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Marcinkiewicz-type interpolation theorem for discrete net spaces

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In this paper, we investigate the interpolation properties of discrete net spaces $n_{p,q}(M)$ and examine their applications to the analysis of linear operators acting on these spaces. These spaces are characterized by the property that, for monotonically non-increasing sequences, the norm in $n_{p,q}(M)$ coincides with the norm of the discrete Lorentz space $l_{p,q}(M)$. At the same time, unlike Lorentz spaces, these spaces $n_{p,q}(M)$ may contain sequences that do not tend to zero, making them suitable for the study of more general function spaces and operator classes. The main result of this paper is an analogue of Marcinkiewicz-type interpolation theorem for discrete net spaces $n_{p,q}(M)$, which offers a powerful tool to study the boundedness of linear operators within this framework. By extending classical interpolation techniques to discrete nets, the theorem enables researchers to derive strong-type estimates for operators based on weak-type estimates on local nets. Consequently, this approach provides a unified framework for obtaining boundedness results, demonstrating the utility of discrete net spaces in analyzing operators within harmonic analysis. These findings contribute significantly to understanding the structural properties of discrete net spaces. Furthermore, they introduce innovative tools for applications in harmonic analysis, operator theory, and related mathematical fields where such spaces naturally arise, ultimately paving the way for advanced theoretical developments and broader analytical applications.

Keywords: net spaces, discrete net spaces, Lorentz space, Marcinkiewicz-type interpolation theorem, real interpolation method, linear operators, local nets, global nets.

2020 Mathematics Subject Classification: 46B70.

Introduction

Let S be the set of all finite sets of indices from $\mathbb{Z}^n$. For a fixed set $M \subset S$ we define the space $n_{p,q}(M)$ $(0 < p, q \leq \infty)$ as the set of sequences $a = \{a_m\}_{m\in\mathbb{Z}^n}$ with quasinorm for $0 < p < \infty$, $0 < q < \infty$

$$\|a\|_{n_{p,q}(M)} = \left(\sum_{k=1}^{\infty} k^{\frac{q}{p}-1} (\bar{a}_k(M))^q\right)^{\frac{1}{q}},$$

and for $q = \infty$, $0 < p \leq \infty$

$$\|a\|_{n_{p,\infty}(M)} = \sup_{1\leq k<\infty} k^{\frac{1}{p}} \bar{a}_k(M),$$

where

$$\bar{a}_k(M) = \sup_{\substack{e\in M \\ |e|\geq k}} \frac{1}{|e|} \left|\sum_{m\in e} a_m\right|,$$

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where $|e|$ is the number of indices in e. Here, $\bar{a}_k(M)$ denotes a discrete analogue of the averaging of a function and is defined as the supremum of the average sums of the elements a_m over all $e \in M$.

These spaces, introduced in [1], are called net spaces. Interpolation theory plays a fundamental role in the analysis of operators in functional spaces [2, 3].

The theory of net spaces has been extensively developed in recent decades, finding applications in harmonic analysis, Fourier multipliers, and operator theory. Various embedding and interpolation properties of net spaces on lattices and compact homogeneous manifolds were investigated in [4, 5]. In particular, [4] analyzes $L_p - L_q$ Fourier multipliers on locally compact groups, providing precise boundedness results, while [5] focuses on net spaces on lattices and establishes Hardy–Littlewood type inequalities along with their converses. The general structure and foundational properties of net spaces were systematically developed in [6], providing a theoretical basis for subsequent applications in analysis and operator theory. Net spaces also play a significant role in the study of stochastic processes and interpolation methods. Collectively, these works demonstrate that net spaces offer a unifying and flexible framework for studying inequalities, operator boundedness, interpolation, and stochastic processes. Modern developments in interpolation theory further explore multi-space frameworks with functional parameters, providing tools that can be adapted to generalized sequences, grand net spaces, and other non-standard function spaces, thereby bridging classical analysis with contemporary operator theory.

Recently, generalizations of net spaces, including grand net spaces, have been introduced [7]. These spaces provide a comprehensive framework for analyzing boundedness properties of integral operators and for studying interpolation results in non-standard function spaces [8–10].

In addition, results on trigonometric Fourier series and convolution inequalities in $\lambda_{p,q}$ and anisotropic Lorentz spaces were obtained in [11, 12], which are closely related to interpolation properties of net spaces.

