

On limited and almost limited operators between Banach lattices

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This article investigates theoretical questions concerning the class of (almost) limited operators on Banach lattices, focusing on their precise relations to L -weakly compact, semi-compact, and Dunford–Pettis operators. The authors establish several conditions under which bounded operators are AM-compact and limited, and characterize when regular operators are almost limited. Furthermore, several new notions of collectively qualified sets of operators are introduced. Using results obtained for individual operators, the authors derive new structural results concerning these collectively qualified sets. The lattice approximation property for collectively M -weakly compact families of operators is also thoroughly investigated. Finally, the paper examines some related topics in functional analysis, focusing on the specific behavior of various kinds of operators acting between Banach lattices and Banach spaces. The presented theorems extend existing results, providing deeper insights into the topological and order properties of bounded operators, and establishing new connections between collective compactness and lattice structures.

Keywords: Banach lattice, Banach space, Schur property, limited set, L -weakly compact set, semi-compact operator, Dunford–Pettis operator, collectively qualified set of operators, enveloping norm.

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Introduction

Limited operators and their relations to $L(M)$ -weakly compact, semi-compact, Dunford–Pettis, and other classes of operators attract permanent attention of many researchers (see, for example, [1–3] and references therein). Collectively qualified families of operators were studied recently in [4–6]. The aim of this work is to find conditions to ensure inclusion relation among major classes of operators that are in use. In trying to do so, we also touch upon properties of Banach lattices and note a method of producing new classes of operators in them. Finally, the collective versions of the results obtained for a single operator are investigated.

Throughout the paper, all vector spaces are real, operators are linear; letters X , Y , and Z denote Banach spaces; E , F , and G denote Banach lattices. A subset C of X is called *bounded* if C is norm bounded. We denote by B_X the closed unit ball of X , by $L(X, Y)$ ($K(X, Y)$, $W(X, Y)$) the space of bounded (compact, weakly compact) operators from X to Y , by Id_X the identity operator on X , by E_+ the positive cone of E , by $\text{sol}(A) = \bigcup_{a \in A} [-|a|, |a|]$ the solid hull of $A \subseteq E$, by $x_\alpha \downarrow 0$ a decreasing net (x_α) with $\inf_\alpha x_\alpha = 0$, and by E^a the order continuous part of E .

The following well-known lemma will be permanently used below.

Lemma 1. ([7, Theorem 4.59] and [8, 2.4.14]) The following assertions are equivalent:

1. The norm on E^* is order continuous.
2. E^* is a KB-space.
3. Every disjoint bounded sequence in E is weakly null.

Definition 1. A bounded subset C

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- 1) of X is *limited* if each w^* -null sequence in X^* is uniformly null on C ;
- 2) of X is *Dunford-Pettis* or a *DP set* if each weakly null sequence in X^* is uniformly null on C ;
- 3) of E is *L-weakly compact* or an *Lwc set* if each disjoint sequence in $\text{sol}(C)$ is norm null;
- 4) of E is *almost order bounded* if, for each $\varepsilon > 0$ there exist $u_\varepsilon \in E_+$ satisfying $C \subseteq [-u_\varepsilon, u_\varepsilon] + \varepsilon B_X$.

It is well known that B_X is not limited unless $\dim(X) < \infty$ (however, B_{c_0} is a limited subset of ℓ^∞ by Phillips' Lemma [7, Theorem 4.67]). For every Lwc subset A of E , we have $A \subseteq E^a$ [8, p. 212]. The following technical lemma is an extension of the Grothendieck theorem.

Lemma 2. Let C be a bounded subset of X (or E). If, for each $\varepsilon > 0$, there exists

- 1) a relatively compact subset $C_\varepsilon \subseteq X$, $C \subseteq C_\varepsilon + \varepsilon B_X$, then C is relatively compact;
- 2) a relatively weakly compact $C_\varepsilon \subseteq X$, $C \subseteq C_\varepsilon + \varepsilon B_X$, then C is relatively weakly compact;
- 3) a limited $C_\varepsilon \subseteq E$, $C \subseteq C_\varepsilon + \varepsilon B_E$, then C is limited;
- 4) an almost limited $C_\varepsilon \subseteq E$, $C \subseteq C_\varepsilon + \varepsilon B_E$, then C is almost limited.

Item 1 of Lemma 2 is a standard fact. For item 2, we refer to [7, Theorem 3.44]. Items 3 and 4 are contained in [9, Lemma 2.2].

The following two definitions will be permanently used through the paper.

Definition 2. A Banach space X has:

- 1) the *Dunford-Pettis property* (or $X \in (\text{DPP})$) if $f_n(x_n) \rightarrow 0$ for each weakly null (x_n) in X and each weakly null (f_n) in X^* [7, p. 341];
- 2) the *Gelfand-Phillips property* (or $X \in (\text{GPP})$) if limited subsets of X are relatively compact [10].

Separable Banach spaces and reflexive Banach spaces are in (GPP) by [10]. The closed unit ball of ℓ^∞ is almost limited, but the closed unit ball of c_0 is not almost limited.

Definition 3. A Banach lattice E has:

- 1) the *dual disjoint w^* -property* (or $E \in (\text{DDw}^*\text{P})$) if every disjoint bounded sequence in E^* is w^* -null;
- 2) the *positive Schur property* (or $E \in (\text{PSP})$) if each w -null sequence in E_+ is norm null [11];
- 3) the *dual disjoint Schur property* (or $E \in (\text{DDSP})$) if each disjoint w^* -null sequence in E^* is norm null [9, Definition 3.2];
- 4) the *property (d)* (or $E \in (\text{d})$) if, for every disjoint w^* -null (f_n) in E^* , the sequence $(|f_n|)$ is also w^* -null [12, Definition 1];
- 5) the *positive Grothendieck property* (or $E \in (\text{PGP})$) if each w^* -null sequence in E_+^* is weakly null [11, p. 760];
- 6) the *disjoint Grothendieck property* (or $E \in (\text{DGP})$) if disjoint w^* -null sequences in E^* are w -null.

The disjoint Grothendieck property was studied in [13] under the name “weak Grothendieck property”. The following assertion illustrates relations of the dual disjoint w^* -property with another ones.

Assertion 1. For every Banach lattice E the following holds:

$$E \in (\text{DGP}) \cap (\text{DDw}^*\text{P}) \implies E^{**} \text{ is a KB-space} \implies E \in (\text{DDw}^*\text{P}).$$

Proof. Let $E \in (\text{DGP}) \cap (\text{DDw}^*\text{P})$ but E^{**} is not a KB-space. Then, by Lemma 1, there is a disjoint bounded (f_n) in E^* which is not weakly null. However $f_n \xrightarrow{w^*} 0$ because $E \in (\text{DDw}^*\text{P})$, and hence $f_n \xrightarrow{w} 0$ as $E \in (\text{DGP})$. The obtained contradiction proves the first implication.

For the second implication, let E^{**} be KB. By Lemma 1, every disjoint (f_n) in B_{E^*} is weakly null and hence w^* -null. It follows $E \in (\text{DDw}^*\text{P})$. $\square$

Recall that E has *sequentially w^* -continuous lattice operations* whenever $f_n \xrightarrow{w^*} 0$ in E^* implies $|f_n| \xrightarrow{w^*} 0$. Any AM-space with unit has sequentially w^* -continuous lattice operations. The property (d) is weaker than the sequential w^* -continuity of the lattice operations, e.g. in ℓ^∞ . It was proved in

[11, Proposition 1.4] that every Dedekind σ -complete E has the property (d) [9, Corollary 2.4], and it was observed in [11, Remark 1.5] that $\ell^\infty/c_0 \in (d)$ but ℓ^∞/c_0 is not Dedekind σ -complete. The lattice $L^p[0, 1]$, $1 \leq p \leq \infty$, has the property (d), whereas $C[0, 1] \notin (d)$ (Example 1).

The following assertion [9, Theorem 3.3] describes the impact of the property (d) on dual Schur properties.

Assertion 2. The following holds for Banach lattices:

1. $(\text{DDSP}) \subseteq (\text{DPSP})$.
2. $((\text{DDSP}) \cap (d)) = ((\text{DPSP}) \cap (d))$.

Clearly $((\text{PGP}) \cap (d)) \subseteq ((\text{DGP}) \cap (d))$. For this inclusion, the property (d) is essential, as $C[0, 1] \in (\text{PGP}) \setminus (\text{DGP})$ by Example 1 below. We have no example of $E \in (\text{DGP}) \setminus (\text{PGP})$. The next example shows that the property (d) is essential in Assertion 2(2).

