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Higher-order balancing numbers: new sequences, recurrence relations, generating functions and identities

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In this article, we study a novel extension of the classic balancing numbers, referred to as the higher-order balancing numbers and denoted by $B_n^{(k)}$. This sequence is analogous to the higher-order Fibonacci numbers and follows the same recurrence relation as the balancing sequence itself. The case $k = 1$ gives the classic balancing numbers (A001109) and for $k = 2$ gives the sequence A029547, thus establishing a direct link to existing number sequences. Here, we first establish the Binet-like formula and then, with its help, present various algebraic properties of this newly introduced sequence, such as recurrence relations, generating functions (both ordinary and exponential), partial sums, binomial sums, combined identities, and more. We also obtain the limiting ratio and establish several well-known identities, including Catalan's identity, d'Ocagane's identity, Vajda's identity, Honsberger's identity, using the Binet-like formula. Finally, we give some mixed identity and series sum formulae. In this study, the obtained identities and algebraic properties are expressed in terms of the existing balancing and Lucas-balancing numbers.

Keywords: balancing numbers, Binet's formula, partial sums, binomial sums, Vajda's identity, binomial transform, generating functions, recurrence relations.

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Introduction

In number theory, the Fibonacci numbers emerges as celebrity kind numbers. Beside Fibonacci numbers, in number theory Lucas, Pell, Jacobsthal, Mersenne, Leonardo, Perrin, Padovan, etc. also follows the same pattern of study and opens an area of research for further investigation. One of such fascinating number sequences is the balancing sequence. The concept of balancing numbers (and balancers) was originally introduced in 1999 by Behera et al. [1]. Let us recall some important properties of it. A natural number n is said to be balancing number with balancer r if it satisfy the Diophantine equation

$$1 + 2 + 3 + ... + (n - 1) = (n + 1) + (n + 2) + ... + (n + r).$$

Thus the balancing numbers $\{B_n\}_{n\geq 0}$ are defined recursively as

$$B_{n+2} = 6B_{n+1} - B_n \quad \text{with} \quad B_0 = 0,\ B_1 = 1.$$

Another sequence close to balancing numbers is "Lucas-balancing numbers $\{C_n\}$" which is defined as $C_n = \sqrt{8B_n^2 + 1}$ and that satisfy the same recurrence relation but with initial condition $C_0 = 1,\ C_1 = 3$. The first few values of these sequence are:

n	**0**	**1**	2	3	4	5	6	7	8	...
B_n	**0**	**1**	6	35	204	1189	6930	40391	235416	...
C_n	**1**	**3**	17	99	577	3363	19601	114243	665857	...

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The closed form formula for the above sequences are given as

$$B_n = \frac{\lambda_1^n - \lambda_2^n}{4\sqrt{2}} \quad \text{and} \quad C_n = \frac{\lambda_1^n + \lambda_2^n}{2}, \tag{1}$$

where $\lambda_1 = 3 + \sqrt{8}$ and $\lambda_2 = 3 - \sqrt{8}$ are the roots of the characteristic equation $x^2 - 6x + 1 = 0$ and hold the following relations:

$$\lambda_1 + \lambda_2 = 6, \quad \lambda_1^2 + \lambda_2^2 = 34, \quad \lambda_1 - \lambda_2 = 2\sqrt{8} \quad \text{and} \quad \lambda_1\lambda_2 = 1.$$

Some identities involving the balancing and Lucas-balancing numbers which we need later are given in the following lemma.

Lemma 1. For integers m and n, we have

1. $B_{n-m}B_{n+m} - B_n^2 = -B_m^2$.
2. $C_{n-m}C_{n+m} - C_n^2 = \frac{1}{2}(C_{2m} - 1)$.
3. $C_{2n} = 2C_n^2 - 1$ and $C_n^2 = 8B_n^2 + 1$.

Theorem 1. For positive integer k and $n \geq 0$, we have

$$B_{k(n+1)} = \lambda_1^k B_{kn} + \lambda_2^{kn} B_k. \tag{2}$$

Proof. From Binet-like formula (1), the LHS can be written as

$$\begin{aligned} B_{k(n+1)} &= \frac{\lambda_1^{k(n+1)} - \lambda_2^{k(n+1)}}{4\sqrt{2}} \\ &= \frac{\lambda_1^{k(n+1)} - \lambda_2^{k(n+1)} + \lambda_1^k\lambda_2^{kn} - \lambda_1^k\lambda_2^{kn}}{4\sqrt{2}} \\ &= \frac{\lambda_1^k(\lambda_1^{kn} - \lambda_2^{kn}) + \lambda_2^{kn}(\lambda_1^k - \lambda_2^k)}{4\sqrt{2}} \\ &= \lambda_1^k B_{kn} + \lambda_2^{kn} B_k. \end{aligned}$$

□

In the next section, we first define the higher-order balancing numbers, and then we study their algebraic properties. In recent years, several articles have appeared on higher-order sequences associated with a famous number sequence, where the authors presented study on their algebraic properties. The study begins with the earlier work of Randić et al. (1996) [2], where the authors proposed higher-order Fibonacci numbers and investigated their properties in the context of applications in chemistry. Later, Ozvatan [3] studied a generalization of these higher-order Fibonacci numbers and obtained several algebraic properties.

Recently, Uysal and Özkan [4, 5] studied the quaternion algebra of higher-order Jacobsthal–Lucas numbers. Kızılateş [6] studied the hypercomplex numbers whose components are higher-order Fibonacci numbers while in Özimamoğlu [7] they considered higher-order Pell numbers. Kızılateş in their recent study [8], presented a comprehensive survey on the generalization of hybrid numbers (polynomial) with higher-order generalized Fibonacci polynomials. Kumari et al. [9] studied the algebra of quaternions and octonions of the higher-order Mersenne numbers. A similar concept was applied in [10] to study new sequences of balancing numbers. A polynomial version of the balancing numbers and their algebraic properties were studied by Ray [11] and Frontczak [12]. Some recent developments and applications of balancing numbers are due to Ray [13,14], Frontczak [12,15,16], Liptai [17], Özkoc [18], Panda [19], Prasad et al. [10, 20], etc. Similar to the above work, many algebraic properties of our proposed new sequences can be investigated after this study.

1 *Higher-order balancing numbers*

Throughout the article, we adopt the symbols $\{B_n\}$, $\{C_n\}$ and $\{B_n^{(k)}\}$ for the n-th balancing, Lucas-balancing and higher-order balancing numbers, respectively.

