Bulletin of the Karaganda University. Mathematics Series, No. 2(122), 2026, pp. 65–79

https://doi.org/10.31489/2026M2/65-79 *Research article*

On the n-inner product spaces from the perspective of its quotient spaces

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In this paper, we investigate several topological properties of n-inner product spaces with respect to the inner products and norms defined on the quotient spaces we constructed. The construction is carried out with respect to a set of n linearly independent vectors, ensuring a consistent analytical framework. This construction was performed in several ways, each resulting in multiple quotient spaces. On each of these quotient spaces, we defined an inner product, along with the corresponding induced norm. Quotient spaces with similar structures are grouped into equivalence classes, thereby yielding several classes of quotient spaces. Within this framework, several topological aspects, including weak convergence, strong convergence, Cauchy sequences, and completeness, are examined with respect to classes of quotient spaces. Consequently, for each aspect, multiple definitions are formulated relative to these classes. We showed that the various definitions associated with a given topological property are equivalent to one another, regardless of the class of quotient spaces we used. Finally, the minimal number of norms within a given class required for an effective investigation of these properties is determined, thereby contributing to a more efficient and non-redundant analytical framework.

Keywords: class of quotient space, inner product, n-inner product space, n-normed space, norm, quotient space, sequence, topology property.

2020 Mathematics Subject Classification: 46B15, 46C50, 46B25.

Introduction

An n-normed space is a generalization of a normed space. This concept was initially introduced by S. Gähler in the 1960's. He subsequently published this concept in a series of papers; see [1] for the concept of 2-normed spaces and [2–4] for generalized metric spaces. Let n be a nonnegative integer and X be a real vector space ($\dim(X) \geq n$). A function $\|\cdot, \dots, \cdot\| : X^n \to \mathbb{R}$ which satisfies the following conditions:

N1. $\|x_1, \dots, x_n\| = 0$ if and only if $x_1, \dots, x_n$ are linearly dependent;
N2. $\|x_1, \dots, x_n\|$ is invariant under permutation;
N3. $\|\alpha x_1, \dots, x_n\| = |\alpha| \|x_1, \dots, x_n\|$ for any $\alpha \in \mathbb{R}$;
N4. $\|x_1 + x_1', \dots, x_n\| \leq \|x_1, \dots, x_n\| + \|x_1', \dots, x_n\|$,

is called an n-norm. The pair $(X, \|\cdot, \dots, \cdot\|)$ is called an n-normed space.

Let $(X, \langle\cdot, \cdot\rangle)$ be an inner product space with $\dim(X) \geq n$,

$$\|x_1 \dots, x_n\|_s = \sqrt{|det(\langle x_i, x_j\rangle)|} = \begin{vmatrix} \langle x_1, x_1\rangle & \langle x_1, x_2\rangle & \cdots & \langle x_1, x_n\rangle \\ \langle x_2, x_1\rangle & \langle x_2, x_2\rangle & \cdots & \langle x_2, x_n\rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle x_n, x_1\rangle & \langle x_n, x_2\rangle & \cdots & \langle x_n, x_n\rangle \end{vmatrix}^{\frac{1}{2}},$$

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Received: 6 April 2025; *Accepted:* 30 March 2026.

defines an n-norm on X. This n-norm is called the standard n-norm. Since then, the theory has been further developed by many researchers; see, for instance, [5–7], which focus on structural and functional analytic aspects of n-normed spaces, and [8–10], which address properties of norms and the topology of n-normed spaces.

On the other hand, we also know that the concept of n-inner product spaces is a generalization of the concept of inner product spaces. For $n = 2$, the concept of 2-inner product space was introduced by Diminnie, Gahler, and White in the 1970s [11]. Misiak then developed the concept of n-inner product spaces for $n \geq 2$ in 1989 [12].

Let n be a nonnegative integer and X be a real vector space with $\dim X \geq n$ ($\dim X$ may be infinite). A function $\langle \cdot, \cdot | \cdot, \ldots, \cdot \rangle : X^{n+1} \to \mathbb{R}$ that satisfies

I1. $\langle x_1, x_1 | x_2, \ldots, x_n \rangle \geq 0$;
$\langle x_1, x_1 | x_2, \ldots, x_n \rangle = 0$, if and only if $x_1, x_2, \ldots, x_n$ are linearly dependent;
I2. $\langle x_1, x_1 | x_2, \ldots, x_n \rangle = \langle x_{i_1}, x_{i_1} | x_{i_2}, \ldots, x_{i_n} \rangle$, for every permutation $(i_1, \ldots, i_n)$ of $(1, \ldots, n)$;
I3. $\langle x, y | x_2, \ldots, x_n \rangle = \langle y, x | x_2, \ldots, x_n \rangle$;
I4. $\langle \alpha x, y | x_2, \ldots, x_n \rangle = \alpha \langle x, y | x_2, \ldots, x_n \rangle$;
I5. $\langle x + x', y | x_2, \ldots, x_n \rangle = \langle x, y | x_2, \ldots, x_n \rangle + \langle x', y | x_2, \ldots, x_n \rangle$,

is called an n-inner product. The pair $(X, \langle \cdot, \cdot | \cdot, \ldots, \cdot \rangle)$ is called an n-inner product space.

Let $(X, \langle \cdot, \cdot \rangle)$ be an inner product space with $\dim(X) \geq n$, we define an n-inner product on X as

$$\langle u, v | x_2, \ldots, x_n \rangle_S := \begin{vmatrix} \langle u, v \rangle & \langle u, x_2 \rangle & \cdots & \langle u, x_n \rangle \\ \langle x_2, v \rangle & \langle x_2, x_2 \rangle & \cdots & \langle x_2, x_n \rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle x_n, v \rangle & \langle x_n, x_2 \rangle & \cdots & \langle x_n, x_n \rangle \end{vmatrix}$$

we called it the standard n-inner product on X.

Various aspects of n-inner product spaces have been studied by many researchers. The reader may see, for instance, [13–15], which focus on structural and functional analytic properties of n-inner product spaces, and [16–18], which deal with generalized and extended settings, including fuzzy and completion aspects.

On an n-inner product space $(X, \langle \cdot, \cdot | \cdot, \ldots, \cdot \rangle)$ we can define an n-norm that induced by the n-inner product, defined by

$$\|x_1, x_2, \ldots, x_n\| = \langle x_1, x_1 | x_2, \ldots, x_n \rangle^{\frac{1}{2}} \quad \text{for any} \quad x_1, \ldots, x_n \in X.$$

One can see that the standard n norm is an induced norm with respect to the standard n-inner product. Moreover, in an n-inner product space $(X, \langle \cdot, \cdot | \cdot, \ldots, \cdot \rangle)$, we have Cauchy–Schwarz inequality

$$|\langle x, y | x_2, \ldots, x_n \rangle| \leq \|x, x_2, \ldots, x_n\| \|y, x_2, \ldots, x_n\|,$$

with $\|\cdot, \ldots, \cdot\|$ is an induced n-norm.