Related problems for Morrey-type spaces and metric interpolation frameworks were studied in [13–15]. These developments emphasize the importance of interpolation methods for operators in generalized function spaces.

Discrete net spaces are closely related to discrete Morrey spaces:

$$m_p^\lambda = \left\{ a = \{a_k\}_{k\in\mathbb{Z}} : \quad \sup_{m\in\mathbb{N}} \sup_{k\in\mathbb{Z}} \frac{1}{m^\lambda} \left(\sum_{r=k}^{k+m} |a_r|^p \right)^{\frac{1}{p}} < \infty \right\}.$$

In the case when $a = \{a_k\}_{k\in\mathbb{Z}},\ a_k \geq 0$, for $\lambda = n\left(1 - \frac{1}{p}\right)$

$$\|a\|_{n_{p,\infty}(M)} \asymp \|a\|_{m_1^\lambda}.$$

Recent investigations consider intermediate and weak discrete Morrey spaces, highlighting new inclusion and interpolation behaviors [16], and generalized local Morrey spaces with applications to Calderon-Zygmund operators [17].

Interpolation properties of Morrey spaces were studied in several works, showing that this scale is not closed under the real interpolation method [18, 19]. Further developments include complex interpolation methods and their applications to Morrey-type and related function spaces [20,21]. At the same time, Marcinkiewicz-type interpolation theorems were established for Morrey-type spaces [22,23] and similar ideas were applied to net spaces and their discrete analogues [24, 25].

Given functions F and G, in this paper $F \lesssim G$ means that $F \leq c\ G$ (or $c\ F \geq G$), where c is a positive number, depending only on numerical parameters, that may be different on different occasions. Moreover, $F \asymp G$ means that $F \lesssim G$ and $G \lesssim F$.

1 Main result

Let $b > 1$. The parametric family $G_b = \{G_k\}_{k\in\mathbb{N}}$ will be called a local net in $\mathbb{Z}^n$, if

$$G_k \hookrightarrow G_{k+1} \quad \text{and} \quad |G_k| = b^k.$$

Here $|G_k|$ is the number of elements in the set G_k.

The set $F_{G_b} = \{G_k + x\}_{k\in\mathbb{N},x\in\mathbb{Z}^n}$ will be called the global net generated by the net G.

Example. The set of cubes with edge length 2^k, $k \in \mathbb{N}$ in $\mathbb{Z}^n$ is a global net generated by the net of concentric cubes in $\mathbb{Z}^n$ with edge lengths 2^k, $k \in \mathbb{N}$.

We will use the classical Hardy inequalities for discrete sequences, which we formulate as the following lemma.

Lemma 1 (Hardy's inequality). Let $\alpha > 0$, $0 < q, h \leq \infty$ and let the sequence $\{d_k\}_{k\in\mathbb{N}}$ satisfy the following condition for some $\delta > 1$

$$\frac{d_{k+1}}{d_k} \geq \delta, \qquad k = 2, 3, \dots \tag{1}$$

Then the following inequalities hold:

$$\left(\sum_{k=0}^{\infty}\left(d_k^{-\alpha}\left(\sum_{r=0}^{k}|b_r|^h\right)^{\frac{1}{h}}\right)^q\right)^{\frac{1}{q}} \leq c_{\alpha,q}\left(\sum_{k=0}^{\infty}\left(d_k^{-\alpha}|b_k|\right)^q\right)^{\frac{1}{q}},$$

$$\left(\sum_{k=0}^{\infty}\left(d_k^{\alpha}\left(\sum_{r=k}^{\infty}|b_r|^h\right)^{\frac{1}{h}}\right)^q\right)^{\frac{1}{q}} \leq c_{\alpha,q}\left(\sum_{k=0}^{\infty}\left(d_k^{\alpha}|b_k|\right)^q\right)^{\frac{1}{q}}.$$

Proof. Let $0 < h \leq q \leq \infty$, $0 < \varepsilon < \alpha$. We will use Holder's inequality.

