)w^1(t)-nw^1(t)\|_{\aleph} \le \frac{2^n(L_1+L_2)^n(M_2+M_3)}{(\mu+L)^{n+1}}, \quad m=1,2,$$

and

$$e^{-Lt}\|(n+1)w^m(t)\|_{\aleph}$$

$$\le M_1 + \frac{M_2+M_3}{\mu+L} + \frac{2(L_1+L_2)(M_2+M_3)}{(\mu+L)^2} + \cdots + \frac{2^n(L_1+L_2)^n(M_2+M_3)}{(\mu+L)^{n+1}}, \quad m=1,2$$

by induction. It follows from this and formula (10) that

$$e^{-Lt}\|w^m(t)\|_{\aleph} \le \|0w^m(t)\|_{\aleph} + \sum_{i=0}^{\infty} e^{-Lt}\|(i+1)w^m(t)-iw^m(t)\|_{\aleph}$$

$$\le M_1 + \sum_{i=0}^{\infty}\frac{2^i(L_1+L_2)^i(M_2+M_3)}{(\mu+L)^{i+1}}, \quad m=1,2$$

which proves the existence of a BS of problem (1) in norm $C_L([0,b],\aleph)$. From this, it follows the existence of a BS of problem (1) in norm $C([0,b],\aleph)$. Theorem 1 is proved.

Now, consider the applications of Theorem 1.

First, we investigate the initial-boundary value problem for one-dimensional system of SPDEs

$$\begin{cases} \frac{\partial \Psi^1(t,z)}{\partial t} - \left(a(z)\Psi_z^1(t,z)\right)_z - \beta\left(a(-z)\Psi_z(t,-z)\right)_z + (\delta+\mu)\Psi^1(t,z) \\ = -F(t,z;\Psi^1(t,z),\Psi^2(t,z)), \\ \frac{\partial \Psi^2(t,z)}{\partial t} - \left(a(z)\Psi_z^2(t,z)\right)_z - \beta\left(a(-z)\Psi_z(t,-z)\right)_z + (\delta+\mu+\xi)\Psi^2(t,z) \\ = F(t,z;\Psi^1(t,z),\Psi^2(t,z)) - \Upsilon(t,z;\Psi^2(t,z)), \\ \frac{\partial \Psi^3(t,z)}{\partial t} - \left(a(z)\Psi_z^3(t,z)\right)_z - \beta\left(a(-z)\Psi_z(t,-z)\right)_z + (\delta+\mu)\Psi^3(t,z) \\ = \Upsilon(t,z;\Psi^2(t,z)),\ t\in(0,b),\ -d<z<d, \\ \Psi^m(t,\pm d)=0,\ t\in[0,b],\ m=\overline{1,3}, \\ \Psi^m(0,z)=\psi^m(z),\ \psi^m(\pm d)=0,\ z\in[-d,d],\ m=\overline{1,3}, \end{cases} \tag{11}$$

where $a(z)$ and $\psi(z)$ are given sufficiently smooth functions. Here, $\delta>0$ is a sufficiently large number. We will suppose that $a \geq a(z) = a(-z) \geq \delta > 0,\ \delta - a|\beta| \geq 0$.

Theorem 2. Suppose the following conditions are satisfied:

1. For each $m=\overline{1,3}$, the initial function ψ^m belongs to the Sobolev space $W_2^2[-d,d]$, and

$$\|\psi^m\|_{W_2^2[-d,d]} \leq M_1.$$

2. The function

$$F : [0,b]\times[-d,d]\times L_2[-d,d]\times L_2[-d,d] \to L_2[-d,d]$$

is continuous in the time variable t and satisfies the uniform bound

$$\|F(t,\cdot,w(t,\cdot),u(t,\cdot))\|_{L_2[-d,d]} \leq M_2$$

for all $(t,\cdot,w,u)\in[0,b]\times L_2[-d,d]\times L_2[-d,d]$. Moreover, F satisfies a Lipschitz condition uniformly in t:

$$\|F(t,\cdot,w,u)-F(t,\cdot,p,q)\|_{L_2[-d,d]} \leq L_1\left(\|w-p\|_{L_2[-d,d]} + \|u-q\|_{L_2[-d,d]}\right).$$

3. The function

$$\Upsilon : [0,b]\times[-d,d]\times L_2[-d,d] \to L_2[-d,d]$$

is continuous in t and satisfies the uniform bound:

$$\|\Upsilon(t,\cdot,w(t,\cdot))\|_{L_2[-d,d]} \leq M_3$$

for all $(t,w)\in[0,b]\times L_2[-d,d]$. Additionally, Υ satisfies a Lipschitz condition uniformly in t:

$$\|\Upsilon(t,\cdot,w)-\Upsilon(t,\cdot,u)\|_{L_2[-d,d]} \leq L_2\|w-u\|_{L_2[-d,d]}.$$

Here and in the sequel, the constants L_m (for $m=1,2$) and M_m (for $m=\overline{1,3}$) are assumed to be positive.

Then, under the above assumptions, there exists a unique solution $\Psi(t,z) = \left(\Psi^1(t,z),\Psi^2(t,z),\Psi^3(t,z)\right)^T$ to problem (11), which is bounded in the space $\mathbb{C}^3(L_2[-d,d]) = C(L_2[-d,d])\times C(L_2[-d,d])\times C(L_2[-d,d])$.

The proof of Theorem 2 is based on the abstract Theorem 1, the self-adjointness and positivity in $L_2[-d,d]$ of a differential operator A^z defined by the formula

$$A^z\omega(z) = -\left(a(z)\omega_z(z)\right)_z - \beta\left(a(-z)\omega_z(-z)\right)_z + \delta\omega(z)$$

with the domain $D(A^z) = \left\{\omega \in W_2^2\left[-d,d\right] : \omega\left(-d\right) = \omega\left(d\right) = 0\right\}$ [13] and on the estimate

$$\|\exp\{-tA^z\}\|_{L_2[-d,d]\to L_2[-d,d]} \leq 1, \ \ t \geq 0.$$

Second, we study the initial-boundary value problem for a multidimensional system of SPDEs

$$\begin{cases} \frac{\partial \Psi^1(t,z)}{\partial t} - \sum\limits_{r=1}^{n}(a_r(z)\Psi^1_{z_r})z_r + (\delta+\mu)\,\Psi^1(t,z) = -F(t,z;\Psi^1\left(t,z\right),\Psi^2\left(t,z\right)), \\ \frac{\partial \Psi^2(t,z)}{\partial t} - \sum\limits_{r=1}^{n}(a_r(z)\Psi^2_{z_r})z_r + (\delta+\mu+\xi)\,\Psi^2(t,z) \\ = F(t,z;\Psi^1\left(t,z\right),\Psi^2\left(t,z\right)) - \Upsilon(t,z;\Psi^2\left(t,z\right)), \\ \frac{\partial \Psi^3(t,z)}{\partial t} - \sum\limits_{r=1}^{n}(a_r(z)\Psi^3_{z_r})z_r + (\delta+\mu)\,\Psi^3(t,z) \\ = \Upsilon(t,z;\Psi^2\left(t,z\right)),\ t\in(0,b),\ z=(z_1,...,z_n)\in\Omega, \\ \Psi^p(0,z) = \psi^p(z),\ z\in\overline{\Omega},\ p=\overline{1,3}, \\ \Psi^p(t,z)=0,\ t\in[0,b],\ z\in S,\ p=\overline{1,3}, \end{cases} \tag{12}$$

where $a_r(z)$ and $\psi^p(z)$ are given sufficiently smooth functions and $\delta > 0$ is a sufficiently large number and $a_r(z) > 0$. Here, $\Omega \subset R^n$ is an open and bounded domain whose boundary S is smooth and we put $\overline{\Omega} = \Omega \cup S$.

Theorem 3. Suppose the following conditions are satisfied:

1. For each $m = \overline{1,3}$, the initial function ψ^m belongs to the Sobolev space $W_2^2(\overline{\Omega})$, and

$$\|\psi^m\|_{W_2^2(\overline{\Omega})} \leq M_1.$$

2. The function

$$F : [0,b] \times [0,l] \times L_2(\overline{\Omega}) \times L_2(\overline{\Omega}) \to L_2(\overline{\Omega})$$

is continuous in the time variable t, and satisfies the uniform bound:

$$\|F(t,\cdot,w(t,\cdot),u(t,\cdot))\|_{L_2(\overline{\Omega})} \leq M_2$$

for all $(t,w,u) \in [0,b] \times L_2(\overline{\Omega}) \times L_2(\overline{\Omega})$. Moreover, F satisfies a Lipschitz condition uniformly in t:

$$\|F(t,\cdot,w,u) - F(t,\cdot,p,q)\|_{L_2(\overline{\Omega})} \leq L_1\left(\|w-p\|_{L_2(\overline{\Omega})} + \|u-q\|_{L_2(\overline{\Omega})}\right).$$

3. The function

$$\Upsilon : [0,b] \times [0,l] \times L_2(\overline{\Omega}) \to L_2(\overline{\Omega})$$

is continuous in t, and satisfies the uniform bound:

$$\|\Upsilon(t,\cdot,w(t,\cdot))\|_{L_2(\overline{\Omega})} \leq M_3$$

for all $(t,w) \in [0,b] \times L_2(\overline{\Omega})$. Υ satisfies a Lipschitz condition uniformly in t:

$$\|\Upsilon(t,\cdot,w) - \Upsilon(t,\cdot,p)\|_{L_2(\overline{\Omega})} \leq L_2\,\|w-p\|_{L_2(\overline{\Omega})}.$$

Then, under the above assumptions, there exists a unique solution $\Psi(t,z) = \left(\Psi^1(t,z),\Psi^2(t,z),\Psi^3(t,z)\right)^T$ to problem (12), which is bounded in the space $\mathbb{C}^3(L_2(\overline{\Omega})) = C\left(L_2(\overline{\Omega})\right) \times C\left(_2(\overline{\Omega})\right) \times C\left(L_2(\overline{\Omega})\right)$.

The proof of Theorem 3 is based on the abstract Theorem 1, on the self-adjointness and positivity in $L_2(\overline{\Omega})$ of a differential operator A^z defined by the formula

$$A^z\Omega(z) = -\sum_{r=1}^{n}(a_r(z)\Omega_{z_r})_{z_r} + \delta\Omega(z)$$

with domain [12]

$$D(A^z) = \left\{\omega(z) : \omega(z), (a_r(z)\omega_{z_r})_{z_r} \in L_2(\overline{\Omega}),\ 1 \le r \le n,\ \omega(z) = 0,\ z \in S\right\}$$

and on the estimate

$$\| \exp\{-tA^z\}\|_{L_2(\overline{\Omega}) \to L_2(\overline{\Omega})} \le 1,\ t \in [0, \infty)$$

and the following theorem on coercivity inequality for the solution of the elliptic problem in $L_2(\overline{\Omega})$ [12].

2 BS of the difference scheme

For the approximate solution of (1) we consider a grid space

$$[0, b]_\tau = \left\{t_k = k\tau,\ k = \overline{1, N},\ N\tau = b\right\}.$$

We consider the first order of accuracy difference scheme

$$\begin{cases} \frac{u_k^1 - u_{k-1}^1}{\tau} + \mu u_k^1 + Au_k^1 = -F(t_k, u_k^1, u_k^2), \\ \frac{u_k^2 - u_{k-1}^2}{\tau} + (\xi + \mu)\, u_k^2 + Au_k^2 = F(t_k, u_k^1, u_k^2) - \Upsilon(t_k, u_k^2), \\ \frac{u_k^3 - u_{k-1}^3}{\tau} + \mu u_k^3 + Au_k^3 = \Upsilon(t_k, u_k^2),\ k = \overline{1, N}, \\ u_0^p = \psi^p,\ p = \overline{1, 3} \end{cases} \tag{13}$$

for the approximate solution of problem (1). The proof method for the basic theorem on the EU of a BS of difference scheme (13) uniformly w.r.t. τ is based on reducing (13) to an equivalent system of semilinear difference equations. Actually, an equivalent system of semilinear difference equations for (13) is

$$\begin{cases} u_k^1 = R^k\psi^1 - \sum_{m=1}^{k} R^{k-m+1} F(t_m, u_m^1, u_m^2)\tau, \\ u_k^2 = R_1^k\psi^2 + \sum_{m=1}^{k} R_1^{k-m+1} \left[F(t_m, u_m^1, u_m^2) - \Upsilon(t_m, u_m^2)\right]\tau, \\ u_k^3 = R^k\psi^3 + \sum_{m=1}^{k} R^{k-m+1}\Upsilon(t_m, u_m^2)\tau,\ k = \overline{1, N} \end{cases} \tag{14}$$

in $\mathbb{C}_\tau^3(\aleph) = C_\tau(\aleph) \times C_\tau(\aleph) \times C_\tau(\aleph)$ and the use of successive approximations. Here and in the future $R_1 = (I + \tau((\mu + \xi) I + A))^{-1}$, $R = (I + \tau(\mu I + A))^{-1}$ and $C_\tau(\aleph)$ stands for the Banach space of mesh functions $w^\tau = \{w_p\}_{p=0}^N$ defined on $[0, b]_\tau$ with values in $\aleph$, equipped with the norm

$$\|w^\tau\|_{C_\tau(\aleph)} = \max_{0 \le p \le N} \|w_p\|_\aleph.$$

The recursive formula for the solution of DS (13) is

$$\begin{cases} \frac{ru_k^1 - ru_{k-1}^1}{\tau} + \mu ru_k^1 + Aru_k^1 = -F(t_k, (r-1)\, u_k^1, (r-1)\, u_k^2), \\ \frac{ru_k^2 - ru_{k-1}^2}{\tau} + (\xi + \mu)\, ru_k^2 + Aru_k^2 = F(t_k, (r-1)\, u_k^1, (r-1)\, u_k^2) - \Upsilon(t_k, (r-1)\, u_k^2), \\ \frac{ru_k^3 - ru_{k-1}^3}{\tau} + \mu ru_k^3 + Aru_k^3 = \Upsilon(t_k, (r-1)\, u_k^2),\ k = \overline{1, N}, \\ ru_0^p = \psi^p,\ p = \overline{1, 3},\ r = 1, 2, ..., \\ 0u_k^p,\ k = \overline{0, N},\ p = \overline{1, 3} \text{ are given.} \end{cases} \tag{15}$$

From (14) and (15) it follows

$$\begin{cases} ru_k^1 = R^k\psi^1 - \sum_{i=1}^{k} R^{k-i+1} F(t_k, (r-1)\, u_k^1, (r-1)\, u_k^2)\tau, \\ ru_k^2 = R_1^k\psi^2 + \sum_{i=1}^{k} R_1^{k-i+1} \left[F(t_k, (r-1)\, u_k^1, (r-1)\, u_k^2) - \Upsilon(t_k, (r-1)\, u_k^2)\right]\tau, \\ ru_k^3 = R^k\psi^3 + \sum_{i=1}^{k} R^{k-m+1}\Upsilon(t_k, (r-1)\, u_k^2)\tau,\ k = \overline{1, N},\ r = 1, 2, ..., \\ 0u_k^p,\ k = \overline{0, N},\ p = \overline{1, 3} \text{ are given.} \end{cases} \tag{16}$$

Theorem 4. Let the assumptions of Theorem 1 be satisfied and

$$\mu + \delta > 2\left(L_1 + L_2\right).$$

Then, there exists a unique BS $u^\tau = \{u_k\}_{k=0}^N$ of difference problem (13) in $\mathbb{C}_\tau^3(\aleph)$ uniformly w.r.t. τ.

Proof. Since u_k^3 does not appear in equations for $\frac{u_k^m - u_{k-1}^m}{\tau}, m = 1, 2$, it is sufficient to analyze the behaviors of solutions u_k^1 and u_k^2 of (13). According to the recursive approximation method (16), we get

$$u_k^m = {}_0u_k^m + \sum_{i=0}^{\infty}\left[{}_{(i+1)}u_k^m - {}_iu_k^m\right], \quad m = 1, 2, \tag{17}$$

where

$$ {}_0u_k^m = \begin{cases} R^k\psi^m, & m = 1, 3, \\ R_1^k\psi^2, & m = 2. \end{cases} \tag{18}$$

Applying the spectral representation of the self-adjoint positive definite operator in a Hilbert space, we get the following estimates

$$\|R\|_{\aleph\to\aleph} \leq \frac{1}{1+\tau\left(\mu+\delta\right)}, \quad \|R_1\|_{\aleph\to\aleph} \leq \frac{1}{1+\tau\left(\mu+\delta+\xi\right)}. \tag{19}$$

From formula (18) and estimates (19) it follows that

$$\|{}_0u_k^m\|_\aleph \leq \|\psi^m\|_\aleph \leq M_1. \tag{20}$$

Using formula (16), estimates (19), we get

$$\|{}_1u_k^1 - {}_0u_k^1\|_\aleph \leq \sum_{m=1}^{k} \|R^{k-m+1}\| \|F(t_m, {}_0u_m^1, {}_0u_m^2)\|_\aleph \tau \leq \frac{M_2}{\mu+\delta},$$

$$\|{}_1u_k^2 - {}_0u_k^2\|_\aleph \leq \sum_{m=1}^{k} \left\|R_1^{k-m+1}\right\| \left[\|F(t_m, {}_0u_m^1, {}_0u_m^2)\|_\aleph + \|\Upsilon(t_m, {}_0u_m^2)\|_\aleph\right]\tau \leq \frac{M_2+M_3}{\mu+\delta+\xi}$$

for any $k = \overline{1, N}$. Using the triangle inequality and estimate (20), we get

$$\|{}_1u_k^1\|_\aleph \leq M_1 + \frac{M_2+M_3}{\mu+\delta}, \quad \|{}_1u_k^2\|_\aleph \leq M_1 + \frac{M_2+M_3}{\mu+\delta}$$

for each $k = \overline{1, N}$. Using formula (16), estimates (19), (6) and (7), we can write

$$\|{}_2u_k^1 - {}_1u_k^1\|_\aleph \leq \sum_{m=1}^{k} \|R^{k-m+1}\| \|F(t_m, {}_1u_m^1, {}_1u_m^2) - F(t_m, {}_0u_m^1, {}_0u_m^2)\|_\aleph \tau \leq \frac{2\left(L_1+L_2\right)\left(M_2+M_3\right)}{\left(\mu+\delta\right)^2},$$

$$\|{}_2u_k^2 - {}_1u_k^2\|_\aleph \leq \sum_{m=1}^{k} \|R_1^{k-m+1}\| \|F(t_m, {}_1u_m^1, {}_1u_m^2) - F(t_m, {}_0u_m^1, {}_0u_m^2)\|_\aleph \tau$$

$$+ \sum_{m=1}^{k} \|R_1^{k-m+1}\| \|\Upsilon(t_m, {}_1u_m^2) - \Upsilon(t_m, {}_0u_m^2)\|_\aleph \tau \leq \frac{2\left(L_1+L_2\right)\left(M_2+M_3\right)}{\left(\mu+\delta+\xi\right)^2}$$

for any $k = \overline{1, N}$. Then,

$$\|2u_k^1\|_{\aleph} \leq M_1 + \frac{M_2 + M_3}{\mu + \delta} + \frac{2\left(L_1 + L_2\right)\left(M_2 + M_3\right)}{\left(\mu + \delta\right)^2},$$

$$\|2u_k^2\|_{\aleph} \leq M_1 + \frac{M_2 + M_3}{\mu + \delta} + \frac{2\left(L_1 + L_2\right)\left(M_2 + M_3\right)}{\left(\mu + \delta\right)^2}$$

for each $k = \overline{1, N}$. Let

$$\|nu_k^m - (n-1)u_k^m\|_{\aleph} \leq \frac{2^{n-1}\left(L_1 + L_2\right)^{n-1}\left(M_2 + M_3\right)}{\left(\mu + \delta\right)^n}, \quad m = 1, 2.$$

Using formula (16), estimates (19), (7) and (6), we get

$$\|\left(n+1\right)u_k^1 - nu_k^1\|_{\aleph} \leq \sum_{m=1}^{k} \|R^{k-m+1}\|\|F(t_m, nu_m^1, (n-1)\,u_m^2) - F(t_m, nu_m^1, (n-1)\,u_m^2)\|_{\aleph}\tau$$

$$\leq \frac{\left(2\left(L_1 + L_2\right)\right)^n \left(M_2 + M_3\right)}{\left(\mu + \delta\right)^{n+1}},$$

$$\|\left(n+1\right)u_k^2 - nu_k^2\|_{\aleph} \leq \sum_{m=1}^{k} \|R_1^{k-m+1}\|\|F(t_m, nu_m^1, nu_m^2) - F(t_m, (n-1)\,u_m^1, (n-1)\,u_m^2)\|_{\aleph}\tau$$

$$+ \sum_{m=1}^{k} \|R_1^{k-m+1}\|\|\Upsilon(t_m, nu_m^2) - \Upsilon(t_m, (n-1)\,u_m^2)\|_{\aleph}\tau$$

$$\leq \frac{\left(2\left(L_1 + L_2\right)\right)^n \left(M_2 + M_3\right)}{\left(\mu + \delta + \xi\right)^{n+1}}$$

for each $k = \overline{1, N}$. Then,

$$\|\left(n+1\right)u_k^1\|_{\aleph} \leq M_1 + \frac{M_2 + M_3}{\mu + \delta} + \frac{2\left(L_1 + L_2\right)\left(M_2 + M_3\right)}{\left(\mu + \delta\right)^2} + \cdots + \frac{\left(2\left(L_1 + L_2\right)\right)^n \left(M_2 + M_3\right)}{\left(\mu + \delta + \xi\right)^{n+1}},$$

$$\|\left(n+1\right)u_k^2\|_{\aleph} \leq M_1 + \frac{M_2 + M_3}{\mu + \delta} + \frac{2\left(L_1 + L_2\right)\left(M_2 + M_3\right)}{\left(\mu + \delta\right)^2} + \cdots + \frac{\left(2\left(L_1 + L_2\right)\right)^n \left(M_2 + M_3\right)}{\left(\mu + \delta\right)^{n+1}}$$

for every $k = \overline{1, N}$. Therefore, for any $n, n \geq 1$, we have that

$$\|\left(n+1\right)u_k^p - nu_k^p\|_{\aleph} \leq \frac{2^n\left(L_1 + L_2\right)^n\left(M_2 + M_3\right)}{\left(\mu + \delta\right)^{n+1}}, \quad p = 1, 2,$$

and

$$\left\|\left(n+1\right)u_k^p\right\|_{\aleph} \leq M_1 + \sum_{r=0}^{n} \frac{2^r\left(L_1 + L_2\right)^r\left(M_2 + M_3\right)}{\left(\mu + \delta\right)^{r+1}}, \quad p = 1, 2$$

by induction. From this and formula (17) it follows that

$$\|u_k^p\|_{\aleph} \leq \|0u_k^p\|_{\aleph} + \sum_{r=0}^{\infty} \|(r+1)u_k^p - ru_k^p\|_{\aleph}$$

$$\leq M_1+\sum_{r=0}^{\infty}\frac{2^r\left(L_1+L_2\right)^r\left(M_2+M_3\right)}{\left(\mu+\delta\right)^{r+1}},\quad p=1,2.$$

This proves the existence of a BS of DS (13) that is bounded in $\mathbb{C}^3_\tau(\aleph)$ uniformly w.r.t. τ. Theorem 4 is proved.

Now, let us consider the applications of Theorem 4. First, the mixed problem (11) for one-dimensional system of SPDEs is considered. The discretization of problem (11) is carried out in two steps.

In the first step, we define the grid space as follows:

$$[-d,d]_h=\{z: z_r=rh,\ n=\overline{-K,K},\ Kh=d\}.$$

We introduce the Hilbert spaces $L_{2h}=L_2([-d,d]_h)$ and $W^2_{2h}=W^2_2([-d,d]_h)$ of the grid functions $\psi^h(z)=\{\psi^r\}_{-K}^K$ defined on $[-d,d]_h$, equipped with the norms

$$\left\|\psi^h\right\|_{L_{2h}}=\left(\sum_{z\in[-d,d]_h}\left|\psi^h(z)\right|^2h\right)^{1/2},$$

$$\left\|\psi^h\right\|_{W^2_{2h}}=\left\|\psi^h\right\|_{L_{2h}}+\left(\sum_{z\in[-d,d]_h}\left|\left(\psi^h\right)_{z\overline{z},r}\right|^2h\right)^{1/2}$$

respectively. To the differential operator A generated by problem (11), we assign the difference operator A^z_h by the formula

$$A^z_h\psi^h(z)=\{-(a(z)\psi_{\overline{z}}(z))_{z,r}-\beta\left(a(-z)\psi_{\overline{z}}(-z)\right)_{z,r}+\delta\psi^r\}_{-K+1}^{K-1},\tag{21}$$

acting in the space of grid functions $\psi^h(z)=\{\psi^r\}_{-K}^K$ satisfying the conditions $\psi^{-K}=\psi^K=0$. With the help of A^z_h, we arrive at the initial value problem

$$\begin{cases}\frac{du^{1h}(t,z)}{dt}+\mu u^{1h}(t,z)+A^z_hu^{1h}(t,z)=-F^h(t,z;u^{1h}(t,z),u^{2h}(t,z)),\\ \frac{du^{2h}(t,z)}{dt}+(\mu+\xi)\,u^{2h}(t,z)+A^z_hu^{2h}(t,z)=F^h(t,z;u^{1h}(t,z),u^{2h}(t,z))\\ -\Upsilon^h(t,z;u^{2h}(t,z)),\\ \frac{du^{3h}(t,z)}{dt}+\mu u^{3h}(t,z)+A^z_hu^{3h}(t,z)=\Upsilon^h(t,z;u^{2h}(t,z)),\ t\in(0,b),\ z\in[-d,d]_h,\\ u^{mh}(0,z)=\psi^m(z),\ m=\overline{1,3},\ z\in[-d,d]_h\end{cases}\tag{22}$$

for an infinite system of semilinear ordinary differential equations. In the second step, we replace problem (22) by DS (13)

$$\begin{cases}\frac{u^1_k-u^1_{k-1}}{\tau}+\mu u^1_k+A^z_hu^1_k=-F^h(t_k,z,u^1_k,u^2_k),\\ \frac{u^2_k-u^2_{k-1}}{\tau}+(\xi+\mu)\,u^2_k+A^z_hu^2_k=F^h(t_k,z,u^1_k,u^2_k)-\Upsilon^h(t_k,z,u^2_k),\\ \frac{u^3_k-u^3_{k-1}}{\tau}+\mu u^3_k+A^z_hu^3_k=\Upsilon^h(t_k,z,u^2_k),\ k=\overline{1,N},\\ u^m_0=\psi^m,\ m=\overline{1,3}.\end{cases}\tag{23}$$

Theorem 5. Let the assumptions of Theorem 2 be satisfied and $\mu+\delta>2\left(L_1+L_2\right)$. Then, there exists a unique solution $u^\tau=\{u_k\}_{k=0}^N$ of DS (23) that is bounded in $\mathbb{C}^3_\tau\left(L_{2h}\right)$ uniformly w.r.t. τ and h.

The proof of Theorem 5 is based on the main Theorem 4 and the symmetry properties of the difference operator A^z_h defined by formula (21).

Second, the initial-boundary value problem (12) for multidimensional system of SPDEs is considered. The discretization of problem (12) is also carried out in two steps. In the first step, let us define the grid sets

$$\overline{\Omega}_h = \left\{ z = z_r = (h_1 r_1, ..., h_n r_n),\ r = (r_1, ..., r_n),\ k = \overline{0, N_i},\ h_i N_i = 1,\ i = 1, ..., n \right\},$$

$$\Omega_h = \overline{\Omega}_h \cap \Omega, \quad S_h = \overline{\Omega}_h \cap S.$$

We introduce the Banach spaces $L_{2h} = L_2(\overline{\Omega}_h)$ and $W_{2h}^2 = W_2^2(\overline{\Omega}_h)$ of the grid functions $\psi^h(z) = \{\psi(h_1 r_1, ..., h_n r_n)\}$ defined on $\overline{\Omega}_h$, equipped with the norms

$$\left\| \psi^h \right\|_{L_{2h}} = \left(\sum_{z \in \overline{\Omega}_h} \left| \psi^h(z) \right|^2 h_1 \cdots h_n \right)^{1/2},$$

$$\left\| \psi^h \right\|_{W_{2h}} = \left\| \psi^h \right\|_{L_{2h}} + \left(\sum_{z \in \overline{\Omega}_h} \sum_{r=1}^{n} \left| \left(\psi^h \right)_{z_r \overline{z}_r, r_r} \right|^2 h_1 \cdots h_n \right)^{1/2}$$

respectively. To the differential operator A generated by problem (12), we assign the difference operator A_h^z by the formula

$$A_h^z u_z^h = -\sum_{r=1}^{n} \left(a_r(z) u_{\overline{z}_r}^h \right)_{z_r, r_r} \tag{24}$$

acting in the space of grid functions $u^h(z)$, satisfying the conditions $u^h(z) = 0$ for all $z \in S_h$. It is known that A_h^z is a self-adjoint positive definite operator in $L_2(\overline{\Omega}_h)$. With the help of A_h^z, we arrive at the initial value problem

$$\begin{cases} \frac{du^{1h}(t,z)}{dt} + \mu u^{1h}(t,z) + A_h^z u^{1h}(t,z) = -F^h(t,z;u^{1h}(t,z), u^{2h}(t,z)), \\ \frac{du^{2h}(t,z)}{dt} + (\mu + \xi) u^{2h}(t,z) + A_h^z u^{2h}(t,z) = F^h(t,z;u^{1h}(t,z), u^{2h}(t,z)) \\ -\Upsilon^h(t,z;u^{2h}(t,z)), \\ \frac{du^{3h}(t,z)}{dt} + \mu u^{3h}(t,z) + A_h^z u^{3h}(t,z) = \Upsilon^h(t,z;u^{2h}(t,z)),\ t \in (0,b),\ z \in \overline{\Omega}_h, \\ u^{mh}(0,z) = \psi^m(z),\ m = \overline{1,3},\ z \in \overline{\Omega}_h \end{cases} \tag{25}$$

for an infinite system of semilinear ordinary differential equations. In the second step, we replace problem (25) by DS (13)

$$\begin{cases} \frac{u_k^1 - u_{k-1}^1}{\tau} + \mu u_k^1 + A_h^z u_k^1 = -F^h(t_k, z, u_k^1, u_k^2), \\ \frac{u_k^2 - u_{k-1}^2}{\tau} + (\xi + \mu) u_k^2 + A_h^z u_k^2 = F^h(t_k, z, u_k^1, u_k^2) - \Upsilon^h(t_k, z, u_k^2), \\ \frac{u_k^3 - u_{k-1}^3}{\tau} + \mu u_k^3 + A_h^z u_k^3 = \Upsilon^h(t_k, z, u_k^2),\ k = \overline{1,N}, \\ u_0^m = \psi^m,\ m = \overline{1,3}. \end{cases} \tag{26}$$

Theorem 6. Let the assumptions of Theorem 3 be satisfied and $\mu + \delta > 2(L_1 + L_2)$. Then, there exists a unique solution $u^\tau = \{u_k\}_{k=0}^N$ of DS (26) that is bounded in $\mathbb{C}_\tau^3(L_{2h})$ uniformly w.r.t. h and τ.

The proof of Theorem 6 is based on Theorem 4 and the symmetry properties of the difference operator A_h^z defined by formula (24) and the theorem on the coercivity inequality of an elliptic problem in L_{2h} [13].

3 Numerical experiments

When analytical methods fail to provide exact solutions or become intractable, numerical methods play a crucial role in obtaining approximate solutions of partial differential equations. Over the years, numerous significant contributions have been made in this area, and various reliable techniques have been developed.

In the present section, we focus on the numerical approximation of the solution to a given initial-boundary value problem. Specifically, we employ a finite DS of first-order accuracy. To solve the resulting discrete system, we apply a modified Gauss elimination method.

Furthermore, we provide an error analysis for both the first-order and second-order accurate DSs, highlighting their performance and convergence behavior. We now consider the following initial-boundary value problem for a system of SPDEs:

$$\begin{cases} \Psi^1_t(t,z) + \nu\Psi^1(t,z) - \beta\Psi^1_{zz}(t,z) \\ \quad = (-1+\nu+\beta)e^{-t}\sin z - \sin\left(\Psi^1(t,z)\Psi^2(t,z) - e^{-2t}\sin^2 z\right), \\ \Psi^2_t(t,z) + (\mu+\xi)\Psi^2(t,z) - d\Psi^2_{zz}(t,z) \\ \quad = (-1+\nu+\xi+d)e^{-t}\sin z, \\ \qquad + \sin\left(\Psi^1(t,z)\Psi^2(t,z) - e^{-2t}\sin^2 z\right) - \cos\left(\Psi^2(t,z) - e^{-t}\sin z\right), \\ \Psi^3_t(t,z) + \nu\Psi^1(t,z) - \gamma\Psi^3_{zz}(t,z) \\ \quad = (-1+\nu+\gamma)e^{-t}\sin z + \cos\left(\Psi^2(t,z) - e^{-t}\sin z\right),\ t\in(0,1),\ z\in(0,\pi), \\ \Psi^m(0,z) = \sin z,\ z\in[0,\pi],\ m=\overline{1,3}, \\ \Psi^m(t,0) = \Psi^m(t,\pi) = 0,\ t\in[0,1],\ m=\overline{1,3}. \end{cases} \tag{27}$$

The spatial variable z may be treated as either discrete or continuous, depending on the context. In all cases, z represents population mobility, such as travel or migration between cities, towns, or even countries.

The exact solution of problem (27) is given by:

$$\Psi^m(t,z) = e^{-t}\sin z, \quad m=\overline{1,3}.$$

We now present a first-order accurate iterative DS for approximating the solution of the initial-boundary value problem (27):

$$\begin{cases} \frac{1}{\tau}\left({}_ru_n^{1,k} - {}_ru_n^{1,k-1}\right) + \nu\, {}_ru_n^{1,k} - \frac{\beta}{h^2}\left({}_ru_{n+1}^{1,k} - 2\,{}_ru_n^{1,k} + {}_ru_{n-1}^{1,k}\right) \\ \quad = (-1+\nu+\beta)e^{-t_k}\sin z_n - \sin\left({}_{(r-1)}u_n^{1,k}\,{}_{(r-1)}u_n^{2,k} - e^{-2t_k}\sin^2 z_n\right), \\ \frac{1}{\tau}\left({}_ru_n^{2,k} - {}_ru_n^{2,k-1}\right) + (\mu+\xi)\,{}_ru_n^{2,k} - \frac{d}{h^2}\left({}_ru_{n+1}^{2,k} - 2\,{}_ru_n^{2,k} + {}_ru_{n-1}^{2,k}\right) \\ \quad = (-1+\nu+\xi+d)e^{-t_k}\sin z_n \\ \qquad + \sin\left({}_{(r-1)}u_n^{1,k}\,{}_{(r-1)}u_n^{2,k} - e^{-2t_k}\sin^2 z_n\right) - \sin\left({}_{(r-1)}u_n^{2,k} - e^{-t_k}\sin z_n\right), \\ \frac{1}{\tau}\left({}_ru_n^{3,k} - {}_ru_n^{3,k-1}\right) + \nu\,{}_ru_n^{3,k} - \frac{\gamma}{h^2}\left({}_ru_{n+1}^{1,k} - 2\,{}_ru_n^{1,k} + {}_ru_{n-1}^{1,k}\right) \\ \quad = (-1+\nu+\gamma)e^{-t_k}\sin z_n + \sin\left({}_{(r-1)}u_n^{2,k} - e^{-t_k}\sin z_n\right), \\ t_k = k\tau,\ k=\overline{1,N},\ N\tau=1,\ z_n=nh,\ n=\overline{1,K-1},\ Kh=\pi, \\ {}_ru_n^{m,0} = \psi^m(z_n),\ {}_ru_0^{m,k} = {}_ru_K^{m,k} = 0,\ k=\overline{0,N}, \\ {}_0u_n^{m,k} \text{ is the initial guess},\ m=\overline{1,3},\ k=\overline{0,N},\ n=\overline{0,K}. \end{cases} \tag{28}$$

To solve the DS (28), we follow the iterative procedure described below. For each time step $k=\overline{0,N-1}$ and spatial index $n=\overline{0,K}$:

1. Initialize iteration with $r = 1$.
2. Assume $(r-1)u_n^{m,k}$ is known for all m.
3. Compute $ru_n^{m,k}$ using the difference equations.
4. If the maximum absolute error between $(r-1)u_n^{m,k}$ and $ru_n^{m,k}$ exceeds a prescribed tolerance, increment $r \to r+1$ and repeat from step 2. Otherwise, accept $ru_n^{m,k}$ as the solution.

The errors of numerical solutions are computed by

$$(rE^m)_K^N = \max_{\overline{k=1,N},\ \overline{n=1,K-1}} \left| \Psi^m(t_k, z_n) - ru_n^{m,k} \right|, \quad m = \overline{1,3},$$

where $\Psi^m(t_k, z_n)$ is the exact solution, and $ru_n^{m,k}$ is the numerical approximation at the grid point (t_k, z_n) for each m.

The results of the error computations for different grid resolutions are presented in Table 1.

Table 1

Maximum error $(rE^m)_K^N$ for different values of $N = K$ and $r = 6$

$(rE^m)_M^N$	$N = K = 20$	$N = K = 40$	$N = K = 80$
$m = 1$	0.0068	0.0032	0.0016
$m = 2$	0.0071	0.0033	0.0016
$m = 3$	0.0073	0.0034	0.0017

As observed in Table 1, when the values of N and K are doubled, the error decreases approximately by a factor of 1/2, which is consistent with the behavior of a first-order accurate finite DS as defined in equation (28). The numerical results confirm both the stability and the accuracy of the proposed DS.

Conclusion

In the present paper, we have established a theorem concerning the EU of a BS for a semilinear system of parabolic equations that models the spread of epidemics with a general semilinear incidence rate. The single-step DS of the w.r.t. for the numerical approximation of the semilinear system has been investigated.

Furthermore, we proved a theorem concerning the EU of a BS for the DS, uniformly w.r.t. the time step τ. The BSs of the semilinear parabolic system and its corresponding numerical scheme were derived. Finally, numerical results were presented for a test problem to illustrate the effectiveness and precision of the proposed DS. Applying methods from this paper and from papers [14] and [15] we can present similar results from this paper for a BS for a semilinear system of delay parabolic equations that models the spread of epidemics with a delay semilinear incidence rate.

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Author Contributions

All authors contributed equally to this work.

Conflict of Interest

The authors declare no conflict of interest.

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On a stable difference scheme for numerically solving a reverse parabolic source identification problem

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This article is devoted to the study of source identification problems for reverse parabolic partial differential equations with nonlocal boundary conditions. The principal aim of the work is to construct and analyze stable difference schemes that can be effectively employed for obtaining approximate solutions of such inverse problems. In particular, attention is focused on the Rothe difference scheme, and stability estimates for the corresponding discrete solutions are rigorously derived. These estimates guarantee the reliability and convergence of the proposed numerical method. A stability theorem for the solution of the difference scheme related to the source identification problem is proved. To establish the well-posedness of the underlying differential problem, the operator-theoretic approach is employed, ensuring a solid analytical foundation for the numerical method. Furthermore, the investigation is extended to an abstract setting for difference schemes, which is then applied to the numerical solution of reverse parabolic equations under boundary conditions of the first kind. This unified framework emphasizes both the theoretical justification and the computational effectiveness of the proposed approach. Finally, the efficiency of the developed method is demonstrated through a numerical illustration with a test example.

Keywords: reverse parabolic equation, inverse problem, difference scheme (DS), partial differential equation (PDE), source identification problem (SIP), self-adjoint positive definite operator (SAPDO), stability estimate, well-posedness.

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Introduction

In recent decades, the importance of SIPs in the mathematical modeling of real-world processes has grown significantly (see [1, 2]). Comprehensive reviews, detailed references, and classifications of recent studies devoted to SIPs for parabolic PDEs can be found in [3–5]. The solvability of various inverse problems for parabolic equations was investigated in [6–8], while the well-posedness of SIPs for hyperbolic–parabolic equations was analyzed in [9]. The work [10] focused on the identification of a space-dependent source term in the heat equation. A numerical algorithm for solving certain SIPs for parabolic equations backward in time was proposed in [11]. The authors of [12] examined the backward-in-time problem for a semilinear system of parabolic equations, whereas [13] developed a regularization technique for the spherically symmetric backward heat conduction problem. Moreover, a numerical approach for the backward heat conduction problem was introduced in [14]. In addition, several stable difference schemes for various direct nonlocal problems associated with reverse parabolic equations have been developed by different researchers (see, for instance, [15, 16] and the references therein).

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We denote by $\mathtt{H}$, a Hilbert space and by $\mathtt{A} : \mathtt{H} \to \mathtt{H}$, a SAPDO such that $\mathtt{A} > \delta \mathtt{I}$ for a real number $\delta > 0$, and $\mathtt{I}$ identity operator. Let $\gamma_k,\ \mu_k,\ k = 1, ..., s$ be given real numbers so that

$$|\mu_1| + ... + |\mu_s|\ < 1, \quad 0 \le \gamma_1 < \gamma_2 < ... < \gamma_s < 1 \tag{1}$$

hold.

We study SIP to search for a pair (v, p) that satisfies reverse parabolic equation

$$\frac{dv}{dt}(t) - \mathtt{A}v(t) = p + g(t), \quad 0 < t < 1 \tag{2}$$

and the following initial condition

$$v(0) = \phi \tag{3}$$

with a nonlocal condition

$$v(1) = \sum_{k\ =\ 1}^{s} \mu_k v(\gamma_k) + \varphi \tag{4}$$

for a given smooth function $g : [0, 1] \to \mathtt{H}$ and elements $\varphi,\ \phi \in \mathtt{H}$.

The well-posedness of the SIP (2)–(4) was established in the paper [17]. The aim of the current study is a stable DS for approximate solution of the SIP (2)–(4), under the assumption (1). Namely, we study the Rothe DS for approximate solution of this SIP and establish stability estimates for its solutions. Subsequently, this approach is employed to obtain stability estimates for the approximate solution of the SIP for a parabolic PDE. A numerical illustration of the test example is carried out.

1 Rothe DS

Denote by $[0, 1]_\tau = \{t_k = k\tau, k = 0, 1, \cdots, N,\ N\tau = 1\}$, the set of uniform grid points for any natural number N.

Let $C([0, 1]_\tau, \mathtt{H})$ denote a linear space of grid functions $\vartheta^\tau = \{\vartheta_k\}_1^N$ taking values in the space $\mathtt{H}$, and let $C_\tau(\mathtt{H}) = C([0, 1]_\tau, \mathtt{H}),\ C_\tau^\alpha(\mathtt{H}) = C^\alpha([0, 1]_\tau, \mathtt{H})$ be the corresponding Banach space of grid functions equipped with the appropriate norms

$$\|\vartheta^\tau\|_{C_\tau(\mathtt{H})} = \max_{1 \le k \le N} \|\vartheta_k\|_{\mathtt{H}},\ \|\vartheta^\tau\|_{C_\tau^\alpha(\mathtt{H})} = \|\vartheta^\tau\|_{C_\tau(\mathtt{H})} + \max_{1 \le k < k+r \le N} (r\tau)^{-\alpha} \|\vartheta_{k+r} - \vartheta_k\|_{\mathtt{H}},$$

where $\alpha \in (0, 1)$ is a given number.

