

Existence, uniqueness, and Ulam–Hyers stability of periodic solutions for a discrete Nicholson’s blowflies equation with proportional delay

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This paper investigates the existence, uniqueness, and stability of periodic solutions for a discrete Nicholson’s blowflies equation with proportional delay. Proportional delay, in which the delay is defined as a fraction of the current time, naturally arises in many discrete dynamical systems, including neural networks and biological population models. The original problem is transformed into an equivalent fixed-point equation by introducing a suitable operator on a Banach space of periodic functions. Sufficient conditions for the existence of N -periodic solutions are established using the Leray–Schauder nonlinear alternative, while uniqueness is obtained through Banach’s fixed-point theorem. In addition, the Ulam–Hyers stability of the periodic solution is investigated, showing that small perturbations in the governing equation lead to only small deviations from the exact solution. To validate the theoretical findings, numerical simulations are carried out using Python. Several representative examples are considered by varying the proportional-delay parameter, delay structure, birth coefficient, and nonlinear coefficient. The numerical results confirm the analytical predictions and clearly illustrate the influence of these parameters on the existence, stability, and qualitative behavior of the periodic solutions, thereby demonstrating the applicability and effectiveness of the proposed theoretical approach.

Keywords: difference equation, Nicholson’s blowflies model, proportional delay, periodic solution, fixed point theorem, Ulam–Hyers stability, existence of solution, uniqueness of solution.

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Introduction

In this paper, we explore the existence, uniqueness, and Ulam–Hyers stability of solutions for a Nicholson’s blowflies difference equation with a proportional delay

$$\psi(k+1) - \psi(k) = -a(k)\psi(k) + b(k)\psi([q\tau(k)])e^{-c(k)\psi([q\tau(k)])}, \quad (1)$$

where $\psi(k)$ is the size of population at the discrete time k , $a(k)$ is the per capita daily adult death rate, $b(k)$ is the maximum per capita daily egg production rate, $[q\tau(k)]$ represents the delay term in the equation, $0 < q < 1$, the symbol $[\cdot]$ denotes the integer part. We assume that

- (i) ψ is N -periodic, $\psi(k) = \psi(k+N)$ for $k \in \mathbb{Z}$, $N \in \mathbb{N}$, $N \geq 1$.
- (ii) a, b, c are nonnegative, N -periodic functions and $0 < a(k) < 1$.
- (iii) $\tau: \mathbb{Z} \rightarrow \mathbb{R}_+$ is N -periodic function with $\tau(k) \leq k$.

Our objective is to analyze the delay difference equation (1) by applying fixed point theorems and conducting a numerical study for some specific cases. This model is a well-known example in population dynamics, particularly for studying insect populations. It was originally introduced by Nicholson in the 1950s to describe the population changes of the Australian sheep blowfly. In [1], Gurney et al. proposed a formal mathematical version of this model, known as the Nicholson blowflies equation, which was based on Nicholson’s experimental data, that is expressed as

$$\psi'(t) = -a\psi(t) + b\psi(t-\tau)e^{-c\psi(t-\tau)},$$

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where $\psi(t)$ is the population size at time t , $a > 0$ is the death rate of the adult flies, $b > 0$ is the maximum per capita daily egg production rate, τ is the time delay representing the developmental period from egg to adult, $1/c$ is the size at which the population reproduces at its maximum rate.

One interesting application of the Nicholson blowflies model is in modeling hematopoiesis, the process of forming blood cells (red blood cells, white blood cells, and platelets) in the bone marrow. In this context, Lasota [2] proposed the following discrete model for red blood cell production based on experimental results:

$$\psi(n+1) - \psi(n) = -a\psi(n) + (c\psi(n))^m e^{-c\psi(n)},$$

where $\psi(n)$ denotes the number of cells at time n , and $a\psi(n)$ the number of cells destroyed in the time interval $[n, n+1]$, $a \in (0, 1)$, c, m are positive constants. The term $(c\psi(n))^m e^{-c\psi(n)}$ represents the number of cells which are produced in the bone marrow during the same period.

This adaptation of the Nicholson blowflies model helps to understand the dynamics of red blood cell production and can provide a better understanding of the regulation and behavior of blood cell populations in the bone marrow.

Recently, the results of Lasota's paper were revisited in the paper [3], where the authors studied this model and applied it to population dynamics with specific seasons, adult survival rates, and density-dependent effects.