Example 1. Consider the Banach lattice $E = C[0, 1]$. Let (f_n) be a positive w^* -null sequence in E^* . It follows from $\|f_n\| = f_n(\mathbf{1}_{[0,1]}) \rightarrow 0$ that $C[0, 1] \in (\text{DPSP})$, and hence $C[0, 1] \in (\text{PGP})$. Take the sequence (r_n) of Rademacher functions on $[0, 1]$ and define a disjoint w^* -null sequence in E^* as follows: $g_n = 2^{n+1} \cdot r_{2n} \cdot \mathbf{1}_{[\frac{1}{2^{n+1}}, \frac{1}{2^n}]}$. Then (g_n) is a disjoint w^* -null sequence in E^* , yet $\|g_n\| = 1$ for all $n \in \mathbb{N}$. Therefore $C[0, 1] \notin (\text{DDSP})$, and hence by Assertion 2, $C[0, 1] \notin (d)$. Note that the set $\{g_n : n \in \mathbb{N}\}$ is not even relatively weakly compact. Indeed, for $y := \sum_{n=1}^\infty r_{2n} \cdot \mathbf{1}_{[\frac{1}{2^{n+1}}, \frac{1}{2^n}]} \in L^\infty[0, 1] \subseteq E^*$, where $y(g_n) \equiv 1$ for all $n \in \mathbb{N}$. It also shows $C[0, 1] \notin (\text{DGP})$.

The following two definitions are permanently used below.

Definition 4. A bounded operator $T : X \rightarrow Y$ is

- 1) *Dunford–Pettis* (or $T \in \text{DP}(X, Y)$) if T takes weakly null sequences to norm null ones;
- 2) *limited* (or $T \in \text{Lim}(X, Y)$) if $T(B_X)$ is a limited subset of Y .
- 3) A bounded operator $T : E \rightarrow Y$ is *almost M-weakly compact* (or $T \in \text{aMwc}(E, Y)$) if $f_n(Tx_n) \rightarrow 0$ for each weakly convergent (f_n) in Y^* and disjoint bounded (x_n) in E [14, Definition 2.2].

For instance, $\text{Id}_{c_0} \in \text{aMwc}(c_0)$. The canonical embedding of c_0 into ℓ^∞ is limited, but not compact. Note also that order intervals of a Banach lattice are not necessarily limited (e.g., $[-\mathbf{1}, \mathbf{1}]$ in c is not limited, where $\mathbf{1}$ denotes the constantly one sequence).

Definition 5. A bounded operator

- 1) $T : E \rightarrow Y$ is *AM-compact* ($T \in \text{AMc}(E, Y)$) if $T[-x, x]$ is relatively compact for every $x \in E_+$.
- 2) $T : E \rightarrow Y$ is *M-weakly compact* ($T \in \text{Mwc}(E, Y)$) if $\|Tx_n\| \rightarrow 0$ for every disjoint bounded (x_n) in E [8, Def. 3.6.9 iv)].
- 3) $T : X \rightarrow F$ is *semi-compact* ($T \in \text{semi-K}(X, F)$) if it carries bounded subsets of X onto almost order bounded subsets of F (or, equivalently, for each $\varepsilon > 0$, there exists $u_\varepsilon \in F_+$ with $T(B_X) \subseteq [-u_\varepsilon, u_\varepsilon] + \varepsilon B_F$).
- 4) $T : X \rightarrow F$ is *L-weakly compact* ($T \in \text{Lwc}(X, F)$) if T carries bounded subsets of X onto Lwc subsets of F .
- 5) $T : X \rightarrow F$ is *almost L-weakly compact* ($T \in \text{aLwc}(X, F)$) if T carries relatively weakly compact subsets of X onto Lwc subsets of F [14, Definition 2.1] (or, equivalently, if $T(X) \subseteq F^a$ and $f_n(Tx_n) \rightarrow 0$ for each w -null (x_n) in X and disjoint (f_n) in F^*).
- 6) $T : X \rightarrow F$ is *limitedly L-weakly compact* ($T \in \text{l-Lwc}(X, F)$) if T maps limited subsets of X onto Lwc subsets of F (or, equivalently, $T^*f_n \xrightarrow{w} 0$ for each disjoint bounded (f_n) in F^*).

The limitedly L(M)wc-operators were defined in [1] and [15] (in the second paper they were named weak L(M)wc-operators). It is well known that $\text{semi-K}(X, F) = \text{Lwc}(X, F)$ for every X , whenever the norm in F is o -continuous. Note that [7, Theorem 5.63] yields a method of producing many classes of operators. For example, if we consider sets of the form $A = T(C)$ with any weakly compact subset C of

X and $B = B_{F^*}$, then we obtain almost L(M)wc operators [14], whereas the choice of sets of the form $A = T[0, x]$ with any order interval $[0, x]$ in E and $B = B_{F^*}$ produces order L(M)wc operators [16].

In Section 1 we prove Theorem 1 that gives a condition on bounded operators to be AM-compact and Theorem 2 that gives condition on bounded operators to be limited. In Section 2, Theorem 12 provides conditions under which every regular operator is almost limited. Theorem 16 of Section 3 gives a tool for lattice approximation for collectively Mwc families of operators.

For unexplained terminology and notation, we refer to [7, 8, 10].

1 The effect of w^* -continuity of lattice operations in the dual

In this section, we study how order continuity of norm in E^* and continuity of lattice operations in E^* (which imply order intervals being limited in E [9]) affect operators. The limited order intervals will play an important role in what follows. So let us recall that lattice operations are weakly continuous if, for every w -null (x_n) in E , $|x_n| \xrightarrow{w} 0$. We need the following lemma [9, Proposition 2.3(1)].

Lemma 3. The lattice operations of E^* are w^* -sequentially continuous iff every almost order bounded subset of E is limited.

Proposition 1. Let $T \in \text{semi-K}(X, F)$.

1. If F has limited order intervals, then $T \in \text{Lim}(X, F)$.
2. If F has compact order intervals, then $T \in \text{K}(X, F)$.
3. If F has o -continuous norm, then $T \in \text{W}(X, F)$.

Proof. 1. We need to show that $T(B_X)$ is limited. Since $T \in \text{semi-K}(X, F)$ then, for every $\varepsilon > 0$, there exists $u_\varepsilon \in F_+$ with $T(B_X) \subseteq [-u_\varepsilon, u_\varepsilon] + \varepsilon B_F$. As each order interval in F is limited, $T(B_X)$ is limited by Lemma 2(3).

2. Similar to 1.

3. Since $F^a = F$, the order intervals in F are w -compact. The rest of the proof is similar to 1. $\square$

Corollary 1. Let E^* be a KB-space and let F have limited order intervals. The following holds:

1. $\text{DP}_+(E, F) \subseteq \text{Lim}(E, F)$.
2. If $E \in (\text{DPP})$, then $\text{W}_+(E, F) \subseteq \text{Lim}(E, F)$.

Proof. 1. Let $T \in \text{DP}_+(E, F)$. By [17, Theorem 3.7.10], $T \in \text{Mwc}(E, F)$. Thus $T \in \text{semi-K}(E, F)$, and hence $T \in \text{Lim}(E, F)$ by Proposition 1(1).

2. If $T \in \text{W}_+(E, F)$ then $T \in \text{DP}(E, F)$ by [7, Theorem 5.82]. The rest of the proof is as in 1. $\square$

Since L-weakly compact, regular M-weakly compact, and positive operators dominated by semi-compact operators are semi-compact ([7, Theorem 5.71, Theorem 5.72]), they are (almost) limited independent of their ranges due to Proposition 1. A semi-compact operator need not be compact, weakly compact, L- or M-weakly compact. The identity operator on ℓ^∞ is semi-compact, but it has none of the compactness properties mentioned.

Corollary 2. Let F have o -continuous norm and limited order intervals. Then, for every E ,

1. each semi-compact operator $T : E \rightarrow F$ is compact;
2. each order bounded Mwc operator $T : E \rightarrow F$ is compact;
3. each Lwc operator $T : E \rightarrow F$ is compact;
4. if E^* has o -continuous norm, then each order bounded aDP operator $T : E \rightarrow F$ is compact.