Let us denote by $\mathtt{R} = (\mathtt{I}\ +\ \tau\ \mathtt{A})^{-1}$ the resolvent of $\mathtt{A}$. Then (see [18]) the estimates

$$\|\mathtt{R}^k\|_{\mathtt{H} \to \mathtt{H}} \le (1 + \delta\tau)^{-k},\ \|\tau \mathtt{R}^k\|_{\mathtt{H} \to \mathtt{H}} \le k^{-1}, \quad k \ge 1 \tag{5}$$

are valid. Let us $l_i = \left[\frac{\gamma_i}{\tau}\right],\ \rho_i = \frac{\gamma_i}{\tau} - l_i,\ i = 1, ..., s.$

Lemma 1. The operator

$$\mathbb{S}_\tau = \mathtt{I} - \left(1 - \sum_{i\ =\ 1}^{s} \mu_i\right) \mathtt{R}^N - \sum_{i\ =\ 1}^{s} \mu_i \mathtt{R}^{N - l_i}$$

has an inverse $\ \mathbb{T}_\tau = \mathbf{S}_\tau^{-1}$ and it is bounded such that:

$$\|\ \mathbb{T}_\tau\ \|_{\mathtt{H} \to \mathtt{H}} \le M. \tag{6}$$

Proof. Since operator $\left(\mathtt{I}-\mathtt{R}^{N}\right)$ and its inverse are bounded, operator $\mathbb{S}_{\tau}$ can be rewritten in the form

$$\mathbb{S}_{\tau}=\left(\mathtt{I}-\mathtt{R}^{N}\right)\left(1+\sum_{i\,=\,1}^{s}\mu_{i}\left(\mathtt{I}-\mathtt{R}^{N}\right)^{-1}\left(\mathtt{R}^{N}-\mathtt{R}^{N-l_{i}}\right)\right)=\left(\mathtt{I}-\mathtt{R}^{N}\right)\mathbb{Q}_{\tau}.$$

Hence, to complete the proof it is sufficient to prove that the operator $\mathbb{Q}_{\tau}$ is invertible and $\mathbf{Q}_{\tau}^{-1}$ is bounded. Spectral resolution of a SAPD operator (see [19]) and the assumption (1) give us

$$\|\,\mathbb{Q}_{\tau}^{-1}\,\|_{\mathtt{H}\to\mathtt{H}}\leq \sup_{\delta<\lambda<\infty}\frac{1}{\left|1+\sum\limits_{i\,=\,1}^{s}\mu_{i}\left(1-\ (1+\tau\lambda)\ ^{-N}\right)^{-1}\left((1\,+\,\tau\,\lambda)^{-N}-(1\,+\,\tau\,\lambda)^{-(N-l_{i})}\right)\right|}\leq\frac{1}{1-\sum\limits_{i\,=\,1}^{s}|\mu_{i}|}\leq M_{1}.$$

Therefore, the proof of Lemma 1 is complete. □

1.1 Stable DS

Now, we the consider the Rothe DS

$$\begin{cases}\tau^{-1}(\vartheta_{k}-\vartheta_{k-1})+\mathtt{A}\vartheta_{k-1}=g_{k}+p,\ g_{k}=g(t_{k}),\ 1\leq k\leq N,\\[2ex] \vartheta_{N}-\sum\limits_{i\,=\,1}^{s}\mu_{i}\ \vartheta_{l_{i}}=\varphi,\ \vartheta_{0}=\phi,\end{cases}\tag{7}$$

of approximate solution of the problem (2)-(3).

We now derive the solution of problem (7). One can see that a unique solution of the difference problem

$$\begin{cases}\tau^{-1}(\vartheta_{k}-\vartheta_{k-1})+\mathtt{A}\vartheta_{k-1}=g_{k}+p,\ 1\leq k\leq N,\\[2ex] \quad\vartheta_{N}\text{ is given}\end{cases}$$

exists and the formula

$$v_{k}=\mathtt{R}^{N-k}v_{N}+\tau\sum_{j=k+1}^{N}\mathtt{R}^{j-k}\left(p+g_{j}\right),\quad 0\leq k\leq N-1\tag{8}$$

holds. Applying formula (8) and the corresponding conditions, we get

$$\mathtt{R}^{N}\vartheta_{N}+\sum_{j=1}^{N}\mathtt{R}^{j}p\tau=\phi-\sum_{j=1}^{N}\mathtt{R}^{j}g_{j}\tau,$$

and

$$\left(\mathtt{I}-\sum_{i\,=\,1}^{s}\mu_{i}\mathtt{R}^{N-l_{i}}\right)\vartheta_{N}\ -\sum_{i\,=\,1}^{s}\mu_{i}\sum\nolimits_{j=l_{i}+1}^{N}\mathtt{R}^{j-l_{i}}p\tau=\sum_{i\,=\,1}^{s}\mu_{i}\sum\nolimits_{j=l_{i}+1}^{N}\mathtt{R}^{j-l_{i}}g_{j}\tau\ +\varphi.$$

Since

$$\sum_{j=1}^{N}\mathtt{R}^{j}\tau=\mathtt{A}^{-1}\ (\mathtt{I}-\mathtt{R}^{N}),\ \sum_{j=l_{i}+1}^{N}\mathtt{R}^{j-l_{i}}\tau=\mathtt{A}^{-1}\ (\mathbb{I}-\mathtt{R}^{N-l_{i}}),$$

we have that

$$\mathtt{R}^{N}\vartheta_{N}+\ (\mathtt{I}-\mathtt{R}^{N})\mathtt{A}^{-1}p=\phi-\sum_{j=1}^{N}\mathtt{R}^{j}g_{j}\tau\tag{9}$$

and

$$\left(\mathtt{I}-\sum_{i=1}^{s}\mu_i\mathtt{R}^{N-l_i}\right)\vartheta_N-\sum_{i=1}^{s}\mu_i(\mathtt{I}-\mathtt{R}^{N-l_i})\mathtt{A}^{-1}p=\sum_{i=1}^{s}\mu_i\sum_{j=l_i+1}^{N}\mathtt{R}^{j-l_i}g_j\tau+\varphi. \quad (10)$$

The determinant operator Δ for the system of equations (9) and (10) is defined by

$$\Delta=-\mathtt{R}^N\sum_{i=1}^{s}\mu_i(\mathtt{I}-\mathtt{R}^{N-l_i})-(\mathtt{I}-\mathtt{R}^N)\left(\mathtt{I}-\sum_{i=1}^{s}\mu_i\mathtt{R}^{N-l_i}\right)$$

$$=\mathtt{R}^N\sum_{i=1}^{s}\mu_i-\mathtt{I}+\mathtt{R}^N+\sum_{i=1}^{s}\mu_i\mathtt{R}^{N-l_i}=-\left[\mathtt{I}-\left(1-\sum_{i=1}^{s}\mu_i\right)\mathtt{R}^N-\sum_{i=1}^{s}\mu_i\mathtt{R}^{N-l_i}\right].$$

Using Lemma 1, we get

$$\vartheta_N=T_\tau\left\{\left(\phi-\sum_{j=1}^{N}\mathtt{R}^j g_j\tau\right)(\mathtt{I}-\mathtt{R}^N)\right.$$

$$\left.-\left(\sum_{i=1}^{s}\mu_i\sum_{j=l_i+1}^{N}\mathtt{R}^{j-l_i}g_j\tau+\varphi\right)\left(\mathtt{I}-\sum_{i=1}^{s}\mu_i\mathtt{R}^{N-l_i}\right)\right\} \quad (11)$$

and

$$\mathtt{A}^{-1}p=T_\tau\left\{\left(\sum_{i=1}^{s}\mu_i\sum_{j=l_i+1}^{N}\mathtt{R}^{j-l_i}g_j\tau+\varphi\right)\mathtt{R}^N-\left(\phi-\sum_{j=1}^{N}\mathtt{R}^j g_j\tau\right)\left(\mathtt{I}-\sum_{i=1}^{s}\mu_i\mathtt{R}^{N-l_i}\right)\right\}. \quad (12)$$

Therefore, DS (7) is uniquely solvable and defined by the formulas (8), (11) and (12).

Theorem 1. For the solution $(\{\vartheta_k\}_{k=1}^N, p)$ of problem (7) in $C_\tau(\mathtt{H})\times\mathtt{H}$, the following stability estimates

$$\| p\|_{\mathtt{H}}\leq \mathbb{M}_\delta\left(\|\mathtt{A}\phi\|_{\mathtt{H}}+\|\mathtt{A}\varphi\|_{\mathbb{H}}+\alpha^{-1}\|\{g_k\}_{k=1}^N\|_{C_\tau^\alpha(\mathtt{H})}\right), \quad (13)$$

$$\|\{\vartheta_k\}_{k=1}^N\|_{C_\tau(\mathtt{H})}\leq \mathbb{M}_\delta\left(\|\phi\|_{\mathtt{H}}+\|\varphi\|_{\mathtt{H}}+\|\{g_k\}_{k=1}^N\|_{C_\tau(\mathtt{H})}\right) \quad (14)$$

hold, where the value of $\mathbb{M}_\delta$ does not depend on $\tau,\alpha,\ \phi,\varphi$, and $\{g_k\}_{k=1}^N$.

Proof. From (12) it follows that

$$p=T_\tau\left\{\mathtt{A}\varphi-\mathbb{A}\mathbb{R}^N\phi-\tau\sum_{j=1}^{N}\mathbb{A}\mathbb{R}^{N-j+1}(g_j-g_N)-(\mathtt{I}-\mathtt{R}^N-\sum_{i=1}^{s}\mu_i(\mathtt{I}-\mathtt{R}^{l_i}))g_N\right.$$

$$\left.-\tau\sum_{i=1}^{s}\mu_i\left(\sum_{j=1}^{l_i}\mathbb{A}\,\mathbb{R}^{l_i-j+1}(g_j-g_N)\right)\right\}.$$

Applying to the right side of the last formula the Cauchy–Shwarz and triangle inequalities and estimates (5), (6), one can obtain estimate (13):

$$\|p\|_{\mathtt{H}}\leq\|T_\tau\|_{\mathtt{H}\to\mathtt{H}}\left(\|\mathtt{A}\varphi\|_{\mathtt{H}}+\left\|\mathbb{A}\mathbb{R}^N\right\|_{\mathtt{H}\to\mathtt{H}}\|\phi\|_{\mathtt{H}}+\sum_{j=1}^{N-1}\left\|\mathbb{A}\mathbb{R}^{N-j+1}\right\|_{\mathbb{H}\to\mathtt{H}}\|g_j-g_N\|_{\mathtt{H}}\tau\right.$$

$$\left.+\left(1+\left\|\mathtt{R}^N\right\|_{\mathtt{H}\to\mathtt{H}}+\sum_{i=1}^{s}|\mu_i|\left(1+\left\|\mathtt{R}^{l_i}\right\|_{\mathtt{H}\to\mathtt{H}}\right)\right)\|g_N\|_{\mathbb{H}}\right)$$

$$\leq\mathbb{M}_\delta\left(\|\phi\|_{\mathtt{H}}+\|\mathtt{A}\varphi\|_{\mathtt{H}}+\alpha^{-1}\|\{g_k\}_{k=1}^N\|_{C_\tau^\alpha(\mathtt{H})}\right).$$

Using relation (8), the triangle inequality and the estimates (5), (6), we show that

$$\|\vartheta_k\|_{\mathtt{H}} \leq \left\| \mathtt{R}^{\ k}\right\|_{\mathtt{H}\to\mathtt{H}} \|\phi\|_{\mathtt{H}} + \tau\sum_{j=1}^{k}\left\| \mathtt{R}^{\ k-j+1}\right\|_{\mathtt{H}\to\mathtt{H}} \|g_j\|_{\mathtt{H}}$$

$$+ \left(1+\left\| \mathtt{R}^{\ k}\right\|_{\mathtt{H}\to\mathtt{H}}\right)\|T_\tau\|_{\mathtt{H}\to\mathtt{H}}\Big\{\|\varphi\|_{\mathtt{H}}+\left\|\mathtt{R}^N\right\|_{\mathtt{H}\to\mathtt{H}}\|\phi\|_{\mathtt{H}}+\tau\ \sum_{j=1}^{N}\left\|\mathtt{R}^{N-j+1}\right\|_{\mathtt{H}\to\mathtt{H}}\|g_j\|_{\mathtt{H}}$$

$$+\tau\sum_{i\ =\ 1}^{s}|\mu_i|\sum_{j=1}^{l_i}\left\|\mathtt{R}^{l_i-j+1}\right\|_{\mathtt{H}\to\mathtt{H}}\|g_j\|_{\mathtt{H}}\Big\}\leq \mathbb{M}_\delta\left(\ \|\phi\|_{\mathtt{H}}+\|\ A\varphi\ \|_{\mathtt{H}}+\alpha^{-1}\ \|\ \{g_k\}_{k=1}^{N}\ \|_{C_\tau^\alpha(\mathtt{H})}\right)$$

for any index k. From that, the estimate (14) follows.

2 The boundary value problem and its approximation

Let $\Omega=(0,l)^n\subset\mathtt{R}^n$, $S=\partial\Omega,\overline{\Omega}=\Omega\cup S$ and (1) holds. Assume that $\phi\in\mathtt{L}_2(\Omega)$, $\varphi\in\mathtt{W}_2^2(\Omega)$ and $g\ \in C^\alpha(\mathtt{L}_2(\Omega))$, a_r are smooth functions such that $\forall x\in\Omega, a_r(x)\geq a_0>0,\ r=1,...,n,\ \sigma$ is a given positive real number.

Let us consider in $[0,1]\times\overline{\Omega}$, SIP for a multi-dimensional reverse parabolic PDE with the Dirichlet-type boundary condition

$$\begin{cases}\vartheta_t(x,t)+\sum\limits_{i=1}^{n}(a_i(x)\vartheta_{x_i}(x,t))_{x_i}-\sigma\vartheta(x,t)=g(x,t)+p(x),\quad 0<t<1,\ x=(x_1,\cdots,x_n)\in\Omega,\\ \vartheta(x,0)=\phi(x),\ \vartheta(x,1)=\sum\limits_{k\ =\ 1}^{s}\mu_k\vartheta(x,\gamma_k)+\varphi(x),\quad x\in\overline{\Omega},\\ \vartheta(x,t)=0,\quad 0\leq t\leq 1,\ x\in S.\end{cases}\tag{15}$$

The well-posedness of the SIP (15) was established in the paper [17].

Now, we will discretize SIP (15) in two steps. Let us take $h_rM_r=\ l,\ r=1,\cdots,n$. In the first step, we define the grid spaces $\widetilde{\Omega}_h=\{x=x_m=\ (h_1m_1,\cdots,h_nm_n);\ m=(m_1,\cdots,m_n),m_r=0,\cdots,M_r\}$, $\Omega_h=\widetilde{\Omega}_h\cap\Omega,\ S_h\ =\ \widetilde{\Omega}_h\cap S$ and the difference operator A_h^x by

$$\mathtt{A}_h^x\ \vartheta^h(x)=-\sum_{r=\ 1}^{n}\left(a_r(x)\vartheta_{\overline{x}_r}^h(x)\right)_{x_r,j_r}+\sigma\vartheta^h(x)$$

whose domain consists of all grid functions $\vartheta^h(x)$ satisfying the homogeneous boundary conditions $\vartheta^h(x)=0$ for all $x\in S_h$.

By using $\mathtt{A}_h^x$, we arrive at some infinite system of ordinary differential equations. Then, in the second step of discretization, we obtain the first-order of ADS

$$\begin{cases}\tau^{-1}\ \left(\vartheta_k^h(x)-\vartheta_{k-1}^h(x)\right)-\mathtt{A}_h^x\vartheta_{k-1}^h(x)=f^h(t_k,x)+p^h(x),\ t_k=\tau k,\quad 1\leq k\leq N,\ x\in\widetilde{\Omega}_h,\\ \vartheta_0^h(x)=\phi^h(x),\vartheta_N^h(x)=\sum\limits_{i\ =\ 1}^{s}\mu_i\ \vartheta_{l_i}^h(x)+\varphi^h(x),\quad x\in\widetilde{\Omega}_h,\ l_i=\left[\frac{s_i}{\tau}\right],\ i=1,...,s.\end{cases}\tag{16}$$

Let $\mathtt{L}_{2h}=\mathtt{L}_2(\widetilde{\Omega}_h)$ and $\mathtt{W}_{2h}^2=\mathtt{W}_2^2(\widetilde{\Omega}_h)$ be spaces of the grid functions $\vartheta^h(x)=\{\vartheta(h_1m_1,\cdots,h_nm_n)\}$

defined on $\widetilde{\Omega}_h$, equipped with the norms

$$\left\|\vartheta^h\right\|_{\mathtt{L}_{2h}} = \left(\sum_{x\in\widetilde{\Omega}_h} |\vartheta^h(x)|^2 h_1\cdots h_n\right)^{1/2}, \quad \left\|\vartheta^h\right\|_{\mathtt{W}^2_{2h}} = \left\|\vartheta^h\right\|_{L_{2h}}$$

$$+\left(\sum_{x\in\widetilde{\Omega}_h}\sum_{r=1}^n \left|(\vartheta^h)_{x_r}\right|^2 h_1\cdots h_n\right)^{1/2} + \left(\sum_{x\in\widetilde{\Omega}_h}\sum_{r=1}^n \left|(\vartheta^h(x))_{x_r\overline{x_r},\, m_r}\right|^2 h_1\cdots h_n\right)^{1/2}.$$

Denote by $\mathcal{C}_\tau(\mathtt{L}_{2h}) = \mathcal{C}([0,1]_\tau, \mathtt{L}_{2h})$, the Banach space of $\mathtt{L}_{2h}$-valued grid functions $\vartheta^\tau = \{\vartheta_k\}_1^N$ with norm

$$\|\vartheta^\tau\|_{\mathcal{C}_\tau(\mathtt{L}_{2h})} = \max_{1\le k\le N} \|\vartheta_k\|_{\mathtt{L}_{2h}}.$$

Let $\mathcal{C}^\alpha(\mathtt{L}_{2h}) = \mathcal{C}^\alpha([0,1]_\tau, \mathtt{L}_{2h})$ and $\mathcal{C}^\alpha_\tau(\mathbb{L}_{2h}) = \mathcal{C}^\alpha_\tau([0,1]_\tau, \mathtt{L}_{2h})$ be respectively the Hölder space and the weighted Hölder space with the norms defined by (1) for $\mathbb{H} = \mathtt{L}_{2h}$.

Theorem 2. Assume that τ and $|h| = \sqrt{h_1^2 + \cdots + h_n^2}$ are sufficiently small positive numbers, $\phi^h \in \mathtt{L}_{2h}$, $\varphi^h \in D(\mathtt{A}_h^x)$, $\left\{g_k^h\right\}_1^N \in \mathcal{C}^\alpha_\tau(\mathtt{L}_{2h})$. Then, for the solutions of DS (16), the following stability estimates hold:

$$\left\|p^h\right\|_{\mathcal{C}_\tau(L_{2h})} \le \mathbb{M}_\delta \left(\left\|\phi^h\right\|_{L_{2h}} + \left\|\mathtt{A}_h^x\varphi^h\right\|_{L_{2h}} + \alpha^{-1}\left\|\left\{g_k^h\right\}_1^N\right\|_{\mathcal{C}^\alpha_\tau(L_{2h})}\right),$$

$$\left\|\left\{\vartheta_k^h\right\}_1^N\right\|_{\mathcal{C}_\tau(L_{2h})} \le \mathbb{M}_\delta \left(\left\|\phi^h\right\|_{L_{2h}} + \left\|\varphi^h\right\|_{L_{2h}} + \left\|\left\{g_k^h\right\}_1^N\right\|_{\mathcal{C}_\tau(L_{2h})}\right),$$

where M_δ is independent of τ, $\phi^h(x)$, $\varphi^h(x)$, and $g_k^h(x)$, $k = 1, \cdots, N-1$.

The proof of Theorem 2 is based on estimates **(**13**)**, (14), the theorem on the coercivity inequality for the solution of the elliptic difference problem in L_{2h} ([20]) and the triangle inequality.

3 Numerical algorithm and example

In $[0,\pi]\times[0,1]$, we consider a test example to search for a pair of functions $(p(x), \nu(x,t))$ for SIP of reverse parabolic equation so that

$$\begin{cases} \nu_t(x,t) + (1+3x)^2\,\nu_{xx}(x,t) + 6\,(1+3x)\,\nu_x(x,t) - \nu(x,t) = p(x) + g(x,t), \ 0<x<\pi, \ 0<t<1, \\ \nu(x,0) = \ \phi(x), \ \nu(x,1) = \sum\limits_{k=1}^{3} \mu_k\ \nu(x,s_k) + \varphi(x), \quad 0\le x\le\pi, \\ \nu(0,t) = 0, \ \nu(1,t) = 0, \ 0\le t\le 1. \end{cases} \tag{17}$$

Here $\zeta(x) = \sin(x)$, $\phi(x) = \zeta(x)$, $\mu_1 = \mu_2 = \mu_3 = \frac{1}{6}$, $s_1 = 0.3$, $s_2 = 0.5$, $s_3 = 0.7$, $g(x,t) = \left(\left(-4-(1+3x)^2\right)\zeta(x) + 6\,(1+3x)\cos(x)\right)e^{-3t}$, $\varphi(x) = \left(1 - \frac{1}{6}\left(e^{-0.9} + e^{-1.5} + e^{-2.1}\right)\right)\zeta(x)$. The exact solution is $\left(\zeta(x), e^{-3t}\zeta(x)\right)$.

We use the algorithm to solve (17). It contains three steps. In the first step we search for solution of an auxiliary direct problem without source

$$\begin{cases} \omega_t(x,t) + (1+3x)^2\,\omega_{xx}(x,t) + 6\,(1+3x)\,\omega_x(x,t) - \omega(x,t) \\ = (1+3x)^2\,\phi_{xx}(x,t) + 6\,(1+3x)\,\phi_x(x,t) + g(x,t),\ 0<t<1,\ 0<x<\pi, \\ \omega(x,1) - \sum\limits_{k=1}^{3} \mu_k \omega(x,s_k) \ = \ \varphi(x),\ \ 0 \le x \le \pi, \\ \omega(0,t) = 0,\ \omega(1,t) = 0,\ 0 \le t \le 1. \end{cases} \tag{18}$$

Later, in the second step, we find a source function using the formula

$$p(x) = (1+3x)^2\,\omega_{xx}(x,0) + 6\,(1+3x)\,\omega_x(x,0) - \omega(t,0).$$

Finally, in the third step we put in the right side of the reverse parabolic PDE and solve it to get the solution $\nu(x,t)$,

Applying (16), we have the following DS

$$\begin{cases} \tau^{-1}\left(\omega_k^n - \omega_{k-1}^n\right) + (1+3x_n)^2\,h^{-2}\left(\omega_{k-1}^{n+1} - 2\omega_{k-1}^n + \omega_{k-1}^{n-1}\right) \\ +6\,(1+3x_n)\,(2h)^{-1}\left(\omega_{k-1}^{n+1} - \omega_{k-1}^{n-1}\right) - \omega_{k-1}^n = (1+3x_n)^2 h^{-2}\left(\phi^{n+1} - 2\phi^n + \phi^{n-1}\right) \\ +6\,(1+3x_n)\,(2h)^{-1}\left(\phi^{n+1} - \phi^{n-1}\right) + g_k^n, t_k = k\,\tau,\ 1 \le k \le N,\ \ x_n = n\,h,\ 1 \le n \le M-1, \\ \omega_N^n - \frac{1}{6}\left(\omega_{l_1} + \omega_{l_2} + \omega_{l_3}\right) = \varphi^n,\ 0 \le n \le M, \\ \omega_k^0 = 0,\ \omega_k^M = 0,\ 0 \le k \le N \end{cases} \tag{19}$$

for approximate solution (18). The approximate value of p at grid points x_n is calculated using the formula

$p_n = (1+3x_n)\ h^{-2}\left(\omega_0^{n+1} - 2\omega_0^n + \omega_0^{n-1}\right) + \ 6\,(1+3x_n)\,(2h)^{-1}\left(\omega_0^{n+1} - \omega_0^{n-1}\right) - \ \omega_0^n,$
$n = 1, ..., M-1.$

DS (19) can be rewritten in the matrix form

$$A_n\omega_{n+1} + B_n\omega_n + C_n\,\omega_{n-1} = I\theta_n,\ n = 1,\cdots,M-1,\ \omega_0 = \overrightarrow{0},\ \omega_M = \overrightarrow{0}. \tag{20}$$

Here, $\omega_n = \left[\ {\omega_0}^n \ \cdots \ \omega_N^0\ \right]^t$, $\omega^{n\pm1} = \left[\ {\omega_0}^{n\pm1} \ \cdots \ \omega_N^{n\pm1}\ \right]^t$, $\theta_n = \left[\ {\theta_0}^n \ \cdots \ \theta_N^{\ n}\ \right]^t$ are $(N+1)\times1$ column vectors, A_n, B_n, C_n are $(N+1)^2$ square matrices, I is the $(N+1)^2$ identity matrix,

$$A_n = \begin{bmatrix} 0 & \ldots & 0 & 0 \\ & & & 0 \\ & a_n I & & \vdots \\ & & & 0 \end{bmatrix},\quad C_n = \begin{bmatrix} 0 & \ldots & 0 & 0 \\ & & & 0 \\ & c_n I & & \vdots \\ & & & 0 \end{bmatrix},$$

$$B_n = \begin{bmatrix} 0 & 0 & 0 & \cdots & -\frac{1}{6} & \cdots & 0 & 0 & 1 \\ d & b_n & 0 & \cdots & 0 & \cdots & 0 & 0 & 0 \\ 0 & d & b_n & \cdots & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \cdots & \cdots & \cdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & \cdots & d & b_n & 0 \\ 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & d & b_n \end{bmatrix},$$

$$a_n = (1+3x_n)^2 \ h^{-2} + \ 6\,(1+3x_n)\,(2h)^{-1}\,, \ b_n = -1 - \tfrac{1}{\tau} - 2\,(1+3x_n)^2 \ h^{-2},$$
$$c_n = (1+3x_n)^2 \ h^{-2} - 6\,(1+3x_n)\,(2h)^{-1}\,, \ d = -\tfrac{1}{\tau},$$

$$\theta_0^n = \psi_n, \ n = 1, \cdots, M-1,$$
$$\theta_k^n = g(t_k, x_n) + \ (1+3x_n)^2\, h^{-2}\left(\phi^{n+1} \ - 2\phi^n \ + \phi^{n-1}\ \right)$$
$$+6\,(1+3x_n)\,(2h)^{-1}\left(\phi^{n+1} \ - \phi^{n-1}\ \right), k = 1, \ \cdots, N, \ n = 1, \cdots, M-1.$$

We use the modified Gauss elimination method to solve (20).

Numerical results are carried out using MATLAB. The numerical solutions of DS are evaluated for distinct values of (N, M). ω_n^k represents the numerical value of $\nu(t,x)$ at $(t,x) = (t_k, x_n)$ and p_n is the numerical value of $p(x)$ at $x = x_n$. The errors in the numerical solutions are computed by

$$E\nu = \max_{0\le k\le N}\left(\sum_{n=1}^{M-1} |\nu(x_n, t_k) - \nu_k^n|^2\, h\right)^{\frac{1}{2}},$$
$$Ep_M = \left(\sum_{n=1}^{M-1} |p(x_n) - p_n|^2\, h\right)^{\frac{1}{2}}.$$

In Table 1 we give the error between the exact solution and the numerical solution of the difference scheme for distinct values of N and M. The table demonstrates that doubling the grid resolution results in approximately a twofold reduction in error.

Table 1

Error analysis

DS | $N = M$	20	40	80
$E\nu$	1.308×10^{-2}	6.723×10^{-3}	3.447×10^{-3}
Ep	1.432×10^{-2}	7.012×10^{-3}	3.465×10^{-3}

Conclusion

In this work we consider SIPs for reverse parabolic PDEs with initial and nonlocal boundary conditions. The main goal is to develop and analyze stable difference schemes, particularly the Rothe scheme, for accurate numerical solutions. Stability estimates are rigorously proved, ensuring reliability and convergence. The well-posedness of the problem is established, providing a strong analytical basis. The study also extends to an abstract setting of difference schemes and applies the results to reverse parabolic equations with first-kind boundary conditions. Numerical experiments confirm the effectiveness of the proposed method.

In future work, we plan to construct and analyze high-order accurate and stable difference schemes for the approximate solution of such SIPs.

Author Contributions

All authors contributed equally to this work.

Conflict of Interest

The authors declare no conflict of interest.

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https://doi.org/10.31489/2025M4/95-106 *Research article*

Inequalities for analytic functions associated with hyperbolic cosine function

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In this paper, we investigate the geometric properties of a specific subclass of analytic functions satisfying the condition $f'(z) \prec cosh(\sqrt{z})$ meaning that the function $f'(z)$ is subordinate to the function $cosh(\sqrt{z})$. Also, we focus on deriving sharp inequalities for Taylor coefficients, particularly for b_2 and the modulus of the second derivative $f''(z)$. Utilizing the Schwarz lemma, both on the unit disc and on its boundary, we provide essential insights into the distortion and growth behaviors of these functions. The paper demonstrates the sharpness of these inequalities through extremal functions and applies the Julia–Wolff lemma to establish boundary behavior results. These findings contribute significantly to the understanding of the analytic functions associated with the hyperbolic cosine function, with potential applications in geometric function theory. It is considered that the extremal functions obtained in this study could be potential hyperbolic activation functions in neural network architectures. This perspective builds a conceptual bridge between geometric function theory and artificial intelligence, indicating that insights from complex analysis can inspire the development of more effective and theoretically grounded activation mechanisms in deep learning. Empirical evaluation of architectures built with novel activation functions may be considered as potential future work.

Keywords: Schwarz estimate, angular derivative, the principle of subordination, activation function, extremal function, analytic function, Julia–Wolff lemma, angular limit, Schwarz lemma at the boundary, the unit disc

2020 Mathematics Subject Classification: 30C80.

Introduction

Let $\mathcal{A}$ represent the class of functions of the form $f(z) = z + b_2 z^2 + b_3 z^3 + ...$, analytic in the unit disc $D = \{z : |z| < 1\}$. Also, let $\mathcal{W}$ be the subclass of $\mathcal{A}$ satisfying the condition

$$f'(z) \prec \cosh\sqrt{z},$$

where the symbol "$\prec$" indicates the principle of subordination [1]. Also, we choose the branch of the square root function so that

$$\cosh\sqrt{z} = 1 + \frac{z}{2!} + \frac{z^2}{4!} + \frac{z^3}{6!} + ...$$

The conformal mapping $\cosh\sqrt{z} : D \to \mathbb{C}$ maps the unit disc D onto the region

$$\left\{\alpha \in \mathbb{C} : \left|\ln\left(\alpha + \sqrt{\alpha^2 - 1}\right)\right| < 1\right\}$$

defined on the principal branch of the logarithm and the square root function [2].

Determining the upper bound for Taylor coefficients has been a key area of focus in understanding geometric properties, offering important insights into different subclasses of $\mathcal{W}$. In this section, we

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establish an upper bound for b_2, a coefficient of the function $f(z)$. To achieve this, we will apply the Schwarz lemma. Furthermore, the following section will evaluate the modulus of the second derivative of $f(z)$ from below, requiring the use of the Schwarz lemma on the boundary.

The Schwarz lemma asserts that for any analytic function $p(z)$ mapping the unit disc D into itself and satisfying $p(0) = 0$, $|p(z)| \leq |z|$ for $z \in D$ and $|p'(0)| \leq 1$. In simpler terms, it asserts that if a function maps the unit disc into itself and maps its origin to the origin, then the function cannot magnify the distances between points inside the disc by more than 1 [1].

Extending the Schwarz lemma to the boundary of the unit disc offers deep insights into the behavior of analytic functions close to this boundary, proving especially beneficial when studying functions that approach the disc's edge. In this paper, we set out to examine the application and significance of this remarkable theorem to various classes of functions. The Schwarz lemma implies the boundary Schwarz lemma $|p'(1)| \geq 1$. Osserman and Unkelbach [3,4] showed that in this case, we have in fact

$$\left|p'(1)\right| \geq 1 + \frac{1 - |p'(0)|}{1 + |p'(0)|} = \frac{2}{1 + |p'(0)|},$$

where p satisfies the conditions of the Schwarz lemma, p extends continuously to the boundary point $1 \in \partial D = \{z : |z| = 1\}$, $|p(1)| = 1$ and $p'(1)$ exists. These inequalities are sharp. In mathematical literature, these inequalities and their generalizations are topics of continuous discussion and hold great importance in the geometric theory of functions [5–7]. Some properties of analytic function classes related to the Jack and the Schwarz lemmas were studied in [8]. In [9], a new bound for the Schwarz inequality was obtained for analytic functions mapping the unit disk onto itself.

If we use the principle of subordination for the class we defined above, there exists a Schwarz function $p(z)$ such that

$$f'(z) = \cosh\sqrt{p(z)}.$$

Here, the function $p(z)$ meets the criteria of the Schwarz lemma [1]. Therefore, applying the Schwarz lemma, we derive

$$f''(z) = \frac{p'(z)}{2\sqrt{p(z)}} \sinh\sqrt{p(z)}$$

and

$$f''(0) = \frac{p'(0)}{2} \lim_{z\to 0} \frac{\sinh\sqrt{p(z)}}{\sqrt{p(z)}}.$$

Therefore, we have

$$f''(0) = \frac{p'(0)}{2}$$

and

$$\left|f''(0)\right| \leq \frac{1}{2}.$$

We will now demonstrate that the final inequality is sharp. Let

$$f(z) = 2\sqrt{z}\sinh\sqrt{z} - 2\cosh\sqrt{z} + 2.$$

Then,

$$f'(z) = \cosh\sqrt{z},$$

$$f''(z) = \frac{1}{2\sqrt{z}} \frac{\sinh\sqrt{z}}{\sqrt{z}}$$

and

$$\left|f''(0)\right| = \frac{1}{2}.$$

Lemma 1. If $f \in \mathcal{W}$, then

$$\left|f''(0)\right| \leq \frac{1}{2}.$$

This result is sharp, as demonstrated by the extremal function

$$f(z) = 2\sqrt{z}\sinh\sqrt{z} - 2\cosh\sqrt{z} + 2.$$

The subsequent lemma, referred to as the Julia–Wolff lemma, is required for the following discussion [10].

Lemma 2 (Julia–Wolff lemma). If p is an analytic function in the unit disc D with $p(0) = 0$ and $p(D) \subset D$, and additionally, p has an angular limit $p(1)$ at $1 \in \partial D$ where $|p(1)| = 1$, then the angular derivative $p'(1)$ exists and $1 \leq |p'(1)| \leq \infty$.

1 *Main results*

This section focuses on examining the second derivative of the analytic function $f(z)$. During this analysis, we will derive stronger inequalities by considering the coefficients of the Taylor series expansion of $f(z)$. Additionally, we will provide an inequality that demonstrates the relationship between these coefficients.

Theorem 1. Let $f(z) \in \mathcal{W}$. Suppose that, for $1 \in \partial D$, f has an angular limit $f(1)$ at 1, $f'(1) = \cosh 1$. Then we have the inequality

$$\left|f''(1)\right| \geq \frac{\sinh 1}{2}. \tag{1}$$

This result is sharp, with equality for the function

$$f(z) = 2\sqrt{z}\sinh\sqrt{z} - 2\cosh\sqrt{z} + 2.$$

Proof. Let

$$f'(z) = \cosh\sqrt{p(z)}.$$

Then,

$$f''(z) = \frac{p'(z)}{2\sqrt{p(z)}}\sinh\sqrt{p(z)},$$

$$f''(1) = \frac{p'(1)}{2\sqrt{p(1)}}\sinh\sqrt{p(1)}$$

and

$$f''(1) = \frac{p'(1)}{2}\sinh 1.$$

Since the function $p(z)$ satisfies the conditions of the Schwarz lemma at the boundary, we obtain

$$1 \leq \left|p'(1)\right| = \frac{2\left|f''(1)\right|}{\sinh 1}$$

and

$$\left|f''(1)\right| \geq \frac{\sinh 1}{2}.$$

Now, we will prove that inequality (1) is sharp. Let

$$f(z) = 2\sqrt{z}\sinh\sqrt{z} - 2\cosh\sqrt{z} + 2.$$

Then,

$$f''(z) = \frac{1}{2\sqrt{z}} \frac{\sinh \sqrt{z}}{\sqrt{z}}$$

and

$$\left| f''(1) \right| = \frac{\sinh 1}{2}.$$

□

Theorem 2. Assuming the same conditions as in Theorem 1, we obtain

$$\left| f''(1) \right| \geq \frac{\sinh 1}{1 + 2\left| f''(0) \right|}. \tag{2}$$

Inequality (2) is sharp, achieving equality for the function

$$f(z) = 2\sqrt{z} \sinh \sqrt{z} - 2\cosh \sqrt{z} + 2.$$

Proof. Let $p(z)$ be as defined in the proof of Theorem 1. Thus, by the Schwarz lemma on the boundary,

$$\frac{2}{1 + \left| p'(0) \right|} \leq \left| p'(1) \right| = \frac{2\left| f''(1) \right|}{\sinh 1}.$$

Since

$$\left| p'(0) \right| = 2\left| f''(0) \right|,$$

we take

$$\frac{2}{1 + 2\left| f''(0) \right|} \leq \frac{2\left| f''(1) \right|}{\sinh 1}$$

and

$$\left| f''(1) \right| \geq \frac{\sinh 1}{1 + 2\left| f''(0) \right|}.$$

Next, we will demonstrate that inequality (2) is sharp. Consider

$$f(z) = 2\sqrt{z} \sinh \sqrt{z} - 2\cosh \sqrt{z} + 2.$$

Then, we have

$$\left| f''(1) \right| = \frac{\sinh 1}{2}.$$

However, we also have

$$z + b_2 z^2 + b_3 z^3 + ... = 2\sqrt{z} \sinh \sqrt{z} - 2\cosh \sqrt{z} + 2,$$

$$1 + 2b_2 z + 3b_3 z^2 + ... = \cosh \sqrt{z}$$

and

$$2b_2 + 6b_3 z + ... = \frac{1}{2} \frac{\sinh \sqrt{z}}{\sqrt{z}}.$$

Upon taking the limit as z approaches 0 in the final equation, we find that $b_2 = \frac{1}{4}$. Consequently, this yields

$$\frac{\sinh 1}{1 + 2\left| f''(0) \right|} = \frac{\sinh 1}{2}.$$

□

Theorem 3. Under the conditions of Theorem 1, we obtain

$$\left|f'(1)\right| \geq \frac{\sinh 1}{2}\left(1+\frac{2\left(1-4\left|b_2\right|\right)^2}{1-16\left|b_2\right|^2+\left|6b_3-\frac{2}{3}b_2^2\right|}\right). \tag{3}$$

The bound is sharp with the extremal function given by

$$f(z)=\sinh z.$$

Proof. Let $p(z)$ denote the same function as in the proof of Theorem 1 and let $u(z)=z$. According to the maximum principle, for every $z \in D$, it follows that $|p(z)| \leq |u(z)|$. Therefore, $h(z)=\frac{p(z)}{u(z)}$ is an analytic function and $|h(z)|<1$ for $|z|<1$. By Taylor expansion of the function $p(z)$, we have $p(z)=c_1z+c_2z^2+c_3z^3+\dots$. Thus, we take

$$h(z)=\frac{p(z)}{u(z)}=\frac{c_1z+c_2z^2+c_3z^3+\ \dots}{z}=c_1+c_2z+c_3z^2+\ \dots\ ,$$

$$|h(0)|=|c_1|$$

and

$$\left|h'(0)\right|=|c_2|\,.$$

Through straightforward computations, we obtain

$$1+2b_2z+3b_3z^2+\dots=\cosh\sqrt{p(z)}=1+\frac{p(z)}{2!}+\frac{p(z)^2}{4!}+\frac{p(z)^3}{6!}+\ \dots\ ,$$

$$\begin{aligned} 2b_2z+3b_3z^2+\dots &= \frac{c_1z+c_2z^2+c_3z^3+\dots}{2!}+\frac{\left(c_1z+c_2z^2+c_3z^3+\ \dots\ \right)^2}{4!}+\ \dots \\ &= \frac{1}{2!}z\left(c_1+c_2z+c_3z^2+\dots\right)+\frac{1}{4!}z^2\left(c_1+c_2z+c_3z^2+\dots\right)^2+\dots, \end{aligned}$$

$$2b_2+3b_3z+\dots=\frac{1}{2!}\left(c_1+c_2z+c_3z^2+\dots\right)+\frac{1}{4!}z\left(c_1+c_2z+c_3z^2+\dots\right)^2+\dots,$$

$$b_2=\frac{1}{4}c_1$$

and

$$3b_3=\frac{1}{2}c_2+\frac{1}{24}c_1^2.$$

Thus, based on the expression for $h(z)$, we have

$$|h(0)|=4\left|b_2\right| \tag{4}$$

and

$$\left|h'(0)\right|=\left|6b_3-\frac{2}{3}b_2^2\right|.$$

The combined function

$$w(z)=\frac{h(z)-h(0)}{1-\overline{h(0)}h(z)}$$

is analytic in the unit disc D, $w(0)=0$, $|w(z)|<1$ for $z\in D$ and $|w(1)|=1$ for $1\in\partial D$. By the Schwarz lemma on the boundary, we obtain

$$\frac{2}{1+|w'(0)|}\leq\left|w'(1)\right|=\frac{1-|h(0)|^2}{\left|1-\overline{h(0)}h(1)\right|^2}\left|h'(1)\right|\leq\frac{1+|h(0)|}{1-|h(0)|}\left(\left|p'(1)\right|-\left|u'(1)\right|\right).$$

Since

$$\left|w'(0)\right|=\frac{|h'(0)|}{1-|h(0)|^2}=\frac{\left|6b_3-\frac{2}{3}b_2^2\right|}{1-16\left|b_2\right|^2},$$

we take

$$\frac{2}{1+\frac{\left|6b_3-\frac{2}{3}b_2^2\right|}{1-16|b_2|^2}}\leq\frac{1+4\left|b_2\right|}{1-4\left|b_2\right|}\left(\frac{2\left|f''(1)\right|}{\sinh 1}-1\right)$$

and

$$\left|f'(1)\right|\geq\frac{\sinh 1}{2}\left(1+\frac{2\left(1-4\left|b_2\right|\right)^2}{1-16\left|b_2\right|^2+\left|6b_3-\frac{2}{3}b_2^2\right|}\right).$$

We will now demonstrate that the inequality (3) achieves equality. Consider

$$f(z)=\sinh z.$$

Then,

$$f'(z)=\cosh z$$

and

$$f''(1)=\sinh 1.$$

On the other hand, we have

$$z+b_2z^2+b_3z^3+...=\sinh z=z+\frac{z^3}{3!}+\frac{z^5}{5!}+...,$$

$$b_2z^2+b_3z^3+...=\frac{z^3}{3!}+\frac{z^5}{5!}+...$$

and

$$b_2+b_3z+...=\frac{z}{3!}+\frac{z^3}{5!}+\ ...\ .$$

Passing to the limit $(z\to 0)$ in the last equality yields $b_2=0$. Similarly, using straightforward calculations, we obtain $b_2=\frac{1}{3!}$. Therefore, we obtain

$$\frac{\sinh 1}{2}\left(1+\frac{2\left(1-4\left|b_2\right|\right)^2}{1-16\left|b_2\right|^2+\left|6b_3-\frac{2}{3}b_2^2\right|}\right)=\sinh 1.$$

□

Theorem 4. Let $f\in\mathcal{W}$ and $f(z)-z$ has no critical point in D except $z=0$ and $b_2>0$. Then

$$\left|3b_3-\frac{1}{3}b_2^2\right|\leq 4\left|b_2\ln\left(4b_2\right)\right|. \tag{5}$$

This result is sharp.