1 Results

1.1 Construction of Quotient Spaces

Let $(X, \langle \cdot, \cdot | \cdot, \ldots, \cdot \rangle)$ be an n-inner product space. First, choose a linearly independent set, namely $A = \{a_1, a_2, \ldots, a_n\}$ in X. Fix a $j \in \{1, 2, \ldots, n\}$ and consider a set $A \setminus \{a_j\}$. We define a subspace generated by $A \setminus \{a_j\}$

$$A_j = \text{span } A \setminus \{a_j\} = \left\{ \sum_{i=1, i \neq j}^{n} \gamma_i a_i \ ; \ \gamma_i \in \mathbb{R} \right\}.$$

For any $x \in X$, we define the coset of A_j by

$$\overline{x} = \left\{ x + \sum_{i=1, i \neq j}^{n} \gamma_i a_i \ ; \ \gamma_i \in \mathbb{R} \right\}.$$

We have $\overline{0} = A_j$. Moreover, $\overline{x} = \overline{y}$ if and only if $x - y \in \text{span}\ A \setminus \{a_j\} = A_j$. Next, we define a quotient space of X as

$$X_j = X/A_j = \{\overline{x} \ : \ x \in X\}.$$

It is easy to see that the addition and scalar multiplication apply in X_j. Consider a function $\langle \cdot, \cdot \rangle_j : X_j^2 \to \mathbb{R}$ defined by

$$\langle \overline{x}, \overline{y} \rangle_j = \langle x, y | a_1, \ldots, a_{j-1}, a_{j+1}, \ldots, a_n \rangle. \tag{1}$$

The following theorem shows that X_j equipped with $\langle x, y \rangle_j$ is an inner product space.

Theorem 1. Let $(X, \langle \cdot, \cdot | \cdot, \ldots, \cdot \rangle)$ be an n-inner product space and $A = \{a_1, a_2, \ldots, a_n\}$ in X be a linearly independent set. Then $(X_j, \langle \cdot, \cdot \rangle_j)$ is an inner product space, with $\langle \cdot, \cdot \rangle_j$ is a function defined in (1).

Proof. We show that the function defined in (1) is an inner product. Using the properties of the n-inner product, we have

$$\langle \overline{x}, \overline{x} \rangle_j = \langle x, x | a_1, \ldots, a_{j-1}, a_{j+1}, \ldots, a_n \rangle \geq 0$$

for all $\overline{x} \in X_j$. Moreover, if $\langle \overline{x}, \overline{x} \rangle_j = 0$, then $\langle x, x | a_1, \ldots, a_{j-1}, a_{j+1}, \ldots, a_n \rangle = 0$. This implies that $x, a_1, \ldots, a_{j-1}, a_{j+1}, \ldots, a_n$ are linearly dependent. As a consequence

$$x = \sum_{i=1, i \neq j}^{n} \gamma_i a_i,$$

since $a_1, a_2, \ldots, a_n$ are linearly independent, we have $x \in A_j$ which leads to $\overline{x} = \overline{0}$ as a result. Conversely, if $\overline{x} = \overline{0}$ it is obvious that

$$\langle \overline{x}, \overline{x} \rangle_j = \langle \overline{0}, \overline{0} \rangle_j = \langle 0, 0 | a_1, \ldots, a_{j-1}, a_{j+1}, \ldots, a_n \rangle = 0.$$

Next, for any $\alpha \in \mathbb{R}$, and $\overline{x}, \overline{y}, \overline{z} \in X_j$, we have

$$\begin{aligned} \langle \alpha \overline{x}, \overline{y} \rangle_j &= \langle \alpha x, y | a_1, \ldots, a_{j-1}, a_{j+1}, \ldots, a_n \rangle \\ &= \alpha \langle x, y | a_1, \ldots, a_{j-1}, a_{j+1}, \ldots, a_n \rangle \\ &= \alpha \langle \overline{x}, \overline{y} \rangle_j. \end{aligned}$$

It is easy to see that using properties of the n-inner product we also have $\langle \overline{x}, \overline{y} \rangle_j = \langle \overline{y}, \overline{x} \rangle_j$ and $\langle \overline{x} + \overline{y}, \overline{z} \rangle_j = \langle \overline{x}, \overline{z} \rangle_j + \langle \overline{y}, \overline{z} \rangle_j$. □

Recall that on an inner product space $(X, \langle \cdot, \cdot \rangle)$ one can define an induced norm from an inner product defined by $\|x\| = \langle x, x \rangle^{\frac{1}{2}}$ for any $x \in X$. As a consequence, we can define a norm on X_j induced by the inner product defined on (1). For all $\overline{x} \in X_j$ we have

$$\begin{aligned} \|\overline{x}\|_j &= \langle x, x \rangle_j^{\frac{1}{2}} \\ &= \langle x, x | a_1, \ldots, a_{j-1}, a_{j+1}, \ldots, a_n \rangle^{\frac{1}{2}} \\ &= \|x, a_1, \ldots, a_{j-1}, a_{j+1}, \ldots, a_n\|. \end{aligned} \tag{2}$$

In an inner product space $(X, \langle\cdot,\cdot\rangle)$, we have Cauchy Schwarz inequality. For any $x, y \in X$

$$|\langle x, y\rangle| \leq \|x\|\|y\|,$$

with $\|\cdot\|$ is the induced norm.

Corollary 1. Let $(X, \langle\cdot,\cdot|\cdot,\ldots,\cdot\rangle)$ be an n-inner product space and $A = \{a_1, a_2, \ldots, a_n\}$ in X be a linearly independent set. Then $(X_j, \|\cdot\|_j)$ is a normed space, with $\|\cdot\|_j$ is an induced norm defined in (2).

The above corollary is a direct corollary from Theorem 1 and equation (2).

The quotient space X_j with a fixed j was constructed with respect to a linearly independent set A by "eliminating" one vector of A. Note that, we can choose any linearly independent set consisting of n vectors to substitute for A. Using this construction, we can get n quotient spaces which are also inner product spaces. We collect these quotient spaces in a set and name it class-1.

Recall that $\overline{x} = \overline{y}$ if and only if $x - y \in \text{span } A \setminus \{a_j\} = A_j$ for a $j \in \{1, \ldots, n\}$. We will examine this case when it applies in all quotient space.

Lemma 1. Let $(X, \langle\cdot,\cdot|\cdot,\ldots,\cdot\rangle)$ be an n-inner product space and $A = \{a_1, a_2, \ldots, a_n\}$ in X be a linearly independent set. Then $x = y$ if and only if $x - y \in \text{span } A \setminus \{a_j\} = A_j$ for all $j \in \{1, \ldots, n\}$.