Nicholson's blowflies difference equations, as well as differential equations, have seen significant progress in the last decades. In the paper [4], the authors studied the following discrete Nicholson's blowflies type system:

$$\begin{aligned}\psi_1(k+1) &= a_{11}(k)\psi_1(k) + a_{12}(k)\psi_2(k) \\ &\quad + b(k)(\psi_1(k-\tau) + \psi_2(k-\tau))^m \\ &\quad e^{-c(k)(\psi_1(k-\tau) + \psi_2(k-\tau))}, \\ \psi_2(k+1) &= a_{21}(k)\psi_1(k) + a_{22}(k)\psi_2(k) \\ &\quad + b(k)(\psi_1(k-\tau) + \psi_2(k-\tau))^m \\ &\quad e^{-c(k)(\psi_1(k-\tau) + \psi_2(k-\tau))},\end{aligned}$$

where $m > 1$ is a constant, τ is a nonnegative integer, and a_{ij} , $i, j = 1, 2$, b and c are all N -periodic functions from $\mathbb{Z}$ to $\mathbb{R}$. They proved the existence of three nonnegative periodic solutions by using the Leggett–Williams fixed point theorem.

In the paper [5], the authors analyzed the global exponential convergence of Nicholson's blowflies model with proportional delays,

$$\psi'(t) = -a\psi(t) + \sum_{i=1}^m b_i(t)\psi(\tau_i t) e^{-c_i(t)\psi(\tau_i t)}, \quad t \geq 0.$$

Here $(\tau_i t)$ is the proportional delay term with $0 < \tau_i < 1$.

Proportional delays arise naturally in many mathematical and engineering models, particularly in neural networks and biological systems, where the delay depends on a proportion of time rather than a fixed constant. Since such delays are time-dependent and unbounded, their analysis is considerably more challenging. Foundational results on difference equations and delayed dynamical systems can be found in [6, 7]. Stability issues for delayed systems have been investigated in [8–10]. Related exponential stability results for neural networks with proportional delays were obtained in [11]. Nicholson-type models with proportional delays and convergence properties were studied in [12, 13]. The existence of periodic and almost periodic solutions for Nicholson-type systems was examined in [14–16]. Related

results for Nicholson-type models with nonlinear mortality terms and fractional dynamics can be found in [17, 18].

Most existing studies on Nicholson-type equations have focused on periodicity, stability, and extinction properties. Almost periodic solutions and related dynamics were investigated in [19–21]. Further results on periodic and almost periodic solutions were obtained in [22, 23]. Stability and extinction properties were analyzed in [24, 25]. However, the Ulam–Hyers stability of discrete Nicholson equations with proportional delays and integer-part arguments has not yet been studied.

In this paper, we investigate a discrete periodic Nicholson’s blowflies model with proportional delay of the form $\lfloor q\tau(k) \rfloor$, $0 < q < 1$.

The combination of the proportional parameter, the time-dependent delay function, and the integer-part operator produces a nonsmooth and piecewise constant delay structure, which increases the difficulty of the analysis.

By using the Leray–Schauder nonlinear alternative and Banach’s contraction principle, we establish the existence, uniqueness, and Ulam–Hyers stability of N -periodic solutions. Numerical simulations are also presented to illustrate the influence of the proportional-delay parameter and the delay functions on the dynamics of the system.

1 Analysis of the problem

Set $\theta(k) = 1 - a(k)$, and

$$\Lambda(k) = \prod_{s=0}^{k-1} \theta(s) = \theta(0) \theta(1) \dots \theta(k-1),$$

and

$$\Lambda^{-1}(k) = \prod_{s=0}^{k-1} (\theta(s))^{-1},$$

then equation (1) can be written as

$$\psi(k+1) = \theta(k) \psi(k) + b(k) \psi(\lfloor q\tau(k) \rfloor) e^{-c(k)\psi(\lfloor q\tau(k) \rfloor)}. \quad (2)$$

Since $0 < a(k) < 1$, then $0 < \theta(k) < 1$ and hence $0 < \Lambda(k) < 1$.