Definition 1. Let $k \in \mathbb{N}$, then the higher-order balancing numbers $\{B_n^{(k)}\}$ is defined as

$$B_n^{(k)} = \frac{B_{kn}}{B_k}, \quad n = 0, 1, 2, ... \tag{3}$$

From Definition 1, note that the following identities hold obviously:

1. $B_0^{(k)} = 0$ and $B_1^{(k)} = 1$.
2. $B_2^{(k)} = \lambda_1^k + \lambda_2^k = \ 2C_k$.

For different values of k, some higher-order balancing numbers are listed in Table 1:

Table 1

List of some higher-order balancing numbers ($B_n^{(k)}$)

Numbers	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$
$B_0^{(k)}$	0	0	0	0	0
$B_1^{(k)}$	1	1	1	1	1
$B_2^{(k)}$	6	34	198	1154	6726
$B_3^{(k)}$	35	1155	39203	1331715	45239075
$B_4^{(k)}$	204	39236	7761996	1536797956	304278011724
$B_5^{(k)}$	1189	1332869	1536836005	1773463509509	2046573861616547

Note that for $k = 2$, relation (3) gives a new sequence $B_n^{(2)} = (1/6)B_{2n}$ which generates the terms $1, 34, 1155, 39236, 1332869, 45278310, 1538129671, 52251130504, \ldots$, indexed as A029547 on OEIS and satisfy the recurrence relation $a_{n+1} = 34a_n - a_{n-1}$, with $a_{-1} = 0$, $a_0 = 1$.

Theorem 2. The sequence $\{B_n^{(k)}\}$ satisfies the following recurrence relation

$$B_{n+2}^{(k)} - (\lambda_1^k + \lambda_2^k)B_{n+1}^{(k)} + B_n^{(k)} = 0, \quad \text{with} \ \ B_0^{(k)} = 0, \ B_1^{(k)} = 1. \tag{4}$$

Proof. Dividing both sides of (2) by B_k and then using Definition 1, we get

$$B_{n+1}^{(k)} = \lambda_1^k B_n^{(k)} + \lambda_2^{kn}. \tag{5}$$

Now replacing n by $n+1$ in (5) yields

$$B_{n+2}^{(k)} = \lambda_1^k B_{n+1}^{(k)} + \lambda_2^{k(n+1)}. \tag{6}$$

Multiplying both sides of (6) by λ_2^{-k}, we get

$$B_{n+2}^{(k)}\lambda_2^{-k} = \lambda_1^k B_{n+1}^{(k)}\lambda_2^{-k} + \lambda_2^{kn}. \tag{7}$$

Thus, subtracting (5) from (7) gives the required result. □

Now we give the Binet's formula for higher-order balancing numbers and using the Binet's formula, we prove some algebraic identities for this sequence.

1.1 Binet's formula and some identities

For a fixed positive integer k, the characteristic equation of recurrences (4) is $x^2-(\lambda_1^k+\lambda_2^k)x+1=0$, whose roots are λ_1^k and λ_2^k. Thus, the Binet's formula for the higher-order balancing numbers $B_n^{(k)}$ is given by

$$B_n^{(k)} = \frac{\lambda_1^{kn} - \lambda_2^{kn}}{\lambda_1^k - \lambda_2^k}. \tag{8}$$

Theorem 3. The limiting ratio of higher-order balancing numbers is λ_1^k, i.e.,

$$\lim_{n\to\infty} \frac{B_{n+1}^{(k)}}{B_n^{(k)}} = \lambda_1^k.$$

Proof. From Binet's formula (8), we can write

$$\begin{aligned}
\lim_{n\to\infty} \frac{B_{n+1}^{(k)}}{B_n^{(k)}} &= \lim_{n\to\infty} \left(\frac{\frac{(\lambda_1^k)^{n+1}-(\lambda_2^k)^{n+1}}{\lambda_1^k-\lambda_2^k}}{\frac{(\lambda_1^k)^{n}-(\lambda_2^k)^{n}}{\lambda_1^k-\lambda_2^k}} \right) \\
&= \lim_{n\to\infty} \left(\frac{(\lambda_1^k)^{n+1} - (\lambda_2^k)^{n+1}}{(\lambda_1^k)^{n} - (\lambda_2^k)^{n}} \right) \\
&= \lim_{n\to\infty} \left(\frac{(\lambda_1^k)^{n}\lambda_1^k - (\lambda_2^k)^{n}\lambda_2^k}{(\lambda_1^k)^{n} - (\lambda_2^k)^{n}} \right) \\
&= \lim_{n\to\infty} \left(\frac{(\lambda_1^k)\left(\frac{(\lambda_2^k)^{n+1}}{(\lambda_1^k)^{n+1}} - 1\right)}{\left(\frac{(\lambda_2^k)^{n}}{(\lambda_1^k)^{n}} - 1\right)} \right).
\end{aligned}$$

Now taking into account that $\lim_{n\to\infty}\left(\frac{(\lambda_2^k)^{n+1}}{(\lambda_1^k)^{n+1}}\right) = 0$ and $\lim_{n\to\infty}\left(\frac{(\lambda_2^k)^{n}}{(\lambda_1^k)^{n}}\right) = 0$ as $|\frac{\lambda_2}{\lambda_1}| < 1$ for $k \in \mathbb{N}$, we have

$$\lim_{n\to\infty} \left(\left(\frac{(\lambda_2^k)^{n+1}}{(\lambda_1^k)^{n+1}} \right) - 1 \right) = -1.$$

So the original limit simplifies to

$$\lim_{n\to\infty} \frac{B_{n+1}^{(k)}}{B_n^{(k)}} = \lambda_1^k. \qquad \square$$

Theorem 4 (Catalan's identity)*.* For $n \geq r$, we have

$$B_{n-r}^{(k)} B_{n+r}^{(k)} - \left(B_n^{(k)}\right)^2 = -\left(B_r^{(k)}\right)^2.$$