$$\left(\sum_{k=0}^{\infty}\left(d_k^{-\alpha}\left(\sum_{r=0}^{k}|b_r|^h\right)^{\frac{1}{h}}\right)^q\right)^{\frac{1}{q}} \leq \left(\sum_{k=0}^{\infty}\left(d_k^{-\alpha}\left(\sum_{r=0}^{k}\left(d_r^{-\varepsilon}|b_r|\right)^q\right)^{\frac{1}{q}}\left(\sum_{r=0}^{k}d_r^{\varepsilon\tau}\right)^{\frac{1}{\tau}}\right)^q\right)^{\frac{1}{q}},$$

where $\frac{1}{\tau} = \frac{1}{h} - \frac{1}{q}$. From the condition (1) we have $\sum_{r=0}^{k} d_r^{\varepsilon\tau} \asymp d_k^{\varepsilon\tau}$. Therefore

$$\left(\sum_{k=0}^{\infty}\left(d_k^{-\alpha}\left(\sum_{r=0}^{k}|b_r|^h\right)^{\frac{1}{h}}\right)^q\right)^{\frac{1}{q}} \lesssim \left(\sum_{k=0}^{\infty}d_k^{(\varepsilon-\alpha)q}\sum_{r=0}^{k}\left(d_r^{-\varepsilon}|b_r|\right)^q\right)^{\frac{1}{q}}$$

$$= \left(\sum_{r=0}^{\infty}\left(d_k^{-\varepsilon}|b_r|\right)^q\sum_{k=r}^{\infty}d_r^{(\varepsilon-\alpha)q}\right)^{\frac{1}{q}} \lesssim \left(\sum_{r=0}^{\infty}\left(d_r^{-\alpha}|b_r|\right)^q\right)^{\frac{1}{q}}.$$

Likewise,

$$\left(\sum_{k=0}^{\infty}\left(d_k^{\alpha}\left(\sum_{r=k}^{\infty}|b_r|^h\right)^{\frac{1}{h}}\right)^q\right)^{\frac{1}{q}} \leq \left(\sum_{k=0}^{\infty}\left(d_k^{\alpha}\left(\sum_{r=k}^{\infty}(d_r^{\varepsilon}|b_r|)^q\right)^{\frac{1}{q}}\left(\sum_{r=k}^{\infty}d_r^{-\varepsilon\tau}\right)^{\frac{1}{\tau}}\right)^q\right)^{\frac{1}{q}}$$

$$\lesssim \left(\sum_{k=0}^{\infty}d_k^{(-\varepsilon+\alpha)q}\sum_{r=k}^{\infty}(d_r^{\varepsilon}|b_r|)^q\right)^{\frac{1}{q}}$$

$$= \left(\sum_{r=0}^{\infty} (d_k^{\varepsilon}|b_r|)^q \sum_{k=0}^{r} d_r^{(-\varepsilon+\alpha)q}\right)^{\frac{1}{q}} \lesssim \left(\sum_{r=0}^{\infty} (d_r^{\alpha}|b_r|)^q\right)^{\frac{1}{q}}.$$

Now let $0 < q < h \leq \infty$. Using Jensen's inequality, we obtain

$$\left(\sum_{k=0}^{\infty}\left(d_k^{-\alpha}\left(\sum_{r=0}^{k}|b_r|^h\right)^{\frac{1}{h}}\right)^q\right)^{\frac{1}{q}} \leq \left(\sum_{k=0}^{\infty} d_k^{-\alpha q}\sum_{r=0}^{k}|b_r|^q\right)^{\frac{1}{q}}$$

$$= \left(\sum_{r=0}^{\infty}|b_r|^q \sum_{k=r}^{\infty} d_k^{-\alpha q}\right)^{\frac{1}{q}} \asymp \left(\sum_{r=0}^{\infty}\left(d_r^{-\alpha}|b_r|\right)^q\right)^{\frac{1}{q}}.$$

The second inequality also follows from Jensen's inequality.
Lemma 1 is proved. □

Theorem 1. Let $G = \{G_t\}_{t>0}$ be a local net, and let $F = \bigcup_{x\in\mathbb{Z}^n}(G+x)$ be the global net generated by the net G. Assume that $0 < p_0 < p_1 < \infty$ and $0 < q_0 \leq q_1 \leq \infty$, $0 < \theta < 1$, $1 \leq \tau \leq \infty$,

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \qquad \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}.$$

If a linear operator T satisfies the following inequalities for some constants $M_0, M_1 > 0$

$$\|Ta\|_{n_{q_i,\infty}(G+x)} \leq M_i\|a\|_{n_{p_i,1}(G+x)}, \quad x \in \mathbb{Z}^n,\ a \in n_{p_i,1}(G+x),\ i = 0,1, \tag{2}$$

then for any $a \in n_{p,\tau}(F)$, the following inequality holds:

$$\|Ta\|nq,\tau(F) \leq c\max\{M_0, M_1\}\|a\|np,\tau(F),$$

where the constant $c > 0$ depends only on the parameters $p_0,\ p_1,\ q_0,\ q_1,\ p,\ q,\ \tau,\ \theta$.