Proof. Given that $b_2 > 0$ in the expression of the function $f(z)$, and considering inequality (4), assuming that $f(z) - z$ has no critical point in D except $z = 0$, we denote the analytic branch of the logarithm by $\ln h(z)$, normalized under the condition

$$\ln h(0) = \ln(4b_2) < 0.$$

The fractional function

$$\Theta(z) = \frac{\ln h(z) - \ln h(0)}{\ln h(z) + \ln h(0)}$$

is analytic in the unit disc D, $|\Theta(z)| < 1$ for $z \in D$ and $\Theta(0) = 0$. By the Schwarz lemma, we obtain

$$1 \geq \left|\Theta'(0)\right| = \frac{|2\ln h(0)|}{|\ln h(0) + \ln h(0)|^2}\left|\frac{h'(0)}{h(0)}\right| = \frac{-1}{2\ln h(0)}\left|\frac{h'(0)}{h(0)}\right| = -\frac{\left|6b_3 - \frac{2}{3}b_2^2\right|}{8b_2\ln(4b_2)}$$

and

$$\left|3b_3 - \frac{1}{3}b_2^2\right| \leq 4b_2\ln(4b_2).$$

Now, we will show that inequality (5) is sharp. Let

$$p(z) = ze^{\frac{1+z}{1-z}\ln 4b_2} = zg(z),$$

where $g(z) = e^{\frac{1+z}{1-z}\ln 2b_1}$. Thus, we have

$$g(z) \;=\; \frac{p(z)}{z} = \frac{c_1 z + c_2 z^2 + c_3 z^3 + ...}{z} = c_1 + c_2 z + c_3 z^2 + \;...\;.$$

Then

$$g(0) = c_1 = 4b_2, \quad g'(0) = c_2 = 6b_3 - \frac{2}{3}b_2^2.$$

Following straightforward computations, we obtain

$$g'(z) = \frac{2}{(1-z)^2}\ln(4b_2)\, e^{\frac{1+z}{1-z}\ln(4b_2)}$$

and

$$g'(0) = 8b_2\ln(4b_2).$$

Thus, we obtain

$$\left|3b_3 - \frac{1}{3}b_2^2\right| = 4\left|b_2\ln(4b_2)\right|.$$

□

Based on the findings from [5], this theorem derives the modulus of the function's derivative at point 1 by considering its Taylor expansions around two points.

Theorem 5. Let $f \in \mathcal{W}$ and $f'(a) = 1$ for $0 < |a| < 1$. Suppose that for $1 \in \partial D$, f has an angular limit $f(1)$ at 1, $f'(1) = \cosh 1$. Then, we have the inequality

$$|f''(1)| \geq \frac{\sinh 1}{2}\Bigg(1 + \frac{1-|a|^2}{|1-a|^2} + \frac{|a|-2|f''(0)|}{|a|+2|f''(0)|} \times \left[1 + \frac{|a|^2+4|f''(0)||f''(a)|\left(1-|a|^2\right)-2|f''(a)|\left(1-|a|^2\right)-2|f''(0)|}{|a|^2+4|f''(0)||f''(a)|\left(1-|a|^2\right)+2|f''(a)|\left(1-|a|^2\right)+2|f''(0)|}\frac{1-|a|^2}{|1-a|^2}\right]\Bigg). \tag{6}$$

Inequality (6) is sharp, with equality for each possible value of $|f''(0)|$ and $|f''(a)|$.

Proof. According to the Schwarz–Pick lemma [1], we have

$$\left|\frac{s(z)-s(a)}{1-\overline{s(a)}s(z)}\right| \leq \left|\frac{z-a}{1-\overline{a}z}\right| = |\rho(z)|$$

and

$$|s(z)| \leq \frac{|s(a)|+|\rho(z)|}{1+|s(a)|\,|\rho(z)|}, \tag{7}$$

where $s: D \to D$ is an analytic function and $a \in D$. If $t: D \to D$ is analytic and $0 < |a| < 1$, letting $s(z) = \frac{t(z)-t(0)}{z\left(1-\overline{t(0)}t(z)\right)}$ in (7), we obtain

$$\left|\frac{t(z)-t(0)}{z\left(1-\overline{t(0)}t(z)\right)}\right| \leq \frac{\left|\frac{t(a)-t(0)}{a\left(1-\overline{t(0)}t(a)\right)}\right| + |\rho(z)|}{1+\left|\frac{t(a)-t(0)}{a\left(1-\overline{t(0)}t(a)\right)}\right| |\rho(z)|}$$

and

$$|t(z)| \leq \frac{|t(0)| + |z|\,\frac{|A|+|\rho(z)|}{1+|A||\rho(z)|}}{1+|t(0)|\,|z|\,\frac{|A|+|\rho(z)|}{1+|A||\rho(z)|}}, \tag{8}$$

where

$$A = \frac{t(a)-t(0)}{a\left(1-\overline{t(0)}t(a)\right)}.$$

If we take

$$t(z) = \frac{p(z)}{z\frac{z-a}{1-\overline{a}z}},$$

then, we have

$$t(0) = \frac{p'(0)}{-a}, t(a) = \frac{p'(a)\left(1-|a|^2\right)}{a}$$

and

$$A = \frac{\frac{p'(a)\left(1-|a|^2\right)}{a} + \frac{p'(0)}{a}}{a\left(1+\overline{\frac{p'(0)}{a}}\frac{p'(a)\left(1-|a|^2\right)}{a}\right)},$$

where $|A| \leq 1$. Let $|t(0)| = b$ and let

$$B = \frac{\left|\frac{p'(a)\left(1-|a|^2\right)}{a}\right| + \left|\frac{p'(0)}{a}\right|}{|a|\left(1+\left|\frac{p'(0)}{a}\right|\left|\frac{p'(a)\left(1-|a|^2\right)}{a}\right|\right)}.$$

From (8), we obtain

$$|p(z)| \leq |z|\,|\rho(z)|\,\frac{b+|z|\,\frac{B+|\rho(z)|}{1+B|\rho(z)|}}{1+b\,|z|\,\frac{B+|\rho(z)|}{1+B|\rho(z)|}}$$

and

$$\frac{1-|p(z)|}{1-|z|} \geq \frac{1+b\,|z|\,\frac{B+|\rho(z)|}{1+B|\rho(z)|} - b\,|z|\,|\rho(z)| - |z|^2\,|\rho(z)|\,\frac{B+|\rho(z)|}{1+B|\rho(z)|}}{(1-|z|)\left(1+b\,|z|\,\frac{B+|\rho(z)|}{1+B|\rho(z)|}\right)} = I.$$

Let $V(z)=1+b\left|z\right|\frac{B+|\rho(z)|}{1+B|\rho(z)|}$ and $R(z)=1+B\left|\rho(z)\right|$. Considering the functions $V(z)$ and $R(z)$ in the earlier inequality, we obtain

$$I=\frac{1}{V(z)R(z)}\left\{\frac{1-|z|^2\left|\rho(z)\right|}{1-|z|}+B\left|\rho(z)\right|\frac{1-|z|^2}{1-|z|}+bB\left|z\right|\frac{1-\left|\rho(z)\right|^2}{1-|z|}\right\}. \tag{9}$$

Since

$$\lim_{z\to 1}V(z)=\lim_{z\to 1}\left(1+b\left|z\right|\frac{B+|\rho(z)|}{1+B\left|\rho(z)\right|}\right)=1+b,$$

$$\lim_{z\to 1}R(z)=\lim_{z\to 1}\left(1+B\left|\rho(z)\right|\right)=1+B$$

and

$$1-\left|\rho(z)\right|^2=1-\left|\frac{z-a}{1-\overline{a}z}\right|^2=\frac{\left(1-|a|^2\right)\left(1-|z|^2\right)}{\left|1-\overline{a}z\right|^2},$$

passing to the angular limit in (9) yields

$$\left|p'(1)\right|\geq 1+\frac{1-|a|^2}{|1-a|^2}+\frac{1-b}{1+b}\left[1+\frac{1-B}{1+B}\frac{1-|a|^2}{|1-a|^2}\right].$$

Moreover, since

$$\frac{1-b}{1+b}=\frac{1-|t(0)|}{1+|t(0)|}=\frac{1-\left|\frac{p'(0)}{a}\right|}{1+\left|\frac{p'(0)}{a}\right|}=\frac{|a|-|p'(0)|}{|a|+|p'(0)|}=\frac{|a|-2\left|f''(0)\right|}{|a|+2\left|f''(0)\right|},$$

$$\frac{1-B}{1+B}=\frac{1-\dfrac{\left|\frac{p'(a)\left(1-|a|^2\right)}{a}\right|+\left|\frac{p'(0)}{a}\right|}{|a|\left(1+\left|\frac{p'(0)}{a}\right|\left|\frac{p'(a)\left(1-|a|^2\right)}{a}\right|\right)}}{1+\dfrac{\left|\frac{p'(a)\left(1-|a|^2\right)}{a}\right|+\left|\frac{p'(0)}{a}\right|}{|a|\left(1+\left|\frac{p'(0)}{a}\right|\left|\frac{p'(a)\left(1-|a|^2\right)}{a}\right|\right)}},$$

$$\frac{1-B}{1+B}=\frac{|a|^2+2\left|f''(0)\right|2\left|f''(a)\right|\left(1-|a|^2\right)-2\left|f''(a)\right|\left(1-|a|^2\right)-2\left|f''(0)\right|}{|a|^2+2\left|f''(0)\right|2\left|f''(a)\right|\left(1-|a|^2\right)+2\left|f''(a)\right|\left(1-|a|^2\right)+2\left|f''(0)\right|}$$

and

$$\frac{1-m}{1+m}=\frac{|a|^2+4\left|f''(0)\right|\left|f''(a)\right|\left(1-|a|^2\right)-2\left|f''(a)\right|\left(1-|a|^2\right)-2\left|f''(0)\right|}{|a|^2+4\left|f''(0)\right|\left|f''(a)\right|\left(1-|a|^2\right)+2\left|f''(a)\right|\left(1-|a|^2\right)+2\left|f''(0)\right|},$$

we obtain

$$\left|p'(1)\right|\geq 1+\frac{1-|a|^2}{|1-a|^2}+\frac{|a|-2\left|f''(0)\right|}{|a|+2\left|f''(0)\right|}$$

$$\times\left[1+\frac{|a|^2+4\left|f''(0)\right|\left|f''(a)\right|\left(1-|a|^2\right)-2\left|f''(a)\right|\left(1-|a|^2\right)-2\left|f''(0)\right|}{|a|^2+4\left|f''(0)\right|\left|f''(a)\right|\left(1-|a|^2\right)+2\left|f''(a)\right|\left(1-|a|^2\right)+2\left|f''(0)\right|}\frac{1-|a|^2}{|1-a|^2}\right].$$

From the definition of $p(z)$, we have

$$\left|p'(1)\right| = \frac{2\left|f''(1)\right|}{\sinh 1}.$$

We thus obtain inequality (6).

Let a be any real number in the interval $(-1, 0)$, and let c and d be arbitrary real numbers such that $0 < c = |p'(0)| < |a|$, $0 < d = |p'(a)| < \frac{|a|}{\left(1-|a|^2\right)}$ to show that inequality (6) is sharp. Let

$$\mathbb{T} = \frac{\frac{c}{a} + \frac{d\left(1-|a|^2\right)}{a}}{a\left(1 + cd\frac{1-|a|^2}{a^2}\right)} = \frac{1}{a^2}\frac{d\left(1-|a|^2\right)+c}{1+cd\frac{1-|a|^2}{a^2}}.$$

Consider the function

$$p(z) = z\frac{z-a}{1-\overline{a}z}\frac{\frac{-c}{a} + z\frac{\mathbb{T}+\rho(z)}{1+\rho(z)\mathbb{T}}}{1-\frac{c}{a}z\frac{\mathbb{T}+\rho(z)}{1+\rho(z)\mathbb{T}}}. \tag{10}$$

From equation (10), after performing straightforward calculations, we derive $p'(0) = c$ and $p'(a) = d$. Therefore, we obtain

$$p'(1) = 1 + \frac{1-a^2}{(1-a)^2} + \frac{a+c}{a-c}\left(1 + \frac{a^2+cd\left(1-|a|^2\right)-d\left(1-|a|^2\right)-c}{a^2+cd\left(1-|a|^2\right)+d\left(1-|a|^2\right)+c}\frac{1-a^2}{(1-a)^2}\right).$$

By selecting appropriate signs for the numbers a, c and d, we can infer from the final equation that inequality (6) is sharp. □

2 Conclusions and discussion

In this paper, geometric properties of a specific subclass of analytic functions satisfying the condition $f'(z) \prec cosh\sqrt{z}$ are investigated. Considering the Schwarz lemma and the boundary Schwarz lemma, significant results on distortion and growth behaviours of these functions have been obtained. Accordingly, two extremal functions have been based on the results of theorems presented in this paper.

The extremal functions obtained in this paper have been considered as activation functions for artificial neural networks. There are already studies in the literature that examine the use of extremal functions as activation functions [11, 12]. In [11], the authors propose a complex-valued activation function obtained using the Schwarz lemma. The authors stated that effective results have been obtained in both classification and function approximation problems according to simulation results. In [12], similar functions obtained in this study are presented as activation functions.

There are also various studies that propose hyperbolic functions to be used as activation functions, which is also valid for our study [13–15]. In one of the recent studies, hyperbolic sine has been used for deep learning in Tensorflow and Keras [13]. In [14], hyper-sinh-convolutional network has been proposed for early detection of Parkinson's disease from spiral drawings. Husein et al. used a hyperbolic activation function to achieve effective instance image retrieval [15]. In our study, we present two hyperbolic activation functions, defined as $g(z) = sinhz$ and $q(z) = 2\sqrt{z}sin\sqrt{z} - 2\cosh\sqrt{z} + 2$. At this point, it is worth noting that the activation functions defined in our study are not arbitrarily selected but they emerge as intuitive outcomes of the problem addressed in this study.

In conclusion, this paper aims to strengthen the connection between complex analysis and artificial intelligence in this paper by introducing the use of extremal functions within neural network architectures. We consider that the obtained results show that mathematical findings from geometric function theory can inspire new directions in neural network research. Empirical evaluation of the new activation functions across various learning tasks and architectures to fully assess their practical impact and limitations can be considered as potential future work.

Author Contributions

T. Azeroğlu and B. N. Örnek were responsible for developing the theoretical framework that underpins the study. T. Düzenli contributed significantly to preparation of the manuscript and provided valuable input to the Conclusions and Discussion section. All authors jointly revised the manuscript and approved the final submission.

Conflict of Interest

The authors declare no conflict of interest.

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Analysis and classification of fixed points of operators on a simplex

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This paper investigates the dynamical behavior of Lotka–Volterra type operators defined on the four and five dimensional simplexes, focusing on their fixed points and structural representation through directed graphs (tournaments). For several classes of such operators, we derive algebraic and combinatorial conditions under which the configuration of fixed points exhibits transitive, cyclic, or homogeneous structures. Using methods from algebraic graph theory, Lyapunov stability theory, and Young's inequality, explicit criteria are established for the existence, uniqueness, and stability of interior and boundary fixed points. A detailed analysis is provided for the class of operators whose associated skew-symmetric matrices are in general position. The connection between the minors of these matrices and the orientation of arcs in the tournament is clarified, revealing how dynamical transitions correspond to changes in tournament type. Furthermore, we demonstrate that under certain parameter regimes, fixed points coincide with evolutionarily stable strategies (ESS) in replicator dynamics, thus bridging discrete population models and evolutionary game theory. The obtained results enrich the theory of quadratic stochastic and Lotka–Volterra operators, providing new insights into nonlinear mappings on simplexes, combinatorial dynamics, and applications to models of interacting populations.

Keywords: Lotka–Volterra mapping, simplex dynamics, fixed points, replicator dynamics, evolutionary stability, directed graphs, tournaments, cyclic structures, Lyapunov function, nonlinear systems.

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Introduction

A number of applied studies are devoted to the investigation of dynamical systems — both continuous and discrete — as well as systems involving fractional-order derivatives [1–3]. To this day, all three types of systems remain relevant; however, they differ in the methods of analysis and in the nature of the results obtained [4–6]. The application areas of such models are wide-ranging and include medicine (covering problems in epidemiology, oncology, and population genetics), ecology, economics, computer virology, and many others [7–9]. Building on these applications, we now turn to the theoretical foundations of a particular class of discrete dynamical systems — the so-called quadratic stochastic operators — which play a central role in many models, especially in population genetics and game dynamics.

Let us start by recalling the known facts that we will rely on in the article, as well as recalling the works of some authors on its topic. It is known that [10], a $(m-1)$-dimensional standard simplex in $\mathbb{R}^m$ is defined as the relation

$$S^{m-1} = \{x = (x_1, ..., x_m) : \ x_i \geq 0, \ \sum_{i=1}^{m} x_i = 1\} \subset \mathbb{R}^m.$$

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It is easy to verify that S^{m-1} is a convex and compact subset of $\mathbb{R}^m$.

A class of mappings defined on S^{m-1} known as *quadratic stochastic operators* was introduced by Bernstein [11] and further developed by R.N. Ganikhodzhaev in [12, 13]. Such mappings are defined by a set of coefficients $P_{ij,k}$ for $i, j, k = 1, \dots, m$, satisfying the conditions

$$P_{ij,k} = P_{ji,k} \geq 0, \quad \sum_{k=1}^{m} P_{ij,k} = 1,$$

and act according to the equations

$$x'_k = (Vx)_k = \sum_{i,j=1}^{m} P_{ij,k} x_i x_j, \quad k = 1, \dots, m.$$

This mapping was introduced by R.N. Ganikhodzhaev in [12].

Definition 1. A quadratic stochastic mapping is called a Lotka–Volterra mapping if the inheritance coefficients satisfy the condition $P_{ij,k} = 0$ for all $k \notin \{i, j\}$.

It is known (see [14]) that any Lotka–Volterra mapping defined on S^{m-1} can be represented as

$$x'_k = x_k(1 + \sum_{i=1}^{m} a_{ki} x_i), \quad k = 1, \dots, m, \tag{1}$$

where

$$a_{ki} = \begin{cases} 2P_{ik,k} - 1, & \text{if } i \neq k, \\ 0, & \text{if } i = k, \end{cases} \quad \text{with } |a_{ki}| \leq 1, \quad k, i = 1, \dots, m. \tag{2}$$

Here, $A = (a_{ki})$ is a real skew-symmetric matrix, satisfying $A = -A^T$, where A^T denotes the transpose of A.

Definition 2. [15] A skew-symmetric matrix is called a matrix of general position if all of its principal minors of even order are nonzero.

Since $a_{ki} = -a_{ik}$, all off-diagonal entries are antisymmetric. In particular, $a_{ki} \neq 0$ for $i \neq k$ if and only if the corresponding $P_{ij,k} \neq \frac{1}{2}$.

It is known that each skew-symmetric matrix in general position can be associated with a complete-oriented graph (tournament) [15].

Let $A = (a_{ki})$ be a skew-symmetric matrix in general position associated with Lotka–Volterra mapping (1), where the coefficients satisfy conditions (2). We place m points on a plane and label them $1, 2, \dots, m$. For each pair of distinct indices $i \neq k$, we draw a directed edge from vertex i to vertex k if $a_{ik} > 0$ (equivalently, $a_{ki} < 0$).

This construction defines a well-posed directed graph. We then call the constructed graph the tournament of dynamic system (1) with the skew-symmetric matrix $A = (a_{ki})$ and denote it by T_m.

A directed graph is called a tournament if, for every pair of distinct vertices i and k, exactly one of the edges (i, k) or (k, i) is present. A graph in which every two vertices are connected by an edge is called a complete graph. If each edge of a complete graph is assigned a direction, the resulting directed graph is a tournament [16–18].

Two tournaments are said to be isomorphic if there exists a bijection between their vertex sets that preserves the direction of all edges.

It is known that there are 12 pairwise non-isomorphic tournaments with 5 vertices [17].

A tournament is called *strong* if, for any two vertices, there exists a directed path from one to the other. Among the 12 tournaments with 5 vertices, 6 are strong [15].

A tournament is said to be *transitive* if it contains no strong subtournaments. Equivalently, a tournament is transitive if it does not contain any directed cycles. Among the tournaments with 5 vertices, exactly 1 is transitive, 6 are strong, and the remaining 5 are neither strong nor transitive.

Definition 3. [15] A tournament is homogeneous if every sub-tournament is either strong or transitive.

In this paper, we study the structure of the set of fixed points (referred to as the *card of fixed points*) and characterize the fixed points of strong and homogeneous tournaments.

Every face of the simplex S^{m-1} is invariant under the Lotka–Volterra mapping, and the restriction V to this face is also a Lotka-Volterra mapping [12–14].

In recent works [19–21] Lotka–Volterra mappings have been studied from the perspective of dynamical systems, population genetics, and game theory. A particularly fruitful approach is to analyze their fixed points and dynamical behavior via combinatorial structures such as tournaments and their geometric realizations on simplex [22–24]. Lotka–Volterra mappings are popular in modeling the spread of viral diseases. In [25–27], degenerate Lotka–Volterra mappings and their applications were considered.

In this paper, we focus on the structure of the set of fixed points — referred to as the *card of fixed points* — for various types of Lotka–Volterra operators V. We pay special attention to operators corresponding to strong and homogeneous tournaments. Also explore conditions for the existence of fixed points on the interior and the faces of the simplex, as well as criteria for their stability and evolutionary significance.

Additionally, we establish links with replicator dynamics and evolutionary game theory, including conditions under which fixed points of the system can be interpreted as evolutionary stable strategies (ESS).

1 *Card of fixed points*

Introduce the following notation:

$$P_\alpha = \{x \in \Gamma_\alpha : A_\alpha x \geq 0\}, \qquad Q_\alpha = \{x \in \Gamma_\alpha : A_\alpha x \leq 0\},$$

where Γ_α denotes the face of the simplex S^{m-1} corresponding to the index set $\alpha \subset I = \{1, 2, \ldots, m\}$, and A_α is the submatrix of A corresponding to the indices in α.

It is known [14], each of the sets P_α and Q_α contains a unique fixed point. In some cases, it is possible that $P_\alpha = Q_\alpha$.

The set of all fixed points of the operator V, $\mathrm{Fix}(V) = \{x \in S^{m-1} : Vx = x\}$ can be represented as a set of points in a plane. For each $\alpha \subset I$, the fixed point P_α is connected to the fixed point Q_α by a directed arc pointing from P_α to Q_α. The resulting directed graph is called the *card of fixed points* of the operator V, and is denoted by G_V [14, 15].

Definition 4. Two fixed points (vertices of the graph G_V) $x(\alpha)$ and $x(\beta)$ are called *adjacent* if the following conditions hold:

1. $|\alpha| = |\beta|$,
2. $|\alpha \cap \beta| = |\alpha| - 1$,

where $|\alpha|$ denotes the number of elements in $\alpha \subset I = \{1, 2, \ldots, m\}$.

In other words, $x(\alpha)$ and $x(\beta)$ correspond to faces of the same dimension and their supports differ by exactly one index.

For example, all vertices of the simplex (corresponding to one-element subsets) are pairwise adjacent. However, the fixed points $x(\{2, 3, 5\})$ and $x(\{1, 2, 4\})$ are not adjacent.

Theorem 1. Any two adjacent vertices in the graph G_V are connected by a directed arc.

Proof. Let $x(\alpha)$ and $x(\beta)$ be adjacent vertices of G_V, corresponding to the subsets $\alpha, \beta \subset I = \{1, 2, \ldots, m\}$. By definition of adjacency, $|\alpha| = |\beta|$, and $|\alpha \cap \beta| = |\alpha| - 1$. Let $\gamma = \alpha \cup \beta$, so that $|\gamma| = |\alpha| + 1$.

Let us denote $\gamma = \{i_1, i_2, \ldots, i_t\}$, with $t = |\gamma|$. Then, without loss of generality, we may assume

$$\alpha = \{i_2, i_3, \ldots, i_t\}, \quad \beta = \{i_1, i_2, \ldots, i_{t-1}\}.$$

Now consider the restriction of the mapping V to the face $\Gamma_\gamma \subset S^{m-1}$. Since $x(\alpha)$ and $x(\beta)$ lie in Γ_γ, we consider the action of the submatrix A_γ from the skew-symmetric matrix A on the face Γ_γ.

Recall the property of Lotka–Volterra mappings on invariant faces: for a fixed point $x \in \Gamma_\gamma$,

$$\operatorname{supp} x \cap \operatorname{supp}(A_\gamma x) = \varnothing, \quad \operatorname{supp} x \cup \operatorname{supp}(A_\gamma x) = \gamma.$$

That is, the nonzero coordinates of $A_\gamma x$ are complementary to the support of x within γ.

Applying this to $x(\alpha)$, which has support $\alpha = \{i_2, \ldots, i_t\}$, we obtain that $(A_\gamma x(\alpha))_{i_1} \neq 0$, and all other coordinates of $A_\gamma x(\alpha)$ vanish. Similarly, since $\beta = \{i_1, \ldots, i_{t-1}\}$, the only nonzero coordinate of $A_\gamma x(\beta)$ is $(A_\gamma x(\beta))_{i_t} \neq 0$.

We now consider the signs of these nonzero coordinates. If

$$\operatorname{sign}(A_\gamma x(\alpha))_{i_1} \cdot \operatorname{sign}(A_\gamma x(\beta))_{i_t} < 0,$$

then, the directions of the corresponding arcs go from one to the other, and $x(\alpha)$ and $x(\beta)$ form a directed pair (P_α, Q_α), meaning they are connected by an arc in G_V.

If the signs are the same, then both $x(\alpha)$ and $x(\beta)$ would have outgoing arcs in the same direction on the face Γ_γ, which contradicts the uniqueness of the sink (i.e., the unique point with all incoming arcs) in the fixed point diagram on Γ_γ.

Hence, in either case, the pair $(x(\alpha), x(\beta))$ must be connected by a directed arc in G_V. □

2 Main results

Consider the general form of the Lotka–Volterra operator V_1:

$$V_1 : \begin{cases} x_1' = x_1(1 - a_{12}x_2 - a_{13}x_3 - a_{14}x_4 + a_{15}x_5), \\ x_2' = x_2(1 + a_{12}x_1 - a_{23}x_3 - a_{24}x_4 - a_{25}x_5), \\ x_3' = x_3(1 + a_{13}x_1 + a_{23}x_2 - a_{34}x_4 - a_{35}x_5), \\ x_4' = x_4(1 + a_{14}x_1 + a_{24}x_2 + a_{34}x_3 - a_{45}x_5), \\ x_5' = x_5(1 - a_{15}x_1 + a_{25}x_2 + a_{35}x_3 + a_{45}x_4). \end{cases} \tag{3}$$

The operator V_1 corresponds to the strong and homogeneous tournament shown in Figure 1.

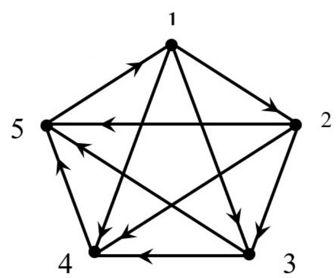

Figure 1. The tournament associated with the operator V_1

The corresponding skew-symmetric matrix $A_1 = (a_{ij})$ associated with mapping (3) has the form:

$$A_1 = \begin{pmatrix} 0 & -a_{12} & -a_{13} & -a_{14} & a_{15} \\ a_{12} & 0 & -a_{23} & -a_{24} & -a_{25} \\ a_{13} & a_{23} & 0 & -a_{34} & -a_{35} \\ a_{14} & a_{24} & a_{34} & 0 & -a_{45} \\ -a_{15} & a_{25} & a_{35} & a_{45} & 0 \end{pmatrix}.$$

In order for the operator V_1 to correspond to a matrix in general position, it is required that all even-order principal minors of the matrix A_1 be nonzero.

For second-order minors, the condition $a_{ki} > 0$ ensures their positivity. Calculating the principal minors of order four (there are five such minors), we obtain:

$$\Delta_1^{11} = (a_{23}a_{45} + a_{25}a_{34} - a_{24}a_{35})^2, \quad \Delta_2^{22} = (a_{15}a_{34} + a_{14}a_{35} - a_{13}a_{45})^2,$$
$$\Delta_3^{33} = (a_{15}a_{24} + a_{14}a_{25} - a_{12}a_{45})^2, \quad \Delta_4^{44} = (a_{15}a_{23} + a_{13}a_{25} - a_{12}a_{35})^2,$$
$$\Delta_5^{55} = (a_{14}a_{23} + a_{12}a_{34} - a_{13}a_{24})^2.$$

Since the matrix A_1 is in general position, all even-order principal minors are nonzero, i.e., $\Delta_i^{ii} \neq 0$ for all $i = 1, \dots, 5$.

Let us define the expressions inside the squares as:

$$\Delta_1 = a_{23}a_{45} + a_{25}a_{34} - a_{24}a_{35},$$
$$\Delta_2 = a_{15}a_{34} + a_{14}a_{35} - a_{13}a_{45},$$
$$\Delta_3 = a_{15}a_{24} + a_{14}a_{25} - a_{12}a_{45},$$
$$\Delta_4 = a_{15}a_{23} + a_{13}a_{25} - a_{12}a_{35},$$
$$\Delta_5 = a_{14}a_{23} + a_{12}a_{34} - a_{13}a_{24}.$$

Theorem 2. If $\Delta_2, \Delta_3, \Delta_4 > 0$, then the card of the fixed point operator V_1 is transitive (Figure 2)

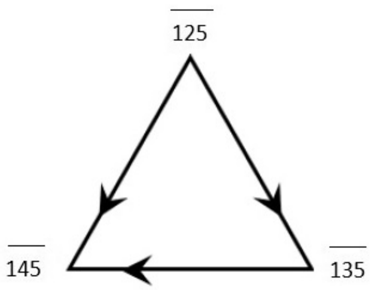

Figure 2. The transitive card of the fixed point

Proof. As shown in Figure 1, the tournament contains three cyclic triples: $\overline{125}, \overline{135}, \overline{145}$. These correspond to the following fixed points:

$$M_{125} = \left(\frac{a_{25}}{a_{12} + a_{15} + a_{25}}, \frac{a_{15}}{a_{12} + a_{15} + a_{25}}, 0, 0, \frac{a_{12}}{a_{12} + a_{15} + a_{25}} \right),$$
$$M_{135} = \left(\frac{a_{35}}{a_{13} + a_{15} + a_{35}}, 0, \frac{a_{15}}{a_{13} + a_{15} + a_{35}}, 0, \frac{a_{13}}{a_{13} + a_{15} + a_{35}} \right),$$
$$M_{145} = \left(\frac{a_{45}}{a_{14} + a_{15} + a_{45}}, 0, 0, \frac{a_{15}}{a_{14} + a_{15} + a_{45}}, \frac{a_{14}}{a_{14} + a_{15} + a_{45}} \right),$$

where all coefficients are assumed to be positive.

Now, define the following functions:

$$\varphi_{125}(x) = (x_1^{a_{25}} x_2^{a_{15}} x_5^{a_{12}})^{\frac{1}{a_{12}+a_{15}+a_{25}}}, \quad \varphi_{135}(x) = (x_1^{a_{35}} x_3^{a_{15}} x_5^{a_{13}})^{\frac{1}{a_{13}+a_{15}+a_{35}}},$$

$$\varphi_{145}(x) = (x_1^{a_{45}} x_4^{a_{15}} x_5^{a_{14}})^{\frac{1}{a_{14}+a_{15}+a_{45}}}.$$

We now apply "Young's inequality" [28], which states that for any $c_k \geq 0$, $p_k \geq 0$, such that $\sum\limits_{k=1}^{m} p_k = 1$, the following holds:

$$\prod_{k=1}^{m} c_k^{p_k} \leq \sum_{k=1}^{m} c_k p_k.$$

Using this, one derives the following estimates:

$$\varphi_{125}(Vx) \leq \frac{\varphi_{125}(x)}{\Delta_{125}} \left(\Delta_{125} - \Delta_4 x_3 - \Delta_3 x_4\right), \tag{4}$$

$$\varphi_{135}(Vx) \leq \frac{\varphi_{135}(x)}{\Delta_{135}} \left(\Delta_{135} + \Delta_4 x_2 - \Delta_2 x_4\right), \tag{5}$$

$$\varphi_{145}(Vx) \leq \frac{\varphi_{145}(x)}{\Delta_{145}} \left(\Delta_{145} + \Delta_3 x_2 + \Delta_2 x_3\right). \tag{6}$$

Here, the constants are:

$$\Delta_{125} = a_{12} + a_{15} + a_{25}, \quad \Delta_{135} = a_{13} + a_{15} + a_{35}, \quad \Delta_{145} = a_{14} + a_{15} + a_{45}.$$

We now determine the directions of arcs between the fixed points:

1. "Between M_{125} and M_{135}": In inequalities (4) and (5), the term involving Δ_4 appears with opposite signs. If $\Delta_4 > 0$, then in (4) this term decreases $\varphi_{125}(Vx)$, while in (5) it increases $\varphi_{135}(Vx)$. This implies the direction of the fixed-point flow is $M_{125} \to M_{135}$.

2. "Between M_{135} and M_{145}": In inequalities (5) and (6), Δ_2 appears with opposite signs. If $\Delta_2 > 0$, this implies the direction $M_{135} \to M_{145}$.

3. "Between M_{125} and M_{145}": Comparing (4) and (6), if $\Delta_3 > 0$, the sign of the corresponding term shows the direction $M_{125} \to M_{145}$.

As a result, all three fixed points are connected in a consistent directed order:

$$M_{125} \to M_{135} \to M_{145} \leftarrow M_{125},$$

and the resulting subgraph forms a transitive triangle, as shown in Figure 2. □

Let $Vx = x$, i.e., x is a fixed point of the mapping. The eigenvalues of the Jacobian matrix at the fixed point are found as the solutions of the characteristic equation:

$$\det(J(x) - \lambda E) = 0, \tag{7}$$

where $J(x)$ is the Jacobian matrix of the mapping V evaluated at the fixed point x, and E is the identity matrix.

The nature of the fixed point can be characterized based on the eigenvalues of the Jacobian. To do this, we first introduce some definitions regarding the classification of fixed points [29].

To investigate the nature of fixed points of the mapping, we introduce the following definitions from [29].

Definition 5. A fixed point is called an *attractor* if all eigenvalues of the Jacobian matrix (i.e., the solutions of equation (7)) have modulus strictly less than one.

Definition 6. A fixed point is called a *repeller* if all eigenvalues of the Jacobian matrix have modulus strictly greater than one.

Definition 7. A fixed point is called a *saddle point* if the spectrum of the Jacobian contains eigenvalues with modulus both less than and greater than one. In other words, it is neither an attractor nor a repeller.

Corollary 1. If $\Delta_2, \Delta_3, \Delta_4 > 0$, then the fixed point M_{125} of the operator V_1 is a repeller, the fixed point M_{145} is an attractor, and the fixed point M_{135} is a saddle point.

Proof. Using equation (7), we compute the eigenvalues of the Jacobian matrix at each fixed point.
Let us denote the diagonal entries of the Jacobian matrix at a general point x as:

$$\begin{aligned}
t_1 &= 1 - a_{12}x_2 - a_{13}x_3 - a_{14}x_4 + a_{15}x_5,\\
t_2 &= 1 + a_{12}x_1 - a_{23}x_3 - a_{24}x_4 - a_{25}x_5,\\
t_3 &= 1 + a_{13}x_1 + a_{23}x_2 - a_{34}x_4 - a_{35}x_5,\\
t_4 &= 1 + a_{14}x_1 + a_{24}x_2 + a_{34}x_3 - a_{45}x_5,\\
t_5 &= 1 - a_{15}x_1 + a_{25}x_2 + a_{35}x_3 + a_{45}x_4.
\end{aligned}$$

Then the Jacobian matrix J takes the form:

$$J = \begin{pmatrix} t_1 & -a_{12}x_1 & -a_{13}x_1 & -a_{14}x_1 & -a_{15}x_1\\ a_{12}x_2 & t_2 & -a_{23}x_2 & -a_{24}x_2 & -a_{25}x_2\\ a_{13}x_3 & a_{23}x_3 & t_3 & -a_{34}x_3 & -a_{35}x_3\\ a_{14}x_4 & a_{24}x_4 & a_{34}x_4 & t_4 & -a_{45}x_4\\ -a_{15}x_5 & a_{25}x_5 & a_{35}x_5 & a_{45}x_5 & t_5 \end{pmatrix}.$$

Substituting the coordinates of the fixed point M_{125} into J, we obtain:

$$J(M_{125}) = \begin{pmatrix} 1 & -\frac{a_{12}a_{25}}{\Delta_{125}} & -\frac{a_{13}a_{25}}{\Delta_{125}} & -\frac{a_{14}a_{25}}{\Delta_{125}} & \frac{a_{15}a_{25}}{\Delta_{125}}\\ \frac{a_{12}a_{15}}{\Delta_{125}} & 1 & -\frac{a_{23}a_{15}}{\Delta_{125}} & -\frac{a_{24}a_{15}}{\Delta_{125}} & -\frac{a_{25}a_{15}}{\Delta_{125}}\\ 0 & 0 & 1+\frac{a_{13}a_{25}+a_{23}a_{15}-a_{35}a_{12}}{\Delta_{125}} & 0 & 0\\ 0 & 0 & 0 & 1+\frac{a_{14}a_{25}+a_{24}a_{15}-a_{45}a_{12}}{\Delta_{125}} & 0\\ -\frac{a_{15}a_{12}}{\Delta_{125}} & \frac{a_{25}a_{12}}{\Delta_{125}} & \frac{a_{35}a_{12}}{\Delta_{125}} & \frac{a_{45}a_{12}}{\Delta_{125}} & 1 \end{pmatrix},$$

where $\Delta_{125} = a_{12} + a_{15} + a_{25}$.

From this matrix, two eigenvalues are immediately identified as: $\lambda_1 = 1 + \frac{\Delta_4}{\Delta_{125}}$, $\lambda_2 = 1 + \frac{\Delta_3}{\Delta_{125}}$, corresponding to the diagonal entries.

The remaining eigenvalues are obtained from the characteristic equation for the 3×3 leading principal minor:

$$\begin{vmatrix} 1-\lambda & -\frac{a_{12}a_{25}}{\Delta_{125}} & \frac{a_{15}a_{25}}{\Delta_{125}}\\ \frac{a_{12}a_{15}}{\Delta_{125}} & 1-\lambda & -\frac{a_{25}a_{15}}{\Delta_{125}}\\ -\frac{a_{15}a_{12}}{\Delta_{125}} & \frac{a_{25}a_{12}}{\Delta_{125}} & 1-\lambda \end{vmatrix} = 0.$$

Solving it, we find: $\lambda_{3,4} = 1 \pm i\sqrt{\frac{a_{12}a_{15}a_{35}}{\Delta_{125}}}$, $\lambda_5 = 1$. Thus, the spectrum of the Jacobian at M_{125} is:

$$\sigma(J(M_{125})) = \left\{1,\ 1 + \frac{\Delta_4}{\Delta_{125}},\ 1 + \frac{\Delta_3}{\Delta_{125}},\ 1 \pm i\sqrt{\frac{a_{12}a_{15}a_{35}}{\Delta_{125}}}\right\}.$$

Similarly, we have:

$$\sigma(J(M_{135})) = \left\{1,\ 1-\frac{\Delta_4}{\Delta_{135}},\ 1+\frac{\Delta_2}{\Delta_{135}},\ 1\pm i\sqrt{\frac{a_{13}a_{15}a_{35}}{\Delta_{135}}}\right\},$$
$$\sigma(J(M_{145})) = \left\{1,\ 1-\frac{\Delta_3}{\Delta_{145}},\ 1-\frac{\Delta_2}{\Delta_{145}},\ 1\pm i\sqrt{\frac{a_{14}a_{15}a_{45}}{\Delta_{145}}}\right\}.$$

Assuming $\Delta_2, \Delta_3, \Delta_4 > 0$, we observe:

– for M_{125}: all real parts of the eigenvalues are strictly greater than 1. Hence, M_{125} is a "repeller";

– for M_{135}: one eigenvalue has real part greater than 1, another less than 1. Hence, M_{135} is a "saddle point";

– for M_{145}: all real parts of the eigenvalues are less than 1. Hence, M_{145} is an "attractor". □

Theorem 3. If $\Delta_2, \Delta_4 > 0$ and $\Delta_3 < 0$, then the fixed point card of the operator V_1 is cyclic and, in addition to the fixed points M_{125}, M_{135}, and M_{145}, contains an internal fixed point with all five coordinates nonzero (see Figure 3).

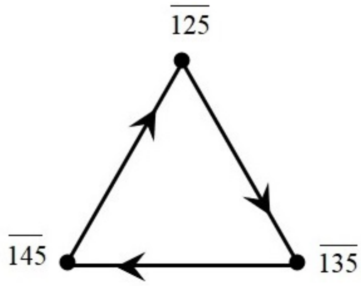

Figure 3. Cyclic structure of the fixed point graph with an additional internal fixed point

Proof. The cyclic structure of the fixed point graph G_V follows from Theorem 2, which characterizes the orientation of arcs between the fixed points M_{125}, M_{135}, and M_{145} depending on the signs of Δ_2, Δ_3, and Δ_4.

When $\Delta_2, \Delta_4 > 0$ and $\Delta_3 < 0$, the inequalities derived in Theorem 2 imply the formation of a cycle:

$$M_{125} \to M_{135} \to M_{145} \to M_{125}.$$

Let $\alpha = \{1,2,3,4,5\}$ denote the full support. Then Γ_α is the interior of the simplex S^4.

Since M_{125}, M_{135}, and M_{145} form a cyclic triple, none of them can serve as the sink (i.e., the unique fixed point Q_α) of the face Γ_α. By the uniqueness of such a point ([15], it follows that Γ_α must contain an additional fixed point M_α, which lies strictly inside the simplex. Hence, all coordinates of M_α are nonzero.

Therefore, under the stated conditions, the graph G_V acquires a cyclic structure and includes an internal fixed point with full support. □

Next, we consider another representative of the Lotka–Volterra mapping and the corresponding tournament.