Proof. 1. Let $T \in \text{semi-K}(E, F)$. By Proposition 1, $T \in \text{Lim}(E, F)$, and so $T(B_E)$ is limited. As $F^a = F$, then F is Dedekind σ -complete. Then $F \in (\text{GPP})$ by [18]. Thus $T(B_E)$ is relatively compact, so $T \in \text{K}(E, F)$.

2. This follows from 1 in view of [7, Theorem 5.72].

3. This follows from 1 in view of [7, Theorem 5.71].

4. If E^* is a KB-space then, by [19, Proposition 2.2], $T \in \text{aDP}(E, F)$ and is order bounded. Hence $T \in \text{Mwc}(E, F)$ and therefore $T \in \text{semi-K}(E, F)$. By 1, $T \in \text{K}(E, F)$. $\square$

By [19, Proposition 2.2], if E^* is a KB-space, $T \in \text{aDP}(E, F)$ and is order bounded, then $T \in \text{semi-K}(E, F)$. Thus it follows easily that if, additionally, $S \in \text{AMc}(F, G)$, then $ST \in \text{K}(E, G)$. Note that not every weakly compact operator is AM-compact, e.g., the identity operator $I_{L^2[0,1]}$ is weakly compact but it is not AM-compact.

Definition 6. An operator $T : E \rightarrow Y$ is called

- 1) *order limited* (or $T \in \text{o-Lim}(E, Y)$) if $T[0, x]$ is limited for each $x \in E_+$;
- 2) *order weakly compact* (or $T \in \text{owc}(E, Y)$) if $T[0, x]$ is relatively weakly compact for each $x \in E_+$.

Theorem 1. Let all order intervals in E be limited. The following holds:

1. $\text{W}(E, Y) \subseteq \text{AMc}(E, Y)$ for all Y .
2. $\text{owc}(E, Y) \subseteq \text{AMc}(E, Y)$ for all Y .
3. If E^* and F both have o-continuous norms, then $\text{L}_+(E, F) \subseteq \text{AMc}(E, F)$.
4. If E^* is a KB-space and Y does not contain c_0 , then $\text{L}(E, Y) \subseteq \text{AMc}(E, Y)$.

Proof. 1. Let $T \in \text{W}(E, Y)$, and let $[-x, x] \subseteq E$. By [7, Theorem 5.38], T factors over a reflexive Z as $T = SR$ with $R \in \text{L}(E, Z)$ and $S \in \text{L}(Z, Y)$. The set $R[-x, x]$ is limited in Z as $[-x, x]$ is limited in E . Since $Z \in (\text{GPP})$ by [10], $R[-x, x]$ is relatively compact in Z , and hence $SR[-x, x]$ is relatively compact in Y . Thus, $T \in \text{AMc}(E, Y)$.

2. Let $T \in \text{owc}(E, Y)$ and let $[-x, x] \subseteq E$. Then T admits a factorization over some $F = F^a$ as $T = SQ$, where $Q : E \rightarrow F$ is a lattice homomorphism and $S \in \text{L}(F, Y)$ by [7, Theorem 5.58]. The set $Q[-x, x]$ is limited as $[-x, x]$ is limited. Since F has o-continuous norm, it is Dedekind σ -complete, hence $F \in (\text{GPP})$ by [18]. Therefore $Q[-x, x]$ is relatively compact in F , and hence $SQ[-x, x]$ is relatively compact in Y . Thus $T \in \text{AMc}(E, Y)$.

3. By [7, Exercise 15, p. 314], T has a factorization $T = SR$ over a Banach lattice G with o-continuous norm. Let $[a, b] \subseteq E$. Then $R[a, b]$ is limited in G . Since $G \in (\text{GPP})$ (by [18]), $R[a, b]$ is relatively compact in G , and hence $SR[a, b]$ is relatively compact in F . Since the interval $[a, b]$ is arbitrary, $T \in \text{AMc}(E, F)$.

4. By [7, Exercise 19, p. 315], the operator T admits a factorization over a reflexive Banach lattice G as $T = SR$, where $R : E \rightarrow G$ and $S : G \rightarrow Y$ are bounded. The rest is similar to proof of 1. $\square$

Corollary 3. Let E have limited order intervals, and $F = F^a$. The following holds.

1. $\text{semi-K}_+(E, F) \subseteq \text{AMc}(E, F)$.
2. $\text{Lwc}(E, F)$ and $\text{Mwc}(E, F)$ are $\text{AMc}(E, F)$.
3. Let $T : E \rightarrow F$ have o-continuous operator norm. Then $T \in \text{AMc}(E, F)$.

Proof. 1. As $T \in \text{semi-K}_+(E, F)$, for each $\varepsilon > 0$, we can find $u_\varepsilon \in F_+$ such that $T(B_E) \subseteq [-u_\varepsilon, u_\varepsilon] + \varepsilon B_F$. Since F has o-continuous norm, the interval $[-u_\varepsilon, u_\varepsilon]$ is weakly compact. Consequently $T \in \text{W}(E, F)$ by Lemma 2(2), and $T \in \text{AMc}(E, F)$ due to Proposition 1.

2. It follows from 1, as L- and M-weakly compact operators are weakly compact.

3. By [7, Theorem 5.68], $T \in \text{Lwc}(E, F)$. The rest follows from 2. $\square$

Corollary 4. Suppose E^* is a KB-space, E has limited order intervals, and $F \in (\text{PSP})$. Then $\text{W}_+(E, F) \subseteq \text{K}(E, F)$.

Proof. Let $T \in \text{W}_+(E, F)$. Then $T \in \text{AMc}(E, F)$ by Proposition 1. On the other hand, $T \in \text{Mwc}(E, F)$ by [20, Theorem 3.3]. As T is positive, it is semi-compact by [7, Theorem 5.72]. Since $F \in (\text{PSP})$, it has o-continuous norm. As E^* and F both have o-continuous norms, $T \in \text{K}(E, F)$ by [17, Theorem 125.5] because of $T \in \text{semi-K}(E, F) \cap \text{AMc}(E, F)$. $\square$

The next result gives some conditions on an operator to be limited.

Definition 7. An operator $T : E \rightarrow Y$ is called *almost Dunford–Pettis* (or $T \in \text{aDP}(E, Y)$) if T maps disjoint weakly null sequences to norm null sequences.

Note that $\text{Id}_{L^1[0,1]}$ is an almost Dunford–Pettis operator but not Dunford–Pettis one.

Theorem 2. The following assertions hold.

1. If E^* is a KB-space and F^* has w^* -continuous lattice operations, then

$$\text{aDP}_+(E, F) \subseteq \text{Lim}(E, F).$$

2. If E^* has w^* -sequentially continuous lattice operations, then, for every F ,

$$\text{Mwc}(E, F) \subseteq \text{Lim}(E, F).$$

Proof. 1. Let $T \in \text{aDP}_+(E, F)$. By [19, Proposition 2.2], $T \in \text{semi-K}(E, F)$. Thus, for each $\varepsilon > 0$, there exists some $u_\varepsilon \in F_+$ such that $T(B_E) \subseteq [-u_\varepsilon, u_\varepsilon] + \varepsilon B_F$. Since F^* has w^* -continuous lattice operations, F has limited order intervals, hence $T(B_E)$ is limited by Lemma 2(3), which means $T \in \text{Lim}(E, F)$.

2. Let $T \in \text{Mwc}(E, F)$ and take $(f_n) \xrightarrow{w^*} 0$ in F^* . We need to show $\|T^*f_n\| \rightarrow 0$. It suffices to prove $|T^*f_n| \xrightarrow{w^*} 0$ and $T^*(f_n)(x_n) \rightarrow 0$ for each norm bounded disjoint sequence (x_n) in E_+ . Since T is bounded, $T^*f_n \xrightarrow{w} 0$ in E^* . w^* -continuity of lattice operations in E^* gives $|T^*f_n| \xrightarrow{w^*} 0$. Consider

$$|T^*(f_n)(x_n)| = |f_n(Tx_n)| \leq \|f_n\| \|Tx_n\|.$$

Since (f_n) is norm bounded and T is an Mwc operator, the claim follows. $\square$

The next example shows that a limited operator need not be Mwc.