Lemma 1. A N -periodic function ψ is a solution of the difference equation (1) if and only if it satisfies

$$\psi(k) = \sum_{s=k}^{k+N-1} \mathfrak{G}(k, s) b(s) \psi(\lfloor q\tau(s) \rfloor) e^{-c(s)\psi(\lfloor q\tau(s) \rfloor)}, \quad k \in \mathbb{Z},$$

where

$$\mathfrak{G}(k, s) = (\Lambda^{-1}(N) - 1)^{-1} \Lambda(k) \Lambda^{-1}(s+1).$$

Proof. Taking into account equality (2) and that $0 < \Lambda(k) < 1$, then by computations it yields

$$\begin{aligned} & \Lambda^{-1}(s+1) \psi(s+1) - \Lambda^{-1}(s) \psi(s) \\ &= \Lambda^{-1}(s+1) \left[\theta(s) \psi(s) + b(s) \psi(\lfloor q\tau(s) \rfloor) e^{-c(s)\psi(\lfloor q\tau(s) \rfloor)} \right] - \Lambda^{-1}(s) \psi(s) \\ &= \Lambda^{-1}(s+1) b(s) \psi(\lfloor q\tau(s) \rfloor) e^{-c(s)\psi(\lfloor q\tau(s) \rfloor)}. \end{aligned}$$

On the other hand, we have for $l \geq k$:

$$\begin{aligned} & \Lambda^{-1}(l+1) \psi(l+1) - \Lambda^{-1}(k) \psi(k) \\ &= \sum_{s=k}^l [\Lambda^{-1}(s+1) \psi(s+1) - \Lambda^{-1}(s) \psi(s)] \\ &= \sum_{s=k}^l \Lambda^{-1}(s+1) b(s) \psi(\lfloor q\tau(s) \rfloor) e^{-c(s)\psi(\lfloor q\tau(s) \rfloor)}, \end{aligned}$$

hence

$$\begin{aligned}\Lambda^{-1}(l+1)\psi(l+1) &= \Lambda^{-1}(k)\psi(k) \\ &+ \sum_{s=k}^l \Lambda^{-1}(s+1)b(s)\psi([q\tau(s)])e^{-c(s)\psi([q\tau(s)])},\end{aligned}$$

for $l = k + N - 1$ yields

$$\begin{aligned}\Lambda^{-1}(k+N)\psi(k+N) &= \Lambda^{-1}(k)\psi(k) \\ &+ \sum_{s=k}^{k+N-1} \Lambda^{-1}(s+1)b(s)\psi([q\tau(s)])e^{-c(s)\psi([q\tau(s)])}.\end{aligned}$$

Taking into account that a and ψ are periodic and

$$\Lambda^{-1}(k+N) = \Lambda^{-1}(k)\Lambda^{-1}(N),$$

then

$$\psi(k) = (\Lambda^{-1}(N) - 1)^{-1} \Lambda(k) \sum_{s=k}^{k+N-1} \Lambda^{-1}(s+1)b(s)\psi([q\tau(s)])e^{-c(s)\psi([q\tau(s)])},$$

thus

$$\psi(k) = \sum_{s=k}^{k+N-1} \mathfrak{G}(k, s)b(s)\psi([q\tau(s)])e^{-c(s)\psi([q\tau(s)])}.$$

The properties of the function $\mathfrak{G}$ is given in the following lemma. □

Lemma 2. $\mathfrak{G}(k, s)$ is nonnegative and for $0 \leq k \leq N - 1$, $k \leq s \leq k + N - 1$, we have

$$A_0^N \leq \mathfrak{G}(k, s) \leq \frac{A_1^N}{A_0^N(1 - A_1^N)},$$

where $A_1 = \max_{0 \leq s \leq N-1} \theta(s)$ and $A_0 = \min_{0 \leq s \leq N-1} \theta(s)$.

Proof. Since $0 < \theta(k) < 1$, $k \in \mathbb{Z}$, then $0 < A_0 \leq \theta(k) \leq A_1 < 1$. From the definition of $\Lambda(N)$, we obtain

$$0 < \Lambda(N) = \prod_{s=0}^{N-1} \theta(s) < 1, \quad A_0^N \leq \Lambda(N) \leq A_1^N.$$

Moreover, we have

$$\mathfrak{G}(k, s) = (1 - \Lambda(N))^{-1} \Lambda(N) \Lambda(k) \Lambda^{-1}(s+1).$$

Since

$$(\Lambda^{-1}(N) - 1)^{-1} = \frac{\Lambda(N)}{1 - \Lambda(N)}$$

we may write

$$\mathfrak{G}(k, s) = \frac{\Lambda(N)}{1 - \Lambda(N)} \prod_{r=k}^s \theta^{-1}(r).$$