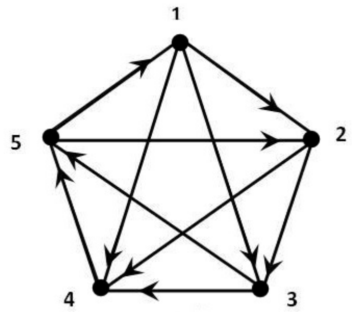

Figure 4. A strong, homogeneous tournament with four cyclic triples

Figure 4 illustrates a strong homogeneous tournament containing four cyclic triples. This tournament corresponds to the Lotka–Volterra operator V_2, defined by:

$$V_2 : \begin{cases} x_1' = x_1(1 - a_{12}x_2 - a_{13}x_3 - a_{14}x_4 + a_{15}x_5), \\ x_2' = x_2(1 + a_{12}x_1 - a_{23}x_3 - a_{24}x_4 + a_{25}x_5), \\ x_3' = x_3(1 + a_{13}x_1 + a_{23}x_2 - a_{34}x_4 - a_{35}x_5), \\ x_4' = x_4(1 + a_{14}x_1 + a_{24}x_2 + a_{34}x_3 - a_{45}x_5), \\ x_5' = x_5(1 - a_{15}x_1 - a_{25}x_2 + a_{35}x_3 + a_{45}x_4). \end{cases} \tag{8}$$

The corresponding skew-symmetric matrix A_2 associated with this operator is given by:

$$A_2 = \begin{pmatrix} 0 & -a_{12} & -a_{13} & -a_{14} & a_{15} \\ a_{12} & 0 & -a_{23} & -a_{24} & a_{25} \\ a_{13} & a_{23} & 0 & -a_{34} & -a_{35} \\ a_{14} & a_{24} & a_{34} & 0 & -a_{45} \\ -a_{15} & -a_{25} & a_{35} & a_{45} & 0 \end{pmatrix}.$$

If we compute all principal minors of order four of the skew-symmetric matrix A_2, we obtain squares of certain expressions. Let these expression denoted by $\Delta_i \neq 0$, for $i = 1, \dots, 5$:

$$\Delta_1 = a_{24}a_{35} - a_{23}a_{45} + a_{25}a_{34}, \quad \Delta_2 = a_{14}a_{35} - a_{13}a_{45} + a_{15}a_{34},$$
$$\Delta_3 = a_{14}a_{25} - a_{15}a_{24} + a_{12}a_{45}, \quad \Delta_4 = a_{12}a_{35} - a_{15}a_{23} + a_{13}a_{25},$$
$$\Delta_5 = a_{12}a_{34} - a_{13}a_{24} + a_{14}a_{23}.$$

The Lotka–Volterra operator V_2 defined in equation (8) admits four cyclic triples: $\overline{135}$, $\overline{145}$, $\overline{235}$, and $\overline{245}$. These cyclic triples correspond to strong sub-tournaments of the tournament on the 4-simplex S^4 (see Figure 4), each containing a unique internal fixed point.

These fixed points are given by:

$$M_{135} = \left(\frac{a_{35}}{a_{13} + a_{15} + a_{35}},\ 0,\ \frac{a_{15}}{a_{13} + a_{15} + a_{35}},\ 0,\ \frac{a_{13}}{a_{13} + a_{15} + a_{35}} \right),$$

$$M_{145} = \left(\frac{a_{45}}{a_{14} + a_{15} + a_{45}},\ 0,\ 0,\ \frac{a_{15}}{a_{14} + a_{15} + a_{45}},\ \frac{a_{14}}{a_{14} + a_{15} + a_{45}} \right),$$

$$M_{235} = \left(0,\ \frac{a_{35}}{a_{23} + a_{25} + a_{35}},\ \frac{a_{25}}{a_{23} + a_{25} + a_{35}},\ 0,\ \frac{a_{23}}{a_{23} + a_{25} + a_{35}} \right),$$

$$M_{245} = \left(0,\ \frac{a_{45}}{a_{24}+a_{25}+a_{45}},\ 0,\ \frac{a_{25}}{a_{24}+a_{25}+a_{45}},\ \frac{a_{24}}{a_{24}+a_{25}+a_{45}}\right),$$

where all coefficients a_{ij} are assumed to be strictly positive.

For the operator V_2, applying Young's inequality yields the following estimates:

$$\begin{aligned}
\varphi_{135}(Vx) &\leq \frac{\varphi_{135}(x)}{\Delta_{135}}\left(\Delta_{135}+\Delta_4 x_2-\Delta_2 x_4\right),\\
\varphi_{145}(Vx) &\leq \frac{\varphi_{145}(x)}{\Delta_{145}}\left(\Delta_{145}+\Delta_3 x_2+\Delta_2 x_3\right),\\
\varphi_{235}(Vx) &\leq \frac{\varphi_{235}(x)}{\Delta_{235}}\left(\Delta_{235}-\Delta_4 x_3-\Delta_3 x_4\right),\\
\varphi_{245}(Vx) &\leq \frac{\varphi_{245}(x)}{\Delta_{245}}\left(\Delta_{245}-\Delta_4 x_3-\Delta_3 x_4\right),
\end{aligned}$$

for all $x \in S^4$, where

$$\Delta_{135} = a_{13}+a_{15}+a_{35}, \quad \Delta_{145} = a_{14}+a_{15}+a_{45}, \quad \Delta_{235} = a_{23}+a_{25}+a_{35}, \quad \Delta_{245} = a_{24}+a_{25}+a_{45}.$$

If the second and fourth even-order principal minors of the skew-symmetric matrix A_2 are nonzero, then A_2 is said to be in general position. In this case, the card of fixed points of the operator V_2 has the structure shown in Figure 5.

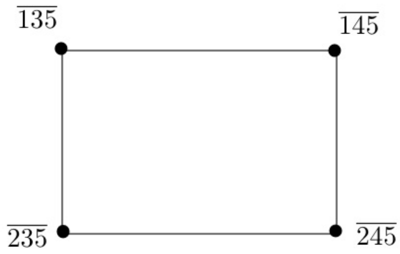

Figure 5. The card of fixed points for the mapping V_2

In the card of fixed points, no directions are initially indicated, as the orientations on the faces of the simplex depend on the signs of the expressions Δ_i, for $i = 1, 2, 3, 4, 5$.

The orientation of a graph refers to assigning a direction (arrow) to each of its edges, i.e., specifying an order for every pair of adjacent vertices. A directed graph, or digraph, is one in which no two vertices are connected by a pair of edges pointing in opposite directions. Thus, every orientation of an undirected graph yields a digraph [17].

For a graph with four vertices, there are $2^4 = 16$ possible orientations. Among these 16 digraphs, some are isomorphic — that is, structurally identical up to a relabeling of vertices. There are exactly four non-isomorphic directed graphs with four vertices that contain a directed cycle. These are illustrated in Figure 6.

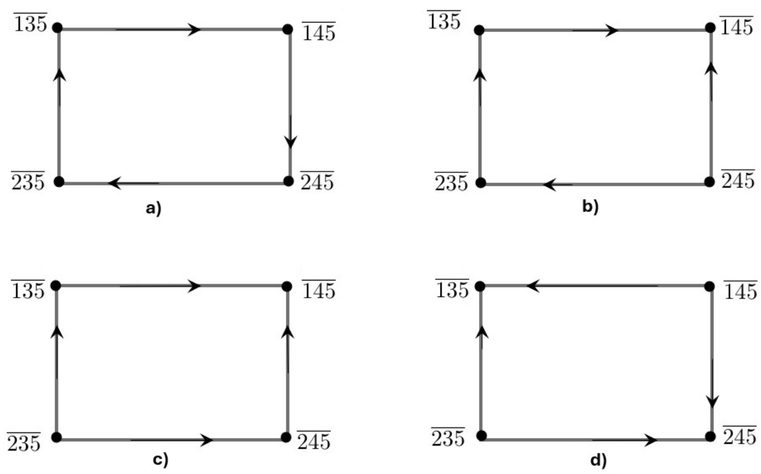

Figure 6. The four non-isomorphic directed graphs

Theorem 4. Let the following conditions hold:

1. If $\Delta_1, \Delta_4 < 0$, then the card of fixed points of the operator V_2 has the structure shown in Figure 6, case a).
2. If $\Delta_1, \Delta_3, \Delta_4 < 0$, then the card of fixed points of the operator V_2 has the structure shown in Figure 6, case b).
3. If $\Delta_3, \Delta_4 < 0$, then the card of fixed points of the operator V_2 has the structure shown in Figure 6, case c).

The proof of Theorem 4 follows directly from Theorems 2 and 3.

Theorem 5. If A_2 is a skew-symmetric matrix in general position, then the card of fixed points of the operator V_2 cannot take the form shown in Figure 6 case d).

Proof. The fact that the card of fixed points of the operator V_2 cannot take the form shown in Figure 6, case d) follows from a uniqueness fakt stated in [15]. Specifically, if the skew-symmetric matrix is in general position, then the sets of points P and Q are each unique [13, 15].

However, in the fixed point diagram shown in Figure 6 case d), there are two P-points, namely $(\overline{145}, \overline{235})$, and two Q-points, namely $(\overline{135}, \overline{245})$, which contradicts this uniqueness. □

Let us consider the mapping $V_3 : S^4 \to S^4$ defined by the following system of equations:

$$V_3 : \begin{cases} x_1^{'} = x_1(1 + a_{12}x_2 + a_{13}x_3 - a_{14}x_4 - a_{15}x_5), \\ x_2^{'} = x_2(1 - a_{12}x_1 + a_{23}x_3 + a_{24}x_4 - a_{25}x_5), \\ x_3^{'} = x_3(1 - a_{13}x_1 - a_{23}x_2 + a_{34}x_4 + a_{35}x_5), \\ x_4^{'} = x_4(1 + a_{14}x_1 - a_{24}x_2 - a_{34}x_3 + a_{45}x_5), \\ x_5^{'} = x_5(1 + a_{15}x_1 + a_{25}x_2 - a_{35}x_3 - a_{45}x_4), \end{cases}$$

where the coefficients satisfy the conditions $0 < a_{ki} \leq 1$ for all i, k.

The strong, homogeneous tournament corresponding to this operator is illustrated in Figure 7.

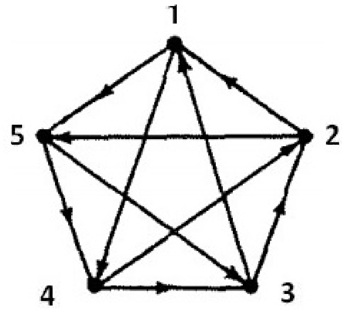

Figure 7. The strong, homogeneous tournament corresponding to the operator V_3

It has five cyclic triples: $\overline{124}, \overline{134}, \overline{135}, \overline{235}, \overline{245}$, each of whose corresponding faces contains exactly one fixed point:

$$M_{124} = \left(\frac{a_{24}}{a_{12}+a_{14}+a_{24}},\ \frac{a_{14}}{a_{12}+a_{14}+a_{24}},\ 0,\ \frac{a_{12}}{a_{12}+a_{14}+a_{24}},\ 0\right),$$
$$M_{134} = \left(\frac{a_{34}}{a_{13}+a_{14}+a_{34}},\ 0,\ \frac{a_{14}}{a_{13}+a_{14}+a_{34}},\ \frac{a_{13}}{a_{13}+a_{14}+a_{34}},\ 0\right),$$
$$M_{135} = \left(\frac{a_{35}}{a_{13}+a_{15}+a_{35}},\ 0,\ \frac{a_{15}}{a_{13}+a_{15}+a_{35}},\ 0,\ \frac{a_{13}}{a_{13}+a_{15}+a_{35}}\right),$$
$$M_{235} = \left(0,\ \frac{a_{35}}{a_{23}+a_{25}+a_{35}},\ \frac{a_{25}}{a_{23}+a_{25}+a_{35}},\ 0,\ \frac{a_{23}}{a_{23}+a_{25}+a_{35}}\right),$$
$$M_{245} = \left(0,\ \frac{a_{45}}{a_{24}+a_{25}+a_{45}},\ 0,\ \frac{a_{25}}{a_{24}+a_{25}+a_{45}},\ \frac{a_{24}}{a_{24}+a_{25}+a_{45}}\right).$$

We use the following notation:

$$\begin{aligned} &\Delta_1 = a_{24}a_{35} - a_{23}a_{45} + a_{25}a_{34}, \quad \Delta_2 = a_{14}a_{35} - a_{15}a_{34} + a_{13}a_{45}, \\ &\Delta_3 = a_{14}a_{25} - a_{12}a_{45} + a_{15}a_{24}, \quad \Delta_4 = a_{12}a_{35} - a_{15}a_{23} + a_{13}a_{25}, \\ &\Delta_5 = a_{13}a_{24} - a_{12}a_{34} + a_{14}a_{23}. \end{aligned} \tag{9}$$

For the operator V_3, we also apply Young's inequality and obtain the following estimates:

$$\varphi_{124}(Vx) \leq \frac{\varphi_{124}(x)}{\Delta_{124}}\left(\Delta_{124} - \Delta_5 x_3 - \Delta_3 x_5\right),$$
$$\varphi_{134}(Vx) \leq \frac{\varphi_{134}(x)}{\Delta_{134}}\left(\Delta_{134} + \Delta_5 x_2 + \Delta_2 x_5\right),$$
$$\varphi_{135}(Vx) \leq \frac{\varphi_{135}(x)}{\Delta_{135}}\left(\Delta_{135} + \Delta_4 x_2 - \Delta_2 x_4\right),$$
$$\varphi_{235}(Vx) \leq \frac{\varphi_{235}(x)}{\Delta_{235}}\left(\Delta_{235} - \Delta_4 x_1 + \Delta_1 x_4\right),$$
$$\varphi_{245}(Vx) \leq \frac{\varphi_{245}(x)}{\Delta_{245}}\left(\Delta_{245} + \Delta_3 x_1 - \Delta_1 x_3\right).$$

for all $x \in S^4$, where

$$\Delta_{124} = a_{12} + a_{14} + a_{24}, \quad \Delta_{134} = a_{13} + a_{14} + a_{34}, \quad \Delta_{135} = a_{13} + a_{15} + a_{35},$$
$$\Delta_{235} = a_{23} + a_{25} + a_{35}, \quad \Delta_{245} = a_{24} + a_{25} + a_{45}.$$

Theorem 6. Let the quantities $\Delta_1, \Delta_2, \Delta_3, \Delta_4, \Delta_5$ be defined as in (9). Then:

1. If $\Delta_1, \Delta_2, \Delta_3, \Delta_4 < 0$ and $\Delta_5 > 0$, then the fixed point card of the operator V_3 contains a Hamiltonian cycle, and the operator admits an internal fixed point with all five coordinates nonzero (see Figure 8, case a)).
2. If $\Delta_2, \Delta_3, \Delta_4 < 0$ and $\Delta_1, \Delta_5 > 0$, then the fixed point card of the operator V_3 takes the form shown in Figure 8, case b).
3. If $\Delta_2, \Delta_4 < 0$ and $\Delta_1, \Delta_3, \Delta_5 > 0$, then the fixed point card of the operator V_3 takes the form shown in Figure 8, case c).
4. If $\Delta_1 \cdot \Delta_2 \cdot \Delta_3 \cdot \Delta_4 \cdot \Delta_5 \neq 0$, then the fixed point card of the operator V_3 cannot take the form shown in Figure 8, case d).

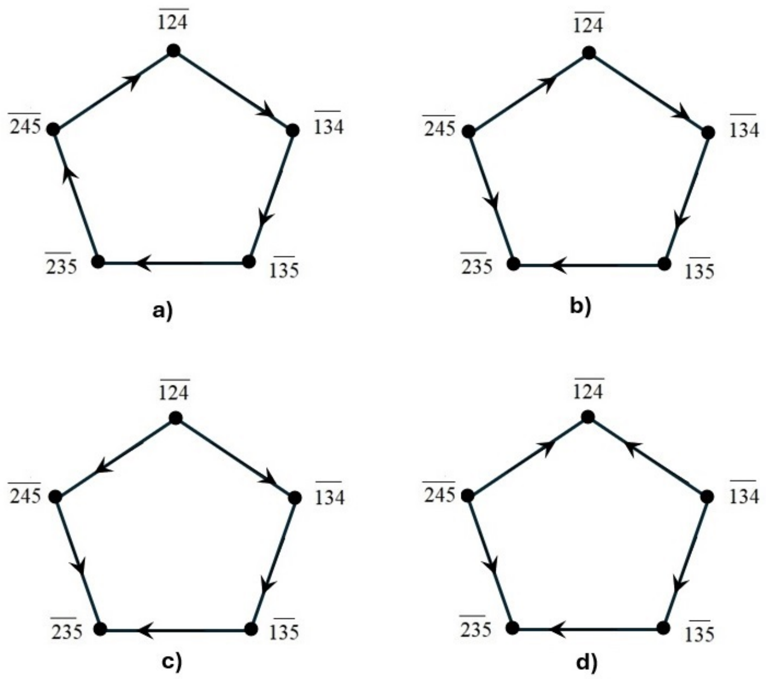

Figure 8. Possible cards of fixed points of the operator V_3

The proof of Theorem 6 follows directly from Theorems 2 and 3.

Theorem 6 characterizes the types of fixed point configurations of the operator V_3 depending on the signs of the expressions Δ_i. In particular, case 1. indicates the existence of an internal fixed point. The following lemma makes this statement precise.

Lemma 1. Let the operator $V_3 : S^4 \to S^4$ be defined by the system

$$V_3(x)_k = x_k \left(1 + \sum_{i=1}^{5} a_{ki} x_i\right), \quad k = 1, \ldots, 5,$$

where $a_{ki} = -a_{ik}$, $x_i \geq 0$, $\sum\limits_{i=1}^{5} x_i = 1$, and Δ_i are the fourth-order principal minors of the skew-symmetric matrix $A = (a_{ij})$. Then:

1. If $\Delta_1, \Delta_2, \Delta_3, \Delta_4 < 0$ and $\Delta_5 > 0$, then the operator V_3 has at least one internal fixed point $x^* \in \mathrm{int}(S^4)$.
2. If at least three of the values Δ_i are positive, then there are no internal fixed points.

Proof. Consider the directed graph (tournament) G_{V_3} corresponding to the operator V_3, where the vertices represent the coordinates x_i, and the direction of the edges is determined by the sign of the coefficients a_{ki}.

1. Existence of an internal point. According to results by Hofbauer J. and Ganikhodzhaev R. [13, 19], if the tournament G_{V_3} contains a Hamiltonian cycle, then the operator V_3 has at least one internal fixed point. This behavior occurs when the fixed points on the faces (e.g., $M_{124}, M_{134}, M_{135}, M_{235}, M_{245}$) are connected by directed transitions forming a cycle. The conditions $\Delta_1, \Delta_2, \Delta_3, \Delta_4 < 0$ and $\Delta_5 > 0$ ensure the required orientation of the transitions between faces, forming a Hamiltonian cycle.

2. Non-existence of an internal point. If at least three of the values Δ_i are positive, the structure of G_{V_3} does not contain a full directed cycle (it becomes either transitive or splits into sub-tournaments). This implies that all trajectories of V_3 are attracted to fixed points on the boundary faces of the simplex, and internal fixed points are either unstable or do not exist. □

3 Connection with replicator dynamics and evolutionary stability

The Lotka–Volterra operators considered in this paper are structurally close to replicator dynamics from evolutionary game theory. In both models, the trajectories are confined to the standard simplex S^{m-1}, and fixed points correspond to stationary population states.

3.1 Replicator dynamics and stability

The replicator equation for a population with m strategies and payoff matrix $A = (a_{ij})$ has the form [30–32]: $\dot{x}_i = x_i\left((Ax)_i - x^\top Ax\right)$, where $x \in S^{m-1}$, and $(Ax)_i$ denotes the fitness of strategy i. A point $x^* \in S^{m-1}$ is a fixed point if all strategies present in x^* have equal fitness: $(Ax^*)_i = x^{*\top}Ax^*$ for all $x_i^* > 0$.

3.2 Evolutionarily stable strategy (ESS)

A point $x^* \in S^{m-1}$ is called an *evolutionarily stable strategy (ESS)* if the following two conditions are satisfied:

1. x^* is a Nash equilibrium: $x^{*\top}Ax^* \geq x^\top Ax^*$ for all $x \in S^{m-1}$;
2. if $x \neq x^*$ and $x^{*\top}Ax = x^{*\top}Ax^*$, then $x^\top Ax < x^{*\top}Ax$.

This means that small deviations from x^* result in lower fitness for mutants, and strategy x^* cannot be invaded.

3.3 Analogy with Lotka–Volterra operators

Consider the discrete Lotka–Volterra operator:

$$x'_k = x_k\left(1 + \sum_{i=1}^{m} a_{ki}x_i\right), \quad k = 1, \dots, m.$$

After normalization and transition to continuous time, this system approximates the replicator form: $\dot{x}_k = x_k\left(\sum\limits_{i=1}^{m} a_{ki}x_i - \Phi(x)\right)$, where $\Phi(x)$ is the average fitness. This supports the interpretation of coefficients a_{ij} as measures of fitness differences or interactions between strategies.

Thus, interior fixed points of the operator V, i.e., those with all coordinates positive, can be interpreted as candidates for ESS.

3.4 Classification of fixed points

Let $M_\alpha \subset S^4$ be a fixed point associated with a face Γ_α defined by a cyclic triple. Then:

- if all eigenvalues of the Jacobian matrix at M_α have modulus less than one, the point is *asymptotically stable* and may be ESS;
- if the point is a saddle or repeller, then it cannot be evolutionarily stable.

Proposition 1. Let x^* be a fixed point of a Lotka–Volterra operator V. Then:

- if x^* is a strict local maximum of a potential function (if one exists), then x^* is an ESS;
- if x^* is a saddle or repeller, then it is not evolutionarily stable.

As an example, we can consider the operator V_2. Under the conditions $\Delta_2, \Delta_3, \Delta_4 < 0$, $\Delta_1, \Delta_5 > 0$, the fixed point structure corresponds to Figure 8, case b), where there exists a unique interior fixed point. If the eigenvalues of the Jacobian matrix at this point all have modulus less than one, the point is asymptotically stable and can be interpreted as an ESS.

The connection with replicator dynamics provides a biological interpretation of the behavior of Lotka–Volterra operators. Attracting interior fixed points behave as stable combinations of strategies or species, while saddle points correspond to unstable ecological or strategic equilibria.

4 Conclusion

In this work, we analyzed the structure of the set of fixed points — referred to as the *card of fixed points* — for Lotka–Volterra type operators defined on the standard simplex S^{m-1}. By associating these nonlinear maps with skew-symmetric matrices in general position, we established a correspondence between the dynamical system and directed graphs, particularly focusing on strong and homogeneous tournaments.

This graph-theoretical interpretation allowed us to classify the qualitative behavior of the system based on the topology of the corresponding tournament - including the presence of Hamiltonian cycles and internal fixed points. Analytical conditions were derived using the signs of even-order principal minors Δ_i, which determine the number and nature of fixed points. Additionally, Young's inequality was applied to obtain upper estimates for the evolution of invariant functions defined on simplex faces.

Beyond theoretical significance, the results of this study find direct applications in several domains where discrete population dynamics are modeled. In evolutionary biology, Lotka–Volterra operators serve as simplified models of frequency-dependent selection, where fixed points correspond to evolutionarily stable strategies (ESS). Interior fixed points represent coexistence states, while saddle points and repellers describe unstable or metastable configurations.

In socio-economic systems, such as market competition, opinion dynamics, or resource allocation, agent interactions can also be described using skew-symmetric structures. In this context, the tournament representation reflects dominance, influence, or preference relations. Therefore, the topological classification of fixed point cards provides insights into long-term system behavior based on interaction patterns.

The proposed approach can be further extended to systems with noise, spatial heterogeneity, or adaptive responses, making it a promising tool for modeling complex real-world phenomena. Future directions may include the development of algorithms to infer tournament structure from empirical data and applying the derived stability criteria to detect equilibrium configurations in evolutionary and economic games.

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Author Contributions

- **Dilfuza Bahramovna Eshmamatova** developed the mathematical model, formulated the main definitions, proved the key results, wrote the introduction and conclusion, interpreted the results in terms of evolutionary dynamics, contributed to the applications section.
- **Mokhbonu Akram khizi Tadzhieva** conducted the theoretical analysis, proved the key results, and derived the main estimates.

All authors contributed equally to this work.

Conflict of Interest

The authors declare no conflict of interest.

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Well-posedness of elliptic-parabolic differential problem with integral condition

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In this paper, we study a class of nonlocal boundary value problems for elliptic-parabolic equations subject to integral-type conditions. Such problems naturally emerge in various physical and engineering contexts, including diffusion processes in composite materials and systems with memory or nonlocal interactions. The model considered involves a mixed-type equation in which the elliptic and parabolic components are coupled through nonlocal boundary terms, while the boundary conditions incorporate integral constraints that generalize the traditional Dirichlet and Neumann formulations. To investigate the solvability of this problem, we employ analytical methods based on the theory of parabolic and elliptic operators in weighted Hölder spaces, which are particularly suitable for handling boundary singularities and ensuring regularity of solutions. We establish the existence, uniqueness, and continuous dependence of solutions on the input data, thereby proving the well-posedness of the problem. Furthermore, we derive coercivity inequalities for solutions of the associated mixed nonlocal boundary problems, which guarantee their stability and provide essential tools for studying related inverse and control problems. The findings extend several classical results and offer a unified approach to the analysis of nonlocal elliptic-parabolic models.

Keywords: elliptic-parabolic equation, nonlocal boundary value problem, integral condition, Hölder spaces, well-posedness, coercivity inequalities, stability, mixed-type differential equations.

2020 Mathematics Subject Classification: 35M12, 39K40.

Introduction

Elliptic partial differential equations play a fundamental role across nearly all branches of mathematics — from harmonic analysis and geometry to Lie theory — and have a wide range of applications in physics and engineering. The well-posedness of local boundary value problems for elliptic equations, along with their various applications, has been extensively studied by numerous researchers [1–3].

Equations of mixed-composite type form an important class of partial differential equations (PDEs) that combine features of different types of equations — typically elliptic, parabolic, and sometimes hyperbolic — within a single formulation [4–6]. These equations often arise in mathematical models describing processes where the nature of the physical phenomenon changes across a domain or depends on certain parameters.

In general, an equation is called mixed type when its classification (elliptic, parabolic, or hyperbolic) varies in different regions of the domain. A mixed-composite type equation extends this idea by coupling different equations or operators — such as elliptic and parabolic ones — through boundary, interface, or integral-type conditions [7].

In mathematical modeling, elliptic equations are paired with local boundary conditions that dictate the solution at the domain's edge. However, traditional boundary conditions may be insufficient for accurately modeling certain processes or phenomena. As a result, nonlocal boundary conditions are often employed in mathematical models of physical, chemical, biological, or environmental processes. These conditions, known as nonlocal boundary conditions, arise when data at the domain's edge cannot be directly observed or when boundary data are dependent on internal data within the domain [8–10].

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Various nonlocal boundary value problems with Samarskii–Ionkin condition for partial differential equations have been investigated by many researchers [11, 12].

Moreover, the identification of partial differential equations (PDEs) arises in numerous applied problems and has been the subject of extensive research [13–15].

The significance of well-posedness (WP) in the analysis of boundary value problems (BVPs) for (PDEs) is widely recognized [16–18].

Considerable attention has been devoted to the study of coercivity inequalities (CIs) arising in nonlocal BVPs for elliptic and parabolic PDEs [19–21].

In this paper, we study the WP of a nonlocal BVP of the form

$$\begin{cases} -\mathcal{U}_{tt}(t) + A\mathcal{U}(t) = g(t), \ t \in (0, d], \\ \mathcal{U}_t(t) - A\mathcal{U}(t) = f(t), \ t \in [-d, 0) \end{cases} \tag{1}$$

with an integral condition $\mathcal{U}(d) = \int_{-d}^{0} \mu(s)\mathcal{U}(s)\, ds + \xi$ in a Hilbert space $\aleph$ with a self-adjoint positive definite operator (SAPDO) A. Here, $\xi \in D(A)$, while $g(t)$ and $f(t)$ are prescribed smooth functions.

The principal result demonstrates the WP of problem (1) in weighted Hölder spaces. New CIs for the solutions of elliptic-parabolic nonlocal BVPs are derived.

1 *The main theorem on the WP of (1)*

Throughout this work, $\aleph$ is a Hilbert space and A is assumed to be a SAPDO satisfying $A \geq \delta I$ for $\delta > \delta_0 > 0$, where I is the identity operator. We also set $V = A^{1/2}$.

First, we present several results that will be needed in the sequel.

Lemma 1. The following estimates hold [22]:

$$\begin{cases} \| V^{\mu} \exp(-tV)\|_{\aleph\to\aleph} \leq (\frac{\mu}{e})^{\mu} t^{-\mu}, \ t \in (0, \infty), \ \mu \in [0, e], \\ \| A^{\mu} \exp(-tA)\|_{\aleph\to\aleph} \leq (\frac{\mu}{e})^{\mu} t^{-\mu}, \ t \in (0, \infty), \ \mu \in [0, e], \\ \| (I - \exp(-2dV))^{-1} \|_{\aleph\to\aleph} \leq M(\delta) \end{cases} \tag{2}$$

for some $M(\delta) \geq 0$.

Lemma 2. Operator

$$(I - V)e^{-2dV} + I + V - 2e^{-dV} \int_{-d}^{0} \mu(s) e^{sV^2} ds$$

has an inverse

$$N = \left((I - V)e^{-2dV} + I + V - 2e^{-dV} \int_{-d}^{0} \mu(s) e^{sV^2} ds \right)^{-1}$$

and the following estimates are fulfilled

$$\| N \|_{\aleph\to\aleph} \leq M(\delta), \quad \| VN \|_{\aleph\to\aleph} \leq M(\delta). \tag{3}$$

The Proof for Lemma 2 relies on the spectral representations of unit SAPDO A [22].

Function $\mathcal{U}(t)$ is said to be a solution of problem (1) it the following conditions are met:

1. $\mathcal{U}(t)$ is twice continuously differentiable on $(0, d]$ and continuously differentiable on $[-d, d]$; the derivatives at the endpoints are understood in the sense of one-sided limits;
2. $\mathcal{U}(t) \in D(A)$ for all $t \in [-d, d]$, and the mapping $t \mapsto A\mathcal{U}(t)$ is continuous on $[-d, d]$;
3. $\mathcal{U}(t)$ satisfies the system and the nonlocal boundary condition in (1).

The function $\mathcal{U}(t)$ fulfilling the above requirements will be referred to as a solution of problem (1) in the space $\mathtt{C}(\aleph) = \mathtt{C}_{-d,d}(\aleph)$, consisting of all continuous functions $\psi(y)$ defined on $[-d, d]$ with values in $\aleph$, with the norm

$$\|\psi\|_{\mathtt{C}_{-d,d}(\aleph)} = \max_{y\in[-d,d]} \|\psi(y)\|_{\aleph}.$$

To derive the formula for solution of problem (1), we will consider the following auxiliary problems

$$\begin{cases} -\mathcal{U}''(t) + A\mathcal{U}(t) = g(t), \ t \in (0, d), \\ \mathcal{U}(0) = \mathcal{U}_0, \ \mathcal{U}(d) = \mathcal{U}_d, \end{cases} \tag{4}$$

$$\begin{cases} \mathcal{U}'(t) - A\mathcal{U}(t) = f(t), \ t \in (-d, 0), \\ \mathcal{U}(0) = \mathcal{U}_0. \end{cases} \tag{5}$$

It is well established (cf. [22]) that, for sufficiently smooth data, problems (4) and (5) admit a unique solution. Moreover, the following relations are valid:

$$\mathcal{U}(t) = \left(I - e^{-2dV}\right)^{-1} \left[\left(e^{-tV} - e^{-(2d-t)V}\right)\mathcal{U}_0 + \left(e^{-(d-t)V} - e^{-(t+d)V}\right)\mathcal{U}_d \right. \tag{6}$$

$$\left. - \left(e^{-(d-t)V} - e^{-(t+d)V}\right)(2V)^{-1} \int_0^d \left(e^{-(d-\theta)V} - e^{-(\theta+d)V}\right) g(\theta)d\theta \right]$$

$$+(2V)^{-1} \int_0^d \left(e^{-|t-\theta|V} - e^{-(t+\theta)V}\right) g(\theta)d\theta, \quad t \in [0, d],$$

$$\mathcal{U}(t) = e^{tA}\mathcal{U}_0 + \int_0^t e^{(t-y)A} f(y)dy, \quad t \in [-d, 0]. \tag{7}$$

Using formula (6), conditions $\mathcal{U}(d) = \int_{-d}^{0} \mu(\theta)\mathcal{U}(\theta)d\theta + \xi$, and $\mathcal{U}'(0+) = \mathcal{U}'(0-)$, we can write

$$\mathcal{U}(d) = \int_{-d}^0 \mu(\theta)e^{\theta A}d\theta\mathcal{U}_0 + \int_{-d}^0 \mu(\theta) \int_0^\theta e^{-(\theta-y)A} f(y)dyd\theta + \xi, \tag{8}$$

$$A\mathcal{U}(0) + f(0) = \left(I - e^{-2dV}\right)^{-1} \left[- V(I + e^{-2dV})\mathcal{U}_0 + 2Ve^{-dV}\mathcal{U}_d \right.$$

$$\left. - e^{-dV} \int_0^d \left(e^{-(d-\theta)V} - e^{-(d+\theta)V}\right) g(\theta)\, d\theta \right] + \int_0^d e^{-\theta V} g(\theta)\, d\theta. \tag{9}$$

Using formulas (8) and (9), we obtain that

$$A\mathcal{U}(0) = \left(I - e^{-2dV}\right)^{-1} \left[- V(I + e^{-2dV})\mathcal{U}_0 \right.$$

$$+2Ve^{-dV}\left\{\int\limits_{-d}^{0}\mu(\theta)e^{\theta A}d\theta\,\mathcal{U}_0+\int\limits_{-d}^{0}\mu(\theta)\int\limits_{0}^{\theta}e^{-(\theta-y)A}f(y)dyd\theta+\xi\right\}$$

$$-e^{-dV}\int\limits_{0}^{d}\left(e^{-(d-\theta)V}-e^{-(d+\theta)V}\right)g(\theta)d\theta\Bigg]+\int\limits_{0}^{d}e^{-\theta V}g(\theta)d\theta-f(0).$$

Since the operator

$$(I-V)e^{-2dV}+I+V-2e^{-dV}\int\limits_{-d}^{0}\mu(\theta)e^{\theta V^2}d\theta$$

has an inverse

$$N=\left((I-V)e^{-2dV}+I+V-2e^{-dV}\int\limits_{-d}^{0}\mu(\theta)e^{\theta V^2}d\theta\right)^{-1},$$

we derive that

$$\mathcal{U}_0=N\left[2e^{-dV}\left(\int\limits_{-d}^{0}\mu(\theta)\int\limits_{0}^{\theta}e^{-(\theta-y)A}f(y)dyd\theta+\xi\right)\right. \quad (10)$$

$$-e^{-dV}\int\limits_{0}^{d}\left(e^{-(d-\theta)V}-e^{-(d+\theta)V}\right)g(\theta)d\theta$$

$$\left.+\left(I-e^{-2dV}\right)V^{-1}\int\limits_{0}^{d}e^{-\theta V}g(\theta)d\theta-\left(I-e^{-2dV}\right)V^{-1}f(0)\right].$$

Hence, the solution of nonlocal BVP (1) is represented by formulas (7), (10), and (9). Now, let us denote by $\mathtt{C}^{\mu}_{-d,d}(\aleph),\ \mu\in(0,1)$, the Banach space obtained by completing the space of smooth $\aleph$-valued function $\psi(y)$ on $[-d,d]$ in the norm

$$\parallel\psi\parallel_{\mathtt{C}^{\mu}_{-d,d}(\aleph)}=\|\psi\|_{\mathtt{C}_{-d,d}(\aleph)}+\sup_{-d<y<y+\Delta y<0}\|\psi(y+\Delta y)-\psi(y)\|_{\aleph}(\Delta y^{-\mu}(-y)^{\mu}$$

$$+\sup_{0<y<y+\Delta y<d}\|\psi(y+\Delta y)-\xi(y)\|_{\aleph}(\Delta y)^{-\mu}(d-y)^{\mu}(y+\Delta y)^{\mu},$$

and denote by $\mathtt{C}^{\mu}_{0,d}(\aleph)$, $\mu\in(0,1)$, the Banach space obtained by completing the space of smooth $\aleph$-valued function $\psi(y)$ on $[0,d]$ in the norm

$$\parallel\psi\parallel_{\mathtt{C}^{\mu}_{0,d}(\aleph)}=\|\psi\|_{\mathtt{C}_{0,d}(\aleph)}+\sup_{0<y<y+\Delta y<d}\|\psi(y+\Delta y)-\psi(y)\|_{\aleph}(\Delta y)^{-\mu}(d-y)^{\mu}(y+\Delta y)^{\mu},$$

finally denote by $\mathtt{C}^{\mu}_{-d,0}(\aleph),\ \mu\in(0,1)$, the Banach space obtained by completion of the set of all smooth $\aleph$-valued functions $\psi(y)$ on $[-d,0]$ in the norm

$$\parallel\psi\parallel_{\mathtt{C}^{\mu}_{-d,0}(\aleph)}=\|\psi\|_{\mathtt{C}_{-d,0}(\aleph)}+\sup_{-d<y<y+\Delta y<0}\|\psi(y+\Delta y)-\psi(y)\|_{\aleph}(\Delta y)^{-\mu}(-y)^{\mu}.$$

Here, $\mathtt{C}_{a,b}(\aleph)$ is defined as the Banach space of all continuous functions $\psi(y)$ defined on $[p,q]$ with values in $\aleph$, endowed with the norm

$$\|\psi\|_{\mathtt{C}_{p,q}(\aleph)} = \max_{y\in[p,q]} \|\psi(y)\|_{\aleph}.$$

Problem (1) is considered *well-posed* in $\mathtt{C}(\aleph)$ if, for every $g(t) \in \mathtt{C}_{0,d}(\aleph)$, $f(t) \in \mathtt{C}_{-d,0}(\aleph)$, and $\xi \in D(A)$, it has a unique solution $\mathcal{U}(t) \in \mathtt{C}(\aleph)$ satisfying the CI

$$\|\mathcal{U}''\|_{\mathtt{C}_{0,d}(\aleph)} + \|\mathcal{U}'\|_{\mathtt{C}_{-d,0}(\aleph)} + \|A\mathcal{U}\|_{\mathtt{C}(\aleph)} \leq M\Big(\|g\|_{\mathtt{C}_{0,d}(\aleph)} + \|f\|_{\mathtt{C}_{-d,0}(\aleph)} + \|A\xi\|_{\aleph}\Big),$$

where M represents a positive constant whose value does not depend on $g(t)$, $f(t)$, and ξ.

The given problem (1) is not well-posed in $\mathtt{C}(\aleph)$ [23]. The WP of BVP (1) can be established by formulating the problem in appropriate function spaces $F(\aleph)$ consisting of smooth $\aleph$-valued functions defined on $[-d,d]$.

A function $\mathcal{U}(t)$ is said to be a solution of problem (1) in $F(\aleph)$ if it satisfies the problem in $\mathtt{C}(\aleph)$ and, moreover, the functions $\mathcal{U}''(t)$ $(t\in[0,d])$, $\mathcal{U}'(t)$ $(t\in[-d,d])$ and $A\mathcal{U}(t)$ $(t\in[-d,d])$ are elements of $F(\aleph)$.

Similarly to the space $\mathtt{C}(\aleph)$, problem (1) is considered *well-posed* in $F(\aleph)$ if the subsequent CI holds:

$$\|\mathcal{U}''\|_{F_{0,d}(\aleph)} + \|\mathcal{U}'\|_{F_{-d,0}(\aleph)} + \|A\mathcal{U}\|_{F(\aleph)} \leq M\Big(\|g\|_{F_{0,d}(\aleph)} + \|f\|_{F_{-d,0}(\aleph)} + \|A\xi\|_{\aleph}\Big),$$

where $M > 0$ denotes a constant that does not depend on $g(t)$, $f(t)$, and ξ.

Setting $F(\aleph) = \mathtt{C}^{\mu}_{0,d}(\aleph) = \mathtt{C}^{\mu}_{0,d}([-d,d],\aleph)$ for $\mu \in (0,1)$, we can formulate our main theorem as follows.

Theorem 1. Suppose $\xi \in D\,(A)$. Then BVP (1) is well-posed in a Hölder space $\mathtt{C}^{\mu}_{0,d}(\aleph)$ and the following CI holds:

$$\left\|\mathcal{U}''\right\|_{\mathtt{C}^{\mu}_{0,d}(\aleph)} + \left\|\mathcal{U}'\right\|_{\mathtt{C}^{\mu}_{-d,0}(\aleph)} + ||A\mathcal{U}||_{\mathtt{C}^{\mu}_{0,d}(\aleph)} \tag{11}$$

$$\leq M(\delta)\left[\mu^{-1}(1-\mu)^{-1}\left[\|f\|_{\mathtt{C}^{\mu}_{-d,0}(\aleph)} + \|g\|_{\mathtt{C}^{\mu}_{0,d}(\aleph)}\right] \;\; + \|A\xi\|_{\aleph}\right],$$

where $M(\delta)$ is a constant that is independent of $g(t)$, $f(t)$, and ξ.

Proof. The CI (11) is derived from the estimate

$$\left\|\mathcal{U}'\right\|_{\mathtt{C}^{\mu}_{-d,0}(\aleph)} + ||A\mathcal{U}||_{\mathtt{C}^{\mu}_{-d,0}(\aleph)} \leq M(\delta)\mu^{-1}(1-\mu)^{-1}\left\|f\right\|_{\mathtt{C}^{\mu}_{-d,0}(\aleph)} + M\left\|A\mathcal{U}_0\right\|_{\aleph} \tag{12}$$

$$\leq M(\delta)\mu^{-1}(1-\mu)^{-1}\left\|f\right\|_{\mathtt{C}^{\mu}_{-d,0}(\aleph)} + M\left\|A\mathcal{U}_0\right\|_{\aleph}$$

for the solution of problem (5) and the estimate

$$\left\|\mathcal{U}''\right\|_{\mathtt{C}^{\mu}_{0,d}(\aleph)} + ||A\mathcal{U}||_{\mathtt{C}^{\mu}_{0,d}(\aleph)} \leq M(\delta)\mu^{-1}(1-\mu)^{-1}\left\|g\right\|_{\mathtt{C}^{\mu}_{0,d}(\aleph)} \tag{13}$$

$$+M(\delta)\left[\|A\mathcal{U}_0\|_{\aleph} \;+ \|A\mathcal{U}_1\|_{\aleph}\right]$$

associated with the solution of BVP (4) and the estimates

$$\|A\mathcal{U}_0\|_{\aleph} \leq M(\delta)\Big[\mu^{-1}(1-\mu)^{-1}\Big[\|f\|_{\mathtt{C}^{\mu}_{-d,0}(\aleph)} + \|g\|_{\mathtt{C}^{\mu}_{0,d}(\aleph)}\Big] + \|A\xi\|_{\aleph}\Big], \tag{14}$$

$$\|A\mathcal{U}_d\|_{\aleph} \leq M(\delta)\Big[\mu^{-1}(1-\mu)^{-1}\Big[\|f\|_{\mathtt{C}^{\mu}_{-d,0}(\aleph)} + \|g\|_{\mathtt{C}^{\mu}_{0,d}(\aleph)}\Big] + \|A\xi\|_{\aleph}\Big] \tag{15}$$

for the solution of BVP (1). Estimates (12) and (13) were obtained in [24]. Applying formula (10), we get

$$A\mathcal{U}_0 = N\left[2V^2e^{-dV}\left(\int\limits_{-d}^{0}\mu(\theta)\int\limits_{0}^{\theta}e^{-(\theta-y)A}f(y)dyd\theta + A\xi\right)\right.$$
$$\left.-V^2e^{-dV}\int\limits_{0}^{d}\left(e^{-(d-\theta)V}-e^{-(d+\theta)V}\right)g(\theta)d\theta\right]$$
$$+NV\left[\left(I-e^{-2dV}\right)\int\limits_{0}^{d}e^{-\theta V}g(\theta)d\theta-\left(I-e^{-2dV}\right)f(0)\right].$$

Therefore, the proof of estimate (14) is based on the triangle inequality and estimates (2), (3). Applying formula (8), we get

$$A\mathcal{U}(d) = \int\limits_{-d}^{0}\mu(\theta)e^{\theta A}d\theta A\mathcal{U}_0 + A\xi$$
$$+\int\limits_{-d}^{0}\mu(\theta)\int\limits_{0}^{\theta}Ae^{-(\theta-y)A}\left(f(y)-f(\theta)\right)dyd\theta+\int\limits_{-d}^{0}\mu(\theta)\left(I-e^{\theta A}\right)f(\theta)d\theta.$$

Therefore, the estimate (15) is proved based on the triangle inequality together with (2) and (3), which completes the proof of Theorem 1.

