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Three-weight Hardy inequalities with iterated operators and generalized kernels

A.A. Kalybay [1], A.M. Temirkhanova[2,*]

[1]*KIMEP University, Almaty, Kazakhstan;*
[2]*L.N. Gumilyov Eurasian National University, Astana, Kazakhstan*
(E-mails: kalybay@kimep.kz, ainura-t@yandex.kz)

The well-known Hardy inequalities, formulated in both continuous and discrete cases, play an important role in mathematical analysis, differential equations and many other branches of mathematics. The original forms of these inequalities were subsequently extended and generalized in various directions, leading to the development of Hardy inequalities as an independent and significant area of research. A central problem in the theory of weighted inequalities is the characterization of conditions under which inequalities involving Hardy-type operators hold. Many cases of weighted estimates for linear integral Hardy-type operators have been considered, and there is a large number of books and scientific articles on this topic. More recently, considerable attention has been given to iterated Hardy-type operators due to their application in Morrey-type spaces. This paper analyzes a class of operators formed by iterating two operators, one of which involves a kernel satisfying conditions that generalize those considered previously. The study examines Hardy-type inequalities associated with these iterated operators and establishes necessary and sufficient conditions for their validity. The characterization of weighted Hardy inequalities involving iterated operators can now be applied to study of bilinear weighted Hardy-type inequalities.

Keywords: integral operator, iterated operator, Hardy-type inequality, weight function, kernel, Lebesgue space, Oinarov condition, Oinarov classes.

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Introduction

For $f \geq 0$, we consider the inequalities

$$\left(\int_0^\infty u(x)\left(\int_0^x\left(\int_t^\infty K(s,t)f(s)\,ds\right)^r w(t)\,dt\right)^{\frac{q}{r}}dx\right)^{\frac{1}{q}} \leq C\left(\int_0^\infty v(x)f^p(x)\,dx\right)^{\frac{1}{p}} \quad (1)$$

and

$$\left(\int_0^\infty u(x)\left(\int_x^\infty\left(\int_0^t K(t,s)f(s)\,ds\right)^r w(t)\,dt\right)^{\frac{q}{r}}dx\right)^{\frac{1}{q}} \leq C\left(\int_0^\infty v(x)f^p(x)\,dx\right)^{\frac{1}{p}}, \quad (2)$$

where u, v, and w are non-negative, measurable, and locally summable weight functions on $I = (0,\infty)$ and $K(\cdot,\cdot)$ is a measurable function referred to as the kernel.

*Corresponding author. *E-mail: ainura-t@yandex.kz*

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Define $L_{p,v} \equiv L_{p,v}(I)$, $1 \le p < \infty$, as the Lebesgue space of measurable functions f on I satisfying

$$\|f\|_{p,v} = \left(\int_0^\infty v(x)|f(x)|^p dx \right)^{\frac{1}{p}} < \infty.$$

Then these Hardy-type inequalities (1) and (2) can be rewritten in the shortened form:

$$\|Tf\|_{q,u} \le C\|f\|_{p,v}, \quad 0 \le f \in L_{p,v}, \tag{3}$$

where T is one of the operators

$$T^+ f(x) = \left(\int_0^x \left(\int_t^\infty K(s,t) f(s) ds \right)^r w(t)\, dt \right)^{\frac{1}{r}},$$

$$T^- f(x) = \left(\int_x^\infty \left(\int_0^t K(t,s) f(s) ds \right)^r w(t)\, dt \right)^{\frac{1}{r}}.$$

When $K(\cdot,\cdot) \equiv 1$, inequalities (1) and (2) have been studied in [1–3] and the references therein. Related results on inequalities (1) and (2) can also be found in [4–6]. In [7], the problem of characterizing inequality (2) for $p = 1$ was established. When $K(\cdot,\cdot)$ satisfies the Oinarov condition $\mathcal{O}$, which states that there exists a constant $h \ge 1$ such that

$$\frac{1}{h}\left(K(x,t) + K(t,s)\right) \le K(x,s) \le h\left(K(x,t) + K(t,s)\right) \tag{4}$$

for $x \ge t \ge s > 0$, inequalities (1) and (2) have been treated in [8–10]. In [8], the simplest case $1 < p \le q < \infty$ and $0 < r < \infty$ is addressed. The papers [9, 10] investigate all possible relations between the summation parameters, though their characterizations depend on the use of an auxiliary function. In [11], both cases $1 < p \le q < \infty$ and $1 < q < p < \infty$ are considered, but with the restriction $r < q$.