Because $\theta(r) \geq A_0$, it follows that

$$\theta^{-1}(r) \leq A_0^{-1},$$

and therefore

$$\prod_{r=k}^s \theta^{-1}(r) \leq A_0^{-(s-k+1)}.$$

Using the inequalities

$$\Lambda(N) \leq A_1^N, \quad 1 - \Lambda(N) \geq 1 - A_1^N,$$

we deduce that

$$\mathfrak{G}(k, s) \leq \frac{A_1^N}{1 - A_1^N} A_0^{-(s-k+1)}.$$

Since $(s - k + 1) \leq N$, we obtain

$$\mathfrak{G}(k, s) \leq \frac{A_1^N}{A_0^N(1 - A_1^N)}.$$

For the lower bound, since $0 < \theta(r) < 1$, we have $\theta^{-1}(r) \geq 1$. Hence,

$$\prod_{r=k}^s \theta^{-1}(r) \geq 1.$$

Using also $\frac{\Lambda(N)}{1-\Lambda(N)} \geq \Lambda(N)$, we get

$$\begin{aligned} \mathfrak{G}(k, s) &= (1 - \Lambda(N))^{-1} \Lambda(N) \Lambda(k) \Lambda^{-1}(s+1) \geq \Lambda(N) \Lambda(k) \Lambda^{-1}(s+1) \\ &= \Lambda(N) \prod_{r=k}^s \theta^{-1}(r) \geq \Lambda(N) \geq A_0^N. \end{aligned}$$

Therefore,

$$A_0^N \leq \mathfrak{G}(k, s) \leq \frac{A_1^N}{A_0^N(1 - A_1^N)}.$$

The proof is complete. □

2 Existence and uniqueness results

We denote by $\mathbb{X}$ the Banach space of all N -periodic functions $\psi: \mathbb{Z} \rightarrow \mathbb{R}$ according to the norm $\|\psi\| = \max_{0 \leq k \leq N-1} |\psi(k)|$. Define the operator $\mathbb{A}$ on $\mathbb{X}$ by

$$\mathbb{A}\psi(k) = \sum_{s=k}^{k+N-1} \mathfrak{G}(k, s) b(s) \psi([q\tau(s)]) e^{-c(s)\psi([q\tau(s)])}.$$

If ψ is a fixed point for the operator $\mathbb{A}$, then it is a solution for the difference equation (1). We have the first existence result:

Theorem 1. Set $b_1 = \max_{0 \leq s \leq N-1} b(s)$, $c_0 = \min_{0 \leq s \leq N-1} c(s)$ and assume that

$$\frac{Nb_1 A_1^N}{A_0^N(1 - A_1^N)} < ec_0, \tag{3}$$

then the difference equation (1) has at least one solution in $\mathbb{X}$.

The proof of Theorem 1 is based on the following fixed point theorem.

Lemma 3 (Leray–Schauder nonlinear alternative). Let E be a Banach space and Ξ a bounded open subset of E , $0 \in \Xi$ and let $\mathcal{A}: \Xi \rightarrow E$ be a completely continuous operator. Then, either there exists $\psi \in \partial\Xi$, $\xi > 1$ such that $\mathcal{A}(\psi) = \xi\psi$, or there exists a fixed point $\psi^* \in \Xi$.

Proof of Theorem 1. Set $\Xi = \{\psi \in \mathbb{X} : \|\psi\| < r\}$, as an open bounded subset in $\mathbb{X}$. In view of the continuity of the functions a, b, c , the operator $\mathbb{A}$ is continuous. Moreover, the space of N -periodic sequences is finite dimensional, hence bounded subsets are relatively compact. Now, let $\psi \in \partial\Xi$, such that $A(\psi) = \xi\psi$, $\xi > 1$; hence, $\psi = \xi^{-1}\mathbb{A}\psi$, $0 < \xi^{-1} < 1$. Using the fact that the function $f(x) = xe^{-cx}$, $x \geq 0$, $c > 0$ satisfies $f(x) \leq \frac{1}{ce}$, and taking into account the estimate (3), we obtain

$$\begin{aligned} |\psi(k)| &= \xi^{-1}|\mathbb{A}\psi(k)| \leq |\mathbb{A}\psi(k)| = \sum_{s=k}^{k+N-1} \mathfrak{G}(k, s)b(s)\psi(\lfloor q\tau(s) \rfloor) e^{-c(s)\psi(\lfloor q\tau(s) \rfloor)} \\ &\leq \frac{Nb_1A_1^N}{A_0^N(1-A_1^N)c_0e} \|\psi\| = \frac{Nb_1A_1^N}{A_0^N(1-A_1^N)c_0e} r < r. \end{aligned}$$

Hence, $\|\psi\| < r$, this contradicts the fact that $\psi \in \partial\Xi$.