Example 2. Suppose E^* is not a KB-space and $F \neq \{0\}$. Then there exists a positive compact (and hence limited) operator $T : E \rightarrow F$ that is not M -weakly compact. Indeed, pick $y \in F_+ \setminus \{0\}$. Since the norm in E^* is not o -continuous, E^* contains a sublattice isomorphic to ℓ^∞ and a sublattice isomorphic to ℓ^1 ; and E contains a sublattice isomorphic to ℓ^1 ([7, Theorem 4.51] and [8, Proposition 2.3.12]). Let $i : \ell^1 \rightarrow E$ be a lattice embedding. Then $i(\ell^1)$ is the range of a positive projection $P : E \rightarrow E$, that is $i(\ell^1) = P(E)$ [8, Proposition 2.3.11]. Define $S : P(E) \rightarrow F$ by

$$S \left(i \left(\sum_{k=1}^{\infty} \beta_k e_k \right) \right) := \left(\sum_{k=1}^{\infty} \beta_k \right) y,$$

and let $T = SP$. Then $T : E \rightarrow F$ is positive. Since S has rank one, T is compact (and hence limited). But $T \notin \text{Mwc}(E, F)$. This follows from $T(i(e_n)) = SP(i(e_n)) = S(i(e_n)) = y$ and so $\|T(i(e_n))\| \not\rightarrow 0$.

In the opposite direction, we have

Theorem 3. If one of the following conditions holds:

1. $\text{Lim}_+(E, F) \subseteq \text{Mwc}(E, F)$ for every F ;
2. $\text{Lim}(E, F) \subseteq \text{aMwc}(E, F)$ for every F ;

then E^* is a KB-space.

Proof. 1. Suppose $F = E$ and E^* is not a KB-space. Then, in view of Lemma 1 and [7, Theorem 4.14], there exists a disjoint sequence $(f_n) \subseteq [0, f] \subseteq E^*$ such that $\|f_n\| \not\rightarrow 0$. Pick $y \in E_+$ such that $\|y\| = 1$ and a functional $g \in E^*$ with $\|g\| = 1$ and $g(y) = 1$. Define a positive operator $T : E \rightarrow E$ by $T(x) = f(x)y$ for $x \in E$. As T is compact, $T \in \text{Lim}(E, E)$. We claim $T \notin \text{Mwc}(E, E)$. By [7, Theorem 5.64], it is enough to show $T^* \notin \text{Lwc}(E^*, E^*)$. Let us observe that $T^*(g) = g(y)f = f$.

Thus (f_n) is a disjoint sequence in $\text{sol}(T^*(B_E))$ with $\|f_n\| \not\rightarrow 0$. Therefore $T^* \notin \text{Lwc}(E^*, E^*)$ and hence $T \notin \text{Mwc}(E, E)$ by [7, Theorem 5.64], by the assumption $\text{Lim}_+(E, E) \subseteq \text{Mwc}(E, E)$. This is a contradiction.

2. Under the assumption, $K(E, F) \subseteq \text{aMwc}(E, F)$, hence E^* is a KB-space by [21, Theorem 2]. $\square$

The following theorem deals with the conditions on operators to be (a)Mwc (compare with [2, Proposition 4.5]).

Theorem 4. Let E^* be a KB-space. Then the following holds.

1. If $F \in (\text{GPP})$ then $\text{Lim}(E, F) \subseteq \text{Mwc}(E, F)$.
2. $\text{Lim}_+(E, F) \subseteq \text{aMwc}(E, F)$ for every F .

Proof. 1. Let $T \in \text{Lim}(E, F)$ and (x_n) be a disjoint bounded sequence in E . Then, by Lemma 1, $x_n \xrightarrow{w} 0$ and hence $Tx_n \xrightarrow{w} 0$. Since $F \in (\text{GPP})$ and the set $\{Tx_n\}_{n=1}^\infty$ is limited, it is relatively compact. Then $\|Tx_n\| \rightarrow 0$, and hence $T \in \text{Mwc}(E, F)$.

2. If $T \in \text{Lim}_+(E, F)$ then $T \in \text{aLim}_+(E, F)$, and hence $T^* \in \text{aDP}_+(F^*, E^*)$ by [12, Theorem 1]. Thus $T^* \in \text{aLwc}_+(F^*, E^*)$ by [14, Proposition 2.4], and so $T \in \text{aMwc}(E, F)$ by [14, Theorem 2.5]. $\square$

The condition that E^* is a KB-space is essential in Theorem 4(1) due to [7, the example before Theorem 5.62].

The operator Id_{ℓ_2} is w-compact but not limited. Theorem 5 in [22] shows that if F is σ -Dedekind complete, then every almost limited $T : E \rightarrow F$ is w-compact iff E is reflexive or $F = F^a$.

Theorem 5. If $W(E, Y) \subseteq \text{Lim}(E, Y)$, then one of the following holds:

1. E^* is a KB-space;
2. $Y \in (\text{DPP})$.

Proof. Let E^* be not a KB-space and $Y \notin (\text{DPP})$. We construct $T \in W(E, Y)$ but $T \notin \text{Lim}(E, Y)$.

Since E^* is not a KB-space, there exists a disjoint sequence (u_n) in $(B_E)_+$ which is not weakly null by Lemma 1. By passing to a subsequence, we may assume that, for some $\theta \in E_+^*$ and $\varepsilon > 0$, $\theta(u_n) > \varepsilon$ for all n . Then the components θ_n of θ in the carriers C_n of u_n form an order bounded disjoint sequence in E_+^* such that $\theta_n(u_n) = \theta(u_n)$ for all n and $\theta_n(u_m) = 0$ if $n \neq m$ by [23, Theorem 116.3]. We define an operator $P : E \rightarrow \ell^1$ by

$$P(x) = \left(\frac{\theta_k(x)}{\theta(u_k)} \right)_{k=1}^\infty \quad (x \in E).$$

The operator P is well defined as

$$\|Px\| = \sum_{k=1}^\infty \left| \frac{\theta_k(x)}{\theta(u_k)} \right| \leq \sum_{k=1}^\infty \frac{|\theta_k(x)|}{\varepsilon} \leq \frac{1}{\varepsilon} \sum_{k=1}^\infty \theta_k(|x|) \leq \frac{1}{\varepsilon} \theta(|x|)$$

holds for each $x \in E$.

As $Y \notin (\text{DPP})$, there is a weakly null sequence (y_n) in Y and a weakly null (hence w^* -null) sequence (f_n) in Y^* such that $(f_n(y_n)) \not\rightarrow 0$. Passing to a subsequence, we may assume that, for some $\epsilon > 0$,

$$|f_n(y_n)| > \epsilon \quad (\forall n \in \mathbb{N}). \quad (1)$$

We define an operator $S : \ell^1 \rightarrow Y$ by $S\alpha = \sum_{k=1}^\infty \alpha_k y_k$. By [7, Theorem 5.26], S is weakly compact.

Then $T = SP \in W(E, Y)$. Suppose T is limited. Then the w^* -null sequence (f_n) is uniformly null on $T(B_E)$. Since $Tu_n = SPu_n = Se_n = y_n$, it follows from (1) that $|f_n(Tu_n)| > \epsilon$ for all n , contradicting uniform convergence to zero of (f_n) on $T(B_E)$. Therefore $T \notin \text{Lim}(E, Y)$, as desired. $\square$

We discuss some composition properties of semi-compact operators.

Proposition 2. Let $F = F^a$, $T \in \text{semi-K}(X, F)$, and $S \in \text{DP}(F, G)$. Then $ST \in \text{K}(X, G)$.

Proof. Since $T \in \text{semi-K}(X, F)$, for each $\varepsilon > 0$, there exists $u_\varepsilon \in F_+$ such that $T(B_X) \subseteq [-u_\varepsilon, u_\varepsilon] + \varepsilon B_F$. Since F has o-continuous norm, $[-u_\varepsilon, u_\varepsilon]$ is relatively weakly compact and hence $T(B_X)$ is weakly compact by Lemma 2(2). In order to show that $ST(B_E)$ is relatively compact, take a sequence (STx_n) in $ST(B_X)$. As $T(B_X)$ is relatively weakly compact then, by [7, Theorem 3.40], there exists a weakly convergent subsequence (Tx_{n_k}) . Since $S \in \text{DP}(F, G)$, the sequence (STx_{n_k}) converges in norm. As the sequence (STx_n) in $ST(B_X)$ is arbitrary, $ST(B_X)$ is compact in G , and hence $ST \in \text{K}(X, G)$. $\square$

If E^* is a KB -space and $F = F^a$ then, for each regular aDP operator T (i.e. there exist $T_1, T_2 \in \text{aDP}_+(E, F)$ such that $T = T_1 - T_2$) and $S \in \text{DP}(F, G)$, we have $ST \in \text{K}(E, G)$. This follows from the fact that each positive aDP operator is semi-compact by [19, Proposition 2.2].