In this paper, we study inequalities (1) and (2) for both cases $1 < p \le q < \infty$ and $1 < q < p < \infty$, but now allowing any $0 < r < \infty$. Moreover, we assume that the kernels $K(\cdot,\cdot)$ belong to the classes $\mathcal{O}_n^{\pm}$, $n \ge 0$, referred to as *Oinarov classes*, which generalize the class of kernels satisfying condition (4).

The importance of studying inequalities (1) and (2) is highlighted in recent papers [2] and [12], which emphasize that, due to numerous applications, this topic has become highly fashionable in the theory of Hardy inequalities. Since papers [2] and [12] thoroughly reveal all applications of these inequalities, we omit their listing here.

Assume that $\frac{1}{p} + \frac{1}{p'} = 1$, $\frac{1}{q} + \frac{1}{q'} = 1$, and $\frac{1}{\mu} = \frac{1}{q} - \frac{1}{p}$. Let the symbol $A \ll B$ denote that $A \le CB$ for some constant $C > 0$, and let the symbol $A \approx B$ denote that $A \ll B \ll A$.

First, we formulate the characterization of the inequalities (1) and (2) for a kernel that satisfies Oinarov's condition (4), which demonstrates the approaches and main ideas of the proof. This case is a special case of the main results presented in Theorems 3 and 4. It is worse to mention that, in this case, it is also possible to include the case $0 < q < 1 \le p < \infty$, which is excluded in the main results.

Theorem 1. Let $1 < q < p < \infty$ and $0 < r < \infty$. Let the kernel $K(\cdot,\cdot)$ satisfy the Oinarov condition $\mathcal{O}$. Then (1) holds if and only if $\mathcal{A} = \max\{A_{q<p}, A_{00}, A_{01}, A_{11}\} < \infty$, where

$$A_{q<p} = \left(\int_0^\infty u(x) \left(\int_x^\infty u(s) ds \right)^{\frac{\mu}{p}} \left(J_{p,r}^-(0,x)\right)^\mu dx \right)^{\frac{1}{\mu}},$$

$$A_{00} = \left(\int_0^\infty \left(\int_0^t u(s) \left(\int_0^s K^r(s,z)w(z)dz \right)^{\frac{q}{r}} ds \right)^{\frac{\mu}{q}} \left(\int_t^\infty v^{1-p'}(s)\, ds \right)^{\frac{\mu}{q'}} v^{1-p'}(t)dt \right)^{\frac{1}{\mu}},$$

$$A_{01} = \left(\int_0^\infty \left(\int_0^t u(s) \left(\int_0^s w(z)dz \right)^{\frac{q}{r}} K^q(t,s)ds \right)^{\frac{\mu}{q}} \left(\int_t^\infty v^{1-p'}(s)\, ds \right)^{\frac{\mu}{q'}} v^{1-p'}(t)dt \right)^{\frac{1}{\mu}},$$

$$A_{11} = \left(\int_0^\infty \left(\int_0^t u(s) \left(\int_0^s w(z)dz \right)^{\frac{q}{r}} ds \right)^{\frac{\mu}{p}} \left(\int_t^\infty K^{p'}(s,t) v^{1-p'}(s)\, ds \right)^{\frac{\mu}{p'}} u(t) \left(\int_0^t w(s)ds \right)^{\frac{q}{r}} dt \right)^{\frac{1}{\mu}}.$$

Moreover, $\mathcal{A} \approx C$, where C is the best constant in (1).

If we split the interior integral on the left-hand side of inequality (1) into two integrals, we obtain

$$\left(\int_0^\infty u(x) \left(\int_0^x \left(\int_t^\infty K(s,t)f(s)\, ds \right)^r w(t)\, dt \right)^{\frac{q}{r}} dx \right)^{\frac{1}{q}}$$

$$\approx \left(\int_0^\infty u(x) \left(\int_0^x \left(\int_t^x K(s,t)f(s)\, ds \right)^r w(t)\, dt \right)^{\frac{q}{r}} dx \right)^{\frac{1}{q}}$$

$$+ \left(\int_0^\infty u(x) \left(\int_0^x \left(\int_x^\infty K(s,t)f(s)\, ds \right)^r w(t)\, dt \right)^{\frac{q}{r}} dx \right)^{\frac{1}{q}}. \tag{5}$$