Proof. Let $a = \{a_m\}_{m\in\mathbb{Z}} \in n_{p,\tau}(F)$, $\gamma > 0$. For any $x \in \mathbb{Z}^n,\ s \in \mathbb{N}$ we define the sequences

$$a_{0,s} = a\chi_{(G_s+x)}, \quad a_{1,s} = a(1 - \chi_{(G_s+x)}),$$

where χ_{G_s+x} denotes the characteristic function of the set G_s+x. It is easy to see that $a_{0,s} \in n_{p_0,1}(G+x)$ and $a_{1,s} \in n_{p_1,1}(G+x)$. Then $a = a_{0,s} + a_{1,s}$ and

$$\sup_{\xi\geq t\gamma}\frac{1}{|G_\xi|}\left|\sum_{m\in G_\xi+x}(Ta)_m\right| \leq \sup_{\xi\geq t\gamma}\frac{1}{|G_\xi|}\left|\sum_{m\in G_\xi+x}(Ta_{0,s})_m\right|$$

$$+\sup_{\xi\geq t\gamma}\frac{1}{|G_s|}\left|\sum_{m\in G_\xi+x}(Ta_{1,s})_m\right| = I_1 + I_2.$$

First, let us estimate I_1. According to inequality (2), we have

$$I_1 = \sup_{\xi\geq t\gamma}\frac{1}{|G_\xi|}\left|\sum_{m\in G_\xi+x}(Ta_{0,s})_m\right|$$

$$\leq b^{-\frac{t\gamma}{q_0}}\sup_{r\in\mathbb{N}} b^{\frac{r}{q_0}}\sup_{\xi\geq r}\frac{1}{|G_\xi|}\left|\sum_{m\in G_\xi+x}(Ta_{0,s})_m\right| = b^{-\frac{t\gamma}{q_0}}\|Ta_{0,s}\|_{n_{q_0,\infty}(G+x)}$$

$$\le M_0 b^{-\frac{t\gamma}{q_0}} \|a_{0,s}\|_{n_{p_0,1}(G+x)}$$

$$= M_0 b^{-\frac{t\gamma}{q_0}} \left(\sum_{r=0}^{s} b^{\frac{r}{p_0}} \sup_{\xi \ge r} \frac{1}{|G_\xi|} \Big| \sum_{m \in G_\xi + x} (a_{0,s})_m \Big| + \sum_{r=s}^{\infty} b^{\frac{r}{p_0}} \sup_{\xi \ge r} \frac{1}{|G_\xi|} \Big| \sum_{m \in G_\xi + x} (a_{0,s})_m \Big| \right).$$

Let $0 < r \le s$, if $\xi \le s$, $m \in G_\xi + x$, we have $a_{0,s}(y) = a_m \chi_{G_s + x} = a_m$, if $\xi > s$, then

$$\Big| \sum_{m \in G_\xi + x} (a_{0,s})_m \Big| = \Big| \sum_{m \in G_\xi + x} a_m \Big|.$$

For the first sum, we have the following:

$$\sum_{r=0}^{s} b^{\frac{r}{p_0}} \sup_{\xi \ge r} \frac{1}{|G_\xi|} \Big| \sum_{m \in G_\xi + x} (a_{0,s})_m \Big|$$

$$\le \sum_{r=0}^{s} b^{\frac{r}{p_0}} \sup_{\xi \ge r} \frac{1}{|G_\xi} \Big| \sum_{m \in G_\xi + x} a_m \Big| \le \sum_{r=0}^{s} b^{\frac{r}{p_0}} \bar{a}(r, F).$$