The square of a semi-compact operator is not compact in general (e.g., Id_c is semi-compact but $\text{Id}_c^2 = \text{Id}_c$ is not compact). This problem was studied in [24, Theorem 2.4]. We give another sufficient condition.

Theorem 6. Let E have limited order intervals.

1. If E has o-continuous norm and $T \in \text{semi-K}_+(E)$, then $T^2 \in \text{K}(E)$.
2. If $T \in \text{semi-K}(E) \cap \text{W}(E)$, then $T^2 \in \text{K}(E)$.

Proof. 1. Let $T \in \text{semi-K}_+(E)$. Then, for $\varepsilon > 0$ there exists $u_\varepsilon \in E_+$ with $T(B_E) \subseteq [-u_\varepsilon, u_\varepsilon] + \varepsilon B_E$, which yields $T^2(B_E) \subseteq [-Tu_\varepsilon, Tu_\varepsilon] + \varepsilon \|T\| B_E$. By the assumption, each $[-Tu_\varepsilon, Tu_\varepsilon]$ is limited, and hence $T^2(B_E)$ is limited by Lemma 2(3). By [18], a Dedekind σ -complete Banach lattice has the Gelfand–Phillips property iff it has o-continuous norm. Therefore $E \in (\text{GPP})$. Then $T^2(B_E)$ is relatively compact, and hence $T^2 \in \text{K}(E)$.

2. As $T \in \text{W}(E)$, it factors over a reflexive space G as $T = SR$, where $R : E \rightarrow G$ and $S : G \rightarrow E$ are bounded by [7, Theorem 5.38].

Let $\varepsilon > 0$. Since $T \in \text{semi-K}(E)$, there exists $u_\varepsilon \in E_+$ such that $T(B_E) \subseteq [-u_\varepsilon, u_\varepsilon] + \varepsilon B_E$. The interval $[-u_\varepsilon, u_\varepsilon]$ is limited in E . Applying R to $[-u_\varepsilon, u_\varepsilon]$, we see that $R[-u_\varepsilon, u_\varepsilon]$ is limited in G . Being reflexive, $G \in (\text{GPP})$, and hence $R[-u_\varepsilon, u_\varepsilon]$ is relatively compact in G . Therefore, $SR[-u_\varepsilon, u_\varepsilon]$ is relatively compact in E and $T^2(B_E) \subseteq SR[-u_\varepsilon, u_\varepsilon] + \varepsilon SR(B_E)$. Since $\varepsilon > 0$ is arbitrary, Lemma 2(1) implies that $T^2(B_E)$ is relatively compact in E , and hence $T^2 \in \text{K}(E)$. $\square$

Proposition 3. $\text{aLwc}(X, F) \subseteq \text{l-Lwc}(X, F)$.

Proof. Let $T \in \text{aLwc}(X, F)$. As one of defining properties of aLwc operators, we have $T(X) \subseteq F^a$. Since F^a has o-continuous norm, $F^a \in (\text{DDw}^*\text{P})$ so that each limited subset in F^a is Lwc by [25, Theorem 2.6], and hence $T \in \text{l-Lwc}(X, F)$. $\square$

Proposition 4. If F has the Schur property then $\text{l-Mwc}(E, F) \subseteq \text{owc}(E, F)$.

Proof. Let $T \in \text{l-Mwc}(E, F)$. By [7, Theorem 5.57], we need to show $\|Tx_n\| \rightarrow 0$ for each order bounded disjoint sequence (x_n) in E . Such a sequence is w-null as it is order bounded and disjoint. Since $T \in \text{l-Mwc}(E, F)$, we get $Tx_n \xrightarrow{w} 0$ and so $\|Tx_n\| \rightarrow 0$ as F has the Schur property. $\square$

A subset A of E is called bi-bounded if $i(A)$ is order bounded in E^{**} , where $i : E \rightarrow E^{**}$ is the canonical embedding. An operator $T : E \rightarrow F$ is b-weakly compact if $T(A)$ is relatively weakly compact for each bi-bounded A in E . A similar proof to the given one in Proposition 4 shows that $\text{l-Mwc}(E, F)$ is actually contained in the narrower set of b-weakly compact operators [26].

2 Almost limited operators

In this section, we focus on almost limited operators and show they contain many interesting classes of operators, and the converse problem of almost limited operators contained in other classes.

A bounded subset A of a Banach space X is called *almost limited* (we write aLim-set) if each disjoint w^* -null sequence in X^* converges uniformly to zero on A . Almost limited sets were introduced in [25]. Let us also note that every order interval in E is aLim iff $E \in (d)$. An operator $T : X \rightarrow F$ is called *almost limited* (we write $T \in \text{aLim}(X, F)$) if $T(B_X)$ is an aLim-set . Almost limited sets and operators were studied in [9, 22, 25].

It was shown in [9, Proposition 4.5] that each order bounded Mwc operator $T : E \rightarrow F$ is almost limited whenever lattice operations in E^* are w^* -sequentially continuous or $F \in (d)$. By [12, Corollary 4], if lattice operations in E^* are w^* -sequentially continuous then $\text{aLim}(X, E) \subseteq \text{Lim}(X, E)$.

Theorem 7. Let E^* be a KB-space. The following holds.

1. If $F \in (d)$ or the lattice operations in E^* are w^* -sequentially continuous then each order bounded aLwc operator $T : E \rightarrow F$ is aLim .
2. If E is reflexive then $\text{aLwc}(E, F) \subseteq \text{aLim}(E, F)$ for every F .

Proof. 1. Take an order bounded $T \in \text{aLwc}(E, F)$. For $T \in \text{aLim}(E, F)$, it suffices to show $f_n(Tx_n) \rightarrow 0$ for every disjoint w^* -null (f_n) in F^* and disjoint (x_n) in $(B_E)_+$ ([9, Theorem 2.7]). Let (f_n) be disjoint w^* -null in F^* and let (x_n) be disjoint in $(B_E)_+$. Since $f_n \xrightarrow{w^*} 0$, it is bounded. As E^* is a KB-space, $x_n \xrightarrow{w} 0$. Since $T \in \text{aLwc}(E, F)$ then $f_n(Tx_n) \rightarrow 0$ by [14, Theorem 2.2].

2. Let $T \in \text{aLwc}(E, F)$, and let (f_n) be disjoint w^* -null in B_{F^*} . We need to show that (f_n) is uniformly null on $T(B_E)$. Assume to the contrary that $\sup_{x \in B_E} |f_n(Tx)| \not\rightarrow 0$ as $n \rightarrow \infty$. Then, for some $\varepsilon > 0$, there exists a sequence (x_{n_k}) in B_E such that

$$|f_{n_k}(Tx_{n_k})| \geq \varepsilon \quad (\forall k \in \mathbb{N}). \quad (2)$$

B_E is weakly compact because E is reflexive. So, $x_{n_{k_j}} \xrightarrow{w} x \in E$ for some further subsequence $(x_{n_{k_j}})$ by the Eberlein–Smulian theorem. Since $T \in \text{aLwc}(E, F)$, then by [14, Theorem 2.2], $f_{n_{k_j}}(Tx_{n_{k_j}}) \rightarrow 0$ which contradicts (2). Therefore (f_n) converges uniformly to zero on $T(B_E)$, and $T \in \text{aLim}(E, F)$. $\square$

Theorem 8. Let $F \in (d)$. Then the following holds:

1. If $F \in (\text{PGP})$, then $\text{span}(\text{aMwc}_+(E, F)) \subseteq \text{aLim}(E, F)$.
2. $l\text{-Lwc}(E, F) \cap \text{Mwc}(E, F) \subseteq \text{aLim}(E, F)$ for every E .

Proof. 1. Let $T \in \text{span}(\text{aMwc}_+(E, F))$. We may suppose T is positive. For proving $T \in \text{aLim}(E, F)$, it suffices to show $f_n(Tx_n) \rightarrow 0$ for every disjoint w^* -null (f_n) in F_+^* and disjoint (x_n) in $(B_E)_+$ ([9, Theorem 2.7]). Let (f_n) be disjoint w^* -null in F_+^* and (x_n) be disjoint in $(B_E)_+$. As $F \in (\text{PGP})$, $f_n \xrightarrow{w} 0$. By Definition 5 (5), $f_n(Tx_n) \rightarrow 0$. Consequently $T \in \text{aLim}(E, F)$.