By Leray–Schauder nonlinear alternative, we conclude that $\mathbb{A}$ has a fixed point in $\bar{\Xi}$ and then the difference equation (1) has at least one solution in $\mathbb{X}$.

The following theorem provides a uniqueness result.

Theorem 2. Assume that

$$\frac{Nb_1A_1^N}{A_0^N(1-A_1^N)} < 1 \quad (4)$$

holds, then the difference equation (1) has a unique nonnegative solution in $\mathbb{X}_+ = \{\psi \in \mathbb{X}, \psi(k) \geq 0, \forall k \in \mathbb{Z}\}$, where $b_1 = \max_{0 \leq s \leq N-1} b(s)$.

Proof. First, let us prove that $\mathbb{A}(\mathbb{X}_+) \subset \mathbb{X}_+$. Let $\psi \in \mathbb{X}_+$. Then $\psi(k) \geq 0, \forall k \in \mathbb{Z}$. Since $b(s) \geq 0, c(s) \geq 0$, and from Lemma 2, $\mathfrak{G}(k, s) \geq 0$. We have $\psi(\lfloor q\tau(s) \rfloor) \geq 0$ and consequently $e^{-c(s)\psi(\lfloor q\tau(s) \rfloor)} > 0$. Therefore,

$$\mathfrak{G}(k, s)b(s)\psi(\lfloor q\tau(s) \rfloor) e^{-c(s)\psi(\lfloor q\tau(s) \rfloor)} \geq 0$$

for every $s \in \{k, k+1, \dots, k+N-1\}$. Summing these nonnegative terms, we obtain $(\mathbb{A}\psi)(k) \geq 0, \forall k \in \mathbb{Z}$. Hence, $\mathbb{A}\psi \in \mathbb{X}_+$, which proves that $\mathbb{A}(\mathbb{X}_+) \subset \mathbb{X}_+$.

Next, since the derivative of the function $f(x) = xe^{-cx}$, $x \geq 0, c > 0$, given by $f'(x) = e^{-cx}(1-cx)$ satisfies $|f'(x)| \leq 1$, then by the mean value theorem it yields $|xe^{-cx} - ye^{-cy}| \leq |x - y|, x, y \geq 0$.

Now, we prove that $\mathbb{A}$ is a contraction. Let $\psi, v \in \mathbb{X}$. Applying (4), the previous estimate and taking Lemma 2 into account, we get

$$\begin{aligned} |\mathbb{A}\psi(k) - \mathbb{A}v(k)| &\leq \sum_{s=k}^{k+N-1} \mathfrak{G}(k, s)b(s) \\ &\quad \times \left| \psi(\lfloor q\tau(s) \rfloor) e^{-c(s)\psi(\lfloor q\tau(s) \rfloor)} - v(\lfloor q\tau(s) \rfloor) e^{-c(s)v(\lfloor q\tau(s) \rfloor)} \right| \\ &\leq \frac{NA_1^Nb_1}{A_0^N(1-A_1^N)} \|\psi - v\|, \end{aligned}$$

thus

$$\|\mathbb{A}\psi - \mathbb{A}v\| \leq \frac{Nb_1A_1^N}{A_0^N(1-A_1^N)} \|\psi - v\|.$$

Thanks to inequality (4) we conclude that $\mathbb{A}$ is a contraction on $\mathbb{X}$. By Banach's fixed-point theorem, the operator $\mathbb{A}$ possesses a unique fixed point in $\mathbb{X}$. Consequently, the difference equation (1) admits a unique nonnegative N -periodic solution. $\square$

3 Ulam–Hyers stability

We recall that an equation is considered Ulam–Hyers stable if, for every approximate solution of the equation, there exists an exact solution that is close to it. More precisely:

Definition 1. The difference equation (1) is Ulam–Hyers stable if there exists a real number $\rho > 0$ such that for each $\epsilon > 0$ and for each solution φ of the inequality

$$|\varphi(k) - \sum_{s=k}^{k+N-1} \mathfrak{G}(k, s) b(s) \varphi([q\tau(s)]) e^{-c(s)\varphi([q\tau(s)])}| \leq \epsilon, \quad (5)$$

there exists a solution ψ of the difference equation (1) with

$$|\varphi(k) - \psi(k)| \leq \rho\epsilon, \quad k \in \mathbb{Z}.$$

Theorem 3. Assume that condition (4) holds, then the equation (1) is Ulam–Hyers stable.