Proof. Using Binet's formula (8) in LHS, we have

$$\begin{aligned}
{B_{n-r}}^{(k)} B_{n+r}^{(k)} - \left(B_n^{(k)}\right)^2 &= \left(\frac{(\lambda_1^k)^{n-r} - (\lambda_2^k)^{n-r}}{\lambda_1^k - \lambda_2^k} \right) \left(\frac{(\lambda_1^k)^{n+r} - (\lambda_2^k)^{n+r}}{\lambda_1^k - \lambda_2^k} \right) - \left(\frac{(\lambda_1^k)^{n} - (\lambda_2^k)^{n}}{\lambda_1^k - \lambda_2^k} \right)^2 \\
&= \frac{1}{(\lambda_1^k - \lambda_2^k)^2} \Big[(\lambda_1^k)^{n-r}(\lambda_1^k)^{n+r} - (\lambda_1^k)^{n-r}(\lambda_2^k)^{n+r} - (\lambda_2^k)^{n-r}(\lambda_1^k)^{n+r} + (\lambda_2^k)^{n-r}(\lambda_2^k)^{n+r} \Big] \\
&\quad - \frac{1}{(\lambda_1^k - \lambda_2^k)^2} \Big[(\lambda_1^k)^{2n} - 2(\lambda_1^k)^{n}(\lambda_2^k)^{n} + (\lambda_2^k)^{2n} \Big] \\
&= \frac{1}{(\lambda_1^k - \lambda_2^k)^2} \Big[\lambda_1^{2kn} - (\lambda_1^k)^{n-r}(\lambda_2^k)^{n+r} - (\lambda_2^k)^{n-r}(\lambda_1^k)^{n+r} + (\lambda_2^k)^{2n} - (\lambda_1^k)^{2n} + 2(\lambda_1^k)^{n}(\lambda_2^k)^{n} - \lambda_2^{2kn} \Big]
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{(\lambda_1^k-\lambda_2^k)^2}\Big[2(\lambda_1^k\lambda_2^k)^n-(\lambda_1^k\lambda_2^k)^n\frac{(\lambda_2^k)^r}{(\lambda_1^k)^r}-(\lambda_1^k\lambda_2^k)^n\frac{(\lambda_1^k)^r}{(\lambda_2^k)^r}\Big]\\
&= \frac{(1)^{kn}}{(\lambda_1^k-\lambda_2^k)^2}\Big(2-\frac{(\lambda_2^k)^r}{(\lambda_1^k)^r}-\frac{(\lambda_1^k)^r}{(\lambda_2^k)^r}\Big) \qquad (\text{since } \lambda_1\lambda_2=1)\\
&= \frac{1}{(\lambda_1^k-\lambda_2^k)^2(\lambda_1\lambda_2)^{kr}}\Big(2(\lambda_1\lambda_2)^{kr}-\lambda_1^{2kr}-\lambda_2^{2kr}\Big)\\
&= -\frac{1}{(\lambda_1^k-\lambda_2^k)^2}(\lambda_1^{kr}-\lambda_2^{kr})^2=-\big(B_r^{(k)}\big)^2.
\end{aligned}
$$

□

Corollary 1 (Cassini's identity). For positive integer n, we have

$$B_{n-1}^{(k)}B_{n+1}^{(k)}-\Big(B_n^{(k)}\Big)^2=-(B_1^k)^2=-1.$$

Theorem 5 (d'Ocagane identity). For integers m, n with $m > n$, we have

$$B_{n+1}^{(k)}B_m^{(k)}-B_n^{(k)}B_{m+1}^{(k)}=B_{m-n}^{(k)}.$$

Proof. Proceeding similar to the above theorem using Binet's formula (8) proves the given identity. □

Theorem 6 (Vajda's identity). For every $n, m, r \geq 0$, we have

$$B_{n+m}^{(k)}B_{n+r}^{(k)}-B_n^{(k)}B_{n+m+r}^{(k)}=B_m^{(k)}B_r^{(k)}.$$

Proof. Substituting Binet's formula (8) in the LHS, we get

$$
\begin{aligned}
&B_{n+m}^{(k)}B_{n+r}^{(k)}-B_n^{(k)}B_{n+m+r}^{(k)}\\
&=\left(\frac{\lambda_1^{k(n+m)}-\lambda_2^{k(n+m)}}{\lambda_1^k-\lambda_2^k}\right)\left(\frac{\lambda_1^{k(n+r)}-\lambda_2^{k(n+r)}}{\lambda_1^k-\lambda_2^k}\right)-\left(\frac{\lambda_1^{kn}-\lambda_2^{kn}}{\lambda_1^k-\lambda_2^k}\right)\left(\frac{\lambda_1^{k(n+m+r)}-\lambda_2^{k(n+m+r)}}{\lambda_1^k-\lambda_2^k}\right).
\end{aligned}
$$

Expanding the product of RHS, it becomes

$$
\begin{aligned}
&=\frac{\Big(\lambda_1^{k(n+m)}-\lambda_2^{k(n+m)}\Big)\Big(\lambda_1^{k(n+r)}-\lambda_2^{k(n+r)}\Big)-\Big(\lambda_1^{kn}-\lambda_2^{kn}\Big)\Big(\lambda_1^{k(n+m+r)}-\lambda_2^{k(n+m+r)}\Big)}{\Big(\lambda_1^k-\lambda_2^k\Big)^2}\\
&=\frac{1}{\Big(\lambda_1^k-\lambda_2^k\Big)^2}\Bigg(\Big(\lambda_1^{k(n+m+n+r)}-\lambda_1^{k(n+m)}\lambda_2^{k(n+r)}-\lambda_2^{k(n+m)}\lambda_1^{k(n+r)}+\lambda_2^{k(n+m+n+r)}\Big)\\
&\quad-\Big(\lambda_1^{k(n+n+m+r)}-\lambda_1^{kn}\lambda_2^{k(n+m+r)}-\lambda_2^{kn}\lambda_1^{k(n+m+r)}+\lambda_2^{k(n+n+m+r)}\Big)\Bigg)\\
&=\frac{1}{\Big(\lambda_1^k-\lambda_2^k\Big)^2}\Bigg(\Big(\lambda_1^{k(2n+m+r)}-\lambda_1^{k(n+m)}\lambda_2^{k(n+r)}-\lambda_2^{k(n+m)}\lambda_1^{k(n+r)}+\lambda_2^{k(2n+m+r)}\Big)\\
&\quad-\Big(\lambda_1^{k(2n+m+r)}-\lambda_1^{kn}\lambda_2^{k(n+m+r)}-\lambda_2^{kn}\lambda_1^{k(n+m+r)}+\lambda_2^{k(2n+m+r)}\Big)\Bigg).
\end{aligned}
$$