Proof. Let $x - y \in \text{span } A \setminus \{a_j\} = A_j$ for all $j \in \{1, \ldots, n\}$. We can write

$$x - y = \alpha_{12}a_2 + \alpha_{13}a_3 + \cdots + \alpha_{1n}a_n; \tag{2.1}$$

$$x - y = \alpha_{21}a_1 + \alpha_{23}a_3 + \cdots + \alpha_{2n}a_n; \tag{2.2}$$

$$\vdots$$

$$x - y = \alpha_{n1}a_1 + \alpha_{n2}a_2 + \cdots + \alpha_{n(n-1)}a_{n-1}, \tag{2.n}$$

with $\alpha_{ij} \in \mathbb{R}$, $i, j \in \{1, \ldots, n\}$. Here, we have n equations. By subtracting equation (2.2) from equation (2.1), we obtain

$$0 = -\alpha_{21}a_1 + \alpha_{12}a_2 + (\alpha_{13} - \alpha_{23})a_3 + \cdots + (\alpha_{1n} - \alpha_{1n})a_n.$$

Since $a_1, \ldots, a_n$ is a linearly independent set, it follows that all the coefficients in the above equation must be 0. Especially we have $\alpha_{12} = 0$. Next, by subtracting equation (2.3) from equation (2.1), we obtain

$$0 = -\alpha_{31}a_1 + (\alpha_{12} - \alpha_{32})a_2 + \alpha_{13}a_3 + (\alpha_{14} - \alpha_{22})a_4 + \cdots + (\alpha_{1n} - \alpha_{1n})a_n.$$

Since $a_1, \ldots, a_n$ is a linearly independent set, it follows that all the coefficients in the above equation must be 0. Especially we have $\alpha_{13} = 0$. By repeating this procedure we have $a_{1j} = 0$. This implies $x - y = 0$ or $x = y$. Conversely, if $x = y$, then $x - y = 0 \in \text{span } A \setminus \{a_j\} = A_j$ for all $j \in \{1, \ldots, n\}$. □

Lemma 1 states that, if the condition $x - y \in A_j$, applies for all $j \in \{1, \ldots, n\}$, then $x = y$. We may conclude that, by considering all quotient spaces, we are able to observe the "real" point rather than its representation as a coset.

Example 1. Let $(\mathbb{R}^n, \langle\cdot,\cdot,\cdot,\cdot\rangle)$ be a 4-inner product space, with $n \geq 4$ and $A = \{a_1, a_2, a_3, a_4\}$ be a linearly independent set. The class-1 of $\mathbb{R}^n$ with respect to A consist of four quotient spaces, namely $\mathbb{R}^n_1, \mathbb{R}^n_2, \mathbb{R}^n_3$ and $\mathbb{R}^n_4$. These quotient spaces are inner product spaces and also normed spaces. Their inner product and norm are defined as

$\langle\overline{x}, \overline{y}\rangle_1 = \langle x, y|a_2, a_3, a_4\rangle$ and its induced norm is $\|\overline{x}\|_1 = \|x, a_2, a_3, a_4\|$;
$\langle\overline{x}, \overline{y}\rangle_2 = \langle x, y|a_2, a_3, a_4\rangle$ and its induced norm is $\|\overline{x}\|_2 = \|x, a_1, a_3, a_4\|$;
$\langle\overline{x}, \overline{y}\rangle_3 = \langle x, y|a_1, a_2, a_4\rangle$ and its induced norm is $\|\overline{x}\|_3 = \|x, a_1, a_2, a_4\|$;
$\langle\overline{x}, \overline{y}\rangle_4 = \langle x, y|a_1, a_2, a_3\rangle$ and its induced norm is $\|\overline{x}\|_4 = \|x, a_1, a_2, a_3\|$, respectively.

We will generalize the above construction by 'eliminating' more vectors of A. For an $m \in \{1, 2, \ldots, n\}$ fix some vectors $a_{j_1}, a_{j_2}, \ldots, a_{j_m} \in A$. Consider the set $A \setminus \{a_{j_1}, \ldots, a_{j_m}\}$ and define a subspace generated by $A \setminus \{a_{j_1}, \ldots, a_{j_m}\}$

$$A_{j_1,\ldots,j_m} = \text{span } A \setminus \{a_{j_1}, \ldots, a_{j_m}\} = \left\{ \sum_{\substack{i=1 \\ i \neq j_1,\ldots j_m}}^{n} \beta_i a_i \ : \ \beta_i \in \mathbb{R} \right\}.$$

For any $x \in X$, the corresponding coset of $A_{j_1,\ldots,j_m}$ is

$$\overline{x} = \left\{ x + \sum_{\substack{i=1 \\ i \neq j_1,\ldots j_m}}^{n} \beta_i a_i \ : \ \beta_i \in \mathbb{R} \right\}.$$

One can see that $\overline{0} = A_{j_1,\ldots,j_m}$. Moreover, $\overline{x} = \overline{y}$, if and only if $x - y \in \text{span } A \setminus \{a_{j_1}, \ldots, a_{j_m}\} = A_{j_1,\ldots,j_m}$. We define a quotient space of X (with respect to $A_{j_1,\ldots,j_m}$)

$$X_{j_1,\ldots,j_m} = X / A_{j_1,\ldots,j_m} = \{\overline{x} \ : \ x \in X\}.$$

The addition and scalar multiplication apply in $X_{j_1,\ldots,j_m}$. Next, we define a function $\langle \cdot, \cdot \rangle_{j_1,\ldots,j_m} : X^2_{j_1,\ldots,j_m} \to \mathbb{R}$ defined by

$$\langle \overline{x}, \overline{y} \rangle_{j_1,\ldots,j_m} = \langle x, y | a_1, \ldots, a_{j_1-1}, a_{j_1+1}, \ldots, a_n \rangle + \cdots + \langle x, y | a_1, \ldots, a_{j_m-1}, a_{j_m+1}, \ldots, a_n \rangle \qquad (3)$$

We can see that each term of equation (3) is an inner product that is defined on each quotient space of class-1 (see equation (1)). Therefore, equation (3) can be written as

$$\langle \overline{x}, \overline{y} \rangle_{j_1,\ldots,j_m} = \langle \overline{x}, \overline{y} \rangle_{j_1} + \cdots + \langle \overline{x}, \overline{y} \rangle_{j_m}. \qquad (4)$$

Moreover, the following theorem states that $\langle \cdot, \cdot \rangle_{j_1,\ldots,j_m}$ is an inner product on $X_{j_1,\ldots,j_m}$.

Theorem 2. Let $(X, \langle \cdot, \cdot | \cdot, \ldots, \cdot \rangle)$ be an n-inner product space and $A = \{a_1, a_2, \ldots, a_n\}$ in X be a linearly independent set. Then $(X_{j_1,\ldots,j_m}, \langle \cdot, \cdot \rangle_{j_1,\ldots,j_m})$ is an inner product space, with $\langle \cdot, \cdot \rangle_{j_1,\ldots,j_m}$ is a function defined in (3).