2 Illustrative examples

We now illustrate several applications of Theorem 1.

Firstly, the nonlocal BVP for an elliptic-parabolic equation

$$\begin{cases} -\mathcal{U}_{tt}-(a(z)\mathcal{U}_z)_z+\delta\mathcal{U}=g(t,z),\ t\in[0,d],\ z\in[0,b] \\ \mathcal{U}_t+(a(z)\mathcal{U}_z)_z-\delta\mathcal{U}=f(t,z),\ t\in[-d,0],\ z\in[0,b], \\ \mathcal{U}(0+,z)=\mathcal{U}(0-,z),\ \mathcal{U}_t(0+,z)=\mathcal{U}_t(0-,z),\ z\in[0,b], \\ \mathcal{U}(t,0)=\mathcal{U}(t,b),\ \mathcal{U}_z(t,0)=\mathcal{U}_z(t,b),\ t\in[-d,d] \end{cases} \tag{16}$$

with the integral condition $\mathcal{U}(d,z)=\int\limits_{-d}^{0}\mu(\Delta t)\,\mathcal{U}(\tau,z)\,d\tau+\xi(z),\ z\in[0,b]$ is considered. Problem (16) admits a unique smooth solution $\mathcal{U}(t,z)$ for smooth functions $a(z)$, with $a(z)=a(0)$ and $a(z)\geq a>0$ for $z\in(0,b)$, and for $g(t,z)$ $(t\in[0,d],\ z\in[0,b])$ and $f(t,z)$ $(t\in[-d,0],\ x\in[0,b])$, where $\delta>0$.

We define the space $L_2[0,b]$ of all square integrable functions $\xi(z)$ defined on $[0,b]$ and the spaces $W_2^1[0,b]$ and $W_2^2[0,b]$ with the norms

$$\|\xi\|_{W_2^1[0,b]}=\|\xi\|_{L_2[0,b]}+\left(\int\limits_0^b|\xi_z|^2\,dz\right)^{1/2},\ \left\|\xi^h\right\|_{W_2^2[0,b]}=\|\xi\|_{L_2[0,b]}+\left(\int\limits_0^b|\xi_{zz}|^2\,dz\right)^{1/2}.$$

This reduces mixed problem (16) to the nonlocal BVP (1) in a Hilbert space $\aleph=L_2[0,b]$ with a SAPDO A given by (16).

Theorem 2. The solution of nonlocal BVP (16) satisfies the CI

$$\| \mathcal{U}_{tt} \|_{\mathtt{C}^{\mu}_{0,d}(L_2(0,b))} + \| \mathcal{U}_t \|_{\mathtt{C}^{\mu}_{-d,0}(L_2(0,b))} + \| \mathcal{U} \|_{\mathtt{C}^{\mu}_{-d,d}(W_2^2(0,b))}$$

$$\leq M_c(\delta)\Big[\mu^{-1}(1-\mu)^{-1}\Big[\| g \|_{\mathtt{C}^{\mu}_{0,d}(L_2(0,b))} + \| f \|_{\mathtt{C}^{\mu}_{-d,0}(L_2(0,b))} \Big] + \|\xi\|_{W_2^2(0,b)}\Big].$$

Here, the constant $M(\delta)$ is independent of the functions $g(t,z)$, $f(t,z)$, and $\xi(z)$.

Proof of Theorem 2 builds upon the theoretical framework developed in Theorem 1, utilizing the symmetry properties of the operator associated with problem (16).

Secondly, let Ω denote the open unit cube in the n-dimensional Euclidean space $\mathbb{R}^n$, defined by $z_k \in (0,1)$ for $k = \overline{1,n}$ with S, so that $\overline{\Omega} = \Omega \cup S$. Within the domain $[-d,d] \times \Omega$, we formulate the BVP for a multi-dimensional mixed problem as follows:

$$\begin{cases} -\mathcal{U}_{tt} - \sum\limits_{r=1}^{n} (a_r(z)\mathcal{U}_{z_r})_{z_r} = g(t,z),\ t \in [0,d],\ z \in \Omega, \\ \mathcal{U}_t + \sum\limits_{r=1}^{n} (a_r(z)\mathcal{U}_{z_r})_{z_r} = f(t,z),\ t \in [-d,0],\ z \in \Omega, \\ \mathcal{U}(0+,z) = \mathcal{U}(0-,z),\ \mathcal{U}_t(0+,z) = \mathcal{U}_t(0-,z),\ z \in \overline{\Omega}, \\ \mathcal{U}(t,z) = 0,\ z \in S,\ [-d,d] \end{cases} \tag{17}$$

with the integral condition $\mathcal{U}(d,z) = \int\limits_{-d}^{0} \mu(\tau)\mathcal{U}(\tau,z)d\tau + \xi(z),\ z \in \overline{\Omega}$. Here, $a_r(z)$ $(z \in \Omega)$, $g(t,z)$ $(t \in (0,d),\ z \in \overline{\Omega})$, and $f(t,z)$ $(t \in (-d,0),\ z \in \overline{\Omega})$ are given smooth functions, with $a_r(z) \geq a > 0$.

We introduce the Hilbert space $L_2(\overline{\Omega})$ consisting of all square-integrable functions $\xi(z)$ defined on $\overline{\Omega}$, endowed with the norm

$$\| \xi \|_{L_2(\overline{\Omega})} = \sqrt{\int \cdots \int\limits_{z \in \overline{\Omega}} |\xi(z)|^2 dz_1 \cdots dz_n}$$

and the Hilbert spaces $W_2^1(\Omega), W_2^2(\Omega)$ defined on Ω, endowed with the norms

$$\|\xi\|_{W_2^1(\Omega)} = \| \xi \|_{L_2(\overline{\Omega})} + \sqrt{\int \cdots \int\limits_{z \in \Omega} \sum_{r=1}^{n} |\xi_{z_r}|^2 \, dz_1 \cdots dz_n}$$

and

$$\left\|\xi^h\right\|_{W_2^2(\Omega)} = \left\|\xi^h\right\|_{L_{2h}} + \sqrt{\int \cdots \int\limits_{z \in \Omega} \sum_{r=1}^{n} \left|\xi_{z_r \overline{z_r}}\right|^2 dz_1 \cdots dz_n}.$$

Problem (17) admits a unique smooth solution $u(t,x)$ for smooth functions $a_r(x)$, $g(t,x)$, and $f(t,x)$. Using this approach, the mixed problem (17) can be reduced to the nonlocal BVP (1) in the Hilbert space $H = L_2(\overline{\Omega})$ with a SAPDO A presented as in (17).

Theorem 3. The solution of nonlocal BVP (17) satisfies the CI

$$\|\mathcal{U}_{tt}\|_{\mathtt{C}^{\mu}_{0,d}(L_2(\overline{\Omega}))} + \|\mathcal{U}_t\|_{\mathtt{C}^{\mu}_{-d,0}(L_2(\overline{\Omega}))} + \|\mathcal{U}\|_{\mathtt{C}^{\mu}_{-d,d}(W_2^2(\Omega))}$$

$$\leq M_c(\delta)\Big[\mu^{-1}(1-\mu)^{-1}\Big[\|g\|_{\mathtt{C}^{\mu}_{0,d}(L_2(\overline{\Omega}))} + \|f\|_{\mathtt{C}^{\mu}_{-d,0}(L_2(\overline{\Omega}))}\Big] + \|\xi\|_{W_2^2(\Omega)}\Big].$$

The proof of Theorem 3 relies on the result given in Theorem 1, together with the symmetry properties of the operator associated with problem (17), and the CI for solutions of elliptic differential problems in $L_2(\overline{\Omega})$ as established in [24].

Conclusion

In the present paper, a nonlocal boundary value problem for an elliptic-parabolic equation subject to an integral condition is investigated. The well-posedness of the problem in weighted Hölder spaces is established. As an application, we derive coercivity inequalities for the solutions of mixed nonlocal boundary value problems associated with elliptic-parabolic equations. By applying the methods developed in this paper and in [25], we can establish the boundedness of solutions to a semilinear elliptic-parabolic equation.

Conflict of Interest

The authors declare no conflict of interest.

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Application of isotropic geometry to the solution of the Monge–Ampere equation

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This paper explores the Monge–Ampere equation in the context of isotropic geometry. The study begins with an overview of the fundamental properties of isotropic space, including its scalar product, distance formula, and the nature of surfaces and curvatures within this geometric framework. A special focus is placed on dual transformations with respect to the isotropic sphere, and the self-inverse property of the dual surface is established. The article formulates the Monge–Ampere equation for isotropic space and studies its invariant solutions under isotropic motions. Several lemmas are proved to demonstrate how solutions transform under linear modifications and isotropic motions. A specific class of Monge–Ampere-type nonlinear partial differential equations is solved analytically using dual transformations and separation of variables. Additionally, translation surfaces and their curvature properties are studied in detail, particularly through the lens of dual curvature. The results demonstrate the deep relationship between curvature invariants and Monge–Ampere-type equations and show how duality simplifies the solution of nonlinear PDEs. These methods can be used for surface reconstruction and modeling in isotropic spaces.

Keywords: isotropic geometry, Monge–Ampere equation, linear transformation, dual transformation, dual surface, curvature invariants, surface reconstruction, Dirichlet problem, PDE.

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Introduction

The Monge–Ampere equation occupies a prominent position in the theory of nonlinear partial differential equations due to its rich mathematical structure and wide applicability in geometric analysis, optimization, and mathematical physics. In classical differential geometry, this equation naturally arises in the context of surface theory, particularly in problems involving the reconstruction of a surface from curvature invariants [1]. A key feature of the Monge–Ampere equation is its close relationship with convex geometry and curvature prescriptions, as first systematically studied by I.Ya. Bakelman in the framework of the generalized Dirichlet problem for convex surfaces [2].

While significant progress has been achieved in Euclidean settings, the exploration of Monge–Ampere-type equations in non-Euclidean geometries, such as isotropic or semi-Riemannian spaces, is relatively recent. Isotropic geometry, which is a limiting case of semi-Euclidean geometry, provides a degenerate metric structure where distances are defined in a directionally dependent manner. This degenerate nature introduces novel phenomena not present in Riemannian or pseudo-Riemannian frameworks, thereby making isotropic geometry a fertile ground for discovering new geometric properties and solving PDEs under non-standard metrics [3].

In his book [4], O'Neill introduced fundamental concepts of semi-Riemannian geometry, from which the notion of isotropic and degenerate metric spaces naturally arises as a special geometric model.

The geometry of isotropic space R_{n+1}^n, as introduced, is characterized by a scalar product that is degenerate not along a single axis. The differential geometry of isotropic space was first studied by K. Strubecker [5, 6]. This leads to a unique classification of surfaces and transformations, including

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duality mappings with respect to the isotropic sphere. The theory of dual surfaces in isotropic space has been actively developed in recent works, including the classification and reconstruction of surfaces via dual curvature invariants [7]. The concept of dual transformation plays a central role in understanding curvature-driven surface generation, a theme that appears throughout this study.

One of the central motivations for the present paper stems from the growing body of research demonstrating that dual transformations in isotropic spaces offer elegant and computationally tractable methods for solving highly nonlinear equations such as the Monge–Ampere equation. Generalizing Lonen's works [8], Artykbaev, Sultanov, and Ismoilov have shown in several studies [9] that the total and mean curvatures of a surface and its dual are closely related, and that this relationship can be used to construct surfaces with prescribed curvature characteristics. The present study builds upon these foundational results and extends them in several directions. In the work by A. Polyanin [10], certain solutions of the Monge–Ampere equation are presented without derivation. In contrast, in this paper, we also explore a method for finding a different type of solution.

Firstly, we investigate the invariant form of the Monge–Ampere equation under isotropic motions and provide a detailed analysis of its solutions under linear perturbations. The result that any solution of the Monge–Ampere equation remains invariant under the addition of linear functions is well-known in classical settings, but here it is adapted and rigorously proven for isotropic geometry, leading to new insights into the geometry of the solution space.

Secondly, we focus on a special class of Monge–Ampere-type equations that arise in the context of translation surfaces in isotropic space. Using the techniques of separation of variables and dual transformation, we derive exact analytical solutions for these equations. In particular, we solve the equation

$$\left(\frac{\partial^2 z}{\partial x \partial y}\right)^2 - \frac{\partial^2 z}{\partial x^2} \cdot \frac{\partial^2 z}{\partial y^2} = f(x)y^2,$$

by assuming a quadratic *ansatz* and reducing the resulting PDE to a system of ODEs. The general solution is expressed in terms of integrals of nonlinear functions and demonstrates the applicability of this approach to constructing explicit surfaces with curvature-driven features.

Thirdly, we introduce and analyze translation surfaces whose total curvature of the dual surface is separable in the form $K^*(x, y) = \varphi(x)\psi(y)$. Using the inverse problem framework, we demonstrate that such dual curvature data uniquely determines the original surface up to an isotropic motion. This result contributes to the general problem of surface reconstruction from curvature invariants and finds relevance in applications such as surface design in computer graphics and shape optimization.

The geometric significance of these results lies in the structure of the isotropic space itself. Unlike Euclidean geometry, where the normal to a surface is uniquely defined by the metric, in isotropic geometry the notion of normality is more subtle. Here, we distinguish between the special normal vector $\vec{n}_m$ and the standard unit normal $\vec{n}$, and we show that the second fundamental form and the total curvature remain invariant under this choice. This confirms earlier findings in [11] and supports the use of duality-based methods for analyzing surface properties.

Furthermore, in the final section of the paper, we consider an application of the Monge–Ampere equation arising in the theory of plasticity and elasticity. A particular nonlinear equation governing large deformations of elastic plates is shown to be a higher-order Monge–Ampere-type equation. We demonstrate how this complex nonlinear equation can be transformed into a linear PDE with constant coefficients by applying dual transformations, and we solve it using separation of variables. The solution process also illustrates how dual mappings can be used not only in geometric but also in physical models.

It is worth noting that similar approaches have been explored by researchers studying special surfaces in isotropic spaces, such as ruled, helicoidal, and Weingarten-type surfaces [12–15]. However, the novelty of the present work lies in the formulation and solution of Monge–Ampere equations specifically in terms of dual curvature data, and the construction of explicit surface representations using integrable systems techniques. The article [16] investigates the parametric and algebraic representations of minimal surfaces in four-dimensional Euclidean space. It presents a generalized form of the Weierstrass–Enneper formula and analyzes the differential-geometric properties, projections, and modeling significance of such surfaces. This approach is closely related to the methods applied in solving the Monge—Ampere equation within isotropic geometry and provides an effective geometric framework for studying related problems

1 *Geometry of isotropic space*

Let $Ox_i\,(i = 1...\,n+1)$ be a coordinate system in affine space A_{n+1}. The scalar product of vectors $\overrightarrow{X}(x_1, x_2, ..., x_{n+1})$ and $\overrightarrow{Y}(y_1, y_2, ..., y_{n+1})$ is defined by the following formula:

$$(\overrightarrow{X}, \overrightarrow{Y}) = \begin{cases} \sum\limits_{i=1}^{n} x_i y_i, & if \;\; \sum\limits_{i=1}^{n} x_i y_i \neq 0, \\ x_{n+1} y_{n+1}, & if \;\; \sum\limits_{i=1}^{n} x_i y_i = 0. \end{cases} \tag{1}$$

Definition 1. An affine space A_{n+1}, in which the scalar product of vectors is calculated using formula (1), is called an isotropic space R^n_{n+1}.

The scalar product (1) is called a degenerate scalar product.

Minkowski space is a pseudo-Euclidean space with index 1. It serves as a geometric framework for the theory of relativity. This space also includes isotropic space as a special case. This can be seen in the following lemma.

Lemma 1. The isotropic space R^n_{n+1} is a subspace of the $(n+2)$-dimensional Minkowski space ${}^1R_{n+2}$ [11].

We define the norm of a vector in isotropic space R^n_{n+1} as the root of the scalar product of a vector $|\overrightarrow{X}| = \sqrt{(\overrightarrow{X}, \overrightarrow{X})}$, and the distances between points are defined as the norm of the vector connecting these points.

If $\overrightarrow{X} - \overrightarrow{Y} = \overrightarrow{AB}$, then the distance between points A and B is calculated using the following formula:

$$d = \begin{cases} \sqrt{\sum\limits_{i=1}^{n} (y_i - x_i)^2}, & if \;\; \sqrt{\sum\limits_{i=1}^{n} (y_i - x_i)^2} \neq 0, \\ |y_{n+1} - x_{n+1}|, & if \;\; x_i = y_i \;\; (i = \overline{1,n}). \end{cases} \tag{2}$$

The hyperplanes in $R^{n-i}_{n+1} (i = 1..n-2)$ can be of two types — isotropic R^{n-i}_{n+1} or Euclidean R_n. Hyperplanes $x_{n+1} = constant$ are Euclidean spaces. If a two-dimensional plane is considered and it is parallel to the Ox_{n+1} axis, then the intrinsic geometry of this plane becomes Galilean. The intrinsic geometry of the Galilean plane is presented in [17].

Since the isotropic space R^n_{n+1} is an affine space, there is an affine coordinate transformation that maintains the distance defined by formula (2). This transformation is called the motion of isotropic space R^n_{n+1} and is given by the following formula [3]:

$$X' = A \cdot X + B, \quad A = \begin{pmatrix} & & & & & 0 \\ & & A_E & & & ... \\ - - - & - - - & - - - & - - - & - - - & 0 \\ h_1 & h_2 & ... & h_{n-1} & h_n & 1 \end{pmatrix}, \tag{3}$$

where $A_E = (a_{ij})_{i,\,j\,=\,1..n}$ is the motion matrix in the Euclidean space R_n, $B^T = (b_1, b_2, ..., b_{n+1})$ is the parallel translation vector, and $(h_1, h_2, ..., h_n, 1)$ is the vector with sliding coordinate components.

If we define a sphere in isotropic space as a set of geometric points equidistant from a given point $(x_1^0, x_2^0, ..., x_n^0, x_{n+1})$, then its equation has the following form:

$$\sum_{i=1}^{n} (x_i - x_i^0)^2 = r^2.$$

We will call this sphere a metric sphere.

Let us consider in the R_{n+1}^n a surface defined by the following vector equation [1]:

$$r(u_1, u_2, ..., u_n) = \Big(x_i(u_1, u_2, ..., u_n) | (u_1, u_2, ..., u_n) \in D \subset R_n,\ i = \overline{1..(n+1)}\Big). \tag{4}$$

The first quadratic form of (4) a surface is defined by analogy with Euclidean space

$$I = ds^2 = \sum_{i=1}^{n}\sum_{j=1}^{n} g_{ij}\, du_i du_j,$$

where g_{ij} are the coefficients of the first quadratic form of the surface and

$$g_{ij} = (\overrightarrow{r'}_{u_i}, \overrightarrow{r'}_{u_j}) = \sum_{k=1}^{n} \left(\frac{\partial x_k}{\partial u_i}, \frac{\partial x_k}{\partial u_j}\right).$$

In the case where $ds^2 = 0$, an additional first quadratic form $ds^2 = dx_{n+1}$ is considered.

Since we mainly consider surfaces with a single-valued projection onto the plane $x_{n+1} = 0$, $ds^2 \neq 0$ for all points of the surface. Therefore, an additional first quadratic form of the surface is not considered.

The normal to the surface is taken to be the only orthogonal vector to all tangent vectors of the surface $\overrightarrow{n_m}(0, 0, ..., 0, 1)$ [9].

By analogy with the Euclidean space, the second quadratic form of the surface is defined as the scalar product of the vector of the second-order differential $d^2\overrightarrow{r}$ by the surface normal.

The surface normal needs a clear definition in order to handle the issues in question. To this end, the following formula can be offered:

The standard, and orthogonal, form of the normals is given by

$$\overrightarrow{n} = \frac{[\overrightarrow{r_{u_1}}, ..., \overrightarrow{r_{u_n}}]}{|[\overrightarrow{r_{u_1}}, ..., \overrightarrow{r_{u_n}}]|},$$

in which $[\overrightarrow{r_{u_1}}, ..., \overrightarrow{r_{u_n}}]$ signifies the vector product.

Since we consider two surface normals (the special normal $\overrightarrow{n_m}$ and the normal $\overrightarrow{n}$), the formula for the second quadratic form will be as follows:

$$II = (d^2 r, \overrightarrow{N}) = \sum_{i,j=1}^{n} D_{ij} du_i du_j.$$

Here, D_{ij} is the coefficient of the second quadratic form, calculated as:

1) $D_{ij} = \frac{\partial^2 x_{n+1}}{\partial u_i \partial u_j}$, if $\overrightarrow{N} = \overrightarrow{n_m}$,

2) $D_{ij} = (r_{u_i u_j}, \overrightarrow{n})$, if $\overrightarrow{N} = \overrightarrow{n}$.

In particular, if the surface is defined by the following equation

$$x_{n+1} = f(x_1, x_2, ..., x_n), \tag{5}$$

where $(x_1, x_2, ..., x_n) \in D \subset R_n$, then

$$II = \sum_{i,j=1}^{n} \frac{\partial^2 x_{n+1}}{\partial x_i \partial x_j} du_i du_j.$$

Hyperplanes parallel to the normal vector are isotropic hyperplanes of the corresponding dimension. In particular, a two-dimensional plane parallel to the normal vector is a two-dimensional isotropic plane, called the Galilean plane [7]. Therefore, the geometry is Galilean in a two-dimensional normal section of the surface. A two-dimensional normal section of the surface is a curve on the Galilean plane.

The curvature of the curve of the normal section is called the normal curvature of the curve on the surface. The normal curvature of the curve on the surface is calculated by the following formula:

$$k_n = \frac{II}{I}.$$

In isotropic space R_{n+1}^n, the second sphere is a surface with constant normal curvature in all directions, given by the following equation:

$$2x_{n+1} = \sum_{i=1}^{n} {x_i}^2. \tag{6}$$

Definition 2. The surface defined by equation (6) is called an isotropic sphere in R_{n+1}^n.

The mean and total curvatures are the main geometric characteristics of a surface. The total curvature of the surface (4) is calculated as:

$$K = \frac{\det \left| (D_{ij})_{i,j=\overline{1.n}} \right|}{\det \left| (g_{ij})_{i,j=\overline{1.n}} \right|}.$$

Lemma 2. The total curvatures of the surface (5), determined by the normal and the special normal, are mutually equal $K = K_m$ [11].

2 *Dual transformation with respect to the isotropic sphere*

Let the surface F be given by the equation (5) and suppose it lies within the isotropic sphere of the space R_{n+1}^n. Consider the set of points obtained via dual mapping of the tangent hyperplanes to the surface F at each of its points, with respect to the isotropic sphere. This set forms a new surface defined as follows.

Definition 3. The surface F^* is called the *dual surface* to the surface F with respect to the isotropic sphere.

If the surface F is regular, then the dual surface F^* is also a surface and is given by the system:

$$\begin{cases} x_i^* = \dfrac{\partial f}{\partial x_i}, \qquad i = 1, \; ... \; , n, \\ x_{n+1}^* = \displaystyle\sum_{i=1}^{n} x_i \cdot \frac{\partial f}{\partial x_i} - f. \end{cases}$$

Theorem 1. The dual image of the surface F^* coincides with the surface F; that is,

$$F^{**} = F.$$

The total curvature of the surface (5) has the form:

$$K = \begin{vmatrix} f_{x_1x_1} & f_{x_1x_2} & \cdots & f_{x_1x_n} \\ f_{x_2x_1} & f_{x_2x_2} & \cdots & f_{x_2x_n} \\ \vdots & \vdots & \ddots & \vdots \\ f_{x_nx_1} & f_{x_nx_2} & \cdots & f_{x_nx_n} \end{vmatrix}.$$

The right-hand side is the Monge–Ampere operator. In isotropic space, the problem of recovering a surface from its total curvature is equivalent to solving the Monge–Ampere equation.

3 Monge–Ampere equation

I.Ya. Bakelman studied the connection between the extrinsic curvature of convex surfaces and the second-order nonlinear Monge–Ampere equation [2]. In this case, I.Ya. Bakelman showed that the solution of the generalized Dirichlet problem for the Monge–Ampere equation exists and is unique by estimating the area of the normal image of the surface. The listed problems were solved only if the domain $D \subset R_2$ is convex where the function is defined. By applying the geometry of the Galilean space, A. Artykbaev solved the problem for the existence and uniqueness of the convex surface for the given extrinsic curvature if the domain $D \subset R_2$ is non-convex [18]. Also, in the article [7], the concept of generalized extrinsic curvature is given, and the existence and uniqueness of the solution to the Monge–Ampere equation in the multi-connected domain is proved. The Monge–Ampere equation in a discrete setting with a special invariant can be observed in the Sharipov's works [19]. In [20, 21], Lions and Urbas established the existence and regularity results for a wide class of fully nonlinear elliptic PDEs. The paper [22], provides a clear and accessible overview of the modern theory of the Monge–Ampere equation. It discusses the notion of Alexandrov (weak) solutions, interior and boundary regularity results, and classical methods developed by Calabi, Cheng–Yau, and Lions. The article also emphasizes the analytical and geometric aspects of the equation, offering valuable insights into the existence and smoothness of convex solutions to Dirichlet-type problems. In this paper, we address the problem of reconstructing a surface in three-dimensional isotropic space by solving the Monge–Ampere equation, using the relationship between the surface equation and the Monge–Ampere equation in isotropic space. To this end, we first introduce the Monge–Ampere equation in three-dimensional space.

It is known that the Monge–Ampere equation is generally as follows:

$$z_{xx}z_{yy} - z_{xy}^2 = \phi\left(x, y, z, z_x, z_y\right).$$

In this case, if $\phi\left(x, y, z, z_x, z_y\right) > 0$, the equation is elliptic and its solution is a convex surface equation. Now, if we consider this equation in the semi-Euclidean space, that is, in the isotropic space, it will be as follows:

$$K\left(x, y\right) = z_{xx}z_{yy} - z_{xy}^2.$$

3.1 General invariant solution

We present some statements related to the solution of the Monge–Ampere equation and motions in isotropic space.

Lemma 3. If the function $z = f(x, y)$ is a solution of the Monge–Ampere equation

$$\det D^2 f = f_{xx}f_{yy} - (f_{xy})^2 = F(x, y),$$

then the function

$$z = f(x, y) + C_1 x + C_2 y + C$$

is also a solution of the same equation.

Proof. This follows from the fact that the Monge–Ampere operator involves only second-order partial derivatives. Since the linear part $C_1 x + C_2 y + C$ vanishes under second-order differentiation of second order, it does not affect the operator:

$$\frac{\partial^2}{\partial x^2}(f + C_1 x + C_2 y + C) = f_{xx}, \quad \frac{\partial^2}{\partial y^2}(f + C_1 x + C_2 y + C) = f_{yy},$$

$$\frac{\partial^2}{\partial x \partial y}(f + C_1 x + C_2 y + C) = f_{xy}.$$

Therefore, the Monge–Ampere determinant remains unchanged. □

Lemma 4. The surface defined by the function

$$z = f(x, y) + C_1 x + C_2 y + C$$

can be obtained from the surface $z = f(x, y)$ by an isotropic motion.

Proof. Consider applying (3) an isotropic motion to the surface, specifically an isotropic shear transformation (translation along the z-axis depending linearly on x and y). This motion is given by:

$$\begin{cases} x' = x, \\ y' = y, \\ z' = Ax + By + z + C, \end{cases} \tag{7}$$

where $A, B, C \in \mathbb{R}$ are constants.

Applying this transformation to the surface $z = f(x, y)$, we obtain:

$$z' = f(x, y) + Ax + By + C,$$

which coincides with the general form $f(x, y) + C_1 x + C_2 y + C$. Hence, the transformation corresponds to (7) a motion in isotropic space. □

Taking Lemmas 3 and 4 into account, we will not consider the linear case in the subsequent solutions. The reason is that, in isotropic space, adding a linear term results in two different solutions representing the same surface, differing only by their position.

4 Analytical solution of a Monge–Ampere-type equation

We study a nonlinear Monge–Ampere-type partial differential equation of the form:

$$\left(\frac{\partial^2 z}{\partial x \partial y}\right)^2 - \frac{\partial^2 z}{\partial x^2} \cdot \frac{\partial^2 z}{\partial y^2} = f(x) y^2. \tag{8}$$

Our aim is to construct general and particular solutions, including transformation invariance and exact construction for a specific case.

We consider a quadratic ansatz in y:

$$z(x, y) = \varphi(x) y^2 + U(x) y + V(x)$$

and compute the necessary derivatives:

$$\frac{\partial^2 z}{\partial x \partial y} = 2\varphi'(x)y + U'(x),$$
$$\frac{\partial^2 z}{\partial x^2} = \varphi''(x)y^2 + U''(x)y + V''(x),$$
$$\frac{\partial^2 z}{\partial y^2} = 2\varphi(x).$$

Substituting into equation (8), we obtain a polynomial in y. Matching coefficients gives the following system:

$$4(\varphi')^2 - 2\varphi\varphi'' = f(x),$$
$$4\varphi' U' - 2\varphi U'' = 0,$$
$$(U')^2 - 2\varphi V'' = 0.$$

Solving this system, we obtain the general solution:

$$w(x,y) = \varphi(x)y^2 + C_1 y \int \varphi^2(x)\, dx + \frac{1}{2}C_1^2 \int_a^x (x-t)\varphi^3(t)\, dt,$$

where $\varphi(x)$ satisfies the nonlinear ODE

$$\varphi\varphi'' = 2(\varphi')^2 - \frac{1}{2}f(x).$$

Particular case: $f(x) = 0$.

We assume $\varphi(x) = A/(x+C)$ and verify that it satisfies:

$$\varphi\varphi'' = 2(\varphi')^2.$$

This leads to a family of solutions of the form:

$$w(x,y) = \frac{A}{x+C}y^2 - \frac{C_1 A^2 y}{x+C} + \frac{C_1^2}{2}\int_a^x (x-t)\left(\frac{A}{t+C}\right)^3 dt.$$

Particular case: $f(x) = x^2$.

We seek $\varphi(x) = ax^n$. The equation

$$a^2 n(n-1)x^{2n-2} = 2a^2 n^2 x^{2n-2} - \frac{1}{2}x^2$$

is satisfied when $n = 2$, yielding

$$a = \pm\frac{1}{2\sqrt{3}}.$$

Hence,

$$\varphi(x) = \frac{1}{2\sqrt{3}}x^2.$$

Using this, we compute the full solution:

$$w(x,y) = \frac{1}{2\sqrt{3}}x^2y^2 + \frac{C_1 y}{60}x^5 \frac{C_1^2}{48\sqrt{3}}\int_a^x (x-t)t^6 dt.$$

5 *Translation surface*

When the surface is uniquely projected onto the Oxy plane in isotropic space, it is given by the parametrization:

$$\overrightarrow{r}(x,y) = x\cdot\overrightarrow{i} + y\cdot\overrightarrow{j} + (f(x)+g(y))\cdot\overrightarrow{k}. \tag{9}$$

In this case, the coefficients of the first fundamental form are: $E=1,\ F=0,\ G=1$, and the coefficients of the second fundamental form are: $L=f''(x),\ M=0,\ N=g''(y)$.

Taking this into account, the formula for the total curvature of the surface can be obtained as:

$$K = f''(x)\cdot g''(y).$$

The total curvature of the dual surface is given by:

$$K^* = \frac{1}{f''(x)\cdot g''(y)}.$$

Let

$$K^* = \varphi(x)\cdot\psi(y)\neq 0$$

be a function defined on the domain $D\subset\mathbb{R}^2$, where $\varphi(x)$ and $\psi(y)$ are continuous, non-vanishing functions.

Lemma 5. If the total curvature of the dual surface is given by $K^*=\varphi(x)\cdot\psi(y)$, then there exists a surface of the form

$$\vec{r}_\lambda(x,y) = x\,\vec{i} + y\,\vec{j} + \left(\int\left[\int\frac{1}{\lambda\varphi(x)}\,dx\right]dx + \int\left[\int\frac{\lambda}{\psi(y)}\,dy\right]dy\right)\vec{k}, \tag{10}$$

for which K^* is the total curvature of its dual surface and $\varphi(x),\psi(y)\in C^2(D)$.

Proof. From the general formula (10) for the total curvature of a dual surface in a translation surface, we have:

$$\frac{1}{f''_{xx}(x)\cdot g''_{yy}(y)} = \varphi(x)\cdot\psi(y).$$

Rewriting, we obtain:

$$\frac{1}{f''_{xx}(x)\cdot\varphi(x)} = g''_{yy}(y)\cdot\psi(y).$$

This leads to the separation of variables as:

$$\frac{1}{f''_{xx}(x)\cdot\varphi(x)} = \lambda = g''_{yy}(y)\cdot\psi(y), \tag{11}$$

where λ is a constant of separation.

Solving (11) these differential equations gives:

$$f_\lambda(x) = \int\left[\int\frac{1}{\lambda\varphi(x)}\,dx + C_1\right]du + C_1',$$

$$g_\lambda(y) = \int\left[\int\frac{\lambda}{\psi(y)}\,dv + C_2\right]dy + C_2'.$$

By substituting the functions $f(x)$ and $g(y)$ into the translation surface equation (9) and omitting their linear parts, we obtain the formula presented in Lemma 5. □

Theorem 2. *(i)* If the surface belongs to a translation surface and the total curvature of the dual surface is $K^* = C_0 = \text{constant} \neq 0$, then the surface has the following equation: $\vec{r}(x,y) = x\,\vec{i} + y\,\vec{j} + \left(\frac{C_0}{2}x^2 + \frac{1}{2C_0}y^2\right)\vec{k}$.

(ii) If the total curvature is given in the form $K^* = \varphi(x)\cdot\psi(y)$, then the surface is given by formula (9).

(iii) However, if the total curvature is a non-separable function, i.e. $K^* = K^*(x,y) \neq \varphi(x)\cdot\psi(y)$, then the problem has no solution in the class of translation surfaces.

Proof. Each case in the theorem is considered separately.

(i) When $K^* = C_0 = \text{constant}$, the result is already established in [8].

(ii) When K^* is separable as $\varphi(x)\cdot\psi(y)$, the theorem follows directly from Lemma 5.

(iii) Finally, when $K^* = K^*(x,y)$ is non-separable, a surface of the form

$$\vec{r}(u,v) = x\,\vec{i} + y\,\vec{j} + (f(x) + g(y))\,\vec{k}$$

has curvature

$$K^*(x,y) = \frac{1}{f''_{xx}(x)} \cdot \frac{1}{g''_{yy}(y)}$$

which is necessarily separable in variables. This contradiction implies that no such transfer surface can exist in the non-separable case. □

6 *Applications of the Monge–Ampere equation*

The Monge–Ampere equation has been widely applied across various scientific fields. Many well-known equations include the Monge–Ampere equation as a structural component. Let us consider one such equation. By doing so, we also address the applicability of the results obtained.

Consider the nonlinear partial differential equation

$$\frac{\partial^2 z}{\partial x^2}\left[\left(\frac{\partial^2 z}{\partial x \partial y}\right)^2 - \frac{\partial^2 z}{\partial x^2}\cdot\frac{\partial^2 z}{\partial y^2}\right] = \frac{\partial^2 z}{\partial y^2}.$$

This equation is relevant in two-dimensional plasticity theory, where $z = f(x,y)$ acts as the generating function. This equation represents a particular case of the nonlinear elastic plate model, describing the bending deformations of a thin elastic plate. It models the variation of elastic energy based on the total curvature of the surface. Due to its nonlinear nature, the equation is suitable for analyzing large deformations. In the absence of external forces, it describes situations where only internal elastic forces are at play.

Let $z = f(x,y)$ be a solution. Then, the following transformed functions also satisfy the same equation:

$$z_1 = \pm C_1^{-2} f(C_1 x + C_2, C_3 y + C_4),$$

where $C_1, \ldots, C_4$ are arbitrary constants.

Let us try to solve the equation. We now define a new function

$$\omega(x,y) = \frac{\partial z}{\partial x}.$$

We consider $\omega(x,y)$ as a surface and move to the surface that is dual to $\omega^*(x,y)$ and use the following 3-dimensional dual transformation:

$$\begin{cases} x^* = \frac{\partial \omega}{\partial x}, \\ y^* = \frac{\partial \omega}{\partial y}, \\ z^* = x\frac{\partial \omega}{\partial x} + y\frac{\partial \omega}{\partial y} - \omega. \end{cases}$$

The goal is to convert the original nonlinear equation into a second-order linear PDE. After performing the transformation, we obtain:

$$(1+x^{*2})^2\frac{\partial^2 z^*}{\partial x^{*2}}+2x^*y^*(1+x^{*2})\frac{\partial^2 z^*}{\partial x^*\partial y^*}+y^{*2}(x^{*2}-1)\frac{\partial^2 z^*}{\partial y^{*2}}=0. \tag{12}$$

This is a hyperbolic partial differential equation. To further simplify it, we use the coordinate transformation:

$$t=\arctan x^*,\quad \xi=\frac{1}{2}\ln(1+x^{*2})-\ln y^*,\quad W=\frac{z^*}{\sqrt{1+x^{*2}}}.$$

Under this change of variables, equation (12) is transformed into a linear PDE with constant coefficients:

$$\frac{\partial^2 W}{\partial t^2}=\frac{\partial^2 W}{\partial \xi^2}-W.$$

We now solve the PDE, using the method of separation of variables. Let

$$W(t,\xi)=T(t)\cdot X(\xi).$$

Substitute into the equation:

$$T''(t)X(\xi)-T(t)X''(\xi)+T(t)X(\xi)=0.$$

Divide both sides by $T(t)X(\xi)$:

$$\frac{T''(t)}{T(t)}-\frac{X''(\xi)}{X(\xi)}+1=0.$$

This implies:

$$\frac{T''(t)}{T(t)}+1=\frac{X''(\xi)}{X(\xi)}=-\lambda.$$

So we obtain two ODEs:

$$\begin{aligned} T''(t)+(\lambda+1)T(t)&=0,\\ X''(\xi)+\lambda X(\xi)&=0.\end{aligned}$$

The general solutions are

$$\begin{aligned} T(t)&=C_1\cos(\sqrt{\lambda+1}\,t)+C_2\sin(\sqrt{\lambda+1}\,t),\\ X(\xi)&=A_1\cos(\sqrt{\lambda}\,\xi)+A_2\sin(\sqrt{\lambda}\,\xi).\end{aligned}$$

Therefore, the general solution to the PDE is

$$W(t,\xi)=\left[A_1\cos(\sqrt{\lambda}\,\xi)+A_2\sin(\sqrt{\lambda}\,\xi)\right]\cdot\left[C_1\cos(\sqrt{\lambda+1}\,t)+C_2\sin(\sqrt{\lambda+1}\,t)\right].$$

We now reverse the transformation steps to reconstruct $\omega(x,y)$.

Recover $Z(X,Y)$, recall that

$$W=\frac{z^*}{\sqrt{1+x^{*2}}}.$$

From this it follows

$$z^*=W\cdot\sqrt{1+x^{*2}}.$$

Using Theorem 1, which states that the dual transformation is self-inverse, we find $\omega(x, y)$:

$$\omega(x, y) = xx^* + yy^* - z^*(x^*, y^*).$$

Integrating $\omega(x, y)$, we get $z = f(x, y)$. Finally, since $\omega = \dfrac{\partial z}{\partial x}$, we integrate:

$$z(x, y) = \int \omega(x, y)\, dx + \phi(y),$$

where $\phi(y)$ is an arbitrary function of y arising from the integration.

Conclusion

In this paper, we investigated the Monge–Ampere equation in three-dimensional isotropic space and demonstrated its strong connection with the geometry of surfaces, dual transformations, and curvature invariants. By leveraging the properties of isotropic geometry, particularly the degenerate metric and dual mappings, we formulated and solved a class of nonlinear Monge–Ampere-type equations.

Using of dual transformation techniques, we linearized a complex nonlinear PDE, solved it analytically using separation of variables, and reconstructed the original surface using the inverse dual transform. The method proved effective in simplifying the solution process and understanding the geometric structure behind the equation.

We also studied translation surfaces and provided conditions under which such surfaces can be constructed from given curvature functions. In particular, we showed that the total curvature of the dual surface imposes strict conditions on the form of the original surface.

The results obtained in this work can serve as a foundation for further research in isotropic differential geometry, geometric PDEs, and applications in computer graphics, elasticity, and geometric modeling. The approach of using duality and curvature invariants offers a powerful framework for the analysis and reconstruction of surfaces governed by Monge–Ampere-type equations.

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Conflict of Interest

The author declare no conflict of interest.

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Rational analogues of Bernstein–Szabados operators on several intervals

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Bernstein polynomials play a very important role in approximation theory, probability theory, computer aided geometric design and many other areas. In 2017 J. Szabados constructed polynomial operators that can be considered as the most natural generalization to several intervals of the classical Bernstein operators. Their main advantages include fixed difference between degrees of the used polynomials and the number of used nodes. Unfortunately, they exist only under strong restrictions on the geometry of intervals (intervals have to form a polynomial inverse image of an interval). The main goal of the paper is to present a rational operator that generalizes J. Szabados' construction, and exists for an arbitrary system of several intervals. Moreover, this construction (unlike J. Szabados') is a linear positive operator. One of the main ingredients in the construction is the fact (which was proved by M.G. Krein, B.Ya. Levin, and A.A. Nudel'man) that an arbitrary finite system of real intervals is the inverse image of an interval by a rational function with precisely one pole at each gap. The approximation properties of such operators are studied as well. Further possible generalizations (of V.S. Videnskii's operators to one interval) are considered.

Keywords: Bernstein polynomials, rational operators, several intervals, inverse images, rate of approximation, linear positive operators, Videnskii rational functions, Ditzian–Totik modulus of continuity.

2020 Mathematics Subject Classification: 41A35, 41A20.

Introduction

Approximation theory and harmonic analysis on several intervals of the real line is an area that attracts attention of many researchers. For example, the asymptotics of Chebyshev polynomials and their norms were studied in papers [1–3]; several related aspects of the theory of orthogonal polynomials can be found in [4–6]; the capacity of several intervals was considered in [7,8]; different approximation problems on several segments were solved in [9–11], among many others.