Taking into account condition (4), from (5) we derive that the validity of (1) is equivalent to the validity of the following three inequalities:

$$\left(\int_0^\infty u(x) \left(\int_0^x \left(\int_t^x K(s,t)f(s)ds \right)^r w(t)\, dt \right)^{\frac{q}{r}} dx \right)^{\frac{1}{q}} \le C_1 \|f\|_{p,v}, \tag{6}$$

$$\left(\int_0^\infty u(x) \left(\int_0^x K^r(x,t)w(t)\, dt \right)^{\frac{q}{r}} \left(\int_x^\infty f(s)\, ds \right)^q dx \right)^{\frac{1}{q}} \le C_2 \|f\|_{p,v}, \tag{7}$$

$$\left(\int_0^\infty u(x) \left(\int_0^x w(t)\, dt \right)^{\frac{q}{r}} \left(\int_x^\infty K(s,x)f(s)\, ds \right)^q dx \right)^{\frac{1}{q}} \le C_3 \|f\|_{p,v}. \tag{8}$$

Inequality (7) is a standard weighted Hardy inequality, which holds if and only if $A_{00} < \infty$ (see, e.g., [13, Theorem 5]). Inequality (8) is a Hardy-type inequality involving a Volterra-type operator

$K^-f(x) = \int\limits_x^\infty K(s,x)f(s)\,ds$, where $K(\cdot,\cdot)$ satisfies the Oinarov condition $\mathcal{O}$. Its validity follows from the conditions $A_{01} < \infty$ and $A_{11} < \infty$, as established in [14].

In [15], it is proved that inequality (6) holds if and only if $A_{q<p} < \infty$, where

$$J^-_{p,r}(\alpha,\beta) = \sup_{f\geq 0} \frac{\left(\int\limits_\alpha^\beta \left(\int\limits_x^\beta K(s,x)f(s)\,ds\right)^r w(x)\,dx\right)^{\frac{1}{r}}}{\|f\|_{p,v,(\alpha,\beta)}}, \quad 0 \leq \alpha < \beta \leq \infty,$$

as mentioned above, can be found in [14] when $K(\cdot,\cdot)$, the kernel of the Volterra-type operator K^-, satisfies the Oinarov condition $\mathcal{O}$.

Unfortunately, inequality (6) has not received as much attention as inequality (1) due to its fewer applications. Consequently, paper [15] has not garnered as many references as those discussing inequality (1), despite the fact that [15] addresses both cases, $1 \leq p \leq q < \infty$ and $0 < q < p < \infty$, $p \geq 1$, for any $0 < r < \infty$, with the results presented in terms of the quantity $J^-_{p,r}$ without imposing any restrictions on the kernel involved. However, (6), being less popular than inequality (1), serves as the basis for characterizing inequality (1), as demonstrated by the splitting in (5).

Let

$$J^+_{p,r}(\alpha,\beta) = \sup_{f\geq 0} \frac{\left(\int\limits_\alpha^\beta \left(\int\limits_\alpha^t K(t,s)f(s)\,ds\right)^r w(t)\,dt\right)^{\frac{1}{r}}}{\|f\|_{p,v,(\alpha,\beta)}}, \quad 0 \leq \alpha < \beta \leq \infty.$$

Similarly, we can derive characterizations for inequality (2) to hold when the kernel of the operator T^- satisfies condition (4).

Theorem 2. Let $1 < q < p < \infty$ and $0 < r < \infty$. Let the kernel $K(\cdot,\cdot)$ satisfy the Oinarov condition $\mathcal{O}$. Then (2) holds if and only if $\mathcal{B} = \max\{B_{q<p}, B_{00}, B_{10}, B_{11}\} < \infty$, where

$$B_{q<p} = \left(\int\limits_0^\infty u(x)\left(\int\limits_0^x u(s)ds\right)^{\frac{\mu}{p}} \left(J^+_{p,r}(x,\infty)\right)^\mu dx\right)^{\frac{1}{\mu}},$$

$$B_{00} = \left(\int\limits_0^\infty \left(\int\limits_t^\infty u(s)\left(\int\limits_s^\infty K^r(z,s)w(z)dz\right)^{\frac{q}{r}} ds\right)^{\frac{\mu}{q}} \left(\int\limits_0^t v^{1-p'}(s)\,ds\right)^{\frac{\mu}{q'}} v^{1-p'}(t)dt\right)^{\frac{1}{\mu}},$$

$$B_{10} = \left(\int\limits_0^\infty \left(\int\limits_t^\infty u(s)\left(\int\limits_s^\infty w(z)dz\right)^{\frac{q}{r}} K^q(s,t)ds\right)^{\frac{\mu}{q}} \left(\int\limits_0^t v^{1-p'}(s)\,ds\right)^{\frac{\mu}{q'}} v^{1-p'}(t)dt\right)^{\frac{1}{\mu}},$$

$$B_{11} = \left(\int\limits_0^\infty \left(\int\limits_t^\infty u(s)\left(\int\limits_s^\infty w(z)dz\right)^{\frac{q}{r}} ds\right)^{\frac{\mu}{p}} \left(\int\limits_0^t K^{p'}(t,s)v^{1-p'}(s)\,ds\right)^{\frac{\mu}{p'}} u(t)\left(\int\limits_t^\infty w(s)ds\right)^{\frac{q}{r}} dt\right)^{\frac{1}{\mu}}.$$

Moreover, $\mathcal{B} \approx C$, where C is the best constant in (2).