Since $\lambda_1\lambda_2 = 1$, so the above quantity reduced to

$$\begin{aligned}
&= \frac{1}{\left(\lambda_1^k - \lambda_2^k\right)^2}\left(\left(\lambda_1^{k(m+r)} + \lambda_2^{k(m+r)} - \lambda_1^{km}\lambda_2^{kr} - \lambda_2^{km}\lambda_1^{kr}\right)\right)\\
&= \frac{1}{\left(\lambda_1^k - \lambda_2^k\right)^2}\left(\left(\lambda_1^{km}(\lambda_1^{kr} - \lambda_2^{kr}) - \lambda_2^{km}(\lambda_1^{kr} - \lambda_2^{kr}\right)\right)\\
&= \left(\frac{\lambda_1^{km} - \lambda_2^{km}}{\lambda_1^k - \lambda_2^k}\right)\left(\frac{\lambda_1^{kr} - \lambda_2^{kr}}{\lambda_1^k - \lambda_2^k}\right)\\
&= B_m^{(k)}B_r^{(k)}.
\end{aligned}$$

Thus, we conclude that

$$B_{n+m}^{(k)}B_{n+r}^{(k)} - B_n^{(k)}B_{n+m+r}^{(k)} = B_m^{(k)}B_r^{(k)}.$$

□

Theorem 7 (Honsberger's identity). For any integers $p,\ n > 0$, we have

$$B_{p-1}^{(k)}B_n^{(k)} + B_p^{(k)}B_{n+1}^{(k)} = \frac{C_{kn}C_{k(p+n)} - C_{k(p-n-1)}}{8B_k^2}.$$

Proof. Taking into account $\lambda_1\lambda_2 = 1$, we obtain

$$\begin{aligned}
&B_{p-1}^{(k)}B_n^{(k)} + B_p^{(k)}B_{n+1}^{(k)}\\
&\quad= \left(\frac{\lambda_1^{k(p-1)} - \lambda_2^{k(p-1)}}{\lambda_1^k - \lambda_2^k}\right)\left(\frac{\lambda_1^{kn} - \lambda_2^{kn}}{\lambda_1^k - \lambda_2^k}\right) + \left(\frac{\lambda_1^{kp} - \lambda_2^{kp}}{\lambda_1^k - \lambda_2^k}\right)\left(\frac{\lambda_1^{k(n+1)} - \lambda_2^{k(n+1)}}{\lambda_1^k - \lambda_2^k}\right)\\
&\quad= \frac{1}{\left(\lambda_1^k - \lambda_2^k\right)^2}\left[\left(\lambda_1^{k(p-1+n)} - \lambda_1^{k(p-1)}\lambda_2^{kn} - \lambda_2^{k(p-1)}\lambda_1^{kn} + \lambda_2^{k(p-1+n)}\right)\right]\\
&\qquad+ \frac{1}{\left(\lambda_1^k - \lambda_2^k\right)^2}\left[\left(\lambda_1^{k(p+1+n)} - \lambda_1^{kp}\lambda_2^{k(n+1)} - \lambda_2^{kp}\lambda_1^{k(n+1)} + \lambda_2^{k(p+1+n)}\right)\right]\\
&\quad= \frac{1}{\left(\lambda_1^k - \lambda_2^k\right)^2}\Big[\lambda_1^{k(p+n)}\left[\lambda_1^{-kn} + \lambda_1^{kn}\right] + \lambda_2^{k(p+n)}\left[\lambda_2^{-kn} + \lambda_2^{kn}\right]\\
&\qquad- \lambda_1^{k(p-1-n)} - \lambda_2^{k(p-n-1)} - \lambda_1^{k(p-n-1)} - \lambda_2^{k(p-n-1)}\Big]\\
&\quad= \frac{1}{32B_k^2}\left[\left(\lambda_1^{k(p+n)} + \lambda_2^{k(p+n)}\right)\left(\lambda_1^{kn} + \lambda_2^{kn}\right) - 2\left(\lambda_1^{k(p-1-n)} + \lambda_2^{k(p-n-1)}\right)\right]\\
&\quad= \frac{C_{kn}C_{k(p+n)} - C_{k(p-n-1)}}{8B_k^2} \qquad \text{(using (1))}.
\end{aligned}$$

□

2 Some identities involving HOBN

Theorem 8 (Generating function). For higher-order balancing sequence, we have

$$\sum_{n=0}^{\infty} B_n^{(k)}x^n = \frac{x}{x^2 - 2C_kx + 1}.$$

Proof. Let $G(x,k)=\sum_{n=0}^{\infty} B_n^{(k)}x^n$ be the generating function for the higher-order balancing sequence. Now multiplying both sides of (4) by x^{n+2} and then taking the summation over 0 to ∞, we get

$$\sum_{n=0}^{\infty} B_{n+2}^{(k)}x^{n+2}-(\lambda_1^k+\lambda_2^k)\sum_{n=0}^{\infty} B_{n+1}^{(k)}x^{n+2}+\sum_{n=0}^{\infty} B_n^{(k)}x^{n+2}=0.$$

Now substituting

$$\sum_{n=0}^{\infty} B_{n+2}^{(k)}x^{n+2}=G(x,k)-B_1^{(k)}x-B_0^{(k)},\quad \sum_{n=0}^{\infty} B_{n+1}^{(k)}x^{n+2}=x[G(x,k)-B_0^{(k)}]$$

$$\text{and}\quad \sum_{n=0}^{\infty} B_n^{(k)}x^{n+2}=x^2G(x,k),$$

we get

$$\begin{gathered}\left(G(x,k)-B_1^{(k)}x-B_0^{(k)}\right)-\big(\lambda_1^k+\lambda_2^k\big)x\left[G(x,k)-B_0^{(k)}\right]+x^2G(x,k)=0,\\ G(x,k)\big[1-\big(\lambda_1^k+\lambda_2^k\big)x+x^2\big]+\big(\lambda_1^k+\lambda_2^k\big)xB_0^{(k)}-B_1^{(k)}x-B_0^{(k)}=0.\end{gathered}$$

Since $B_0^{(k)}=0$ and $B_1^{(k)}=1$, so after simplifications it gives the proposed identity. □

For instance, setting $k=1$ in the above theorem gives the generating function for the balancing sequence, i.e.,

$$G(x,1)=\frac{x}{x^2-6x+1}.$$

Theorem 9. The generating function for odd and even indexed sequences of higher-order balancing numbers $\{B_n^{(k)}\}$ are:

$$\sum_{r=0}^{\infty} B_{2r}^{(k)}x^r=\frac{2xC_k}{x^2-2C_{2k}x+1}\quad\text{and}\quad \sum_{r=0}^{\infty} B_{2r+1}^{(k)}x^r=\frac{x+1}{x^2-2C_{2k}x+1}.$$