The polynomial inverse image method plays an important role in solving a number of problems in this field (see, for example, the survey [12], as well as later works with references to it). The method consists of several steps. Firstly it is necessary to prove the result for a system of intervals, that is a preimage of an interval under polynomial mapping (inverse image of an interval). The next step is to prove the result for arbitrary polynomials on an inverse image of an interval. Finally it is necessary to approximate an arbitrary system of intervals by inverse images, varying some endpoints of the intervals.

Sometimes, for example in polynomial interpolation, slight change of the system of intervals gives dramatically worse the asymptotic behaviour of the Lebesgue constants (see, for example, [13]). In [14] it was proved that even in the case of interpolation by polynomials on several intervals, it is useful to replace the preimage of an interval under polynomial mappings with the preimage of an interval under rational functions with fixed denominator. Then instead of varying the systems of intervals it is possible to vary the poles.

J. Szabados in [15] constructed analogues of Bernstein polynomials on several intervals with similar reproducing and interpolation properties only for the case of polynomial preimages of an interval. More

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precisely, they preserve polynomials up to a fixed degree (like classical Bernstein polynomials preserve polynomials up to degree one) and interpolate at the endpoints of the intervals (like classical Bernstein polynomials interpolate at ± 1.) We refer to [16–18] for the theory of classical Bernstein polynomials.

However, Szabados' operators are not positive, unlike the classical Bernstein polynomials, and polynomial preimages correspond to very special systems of intervals.

The main goal of the paper is to show that the use of rational functions makes it possible to overcome these disadvantages of BS.

Main results

Let $J_s = \cup_{j=1}^{s} I_j$, $I_j = [a_j, b_j]$, $s \geq 1$, be a system of real intervals. More precisely, $0 = a_1 < b_1 < \ldots < a_s < b_s = 1$, and let Π_n be the set of polynomials of degree at most n. Let $C(J_s)$ be the space of continuous functions on J_s with the sup-norm.

J. Szabados' construction works for the case where $J_s = p^{-1}([0,1])$, where $p \in \Pi_m$, $m \geq s$. For $n \in \mathbb{N}$, let $x_{k1} < \ldots < x_{km_k}$ be defined by

$$p(x_{ki}) = \frac{k}{n}, \quad i = 1, \ldots, m_k, \quad k = 0, \ldots, n,$$

where m_k are given explicitly (in most cases they are equal to m, with normalization $p(0) = 0$).

For an arbitrary $f(x) \in C(J_s)$, let

$$\tilde{L}_k(f, x) = \sum_{i=1}^{m_k} f(x_{ki}) \tilde{\ell}_{ki}(x) \in \Pi_{m_k - 1}, \quad k = 0, \ldots, n,$$

be the Lagrange interpolation polynomial with respect to the nodes x_{ki}. J. Szabados' operator is given by

$$BS_n(f, x) = \sum_{k=0}^{n} L_k(f, x) b_{nk}(p(x)), \quad x \in J_s,$$

where

$$b_{nk}(x) = \left(\begin{array}{c} n \\ k \end{array} \right) x^k (1-x)^{n-k}, \quad k = 0, \ldots, n,$$

are the fundamental functions of the Bernstein polynomials.

The main advantage of the operators BS compared to ordinary Bernstein polynomials of an extended function onto $[0,1]$ is that the difference between the number of function values and the degree of the operator is $m - s - 1$, i.e., independent of n, just as in the case of the classic Bernstein polynomials. But BS are not positive operators and the assumption $J_s = p^{-1}([0,1])$ with $s > 1$ is valid for very special systems of intervals only (for J_s in general position it is not satisfied for any $m \geq s$).

Now we will give the construction of rational analogues of operators BS that preserve their main advantages, exist for all J_s and n, and are positive.

From [19] it follows that for any system J_s there exists a polynomial $S \in \Pi_{s-1}$ with exactly one zero at each gap (b_i, a_{i+1}), $i = 1, \ldots, l-1$ such that $J_s = R^{-1}\left([0,1]\right)$, where

$$R(x) = \frac{\prod_{i=1}^{s}(x - a_i)}{S(x)}.$$

Now for $n \in \mathbb{N}$ let $x_{k1} < \ldots < x_{ks}$ be such that

$$R(x_{ki}) = \frac{k}{n}, \quad i = 1, \ldots, s, \quad k = 0, \ldots, n.$$

Then for an arbitrary $f(x) \in C(J_s)$, let

$$L_k(f,x) = \left(\sum_{i=1}^{s} f(x_{ki}) \ell_{ki}^2(x) \right) \Bigg/ \sum_{i=1}^{s} \ell_{ki}^2(x) , \quad k = 0, \ldots, n,$$

where

$$\ell_{ki}(x) = \frac{S(x_{ki})}{S(x)} \prod_{\substack{j=1 \\ j \neq i}}^{m_k} \frac{x - x_{kj}}{x_{ki} - x_{kj}}$$

are the fundamental Lagrange rational functions with the denominator $S(x)$. Rational analogues of Bernstein–Szabados operators are then defined by the formula

$$B_n(f,x) = \sum_{k=0}^{n} L_k(f,x) b_{nk}(R(x)), \quad x \in J_s. \tag{1}$$

Those operators are linear and positive, each term in (1) is a rational function of degree $sn + s - 1$, they preserve constants and interpolate f at the endpoints of J_s.

Now we state an analogue of J. Szabados' convergence estimate for (1). Let

$$\varphi(x) = \sqrt{(x - a_j)(b_j - x)} \quad \text{if} \quad x \in I_j, \; j = 1, \ldots, s,$$

and define the Ditzian–Totik modulus of continuity as

$$\omega_\varphi(f,t) = \sup_{0<h\leq t} \|\Delta_{h\varphi(x)} f(x)\|_{J_s},$$

where the difference is meant to be zero if any of the arguments is outside J_s, and we assume that t is so small that both $x \pm \varphi(x)$ fall into the same interval I_j. Further let

$$V(f) = \sup_{x,y \in J_s} |f(x) - f(y)|.$$

Theorem 1. For an arbitrary $f \in C(J_s)$ we have

$$\|f(x) - B_n(f,x)\|_{J_s} \leq c\omega_\varphi \left(f, \frac{1}{\sqrt{n}} \right) + c \frac{V(f)}{\sqrt{n}}.$$

(Here and in what follows, c always denotes a positive constant depending on J_s, but independent of n, not necessarily the same at each occurrence.)

Proof. The proof goes essentially the same way as in [15, Proof of Theorem 1]. Let $x \in I_j$. Since both operators L_k and the classic Bernstein polynomials reproduce constants, we get

$$|f(x) - B_n(f,x)| \leq c \sum_{k=0}^{n} \sum_{i=1}^{s} |f(x) - f(x_{ki}| \ell_{ki}^2(x) b_{nk}(R(x))$$

$$\leq c \sum_{k=0}^{n} \left\{ \omega_\varphi \left(f, \frac{1}{\sqrt{n}} \right) \left[\frac{\sqrt{n}}{\varphi(x)} (x - x_{kj}) + 1 \right] \ell_{kj}^2(x) + V(f) \sum_{i \neq j} \ell_{ki}^2(x) \right\} b_{nk}(R(x)).$$

We estimate the right-hand side sum for $0 \leq k \leq n/2$; the other part can be handled similarly. Then it is sufficient to consider the case $0 \leq R(x) \leq 4/5$, since for $4/5 \leq y \leq 1$ the estimate

$$\sum_{k=0}^{[n/2]} \left(\frac{n}{k}\right)^{\alpha} b_{nk}(y) \leq n^{\alpha}(4/5)^{n/2}, \quad \alpha \geq 0,$$

was proved in [15].

Let first $k = 0$. Then we have $|\ell_{0i}(x)| \leq c,\ x_{0i} = a_i,\ i = 1, \dots, s$, and

$$\frac{(x - a_i)\ell_{0i}^2(x)}{\varphi(x)} \leq c\frac{x - a_i}{\varphi(x)} \leq c\sqrt{R(x)}.$$

Now let $1 \leq k \leq n/2$.

Because of $|\ell_{kj}(x)| \leq c$ we get

$$\sum_{k=1}^{[n/2]} \ell_{kj}^2(x) b_{nk}(R(x)) \leq c \sum_{k=1}^{[n/2]})b_{nk}(R(x)) \leq c. \tag{2}$$

On the other hand, $|x - x_{kj}| \leq |R(x) - \frac{k}{n}|$, therefore by [15, Lemma 1] applied with $\alpha = 0,\ \beta = 1$ yields

$$\sum_{k=1}^{[n/2]} |x - x_{kj}|\ell_{kj}^2(x) b_{nk}(R(x)) \leq c \sum_{k=1}^{[n/2]} |x - x_{kj}| b_{nk}(R(x))$$

$$\leq \sum_{k=1}^{[n/2]} |R(x) - \frac{k}{n}| \leq c\sqrt{\frac{R(x)}{n}} \leq \frac{\varphi(x)}{\sqrt{n}}. \tag{3}$$

Finally, using $\ell_{ki}^2(x) \leq c\sqrt{\frac{n}{k}}|x - x_{kj}|,\ i \neq j$, and [15, Lemma 1] with $\alpha = 1/2,\ \beta = 1$, we obtain that

$$\sum_{k=1}^{[n/2]} \sum_{i \neq j} \ell_{ki}^2(x) b_{nk}(R(x) \leq c \sum_{k=1}^{[n/2]} \sqrt{\frac{n}{k}} |x - x_{kj}| b_{nk}(R(x))$$

$$\leq c \sum_{k=1}^{[n/2]} \sqrt{\frac{n}{k}} \left| R(x) - \frac{k}{n} \right| b_{nk}(R(x)) \leq \frac{c}{\sqrt{n}}. \tag{4}$$

Combining (2)–(4) completes the proof. □

Remark 1. Substituting

$$L_k^{(1)}(f, x) = \sum_{i=1}^{s} f(x_{ki})\ell_{ki}(x)$$

instead of $L_k(f, x)$ in (1) other operators (denoted by $B_n^{(1)}(f, x)$) can be constructed. They are rational functions of degree $ns + s - 1$, use ns function values, satisfy $B_n^{(1)}(R, x) = R(x)$ for all $x \in J_s$, but don't form positive operators.

Remark 2. V.S. Videnskii in a series of papers (compare also his book [18], and paper [20]) considered a generalization of the classical Bernstein polynomials for rational approximation on $[0, 1]$.

More precisely, Videnskii's operators have the form

$$V_n(f, x) = \sum_{k=0}^{n} f(\tau_{nk})\, u_{nk}(x),$$

where the nodes τ_{nk} are determined by the formulas

$$\phi_n(\tau_{nk}) = \frac{k}{n}, \quad k = 0, 1, .., n,$$

$$\phi_n(x) = \frac{1}{n}\sum_{i=1}^{n} h_{ni}(x),$$

and the rational functions $u_{nk}(x)$, which are analogues of $b_{nk}(x)$ from the classical Bernstein operators, are defined with the help of the generating function as follows:

$$h_{ni}(x) = \frac{\rho_{ni} x}{1 + \rho_{ni} - x}, \quad \rho_{ni} > 0, \quad i = 0, 1, ...,$$

$$g_n(x, y) = \sum_{k=0}^{n} y^k u_{nk}(x),$$

$$g_n(x, y) = \prod_{i=0}^{n-1} \left(h_{ni}(x)\, y + (1 - h_{ni}(x))\right).$$

It is possible to generalize this construction to the case of several intervals by the same method as above.

Conclusion

Bernstein polynomials have many applications in modern science and technology, but up to now there is no complete analogue of them for the case of several (greater than one) intervals of the real axis. In this paper a generalization of Bernstein polynomials to rational functions on several intervals is constructed. Those operators exist for an arbitrary (unlike previously constructed generalizations) system of intervals. Approximation properties of the presented operators are studied as well.

Conflict of interests

There is no conflict of interest.

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Investigation of the solution of a boundary value problem with variable coefficients whose principal part is the Cauchy–Riemann equation

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This study is devoted to obtaining an analytical expression for the solution of a non-local boundary value problem for a linear inhomogeneous differential equation with variable coefficients, which principle part is the Cauchy–Riemann equation. Since the Cauchy–Riemann equation is a first-order elliptic equation, the problem formulated with a classical boundary condition in a finite domain is ill-posed. Defining a boundary condition for a first-order elliptic equation within a finite domain requires special investigation. For a first-order elliptic equation in the x_1ox_2 plane, a new boundary condition is proposed within a bounded region that is concave in the x_2 direction, and an expression for the solution is obtained. For this purpose, using the fundamental solution of the principal part of the equation, the main relation consisting of two parts is obtained, the first part yields an arbitrary solution to the equation, and the second part gives the boundary values of the solution representing the necessary conditions. Utilizing these necessary and specified boundary conditions, a system of Fredholm integral equations of the second kind with a singular kernel is constructed to find a solution, and a method for elimination the singularity in the solution is proposed.

Keywords: first-order elliptic equation, Cauchy–Riemann equation, embroidery condition, nonlocal boundary condition, main relation, Green's second formula, necessary conditions, regularization of singularity.

2020 Mathematics Subject Classification: 35J67.

Introduction

The Dirichlet, Neumann, Poincaré and directional derivative problems for second-order elliptic equations, particularly for the Laplace equation with local boundary conditions, have been widely studied in [1–3]. Since the Cauchy–Riemann equation is a first-order elliptic equation, the problems formulated for it using classical conditions are known to be globally ill-posed.

In [4], the Dirichlet problem for the Cauchy–Riemann equation is studied under the condition that the given function on the boundary satisfies what the authors call the necessary condition, a very rigid condition.

In general, writing out boundary conditions for first-order elliptic equations, as well as proving the correctness of the problem, require special research. Unlike previous works focusing on Dirichlet or Neumann problems, in [5] a unified analytical framework was developed to handle mixed (Robin type) boundary conditions by combining complex analysis and functional analysis methods, thus expanding the applicability of the Cauchy–Riemann boundary problem theory. In [6–8], problems related to the Cauchy–Riemann equation under classical boundary conditions are studied essentially using methods of complex analysis. In [9], the Cauchy–Riemann operator's spectral behavior with homogeneous

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Dirichlet-type boundary conditions was investigated, and the theoretical understanding of its spectral properties was advanced by demonstrating that the operator possesses the Volterra property.

In this study, a new boundary condition for the Cauchy–Riemann equation is proposed and the analytical solution of the problem is reduced to the system of Fredholm integral equations of the second type.

Let D be a region bounded in the x_1ox_2 plane and convex in the x_2 direction as shown in Figure 1. If we project the domain D onto the x_1 axis parallel to the x_2 axis, then the boundary Γ is divided into two parts Γ_1 and Γ_2 ($\Gamma = \Gamma_1 \cup \Gamma_2$, since the domain D is convex in the x_2 direction). The equations of the curves Γ_1 and Γ_2 are given by

$$x_2 = \gamma_k(x_1),\ (k = 1, 2); \quad x_1 \in [a_1, b_1] = pr|_{x_1} D = pr|_{x_1}\Gamma_1 = pr|_{x_1}\Gamma_2$$

and Γ is a Lyapunov curve.

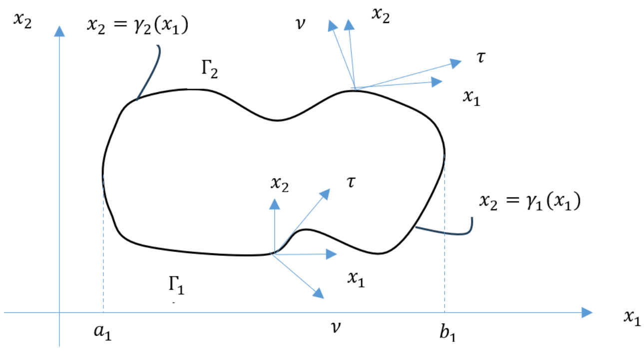

Figure 1. Region D

The following problem in the domain D is considered

$$\frac{\partial u(x)}{\partial x_2} + i\frac{\partial u(x)}{\partial x_1} = a(x)u(x) + f(x), \quad x \in D, \tag{1}$$

$$\alpha_1(x_1)u\left(x_1, \gamma_1(x_1)\right) + \alpha_2(x_1)u\left(x_1, \gamma_2(x_1)\right) = \varphi(x_1), \quad x_1 \in [a_1, b_1]. \tag{2}$$

Here, $a(x)$, $f(x)$, $\alpha_1(x_1)$, $\alpha_2(x_1)$ and $\varphi(x_1)$ are given continuously differentiable functions and $i = \sqrt{-1}$.

Let us denote by ν and τ the outward and tangential normals drawn to boundary of Γ, respectively.

It is known that the fundamental solution of the Cauchy–Riemann equation is

$$U(x-\xi) = \frac{1}{2\pi}\frac{1}{x_2 - \xi_2 + i(x_1 - \xi_2)}, \tag{3}$$

where $x - \xi = (x_1 - \xi_1, x_2 - \xi_2)$.

1 The main relation

We multiply equation (1) by the fundamental solution (3) and integrate over the domain D:

$$\int_D \frac{\partial u(x)}{\partial x_2}U(x-\xi)dx + i\int_D \frac{\partial u(x)}{\partial x_1}U(x-\xi)dx = \int_D a(x)u(x)U(x-\xi)dx + \int_D f(x)U(x-\xi)dx. \tag{4}$$

Applying the Ostrogradsky–Gauss formula to the integral on the left-hand side of equation (4), we get

$$\int_D \frac{\partial u(x)}{\partial x_2}U(x-\xi)dx = \int_\Gamma u(x)U(x-\xi)\cos(\nu, x_2)dx - \int_D u(x)\frac{\partial U(x-\xi)}{\partial x_2}dx, \tag{5}$$

$$\int_D \frac{\partial u(x)}{\partial x_1} U(x-\xi)dx = \int_\Gamma u(x)U(x-\xi)\cos(\nu, x_1)dx - \int_D u(x)\frac{\partial U(x-\xi)}{\partial x_1}dx. \qquad (6)$$

Let us substitute expressions (5) and (6) into equation (4) and write it down as follows

$$\int_\Gamma u(x)U(x-\xi)\cos(\nu, x_2)dx - \int_D u(x)\frac{\partial U(x-\xi)}{\partial x_2}dx + i\Bigg[\int_\Gamma u(x)U(x-\xi)\cos(\nu, x_1)dx$$

$$-\int_D u(x)\frac{\partial U(x-\xi)}{\partial x_1}dx\Bigg] = \int_D a(x)u(x)U(x-\xi)dx + \int_D f(x)U(x-\xi)dx$$

or

$$\int_\Gamma u(x)U(x-\xi)\Big[\cos(\nu, x_2) + i\cos(\nu, x_1)\Big]dx - \int_D a(x)u(x)U(x-\xi)dx - \int_D f(x)U(x-\xi)dx$$

$$= \int_D u(x)\left[\frac{\partial U(x-\xi)}{\partial x_2} + i\frac{\partial U(x-\xi)}{\partial x_1}\right]dx.$$

Since the function $U(x-\xi)$ is a fundamental solution of the principal part of equation (1), we can write the last equation as

$$\int_\Gamma u(x)U(x-\xi)\Big[\cos(\nu, x_2) + i\cos(\nu, x_1)\Big]dx - \int_D a(x)u(x)U(x-\xi)dx - \int_D f(x)U(x-\xi)dx$$

$$= \begin{cases} u(\xi), & \xi \in D, \\ \frac{1}{2}u(\xi), & \xi \in \Gamma. \end{cases} \qquad (7)$$

Main relation (7) we obtained consists of two parts. The first part gives an arbitrary solution of the given equation (1) in the domain D for $\xi \in D$, and the second part gives the necessary conditions for $\xi \in \Gamma$ that gives relation between values of boundary conditions and values of the obtained solution. It should be noted that in the literature expressions of the form (7) are known as necessary conditions derived from the first fundamental relations, similar to Green's second formula, in the study of higher-order equations [2]. Similar methods have also been applied in [10–12] in the process of finding analytical solutions to problems for the Cauchy–Riemann equation with non-local boundary conditions in regions with various geometries.

Expressions of the type (7) are derived from various basic relations, by which all necessary linearly independent conditions can be obtained. As emphasized in [2] while the D'Alembert formula gives the solution of the Cauchy problem for the second-order wave equation, it cannot directly give a solution to the boundary value problem posed for the Laplace equation. That is, since the D'Alembert formula includes it's own initial conditions, then, by writing them down, we obtain a solution to the Cauchy problem from the D'Alembert formula. However, it is not possible to specify the two functions that participate in the Green's II formula obtained for the Laplace equation (they are linearly dependent functions). By specifying one of them, we obtain the Dirichlet problem, and by specifying the other, we obtain the Neumann problem.

In [2], a boundary value problem for the Laplace equation was considered and the expression derived from Green's II formula was called a necessary and sufficient condition. In [10] and [11], for a first-order elliptical equation a new approach to non-local boundary value problem for the Cauchy–Riemann equation was proposed. Paper [11] was devoted to investigation of a new method for investigating of solutions to boundary value problems for first order elliptic equations. Computational aspects of first-order partial differential equations with nonlocal boundary condition were considered in [13].

2 Necessary and sufficient conditions

Now let us single out the necessary and sufficient conditions from the main relation (7):

$$\frac{1}{2}u\left(\xi_1,\gamma_1(\xi_1)\right)=\frac{1}{2\pi}\int_{\Gamma_1}\frac{u(x)}{x_2-\gamma_1(\xi_1)+i(x_1-\xi_1)}\Big[\cos(\nu,x_2)+i\cos(\nu,x_1)\Big]dx$$
$$+\frac{1}{2\pi}\int_{\Gamma_2}\frac{u(x)}{x_2-\gamma_1(\xi_1)+i(x_1-\xi_1)}\Big[\cos(\nu,x_2)+i\cos(\nu,x_1)\Big]dx$$
$$-\frac{1}{2\pi}\int_{D}\frac{a(x)u(x)}{x_2-\gamma_1(\xi_1)+i(x_1-\xi_1)}dx-\frac{1}{2\pi}\int_{D}\frac{f(x)}{x_2-\gamma_1(\xi_1)+i(x_1-\xi_1)}dx$$
$$=\frac{1}{2\pi}\int_{a_1}^{b_1}\frac{u\left(x_1,\gamma_1(x_1)\right)}{\gamma_1(x_1)-\gamma_1(\xi_1)+i(x_1-\xi_1)}\Big[-\cos(x_1,\tau)+i\sin(x_1,\tau)\Big]\frac{dx_1}{\cos(x_1,\tau)}$$
$$+\frac{1}{2\pi}\int_{a_1}^{b_1}\frac{u\left(x_1,\gamma_2(x_1)\right)}{\gamma_2(x_1)-\gamma_1(\xi_1)+i(x_1-\xi_1)}\Big[\cos(x_1,\tau)-i\sin(x_1,\tau)\Big]\frac{dx_1}{\cos(x_1,\tau)}$$
$$-\frac{1}{2\pi}\int_{D}\frac{a(x)u(x)}{x_2-\gamma_1(\xi_1)+i(x_1-\xi_1)}dx-\frac{1}{2\pi}\int_{D}\frac{f(x)}{x_2-\gamma_1(\xi_1)+i(x_1-\xi_1)}dx$$
$$=\frac{1}{2\pi}\int_{a_1}^{b_1}\frac{u\left(x_1,\gamma_1(x_1)\right)[1-i\gamma_1{}'(x_1)]dx_1}{\gamma_1'\left(\sigma_1(x_1,\xi_1)\right)(x_1-\xi_1)+i(x_1-\xi_1)}+\frac{1}{2\pi}\int_{a_1}^{b_1}\frac{u\left(x_1,\gamma_2(x_1)\right)[1-i\gamma_2{}'(x_1)]dx_1}{\gamma_2(x_1)-\gamma_1(\xi_1)+i(x_1-\xi_1)}$$
$$-\frac{1}{2\pi}\int_{D}\frac{a(x)u(x)}{x_2-\gamma_1(\xi_1)+i(x_1-\xi_1)}dx-\frac{1}{2\pi}\int_{D}\frac{f(x)}{x_2-\gamma_1(\xi_1)+i(x_1-\xi_1)}dx, \tag{8}$$
$$\frac{1}{2}u\left(\xi_1,\gamma_2(\xi_1)\right)=\frac{1}{2\pi}\int_{\Gamma_1}\frac{u(x)}{x_2-\gamma_2(\xi_1)+i(x_1-\xi_1)}\Big[\cos(\nu,x_2)+i\cos(\nu,x_1)\Big]dx$$
$$+\frac{1}{2\pi}\int_{\Gamma_2}\frac{u(x)}{x_2-\gamma_2(\xi_1)+i(x_1-\xi_1)}\Big[\cos(\nu,x_2)+i\cos(\nu,x_1)\Big]dx$$
$$-\frac{1}{2\pi}\int_{D}\frac{a(x)u(x)}{x_2-\gamma_2(\xi_1)+i(x_1-\xi_1)}dx-\frac{1}{2\pi}\int_{D}\frac{f(x)}{x_2-\gamma_2(\xi_1)+i(x_1-\xi_1)}dx$$
$$=\frac{1}{2\pi}\int_{a_1}^{b_1}\frac{u\left(x_1,\gamma_1(x_1)\right)}{\gamma_1(x_1)-\gamma_2(\xi_1)+i(x_1-\xi_1)}\Big[-\cos(x_1,\tau)+i\sin(x_1,\tau)\Big]\frac{dx_1}{\cos(x_1,\tau)}$$
$$+\frac{1}{2\pi}\int_{a_1}^{b_1}\frac{u\left(x_1,\gamma_2(x_1)\right)}{\gamma_2(x_1)-\gamma_2(\xi_1)+i(x_1-\xi_1)}\Big[\cos(x_1,\tau)-i\sin(x_1,\tau)\Big]\frac{dx_1}{\cos(x_1,\tau)}$$
$$-\frac{1}{2\pi}\int_{D}\frac{a(x)u(x)}{x_2-\gamma_2(\xi_1)+i(x_1-\xi_1)}dx-\frac{1}{2\pi}\int_{D}\frac{f(x)}{x_2-\gamma_2(\xi_1)+i(x_1-\xi_1)}dx$$
$$=-\frac{1}{2\pi}\int_{a_1}^{b_1}\frac{u\left(x_1,\gamma_1(x_1)\right)[1-i\gamma_1{}'(x_1)]dx_1}{\gamma_1(x_1)-\gamma_2(\xi_1)+i(x_1-\xi_1)}+\frac{1}{2\pi}\int_{a_1}^{b_1}\frac{u\left(x_1,\gamma_2(x_1)\right)[1-i\gamma_2{}'(x_1)]dx_1}{\gamma_2'\left(\sigma_2(x_1,\xi_1)\right)(x_1-\xi_1)+i(x_1-\xi_1)}$$
$$-\frac{1}{2\pi}\int_{D}\frac{a(x)u(x)}{x_2-\gamma_2(\xi_1)+i(x_1-\xi_1)}dx-\frac{1}{2\pi}\int_{D}\frac{f(x)}{x_2-\gamma_2(\xi_1)+i(x_1-\xi_1)}dx. \tag{9}$$

Let us clarify the features of expressions (8) and (9), which we obtained for the necessary and sufficient conditions: To do this, we write equations (8) and (9) as follows:

$$\begin{cases} u\left(\xi_1,\gamma_1(\xi_1)\right)=-\frac{1}{\pi}\int_{a_1}^{b_1}\frac{u(x_1,\gamma_1(x_1))[1-i\gamma_1{}'(x_1)]dx_1}{(x_1-\xi_1)\gamma_1'(\sigma_1(x_1,\xi_1))+i}+\Re_1, \\ \\ u\left(\xi_1,\gamma_2(\xi_1)\right)=\frac{1}{\pi}\int_{a_1}^{b_1}\frac{u(x_1,\gamma_2(x_1))[1-i\gamma_2{}'(x_1)]dx_1}{(x_1-\xi_1)\gamma_2'(\sigma_2(x_1,\xi_1))+i}+\Re_2. \end{cases} \tag{10}$$

In expression (10), $\Re_1$, and $\Re_2$ denote the sum of the regular terms of expressions (8) and (9). It is easy to see that

$$\frac{[1-i\gamma_k{}'(x_1)]}{\gamma_k^{'}\left(\sigma_k(x_1,\xi_1)\right)+i}=\frac{[1-i\gamma_k{}'(x_1)]}{\gamma_k^{'}\left(\sigma_k(x_1,\xi_1)\right)+i}+i-i$$

$$=-i+\frac{[1-i\gamma_k{}'(x_1)]+i\left(\gamma_k^{'}\left(\sigma_k(x_1,\xi_1)\right)+i\right)}{\gamma_k^{'}\left(\sigma_k(x_1,\xi_1)\right)+i}=-i+\frac{1-i\gamma_k{}'(x_1)+i\gamma_k^{'}\left(\sigma_k(x_1,\xi_1)\right)+i^2}{\gamma_k^{'}\left(\sigma_k(x_1,\xi_1)\right)+i}$$

$$=-i+i\frac{\gamma_k^{'}\left(\sigma_k(x_1,\xi_1)\right)-\gamma_k{}'(x_1)}{\gamma_k^{'}\left(\sigma_k(x_1,\xi_1)\right)+i},\quad k=1,2;\ x_1\in[a_1,b_1]. \tag{11}$$

Since the points $\sigma_k(x_1,\xi_1)$ lie between x_1 and ξ_1, when x_1 and ξ_1 coincide, the point $\sigma_k(x_1,\xi_1)$ also coincides with them. Therefore, when $x_1-\xi_1\longrightarrow 0$, $\gamma_k^{'}\left(\sigma_k(x_1,\xi_1)\right)-\gamma_k{}'(x_1)=0$, $k=1,2$.

If we substitute expression (11) into (10), we get

$$\begin{cases} u\left(\xi_1,\gamma_1(\xi_1)\right)=\frac{i}{\pi}\int_{a_1}^{b_1}\frac{u(x_1,\gamma_1(x_1))}{(x_1-\xi_1)}dx_1+\Re_3, \\ u\left(\xi_1,\gamma_2(\xi_1)\right)=-\frac{i}{\pi}\int_{a_1}^{b_1}\frac{u(x_1,\gamma_2(x_1))}{(x_1-\xi_1)}dx_1+\Re_4, \end{cases} \tag{12}$$

where $\Re_3$ and $\Re_4$ denote the sum of the regular integrals corresponding to expressions (10) and (11), respectively.

3 Regularization of singularities

Taking into account boundary condition (2), we write the following linear combination from (12)

$$\alpha_1(\xi_1)u\left(\xi_1,\gamma_1(\xi_1)\right)-\alpha_2(\xi_1)u\left(\xi_1,\gamma_2(\xi_1)\right)$$

$$=\frac{i}{\pi}\int_{a_1}^{b_1}\frac{\alpha_1(\xi_1)u\left(x_1,\gamma_1(x_1)\right)+\alpha_2(\xi_1)u\left(x_1,\gamma_2(x_1)\right)}{x_1-\xi}dx_1+\alpha_1(\xi_1)\Re_3+\alpha_2(\xi_1)\Re_4$$

$$=\frac{i}{\pi}\int_{a_1}^{b_1}\left\{\Big[\left(\alpha_1(\xi_1)-\alpha_1(x_1)\right)+\alpha_1(x_1)\Big]u\left(x_1,\gamma_1(x_1)\right)\right.$$

$$\left.+\Big[\left(\alpha_2(\xi_1)-\alpha_2(x_1)\right)+\alpha_2(x_1)\Big]u\left(x_1,\gamma_2(x_1)\right)\right\}\frac{dx_1}{x_1-\xi_1}$$

$$=\frac{i}{\pi}\int_{a_1}^{b_1}\frac{\alpha_1(x_1)u\left(x_1,\gamma_1(x_1)\right)+\alpha_2(x_1)u\left(x_1,\gamma_2(x_1)\right)}{x_1-\xi_1}dx_1+\alpha_1(\xi_1)\Re_3+\alpha_2(\xi_1)\Re_4$$

$$=\frac{i}{\pi}\int_{a_1}^{b_1}\frac{\varphi(x_1)}{x_1-\xi_1}dx_1+\alpha_1(\xi_1)\Re_3+\alpha_2(\xi_1)\Re_4. \tag{13}$$

If the functions $\alpha_1(x_1)$ and $\alpha_2(x_1)$ belong to the Hölder class, then we can say that the limit in expression (13) exists in the Cauchy sense. Since $x_1-\xi_1=0$, $\alpha_k(\xi_1)-\alpha_k(x_1)=0$, $k=1,2$. In this case the resulting singular integral no longer contains the unknown function.

Note 1. If the function $\varphi(x_1)$ on the right-hand side of boundary condition (2) satisfies the following conditions:

$$\varphi(a_1)=\varphi(b_1)=0,\quad \varphi(x_1)\in C^{(1)}[a_1,b_1], \tag{14}$$

then the singular limit in (13) will also exist in the usual sense.

Theorem 1. Assume that the following conditions are satisfied:
(i) A bounded in plane D is convex in the direction x_2, and the boundary Γ is a Lyapunov curve;
(ii) $a(x)$, $f(x)$ are continuous functions;
(iii) $\alpha_1(x_1)$, $\alpha_2(x_1)$ belong to the Hölder class, and the function $\varphi(x)$ satisfies condition (14).
Then expression (13) is regular.

4 Fredholm property of the problem

Now, taking into account boundary condition (2), together with the regular expression (13), we obtain the following system of algebraic equations

$$\begin{cases} \alpha_1(\xi_1)u\left(\xi_1,\gamma_1(\xi_1)\right)+\alpha_2(\xi_1)u\left(\xi_1,\gamma_2(\xi_1)\right)=\varphi(\xi_1), \\ \alpha_1(\xi_1)u\left(\xi_1,\gamma_1(\xi_1)\right)-\alpha_2(\xi_1)u\left(\xi_1,\gamma_2(\xi_1)\right)=\frac{i}{\pi}\int_{a_1}^{b_1}\frac{\varphi(x_1)}{x_1-\xi_1}dx_1+\alpha_1(\xi_1)\Re_3+\alpha_2(\xi_1)\Re_4. \end{cases} \tag{15}$$

From this it follows

$$\begin{cases} u\left(\xi_1,\gamma_1(\xi_1)\right)=\frac{\varphi(\xi_1)}{2\alpha_1(\xi_1)}+\frac{i}{2\alpha_1(\xi_1)\pi}\int_{a_1}^{b_1}\frac{\varphi(x_1)}{(x_1-\xi_1)}dx_1+\Re_3, \\ u\left(\xi_1,\gamma_2(\xi_1)\right)=\frac{\varphi(\xi_1)}{2\alpha_2(\xi_1)}-\frac{i}{2\alpha_2(\xi_1)\pi}\int_{a_1}^{b_1}\frac{\varphi(x_1)}{(x_1-\xi_1)}dx_1+\Re_4. \end{cases} \tag{16}$$

If the conditions

$$\alpha_k(x_1)\neq 0, \quad k=1,2 \tag{17}$$

are satisfied, then expressions (16) give a system of integral equations with a regular Fredholm kernel of the second kind for the boundary values of the unknown function in problem (1), (2). This kernel does not include the integral of the sought function over the domain D. Thus, we show that problem (1), (2) has the Fredholm property.

Theorem 2. If the conditions of Theorem 1 and (17) are satisfied, then problem (1), (2) has the Fredholm property.

5 Solution of the boundary value problem

If we solve the system of integral equations (15), then for the functions $u\left(\xi_1,\gamma_k(\xi_1)\right)$, $(k=1,2)$ we obtain certain expressions depending on the expression

$$\int_D \frac{a(x)u(x)}{x_2-\gamma_2(\xi_1)+i(x_1-\xi_1)}dx.$$

By writing these expressions on the left side of the main relations (7), from the first component of the main relation for the function $u(x)$ we obtain a Fredholm-type integral equation of the second kind with a regular kernel. Thus, we obtain a solution to problems (1), (2).

Conclusion

Firstly, for a first-order elliptic equation with variable coefficients whose main part is the Cauchy–Riemann equation were written out the non-local boundary conditions (constructive) obtained by means of stitching from the boundaries of a plane region bounded and convex in the direction x_2 and divided into two parts, provided that the Carleman's condition on the boundary is satisfied.

The main relation consisting of two parts is obtained, the first of which gives arbitrary solutions of the equation and the second part gives the necessary conditions for $\xi \in \Gamma$ that gives relation between values of boundary conditions and the obtained solution.

For them, in the case of a partial differential equation, a a system of Fredholm integral equations of the second type with a regular kernel is obtained.

For the first time, it proved for a partial differential equation that the solution of the considered boundary problem can be obtained from the Green's formula, and for the problem of an ordinary differential equation from the Lagrange formula.

Author Contributions

All authors contributed equally to this work.

Conflict of Interest

The authors declare no conflict of interest.

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Studying a system of non-local condition hyperbolic equations

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Local boundary value problems for hyperbolic differential equations have been studied in considerable detail. However, the mathematical modeling of a number of real-world processes leads to nonlocal boundary value problems involving nonlinear hyperbolic differential equations, which remain poorly understood. In this paper, we consider a system of hyperbolic equations defined by both point and integral boundary conditions in a rectangular domain. To the best of our knowledge, such a problem is studied here for the first time. We note that this formulation is quite general and encompasses several special cases. The classical Goursat-Darboux problem-a problem with integral boundary conditions in which some boundary conditions are specified as point conditions and others as integral conditions-is derived from this formulation as a particular case. Under natural conditions on the initial data, the necessary conditions for the solvability of a nonlocal boundary value problem are established. A corresponding Green‘s function for the boundary value problem is constructed and the problem is reduced to an equivalent integral equation. Using the principle of contracting Banach maps, conditions for the existence and uniqueness of a solution to the boundary value problem are established. An example is given illustrating the validity of the obtained results.

Keywords: non-local boundary value problems, integral and point boundary conditions, Goursat-Darboux problem, system of hyperbolic equations, existence and uniqueness of solutions, unique solvability, Green‘s function.

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Introduction

Recently, intensive research has been carried out on nonlocal boundary value problems for both ordinary and partial differential equations. The significance of these problems was emphasized in [1]. If, instead of classical boundary conditions, algebraic relations are defined between the values of the unknown function on the boundary and/or inside the domain, such a boundary value problem is referred to as a nonlocal boundary value problem [2–4]. These algebraic relations can be expressed in terms of pointwise values of the unknown function and/or its integral.

Non-local condition boundary value problems arise while constructing mathematical models of processes that occur in atomic and nuclear physics, demography, heating processes and in other fields of natural science. The papers [5, 6] study one-dimensional nonlinear hyperbolic equations given with integral and multipoint boundary conditions. Sufficient conditions for the existence and uniqueness of the problem are found.

In [7–9], a system of hyperbolic equations is investigated under two-point and integral boundary conditions. The Green‘s function for the problem is constructed, the boundary value problem is reduced to an equivalent integral equation, and sufficient conditions for the existence and uniqueness of the solution are obtained.

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In [10–12], a nonlocal problem with integral conditions for a system of hyperbolic equations in a rectangular domain is analyzed. The existence of a unique classical solution and the methods for its construction are discussed.

Kozhanov A.I. and Pulkina L.S. investigated a multidimensional hyperbolic equation with integral boundary conditions in [13].

In [14–16], a nonlocal boundary value problem with an integral condition for a system of hyperbolic equations was considered, and necessary and sufficient conditions for its well-posedness were established.

Papers [17–19] study the existence and uniqueness of strong solutions using methods of functional analysis.

Paper [20] analyzes an optimal control problem with integral boundary conditions.

In the present work, we consider a Goursat–Darboux system with pointwise and integral condition. A necessary condition for the solvability of the problem is proved. The problem considered is reduced to an equivalent equation by means of equivalent transformations. Sufficient conditions for the existence and uniqueness of the solution are found by means of the Banach compressed mapping principle.

1 Problem statement

We consider a non-local problem with integral and pointwise boundary conditions for a Goursat–Darboux system in the domain $Q = [0,T] \times [0,l]$:

$$z_{tx} = f(t,x,z(t,x)), \tag{1}$$

$$Az(0,x) + \int_0^T n(t)z(t,x)dt = \varphi(x), \;\; x \in [0,l], \tag{2}$$

$$Bz(t,0) + \int_0^l m(x)z(t,x)dx = \psi(t), \;\; t \in [0,T]. \tag{3}$$

Here, $z(t,x) = col(z_1(t,x), z_2(t,x), \ldots, z_n(t,x))$ is an unknown n-dimensional vector-function; $f : Q \times R^n \to R^n$ is a given function; $\varphi(x), \psi(t)$ are functions that are differentiable on $[0,T], [0,l]$ respectively. $A, B \in R^{n\times n}$ are the given matrices, $n(t)$ and $m(x)$ are $n\times n$-dimensional matrix functions. $det\left(A + \int_0^T n(t)dt\right) \neq 0$, $det\left(B + \int_0^l m(x)dx\right) \neq 0$. Furthermore, the matrices $A, n(t)$ and $B, m(x)$ are pairwise commutative. So, $A\cdot B = B\cdot A, A\cdot m(x) = m(x)\cdot A, B\cdot n(t) = n(t)\cdot B, m(x)\cdot n(t) = n(t)\cdot m(x)$.

Note that problem (1)–(3) is quite general. For example, if the matrices A and B are both zero, then the problem reduces to one with pure integral conditions. When $A = B = E$ and $n(t) \equiv m(x) \equiv 0$, we obtain the classical Goursat–Darboux problem, and there are other variants.

2 Main results

In the paper, it is shown that for the solvability of problem (1)–(3) the compatability condition of functions $\varphi(x)$ and $\psi(t)$ is satisfied.

Theorem 1. For the solvability of problem (1)–(3), it is necessary that the compatibility condition

$$B\varphi(0) + \int_0^l m(x)\varphi(x)dx = A\psi(0) + \int_0^T n(t)\psi(t)dt$$

is fulfilled.