Theorems 1 and 2 are corollaries of the main results presented later in the paper, where inequalities (1) and (2) are established for operators with more general kernels. We chose to present Theorems 1 and 2 in the Introduction because, for over 30 years since the Oinarov condition $\mathcal{O}$ was introduced in 1991, Hardy-type inequalities have been primarily studied for these kernels. The Oinarov classes $\mathcal{O}_n^{\pm}$, $n \geq 0$, were introduced in 2007 in [16]. However, despite the fact that the conditions for belonging to these classes are significantly weaker than the condition $\mathcal{O}$, they have not been as widely used as the Oinarov classes yet.

Theorem 1 examines the case $1 < q < p < \infty$ and $0 < r < \infty$. Since the case $1 < p \leq q < \infty$ and $0 < r < \infty$ was established in [8] using a similar splitting method, it has not been included in the Introduction. However, it is worth noting that the conditions for the validity of standard Hardy inequality (7) in the case $0 < q < 1 \leq p < \infty$ are known from [17]. When the kernel satisfies the Oinarov condition, the missing characterizations for the validity of inequality (8) in the case $0 < q < 1 \leq p < \infty$ were recently provided in [18]. Furthermore, the condition $A_{q<p} < \infty$ is necessary and sufficient for the validity of inequality (6) when $0 < q < p < \infty$, $p \geq 1$, and $0 < r < \infty$. Since the simultaneous validity of inequalities (6), (8), and (7) ensures the validity of inequality (1), we can easily derive its characterizations for the case $0 < q < 1 \leq p < \infty$ and $0 < r < \infty$ when the kernel satisfies the condition $\mathcal{O}$. The same arguments can be applied to determine the conditions for the validity of inequality (2).

The structure of the paper is organized as follows. Section 1 is devoted to Oinarov's classes. In Section 2, we present our first main result, namely the characterization of inequality (1), and in the next section, we present our second main result, namely the characterization of inequality (2).

1 The Oinarov classes of kernels

In the work [16], R. Oinarov introduced the classes of functions $\mathcal{O}_n^{\pm}$, $n \geq 0$. Let us give the definitions of these classes.

Let $\Omega = \{(x,s) : x \geq s > 0\}$. We define the classes $\mathcal{O}_n^{\pm}$, $n \geq 0$, in a recurrent form as a set of functions $K(\cdot,\cdot)$ that are non-negative and measurable on the set Ω and satisfy certain conditions.

Definition 1. The class $\mathcal{O}_0^{+}$ $\left(\mathcal{O}_0^{-}\right)$ consists of functions of the form $K_0(x,s) \equiv r(s)$ $(K_0(x,s) \equiv r(x))$.

Let the classes $\mathcal{O}_i^{\pm}$ be defined for $i = 0,1,...,n-1$, $n \geq 1$.

Definition 2. A function $K(\cdot,\cdot) \equiv K_n(\cdot,\cdot) \in \mathcal{O}_n^{+}$, $n \geq 1$, if it is non-decreasing in the first argument and there exist non-negative measurable on Ω functions $K_i(\cdot,\cdot)$, $K_{n,i}(\cdot,\cdot)$, $i = 0,1,...,n-1$, and a number $h_n \geq 1$ such that $K_i(\cdot,\cdot) \in \mathcal{O}_i^{+}$, $i = 0,1,...,n-1$, and

$$\frac{1}{h_n}\sum_{i=0}^{n} K_{n,i}(x,t)K_i(t,s) \leq K_n(x,s) \leq h_n \sum_{i=0}^{n} K_{n,i}(x,t)K_i(t,s) \tag{9}$$

for all $x \geq t \geq s > 0$, where $K_{n,n}(\cdot,\cdot) \equiv 1$.