Proof. Let ψ be a solution of the difference equation (1) and φ satisfying inequality (5), then

$$\begin{aligned} |\varphi(k) - \psi(k)| &= \left| \varphi(k) - \sum_{s=k}^{k+N-1} \mathfrak{G}(k, s) b(s) \psi([q\tau(s)]) e^{-c(s)\psi([q\tau(s)])} \right| \\ &\leq \left| \varphi(k) - \sum_{s=k}^{k+N-1} \mathfrak{G}(k, s) b(s) \varphi([q\tau(s)]) e^{-c(s)\varphi([q\tau(s)])} \right| \\ &\quad + \sum_{s=k}^{k+N-1} \mathfrak{G}(k, s) b(s) \left| \varphi([q\tau(s)]) e^{-c(s)\varphi([q\tau(s)])} - \psi([q\tau(s)]) e^{-c(s)\psi([q\tau(s)])} \right| \\ &\leq \epsilon + \frac{Nb_1 A_1^N}{A_0^N (1 - A_1^N)} \|\psi - \varphi\|. \end{aligned}$$

Set $\alpha = \frac{Nb_1 A_1^N}{A_0^N (1 - A_1^N)}$, hence

$$\|\psi - \varphi\| \leq \epsilon + \alpha \|\psi - \varphi\|,$$

that implies

$$\|\psi - \varphi\| \leq \frac{\epsilon}{1 - \alpha}.$$

Thus equation (1) is Ulam–Hyers stable, where $\rho = 1/(1 - \alpha)$. $\square$

4 Numerical simulation

To illustrate the theoretical results, we consider equation (1) with period $N = 4$. The coefficients are chosen as periodic functions satisfying assumptions (i)–(iii), namely

$$\begin{aligned} a(k) &= \frac{2}{3} - \frac{1}{6}(k \bmod 4), \quad b(k) = 0.001(k \bmod 4), \\ c(k) &= 1 + (k \bmod 4), \quad \tau(k) = \min(k, k \bmod 4). \end{aligned}$$

These choices ensure that $a(k) \in (0, 1)$, $b(k) \geq 0$, and $\tau(k) \leq k$, so that all assumptions of the theoretical analysis are satisfied. Moreover,

$$\theta(k) = \frac{1}{3} + \frac{1}{6}(k \bmod 4), \quad A_1 = \frac{5}{6}, \quad A_0 = \frac{1}{3}, \quad b_1 = 0.0015, \quad b_0 = 0,$$

and

$$\frac{Nb_1 A_1^N}{A_0^{N+1}(1 - A_1^N)} = 0.45268 < 1.$$

Hence, equation (1) admits a unique positive periodic solution in the space $\mathbb{X}$. In addition, the solution is Ulam–Hyers stable with $\rho = 1.8271$.

The periodic solution is computed iteratively through the operator scheme $\psi_{n+1} = \mathbb{A}\psi_n$, starting from a positive periodic initial function. To analyze the influence of the proportional delay parameter, we consider the values $q = 0.2$, $q = 0.4$, and $q = 0.8$, together with the two delay functions

$$\tau_1(k) = \begin{cases} 0, & k \equiv 0 \pmod{4}, \\ 0, & k \equiv 1 \pmod{4}, \\ 1, & k \equiv 2 \pmod{4}, \\ 2, & k \equiv 3 \pmod{4}, \end{cases} \quad \tau_2(k) = \min(k, k \pmod{4}).$$

For $q = 0.2$, the solutions associated with $\tau_1(k)$ and $\tau_2(k)$ are almost indistinguishable, as shown in Figure 1. Both solutions exhibit the same positive 4-periodic oscillatory behavior with nearly identical amplitudes and phases. This indicates that, for small proportional delays, the influence of the delay structure on the dynamics is weak.