Proof. Using Binet's formula of $B_{2r}^{(k)}$, we write

$$\begin{aligned}\sum_{r=0}^{\infty} B_{2r}^{(k)}x^r&=\sum_{r=0}^{\infty}\frac{\lambda_1^{2kr}-\lambda_2^{2kr}}{\lambda_1^k-\lambda_2^k}x^r=\frac{1}{\lambda_1^k-\lambda_2^k}\left(\sum_{r=0}^{\infty}\lambda_1^{2kr}x^r-\lambda_2^{2kr}x^r\right)\\ &=\frac{1}{\lambda_1^k-\lambda_2^k}\left[\frac{1}{1-\lambda_1^{2k}x}-\frac{1}{1-\lambda_2^{2k}x}\right]\\ &=\frac{1}{\lambda_1^k-\lambda_2^k}\left[\frac{x(\lambda_1^{2k}-\lambda_2^{2k})}{x^2-x(\lambda_1^{2k}+\lambda_2^{2k})+1}\right]\\ &=\frac{xB_2^{(k)}}{x^2-2C_{2k}x+1}=\frac{2xC_k}{x^2-2C_{2k}x+1}.\end{aligned}$$

In a similar fashion, the second identity can be also obtained. □

Theorem 10. The exponential generating function $E(x,k)$ for higher-order balancing numbers is

$$E(x,k)=\frac{e^{\lambda_1^k x}-e^{\lambda_2^k x}}{4\sqrt{2}B_k}.$$

Proof. Using Binet's formula (8) in exponential generating function $E(x,k)=\sum_{n=0}^{\infty}\frac{B_n^{(k)}x^n}{n!}$, we get

$$\begin{aligned}E(x,k)&=\sum_{n=0}^{\infty}\Big(\frac{\lambda_1^{kn}-\lambda_2^{kn}}{\lambda_1^k-\lambda_2^k}\Big)\frac{x^n}{n!}\\&=\frac{1}{\lambda_1^k-\lambda_2^k}\Big(\sum_{n=0}^{\infty}\frac{(\lambda_1^k x)^n}{n!}-\sum_{n=0}^{\infty}\frac{(\lambda_2^k x)^n}{n!}\Big)\\&=\frac{e^{\lambda_1^k x}-e^{\lambda_2^k x}}{\lambda_1^k-\lambda_2^k}.\end{aligned}$$

Substituting $\lambda_1^k-\lambda_2^k=4\sqrt{2}B_k$ proves the required identity. □

Theorem 11. The Poisson generating function for the higher-order balancing numbers is

$$\sum_{r=0}^{\infty}B_r^{(k)}\frac{x^{-r}}{r!}=\frac{e^{\frac{\lambda_1^k}{x}}-e^{\frac{\lambda_2^k}{x}}}{4\sqrt{2}B_k}.$$

Proof. In the previous theorem replacing x by x^{-1} gives the required identity. □

2.1 Partial sum and Binomial sum

Theorem 12 (Partial sum). For $k,n\in\mathbb{N}$, we have

$$\sum_{i=1}^{n}B_i^{(k)}=\frac{B_n^{(k)}-B_{n+1}^{(k)}+1}{2(1-C_k)}.$$

Proof. Using Binet's formula of the higher-order balancing numbers in $\sum_{i=1}^{n}B_i^{(k)}$, we have

$$\begin{aligned}&\sum_{i=1}^{n}\frac{1}{\lambda_1^k-\lambda_2^k}(\lambda_1^{ki}-\lambda_2^{ki})\\&=\frac{1}{\lambda_1^k-\lambda_2^k}\left(\lambda_1^k\frac{(\lambda_1^k)^n-1}{\lambda_1^k-1}-\lambda_2^k\frac{(\lambda_2^k)^n-1}{\lambda_2^k-1}\right)\\&=\frac{1}{\lambda_1^k-\lambda_2^k}\left[\frac{\big((\lambda_1^k)^{n+1}\lambda_2^k-\lambda_1^k\lambda_2^k-(\lambda_1^k)^{n+1}+\lambda_1^k\big)-\big((\lambda_2^k)^{n+1}\lambda_1^k-\lambda_1^k\lambda_2^k-(\lambda_2^k)^{n+1}+\lambda_2^k\big)}{(\lambda_1^k-1)(\lambda_2^k-1)}\right]\\&=\frac{1}{\lambda_1^k-\lambda_2^k}\left[\frac{\Big((\lambda_1^k)^n-(\lambda_2^k)^n\Big)-\Big((\lambda_1^k)^{n+1}-(\lambda_2^k)^{n+1}\Big)+\Big(\lambda_1^k-\lambda_2^k\Big)}{(\lambda_1\lambda_2)^k-\lambda_1^k-\lambda_2^k+1}\right]\\&=\frac{B_n^{(k)}-B_{n+1}^{(k)}+1}{2(1-C_k)}.\end{aligned}$$

□

Theorem 13 (Partial sum with even indexes). For $k,n\in\mathbb{N}$, we have

$$\sum_{i=1}^{n}B_{2i}^{(k)}=\frac{B_2^{(k)}+B_{2n}^{(k)}-B_{2(n+1)}^{(k)}}{2-2C_{2k}}.$$

Proof. We write

$$\sum_{i=1}^{n} B_{2i}^{(k)} = \sum_{i=1}^{n} \frac{1}{\lambda_1^k - \lambda_2^k}\left(\lambda_1^{2ki} - \lambda_2^{2ki}\right)$$

$$= \frac{1}{\lambda_1^k - \lambda_2^k}\left(\lambda_1^{2k}\frac{(\lambda_1^{2k})^n - 1}{\lambda_1^{2k} - 1} - \lambda_2^{2k}\frac{(\lambda_2^{2k})^n - 1}{\lambda_2^{2k} - 1}\right).$$

After simplifying using Binet's formula, we get

$$\sum_{i=1}^{n} B_{2i}^{(k)} = \frac{B_2^{(k)} + B_{2n}^{(k)} - B_{2(n+1)}^{(k)}}{2 - 2C_{2k}}.$$