Definition 3. A function $K(\cdot,\cdot) \equiv K_n(\cdot,\cdot) \in \mathcal{O}_n^{-}$, $n \geq 1$, if it is non-increasing in the second argument and there exist non-negative measurable on Ω functions $K_i(\cdot,\cdot)$, $K_{i,n}(\cdot,\cdot)$, $i = 0,1,...,n-1$, and a number $\bar{h}_n \geq 1$ such that $K_i(\cdot,\cdot) \in \mathcal{O}_i^{-}$, $i = 0,1,...,n-1$, and

$$\frac{1}{\bar{h}_n}\sum_{i=0}^{n} K_i(x,t)K_{i,n}(t,s) \leq K_n(x,s) \leq \bar{h}_n \sum_{i=0}^{n} K_i(x,t)K_{i,n}(t,s) \tag{10}$$

for all $x \geq t \geq s > 0$, where $K_{n,n}(\cdot,\cdot) \equiv 1$.

Let us present some examples. It is easy to see that the kernel $I_\alpha(x,s) = (x-s)^\alpha$, $\alpha > 0$, satisfies the Oinarov condition $\mathcal{O}$. However, if we slightly modify it to the form $K_1(x,s) = (f(x) - g(s))^\alpha$, where $f(\cdot)$ is a non-negative function and $g(\cdot)$ is a non-negative increasing function, then it does not satisfy (5). Indeed, since for all $x \geq t \geq s > 0$, we have

$$K_1(x,s) \approx (f(x) - g(t))^\alpha + (g(t) - g(s))^\alpha = K_1(x,t) + K_{0,1}(t,s),$$

where $G(t,s) = (g(t) - g(s))^\alpha$ is taken as $K_{0,1}(t,s)$, i.e., condition (10) holds for $n = 1$. Thus, $K_1(x,s) = (f(x) - g(s))^\alpha \in \mathcal{O}_1^-$, but it does not satisfy the Oinarov condition $\mathcal{O}$.

One more kernel $W(x,s) = \int\limits_s^x w(t)dt$ satisfies the Oinarov condition $\mathcal{O}$. Let us modify it by multiplying the weight $w(t)$ by the kernel $K_1(t,s) = (f(t) - g(s))^\alpha$ from the previous example, so that the new kernel takes the form $K_2(x,s) = \int\limits_s^x (f(t) - g(s))^\alpha w(t)\,dt$. Then, for all $x \geq z \geq s > 0$, we have

$$K_2(x,s) = \int\limits_s^z (f(t) - g(s))^\alpha w(t)\,dt + \int\limits_z^x (f(t) - g(s))^\alpha w(t)\,dt$$

$$\approx K_2(z,s) + \int\limits_z^x (f(t) - g(z))^\alpha w(t)\,dt + (g(z) - g(s))^\alpha \int\limits_z^x w(t)\,dt$$

$$= K_2(x,z) + W(x,z)K_{1,2}(z,s) + K_2(z,s),$$

where now $G(z,s) = (g(z) - g(s))^\alpha$ is taken as $K_{1,2}(z,s)$, i.e., the condition (10) holds for $n = 2$. Therefore, $K_2(x,s) \in \mathcal{O}_2^-$.

In general, it was proved in [19] that the kernel

$$K_w(x,s) = \int\limits_s^x K_n(t,s)w(t)\,dt, \quad x \geq t \geq s > 0,$$

belongs to the class $\mathcal{O}_{n+1}^-$ if $K_n(\cdot,\cdot)$ belongs to the class $\mathcal{O}_n^-$, $n \geq 0$, and to the class $\mathcal{O}_{n+1}^+$ if $K_n(\cdot,\cdot)$ belongs to the class $\mathcal{O}_n^+$, $n \geq 0$.

2 *Characterization of inequality* (1)

Theorem 3. Let $0 < r < \infty$ and the kernel $K(\cdot,\cdot)$ belong to the Oinarov class $\mathcal{O}_n^-$, $n \geq 1$.

(i) If $1 < q < p < \infty$, then (1) holds if and only if $\widehat{A} = \max\{A_{q<p}, \widehat{A}_{ji}\} < \infty$, where

$$\widehat{A}_{ji} = \max_{0\leq i\leq n}\max_{0\leq j\leq i}\left(\int\limits_0^\infty\left(\int\limits_0^t K_{j,i}^q(t,s)\,u(s)\left(\int\limits_0^s K_{i,n}^r(s,z)w(z)\,dz\right)^{\frac{q}{r}}ds\right)^{\frac{\mu}{q}}\right.$$

$$\left.\times\left(\int\limits_t^\infty K_j^{p'}(s,t)v^{1-p'}(s)ds\right)^{\frac{\mu}{q'}} d\left(-\int\limits_t^\infty K_j^{p'}(s,t)v^{1-p'}(s)ds\right)\right)^{\frac{1}{\mu}}$$