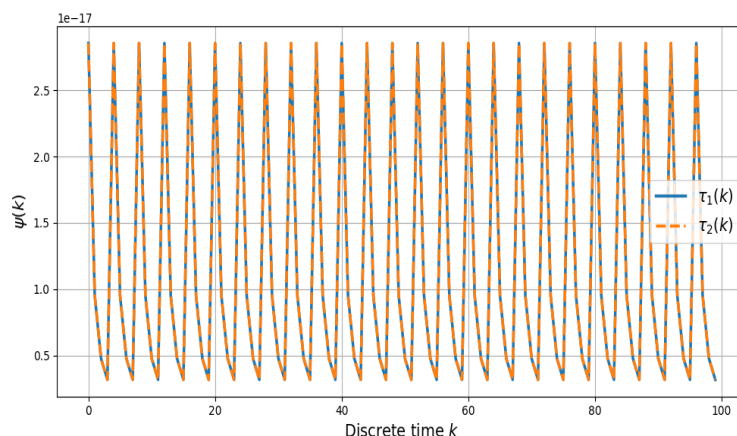


Figure 1. Positive 4-periodic solutions for $q = 0.2$

For $q = 0.4$, the difference between the two delay structures becomes more visible (Fig. 2). The solution corresponding to $\tau_1(k)$ attains larger oscillatory amplitudes, whereas the solution associated with $\tau_2(k)$ remains comparatively smaller. Although both solutions preserve their periodic character, increasing q enhances the contribution of the delayed states and therefore amplifies the sensitivity of the system to the delay structure.

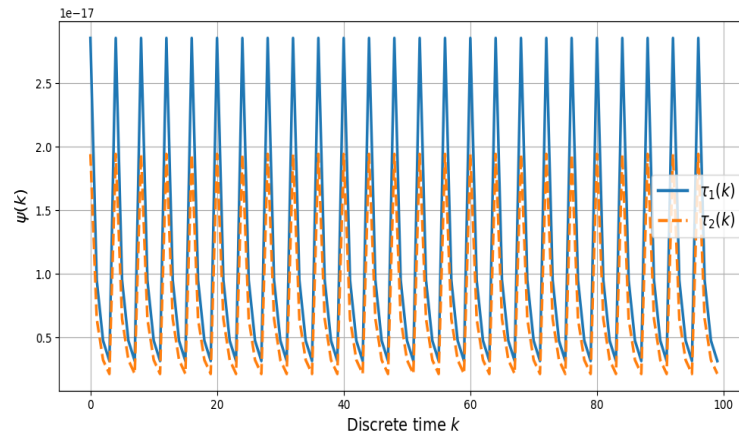


Figure 2. Positive 4-periodic solutions for $q = 0.4$

For $q = 0.8$, the effect of the delay becomes highly pronounced (Fig. 3). The solution associated with $\tau_2(k)$ still exhibits significant oscillations, while the solution generated by $\tau_1(k)$ remains close to zero. This demonstrates that large proportional delays substantially modify the amplitude of the periodic response. Nevertheless, all obtained solutions remain positive, bounded, and 4-periodic, confirming the theoretical predictions.

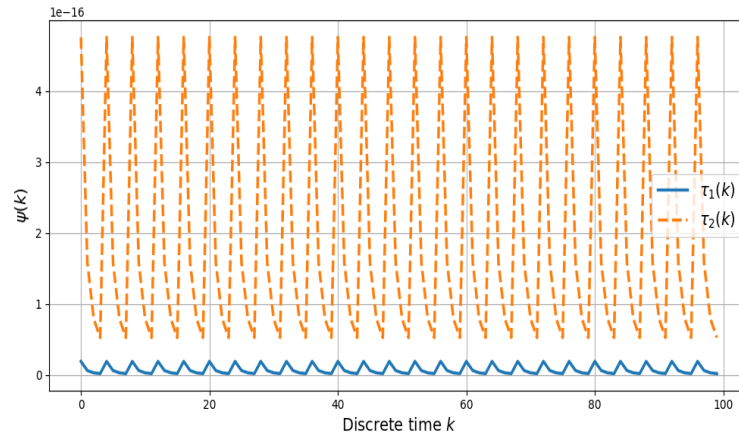


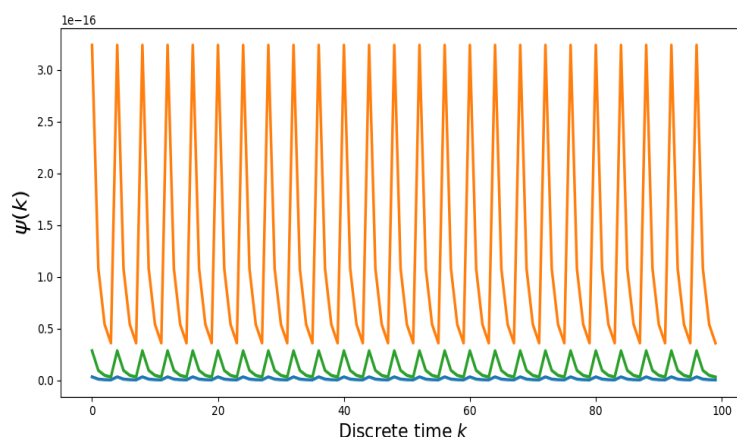
Figure 3. Positive 4-periodic solutions for $q = 0.8$