☐

Theorem 14 (Partial sum with odd indexes). For $k, n \in \mathbb{N}$, we have

$$\sum_{i=0}^{n} B_{2i+1}^{(k)} = \frac{C_{2k(n+1)-1}}{C_{2k} - 1}.$$

Proof. Using the Binet's formula, we have

$$\sum_{i=0}^{n} B_{2i+1}^{(k)} = \sum_{i=0}^{n} \frac{1}{\lambda_1^k - \lambda_2^k}\left(\lambda_1^{k(2i+1)} - \lambda_2^{k(2i+1)}\right)$$

$$= \frac{1}{\lambda_1^k - \lambda_2^k}\left(\lambda_1^{k}\frac{(\lambda_1^{2k})^{n+1} - 1}{\lambda_1^{2k} - 1} - \lambda_2^{k}\frac{(\lambda_2^{2k})^{n+1} - 1}{\lambda_2^{2k} - 1}\right)$$

$$= \frac{1}{(\lambda_1^k - \lambda_2^k)(\lambda_1^k - 1)(\lambda_2^k - 1)}\Big[\Big(\lambda_1^k(\lambda_1^{2k})^{n+1}\lambda_2^{2k} - \lambda_1^k\lambda_2^{2k} - \lambda_1^k(\lambda_1^{2k})^{n+1} + \lambda_1^k\Big)$$
$$- \Big(\lambda_2^k(\lambda_2^{2k})^{n+1}\lambda_1^{2k} - \lambda_1^{2k}\lambda_2^k - \lambda_2^k(\lambda_2^{2k})^{n+1} + \lambda_2^k\Big)\Big]$$

$$= \frac{1}{\lambda_1^k - \lambda_2^k}\left[\frac{\left(\lambda_1^k(\lambda_1^{2k})^n - \lambda_1^k(\lambda_1^{2k})^{n+1} - \lambda_2^k(\lambda_2^{2k})^n + \lambda_2^k(\lambda_2^{2k})^{n+1} + 2(\lambda_1^k - \lambda_2^k)\right)}{(\lambda_1^k - 1)(\lambda_2^k - 1)}\right] \quad (\text{by } \lambda_1\lambda_2 = 1)$$

$$= \frac{1}{\lambda_1^k - \lambda_2^k}\left[\frac{\left((\lambda_1^{2k})^{n+1}(\lambda_2^k - \lambda_1^k) + (\lambda_1^{2k})^{n+1}(\lambda_2^k - \lambda_1^k) + 2(\lambda_1^k - \lambda_2^k)\right)}{(\lambda_1\lambda_2)^k - \lambda_1^k - \lambda_2^k + 1}\right]$$

$$= \frac{C_{2k(n+1)-1}}{C_{2k} - 1}.$$

☐

Theorem 15 (Binomial sum). For the higher-order balancing numbers $\{B_n^{(k)}\}$, we have

1. $\sum_{n=0}^{s-1}\binom{s-1}{n}B_n^{(k)} = \frac{1}{4\sqrt{2}B_k}\left((1+\lambda_1^k)^{s-1} - (1+\lambda_2^k)^{s-1}\right)$.
2. $\sum_{n=0}^{s-1}(-1)^n\binom{s-1}{n}B_n^{(k)} = \frac{1}{4\sqrt{2}B_k}\left((1-\lambda_1^k)^{s-1} - (1-\lambda_2^k)^{s-1}\right)$.

Proof. 1. From Binet's formula (8), we write

$$\sum_{n=0}^{s-1}\binom{s-1}{n}B_n^{(k)} = \sum_{n=0}^{s-1}\binom{s-1}{n}\frac{\lambda_1^{kn} - \lambda_2^{kn}}{\lambda_1^k - \lambda_2^k}$$

$$= \frac{1}{\lambda_1^k - \lambda_2^k}\left(\sum_{n=0}^{s-1}\binom{s-1}{n}\lambda_1^{kn} - \sum_{n=0}^{s-1}\binom{s-1}{n}\lambda_2^{kn}\right)$$

$$= \frac{1}{4\sqrt{2}B_k}\left((1+\lambda_1^k)^{s-1} - (1+\lambda_2^k)^{s-1}\right) \quad (\text{using the Binomial theorem}).$$

2. From Binet's formula (8), we write

$$\begin{aligned}\sum_{n=0}^{s-1}(-1)^n\binom{s-1}{n}B_n^{(k)} &= \sum_{n=0}^{s-1}\binom{s-1}{n}\frac{(-\lambda_1^k)^n-(-\lambda_2^k)^n}{\lambda_1^k-\lambda_2^k}\\ &= \frac{1}{\lambda_1^k-\lambda_2^k}\left(\sum_{n=0}^{s-1}\binom{s-1}{n}(-\lambda_1^k)^n-\sum_{n=0}^{s-1}\binom{s-1}{n}(-\lambda_2^k)^n\right)\\ &= \frac{1}{4\sqrt{2}B_k}\left((1-\lambda_1^k)^{s-1}-(1-\lambda_2^k)^{s-1}\right) \quad \text{(using the Binomial theorem).} \qquad \square\end{aligned}$$

Theorem 16. If m and n are positive integers, then

1. $\sum_{r=0}^{2n}\binom{2n}{r}B_{2r}^{(k)} = \frac{1}{4\sqrt{2}B_k}\left((1+\lambda_1^{2k})^{2n}-(1+\lambda_2^{2k})^{2n}\right)$.
2. $\sum_{r=0}^{2n+1}\binom{2n+1}{r}B_{2r}^{(k)} = \frac{1}{4\sqrt{2}B_k}\left((1+\lambda_1^{2k})^{2n+1}-(1+\lambda_2^{2k})^{2n+1}\right)$.

Proof. 1. From Binet's formula (8), we write

$$\begin{aligned}\sum_{r=0}^{2n}\binom{2n}{r}B_{2r}^{(k)} &= \sum_{r=0}^{2n}\binom{2n}{r}\frac{\lambda_1^{2kr}-\lambda_2^{2kr}}{\lambda_1^k-\lambda_2^k}\\ &= \frac{1}{\lambda_1^k-\lambda_2^k}\left(\sum_{r=0}^{2n}\binom{2n}{r}\lambda_1^{2kr}-\sum_{r=0}^{2n}\binom{2n}{r}\lambda_2^{2kr}\right)\\ &= \frac{1}{\lambda_1^k-\lambda_2^k}\left((1+\lambda_1^{2k})^{2n}-(1+\lambda_2^{2k})^{2n}\right) \quad \text{(using the Binomial theorem)}\\ &= \frac{1}{4\sqrt{2}B_k}\left((1+\lambda_1^{2k})^{2n}-(1+\lambda_2^{2k})^{2n}\right).\end{aligned}$$

2. From Binet's formula (8), we write

$$\begin{aligned}\sum_{r=0}^{2n+1}\binom{2n+1}{r}B_{2r}^{(k)} &= \sum_{r=0}^{2n+1}\binom{2n+1}{r}\frac{\lambda_1^{2kr}-\lambda_2^{2kr}}{\lambda_1^k-\lambda_2^k}\\ &= \frac{1}{\lambda_1^k-\lambda_2^k}\left(\sum_{r=0}^{2n+1}\binom{2n+1}{r}\lambda_1^{2kr}-\sum_{r=0}^{2n+1}\binom{2n+1}{r}\lambda_2^{2kr}\right)\\ &= \frac{1}{\lambda_1^k-\lambda_2^k}\left((1+\lambda_1^{2k})^{2n+1}-(1+\lambda_2^{2k})^{2n+1}\right) \quad \text{(using the Binomial theorem)}\\ &= \frac{1}{4\sqrt{2}B_k}\left((1+\lambda_1^{2k})^{2n+1}-(1+\lambda_2^{2k})^{2n+1}\right). \qquad \square\end{aligned}$$

With the help of Binet's formula, the following identities can be easily established.