For the second sum, we have

$$\sum_{r=s}^{\infty} b^{\frac{r}{p_0}} \sup_{\xi \ge r} \frac{1}{|G_\xi|} \Big| \sum_{m \in G_\xi + x} (a_{0,s})_m \Big| = \sum_{r=s}^{\infty} b^{\frac{r}{p_0}} \sup_{\xi \ge r} \frac{1}{|G_\xi|} \Big| \sum_{m \in G_s + x} a_m \Big|$$

$$= \Big| \sum_{m \in G_s + x} a_m \Big| \sum_{r=s}^{\infty} b^{\frac{r}{p_0}} \sup_{\xi \ge r} \frac{1}{|G_{\xi|}} = \Big| \sum_{m \in G_s + x} a_m \Big| \sum_{r=s}^{\infty} b^{\frac{r}{p_0} - 1}$$

$$\le b^{\frac{s}{p_0}} \frac{1}{|G_s|} \Big| \sum_{m \in G_s + x} a_m \Big| \le b^{\frac{s}{p_0}} \bar{a}(b^s, F).$$

Thus, we obtain

$$I_1 \lesssim M_0 b^{-\frac{t\gamma}{q_0}} \left(\sum_{r=0}^{s} b^{\frac{r}{p_0}} \bar{a}(r, F) + b^{\frac{s}{p_0}} \bar{a}(s, F) \right).$$

Similarly, we estimate I_2. Applying inequality (2), we obtain

$$I_2 = \sup_{s \ge t\gamma} \frac{1}{|G_s|} \Big| \sum_{m \in G_\xi + x} (Ta_{1,s})_m \Big|$$

$$\le b^{-\frac{t\gamma}{q_1}} \sup_{r \in \mathbb{N}} b^{\frac{r}{q_1}} \sup_{s \ge r} \frac{1}{|G_s|} \Big| \sum_{m \in G_s + x} (Ta_{1,s})_m \Big| = b^{-\frac{t\gamma}{q_1}} \|(Ta_{1,s})_m\|_{n_{q_1,\infty}(G+x)}$$

$$\le M_1 b^{-\frac{t\gamma}{q_1}} \|a_{1,s}\|_{n_{p_1,1}(G+x)} = M_1 b^{-\frac{t\gamma}{q_1}} \left(\sum_{r=0}^{\infty} b^{\frac{r}{p_1}} \sup_{s \ge r} \frac{1}{|G_s|} \Big| \sum_{m \in G_s + x} (a_{1,s})_m \Big| \right)$$

$$= M_1 b^{-\frac{t\gamma}{q_1}} \left(\sum_{r=0}^{s} b^{\frac{r}{p_1}} \sup_{\xi \ge r} \frac{1}{|G_\xi|} \Big| \sum_{m \in G_\xi + x} (a_{1,s})_m \Big| + \sum_{r=s}^{\infty} b^{\frac{r}{p_1}} \sup_{\xi \ge r} \frac{1}{|G_\xi|} \Big| \sum_{m \in G_\xi + x} (a_{1,s})_m \Big| \right)$$

$$= M_1 b^{-\frac{t\gamma}{q_1}} \Big(J_1 + J_2 \Big).$$

To estimate J_1 and J_2, we note that

$$\sum_{m\in G_\xi+x}(a_{1,s})_m=\begin{cases}0,\ \xi\le s,\\ \sum_{m\in(G_\xi+x)\setminus(G_s+x)}a_m,\ \xi>s\end{cases}$$

$$=\begin{cases}0\quad\text{for}\quad\xi\le s,\\ \left|\sum_{m\in G_\xi+x}a_m-\sum_{m\in G_s+x}a_m\right|\le\left|\sum_{m\in G_\xi+x}a_m\right|+\left|\sum_{m\in G_s+x}a_m\right|\ \text{for}\quad\xi>s.\end{cases}$$

Next, using this estimate, we have

$$J_1\le\sum_{r=0}^{s}b^{\frac{r}{p_1}}\sup_{\xi\ge r}\frac{1}{|G_\xi|}\left(\left|\sum_{m\in}a_m\right|+\left|\sum_{m\in G_s+x}a_m\right|\right)$$

$$\le\sum_{r=0}^{s}b^{\frac{r}{p_1}}\left(\bar a(b^s,F)+\left|\sum_{m\in G_s+x}a_m\right|\sup_{\xi\ge r}\frac{1}{|G_\xi|}\right)\le2\bar a(b^s,F)\sum_{r=0}^{s}b^{\frac{r}{p_1}}=2p_1b^{\frac{s}{p_1}}\bar a(b^s,F)$$

and

$$J_2\le\sum_{r=s}^{\infty}b^{\frac{r}{p_1}}\sup_{\xi\ge r}\frac{1}{|G_\xi|}\left(\left|\sum_{m\in G_\xi+x}a_m\right|+\left|\sum_{m\in G_s+x}a_m\right|\right)$$