Proof. We show that $\langle \cdot, \cdot \rangle_{j_1,\ldots,j_m}$ is an inner product on $X_{j_1,\ldots,j_m}$. It is easy to see that

$$\langle \overline{x}, \overline{x} \rangle_{j_1,\ldots,j_m} = \langle \overline{x}, \overline{x} \rangle_{j_1} + \cdots + \langle \overline{x}, \overline{x} \rangle_{j_m} \geq 0.$$

Moreover, if $x = 0$, then it is obvious that $\langle \overline{x}, \overline{x} \rangle_{j_1,\ldots,j_m} = 0$. Conversely, if $\langle \overline{x}, \overline{x} \rangle_{j_1,\ldots,j_m} = 0$, then each term of the above equation is also 0. Based on equation (3), for each term of it we obtain

$$x = \sum_{i=1, i \neq j_1}^{n} \gamma_{1i} a_i, \quad \gamma_{1i} \in \mathbb{R}$$

$$\vdots$$

$$x = \sum_{i=1, i \neq j_m}^{n} \gamma_{mi} a_i, \quad \gamma_{mi} \in \mathbb{R}$$

with $j_1, \ldots, j_m \in \{1, \ldots, n\}$. In other words, x is linearly dependent to $A \setminus \{a_{j_k}\}$ for each $k \in \{1, 2, \ldots, m\}$. It implies x is linearly dependent to $A \setminus \{a_{j_1} \ldots, a_{j_m}\}$. We can write it as

$$\sum_{\substack{i=1 \\ i \neq j_1, \ldots j_m}}^{n} \beta_i a_i \ : \ \beta_i \in \mathbb{R}.$$

This leads to $x \in \text{span } A \setminus \{a_{j_1}, \ldots, a_{j_m}\}$ or $\overline{x} = \overline{0}$.

Moreover, it is straightforward to verify that we also have $\langle \overline{x}, \overline{y} \rangle_{j_1,\ldots,j_m} = \langle \overline{y}, \overline{x} \rangle_{j_1,\ldots,j_m}$, $\langle \alpha\overline{x}, \overline{y} \rangle_{j_1,\ldots,j_m} = \alpha \langle \overline{x}, \overline{y} \rangle_{j_1,\ldots,j_m}$ for any $\alpha \in \mathbb{R}$, and $\langle \overline{x} + \overline{y}, \overline{z} \rangle_{j_1,\ldots,j_m} = \langle \overline{x}, \overline{z} \rangle_{j_1,\ldots,j_m} + \langle \overline{y}, \overline{z} \rangle_{j_1,\ldots,j_m}$. □

Using the relation between an n-inner product and an n-norm on equation (2), we define a norm that is induced by the inner product defined on equation (3)

$$\begin{aligned} \|\overline{x}\|_{j_1,\ldots,j_m} &= \langle \overline{x}, \overline{x} \rangle_{j_1,\ldots,j_m}^{\frac{1}{2}} \\ &= (\langle x, x | a_1, \ldots, a_{j_1-1}, a_{j_1+1}, \ldots, a_n \rangle + \ldots \\ &\quad + \langle x, x | a_1, \ldots, a_{j_m-1}, a_{j_m+1}, \ldots, a_n \rangle)^{\frac{1}{2}} \\ &= \left(\|x, a_1, a_1, \ldots, a_{j_1-1}, a_{j_1+1}, \ldots, a_n\|^2 + \ldots \right. \\ &\quad \left. + \|x, a_1, \ldots, a_{j_m-1}, a_{j_m+1}, \ldots, a_n\|^2 \right)^{\frac{1}{2}} \\ &= \left(\|\overline{x}\|_{j_1}^2 + \cdots + \|\overline{x}\|_{j_m}^2 \right)^{\frac{1}{2}}. \end{aligned} \tag{5}$$

We will have the same result using equation (4). Then we obtain the following corollary.

Corollary 2. Let $(X, \langle \cdot, \cdot | \cdot, \ldots, \cdot \rangle)$ be an n-inner product space and $A = \{a_1, a_2, \ldots, a_n\}$ in X be a linearly independent set. Then $(X_{j_1,\ldots,j_m}, \| \cdot \|_{j_1,\ldots,j_m})$ is a normed space, with $\| \cdot \|_{j_1,\ldots,j_m}$ is an induced norm defined in (5).

The corollary above is a direct corollary from Theorem 2 and equation (5).

Furthermore, using this construction, we can get $\binom{n}{m}$ quotient spaces which are also inner product spaces. We collect these quotient spaces in a set and name it class-m. Since we construct these quotient spaces with respect to a linearly independent set containing n vectors $A = \{a_1, \ldots, a_n\}$, we can have n classes of quotient spaces. These inner products and norms that we defined will be our tools to investigate some aspects in n-inner product spaces.

In particular, for $m = n$ we actually observe an n-inner product space as an inner product space. The class-n will contain itself as a quotient space. The inner product on $(X, \langle \cdot, \cdot | \cdot, \ldots, \cdot \rangle)$ will be defined by

$$\langle \overline{x}, \overline{y} \rangle_{1,\ldots,n} = \sum \langle x, y | a_{j_2}, \ldots, a_{j_n} \rangle,$$

the sum is taken over $\{j_2, \ldots, j_n\} \subset \{1, \ldots, n\}$. Then $(X, \langle \cdot, \cdot \rangle_{1,\ldots,n})$ is an inner product space. This means $(X, \| \cdot \|_{1,\ldots,n})$ is a normed space with the induced norm defined by

$$\|\overline{x}\|_{1,\ldots,n} = \left(\sum \|x, a_{j_2}, \ldots, a_{j_n}\|^2 \right)^{\frac{1}{2}},$$

the sum is taken over $\{j_2, \ldots, j_n\} \subset \{1, \ldots, n\}$.

Lemma 2. Let $(X, \langle \cdot, \cdot | \cdot, \ldots, \cdot \rangle)$ be an n-inner product space and $A = \{a_1, a_2, \ldots, a_n\}$ in X be a linearly independent set. Then $x = y$ if and only if $x - y \in \text{span } A \setminus \{a_{j_1}, \ldots, a_{j_m}\} = A_{j_1,\ldots,j_m}$ for all $\{j_1, \ldots, j_m\} \subset \{1, \ldots, n\}$.

Proof. The proof is similar to proof of Lemma 1. The readers can see that on Lemma 1 we have n equations. Here, we have $\binom{n}{m}$ equations. By subtracting from each of the subsequent equations the first equation, we obtain $x = y$. $\square$

Similar to Lemma 1, if the condition $x - y \in A_{j_1,\dots,j_m}$ holds for all $\{j_1,\dots,j_m\} \subset \{1,\dots,n\}$, then it follows that $x = y$. Therefore, by considering all quotient spaces, we are able to identify the underlying element itself rather than merely its representation as a coset.