Theorem 17. For $n \geq 0$, the following identitites are verified:

1. $B_{-n}^{(k)} = -B_n^{(k)}$.
2. $B_{2n}^{(k)} = 2C_{kn}B_n^{(k)}$.
3. $B_{3n}^{(k)} = 32B_n^{(k)}(B_{kn}^2+3)$.
4. $B_{n\pm r}^{(k)} = B_n^{(k)}C_{kr} \pm B_r^{(k)}C_{kn}$.
5. $B_{n+r}^{(k)} + B_{n-r}^{(k)} = 2C_{kr}B_n^{(k)}$.
6. $B_{n+r}^{(k)} - B_{n-r}^{(k)} = 2C_{kn}B_r^{(k)}$.
7. $(B_{n+r}^{(k)})^2 - (B_{n-r}^{(k)})^2 = 4C_{kn}C_{kr}B_n^{(k)}B_r^{(k)}$.
8. $B_{n+r}^{(k)}B_{n-r}^{(k)} = (B_n^{(k)})^2 - (B_r^{(k)})^2$.

Proof. 1. Since $B_{-nk} = -B_{nk}$, so

$$B^{(k)}_{-n} = \frac{B_{-nk}}{B_k} = -\frac{B_{nk}}{B_k} = -B^{(k)}_n.$$

2. Using $B_{2k} = 2B_kC_k$, we can write $B^{(k)}_{2n} = \dfrac{B_{2kn}}{B_k} = 2C_{kn}\dfrac{B_{kn}}{B_k} = 2C_{kn}B^{(k)}_n$.

3. Taking into account the identity $B_{3k} = 32B^3_k + 3B_k$, we have

$$B^{(k)}_{3n} = \frac{B_{3nk}}{B_k} = \frac{32B^3_{kn} + 3B_{kn}}{B_k} = 32B^{(k)}_n(B^2_{kn} + 3).$$

For identity 4, 5 and 6, along with Definition 1, use $B_{m\pm n} = B_mC_n \pm C_mB_n)$, $B_{a+b}+B_{a-b} = 2B_aC_b$ and $B_{a+b} - B_{a-b} = 2C_aB_b$, respectively.

7. Using $B_{a+b} + B_{a-b} = 2B_aC_b$ and $B_{a+b} - B_{a-b} = 2C_aB_b$, we have

$$\begin{aligned}(B^{(k)}_{n+r})^2 - (B^{(k)}_{n-r})^2 &= \frac{(B_{k(n+r)})^2 - (B_{k(n-r)})^2}{(B_k)^2}\\ &= \frac{(B_{kn+kr} + B_{kn-kr})(B_{kn+kr} - B_{kn-kr})}{(B_k)^2}\\ &= \frac{4B_{kn}B_{kr}C_{kn}C_{kr}}{(B_k)^2}\\ &= 4C_{kn}C_{kr}B^{(k)}_nB^{(k)}_r.\end{aligned}$$

8. Since $B_{a+b}B_{a-b} = B^2_a - B^2_b$, so

$$B^{(k)}_{n+r}B^{(k)}_{n-r} = \frac{B_{kn+kr}B_{kn-kr}}{B^2_k} = \frac{B^2_{kn} - B^2_{kr}}{B^2_k} = (B^{(k)}_n)^2 - (B^{(k)}_r)^2. \qquad \square$$

Theorem 18. If m and n are positive integers, then

1. $\sum_{r=1}^n B^{(k)}_{2mr} = \dfrac{B^{(k)}_{2mn} - B^{(k)}_{2m(n+1)} + B^{(k)}_{2m}}{2 - 2C_{2mk}}$.
2. $\sum_{r=1}^n (-1)^r B^{(k)}_{2mr} = \dfrac{(-1)^n\left(B^{(k)}_{2mn} + B^{(k)}_{2m(n+1)}\right) - B^{(k)}_{2m}}{2 + 2C_{2mk}}$.
3. $\sum_{r=1}^n (B^{(k)}_{mr})^2 = \dfrac{1}{C_{2k} - 1}\left(\dfrac{C_{2mk} + C_{2mkn} - C_{2mk(1+n)} - 1}{2(1 - C_{2mk})} - n\right)$.
4. $\sum_{r=1}^n (-1)^r (B^{(k)}_{mr})^2 = \begin{cases} \dfrac{1}{2(C_{2k} - 1)}\left[\frac{(-1)^n(C_{2mkn}+C_{2mk(n+1)})-C_{2mk}}{C_{2mk}+1}\right] & : n = \text{even}\\ \dfrac{1}{2(C_{2k} - 1)}\left[\frac{(-1)^n(C_{2mkn}+C_{2mk(n+1)})-C_{2mk}}{C_{2mk}+1} + 2\right] & : n = \text{odd}.\end{cases}$