$$\approx \max_{0\le i\le n}\max_{0\le j\le i}\left(\int\limits_0^\infty \left(\int\limits_t^\infty K_j^{p'}(s,t)v^{1-p'}(s)ds\right)^{\frac{\mu}{p'}}\right.$$
$$\times\left(\int\limits_0^t K_{j,i}^q(t,s)\,u(s)\left(\int\limits_0^s K_{i,n}^r(s,z)w(z)\,dz\right)^{\frac{q}{r}}ds\right)^{\frac{\mu}{p}}$$
$$\left.\times d\left(\int\limits_0^t K_{j,i}^q(t,s)\,u(s)\left(\int\limits_0^s K_{i,n}^r(s,z)w(z)\,dz\right)^{\frac{q}{r}}ds\right)\right)^{\frac{1}{\mu}}.$$

Moreover, $\widehat{A}\approx C$, where C is the best constant in (1).

(ii) If $1<p\le q<\infty$, then (1) holds if and only if $\widetilde{A}=\max\{A_{p\le q},\widetilde{A}_{in}\}<\infty$, where

$$A_{p\le q}=\sup_{z>0}\left(\int\limits_z^\infty u(x)\,dx\right)^{\frac{1}{q}}J_{p,r}^-(0,z),$$

$$\widetilde{A}_{in}=\sup_{z>0}\max_{0\le i\le n}\left(\int\limits_z^\infty v^{1-p'}(x)\left(\int\limits_0^z K_i^q(x,s)u(s)\left(\int\limits_0^s K_{i,n}^r(s,t)w(t)\,dt\right)^{\frac{q}{r}}ds\right)^{\frac{p'}{q}}dx\right)^{\frac{1}{p'}}$$

$$\approx\sup_{z>0}\max_{0\le i\le n}\left(\int\limits_0^z\left(\int\limits_z^\infty K_i^{p'}(x,s)v^{1-p'}(x)dx\right)^{\frac{q}{p'}}u(s)\left(\int\limits_0^s K_{i,n}^r(s,t)w(t)\,dt\right)^{\frac{q}{r}}ds\right)^{\frac{1}{q}}.$$

Moreover, $\widetilde{A}\approx C$, where C is the best constant in (1).

Proof. Applying (10) to the second term in (5), we get

$$\left(\int\limits_0^\infty u(x)\left(\int\limits_0^x\left(\int\limits_x^\infty K(s,t)f(s)\,ds\right)^r w(t)\,dt\right)^{\frac{q}{r}}dx\right)^{\frac{1}{q}}$$

$$\approx\sum_{i=0}^n\left(\int\limits_0^\infty u(x)\left(\int\limits_0^x K_{i,n}^r(x,t)w(t)\,dt\right)^{\frac{q}{r}}\left(\int\limits_x^\infty K_i(s,x)f(s)\,ds\right)^q dx\right)^{\frac{1}{q}}.$$

Thus, the validity of inequality (1) is equivalent to the simultaneous validity of inequality (6) and the following $n+1$ inequalities:

$$\left(\int\limits_0^\infty u(x)\left(\int\limits_0^x K_{i,n}^r(x,t)w(t)\,dt\right)^{\frac{q}{r}}\left(\int\limits_x^\infty K_i(s,x)f(s)\,ds\right)^q dx\right)^{\frac{1}{q}}\le C_i'\,\|f\|_{p,v}, \qquad (11)$$

where $K_i(\cdot,\cdot)$ belongs to the class O_i^-, $i=0,1,...,n$.

In the case $1 < q < p < \infty$, it follows from [15, Theorem 3.1] that inequality (6) holds if and only if $A_{q<p} < \infty$, while from [20, Theorem 14] it follows that inequalities (11) hold if and only if $\widehat{A}_{ji} < \infty$. Additionally, by combining the best constant C_1 of inequality (6) and the best constants C_i', $i = 0, 1, ..., n$, of inequalities (11), we obtain $\widehat{A} \approx C$.

Similarly, in the case $1 < p \leq q < \infty$, it follows from [15, Theorem 3.1] that inequality (6) holds if and only if $A_{p\leq q} < \infty$, while from [16, Theorem 6] it follows that inequalities (11) hold if and only if $\widetilde{A}_{in} < \infty$. Moreover, in this case, $\widetilde{A} \approx C$. □

3 Characterization of inequality (2)

Theorem 4. Let $0 < r < \infty$ and the kernel $K(\cdot,\cdot)$ belong to the Oinarov class $\mathcal{O}_n^+$, $n \geq 1$.