Example 2. Let $(\mathbb{R}^n, \langle\cdot,\cdot,\cdot,\cdot\rangle)$ be a 4-inner product space, with $n \geq 4$ and $A = \{a_1,a_2,a_3,a_4\}$ be a linearly independent set. The class-3 of $\mathbb{R}^n$ with respect to A consist of four quotient spaces, namely $\mathbb{R}^n_{1,2,3}, \mathbb{R}^n_{1,2,4}, \mathbb{R}^n_{1,3,4}$ and $\mathbb{R}^n_{2,3,4}$. These quotient spaces are inner product spaces and also normed spaces. Their inner product and norm are defined as

$\langle\overline{x},\overline{y}\rangle_{1,2,3} = \langle x,y|a_2,a_3,a_4\rangle + \langle x,y|a_1,a_3,a_4\rangle + \langle x,y|a_1,a_2,a_4\rangle$ and its induced norm is $\|x\|_{1,2,3} = \left(\|x,a_2,a_3,a_4\|^2 + \|x,a_1,a_3,a_4\|^2 + \|x,a_1,a_2,a_4\|^2\right)^{\frac{1}{2}}$;

$\langle\overline{x},\overline{y}\rangle_{1,2,4} = \langle x,y|a_2,a_3,a_4\rangle + \langle x,y|a_1,a_3,a_4\rangle + \langle x,y|a_1,a_2,a_3\rangle$ and its induced norm is $\|x\|_{1,2,4} = \left(\|x,a_2,a_3,a_4\|^2 + \|x,a_1,a_3,a_4\|^2 + \|x,a_1,a_2,a_3\|^2\right)^{\frac{1}{2}}$;

$\langle\overline{x},\overline{y}\rangle_{1,3,4} = \langle x,y|a_2,a_3,a_4\rangle + \langle x,y|a_1,a_2,a_4\rangle + \langle x,y|a_1,a_2,a_3\rangle$ and its induced norm is $\|x\|_{1,3,4} = \left(\|x,a_2,a_3,a_4\|^2 + \|x,a_1,a_2,a_4\|^2 + \|x,a_1,a_2,a_3\|^2\right)^{\frac{1}{2}}$;

$\langle\overline{x},\overline{y}\rangle_{2,3,4} = \langle x,y|a_1,a_3,a_4\rangle + \langle x,y|a_1,a_2,a_4\rangle + \langle x,y|a_1,a_2,a_3\rangle$ and its induced norm is $\|x\|_{2,3,4} = \left(\|x,a_1,a_3,a_4\|^2 + \|x,a_1,a_2,a_4\|^2 + \|x,a_1,a_2,a_3\|^2\right)^{\frac{1}{2}}$.

Here one can easily see that each term of the summation in each defined inner product is an inner product of a quotient space in class-1. For example we can write $\langle\overline{x},\overline{y}\rangle_{1,2,3} = \langle\overline{x},\overline{y}\rangle_1 + \langle\overline{x},\overline{y}\rangle_2 + \langle\overline{x},\overline{y}\rangle_3$, while for the induced norm we can write $\|x\|_{1,2,3} = \left(\|x\|_1^2 + \|x\|_2^2 + \|x\|_3^2\right)^{\frac{1}{2}}$.

Remark 1. We aim to interpret a point in an n-inner product space $(X, \langle\cdot,\cdot|\cdot,\dots,\cdot\rangle)$ as a "real point" not as a coset. We do so by observing all of its quotient space. For example, if we investigate $0 \in X$ then we can investigate $\overline{0}$ in each $X_{j_1,\dots,j_m}$ on class-m for all $\{j_1,\dots,j_m\} \subset \{1,\dots,n\}$. In other words we observe

$$\bigcap \overline{0}_{j_1,\dots,j_m}$$

to get 0 as a point, with $\overline{0}_{j_1,\dots,j_m} \in X_{j_1,\dots,j_m}$. The intersection is taken by $\{j_1,\dots,j_m\} \in \{1,\dots,n\}$. It applies to each vector in X that we observe.

Moreover, we will investigate some aspects of n-inner product space using these inner products and norms of classes of quotient spaces.

As we mentioned before, that all the quotient spaces constructed above are done with respect to a linearly independent set containing n vectors. We can choose any n linearly independent vectors. From here on, we will not mention the set explicitly, unless it is necessary.

1.2 *Some Topology Properties*

We start this section by defining weakly convergent sequences in an n-inner product space using the inner product defined earlier.

Definition 1. Let $(X, \langle\cdot,\cdot|\cdot,\dots,\cdot\rangle)$ be an n-inner product space. A sequence $\{x_k\} \subset X$ is said to be weakly convergent with respect to class-m to x if

$$\lim_{k\to\infty} \langle\overline{x_k},\overline{y}\rangle_{j_1,\dots,j_m} = \langle\overline{x},\overline{y}\rangle_{j_1,\dots,j_m}$$

for all $\overline{y} \in X_{j_1,\dots,j_m}$ and $\{j_1,\dots,j_m\} \subset \{1,\dots,n\}$.

The point $x \in X$ is called the limit point of the sequence x_n. One can see that this definition is well defined. We consider the sequence in all quotient spaces. Similar to Lemma 2, this implies that the limit point is identified as the underlying element itself, rather than as a coset.

Moreover, there are n classes of quotient spaces, then the above definition gives n types of weak convergence. The following theorem states that for any two classes we choose ($m \in \{1, \ldots, n\}$), the definitions are equivalent.

Theorem 3. Let $(X, \langle \cdot, \cdot | \cdot, \ldots, \cdot \rangle)$ be an n-inner product space. A sequence $\{x_k\} \subset X$ is a weakly convergent sequence with respect to class-m_1 if and only if it is a weakly convergent sequence with respect to class-m_2, with $m_1, m_2 \in \{1, \ldots, n\}$.