$$\le\sum_{r=s}^{\infty}b^{\frac{r}{p_1}}\left(\bar a(b^s,F)+\left|\sum_{m\in G_s+x}a_m\right|\sup_{\xi\ge r}\frac{1}{|G_\xi|}\right)\le\sum_{r=s}^{\infty}b^{\frac{r}{p_1}}\bar a(b^r,F)$$

$$+\left|\sum_{m\in G_s+x}a_m\right|\sum_{r=s}^{\infty}b^{\frac{r}{p_1}}\sup_{\xi\ge r}\frac{1}{|G_\xi|}\asymp\sum_{r=s}^{\infty}b^{\frac{r}{p_1}}\bar a(b^r,F)+\left|\sum_{m\in G_s+x}a_m\right|b^{\frac{s}{p_1}}$$

$$\lesssim\sum_{r=s}^{\infty}b^{\frac{r}{p_1}}\bar a(b^r,F)+b^{\frac{s}{p_1}}\bar a(b^s,F).$$

Combining the obtained estimates, we obtain the following estimate

$$I_2\lesssim M_1b^{\frac{t\gamma}{q_1}}\left(\sum_{r=s}^{\infty}b^{\frac{r}{p_1}}\bar a(b^r,F)+b^{\frac{s}{p_1}}\bar a(b^s,F)\right).$$

Thus, we got

$$\sup_{\xi\ge t\gamma}\frac{1}{|G_s|}\left|\sum_{m\in G_\xi+x}(Ta)_m\right|\lesssim M_0b^{-\frac{t\gamma}{q_0}}\left(\sum_{r=0}^{s}b^{\frac{r}{p_0}}\bar a(b^r,F)+b^{\frac{s}{p_0}}\bar a(a^s,F)\right)$$

$$+M_1b^{-\frac{t\gamma}{q_1}}\left(\sum_{r=s}^{\infty}b^{\frac{r}{p_1}}\bar a(b^r,F)+b^{\frac{s}{p_1}}\bar a(b^s,F)\right).$$

Let $\gamma=\left(\frac{1}{p_0}-\frac{1}{p_1}\right)\Big/\left(\frac{1}{q_0}-\frac{1}{q_1}\right)$. Using the equivalent normalization for $d=b^\gamma$

$$\|Ta\|_{n_{q,\tau}(F)}\asymp\left(\sum_{t=0}^{\infty}\left(d^{\frac{t}{q}}\sup_{\substack{|G_\xi|\ge d^t\\ x\in\mathbb{Z}^n}}\frac{1}{|G_\xi|}\left|\sum_{m\in G_\xi+x}a_m\right|\right)^{\tau}\right)^{\frac{1}{\tau}}.$$

Since $|G_\xi| = b^\xi$, from the inequality $|G_\xi| \geq d^t$ it follows that $b^\xi \geq d^t = b^{t\gamma}$ which implies $\xi \geq t\gamma$. Let $s = t$, Then, taking into account the obtained inequalities, we have

$$\|Ta\|_{n_{q,\tau}(F)} \asymp \left(\sum_{t=0}^{\infty} \left(d^{\frac{t}{q}} \sup_{\substack{\xi \geq t\gamma \\ x \in \mathbb{Z}^n}} \frac{1}{|G_\xi|} \Big| \sum_{m \in G_\xi + x} a_m \Big| \right)^\tau \right)^{\frac{1}{\tau}}$$

$$\lesssim M_0 A_1 + M_0 A_2 + M_1 A_3 + M_1 A_4,$$

where, taking into account that

$$\gamma \left(\frac{1}{q} - \frac{1}{q_0} \right) = -\theta \left(\frac{1}{p_0} - \frac{1}{p_1} \right)$$

and

$$\gamma \left(\frac{1}{q} - \frac{1}{q_1} \right) = (1-\theta) \left(\frac{1}{p_0} - \frac{1}{p_1} \right).$$

$$A_1 = \left(\sum_{t=0}^{\infty} \left(d^{t(\frac{1}{q} - \frac{1}{q_0})} \sum_{r=0}^{t} b^{\frac{r}{p_0}} \bar{a}(b^r, F) \right)^\tau \right)^{\frac{1}{\tau}} = \left(\sum_{t=0}^{\infty} \left(b^{-\theta t \left(\frac{1}{p_0} - \frac{1}{p_1} \right)} \sum_{r=0}^{t} b^{\frac{r}{p_0}} \bar{a}(b^r, F) \right)^\tau \right)^{\frac{1}{\tau}},$$