(i) If $1 < q < p < \infty$, then (2) holds if and only if $\widehat{B} = \max\{B_{q<p}, \widehat{B}_{ij}\} < \infty$, where

$$\widehat{B}_{ij} = \max_{0\leq i\leq n}\max_{0\leq j\leq i}\left(\int_0^\infty\left(\int_t^\infty K_{i,j}^q(s,t)\,u(s)\left(\int_s^\infty K_{n,i}^r(z,s)w(z)\,dz\right)^{\frac{q}{r}}ds\right)^{\frac{\mu}{q}}\right.$$

$$\left.\times\left(\int_0^t K_j^{p'}(t,s)v^{1-p'}(s)ds\right)^{\frac{\mu}{q'}} d\left(\int_0^t K_j^{p'}(t,s)v^{1-p'}(s)ds\right)\right)^{\frac{1}{\mu}}$$

$$\approx \max_{0\leq i\leq n}\max_{0\leq j\leq i}\left(\int_0^\infty\left(\int_0^t K_j^{p'}(t,s)v^{1-p'}(s)ds\right)^{\frac{\mu}{p'}}\right.$$

$$\times\left(\int_t^\infty K_{i,j}^q(s,t)\,u(s)\left(\int_s^\infty K_{n,i}^r(z,s)w(z)\,dz\right)^{\frac{q}{r}}ds\right)^{\frac{\mu}{p}}$$

$$\left.\times d\left(-\int_t^\infty K_{i,j}^q(s,t)\,u(s)\left(\int_s^\infty K_{n,i}^r(z,s)w(z)\,dz\right)^{\frac{q}{r}}ds\right)\right)^{\frac{1}{\mu}}.$$

Moreover, $\widehat{B} \approx C$, where C is the best constant in (2).

(ii) If $1 < p \leq q < \infty$, then (2) holds if and only if $\widetilde{B} = \max\{B_{p\leq q}, \widetilde{B}_{ni}\} < \infty$, where

$$B_{p\leq q} = \sup_{z>0}\left(\int_0^z u(x)\,dx\right)^{\frac{1}{q}} J_{p,r}^+(z,\infty),$$

$$\widetilde{B}_{ni} = \sup_{z>0}\max_{0\leq i\leq n}\left(\int_0^z v^{1-p'}(x)\left(\int_z^\infty K_i^q(s,x)u(s)\left(\int_s^\infty K_{n,i}^r(t,s)w(t)\,dt\right)^{\frac{q}{r}}ds\right)^{\frac{p'}{q}}dx\right)^{\frac{1}{p'}}$$

$$\approx \sup_{z>0}\max_{0\leq i\leq n}\left(\int_z^\infty\left(\int_0^z K_i^{p'}(s,x)v^{1-p'}(x)dx\right)^{\frac{q}{p'}}u(s)\left(\int_s^\infty K_{n,i}^r(t,s)w(t)\,dt\right)^{\frac{q}{r}}ds\right)^{\frac{1}{q}}.$$

Moreover, $\widetilde{B} \approx C$, where C is the best constant in (2).

Proof. Splitting the interior integral on the left-hand side of inequality (2), we get

$$\left(\int_0^\infty u(x)\left(\int_x^\infty\left(\int_0^t K(t,s)f(s)\,ds\right)^r w(t)\,dt\right)^{\frac{q}{r}} dx\right)^{\frac{1}{q}}$$

$$\approx \left(\int_0^\infty u(x)\left(\int_x^\infty\left(\int_0^x K(t,s)f(s)\,ds\right)^r w(t)\,dt\right)^{\frac{q}{r}} dx\right)^{\frac{1}{q}}$$

$$+ \left(\int_0^\infty u(x)\left(\int_x^\infty\left(\int_x^t K(t,s)f(s)\,ds\right)^r w(t)\,dt\right)^{\frac{q}{r}} dx\right)^{\frac{1}{q}}. \tag{12}$$

Applying (9) to the first term in (12), we deduce

$$\left(\int_0^\infty u(x)\left(\int_x^\infty\left(\int_0^x K(t,s)f(s)\,ds\right)^r w(t)\,dt\right)^{\frac{q}{r}} dx\right)^{\frac{1}{q}}$$

$$\approx \sum_{i=0}^n \left(\int_0^\infty u(x)\left(\int_x^\infty K^r_{n,i}(t,x)w(t)\,dt\right)^{\frac{q}{r}}\left(\int_0^x K_i(x,s)f(s)\,ds\right)^q dx\right)^{\frac{1}{q}}.$$