Proof. It suffices to show that a sequence $\{x_k\} \subset X$ is a weakly convergent sequence with respect to class-1 if and only if it is a weakly convergent sequence with respect to class-m, with $m \in \{1, \ldots, n\}$. Let $\{x_k\}$ be a weakly convergent sequence with respect to class-1, then for any $\varepsilon > 0$, there is an $N_1 \in \mathbb{N}$ such that for $k \geq N_1$ we have

$$\lim_{k\to\infty} \langle \overline{x_k}, \overline{y} \rangle_1 = \langle \overline{x}, \overline{y} \rangle_1$$

$$\vdots$$

$$\lim_{k\to\infty} \langle \overline{x_k}, \overline{y} \rangle_1 = \langle \overline{x}, \overline{y} \rangle_n$$

from these, we obtain

$$\begin{aligned} \lim_{k\to\infty} \langle \overline{x_k}, \overline{y} \rangle_{j_1,\ldots,j_m} &= \lim_{k\to\infty} \langle \overline{x_k}, \overline{y} \rangle_{j_1} + \cdots + \lim_{k\to\infty} \langle \overline{x_k}, \overline{y} \rangle_{j_m} \\ &= \langle \overline{x}, \overline{y} \rangle_{j_1} + \cdots + \langle \overline{x}, \overline{y} \rangle_{j_m} \\ &= \langle \overline{x}, \overline{y} \rangle_{j_1,\ldots,j_m} \end{aligned}$$

for any $\{j_1, \ldots, j_m\} \subset \{1, \ldots, n\}$. This implies that $\{x_k\}$ is a weakly convergent sequence with respect to class-m. Conversely, let $\{x_k\}$ be a weakly convergent sequence with respect to class-m, then we have

$$\lim_{k\to\infty} \langle \overline{x_k}, \overline{y} \rangle_{j_1,\ldots,j_m} = \langle \overline{x}, \overline{y} \rangle_{j_1,\ldots,j_m} \tag{6}$$

for any $\{j_1, \ldots, j_m\} \subset \{1, \ldots, n\}$. From (6) we actually have $\binom{n}{m}$ equations. As a consequence, we have

$$\lim_{k\to\infty} \langle \overline{x_k}, \overline{y} \rangle_i = \langle \overline{x}, \overline{y} \rangle_i$$

for any $i \in \{1, \ldots, n\}$. Therefore, $\{x_k\}$ is a weakly convergent sequence with respect to class-1. This ends the proof. □

Note that The conclusion follows by a standard elimination argument on (6). Moreover, the proof of Theorem (3) used class-1 as a bridge to connect all other classes. Later, we will find some theorems that will be proved using the same technique.

Proposition 1. Let x_k be a weakly convergent sequence with respect to class-m, then its limit point is unique.

Proof. Let x, x' be the limit points of x_k, then

$$\langle \overline{x}, \overline{y} \rangle_{j_1,\ldots,j_m} = \lim_{k\to\infty} \langle \overline{x_k}, \overline{y} \rangle_{j_1,\ldots,j_m} = \langle \overline{x'}, \overline{y} \rangle_{j_1,\ldots,j_m}.$$

Here we have

$$\langle \overline{x}, \overline{y} \rangle_{j_1,\dots,j_m} = \langle \overline{x'}, \overline{y} \rangle_{j_1,\dots,j_m}$$

for any $\{j_1, \dots, j_m\} \subset \{1, \dots, n\}$. This implies

$$\langle \overline{x} - \overline{x'}, \overline{y} \rangle_{j_1,\dots,j_m} = 0.$$

Since it is applied for all $\overline{y} \in X_{j_1,\dots,j_m}$ and $\{j_1, \dots, j_m\} \subset \{1, \dots, n\}$, we have $x = x'$. We say the limit point is unique. □

In Proposition 1, we consider all quotient spaces simultaneously. This implies that $x - x' = 0$, and hence $x = x'$. Next, we give a definition of a strongly convergent sequence in an n-inner product space with respect to a class of quotient spaces.

Definition 2. Let $(X, \langle \cdot, \cdot | \cdot, \dots, \cdot \rangle)$ be an n-inner product space. A sequence $\{x_k\} \subset X$ is said to be strongly convergent with respect to class-m to x if

$$\lim_{k \to \infty} \| \overline{x_k} - \overline{x} \|_{j_1,\dots,j_m} = 0$$

for any $\{j_1, \dots, j_m\} \subset \{1, \dots, n\}$.

Definition 2 gives us n types of strongly convergent sequences, since we have n classes of quotient spaces. The following theorem states that all the strongly convergent sequence types are equivalent for any class we choose.

Theorem 4. Let $(X, \langle \cdot, \cdot | \cdot, \dots, \cdot \rangle)$ be an n-inner product space. A sequence $\{x_k\} \subset X$ is a strongly convergent sequence with respect to class-m_1 if and only if it is a strong convergent sequence with respect to class-m_2, with $m_1, m_2 \in \{1, \dots, n\}$.

Proof. We prove this theorem by using class-1 as a bridge. Let $\{x_k\} \in X$ is a sequence that strongly converges with respect to class-m to a point x. Based on the definition we have

$$\lim_{k \to \infty} \| \overline{x_k} - \overline{x} \|_{j_1,\dots,j_m} = \lim_{k \to \infty} \| \overline{x_k} - \overline{x} \|_{j_1} + \dots + \lim_{k \to \infty} \| \overline{x_k} - \overline{x} \|_{j_m} = 0$$

for any $\{j_1, \dots, j_m\} \subset \{1, \dots, n\}$. Therefore, each term of the summation in the above equation equals 0. We write

$$\lim_{k \to \infty} \| \overline{x_k} - \overline{x} \|_i = 0$$

for any $i \in \{1, \dots, n\}$. This implies, the sequence $\{x_k\}$ is a sequence that strongly converges with respect to class-1 to a point $x \in X$. Conversely, let $\{x_k\} \in X$ is a sequence that strongly converges with respect to class-1 to a point x. We have

$$\lim_{k \to \infty} \| \overline{x_k} - \overline{x} \|_i = 0$$

for any $i, m \in \{1, \dots, n\}$. Then we have

$$\lim_{k \to \infty} \| \overline{x_k} - \overline{x} \|_{j_1,\dots,j_m} = \lim_{k \to \infty} \| \overline{x_k} - \overline{x} \|_{j_1} + \dots + \lim_{k \to \infty} \| \overline{x_k} - \overline{x} \|_{j_m} = 0$$

for any $\{j_1, \dots, j_m\} \subset \{1, \dots, n\}$. With this, we know that the sequence $\{x_k\}$ is a sequence that strongly converges with respect to class-m to a point $x \in X$, with $m \in \{1, \dots, n\}$. This proves the theorem. □

Proposition 2. If $\{x_k\}$ is a strongly convergent sequence with respect to class-m, then its limit point is unique.

Proof. Let x, x' be the limit points of x_k. Then we have

$$\lim_{k\to\infty} \|\overline{x_k} - \overline{x}\|_{j_1,\ldots,j_m} = \lim_{k\to\infty} \|\overline{x_k} - \overline{x'}\|_{j_1,\ldots,j_m}.$$

One can see that $\overline{x} = \overline{x'}$ this applies in each quotient space $X_{j_1,\ldots,j_m}$ for any $\{j_1,\ldots,j_m\}$. As a result we have $x = x'$. □

The readers will realize that in the above proof, we use Remark 1 on the last part. Note that we can investigate a convergent sequence (either strongly or weakly) with respect to any class of quotient spaces. The limit point of the sequence will be the same. Next, we will observe a Cauchy sequence on n-normed spaces using tools that we have.