2. Let $T \in l\text{-Lwc}(E, F) \cap \text{Mwc}(E, F)$. We need to show $T(B_E)$ is almost limited. It suffices to show $f_n(Tx_n) \rightarrow 0$ for every disjoint (x_n) in $(B_E)_+$ and every disjoint w^* -null (f_n) in B_{F^*} [9, Theorem 2.7]. Let (x_n) be disjoint in $(B_E)_+$ and (f_n) be disjoint w^* -null in B_{F^*} . Since $T \in l\text{-Lwc}(E, F)$, then $(T^*f_n) \xrightarrow{w^*} 0$ in E^* by [1, Theorem 2.7]. As $f_n \xrightarrow{w^*} 0$, $\|f_n\| \leq M$ for all n . Since $T \in \text{Mwc}$, then $\|Tx_n\| \rightarrow 0$, and hence

$$|f_n(Tx_n)| \leq \|f_n\| \cdot \|Tx_n\| \leq M \cdot \|Tx_n\|$$

implies $f_n(Tx_n) \rightarrow 0$ as desired. $\square$

The following presents conditions for almost limited operators to be L-weakly compact.

Theorem 9. The following are equivalent.

1. F has o-continuous norm.
2. For every X , $\text{aLim}(X, F) \subseteq \text{Lwc}(X, F)$.
3. For every E , each positive rank one operator $T : E \rightarrow F$ is Lwc.

Proof. $1 \implies 2$: Let $T \in \text{aLim}(X, F)$. Then $T(B_X)$ is an aLim-set in F . By [25, Theorem 2.6], $T(B_X)$ is an Lwc-set, and hence $T \in \text{Lwc}(X, F)$.

$2 \implies 3$: It is trivial.

$3 \implies 1$: Since, for each $y \in F_+$, there exists a positive rank one operator $T : E \rightarrow F$ such that $T(E) = \text{span}(y)$, then $y \in F^a$ by [7, Theorem 5.66], and hence F has o-continuous norm. $\square$

Order continuity of the norm in F is essential above as the identity of ℓ^∞ is almost limited but it is neither M- or L-weakly compact.

Theorem 10. Let E be an AL-space and let F have o-continuous norm. Then each $T \in \text{aLim}(E, F)$ has a modulus T which is almost limited.

Proof. As $F^a = F$ and T is almost limited, then T is Lwc. Hence T has the modulus $|T|$ that is Lwc by [20, Theorem 2.4]. Therefore $|T|$ is almost limited by [25, Theorem 2.6]. $\square$

However [20, Theorem 2.2] shows that there is a regular compact (hence limited) operator from $E = L^2[0, 1]$ to $F = c(L^2[0, 1])$ which is both M- and L-weakly compact whose modulus does not exist. The identity of ℓ^∞ is almost limited but it is neither M- nor L-weakly compact.

Theorem 11. Suppose E is infinite dimensional and has o-continuous norm, and $F \in (d)$. If $\text{aLim}(E, F) \subseteq \text{Mwc}(E, F)$, then E^* and F both have order continuous norms.

Proof. If $\text{aLim}(E, F) \subseteq \text{Mwc}(E, F)$, then $K(E, F) \subseteq \text{Mwc}(E, F)$. The fact that the norm in E^* is o-continuous follows from [21, Theorem 2].

Suppose now the norm of F is not o-continuous. Then there exist some $u \in F_+$ and a disjoint sequence (u_n) in $[0, u]$ such that $\|u_n\| \not\rightarrow 0$. We may assume $\|u_n\| = 1$ for all n . Since $\dim(E) = \infty$, there exists a disjoint normalized sequence (x_n) in E_+ . By [23, Theorem 116.3 (iii)], there exists a disjoint sequence (g_n) in $(B_{E^*})_+$ such that $\frac{1}{2} \leq g_n(x_n) \leq 1$ for all n and $g_n(x_m) = 0$ for $n \neq m$. Since E has o-continuous norm, then $g_n \xrightarrow{w^*} 0$ by [23, Theorem 116.2].

Define positive operator $R : E \rightarrow c_0$ by $Rx = (g_k(x))_{k=1}^\infty$. Clearly $R(B_E) \subseteq B_{c_0}$. Define positive operator $S : c_0 \rightarrow F$ by $S\alpha = \sum_n \alpha_n u_n$. Then $S(B_{c_0}) \subseteq [-u, u]$, and hence $SR(B_E) \subseteq [-u, u]$. As order intervals in F are almost limited by the (d)-property of F , $SR(B_E)$ is almost limited, and $SR \in \text{aLim}(E, F)$. Since $SR(x_n) = S(g_n(x_n) \cdot e_n) = g_n(x_n) \cdot u_n$, then $\|SR(x_n)\| \geq |g_n(x_n)| \geq \frac{1}{2}$ for all n . Therefore $SR \notin \text{Mwc}(E, F)$. A contradiction. $\square$

An operator $T : E \rightarrow F$ is called *positively limited* if $\|T^*f_n\| \rightarrow 0$ for each w^* -null (f_n) in F_+^* . We denote the set of such operators by $\text{p-Lim}(E, F)$. We need the following lemma [27, Theorem 2.2].

Lemma 4. $T : E \rightarrow F$ is positively limited iff $T^*(y_n^*)(x_n) \rightarrow 0$ for each $y_n^* \xrightarrow{w^*} 0$ in F_+^* and every disjoint bounded (x_n) in E_+ .

Theorem 12. Suppose E^* and F both have o-continuous norms, F has limited order intervals, and each disjoint bounded sequence of E is order bounded. Then each regular operator $T : E \rightarrow F$ is almost limited.

Proof. Suppose $T : E \rightarrow F$ is positive. T admits a factorization over a Banach lattice G with G and G^* both have o-continuous norms, say $T = SR$, where R and S are both positive as in [7, Problem 15 on page 314]. We claim R is positively limited. Let (z_n^*) be a positive w^* -null sequence in G^* and (x_n) be a positive disjoint bounded sequence in E . By the assumption, there exists x with $0 \leq x_n \leq x$ for each n . As R is positive, $0 \leq Rx_n \leq Rx$. Thus $0 \leq R^*(z_n^*)(x_n) \leq z_n^*(Rx_n) \leq z_n^*(Rx)$ which shows $R^*(z_n^*)(x_n) \rightarrow 0$. By Lemma 4, we conclude that R is positively limited.

Now we show $T = SR \in \text{aLim}(E, F)$. Let (u_n^*) be disjoint w^* -null in F^* . Since F^* has w^* -continuous lattice operations, $|u_n^*| \xrightarrow{w^*} 0$ in F^* . As R is positively limited, $\|R^*S^*|u_n^*|\| \rightarrow 0$. Since $R^*|S^*u_n^*| \leq R^*S^*|u_n^*|$, we have $|R^*S^*u_n^*| \leq R^*|S^*u_n^*| \leq R^*S^*|u_n^*|$. Resulting that $\|R^*S^*u_n^*\| = \|T^*u_n\| \rightarrow 0$. It follows $T \in \text{aLim}(E, F)$. $\square$

It is worth noting that both conditions on E in Theorem 12 can be replaced by a single condition that E is an AM-space in which every disjoint bounded sequence is order bounded. Indeed, by the Abramovich's result [28], if E is not isomorphic to an AM-space then E contains a positive disjoint normalized sequence (x_n) such that $\|\bigvee_{n=1}^m x_n\| \rightarrow \infty$, and hence such a sequence (x_n) cannot be order bounded [28]. Thus, E must be (isomorphic to) an AM-space, and hence the norm in E^* is o -continuous. Also note that this condition on E is weaker than the condition that E is an AM-space with unit, as the space $C_\omega(\Gamma)$ of real-valued bounded countably supported functions on an uncountable set Γ still satisfies the condition that every disjoint bounded sequence is order bounded, yet $C_\omega(\Gamma)$ has not got even a weak unit.

Theorem 13. Let $T : E \rightarrow E$ be a positive operator in the ideal center of E . Suppose $E \in (\text{DPSP})$. Then T is positively limited.