Proof. Let us find the solution of equation (1) as follows:

$$z(t,x) = a(t) + b(x) + \int\limits_0^t \int\limits_0^x f(\tau, s, z(\tau, s)) d\tau\, ds, \tag{4}$$

where the functions $a(t)$ and $b(x)$ are unknown differentiable functions and are determined in the intervals $[0,T], [0,l]$, respectively. We require that the function determined by equality (4) satisfies boundary conditions (2) and (3). Then, we obtain the relations

$$A\left[a(0) + b(x)\right] + \int\limits_0^T n(t) \left[a(t) + b(x) + \int\limits_0^t \int\limits_0^x f(\tau, s, z(\tau, s)) d\tau\, ds\right] dt$$

$$= Aa(0) + \int\limits_0^T n(t)a(t)dt + \left(A + \int\limits_0^T n(t)dt\right) b(x)$$

$$+ \int\limits_0^T n(t) \int\limits_0^t \int\limits_0^x f(\tau, s, z(\tau, s)) d\tau ds\, dt = \varphi(x), \quad x \in [0,l]. \tag{5}$$

$$B\left[a(t) + b(0)\right] + \int\limits_0^l m(x) \left[a(t) + b(x) + \int\limits_0^t \int\limits_0^x f(\tau, s, z(\tau, s)) d\tau\, ds\right] dx$$

$$= \left(B + \int\limits_0^l m(x)dx\right) a(t) + Bb(0) + \int\limits_0^l m(x)b(x)dx$$

$$+ \int\limits_0^l m(x) \int\limits_0^t \int\limits_0^x f(\tau, s, z(\tau, s)) d\tau ds\, dx = \psi(t), \quad t \in [0,T]. \tag{6}$$

Applying conditions (3) to relation (5) and conditions (2) to relation (6), we obtain

$$B\left[Aa(0) + \int\limits_0^T n(t)a(t)dt + \left(A + \int\limits_0^T n(t)dt\right) b(0)\right]$$

$$+ \int\limits_0^l m(x) \left[Aa(0) + \int\limits_0^T n(t)a(t)dt + \left(A + \int\limits_0^T n(t)dt\right) b(x)\right] dx$$

$$+ \int\limits_0^T \int\limits_0^l n(t)m(x) \int\limits_0^t \int\limits_0^x f(\tau, s, z(\tau, s)) d\tau\, ds\, dt\, dx = B\varphi(0) + \int\limits_0^l m(x)\varphi(x)dx,$$

$$A\left[\left(B + \int\limits_0^l m(x)dx\right) a(0) + Bb(0) + \int\limits_0^l m(x)b(x)dx\right]$$

$$+\int_0^T n(t)\left[\left(B+\int_0^l m(x)dx\right)a(t)+\left(Bb(0)+\int_0^l m(x)b(x)dx\right)\right]dt$$

$$+\int_0^T\int_0^l n(t)m(x)\int_0^t\int_0^x f(\tau,s,z(\tau,s))d\tau\,ds\,dt\,dx = A\psi(0)+\int_0^T n(t)\psi(t)dt.$$

From this we obtain

$$\left(Aa(0)+\int_0^T n(t)a(t)dt\right)\left(B+\int_0^l m(x)dx\right)+\left(A+\int_0^T n(t)dt\right)\left(Bb(0)+\int_0^l m(x)b(x)dx\right)$$

$$+\int_0^T\int_0^l n(t)m(x)\int_0^t\int_0^x f(\tau,s,z(\tau,s))d\tau\,ds\,dt\,dx = B\varphi(0)+\int_0^l m(x)\varphi(x)dx,$$

$$\left(Aa(0)+\int_0^T n(t)a(t)dt\right)\left(B+\int_0^l m(x)dx\right)+\left(Bb(0)+\int_0^l m(x)b(x)dx\right)\left(A+\int_0^T n(t)dt\right)$$

$$+\int_0^T\int_0^l n(t)m(x)\int_0^t\int_0^x f(\tau,s,z(\tau,s))d\tau\,ds\,dt\,dx = A\psi(0)+\int_0^T n(t)\psi(t)dt.$$

The right hand side equality is obtained from the left-hand side equality. □

In this paper, we construct the Green function for problem (1)–(3). We note that problem (1)–(3) is reduced to an equivalent integral equation.

Theorem 2. The equivalent integral equation for the problem (1)–(3) is as follows

$$z(t,x)=\left(B+\int_0^l m(x)dx\right)^{-1}\psi(t)+\left(A+\int_0^T n(t)dt\right)^{-1}\varphi(x)$$

$$-\left(B+\int_0^l m(x)dx\right)^{-1}\left(A+\int_0^T n(t)dt\right)^{-1}\left(B\varphi(0)+\int_0^l m(x)\varphi(x)dx\right)$$

$$+\int_0^T\int_0^l G(t,x,\tau,s)f(\tau,s,z)d\tau\,ds, \tag{7}$$

where

$$G(t,x,\tau,s)=\left(B+\int_0^l m(x)dx\right)^{-1}\left(A+\int_0^T n(t)dt\right)^{-1}$$

$$\times\begin{cases} \left(A+\int\limits_0^\tau n(\alpha)\,d\alpha\right)\left(B+\int\limits_0^s m(\beta)\,d\beta\right), & 0\le\tau\le t,\, 0\le s\le x,\\ -\left(A+\int\limits_0^\tau n(\alpha)\,d\alpha\right)\int\limits_s^l m(\beta)\,d\beta, & 0\le\tau\le t,\, x<s\le l,\\ -\left(B+\int\limits_0^s m(\beta)\,d\beta\right)\int\limits_\tau^T n(\alpha)\,d\alpha, & t<\tau\le T, 0\le s\le x,\\ \int\limits_\tau^T n(\alpha)\,d\alpha\int\limits_s^l m(\beta)\,d\beta, & t<\tau\le T, x<s\le l. \end{cases}$$

Proof. The unknown functions $a(t)$ and $b(x)$ can be considered as solutions to a system of linear algebraic equations defined by equalities (5) or (6). This system is of the n-th order. The sought functions $a(t)$ and $b(x)$ have dimension $2n$. It is clear that this system has an infinite set of solutions. We fix an arbitrary solution. Let

$$Aa(0)+\int\limits_0^T n(t)a(t)dt=0$$

be an arbitrary solution.

Then, from equality (5), we find

$$b(x)=\left(A+\int\limits_0^T n(t)dt\right)^{-1}\varphi(x)-\left(A+\int\limits_0^T n(t)dt\right)^{-1}\int\limits_0^T n(t)\int\limits_0^t\int\limits_0^x f(\tau,s,z(\tau,s))d\tau\,ds\,dt, \qquad (8)$$

$$b(0)=\left(A+\int\limits_0^T n(t)dt\right)^{-1}\varphi(0).$$

Taking the equalities $b(x)$ and $b(0)$ into account in equality (6), we get

$$\left(B+\int\limits_0^l m(x)dx\right)a(t)+\left(A+\int\limits_0^T n(t)dt\right)^{-1}B\varphi(0)$$

$$+\int\limits_0^l m(x)\left(A+\int\limits_0^T n(t)dt\right)^{-1}\varphi(x)dx-\int\limits_0^l m(x)\left(A+\int\limits_0^T n(t)dt\right)^{-1}$$

$$\times\int\limits_0^T n(t)\int\limits_0^t\int\limits_0^x f(\tau,s,z(\tau,s))d\tau\,ds\,dt\,dx+\int\limits_0^l m(x)\int\limits_0^t\int\limits_0^x f(\tau,s,z(\tau,s))d\tau\,ds\,dx=\psi(t).$$

Hence,

$$a(t)=\left(B+\int\limits_0^l m(x)dx\right)^{-1}\psi(t)-\left(B+\int\limits_0^l m(x)dx\right)^{-1}\left(A+\int\limits_0^T n(t)dt\right)^{-1}B\varphi(0)$$

$$-\left(B+\int\limits_0^l m(x)dx\right)^{-1}\left(A+\int\limits_0^T n(t)dt\right)^{-1}\int\limits_0^l m(x)\varphi(x)dx$$

$$+\left(B+\int_0^l m(x)dx\right)^{-1}\left(A+\int_0^T n(t)dt\right)^{-1}\times\int_0^l\int_0^T m(x)n(t)\int_0^t\int_0^x f(\tau,s,z(\tau,s))d\tau\,ds\,dt\,dx$$

$$-\left(B+\int_0^l m(x)dx\right)^{-1}\int_0^l m(x)\int_0^t\int_0^x f(\tau,s,z(\tau,s))d\tau\,ds\,dx. \tag{9}$$

Taking into account equalities (8) and (9) obtained for functions $b(x)$ and $a(t)$ in equality (4), we have

$$z(t,x)=\left(B+\int_0^l m(x)dx\right)^{-1}\psi(t)+\left(A+\int_0^T n(t)dt\right)^{-1}\varphi(x)$$

$$-\left(B+\int_0^l m(x)dx\right)^{-1}\left(A+\int_0^T n(t)dt\right)^{-1}\left[B\varphi(0)+\int_0^l m(x)\varphi(x)dx\right]$$

$$+\left(B+\int_0^l m(x)dx\right)^{-1}\left(A+\int_0^T n(t)dt\right)^{-1}\int_0^l\int_0^T m(x)n(t)\int_0^t\int_0^x f(\tau,s,z(\tau,s))d\tau\,ds\,dt\,dx$$

$$-\left(B+\int_0^l m(x)dx\right)^{-1}\int_0^l m(x)\int_0^t\int_0^x f(\tau,s,z(\tau,s))d\tau\,ds\,dx$$

$$-\left(A+\int_0^T n(t)dt\right)^{-1}\int_0^T n(t)\int_0^t\int_0^x f(\tau,s,z(\tau,s))d\tau\,ds\,dt+\int_0^t\int_0^x f(\tau,s,z(\tau,s))d\tau\,ds,\quad (t,x)\in Q. \tag{10}$$

We make the same transformations in equality (10) as follows

$$\int_0^T n(t)\int_0^t\int_0^x f(\tau,s,z(\tau,s))d\tau\,ds\,dt=\int_0^T\int_0^x\left(\int_t^T n(\tau)d\tau\right)f(t,s,z(t,s))dt\,ds,$$

$$\int_0^l m(x)\int_0^t\int_0^x f(\tau,s,z(\tau,s))d\tau\,ds\,dx=\int_0^l\int_0^t\left(\int_x^l m(s)ds\right)f(\tau,x,z(\tau,x))d\tau\,dx,$$

$$\int_0^l\int_0^T m(x)n(t)\int_0^t\int_0^x f(\tau,s,z(\tau,s))d\tau\,ds\,dt\,dx=\int_0^T\int_0^l\left(\int_t^T n(\tau)d\tau\int_x^l m(s)ds\right)f(t,x,z(t,x))dt\,dx.$$

Taking into account these expressions in equality (10), we can write

$$z(t,x)=\left(B+\int_0^l m(x)dx\right)^{-1}\psi(t)+\left(A+\int_0^T n(t)dt\right)^{-1}\varphi(x)$$

$$-\left(B+\int_0^l m(x)dx\right)^{-1}\left(A+\int_0^T n(t)dt\right)^{-1}\left[B\varphi(0)+\int_0^l m(x)\varphi(x)dx\right]$$

$$+\left(B+\int_0^l m(x)dx\right)^{-1}\left(A+\int_0^T n(t)dt\right)^{-1}\int_0^T\int_0^l\left[\int_t^T n(\tau)d\tau\int_x^l m(s)ds\right]f(t,x,z(t,x))dt\,dx$$

$$-\left(B+\int_0^l m(x)dx\right)^{-1}\int_0^l\int_0^t\left(\int_x^l m(s)ds\right)f(\tau,x,z(\tau,x))d\tau\,dx$$

$$-\left(A+\int_0^T n(t)dt\right)^{-1}\int_0^T\int_0^x\left(\int_t^T n(\tau)d\tau\right)f(t,s,z(t,s))dt\,ds$$

$$+\int_0^t\int_0^x f(\tau,s,z(\tau,s))d\tau\,ds,\quad (t,x)\in Q. \tag{11}$$

From equality (11) we obtain

$$z(t,x)=\left(B+\int_0^l m(x)dx\right)^{-1}\psi(t)+\left(A+\int_0^T n(t)dt\right)^{-1}\varphi(x)$$

$$-\left(B+\int_0^l m(x)dx\right)^{-1}\left(A+\int_0^T n(t)dt\right)^{-1}\left(B\varphi(0)+\int_0^l m(x)\varphi(x)dx\right)$$

$$+\int_0^t\int_0^x\Bigg[E-\left(A+\int_0^T n(t)dt\right)^{-1}\int_\tau^T n(\alpha)d\alpha-\left(B+\int_0^l m(x)dx\right)^{-1}\int_s^l m(\beta)d\beta$$

$$+\left(B+\int_0^l m(x)dx\right)^{-1}\left(A+\int_0^T n(t)dt\right)^{-1}\int_\tau^T n(\alpha)d\alpha\int_s^l m(\beta)d\beta\Bigg]f(\tau,s,z(\tau,s))d\tau\,ds$$

$$+\int_0^t\int_x^l\Bigg[\left(B+\int_0^l m(x)dx\right)^{-1}\left(A+\int_0^T n(t)dt\right)^{-1}\int_\tau^T n(\alpha)d\alpha\int_s^l m(\beta)d\beta$$

$$-\left(B+\int_0^l m(x)dx\right)^{-1}\int_s^l m(\beta)d\beta\Bigg]f(\tau,s,z(\tau,s))d\tau\,ds$$

$$+\int_t^T\int_0^x\Bigg[\left(B+\int_0^l m(x)dx\right)^{-1}\left(A+\int_0^T n(t)dt\right)^{-1}\int_\tau^T n(\alpha)d\alpha\int_s^l m(\beta)d\beta$$

$$-\left(A+\int_0^T n(t)dt\right)^{-1}\int_\tau^T n(\alpha)d\alpha\Bigg]f(\tau,s,z(\tau,s))d\tau ds$$

$$+\int\limits_t^T\int\limits_x^l\left[\left(B+\int\limits_0^l m(x)dx\right)^{-1}\left(A+\int\limits_0^T n(t)dt\right)^{-1}\int\limits_\tau^T n(\alpha)d\alpha\int\limits_s^l m(\beta)d\beta\right]$$
$$\times f(\tau,s,z(\tau,s))d\tau ds,\ \ (t,x)\in Q. \tag{12}$$

Given equality (12), we can write:

$$E-\left(A+\int\limits_0^T n(t)dt\right)^{-1}\int\limits_\tau^T n(\alpha)d\alpha-\left(B+\int\limits_0^l m(x)dx\right)^{-1}\int\limits_s^l m(\beta)d\beta$$
$$+\left(B+\int\limits_0^l m(x)dx\right)^{-1}\left(A+\int\limits_0^T n(t)dt\right)^{-1}\int\limits_\tau^T n(\alpha)d\alpha\int\limits_s^l m(\beta)d\beta$$
$$=\left(B+\int\limits_0^l m(x)dx\right)^{-1}\left(A+\int\limits_0^T n(t)dt\right)^{-1}\times\left(A+\int\limits_0^\tau n(\alpha)d\alpha\right)\left(B+\int\limits_0^s m(\beta)d\beta\right),$$
$$\left(B+\int\limits_0^l m(x)dx\right)^{-1}\left(A+\int\limits_0^T n(t)dt\right)^{-1}\int\limits_\tau^T n(\alpha)d\alpha\int\limits_s^l m(\beta)d\beta$$
$$-\left(B+\int\limits_0^l m(x)dx\right)^{-1}\int\limits_s^l m(\beta)d\beta$$
$$=-\left(B+\int\limits_0^l m(x)dx\right)^{-1}\left(A+\int\limits_0^T n(t)dt\right)^{-1}\left[\left(A+\int\limits_0^\tau n(\alpha)d\alpha\right)\int\limits_s^l m(\beta)d\beta\right],$$
$$\left(B+\int\limits_0^l m(x)dx\right)^{-1}\left(A+\int\limits_0^T n(t)dt\right)^{-1}\int\limits_\tau^T n(\alpha)d\alpha\int\limits_s^l m(\beta)d\beta$$
$$-\left(A+\int\limits_0^T n(t)dt\right)^{-1}\int\limits_\tau^T n(\alpha)d\alpha$$
$$=-\left(B+\int\limits_0^l m(x)dx\right)^{-1}\left(A+\int\limits_0^T n(t)dt\right)^{-1}\left[\left(B+\int\limits_0^s m(\beta)d\beta\right)\int\limits_\tau^T n(\alpha)d\alpha\right].$$

As a result, we obtain equation (7)

$$z(t,x)=\left(B+\int\limits_0^l m(x)dx\right)^{-1}\psi(t)+\left(A+\int\limits_0^T n(t)dt\right)^{-1}\varphi(x)$$
$$-\left(B+\int\limits_0^l m(x)dx\right)^{-1}\left(A+\int\limits_0^T n(t)dt\right)^{-1}\left[B\varphi(0)+\int\limits_0^l m(x)\varphi(x)dx\right]$$

$$+\int_0^t\int_0^x\left(B+\int_0^l m(x)dx\right)^{-1}\left(A+\int_0^T n(t)dt\right)^{-1}$$

$$\times\left(A+\int_0^\tau n(\alpha)d\alpha\right)\left(B+\int_0^s m(\beta)d\beta\right)f(\tau,s,z(\tau,s))d\tau\,ds$$

$$-\int_0^t\int_x^l\left(B+\int_0^l m(x)dx\right)^{-1}\left(A+\int_0^T n(t)dt\right)^{-1}$$

$$\times\left[\left(A+\int_0^\tau n(\alpha)d\alpha\right)\int_s^l m(\beta)d\beta\right]f(\tau,s,z(\tau,s))d\tau\,ds$$

$$-\int_t^T\int_0^x\left(B+\int_0^l m(x)dx\right)^{-1}\left(A+\int_0^T n(t)dt\right)^{-1}$$

$$\times\left[\left(B+\int_0^s m(\beta)d\beta\right)\int_\tau^T n(\alpha)d\alpha\right]f(\tau,s,z(\tau,s))d\tau\,ds$$

$$+\int_t^T\int_x^l\left(B+\int_0^l m(x)dx\right)^{-1}\left(A+\int_0^T n(t)dt\right)^{-1}\left[\int_\tau^T n(\alpha)d\alpha\int_s^l m(\beta)d\beta\right]$$

$$\times f(\tau,s,z(\tau,s))d\tau\,ds,\quad (t,x)\in Q. \tag{13}$$

In this equality, having determined the matrix-function $G(t,x,\tau,s)$, we proved the first part of the theorem. We now calculate the derivative of the function $z(t,x)$ determined by equality (13) with respect to t and x

$$z_{tx}(t,x)=\frac{\partial^2}{\partial t\partial x}\left[\left(B+\int_0^l m(x)dx\right)^{-1}\psi(t)+\left(A+\int_0^T n(t)dt\right)^{-1}\varphi(x)\right.$$

$$\left.-\left(B+\int_0^l m(x)dx\right)^{-1}\left(A+\int_0^T n(t)dt\right)^{-1}\left[B\varphi(0)+\int_0^l m(x)\varphi(x)dx\right]\right]$$

$$+\frac{\partial^2}{\partial t\partial x}\left[\int_0^t\int_0^x\left(B+\int_0^l m(x)dx\right)^{-1}\left(A+\int_0^T n(t)dt\right)^{-1}\right.$$

$$\left.\times\left(A+\int_0^\tau n(\alpha)d\alpha\right)\left(B+\int_0^s m(\beta)d\beta\right)f(\tau,s,z(\tau,s))d\tau\,ds\right]$$

$$-\frac{\partial^2}{\partial t\partial x}\left[\int_0^t\int_x^l\left(B+\int_0^l m(x)dx\right)^{-1}\left(A+\int_0^T n(t)dt\right)^{-1}\right.$$

$$\times \left[\left(A+\int\limits_0^{\tau} n(\alpha)d\alpha \right) \int\limits_s^l m(\beta)d\beta \right] f(\tau,s,z(\tau,s))d\tau\, ds \Bigg]$$

$$-\frac{\partial^2}{\partial t\partial x}\left[\int\limits_t^T\int\limits_0^x \left(B+\int\limits_0^l m(x)dx\right)^{-1}\left(A+\int\limits_0^T n(t)dt\right)^{-1}\right.$$

$$\times \left[\left(B+\int\limits_0^{s} m(\beta)d\beta \right) \int\limits_{\tau}^T n(\alpha)d\alpha \right] f(\tau,s,z(\tau,s))d\tau ds \Bigg]$$

$$+\frac{\partial^2}{\partial t\partial x}\int\limits_t^T\int\limits_x^l \left(B+\int\limits_0^l m(x)dx\right)^{-1}\left(A+\int\limits_0^T n(t)dt\right)^{-1}$$

$$\times \left[\int\limits_{\tau}^T n(\alpha)d\alpha \int\limits_s^l m(\beta)d\beta \right] f(\tau,s,z(\tau,s))d\tau\, ds$$

$$= \left(B+\int\limits_0^l m(x)dx\right)^{-1}\left(A+\int\limits_0^T n(t)dt\right)^{-1}\left[AB+B\int\limits_0^t n(\alpha)d\alpha + A\int\limits_0^x m(\beta)d\beta\right.$$

$$+\int\limits_0^t n(\alpha)d\alpha\int\limits_0^x m(\beta)d\beta + A\int\limits_x^l m(\beta)d\beta + \int\limits_0^t n(\alpha)d\alpha\int\limits_x^l m(\beta)d\beta$$

$$\left. +B\int\limits_t^T n(\alpha)d\alpha + \int\limits_0^x m(\beta)d\beta\int\limits_t^T n(\alpha)d\alpha + \int\limits_t^T n(\alpha)d\alpha\int\limits_x^l m(\beta)d\beta\right]$$

$$\times f(t,x,z(t,x)) = f(t,x,z(t,x)).$$

We now show that the function defined by equation (11) satisfies the non-local boundary conditions (2) and (3), with

$$A\left[\left(B+\int\limits_0^l m(x)dx\right)^{-1}\psi(0)+\left(A+\int\limits_0^T n(t)dt\right)^{-1}\varphi(x)\right.$$

$$-\left(B+\int\limits_0^l m(x)dx\right)^{-1}\left(A+\int\limits_0^T n(t)dt\right)^{-1}\left[B\varphi(0)+\int\limits_0^l m(x)\varphi(x)dx\right]$$

$$+\left(B+\int\limits_0^l m(x)dx\right)^{-1}\left(A+\int\limits_0^T n(t)dt\right)^{-1}$$

$$\times\int\limits_0^T\int\limits_0^l \left[\int\limits_t^T n(\tau)d\tau\int\limits_x^l m(s)ds\right] f(t,x,z(t,x))dt\,dx$$

$$-\left(A+\int_0^T n(t)dt\right)^{-1}\int_0^T\int_0^x\left(\int_t^T n(\tau)d\tau\right)f(t,s,z(t,s))dtds\Bigg]$$

$$+\int_0^T n(t)\Bigg[\left(B+\int_0^l m(x)dx\right)^{-1}\psi(t)+\left(A+\int_0^T n(t)dt\right)^{-1}\varphi(x)$$

$$-\left(B+\int_0^l m(x)dx\right)^{-1}\left(A+\int_0^T n(t)dt\right)^{-1}\left[B\varphi(0)+\int_0^l m(x)\varphi(x)dx\right]$$

$$+\left(B+\int_0^l m(x)dx\right)^{-1}\left(A+\int_0^T n(t)dt\right)^{-1}$$

$$\times\int_0^T\int_0^l\left[\int_\tau^T n(\tau)d\tau\int_x^l m(s)ds\right]f(t,x,z(t,x))dt\,dx$$

$$-\left(B+\int_0^l m(x)dx\right)^{-1}\int_0^l\int_0^t\left(\int_x^l m(s)ds\right)f(\tau,x,z(\tau,x))d\tau\,dx$$

$$-\left(A+\int_0^T n(t)dt\right)^{-1}\int_0^T\int_0^x\left[\int_t^T n(\tau)d\tau\right]f(t,s,z(t,s))dtds+\int_0^t\int_0^x f(\tau,s,z(\tau,s))d\tau\,ds\Bigg]dt$$

$$=\left(B+\int_0^l m(x)dx\right)^{-1}\left[A\psi(0)+\int_0^T n(t)\psi(t)dt\right]$$

$$-\left(B+\int_0^l m(x)dx\right)^{-1}\left(A+\int_0^T n(t)dt\right)^{-1}B\varphi(0)\left(A+\int_0^T n(t)dt\right)$$

$$+\left(A+\int_0^T n(t)dt\right)^{-1}\varphi(x)\left(A+\int_0^T n(t)dt\right)-\left(A+\int_0^T n(t)dt\right)^{-1}$$

$$\times\left(A+\int_0^T n(t)dt\right)\int_0^T\int_0^x\left(\int_t^T n(\tau)d\tau\right)f(t,s,z(t,s))dt\,ds$$

$$-\left(B+\int_0^l m(x)dx\right)^{-1}\left(A+\int_0^T n(t)dt\right)^{-1}\left(A+\int_0^T n(t)dt\right)$$

$$\times\int_0^l m(x)\varphi(x)dx+\left(B+\int_0^l m(x)dx\right)^{-1}\left(A+\int_0^T n(t)dt\right)^{-1}$$

$$\times\left(A+\int_0^T n(t)dt\right)\int_0^T\int_0^l\left[\int_t^T n(\tau)d\tau\int_x^l m(s)ds\right]f(t,x,z(t,x))dt\,dx$$

$$-\left(B+\int_0^l m(x)dx\right)^{-1}\int_0^T n(t)\int_0^l\int_0^t\left(\int_x^l m(s)ds\right)f(\tau,x,z(\tau,x))d\tau\,dx\,dt$$

$$+\int_0^T\int_0^x\left(\int_t^T n(\tau)d\tau\right)f(t,s,z(t,s))dtds$$

$$=\left(B+\int_0^l m(x)dx\right)^{-1}\left[A\psi(0)+\int_0^T n(t)\psi(t)dt\right]-\left(B+\int_0^l m(x)dx\right)^{-1}B\varphi(0)+\varphi(x)$$

$$-\int_0^T\int_0^x\left(\int_t^T n(\tau)d\tau\right)f(t,s,z(t,s))dt\,ds$$

$$-\left(B+\int_0^l m(x)dx\right)^{-1}\int_0^l m(x)\varphi(x)dx+\left(B+\int_0^l m(x)dx\right)^{-1}$$

$$\times\int_0^T\int_0^l\left[\int_t^T n(\tau)d\tau\int_x^l m(s)ds\right]f(t,x,z(t,x))dtdx$$

$$-\left(B+\int_0^l m(x)dx\right)^{-1}\int_0^T n(t)\int_0^l\int_0^t\left(\int_x^l m(s)ds\right)f(\tau,x,z(\tau,x))d\tau\,dx\,dt$$

$$+\int_0^T\int_0^x\left(\int_t^T n(\tau)d\tau\right)f(t,s,z(t,s))dtds$$

$$=\left(B+\int_0^l m(x)dx\right)^{-1}\left[\left(A\psi(0)+\int_0^T n(t)\psi(t)dt\right)-\left(B\varphi(0)+\int_0^l m(x)\varphi(x)dx\right)\right]$$

$$+\varphi(x)=\varphi(x).$$

In a similar way, we can show that the point-wise and integral boundary condition

$$Bz(t,0)+\int_0^l m(x)z(t,x)dx=\psi(t),\quad t\in[0,T]$$

is satisfied. □

3 Existence and uniqueness

It is seen from the proved theorem that problem (1)–(3) is equivalent to the integral equation

$$z(t,x)=\left(B+\int\limits_0^l m(x)dx\right)^{-1}\psi(t)+\left(A+\int\limits_0^T n(t)dt\right)^{-1}\varphi(x)$$

$$-\left(B+\int\limits_0^l m(x)dx\right)^{-1}\left(A+\int\limits_0^T n(t)dt\right)^{-1}\left[B\varphi(0)+\int\limits_0^l m(x)\varphi(x)dx\right]$$

$$+\int\limits_0^T\int\limits_0^l G(t,x,\tau,s)f(\tau,s,z)d\tau\, ds. \tag{14}$$

In order to prove the existence and uniqueness of the solution to problem (1)–(3) we determine the operator $P : C\left(Q; R^n\right) \to C\left(Q; R^n\right)$ as follows:

$$(Pz)(t,x)=\left(B+\int\limits_0^l m(x)dx\right)^{-1}\psi(t)+\left(A+\int\limits_0^T n(t)dt\right)^{-1}\varphi(x)$$

$$-\left(B+\int\limits_0^l m(x)dx\right)^{-1}\left(A+\int\limits_0^T n(t)dt\right)^{-1}\left(B\varphi(0)+\int\limits_0^l m(x)\varphi(x)dx\right)$$

$$+\int\limits_0^T\int\limits_0^l G(t,x,\tau,s)f(\tau,s,z)d\tau\, ds.$$

It is known that solving problem (1)–(3) or integral equation (14) is equivalent to finding the fixed point of the operator P. In other words, problem (1)–(3) has a solution if and only if the operator P has a fixed point.

Theorem 3. Assume that the following conditions hold:

$$\left|f(t,x,z_2)-f(t,x,z_1)\right|\le M\left|z_2-z_1\right|,\quad \forall\,(t,x)\in Q,\ \ z_1,z_2\in R^n,\ \ M\ge 0 \tag{15}$$

and

$$L=lTSM<1, \tag{16}$$

where

$$S=\max_{Q\times Q}\|G(t,x,\tau,s)\|.$$

Then, problem (1)–(3) has a unique solution in Q.

Proof. Denote

$$N=\max_{Q}\left|\left(B+\int\limits_0^l m(x)dx\right)^{-1}\psi(t)+\left(A+\int\limits_0^T n(t)dt\right)^{-1}\varphi(x)\right.$$

$$-\left(B+\int_0^l m(x)dx\right)^{-1}\left(A+\int_0^T n(t)dt\right)^{-1}\left(B\varphi(0)+\int_0^l m(x)\varphi(x)dx\right)\Bigg|,$$

$$\max_{(t,x)\in Q}|f(t,x,0)| = M_f$$

and choose $r \geq \frac{N+M_f TS}{1-L}$. We show that the relation $PB_r \subset B_r$ holds, where

$$B_r = \{x \in C(Q,R^n)\ :\ \|z\| \leq r\}.$$

For arbitrary $z \in B_r$, we have

$$\|Pz(t,x)\| \leq N + \int_0^T \int_0^l |G(t,x,\tau,s)|\left(|f(\tau,s,z(\tau,s)-f(\tau,s,0)|+|f(\tau,s,0)|\right)d\tau\, ds$$

$$\leq N + S\int_0^T \int_0^l \left(M|z|+M_f\right)dt\, dx \leq N + SMrTl + M_f TlS \leq \frac{N+M_f TS}{1-L} \leq r.$$

On the other hand, from condition (15) we obtain that for arbitrary $z_1, z_2 \in B_r$ the relation

$$|Pz_2 - Pz_1| \leq \int_0^T \int_0^l |G(t,x,\tau,s)|\left(|f(\tau,s,z_2(\tau,s)-f(\tau,s,z_1(\tau,s)|\right)$$

$$\leq S\int_o^T \int_0^l M|z_2(t,x)-z_1(t,x)|dtdx \leq MSTl\max_Q|z_2(t,x)-z_1(t,x)| \leq MSTl\|z_2-z_1\|$$

holds. Hence, we obtain

$$\|Pz_2 - Pz_1\| \leq L\|z_2 - z_1\|.$$

Taking condition (16) into account we obtain that the operator P is compressive. So, problem (1)–(3) has a unique solution. □

4 Application of the obtained results

To illustrate the obtained results, let us consider the system of hyperbolic equations

$$\begin{cases} z_{1tx}(t,x) = 0.1\cos z_2(t,x), \\ z_{2tx}(t,x) = \frac{|z_1(t,x)|}{10(1+|z_1(t,x)|)}, \end{cases} \quad (t,x)\in[0,1]\times[0,1]. \tag{17}$$

Assume that the following boundary conditions are satisfied

$$\begin{cases} 2z_1(0,x) + \int_0^1 tz_1(t,x)dt = x^2, \\ z_2(0,x) = 1, \end{cases} \quad x \in [0,1]. \tag{18}$$

$$\begin{cases} 2z_1(0,x) + \int_0^1 xz_1(t,x)dt = t^2, \\ z_2(t,0) = 1, \end{cases} \quad t \in [0,1]. \tag{19}$$

Make the following notation:

$$A = B = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}, \; n(t) = \begin{pmatrix} t & 0 \\ 0 & 0 \end{pmatrix}, \; m(x) = \begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix},$$

$$\varphi(x) = \begin{pmatrix} x^2 \\ 1 \end{pmatrix}, \; \psi(t) = \begin{pmatrix} t^2 \\ 1 \end{pmatrix}.$$

Then, conditions (18), (19) can be written as follows:

$$\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} z_1(0,x) \\ z_2(0,x) \end{pmatrix} + \int\limits_0^1 \begin{pmatrix} t & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} z_1(t,x) \\ z_2(t,x) \end{pmatrix} dt = \begin{pmatrix} x^2 \\ 1 \end{pmatrix}, \quad x \in [0,1].$$

$$\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} z_1(t,0) \\ z_2(t,0) \end{pmatrix} + \int\limits_0^1 \begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} z_1(t,x) \\ z_2(t,x) \end{pmatrix} dx = \begin{pmatrix} t^2 \\ 1 \end{pmatrix}, \quad t \in [0,1].$$

$$A + \int\limits_0^1 n(t)dt = \begin{pmatrix} 2.5 & 0 \\ 0 & 1 \end{pmatrix}, \quad B + \int\limits_0^1 m(x)dx = \begin{pmatrix} 2.5 & 0 \\ 0 & 1 \end{pmatrix},$$

$$\left(B + \int\limits_0^l m(x)dx \right)^{-1} = \left(A + \int\limits_0^T n(t)dt \right)^{-1} = \begin{pmatrix} 0.4 & 0 \\ 0 & 1 \end{pmatrix}.$$

Taking them into account:

$$G(t,x,\tau,s) = \begin{cases} \begin{pmatrix} 0.16\left(2+\frac{\tau^2}{2}\right)\left(2+\frac{s^2}{2}\right) & 0 \\ 0 & 1 \end{pmatrix}, & 0 \le \tau \le t, \, 0 \le s \le x, \\ -\begin{pmatrix} 0.16\left(2+\frac{\tau^2}{2}\right)\frac{s^2}{2} & 0 \\ 0 & 0 \end{pmatrix}, & 0 \le \tau \le t, x < s \le 1, \\ -\begin{pmatrix} 0.16\frac{\tau^2}{2}\left(2+\frac{s^2}{2}\right), & 0 \\ 0 & 0 \end{pmatrix}, & t \le \tau \le 1, 0 < s \le x, \\ \begin{pmatrix} 0.16\frac{\tau^2}{2} \cdot \frac{s^2}{2} & 0 \\ 0 & 0 \end{pmatrix}, & t \le \tau \le 1, x < s \le 1. \end{cases}$$

Let us estimate the main parameters of the boundary value problem (17)–(19). We have that the following estimate holds for the norm of the Green function $\max \|G(t,x,\tau,s)\| \le 1$; the Lipschitz constant $M = 0.1$, and the compression parameter $L = 1 \cdot 1 \cdot 0.1 \cdot 1 = 0.1 < 1$. So, all the conditions of Theorem 3 are fulfilled and the boundary value problem (17)–(19) has a unique solution.

Conclusion

The present work studied a system of hyperbolic equations with non-local condition. Boundary conditions are rather general. In the special case, it contains the classical Goursat–Darboux problem, "pure" integral conditions, a boundary value problem whose part of the conditions is pointwise, the other part is in integral form, and other cases.

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All authors contributed equally to this work.

Conflict of Interest

The authors declare no conflict of interest.

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An analogue of Leibniz's rule for Hadamard derivatives and their application

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This paper explores new analogues of the Leibniz rule for Hadamard and Caputo–Hadamard fractional derivatives. Unlike classical derivatives, fractional ones have a strong nonlocal character, meaning that the value of the derivative at a given point depends on the entire history of the function. Because of this nonlocality, the standard product rule cannot be directly applied. The study develops refined formulas for differentiating the product of two functions, which include additional integral terms representing memory effects inherent to fractional calculus. The paper also establishes a series of inequalities that make it possible to estimate the fractional derivatives of nonlinear expressions, such as powers of a function, through the derivative of the function itself. In particular, it is shown that a specific inequality holds for positive functions that relates the fractional derivative of the function power to the function product and its fractional derivative. These theoretical results are of great importance for the study of linear and nonlinear fractional diffusion equations. They provide useful tools for proving the existence, uniqueness, and stability of their solutions and for deriving a priori estimates that describe the qualitative behavior of such systems.

Keywords: linear and nonlinear diffusion equation, Hadamard-type time fractional derivative, Hadamard time fractional derivative, Mittag-Leffler function, a priori estimates, Leibniz rule, porous medium equation, Gronwall inequality.

2020 Mathematics Subject Classification: 35R11, 35A02.

Introduction

In the theory of differential calculus, the Leibniz's rule is one of the most important rules. Leibniz's rule states that: for two differentiable functions $u(x)$ and $v(x)$, the derivative of their product $u(x)v(x)$ is given by

$$\frac{d}{dx}(u(x)v(x)) = u'(x)v(x) + u(x)v'(x). \tag{1}$$

The Leibniz's rule is applied to many problems in PDEs, including a priori estimates for solutions to linear and nonlinear parabolic problems.

However, in the case of fractional derivatives, it is not possible to obtain a simple expression analogous to (1). Tarasov [1] demonstrated that the formula

$$D^{\alpha}(u(x)v(x)) = D^{\alpha}u(x)v(x) + u(x)D^{\alpha}v(x)$$

α is an integer. This limitation arises from the inherently nonlocal nature of fractional derivatives. Nevertheless, various analogues of the classical The Leibniz rule for fractional derivatives have been deve-loped in the literature. In particular, the foundations of fractional calculus and the main properties of fractional operators, including Hadamard-type derivatives, were systematically presented in the

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monographs [2–4]. Further generalizations of the Leibniz formula for fractional derivatives of different types were obtained in [5–8], where both analytical and operator approaches were discussed. The results concerning fractional diffusion equations and applications of fractional Leibniz-type rules to boundary and initial value problems can be found in [9–11]. For example, in [9] Alsaedi, Ahmad and Kirane obtained an analogue of the Leibniz's rule in the following form:

$$\begin{aligned} D^{\alpha}(uv)(t) &= u(t)D^{\alpha}v(t) + v(t)D^{\alpha}u(t) \\ &\quad - \frac{u(t)v(t)}{\Gamma(1-\alpha)t^{\alpha}} - \frac{\alpha}{\Gamma(1-\alpha)}\int_0^t \frac{(u(s)-u(t))(v(s)-v(t))}{(t-s)^{1+\alpha}}ds, \end{aligned}$$

where D^{α} is the Riemann-Liouville fractional derivative of order $\alpha \in (0,1)$:

$$D^{\alpha}u(t) = \frac{1}{(\Gamma(1-\alpha)}\frac{d}{dt}\int_0^t (t-s)^{-\alpha}u(s)ds.$$

Later in [10], Cuesta et al. extended this formula to the Riemann-Liouville fractional derivative of variable order $\alpha(t) \in (0,1)$, $t > 0$. This makes fractional calculus particularly relevant in fields such as physics, biology, materials science, and economics, where traditional approaches are insufficient to describe real-world phenomena. The application of fractional models in continuum mechanics and physical systems was discussed in [12, 13], while the classical foundations of fractional calculus were established in [14, 15]. Further developments related to anomalous diffusion processes and boundary value problems in mathematical physics were presented in [16, 17].

In recent years, there has been a growing interest in the study of both linear and nonlinear differential equations involving Hadamard and Hadamard-type fractional derivatives. Fundamental results on the theory and applications of such derivatives can be found in [15, 18, 19]. Theoretical and numerical studies addressing the well-posedness, regularity, and stability of related equations are provided in [20–22]. Moreover, generalized forms of the Leibniz-type rule for Hadamard fractional operators and their applications to extremum principles have been explored in [23–25]. In [24], it was proved that the Hadamard multi-index fractional diffusion problem has at most one classical solution, and this solution depends continuously on its initial boundary conditions. In [25], Kirane and Torebek obtained new estimates for the fractional Hadamard derivatives of a function at its extreme points, and using the extremum principle, showed that linear and nonlinear fractional diffusion equations with initial-boundary conditions have at most one classical solution, and this solution continuously depends on the initial and boundary conditions. For Hadamard fractional differential equations with initial boundary conditions involving a fractional Laplace operator, Wang, Ren, and Baleanu [24] applied the maximum principle and obtained certain existence and uniqueness results.

In [26], the authors have given a small generalization of the Gronwall inequality, which they used to study a solution to a generalized Cauchy-type problem with a Hilfer–Hadamard-type fractional derivative. The Leibniz's rule for fractional derivatives of constant order was introduced in [9] as an extension of the classical product rule for integer-order derivatives. This differentiation rule (as well as other fractional rules found in the literature) includes additional terms that account for the non-local nature of fractional derivatives, particularly in the case of fractional derivatives of variable order (FDVO). The authors present a contemporary proof of the maximum principle applicable to the linear and nonlinear Riemann–Liouville fractional diffusion equations using the following inequality, for any integer $p \geq 2$ and $u \geq 0$

$$D^{\alpha}_{a+,t}u^p \leq pu^{p-1}D^{\alpha}_{a+,t}u, \quad \begin{cases} \text{for } p \text{ even}, \\ \text{for } p \text{ odd whenever}. \end{cases} \tag{2}$$

In [10], the authors further advance this concept by extending this property to fractional derivatives with a variable order $\alpha(t)$. They derive a Leibniz inequality and an integration by parts formula. They also studied an initial value problem with their time variable order fractional derivative and present a regularity result for it, and study its on the asymptotic behavior.

Motivated by the neet to explore in the context of Hadamard derivatives, we embarked on an investigation of the Leibniz inequality for both linear and nonlinear diffusion equations. After establishing inequality (2) for Hadamard and Hadamard-type fractional derivatives using the Gronwall inequality, we explored a priori decay estimates for the solutions.

Our main results are given in the following form:

Lemma 1. Let u, v satisfy the following condition

$$u \in AC[a,T] \text{ and } v \in AC[a,T],\ 0 < \alpha \leq 1.$$

Then, the following holds true

$$\begin{aligned} D^{\alpha}_{a+,t}[uv](t) &= u(t)D^{\alpha}_{a+,t}v(t) + v(t)D^{\alpha}_{a+,t}u(t) \\ &- \frac{u(t)v(t)}{\Gamma(1-\alpha)\left(\log\frac{t}{a}\right)^{\alpha}} - \frac{\alpha}{\Gamma(1-\alpha)}\int_a^t \frac{(u(s)-u(t))(v(s)-v(t))}{s\left(\log\frac{t}{s}\right)^{1+\alpha}}ds. \end{aligned}$$

This leads to the following cases.

Corollary 1. If u and v have the same signs, then

$$D^{\alpha}_{a+,t}(uv)(t) \leq u(t)D^{\alpha}_{a+,t}v(t) + v(t)D^{\alpha}_{a+,t}u(t). \tag{3}$$

Let $u \in AC[a,T]$ and $0 < \alpha \leq 1$. Applying $u = v$ in inequality (3), we get the following statement

$$2u(t)D^{\alpha}_{a+,t}u(t) \geq D^{\alpha}_{a+,t}u^2(t). \tag{4}$$

Then

$$D^{\alpha}_{a+,t}u^p \leq pu^{p-1}D^{\alpha}_{a+,t}u, \tag{5}$$

where $p \geq 2$ and $u \geq 0$. Using mathematical induction we can prove inequality (5).

Lemma 2. Let u, v satisfy the following condition

$$u \in AC[a,T] \text{ and } v \in AC[a,T],\ 0 < \alpha \leq 1.$$

Then, the following holds true

$$\begin{aligned} {}^{C}_{H}D^{\alpha}_{a+,t}[uv](t) &= u(t){}^{C}_{H}D^{\alpha}_{a+,t}v(t) + v(t){}^{C}_{H}D^{\alpha}_{a+,t}u(t) \\ &- \frac{(u(a)-u(t))(v(a)-v(t))}{\Gamma(1-\alpha)\left(\log\frac{t}{a}\right)^{\alpha}} \\ &- \frac{\alpha}{\Gamma(1-\alpha)}\int_a^t \frac{(u(s)-u(t))(v(s)-v(t))}{s\left(\log\frac{t}{s}\right)^{1+\alpha}}ds. \end{aligned}$$

This leads to the following cases.