Definition 3. Let $(X, \langle\cdot,\cdot|\cdot,\ldots,\cdot\rangle)$ be an n-inner product space. A sequence $\{x_k\} \subset X$ is said to be weakly Cauchy with respect to class-m if

$$\lim_{k,l\to\infty} \langle\overline{x_k} - \overline{x_l}, \overline{y}\rangle_{j_1,\ldots,j_m} = 0$$

for all $\overline{y} \in X_{j_1,\ldots,j_m}$ and $\{j_1,\ldots,j_m\} \subset \{1,\ldots,n\}$.

Similar to the weakly convergent sequence, there are n types of the weakly convergent sequence since we have n classes of quotient spaces. We also have all the types of weakly convergent sequence for any class we choose are equivalent. It is stated in the following theorem.

Theorem 5. Let $(X, \langle\cdot,\cdot|\cdot,\ldots,\cdot\rangle)$ be an n-inner product space. A sequence $\{x_k\} \subset X$ is a weakly Cauchy sequence with respect to class-m_1 if and only if it is a weakly Cauchy sequence with respect to class-m_2, with $m_1, m_2 \in \{1,\ldots,n\}$.

Proof. This proof is analogous to the proof of Theorem 3. □

The readers will easily see that the proof of Theorem 5 also uses class-1 as a bridge. Moreover, the following property establishes that weak convergence implies the weak Cauchy property.

Corollary 3. If $\{x_k\}$ is a weakly convergent sequence with respect to class-m, then it is also weakly Cauchy with respect to class-m for an $m \in \{1,\ldots,n\}$.

Proof. Using the Cauchy-Schwarz inequality, we have

$$|\langle\overline{x_k} - \overline{x}, \overline{y}\rangle_{j_1,\ldots,j_m}| \leq \|\overline{x_k} - \overline{x}\|_{j_1,\ldots,j_m} \|\overline{y}\|_{j_1,\ldots,j_m}$$

for any $\{j_1,\ldots,j_m\} \subset \{1,\ldots,n\}$.

Since the right-hand side tends to 0 as k tends to ∞, it applies to the left-hand side. We conclude that weakly convergent implies weakly Cauchy. □

If the converse is true, then the space is called weakly n-Hilbert Space.

Definition 4. Let $(X, \langle\cdot,\cdot|\cdot,\ldots,\cdot\rangle)$ be an n-inner product space. A sequence $\{x_k\} \subset X$ is said to be strongly Cauchy with respect to class-m if

$$\lim_{k,l\to\infty} \|\overline{x_k} - \overline{x_l}\|_{j_1,\ldots,j_m} = 0$$

for any $\{j_1,\ldots,j_m\} \subset \{1,\ldots,n\}$.

Since there are n classes of quotient spaces, we also have n types of strongly convergent sequences. Here we give a theorem about relations among all the types of strongly convergent sequence.

Theorem 6. Let $(X, \langle \cdot, \cdot | \cdot, \ldots, \cdot \rangle)$ be an n-inner product space. A sequence $\{x_k\} \subset X$ is a weakly Cauchy sequence with respect to class-m_1 if and only if it is a weakly Cauchy sequence with respect to class-m_2, with $m_1, m_2 \in \{1, \ldots, n\}$.

Proof. This proof is analogous to the proof of Theorem 3. □

One can see that we prove this theorem using the same method, that is using class-1 as a bridge.

Corollary 4. If $\{x_k\}$ is a strongly convergent sequence with respect to class-m then it is also a strongly Cauchy sequence with respect to class-m, for an $m \in \{1, \ldots, n\}$.

Proof. Let $\{x_k\}$ strongly converges with respect to class-m to a point x. Then we have

$$\lim_{k\to\infty} \|\overline{x_k} - \overline{x}\|_{j_1,\ldots,j_m} = 0, \text{ and } \lim_{l\to\infty} \|\overline{x_l} - \overline{x}\|_{j_1,\ldots,j_m} = 0.$$

On the other hand, we also have

$$\|\overline{x_k} - \overline{x_l}\|_{j_1,\ldots,j_m} \leq \|\overline{x_k} - \overline{x}\|_{j_1,\ldots,j_m} + \|\overline{x_l} - \overline{x_l}\|_{j_1,\ldots,j_m}.$$

The right-hand side tends to 0 as $k, l \to \infty$, so does the left-hand side. This leads us to conclude that $\{x_k\}$ is a strongly Cauchy sequence. □

If the converse of the Corollary 4 is true, then the space is called a strong n-Hilbert space.

Remark 2. Since the strongly convergent and Cauchy imply weakly convergent and Cauchy, respectively, the strongly n-Hilbert space implies weakly n-Hilbert space.

Furthermore, in a class of quotient spaces there is more than one quotient space (except for class-n). Do we have to use all the quotient spaces to investigate the above topological properties? To answer this question, we give a simple example below.

Example 3. Let $(\mathbb{R}^4, \langle \cdot, \cdot | \ldots, \cdot \rangle_S)$ be a 4-inner product space with standard n-inner product space defined by

$$\langle u, v | x_2, x_3, x_4 \rangle_S := \begin{vmatrix} \langle u, v\rangle & \langle u, x_2\rangle & \langle u, x_3\rangle & \langle u, x_4\rangle \\ \langle x_2, v\rangle & \langle x_2, x_2\rangle & \langle x_2, x_3\rangle & \langle x_2, x_4\rangle \\ \langle x_3, v\rangle & \langle x_3, x_2\rangle & \langle x_3, x_3\rangle & \langle x_3, x_4\rangle \\ \langle x_4, v\rangle & \langle x_4, x_2\rangle & \langle x_4, x_3\rangle & \langle x_4, x_4\rangle \end{vmatrix}.$$

Firstly, we choose $A = \{a_1, a_2, a_3, a_4\} = \{(1,0,0,0), (0,1,0,0), (0,0,1,0), (0,0,0,1)\}$, a standard basis vectors set in $\mathbb{R}^4$. In class-2 we have six quotient spaces, namely $\mathbb{R}^4_{1,2}, \mathbb{R}^4_{1,3}, \mathbb{R}^4_{1,4}, \mathbb{R}^4_{2,3}, \mathbb{R}^4_{2,4}, \mathbb{R}^4_{3,4}$. Consider a sequence $x_k = \{(\frac{1}{k}, \frac{1}{k}, \frac{1}{k}, \frac{1}{k}) : k \in \mathbb{N}\}$. We know that the sequence x_n is a (weakly and strongly) convergent sequence. Let $x = (0,0,0,0)$, consider two quotients spaces in class-2, namely $\mathbb{R}^4_{1,3}$ and $\mathbb{R}^4_{2,4}$ with their inner product. Now, we are investigating the sequence using the quotient spaces. We can see that

$$\begin{aligned} \lim_{k\to\infty} \langle \overline{x_k}, \overline{y}\rangle_{1,3} &= \lim_{k\to\infty} \langle x_k, y | a_2, a_3, x_4\rangle_S + \lim_{k\to\infty} \langle x_k, y | a_1, a_2, x_4\rangle_S \\ &= 0 \\ &= \langle \overline{x}, \overline{y}\rangle_{1,3}, \end{aligned}$$

and

$$\begin{aligned} \lim_{k\to\infty} \langle \overline{x_k}, \overline{y}\rangle_{2,4} &= \lim_{k\to\infty} \langle x_k, y | a_1, a_3, x_4\rangle_S + \lim_{k\to\infty} \langle x_k, y | a_1, a_2, x_3\rangle_S \\ &= 0 \\ &= \langle \overline{x}, \overline{y}\rangle_{2,4}. \end{aligned}$$