$$A_2 = \left(\sum_{t=0}^{\infty} \left(d^{t(\frac{1}{q} - \frac{1}{q_0})} b^{\frac{t}{p_0}} \bar{a}(b^t, F) \right)^\tau \right)^{\frac{1}{\tau}} = \left(\sum_{t=0}^{\infty} \left(b^{-\theta t \left(\frac{1}{p_0} - \frac{1}{p_1} \right)} b^{\frac{t}{p_0}} \bar{a}(b^t, F) \right)^\tau \right)^{\frac{1}{\tau}}$$

$$= \left(\sum_{t=0}^{\infty} \left(b^{\frac{t}{p}} \bar{a}(b^t, F) \right)^\tau \right)^{\frac{1}{\tau}} \asymp \|a\|_{n_{p,\tau}(F)}$$

and

$$A_3 = \left(\sum_{t=0}^{\infty} \left(d^{t(\frac{1}{q} - \frac{1}{q_1})} \sum_{r=t}^{\infty} b^{\frac{r}{p_1}} \bar{a}(b^r, F) \right)^\tau \right)^{\frac{1}{\tau}} = \left(\sum_{t=0}^{\infty} \left(b^{(1-\theta) t \left(\frac{1}{p_0} - \frac{1}{p_1} \right)} \sum_{r=t}^{\infty} b^{\frac{r}{p_1}} \bar{a}(b^r, F) \right)^\tau \right)^{\frac{1}{\tau}},$$

$$A_4 = \left(\sum_{t=0}^{\infty} \left(d^{t(\frac{1}{q} - \frac{1}{q_1})} b^{\frac{t}{p_1}} \bar{a}(b^t, F) \right)^\tau \right)^{\frac{1}{\tau}} = \left(\sum_{t=0}^{\infty} \left(b^{(1-\theta) t \left(\frac{1}{p_0} - \frac{1}{p_1} \right)} b^{\frac{t}{p_1}} \bar{a}(b^t, F) \right)^\tau \right)^{\frac{1}{\tau}}$$

$$= \left(\sum_{t=0}^{\infty} \left(b^{\frac{t}{p}} \bar{a}(b^t, F) \right)^\tau \right)^{\frac{1}{\tau}} \asymp \|a\|_{n_{p,\tau}(F)}.$$

To estimate A_1 and A_3 we use the Hardy inequalities from Lemma 1. Thus, we obtain

$$\|Ta\|_{n_{q,\tau}(F)} \leq c \max\{M_0, M_1\} \|a\|_{n_{p,\tau}(F)},$$

where the constant $c > 0$ depends only on the parameters $p_0,\ p_1,\ q_0,\ q_1,\ p,\ q,\ \tau,\ \theta$.

Consequently, we have obtained the desired estimate. The theorem is proved. □

Conclusion

In this paper, we studied the interpolation properties of discrete net spaces for a broad class of nets. We established an analogue of the Marcinkiewicz-type interpolation theorem for linear operators, extending existing results in the theory of interpolation. Our approach builds upon the ideas developed in [22,23], where alternative analogues of Marcinkiewicz-type interpolation theorems for net spaces were obtained [24, 25].

A comparison of our results with previous works shows that our findings provide a new perspective on interpolation in net spaces, refining and generalizing existing methods. The scientific novelty of this work lies in the further development of interpolation techniques adapted to discrete net spaces, expanding their theoretical framework and potential applications.

The practical significance of our results is reflected in their possible applications to harmonic analysis, operator theory, and stochastic processes, where net spaces serve as a fundamental tool. Future research directions include the extension of interpolation results to nonlinear operators, the exploration of stability properties, and the application of net space interpolation to more complex functional spaces and applied problems.

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Author Contributions

All authors contributed equally to this work. E.D. Nursultanov collected and analyzed data, and led manuscript preparation. A.K. Kalidolday served as the principal investigator of the research grant and supervised the research process. A.N. Sharipova contributed to the analysis and interpretation of results and assisted in manuscript preparation. All authors participated in the revision of the manuscript and approved the final submission.

Conflict of Interest

The authors declare no conflict of interest.

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