Corollary 2. If u and v have the same signs, then

$${}^{C}_{H}D^{\alpha}_{a+,t}(uv)(t) \leq u(t){}^{C}_{H}D^{\alpha}_{a+,t}v(t) + v(t){}^{C}_{H}D^{\alpha}_{a+,t}u(t). \tag{6}$$

Then

$${}^{C}_{H}D^{\alpha}_{a+,t}u^p \leq pu^{p-1}{}^{C}_{H}D^{\alpha}_{a+,t}u, \tag{7}$$

where $p \geq 2$ and $u \geq 0$. Applying mathematical induction we can prove inequality (7).

1 Preliminaries

1.1 The weighted space of continuous functions space

Let us consider the weighted space of continuous functions denoted by $C_{\gamma,\log}[a,b]$, where $0 \le \gamma < 1$. A function $f : (a,b] \to \mathbb{R}$ belongs to this space if the function $\left(\log \frac{t}{a}\right)^{\gamma} f(t)$ can be continuously extended to the closed interval $[a,b]$. More precisely,

$$C_{\gamma,\log}[a,b] = \left\{ f : (a,b] \to \mathbb{R} \;\middle|\; \left(\log \frac{t}{a}\right)^{\gamma} f(t) \in C[a,b] \right\}.$$

The norm associated with this space is given by

$$\|f\|_{C_{\gamma,\log}[a,b]} = \left\| \left(\log \frac{t}{a}\right)^{\gamma} f(t) \right\|_{C[a,b]}.$$

It is worth noting that for $\gamma = 0$, this space reduces to the classical space of continuous functions, i.e., $C_{0,\log}[a,b] = C[a,b]$.

For any positive integer n, we work within the Banach space $C^n_{\delta,\gamma}[a,b]$ of functions possessing continuous δ-derivatives up to order $n-1$ on $[a,b]$, and a δ^n-derivative on $(a,b]$ such that $\delta^n f \in C_{\gamma,\log}[a,b]$. The dilation operator is defined as $\delta = t\frac{d}{dt}$. Functions in this space satisfy the norm condition

$$\|f\|_{C^n_{\delta,\gamma}} = \sum_{k=0}^{n-1} \|\delta^k f\|_{C[a,b]} + \|\delta^n f\|_{C_{\gamma,\log}[a,b]} < \infty.$$

In the special case of $n = 0$, the space $C^0_{\delta,\gamma}[a,b]$ coincides with $C_{\gamma,\log}[a,b]$.

Additionally, we make use of the space $AC^n_{\delta}[a,b]$, which consists of functions $f : [a,b] \to \mathbb{C}$ for which the $(n-1)$-th δ-derivative, $\delta^{n-1} f$, belongs to the space of absolutely continuous functions $AC[a,b]$. Explicitly,

$$AC^n_{\delta}[a,b] = \left\{ f : [a,b] \to \mathbb{C} \;\middle|\; \delta^{n-1} f \in AC[a,b] \right\}.$$

It is evident that $AC^1_{\delta}[a,b]$ coincides with $AC[a,b]$.

These functional spaces and operators provide a natural framework for analyzing differential equations involving weighted logarithmic behaviors and dilation-invariant properties, which are especially relevant in the study of nonlocal models and fractional dynamics (see more details [4, 17] and links therein).

Definition 1. [4, p. 110] Let $f \in L^1_{loc}([a,b])$. The Hadamard fractional integral $I^{\alpha}_{a+,t}$ of order $\alpha \in (0,1)$ $(a > 0)$ is defined as

$$I^{\alpha}_{a+,t} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \left(\log \frac{t}{s}\right)^{\alpha-1} f(s) \frac{ds}{s}.$$

Definition 2. [4, p. 111] Let $a > 0$ and $f \in W^1_2([a,b])$. The Hadamard fractional derivative of order $\alpha \in (0,1)$ is defined by

$$D^{\alpha}_{a+,t} f(t) = t\frac{d}{dt} I^{1-\alpha}_{a+,t} f(t) = t\frac{d}{dt} \frac{1}{\Gamma(1-\alpha)} \int_a^t \left(\log \frac{t}{s}\right)^{-\alpha} f(s) \frac{ds}{s}.$$

Property 1. [4, p. 116] Let $0 < \alpha < 1$ and $0 < a, b < \infty$. If $f \in C_{\mu,\log}[a,b]\,(0 \leq \mu < 1)$ and $I^{1-\alpha}_{a+,t} f \in C^1_{\delta,\mu}[a,b]$, then

$$\left(I^{\alpha}_{a+,t} D^{\alpha}_{a+,t} f\right)(t) = f(t) - \frac{\left(I^{1-\alpha}_{a+,t} f\right)(a)}{\Gamma(\alpha)} \left(\log \frac{t}{a}\right)^{\alpha-1}, \ t \in [a,b]$$

holds at any point $t \in (a,b]$.

Definition 3. [4, p. 115] The Hadamard-type fractional derivative of order $\alpha \in (0,1)$ with $a > 0$, then for $f(t) \in AC[a,b]$

$${}^{C}_{H}D^{\alpha}_{a+,t} f(t) = \frac{1}{\Gamma(1-\alpha)} \int_a^t \left(\log \frac{t}{s}\right)^{-\alpha} f'(s) ds.$$

Alternatively, for $u \in C^1[a,t]$ an equivalent representation is

$${}^{C}_{H}D^{\alpha}_{a+,t} u(t) = \frac{t}{\Gamma(1-\alpha)} \frac{d}{dt} \int_a^t \frac{u(s) - u(t)}{s \log(t/s)^{\alpha}} ds$$

Definition 4. [4, p. 42] The Mittag-Leffler function with two parameters is represented as

$$E_{\alpha,\beta}(z) := \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \ (z, \beta \in \mathbb{C}, \ \Re(\alpha) > 0).$$

Lemma 3. [26, Lemma 3.1] Let $\alpha > 0$, $u(t)$, $v(t)$ be nonnegative functions and locally integrable on $0 < a \leq t < T \leq \infty$, and $\mathcal{M}(t)$ is a nonnegative, nondecreasing continuous function defined on $0 < a \leq t < T \leq \infty$, $\mathcal{M}(t) \leq m\,(constant)$

$$u(t) \leq v(t) + \mathcal{M}(t) \int_a^t \left(\log \frac{t}{s}\right)^{\alpha-1} u(s) \frac{ds}{s},$$

then

$$u(t) \leq v(t) + \int_a^t \left[\sum_{k=1}^{\infty} \frac{(\mathcal{M}(t)\Gamma(\alpha))^k}{\Gamma(k\alpha)} \left(\log \frac{t}{s}\right)^{k\alpha-1} \frac{v(s)}{s} \right] ds.$$

Lemma 4. Let a nonnegative absolutely continuous function $y(t)$ satisfy the inequality

$$\partial^{\alpha}_{a,t} y(t) \leq \theta y(t) + \mu(t), \quad 0 < \alpha \leq 1$$

for almost all t in $[a,T]$, where $\theta > 0$ and $\mu(t)$ is an integrable nonnegative function on $[a,T]$. Then

$$y(t) \leq y(a) E_{\alpha,1}\left(\theta \left(\log \frac{t}{a}\right)^{\alpha}\right) + \Gamma(\alpha) E_{\alpha,\alpha}\left(\theta \left(\log \frac{t}{a}\right)^{\alpha}\right) \partial^{-\alpha}_{a,t} \mu(t),$$

where the function $E_{\alpha,\beta}(z)$ is the Mittag-Leffler function.

Remark 1. The case $\alpha = 1$ of Lemma 4 is studied in [17, p. 152].

Proof. Let $\partial^{\alpha}_{a,t} y(t) - \theta y(t) = g(t)$, then

$$y(t) = y(a) E_{\alpha,1}\left(\theta \left(\log \frac{t}{a}\right)^{\alpha}\right) + \int_a^t \left(\log \frac{t}{\tau}\right)^{\alpha-1} E_{\alpha,\alpha}\left(\theta \left(\log \frac{t}{\tau}\right)^{\alpha}\right) \frac{g(\tau)}{\tau} d\tau.$$

By virtue of the inequality $g(t) \le \mu(t)$, the positivity of the Mittag-Leffler function $E_{\alpha,\alpha}\left(\theta\left(\log\frac{t}{\tau}\right)^{\alpha}\right)$ for given parameters, and the growth of the function $E_{\alpha,\alpha}(t)$, from [26], we obtain

$$\begin{aligned} y(t) &\le y(a)E_{\alpha,1}\left(\theta\left(\log\frac{t}{a}\right)^{\alpha}\right) + \int_a^t \left(\log\frac{t}{\tau}\right)^{\alpha-1} E_{\alpha,\alpha}\left(\theta\left(\log\frac{t}{\tau}\right)^{\alpha}\right)\frac{\mu(\tau)}{\tau}d\tau \\ &\le y(a)E_{\alpha,1}\left(\theta\left(\log\frac{t}{a}\right)^{\alpha}\right) + \Gamma(\alpha)E_{\alpha,\alpha}\left(\theta\left(\log\frac{t}{a}\right)^{\alpha}\right)\partial_{a,t}^{-\alpha}\mu(t), \end{aligned}$$

which completes the proof. □

1.2 The proof of the main results

In this subsection, we give a detailed proof of our main results.

The proof of Lemma 1. In view of the expression

$$u(s)v(s) = (u(s)-u(t))(v(s)-v(t)) + u(t)v(s) + u(s)v(t) - u(t)v(t)$$

and the Definition 2

$$D_{a+,t}^{\alpha}[uv](t) = \frac{t}{\Gamma(1-\alpha)}\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\left[\int_a^{t+\varepsilon}\frac{u(s)v(s)}{s\left(\log\frac{t+\varepsilon}{s}\right)^{\alpha}}ds - \int_a^t \frac{u(s)v(s)}{s\left(\log\frac{t}{s}\right)^{\alpha}}ds\right],$$

we arrive at

$$\begin{aligned} D_{a+,t}^{\alpha}[uv](t) = \frac{t}{\Gamma(1-\alpha)}\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\Big[&(\mathcal{I}_1(\varepsilon)-\mathcal{I}_1(0)) + u(t)\left(\mathcal{I}_2(\varepsilon)-\mathcal{I}_2(0)\right) \\ &+ v(t)\left(\mathcal{I}_3(\varepsilon)-\mathcal{I}_3(0)\right) - u(t)v(t)\left(\mathcal{I}_4(\varepsilon)-\mathcal{I}_4(0)\right)\Big], \end{aligned} \tag{8}$$

with

$$\begin{aligned} \mathcal{I}_1(\varepsilon) &= \int_a^{t+\varepsilon}\frac{(u(s)-u(t))(v(s)-v(t))}{s\left(\log\frac{t+\varepsilon}{s}\right)^{\alpha}}ds, \quad \mathcal{I}_2(\varepsilon) = \int_a^{t+\varepsilon}\frac{v(s)}{s\left(\log\frac{t+\varepsilon}{s}\right)^{\alpha}}ds, \\ \mathcal{I}_3(\varepsilon) &= \int_a^{t+\varepsilon}\frac{u(s)}{s\left(\log\frac{t+\varepsilon}{s}\right)^{\alpha}}ds, \quad \mathcal{I}_4(\varepsilon) = \int_a^{t+\varepsilon}\frac{1}{s\left(\log\frac{t+\varepsilon}{s}\right)^{\alpha}}ds. \end{aligned}$$

Hence, $u(t)\left(\mathcal{I}_2(\varepsilon)-\mathcal{I}_2(0)\right)$ and $v(t)\left(\mathcal{I}_3(\varepsilon)-\mathcal{I}_3(0)\right)$ are standard Hadamard derivatives, then

$$\begin{aligned} \frac{tu(t)}{\Gamma(1-\alpha)}\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\left[\int_a^{t+\varepsilon}\frac{v(s)}{s\left(\log\frac{t+\varepsilon}{s}\right)^{\alpha}}ds - \int_a^t\frac{v(s)}{s\left(\log\frac{t}{s}\right)^{\alpha}}ds\right] &= u(t)D_{a+,t}^{\alpha}v(t), \\ \frac{tv(t)}{\Gamma(1-\alpha)}\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\left[\int_a^{t+\varepsilon}\frac{u(s)}{s\left(\log\frac{t+\varepsilon}{s}\right)^{\alpha}}ds - \int_a^t\frac{u(s)}{s\left(\log\frac{t}{s}\right)^{\alpha}}ds\right] &= v(t)D_{a+,t}^{\alpha}u(t). \end{aligned}$$

Similarly, for the last term we have

$$\begin{aligned} u(t)v(t)\left(\mathcal{I}_4(\varepsilon)-\mathcal{I}_4(0)\right) &= \frac{tu(t)v(t)}{\Gamma(1-\alpha)}\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\left[\int_a^{t+\varepsilon}\frac{1}{s\left(\log\frac{t+\varepsilon}{s}\right)^{\alpha}}ds - \int_a^t\frac{1}{s\left(\log\frac{t}{s}\right)^{\alpha}}ds\right] \\ &= \frac{tu(t)v(t)}{\Gamma(1-\alpha)}\cdot\frac{d}{dt}\int_a^t\frac{1}{s\left(\log\frac{t}{s}\right)^{\alpha}}ds \\ &= \frac{u(t)v(t)}{\Gamma(1-\alpha)\left(\log\frac{t}{a}\right)^{\alpha}}. \end{aligned}$$

Now for the most complex term, we apply differentiation under the integral and use integration by parts, which gives

$$\frac{t}{\Gamma(1-\alpha)} \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \left[\int_a^{t+\varepsilon} \frac{(u(s)-u(t))(v(s)-v(t))}{s\left(\log \frac{t+\varepsilon}{s}\right)^{\alpha}} ds - \int_a^t \frac{(u(s)-u(t))(v(s)-v(t))}{s\left(\log \frac{t}{s}\right)^{\alpha}} ds \right]$$
$$= -\frac{\alpha}{\Gamma(1-\alpha)} \int_a^t \frac{(u(s)-u(t))(v(s)-v(t))}{s\left(\log \frac{t}{s}\right)^{1+\alpha}} ds.$$

The combination of integrals in (8) completes the proof. □

The proof of Lemma 2. Similar to the previous Lemma, we now use the decomposition

$$u(s)v(s) - u(t)v(t) = (u(s)-u(t))(v(s)-v(t)) + u(t)(v(s)-v(t)) + v(t)(u(s)-u(t)).$$

Then taking into account Definition 3, we obtain

$${}_H^C D_{a+,t}^{\alpha}[uv](t) = \mathcal{J}_1 + \mathcal{J}_2 + \mathcal{J}_3,$$

where

$$\mathcal{J}_1 = \frac{t}{\Gamma(1-\alpha)} \frac{d}{dt} \int_a^t \frac{(u(s)-u(t))(v(s)-v(t))}{s\left(\log \frac{t}{s}\right)^{\alpha}} ds,$$
$$\mathcal{J}_2 = u(t) \cdot \frac{t}{\Gamma(1-\alpha)} \frac{d}{dt} \int_a^t \frac{v(s)-v(t)}{s\left(\log \frac{t}{s}\right)^{\alpha}} ds,$$
$$\mathcal{J}_3 = v(t) \cdot \frac{t}{\Gamma(1-\alpha)} \frac{d}{dt} \int_a^t \frac{u(s)-u(t)}{s\left(\log \frac{t}{s}\right)^{\alpha}} ds.$$

From the Caputo–Hadamard derivative definition

$${}_H^C D_{a+,t}^{\alpha} v(t) = \frac{t}{\Gamma(1-\alpha)} \frac{d}{dt} \int_a^t \frac{v(s)-v(t)}{s\left(\log \frac{t}{s}\right)^{\alpha}} ds,$$

this yields

$$\mathcal{J}_2 = u(t) \cdot {}_H^C D_{a+,t}^{\alpha} v(t),$$
$$\mathcal{J}_3 = v(t) \cdot {}_H^C D_{a+,t}^{\alpha} u(t).$$

We now calculate

$$\mathcal{J}_1 = \frac{t}{\Gamma(1-\alpha)} \frac{d}{dt} \int_a^t \frac{(u(s)-u(t))(v(s)-v(t))}{s\left(\log \frac{t}{s}\right)^{\alpha}} ds.$$

This term is nonlocal, and it was shown earlier that

$$\frac{t}{\Gamma(1-\alpha)} \frac{d}{dt} \int_a^t \frac{(u(s)-u(t))(v(s)-v(t))}{s\left(\log \frac{t}{s}\right)^{\alpha}} ds$$
$$= -\frac{(u(a)-u(t))(v(a)-v(t))}{\Gamma(1-\alpha)\left(\log \frac{t}{a}\right)^{\alpha}} - \frac{\alpha}{\Gamma(1-\alpha)} \int_a^t \frac{(u(s)-u(t))(v(s)-v(t))}{s\left(\log \frac{t}{s}\right)^{\alpha+1}} ds.$$

Finally, combining the integrals, we complete our proof. □

2 Applications

In this section, due to the obtained results we explored a-priori estimates of the solutions.

2.1 Time-fractional diffusion equations

Let us consider the following time-fractional diffusion equation

$$D^{\alpha}_{a+,t}u = b(t)\Delta_x u + c(t,x)u + f(t,x), \quad (t,x) \in (a,T] \times \Omega := Q, \tag{9}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with regular boundary $\partial\Omega$, and the Dirichlet boundary condition

$$u(t,x) = 0, \quad t > a, \quad x \in \partial\Omega \tag{10}$$

or the Neumann boundary condition

$$\frac{\partial u}{\partial \eta} = 0, \quad t > a, \quad x \in \partial\Omega, \tag{11}$$

where η is the outward normal and the initial condition is

$$\lim_{t \to a} \Gamma(\alpha) \left(\log \frac{t}{a} \right)^{1-\alpha} u(a,x) = u_0(x). \tag{12}$$

Here
(A) $b(t)$ is a nonnegative continuous function;
(B) $||c(t,x)||_{C(([a,T);L^2(\Omega))} = d$;
(C) $||f(t,x)||_{C(([a,T);L^2(\Omega))} = h$.

Theorem 1. Let $u_0 \in L^2(\Omega)$ and statements (A), (B), (C) hold true. If u satisfies (9)–(12) for every $t \in (a,T]$, then

$$||u||_{C_{1-\alpha,\log}((a,T];L^2(\Omega))} \leq K_1(T)\,||u_0||_{L^2(\Omega)} + K_2(T)\,||f||_{C((a,T];L^2(\Omega))},$$

where
$$K_1(T) = \left[\frac{1}{\Gamma(\alpha)} + (2d+1) \left(\log \frac{T}{a} \right)^{\alpha} E_{\alpha,2\alpha} \left((2d+1) \left(\log \frac{T}{a} \right)^{\alpha} \right) \right]$$

and
$$K_2(T) = \left(\log \frac{T}{a} \right) \left[\frac{1}{\Gamma(\alpha+1)} + (2d+1) \left(\log \frac{T}{a} \right)^{\alpha} E_{\alpha,2\alpha+1} \left((2d+1) \left(\log \frac{T}{a} \right)^{\alpha} \right) \right].$$

Proof. Multiplying (9) by u and integrating over Ω, we get

$$\int_{\Omega} (D^{\alpha}_{a+,t}u) u dx = b(t) \int_{\Omega} (\Delta_x u) u dx + \int_{\Omega} c(t,x) u^2 dx + \int_{\Omega} f(t,x) u dx.$$

We begin by integrating by parts and then apply (4) together with Holder's inequality to get

$$\begin{aligned} \frac{1}{2} D^{\alpha}_{a+,t} \int_{\Omega} u^2 dx \leq{}& b(t) \int_{\partial\Omega} u \frac{\partial u}{\partial n} d\sigma - b(t) \int_{\Omega} \nabla u \nabla u dx \\ &+ \int_{\Omega} c(t,x) u^2 dx + \left(\int_{\Omega} |f(t,x)|^2 dx \right)^{\frac{1}{2}} \left(\int_{\Omega} |u|^2 dx \right)^{\frac{1}{2}}. \end{aligned}$$

Taking into account (B) and using $-b(t)\int_\Omega \nabla u \nabla u dx \leq 0$, we have

$$D^\alpha_{a+,t}\int_\Omega u^2dx \leq 2d\int_\Omega u^2dx + 2\Big(\int_\Omega |f(t,x)|^2dx\Big)^{\frac{1}{2}}\Big(\int_\Omega |u|^2dx\Big)^{\frac{1}{2}}.$$

At this stage, applying Young's inequality to the last term of the previous inequality, we deduce that

$$D^\alpha_{a+,t}\int_\Omega u^2dx \leq (2d+1)\int_\Omega u^2dx + \int_\Omega |f(t,x)|^2dx. \tag{13}$$

Let us define $y(t) = ||u(t,\cdot)||^2_{L^2(\Omega)}$ and taking into account (C) in (13), we get the time-fractional differential inequality

$$D^\alpha_{a+,t}y(t) \leq (2d+1)\,y(t) + h. \tag{14}$$

Applying the integral $I^\alpha_{a+,t}$ to both sides of the inequality (14) and using the Property 1, we obtain

$$\begin{aligned} y(t) &\leq \frac{\left(I^{1-\alpha}_{a+,t}y\right)(a)}{\Gamma(\alpha)}\left(\log\frac{t}{a}\right)^{\alpha-1} + \frac{1}{\Gamma(\alpha)}\int_a^t \left(\log\frac{t}{s}\right)^{\alpha-1}\left[(2d+1)y(s)+h\right]\frac{ds}{s} \\ &= \frac{2d+1}{\Gamma(\alpha)}\int_a^t \left(\log\frac{t}{s}\right)^{\alpha-1} y(s)\frac{ds}{s} \\ &+ \underbrace{\frac{\left(I^{1-\alpha}_{a+,t}y\right)(a)}{\Gamma(\alpha)}\left(\log\frac{t}{a}\right)^{\alpha-1} + \frac{h}{\Gamma(1+\alpha)}\left(\log\frac{t}{a}\right)^{\alpha}}_{g(t)}. \end{aligned}$$

Using Lemma 3 to the last estimate, it yields

$$\begin{aligned} y(t) &\leq g(t) + \int_a^t\left[\sum_{k=1}^\infty \frac{(2d+1)^k}{\Gamma(k\alpha)}\left(\log\frac{t}{a}\right)^{k\alpha-1}\frac{g(s)}{s}\right]ds \\ &\overset{j=k+1}{\leq} (2d+1)\int_a^t\left[E_{\alpha,\alpha}\left((2d+1)\left(\log\frac{t}{s}\right)^\alpha\right)\left(\log\frac{t}{s}\right)^{\alpha-1}\frac{g(s)}{s}\right]ds. \end{aligned}$$

Consequently, it follows that

$$\begin{aligned} y(t) &\leq \frac{\left(I^{1-\alpha}_{a+,t}y\right)(a)}{\Gamma(\alpha)}\left(\log\frac{t}{a}\right)^{\alpha-1} + \frac{h}{\Gamma(1+\alpha)}\left(\log\frac{t}{a}\right)^{\alpha} \\ &+ \frac{\left(I^{1-\alpha}_{a+,t}y\right)(a)(2d+1)}{\Gamma(\alpha)}\int_a^t\left[E_{\alpha,\alpha}\left((2d+1)\left(\log\frac{t}{s}\right)^\alpha\right)\left(\log\frac{t}{s}\right)^{2(\alpha-1)}\right]\frac{ds}{s} \\ &+ \frac{(2d+1)h}{\Gamma(1+\alpha)}\int_a^t\left[E_{\alpha,\alpha}\left((2d+1)\left(\log\frac{t}{s}\right)^\alpha\right)\left(\log\frac{t}{s}\right)^{\alpha-1}\left(\log\frac{t}{a}\right)^{\alpha}\right]\frac{ds}{s}. \end{aligned} \tag{15}$$

Applying formula (2.2.51) from [4, p. 86], we have the following calculations

$$\begin{aligned} &\frac{1}{\Gamma(\alpha)}\int_a^t\left[E_{\alpha,\alpha}\left((2d+1)\left(\log\frac{t}{s}\right)^\alpha\right)\left(\log\frac{t}{s}\right)^{2(\alpha-1)}\right]\frac{ds}{s} \\ &= \left(\log\frac{t}{a}\right)^{2\alpha-1}E_{\alpha,2\alpha}\left((2d+1)\left(\log\frac{t}{a}\right)^\alpha\right) \end{aligned} \tag{16}$$

and

$$\frac{1}{\Gamma(\alpha+1)}\int_a^t \left[E_{\alpha,\alpha}\left((2d+1)\left(\log\frac{t}{s}\right)^{\alpha}\right)\left(\log\frac{t}{s}\right)^{\alpha-1}\left(\log\frac{t}{a}\right)^{\alpha}\right]\frac{ds}{s} = \left(\log\frac{t}{a}\right)^{2\alpha} E_{\alpha,2\alpha+1}\left((2d+1)\left(\log\frac{t}{a}\right)^{\alpha}\right). \tag{17}$$

Substituting (16), (17) in to the inequality (15), we obtain

$$\begin{aligned} y(t) \le \left(I_{a+,t}^{1-\alpha}y\right)(a)\left(\log\frac{t}{a}\right)^{\alpha-1}\left[\frac{1}{\Gamma(\alpha)} + (2d+1)\left(\log\frac{t}{a}\right)^{\alpha} E_{\alpha,2\alpha}\left((2d+1)\left(\log\frac{t}{a}\right)^{\alpha}\right)\right] \\ + h\left(\log\frac{t}{a}\right)^{\alpha}\left[\frac{1}{\Gamma(\alpha+1)} + (2d+1)\left(\log\frac{t}{a}\right)^{\alpha} E_{\alpha,2\alpha+1}\left((2d+1)\left(\log\frac{t}{a}\right)^{\alpha}\right)\right]. \end{aligned} \tag{18}$$

By multiplying both sides of (18) by $\left(\log\frac{t}{a}\right)^{1-\alpha}$, we get

$$\begin{aligned} \left(\log\frac{t}{a}\right)^{1-\alpha} y(t) &\le \left(I_{a+,t}^{1-\alpha}y\right)(a)\left[\frac{1}{\Gamma(\alpha)} + (2d+1)\left(\log\frac{t}{a}\right)^{\alpha} E_{\alpha,2\alpha}\left((2d+1)\left(\log\frac{t}{a}\right)^{\alpha}\right)\right] \\ &\quad + h\left(\log\frac{t}{a}\right)\left[\frac{1}{\Gamma(\alpha+1)} + (2d+1)\left(\log\frac{t}{a}\right)^{\alpha} E_{\alpha,2\alpha+1}\left((2d+1)\left(\log\frac{t}{a}\right)^{\alpha}\right)\right] \\ &\le \left(I_{a+,t}^{1-\alpha}y\right)(a)\left[\frac{1}{\Gamma(\alpha)} + (2d+1)\left(\log\frac{T}{a}\right)^{\alpha} E_{\alpha,2\alpha}\left((2d+1)\left(\log\frac{T}{a}\right)^{\alpha}\right)\right] \\ &\quad + h\left(\log\frac{T}{a}\right)\left[\frac{1}{\Gamma(\alpha+1)} + (2d+1)\left(\log\frac{T}{a}\right)^{\alpha} E_{\alpha,2\alpha+1}\left((2d+1)\left(\log\frac{T}{a}\right)^{\alpha}\right)\right]. \end{aligned}$$

Then, we have

$$||u||_{C_{1-\alpha,\log}((a,T];L^2(\Omega))} \le K_1(T)\,||u_0||_{L^2(\Omega)} + K_2(T)\,||f||_{C((a,T],L^2(\Omega))},$$

which gives the desired result. □

2.2 The porous medium equation

Next, we study the porous medium equation

$$D_{a+,t}^{\alpha}u(t,x) = a(t,x)\Delta u^m(t,x) + f(t,x), \quad (t,x) \in (a,T]\times\Omega := Q, \tag{19}$$

with the initial condition

$$\lim_{t\to a}\Gamma(\alpha)\left(\log\frac{t}{a}\right)^{1-\alpha} u(t,x) = \lim_{t\to a}\left(I_{a+}^{1-\alpha}u\right)(t,x) = \phi(x), \quad x\in\Omega \tag{20}$$

and the boundary condition

$$u(t,x) = 0, \quad t > a, \quad x \in \partial\Omega, \tag{21}$$

where $m > 1$ and $a(t,x)$, $f(t,x)$ are nonnegative continuous functions.

Theorem 2. Let $\Omega \subset \mathbb{R}^n$ and $\phi \in L^p(\Omega)$. The function $u \in C_{1-\alpha,\log}((a,T];L^p(\Omega))$ is a solution of problem (19)–(21) and

$$||u||_{C_{1-\alpha,\log}((a,T];L^p(\Omega))} \le K_3(T)\,||\phi)||_{L^p(\Omega)} + K_4(T)\,||f||_{C((a,T];L^p(\Omega))},$$

where

$$K_3(T) = \left[\frac{1}{\Gamma(\alpha)} + M\left(\log\frac{T}{a}\right)^{\alpha} E_{\alpha,2\alpha}\left(M\left(\log\frac{T}{a}\right)^{\alpha}\right)\right],$$

$$K_4(T) = \left(\log\frac{T}{a}\right)\left[\frac{1}{\Gamma(\alpha+1)} + M\left(\log\frac{T}{a}\right)^{\alpha} E_{\alpha,2\alpha+1}\left(M\left(\log\frac{T}{a}\right)^{\alpha}\right)\right].$$

Proof. Multiplying (19) by pu^{p-1} $(p \geq 2)$, and integrating over Ω, we arrive at

$$\int_\Omega pu^{p-1} D^\alpha_{a+} u dx - \int_\Omega a(t,x) pu^{p-1} \Delta u^m dx - \int_\Omega pu^{p-1} f(t,x) dx = 0.$$

In view of the expression

$$\begin{aligned} p\int_\Omega a(t,x) u^{p-1} \Delta u^m dx &= p\int_{\partial\Omega} a(t,x) u^{p-1} u^{m-1} \frac{\partial}{\partial \eta} u d\sigma \\ &\quad - p\int_\Omega (p-1) a(t,x) u^{p-2} u^{m-1} |\nabla u|^2 dx \\ &= -p\int_\Omega (p-1) a(t,x) u^{p-2} u^{m-1} |\nabla u|^2 dx, \end{aligned}$$

it follows that

$$\begin{aligned} \int_\Omega pu^{p-1} D^\alpha_{a+} u dx + p\int_\Omega (p-1) a(t,x) u^{p-2} u^{m-1} |\nabla u|^2 dx \\ - p\int_\Omega u^{p-1} f(t,x) dx = 0. \end{aligned} \tag{22}$$

Applying (7) and the Hölder inequality to (22), we obtain

$$\begin{aligned} \int_\Omega D^\alpha_{a+} u^p dx + \frac{4p(p-1)d_1}{(p+m-1)^2} \int_\Omega \left|\nabla u^{\frac{p+m-1}{2}}\right|^2 dx \\ - p\left(\int_\Omega |f(t,x)|^p dx\right)^{1/p} \left(\int_\Omega u^p dx\right)^{1-1/p} \leq 0. \end{aligned} \tag{23}$$

Using Young's inequality in the last term of (23), it follows that

$$\begin{aligned} \int_\Omega D^\alpha_{a+} u^p dx + \frac{4p(p-1)d_1}{(p+m-1)^2} \int_\Omega \left|\nabla u^{\frac{p+m-1}{2}}\right|^2 dx \\ - \varepsilon^p \int_\Omega |f(t,x)|^p dx - \frac{p-1}{\varepsilon^{\frac{p}{p-1}}} \int_\Omega u^p dx \leq 0, \quad \varepsilon > 0. \end{aligned}$$

Let's make the following notations

$$y(t) = ||u(t,\cdot)||^p_{L^p(\Omega)}, \quad \mathcal{H} = \varepsilon^p ||f(t,\cdot)||^p_{L^p(\Omega)}, \quad M = \frac{p-1}{\varepsilon^{\frac{p}{p-1}}}.$$

Then, we have

$$D^\alpha_{a+} y(t) \leq My(t) + \mathcal{H}. \tag{24}$$

Starting from (24), by performing the same actions as in the proof of the previous theorem, we obtain the following conclusion

$$\begin{aligned} \left(\log\frac{t}{a}\right)^{1-\alpha} y(t) &\leq \left(I^{1-\alpha}_{a+,t} y\right)(a) \left[\frac{1}{\Gamma(\alpha)} + M\left(\log\frac{t}{a}\right)^\alpha E_{\alpha,2\alpha}\left(M\left(\log\frac{t}{a}\right)^\alpha\right)\right] \\ &\quad + \mathcal{H}\left(\log\frac{t}{a}\right)\left[\frac{1}{\Gamma(\alpha+1)} + M\left(\log\frac{t}{a}\right)^\alpha E_{\alpha,2\alpha+1}\left(M\left(\log\frac{t}{a}\right)^\alpha\right)\right] \\ &\leq \left(I^{1-\alpha}_{a+,t} y\right)(a) \left[\frac{1}{\Gamma(\alpha)} + M\left(\log\frac{T}{a}\right)^\alpha E_{\alpha,2\alpha}\left(M\left(\log\frac{T}{a}\right)^\alpha\right)\right] \\ &\quad + \mathcal{H}\left(\log\frac{T}{a}\right)\left[\frac{1}{\Gamma(\alpha+1)} + M\left(\log\frac{T}{a}\right)^\alpha E_{\alpha,2\alpha+1}\left(M\left(\log\frac{T}{a}\right)^\alpha\right)\right]. \end{aligned}$$

Hence, we deduce that

$$||u||_{C_{1-\alpha,\log}((a,T];L^p(\Omega))} \leq K_3(T)\,||\phi||_{L^p(\Omega)} + K_4(T)\,||f||_{C((a,T];L^p(\Omega))}\,,$$

where

$$K_3(T) = \left[\frac{1}{\Gamma(\alpha)} + M\left(\log\frac{T}{a}\right)^{\alpha} E_{\alpha,2\alpha}\left(M\left(\log\frac{T}{a}\right)^{\alpha}\right)\right]$$

and

$$K_4(T) = \left(\log\frac{T}{a}\right)\left[\frac{1}{\Gamma(\alpha+1)} + M\left(\log\frac{T}{a}\right)^{\alpha} E_{\alpha,2\alpha+1}\left(M\left(\log\frac{T}{a}\right)^{\alpha}\right)\right].$$

Therefore, we have proven the statement. □

2.3 Fractional-order diffusion equation

In the next case, we consider the fractional-order diffusion equation

$$ {}^{C}_{H}D^{\alpha}_{a+,t}u = \log\left(\frac{t}{a}\right)\Delta_x u + c(t,x)u + f(t,x), \quad (t,x)\in(a,T]\times\Omega, \tag{25}$$

with the Dirichlet boundary condition

$$u(t,x) = 0, \quad t > a > 0, \quad x\in\partial\Omega \tag{26}$$

and with the Cauchy condition

$$u(a,x) = u_0(x), \tag{27}$$

where the functions $c(t,x), f(t,x)$ satisfy

(A) $\|c(t,x)\|_{C(([a,T);L^2(\Omega))} = d, c(t,x)\leq 0;$

(B) $\|f(t,x)\|_{C(([a,T);L^2(\Omega))} = h.$

Theorem 3. Suppose $u_0\in L^2(\Omega)$ and (A), (B) hold. If the function $u(t,x)$ satisfies the problem (25)–(27) for each $t\in(a,T]$, then the following estimate holds

$$\|u\|_{C((a,T];L^2(\Omega))} \leq K_5(T)\,\|u_0\|_{L^2(\Omega)} + K_6(T)\|f\|_{C((a,T];L^2(\Omega))},$$

where

$$K_5(T) = 1 + (2d+1)\int_a^T\left[E_{\alpha,\alpha}\left((2d+1)\left(\log\frac{t}{s}\right)^{\alpha}\right)\left(\log\frac{t}{s}\right)^{\alpha-1}\right]\frac{ds}{s}$$

and

$$K_6(T) = \left(\log\frac{T}{a}\right)^{\alpha}\left[\frac{1}{\Gamma(\alpha+1)} + (2d+1)\left(\log\frac{T}{a}\right)^{\alpha} E_{\alpha,2\alpha+1}\left((2d+1)\left(\log\frac{T}{a}\right)^{\alpha}\right)\right].$$

Proof. Multiplying each term of equation (25) by the function u and integrating over Ω,

$$\int_\Omega \left({}^{C}_{H}D^{\alpha}_{a+,t}u\right)u dx = \log\left(\frac{t}{a}\right)\int_\Omega(\Delta_x u)\,u dx + \int_\Omega c(t,x)u^2dx + \int_\Omega f(t,x)u dx.$$

Taking into account the estimate (6) and using Hölder's inequality for the last term of the previously mentioned inequality, we arrive at

$$\frac{1}{2}{}^{C}_{H}D^{\alpha}_{a+,t}\int_\Omega u^2dx \leq \log\left(\frac{t}{a}\right)\int_{\partial\Omega}u\frac{\partial u}{\partial n}d\sigma - \log\left(\frac{t}{a}\right)\int_\Omega \nabla u\nabla u dx$$
$$+\int_\Omega c(t,x)u^2dx + \left(\int_\Omega|f(t,x)|^2dx\right)^{\frac{1}{2}}\left(\int_\Omega|u|^2dx\right)^{\frac{1}{2}}.$$

Applying Young's inequality to the last term of the previous inequality and in view of

$$-\log\left(\frac{t}{a}\right)\int_{\Omega}\nabla u\nabla u dx\leq 0$$

with the notation (A), (B) and $y(t)=\|u(t,x)\|^2_{L^2(\Omega)}$, we obtain

$$^{C}_{H}D^{\alpha}_{a+,t}y(t)\leq(2d+1)y(t)+h. \tag{28}$$

By applying the integral ${}_HI^{\alpha}_{a+,t}$ to both sides of inequality (28) and using Property 1, we derive the following expression

$$\begin{aligned} y(t)&\leq y(a)+\frac{2d+1}{\Gamma(\alpha)}\int_a^t\left(\log\frac{t}{s}\right)^{\alpha-1}y(s)\frac{ds}{s}+\frac{h}{\Gamma(\alpha)}\int_a^t\left(\log\frac{t}{s}\right)^{\alpha-1}\frac{ds}{s}\\ &=\underbrace{y(a)+\frac{h}{\Gamma(1+\alpha)}\left(\log\frac{t}{a}\right)^{\alpha}}_{g(t)}+\frac{2d+1}{\Gamma(\alpha)}\int_a^t\left(\log\frac{t}{s}\right)^{\alpha-1}y(s)\frac{ds}{s}. \end{aligned}$$

According to the result of Lemma 3, we deduce that

$$\begin{aligned} y(t)&\leq g(t)+\int_a^t\left[\sum_{k=1}^{\infty}\frac{[(2d+1)\Gamma(\alpha)]^k}{\Gamma(k\alpha)}\left(\log\frac{t}{a}\right)^{k\alpha-1}\frac{g(s)}{s}\right]ds\\ &\overset{j=k+1}{\leq} g(t)+(2d+1)\Gamma(\alpha)\int_a^t\left[E_{\alpha,\alpha}\left((2d+1)\Gamma(\alpha)\left(\log\frac{t}{s}\right)^{\alpha}\right)\left(\log\frac{t}{s}\right)^{\alpha-1}\frac{g(s)}{s}\right]ds. \end{aligned}$$

Therefore, it follows that

$$\begin{aligned} y(t)\leq{}& y(a)+\frac{h}{\Gamma(1+\alpha)}\left(\log\frac{t}{a}\right)^{\alpha}\\ &+y(a)(2d+1)\Gamma(\alpha)\int_a^t\left[E_{\alpha,\alpha}\left((2d+1)\Gamma(\alpha)\left(\log\frac{t}{s}\right)^{\alpha}\right)\left(\log\frac{t}{s}\right)^{\alpha-1}\right]\frac{ds}{s}\\ &+\frac{(2d+1)\Gamma(\alpha)h}{\Gamma(\alpha+1)}\int_a^t\left[E_{\alpha,\alpha}\left((2d+1)\Gamma(\alpha)\left(\log\frac{t}{s}\right)^{\alpha}\right)\left(\log\frac{t}{s}\right)^{2\alpha-1}\right]\frac{ds}{s}. \end{aligned}$$

In view of formula (2.2.51) in [4, p. 86], we arrive at the following:

$$\begin{aligned} &\frac{1}{\Gamma(\alpha+1)}\int_a^t\left[E_{\alpha,\alpha}\left((2d+1)\left(\log\frac{t}{s}\right)^{\alpha}\right)\left(\log\frac{t}{s}\right)^{2\alpha-1}\right]\frac{ds}{s}\\ &=\left(\log\frac{t}{a}\right)^{2\alpha}E_{\alpha,2\alpha+1}\left((2d+1)\left(\log\frac{t}{a}\right)^{\alpha}\right). \end{aligned}$$

It implies that

$$\begin{aligned} y(t)\leq{}& y(a)\left[1+(2d+1)\Gamma(\alpha)\int_a^T\left[E_{\alpha,\alpha}\left((2d+1)\Gamma(\alpha)\left(\log\frac{t}{s}\right)^{\alpha}\right)\left(\log\frac{t}{s}\right)^{\alpha-1}\right]\frac{ds}{s}\right]\\ &+h\left(\log\frac{T}{a}\right)^{\alpha}\left[\frac{1}{\Gamma(\alpha+1)}+(2d+1)\Gamma(\alpha)\left(\log\frac{T}{a}\right)^{\alpha}E_{\alpha,2\alpha+1}\left((2d+1)\Gamma(\alpha)\left(\log\frac{T}{a}\right)^{\alpha}\right)\right]. \end{aligned}$$

Finally, we conclude

$$\|u\|_{C((a,T];L^2(\Omega))} \leq K_5(T)\,\|u_0\|_{L^2(\Omega)} + K_6(T)\|f\|_{C((a,T],L^2(\Omega))},$$

where

$$K_5(T) = 1 + (2d+1)\Gamma(\alpha)\int_a^T \left[E_{\alpha,\alpha}\left((2d+1)\Gamma(\alpha)\left(\log\frac{t}{s}\right)^{\alpha}\right)\left(\log\frac{t}{s}\right)^{\alpha-1}\right]\frac{ds}{s}$$

and

$$K_6(T) = \left(\log\frac{T}{a}\right)^{\alpha}\left[\frac{1}{\Gamma(\alpha+1)} + (2d+1)\left(\log\frac{T}{a}\right)^{\alpha} E_{\alpha,2\alpha+1}\left((2d+1)\left(\log\frac{T}{a}\right)^{\alpha}\right)\right],$$

which completes the proof. □

Conclusion

In this work, we have established new analogues of the Leibniz rule for the Hadamard and Caputo–Hadamard fractional derivatives, taking into account their inherent nonlocal properties. The refined differentiation formulas and derived inequalities provide a deeper understanding of how fractional derivatives interact with nonlinear functions. In particular, the obtained estimates form an analytical foundation for studying fractional diffusion equations of various types. The results can be effectively applied to prove the existence, uniqueness, and stability of solutions, as well as to derive a priori bounds essential for the qualitative analysis of such models. Future research may extend these methods to systems with variable order or to multidimensional fractional operators.

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Conflict of Interest

The authors declare no conflict of interest.

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