Thus, the validity of inequality (2) is equivalent to the simultaneous validity of inequality

$$\left(\int_0^\infty u(x)\left(\int_x^\infty\left(\int_x^t K(t,s)f(s)\,ds\right)^r w(t)\,dt\right)^{\frac{q}{r}} dx\right)^{\frac{1}{q}} \le C''_1\,\|f\|_{p,v} \tag{13}$$

and the following $n+1$ inequalities:

$$\sum_{i=0}^n \left(\int_0^\infty u(x)\left(\int_x^\infty K^r_{n,i}(t,x)w(t)\,dt\right)^{\frac{q}{r}}\left(\int_0^x K_i(x,s)f(s)\,ds\right)^q dx\right)^{\frac{1}{q}} \le C''_i\,\|f\|_{p,v}, \tag{14}$$

where $K_i(\cdot,\cdot)$ belongs to the class O_i^+, $i = 0, 1, ..., n$.

In the case $1 < q < p < \infty$, it follows from [15, Theorem 3.3] that inequality (13) holds if and only if $B_{q<p} < \infty$, while from [20, Theorem 11] it follows that inequalities (14) hold if and only if $\widehat{B}_{ij} < \infty$. Additionally, by combining the best constant C''_1 of inequality (13) and the best constants C''_i, $i = 0, 1, ..., n$, of inequalities (14), we have $\widehat{B} \approx C$.

Similarly, in the case $1 < p \le q < \infty$, it follows from [15, Theorem 3.3] that inequality (13) holds if and only if $B_{p\le q} < \infty$, while from [16, Theorem 5] we have that inequalities (14) hold if and only if $\widetilde{B}_{ni} < \infty$. Moreover, in this case, $\widetilde{B} \approx C$.

□

Remark 1. Let us again consider inequality (1). Suppose that the kernel $K(\cdot,\cdot)$ belongs to the Oinarov class $\mathcal{O}_n^+$, $n \geq 1$, but not to the Oinarov class $\mathcal{O}_n^-$, $n \geq 1$, as stated in the condition of Theorem 3. Then, applying (9) to the second term in (5), we obtain the following inequality:

$$\left(\int_0^\infty u(x) \left(\int_0^x K_i^r(x,t) w(t)\,dt \right)^{\frac{q}{r}} \left(\int_x^\infty K_{n,i}(s,x) f(s)\,ds \right)^q dx \right)^{\frac{1}{q}} \leq C_i \, \|f\|_{p,v}$$

instead of (11). The characterization of Hardy-type inequality (3) when the operator T is given by $K^- f(x) = \int_x^\infty K(s,x) f(s)\,ds$, without any restriction on its kernel $K(\cdot,\cdot)$, remains an open problem. Since, by definition of the class $\mathcal{O}_n^+$, $n \geq 1$, there are no restrictions on the non-negative measurable functions $K_{n,i}(\cdot,\cdot)$, we cannot establish the validity of inequality (1) when the kernel $K(\cdot,\cdot)$ belongs to the Oinarov class $\mathcal{O}_n^+$, $n \geq 1$; instead, we can only do so when it belongs to the class $\mathcal{O}_n^-$, $n \geq 1$. A similar situation arises for inequality (2): we can characterize it only if the kernel $K(\cdot,\cdot)$ belongs to the Oinarov class $\mathcal{O}_n^+$, $n \geq 1$.

Conclusion

The necessary and sufficient conditions for the validity of the three-weight Hardy inequalities with iterated operators and generalized kernels belonging to the Oinarov classes were obtained. The obtained results can be used in harmonic analysis, in the theory of differential and difference equations, as well as in other areas of mathematics. Moreover, the characteristics of weighted Hardy inequalities with iterated operators can now be used to study bilinear weighted Hardy-type inequalities.

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Author Contributions

All authors contributed equally to this work.

Conflict of Interest

The authors declare no conflict of interest.

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*Author Information**

Aigerim Arystankyzy Kalybay — PhD, Candidate of Physical and Mathematical Sciences, Professor, KIMEP University, Almaty, Kazakhstan; e-mail: *kalybay@kimep.kz*; https://orcid.org/0000-0003-3011-1783

Ainur Maralkyzy Temirkhanova (*corresponding author*) — PhD, Associate Professor, L.N. Gumilyov Eurasian National University, Astana, Kazakhstan; e-mail: *ainura-t@yandex.kz*; https://orcid.org/0000-0001-5610-3314

*Authors' names are presented in the following order: first name, middle name (if any), last name.