Proof. For simplicity suppose $0 \leq T \leq I$. Let (f_n) be a positive w^* -null sequence in E^* . Then $0 \leq T^* \leq I$. Hence $0 \leq T^*f_n \leq f_n$ for all n . Since $E \in (\text{DPSP})$, $\|f_n\| \rightarrow 0$ hence $T^*f_n \rightarrow 0$. $\square$

It is easy to see that $\text{p-Lim}(E, F)$ is a vector space.

Proposition 5. Suppose $\|T_n - T\| \rightarrow 0$ and $T_n \in \text{p-Lim}(E, F)$ for all n . Then $T \in \text{p-Lim}(E, F)$.

We shall also use a known result (see [29, Theorem 2.4] and [3, Proposition 1.10]): if P is a norm closed subspace of $L(E, F)$, then $\text{span}(P_+)$ is complete under the enveloping norm $\|T\|_{r-P} = \inf\{\|S\| : \pm T \leq S \in P\}$. Together with Proposition 5, this immediately implies the following

Assertion 3. If P is a norm closed subspace of $L(E, F)$, then $\text{span}(P_+)$ is complete under the enveloping norm $\|T\|_{r-P} = \inf\{\|S\| : \pm T \leq S \in P\}$.

Corollary 5. For every E and F , $\text{span}(\text{p-Lim}_+(E, F))$ is a Banach space under the enveloping norm.

It was shown in [21, Theorem 1] that $K(X, F) \subseteq \text{aLwc}(X, F)$ for all X iff F has o -continuous norm. We extend this to almost limited operators.

Theorem 14. $\text{aLim}(X, F) \subseteq \text{aLwc}(X, F)$ iff F has o -continuous norm.

Proof. The necessity follows from [21, Theorem 1] as $K(X, F) \subseteq \text{aLim}(X, F)$.

For the sufficiency, suppose in contrary that $\text{aLim}(X, F) \subseteq \text{aLwc}(X, F)$ but the norm in F is not o -continuous. Then there exists $y_0 \in F \setminus F^a$. We choose $x_0 \in X$ and $h \in X^*$ such that $h(x_0) = \|x_0\| = 1$ and define $Tx = h(x)y_0$. The operator T is almost limited as it is compact. However $T \notin \text{aLwc}(X, F)$ as $\{Tx_0\} = \{y_0\}$ is not an Lwc-set. A contradiction, as an almost Lwc operator must map X into the order continuous part of F . $\square$

By [21, Theorem 2], $K(E, X) \subseteq \text{aMwc}(E, X)$ iff E^* is a KB-space. We extend this result as follows.

Proposition 6. If E^* is a KB-space then $\text{aLim}_+(E, F) \subseteq \text{aMwc}(E, F)$.

Proof. Let E^* be a KB-space and $T \in \text{aLim}_+(E, F)$. Then $T^* \in \text{aDP}_+(F^*, E^*)$ [12, Theorem 1]. Consequently $T^* \in \text{aLwc}(F^*, E^*)$ by [14, Proposition 2.4]. Then $T \in \text{aMwc}(E, F)$ by [14, Theorem 2.5]. $\square$

Theorem 15. If E^* and F both have o -continuous norms then $\text{span}(\text{aLim}_+(E, F)) \subseteq \text{Mwc}(E, F)$.

Proof. Let $T \in \text{span}(\text{aLim}_+(E, F))$. Suppose $T \geq 0$. Let (x_n) be a disjoint sequence in B_E , we show $\|Tx_n\| \rightarrow 0$. Since E^* is a KB-space, $x_n \xrightarrow{w} 0$ in E . Since (x_n) is disjoint, $|x_n| \xrightarrow{w} 0$ [11, Proposition 1.3], and hence $T|x_n| \xrightarrow{w} 0$ in F . As T is positive, $|Tx_n| \leq T|x_n|$ and $|Tx_n| \xrightarrow{w} 0$ in F . It remains to show $f_n(Tx_n) \rightarrow 0$ for each disjoint bounded sequence (f_n) in F^* . This follows from $|f_n(Tx_n)| \leq \|T^*f_n\| \|x_n\|$, since $\|T^*f_n\| \rightarrow 0$ as T is almost limited. $\square$

We continue with a necessary conditions for positive w-compact operators to be almost limited (see Theorem 5).

Proposition 7. Suppose $E = E^a$ and $W(X, E) \subseteq \text{aLim}(X, E)$. Then E is a KB-space or $X \in (\text{DPP})$.

Proof. Suppose E is not a KB-space. We show $X \in (\text{DPP})$. Since E is not a KB-space, it contains a lattice copy (say F) of c_0 . Then F is a closed linear span of some disjoint sequence (y_n) in E_+ such that $d = \inf_n \|y_n\| > 0$ and $\sup_m \|\sum_{i=1}^m y_i\| < \infty$ by [17, Proposition 0.5.1]. The operator

$$S\beta = \sum_{k=1}^{\infty} \beta_k y_k \quad (\beta = (\beta_k)_k \in c_0)$$

is a lattice homomorphism from c_0 onto F . There exists a positive disjoint (f_n) in B_{E^*} such that $f_n(y_n) \geq d$ for all n and $f_n(y_m) = 0$ if $n \neq m$. Take $B_X \ni x_n \xrightarrow{w} 0$ and $B_{X^*} \ni x_n^* \xrightarrow{w} 0$. Define $R : X \rightarrow c_0$ by $Rx = (x_k^*(x))_k$. Then $R \in W(X, c_0)$ by [7, Theorem 5.26]. Therefore $T = SR \in W(X, E)$ and hence $T \in \text{aLim}(X, E)$ by the hypothesis. Since E has o-continuous norm, the bounded disjoint sequence (f_n) is w^* -null and therefore $\|T^*f_n\| \rightarrow 0$ as $T \in \text{aLim}(X, E)$. Since (x_n) is bounded and $|f_n(Tx_n)| \leq \|x_n\| \cdot \|T^*f_n\|$, these yield $|f_n(Tx_n)| \rightarrow 0$. Thus, it follows from

$$\begin{aligned} |f_n(Tx_n)| &= |f_n(S(R(x_n)))| = \left| f_n \left(\sum_{k=1}^{\infty} [R(x_n)]_k y_k \right) \right| = |f_n([R(x_n)]_n y_n)| = |f_n([(x_k^*(x_n))_k]_n y_n)| = \\ &= |f_n(x_n^*(x_n) y_n)| = |x_n^*(x_n) f_n(y_n)| = |x_n^*(x_n)| \cdot |f_n(y_n)| \geq |x_n^*(x_n)| \cdot d \end{aligned}$$

that $x_n^*(x_n) \rightarrow 0$, and hence $X \in (\text{DPP})$. $\square$

Corollary 6. $[W(X, c_0) \subseteq \text{aLim}(X, c_0)] \implies X \in (\text{DPP})$.

3 Collectively limited and almost limited operator families

Some of the results given previously in the present paper for individual operators have collective versions as we show below following the recent study of collectively qualified families of operators in [4–6]. At the expense, we can give more general results extending results obtained in previous sections.

Definition 8. A norm bounded family $\mathcal{T}$ of operators is

- 1) *collectively compact* (or $\mathcal{T} \in \mathbf{K}(X, Y)$) if $\mathcal{T}B_X = \bigcup_{T \in \mathcal{T}} T(B_X)$ is relatively compact in Y ;
- 2) *collectively limited* (or $\mathcal{T} \in \mathbf{Lim}(X, Y)$) if $\mathcal{T}B_X$ is limited in Y ;
- 3) *collectively weakly compact* (or $\mathcal{T} \in \mathbf{W}(X, Y)$) if $\mathcal{T}B_X$ is relatively weakly compact in Y ;
- 4) *collectively Lwc* (or $\mathcal{T} \in \mathbf{Lwc}(X, F)$) if $\mathcal{T}B_X$ is an Lwc set in F ;
- 5) *collectively Mwc* (or $\mathcal{T} \in \mathbf{Mwc}(E, Y)$) if $\sup_{T \in \mathcal{T}} \|Tx_n\| \rightarrow 0$ for every disjoint (x_n) in B_E ;
- 6) *collectively semi-compact* (or $\mathcal{T} \in \mathbf{semi-K}(X, F)$) if, for each $\varepsilon > 0$ there exists $u \in F_+$ such that $\mathcal{T}B_X \subseteq [-u, u] + \varepsilon B_F$;
- 7) *collectively order bounded* (or $\mathcal{T} \in \mathbf{Lob}(E, F)$) if, for each $a \in E_+$, there exists $b \in F_+$ with $\bigcup_{T \in \mathcal{T}} T[-a, a] \subseteq [-b, b]$;

- 8) *collectively almost Dunford–Pettis* (or $\mathcal{T} \in \mathbf{aDP}(E, F)$) if, for each disjoint $x_n \xrightarrow{w} 0$ on E ,

$$\sup_{T \in \mathcal{T}} \|Tx_n\| \rightarrow 0.$$

We give an example of collectively limited set consisting of compact operators that is not collectively compact.