Using these two quotient spaces, it is sufficient to say that x_k weakly converges to x. We will obtain the same conclusion if we choose another two quotient spaces, namely $\mathbb{R}^4_{1,2}$ and $\mathbb{R}^4_{3,4}$, with each inner product to investigate the same sequence.

On the other hand, consider a sequence $x_t = \{(t,0,0,0) : t \in \mathbb{N}\}$. Obviously, this sequence is not a (weakly or strongly) convergent sequence. Recall that the standard n-norm in $\mathbb{R}^4$ is defined by

$$\|x_1,x_2,x_3,x_4\|_S := \begin{vmatrix} \langle x_1,x_1\rangle & \langle x_1,x_2\rangle & \langle x_1,x_3\rangle & \langle x_1,x_4\rangle \\ \langle x_2,x_1\rangle & \langle x_2,x_2\rangle & \langle x_2,x_3\rangle & \langle x_2,x_4\rangle \\ \langle x_3,x_1\rangle & \langle x_3,x_2\rangle & \langle x_3,x_3\rangle & \langle x_3,x_4\rangle \\ \langle x_4,x_1\rangle & \langle x_4,x_2\rangle & \langle x_4,x_3\rangle & \langle x_4,x_4\rangle \end{vmatrix}.$$

Moreover, choose two quotient spaces $\mathbb{R}^4_{2,3}$ and $\mathbb{R}^4_{3,4}$ with their inner product. This time, we investigate the sequence x_t using its quotient spaces of class-2. We have

$$\begin{aligned} \lim_{t\to\infty}\|\overline{x_t}-\overline{x}\|_{2,3} &= \lim_{t\to\infty}\|x_t-x,a_1,a_3,x_4\|_S + \lim_{t\to\infty}\|x_t-x,a_1,a_2,x_4\|_S \\ &= \lim_{t\to\infty}\|x_t,a_1,a_3,x_4\|_S + \lim_{t\to\infty}\|x_t,a_1,a_2,x_4\|_S \\ &= 0. \end{aligned}$$

The result is 0 since x_t is linearly dependent to a_1. This implies $\|x_t,a_1,a_3,x_4\|_S = 0 = \|x_t,a_1,a_2,x_4\|_S$. We also have

$$\begin{aligned} \lim_{t\to\infty}\|\overline{x_t}-\overline{x}\|_{3,4} &= \lim_{t\to\infty}\|x_t-x,a_1,a_2,x_4\|_S + \lim_{t\to\infty}\|x_t-x,a_1,a_2,x_3\|_S \\ &= \lim_{t\to\infty}\|x_t,a_1,a_2,x_4\|_S + \lim_{t\to\infty}\|x_t,a_1,a_2,x_3\|_S \\ &= 0. \end{aligned}$$

The result is 0 since x_t is linearly dependent to a_1. If we only use these two quotient spaces of class-2, we will say that the sequence x_k strongly converges to x. This leads to a false conclusion. But if we add another quotient space of class-2, namely $\mathbb{R}^4_{1,2}$, we have

$$\begin{aligned} \lim_{t\to\infty}\|\overline{x_t}-\overline{x}\|_{1,2} &= \lim_{t\to\infty}\|x_t-x,a_2,a_3,x_4\|_S + \lim_{t\to\infty}\|x_t-x,a_1,a_3,x_4\|_S \\ &= \lim_{t\to\infty}\|x_t,a_2,a_3,x_4\|_S + \lim_{t\to\infty}\|x_t,a_1,a_3,x_4\|_S \\ &= \lim_{t\to\infty}\|x_t,a_2,a_3,x_4\|_S + 0 \\ &= \lim_{t\to\infty} t^2, \end{aligned}$$

which implies the sequence x_t is not a strongly convergent sequence.

From the above example, we can see that there is a condition on how we can choose quotient spaces such that we can investigate the topological properties. We write it in the following remark.

Remark 3. The above topological properties can be investigated by quotient spaces of the class-m ($m \in \mathbb{N}, m \in \{1,\dots,n\}$) by choosing quotient spaces $X_{j_1,\dots,j_m}$ such that

$$\bigcup\{j_1,\dots,j_m\} \supseteq \{1,\dots,n\}.$$

Moreover, the quotients spaces that we choose on a class-m is at least $\lceil \frac{n}{m} \rceil$. As a consequence, if we use class-1 or class-n of quotient spaces we have to choose all the quotient spaces in it to satisfy the above condition. For other classes, we do not have to choose all quotient spaces.

Conclusion

Aspects of an n-inner product space can be investigated with respect to inner products on norms of its quotient spaces. These inner products and norms are derived from its n-inner product. This approach provides an alternative framework for observing n-inner product spaces. Most researchers viewed some aspects of n-inner product using n-inner product defined on the space (see, for instance [13, 19, 20]). Here we provide a new view point to observe some aspects namely weak and strong convergence, Cauchy sequences, and completeness. We show that these aspects can be defined in several ways, providing additional approaches to study n-inner product spaces. We also show that these definitions are equivalent, so any of them can be used to study the corresponding aspect. We find that we do not need to use all norms or inner products of a class. We give a condition to select minimal norms or inner products to observe an aspect of an n-inner product space. This perspective offers a more flexible and efficient way to study the structure of n-inner product spaces.

Acknowledgments

The authors would like to thank the Ministry of Education, Culture, Research, and Technology for its support of this research. The first and third authors also acknowledge support from Pattimura University. The authors express their gratitude to the editor and anonymous reviewers for their valuable comments that helped improve the quality of this work.

Author Contributions

H. Batkunde constructed the quotient spaces, formulated the concepts and definitions of the topological properties, and proved the main theorems. M. Nur was responsible for verifying the logical flow of the manuscript, checking the proofs, and assisting the first author in preparing the article. M.I. Tilukay contributed to the refinement of the theoretical framework, validation of the results, and improvement of the overall presentation of the manuscript. All authors participated in revising the manuscript and approved the final version for submission. All authors contributed equally to this work.

Conflict of Interest

The authors declare no conflict of interest.

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