Next, we investigate the influence of the periodic birth coefficient. Let us choose

$$b(k) = \mu k \pmod{4},$$

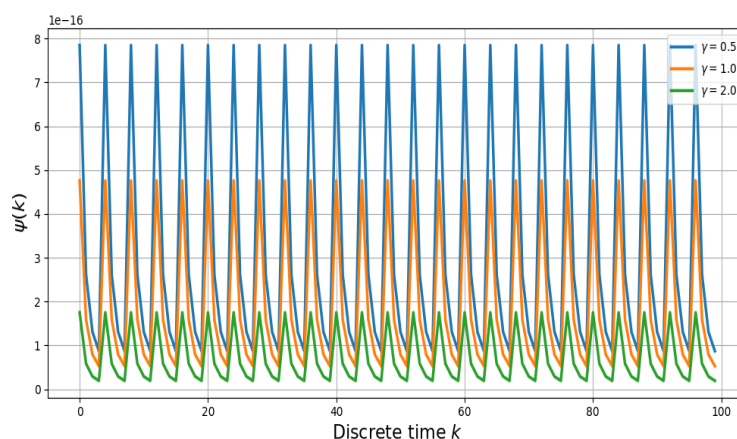
for $\mu = 0.0002$, 0.0005 , and 0.001 , while fixing $q = 0.6$. The results are displayed in Figure 4.

The parameter μ scales the delayed nonlinear birth term $b(k)\psi([q\tau(k)])e^{-c(k)\psi([q\tau(k)])}$, and therefore controls the strength of delayed feedback. For small values of μ , the oscillations remain close to zero, indicating weak reproduction effects. For the intermediate value $\mu = 0.0005$, the oscillatory amplitude becomes maximal, corresponding to the most effective amplification of the dynamics by delayed feedback. When μ increases further to 0.001 , the oscillations decrease again because the exponential damping term dominates the nonlinear response. Thus, the influence of μ is non-monotone due to the saturating behavior of the Nicholson-type nonlinearity xe^{-cx} .

Figure 4. Influence of the birth coefficient $b(k)$

Finally, Figure 5 illustrates the influence of the nonlinear coefficient

$$c(k) = \gamma + (k \bmod 4), \quad \gamma = 0.5, 1, 2.$$

Figure 5. Influence of the nonlinear coefficient $c(k)$

The numerical simulations show that the periodic structure of the solution is preserved for all tested values of γ , while the oscillatory amplitude decreases significantly as γ increases. In particular, $\gamma = 0.5$ produces the largest oscillations, whereas $\gamma = 2$ yields the smallest amplitudes. This behavior is explained by the exponential term $e^{-c(k)\psi([q\tau(k)])}$, which induces stronger damping for larger values of $c(k)$. Consequently, the coefficient $c(k)$ acts as a stabilizing factor that suppresses the magnitude of oscillations without destroying the periodic behavior of the solution.

Conclusion

In this paper, we investigated the existence, uniqueness, and Ulam–Hyers stability of positive periodic solutions for a Nicholson’s blowflies difference equation with proportional delay. The model involves a proportional-delay term combined with the integer-part operator, providing a more flexible framework for describing memory effects in discrete population dynamics. By reformulating the problem as an equivalent fixed-point equation, we established existence and uniqueness result via fixed point theorems. Furthermore, Ulam–Hyers stability of the periodic solution was obtained. Numerical simulations were presented to illustrate the theoretical results and to examine the effects of the

proportional-delay parameter, the delay structure, and the nonlinear coefficients on the qualitative behavior of the solutions. The simulations show that increasing the proportional delay parameter enhances the sensitivity of the dynamics to the delay structure, while larger nonlinear coefficients reduce the oscillation amplitudes due to stronger exponential damping. Future work may focus on asymptotic stability, bifurcation phenomena, and comparisons with classical fixed-delay Nicholson models in order to better understand the role of proportional delays in discrete dynamical systems.

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Author contributions

All authors contributed equally to this work.

Conflict of interest

The authors declare no conflict of interest.

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