Example 3. Let $E = \ell^2$ and $F = \ell^\infty$. Define operators $T_n : E \rightarrow F$ by $T_n(x) = x_n \cdot \mathbf{e}_n$. Then $\mathcal{T} = \{T_n : n = 1, \dots, \infty\} \subseteq K(\ell^2, \ell^\infty)$. The family $\mathcal{T}$ is not collectively compact since $\bigcup_{n=1}^\infty T_n(B_{\ell^2})$ contains the sequence $\mathbf{e}_n$ that is not a convergent sequence in ℓ^∞ . However, $\mathcal{T}$ is collectively limited by the Phillips lemma since $\bigcup_{n=1}^\infty T_n(B_{\ell^2}) \subseteq B_{c_0}$.

Let $\mathcal{T}$ be a norm bounded family of operators. The following proposition gives a method of producing collectively qualified families. We omit its straightforward proof.

Proposition 8. Let $\mathcal{T} \subseteq L(X, F)$ be norm bounded and $S \in L(F, G)$. The following is true.

1. If $S \in K(F, G)$, then $\mathcal{T} \circ S \in \mathbf{K}(E, G)$.
2. If $S \in W(F, G)$, then $\mathcal{T} \circ S \in \mathbf{W}(E, G)$.
3. If $S \in \text{Lim}(F, G)$, then $\mathcal{T} \circ S \in \mathbf{Lim}(E, G)$.
4. If $S \in \mathbf{aLim}(F, G)$, then $\mathcal{T} \circ S \in \mathbf{aLim}(E, G)$.
5. If $S \in \text{Lwc}(F, G)$, then $\mathcal{T} \circ S \in \mathbf{Lwc}(E, G)$.
6. If $\mathcal{T} \in \mathbf{semi-K}(E, F)$ and $S \in \text{Lwc}(F, G)$, then $\mathcal{T} \circ S \in \mathbf{K}(E, G)$.
7. If $\mathcal{T} \in \mathbf{semi-K}(E, F)$ and $S \in \text{owc}(F, G)$, then $\mathcal{T} \circ S \in \mathbf{W}(E, G)$.

The following result is a consequence of Lemma 2.

Proposition 9. For every E the following hold:

1. If $F = F^a$, then $\mathbf{semi-K}(E, F) \subseteq \mathbf{Lwc}(E, F)$.
2. If $F = F^a$, then $\mathbf{semi-K}(E, F) \subseteq \mathbf{W}(E, F)$.
3. If F is discrete and $F = F^a$, then $\mathbf{semi-K}(E, F) \subseteq \mathbf{K}(E, F)$.
4. If F has limited order intervals, then $\mathbf{semi-K}(E, F) \subseteq \mathbf{Lim}(E, F)$.

Proof. 1. The statement follows from the well-known fact that in a Banach lattice with order continuous norm, every almost order bounded set is L-weakly compact.

2. It follows from 1, since every collectively L-weakly compact set is collectively weakly compact by [5, Corollary 2.11].

3. By the assumption, each interval $[-u, u]$ is norm compact. The rest follows from Lemma 2(1).

4. It follows from Lemma 2(3). $\square$

The following theorem is an important tool in lattice approximation for collectively Mwc families of operators.

Theorem 16. Let $\mathcal{T} \in \mathbf{Mwc}(E, F)$ be norm bounded. For each $\varepsilon > 0$, there exists $u \in E_+$ such that $\sup_{T \in \mathcal{T}} \|T[(|x| - u)^+]\| \leq \varepsilon$ holds for all $x \in B_E$.

Proof. Let $\mathcal{T} \in \mathbf{Mwc}(E, F)$. Take $\varepsilon > 0$, $A = B_E$, and consider $\rho(x) = \sup_{T \in \mathcal{T}} \|Tx\|$. The seminorm ρ is norm continuous as $\rho(x) \leq \|T\| \|x\| \leq K \|x\|$ for all $T \in \mathcal{T}$ and some $K > 0$. If (x_n) is a disjoint sequence (x_n) in B_E then $\rho(x_n) \rightarrow 0$ by Definition 8. By [7, Theorem 4.36], there exists $u \in E_+$ such that $\sup_{T \in \mathcal{T}} \|T[(|x| - u)^+]\| = \rho(\text{Id}_E(|x| - u)^+) \leq \varepsilon$ for $x \in B_E$. $\square$

Corollary 7. $\mathbf{Lob}(E, F) \cap \mathbf{Mwc}(E, F) \subseteq \mathbf{semi-K}(E, F)$.

Proof. Let $\mathcal{T} \in \mathbf{L}_{ob}(E, F) \cap \mathbf{Mwc}(E, F)$ and $\varepsilon > 0$. Since $\mathcal{T} \in \mathbf{L}_{ob}(E, F)$, [4, Theorem 2.1] implies that $\mathcal{T}$ is norm bounded. As $\mathcal{T} \in \mathbf{Mwc}(E, F)$, by Theorem 16 there exists $u \in E_+$ with

$$\sup_{T \in \mathcal{T}} \|T(|x| - u)^+\| \leq \varepsilon \quad \text{for all } x \in B_E. \quad (3)$$

If $y \in B_E^+$ and $T \in \mathcal{T}$, then $Ty = T(y \wedge u) + T((y - u)^+) \in T[0, u] + \varepsilon B_F$ by (3). Therefore, for every $x \in B_E$ and $T \in \mathcal{T}$, we have $Tx = T(x^+) - T(x^-) \in T[0, u] - T[0, u] + 2\varepsilon B_F \subseteq T[-u, u] + 2\varepsilon B_F$. Thus, $\bigcup_{T \in \mathcal{T}} T(B_E) \subseteq [-u, u] + 2\varepsilon B_F$ proving $\mathcal{T}$ is collective semicompact. $\square$

Example 4. Consider $T : E \rightarrow F$ be a positive almost Dunford–Pettis operator. Then $[0, T] \in \mathbf{aDP}(E, F)$. Indeed, let (x_n) be a disjoint w-null sequence in E . Then $(|x_n|)$ is also disjoint and w-null. Hence $\|T|x_n|\| \rightarrow 0$. As $|Sx_n| \leq S|x_n| \leq T|x_n|$ for each $S \in [0, T]$, we have $\|Sx_n\| \leq \|T|x_n|\|$ and so $\sup_{S \in [0, T]} \|Sx_n\| \leq \|T|x_n|\| \rightarrow 0$.

The following is a collective version of [19, Proposition 2.1].

Proposition 10. If E^* is a KB-space, then $\mathbf{L}_{ob}(E, F) \cap \mathbf{aDP}(E, F) \subseteq \mathbf{semi-K}(E, F)$.

Proof. Since $\mathcal{T} \in \mathbf{L}_{ob}(E, F)$, then $\mathcal{T}$ is norm bounded by [4, Theorem 2.1]. In view of Corollary 7, it suffices to show $\mathcal{T} \in \mathbf{Mwc}(E, F)$. Let (x_n) be a disjoint sequence in B_E . Since E^* has order continuous norm, $x_n \xrightarrow{w} 0$ in E . Therefore, $\sup_{T \in \mathcal{T}} \|Tx_n\| \rightarrow 0$ as $\mathcal{T} \in \mathbf{aDP}(E, F)$. Thus $\mathcal{T} \in \mathbf{Mwc}(E, F)$. $\square$

Conclusion

This research represents a novel approach to the investigation of the classes of (almost) limited, L-weakly compact, semi-compact, and Dunford–Pettis operators between Banach lattices and Banach spaces. The main scientific contribution lies in the comprehensive analysis of collectively qualified families of operators in Banach lattices. Future study might investigate extensions to operators in locally solid lattices and ordered Banach spaces.

Author contributions

Both authors contributed equally to this work.

Conflict of interest

The authors declare no conflict of interest.

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