Proof. 1. Using the Binet's formula (8), we have

$$\begin{aligned}
\sum_{r=1}^{n} B_{2mr}^{(k)} &= \sum_{r=1}^{n} \frac{1}{\lambda_1^k - \lambda_2^k}(\lambda_1^{2mrk} - \lambda_2^{2mrk}) \\
&= \frac{1}{\lambda_1^k - \lambda_2^k}\left(\lambda_1^{2mk}\frac{(\lambda_1^{2mk})^n - 1}{\lambda_1^{2mk} - 1} - \lambda_2^{2mk}\frac{(\lambda_2^{2mk})^n - 1}{\lambda_2^{2mk} - 1}\right) \\
&= \frac{1}{\lambda_1^k - \lambda_2^k}\left[\frac{\left((\lambda_1^{2mk})^{n+1}\lambda_2^{2mk} - (\lambda_1\lambda_2)^{2mk} - (\lambda_1^{2mk})^{n+1} + \lambda_1^{2mk}\right)}{(\lambda_1^{2mk} - 1)(\lambda_2^{2mk} - 1)}\right] \\
&\quad - \frac{1}{\lambda_1^k - \lambda_2^k}\left[\frac{\left((\lambda_2^{2mk})^{n+1}\lambda_1^{2mk} - (\lambda_1\lambda_2)^{2mk} - (\lambda_2^{2mk})^{n+1} + \lambda_2^{2mk}\right)}{(\lambda_1^{2mk} - 1)(\lambda_2^{2mk} - 1)}\right] \\
&= \frac{1}{\lambda_1^k - \lambda_2^k}\left[\frac{\Big((\lambda_1^{2mk})^n - (\lambda_2^{2mk})^n\Big) - \Big((\lambda_1^{2mk})^{n+1} - (\lambda_2^{2mk})^{n+1}\Big) + \Big(\lambda_1^{2mk} - \lambda_2^{2mk}\Big)}{(\lambda_1\lambda_2)^{2mk} - \lambda_1^{2mk} - \lambda_2^{2mk} + 1}\right] \\
&= \frac{B_{2mn}^{(k)} - B_{2m(n+1)}^{(k)} + B_{2m}^{(k)}}{2 - 2C_{2mk}}.
\end{aligned}$$

2. Using the Binet's formula (8), we have

$$\begin{aligned}
\sum_{r=1}^{n}(-1)^r B_{2mr}^{(k)} &= \sum_{r=1}^{n} \frac{(-1)^r}{\lambda_1^k - \lambda_2^k}(\lambda_1^{2mrk} - \lambda_2^{2mrk}) \\
&= \frac{1}{\lambda_1^k - \lambda_2^k}\left(\sum_{r=1}^{n}(-\lambda_1^{2mk})^r - \sum_{r=1}^{n}(-\lambda_2^{2mk})^r\right) \\
&= \frac{1}{\lambda_1^k - \lambda_2^k}\left(-\lambda_1^{2mk}\frac{1 - (-\lambda_1^{2mk})^n}{1 - (-\lambda_1^{2mk})} - (-\lambda_2^{2mk})\frac{1 - (-\lambda_2^{2mk})^n}{1 - (-\lambda_2^{2mk})}\right).
\end{aligned}$$

After simplifying, we get

$$\sum_{r=1}^{n}(-1)^r B_{2mr}^{k} = \frac{(-1)^n\Big(B_{2mn}^{(k)} + B_{2m(n+1)}^{(k)}\Big) - B_{2m}^{(k)}}{2 + 2C_{2mk}}.$$

3. Using the Binet's formula (8) in $\sum_{r=1}^{n}(B_{mr}^{(k)})^2$, we have

$$\begin{aligned}
\sum_{r=1}^{n}\left(\frac{\lambda_1^{mrk} - \lambda_2^{mrk}}{\lambda_1^k - \lambda_2^k}\right)^2 &= \frac{1}{(\lambda_1^k - \lambda_2^k)^2}\sum_{r=1}^{n}\Big(\lambda_1^{2mkr} + \lambda_2^{2mkr} - 2(\lambda_1\lambda_2)^{mkr}\Big) \\
&= \frac{1}{(\lambda_1^k - \lambda_2^k)^2}\left(\lambda_1^{2mk}\frac{(\lambda_1^{2mk})^n - 1}{\lambda_1^{2mk} - 1} + \lambda_2^{2mk}\frac{(\lambda_2^{2mk})^n - 1}{\lambda_2^{2mk} - 1} - 2n\right).
\end{aligned}$$

After simplifying and rearrangement, it becomes

$$\sum_{r=1}^{n}(B_{mr}^{(k)})^2 = \frac{1}{C_{2k} - 1}\left(\frac{C_{2mk} + C_{2mkn} - C_{2mk(1+n)} - 1}{2(1 - C_{2mk})} - n\right).$$

4. Using the Binet's formula of higher-order balancing number, we have

$$\begin{aligned}\sum_{r=1}^{n}(-1)^r (B_{mr}^{(k)})^2 &= \sum_{r=1}^{n}(-1)^r \left(\frac{\lambda_1^{mrk}-\lambda_2^{mrk}}{\lambda_1^k-\lambda_2^k}\right)^2\\
&= \frac{1}{(\lambda_1^k-\lambda_2^k)^2}\sum_{r=1}^{n}(-1)^r\left(\lambda_1^{2mkr}+\lambda_2^{2mkr}-2(\lambda_1\lambda_2)^{mkr}\right)\\
&= \frac{1}{(\lambda_1^k-\lambda_2^k)^2}\left[\sum_{r=1}^{n}(-\lambda_1^{2mk})^r+\sum_{r=1}^{n}(-\lambda_2^{2mk})^r-2\sum_{r=1}^{n}(-1)^r\right]\\
&= \frac{1}{(\lambda_1^k-\lambda_2^k)^2}\left[(-\lambda_1^{2mk})\frac{(-\lambda_1^{2mk})^n-1}{(-\lambda_1^{2mk})-1}+(-\lambda_2^{2mk})\frac{(-\lambda_2^{2mk})^n-1}{(-\lambda_2^{2mk})-1}-2\sum_{r=1}^{n}(-1)^r\right]\\
&= \frac{1}{2(C_{2k}-1)}\left[\frac{(-1)^n(C_{2mkn}+C_{2mk(n+1)})-C_{2mk}}{C_{2mk}+1}-2\sum_{r=1}^{n}(-1)^r\right].\end{aligned}$$

Since $\sum_{r=1}^{n}(-1)^r=0$ and -1 for $n=$ even and odd, respectively, so

$$\sum_{r=1}^{n}(-1)^r (B_{mr}^{(k)})^2=\begin{cases}\dfrac{1}{2(C_{2k}-1)}\left[\dfrac{(-1)^n(C_{2mkn}+C_{2mk(n+1)})-C_{2mk}}{C_{2mk}+1}\right] & : n=\text{even}\\ \dfrac{1}{2(C_{2k}-1)}\left[\dfrac{(-1)^n(C_{2mkn}+C_{2mk(n+1)})-C_{2mk}}{C_{2mk}+1}+2\right] & : n=\text{odd}.\end{cases}$$

□

Conclusion

In this article, we introduce a new extension of the classical balancing numbers, which generalizes the balancing numbers in a different way. For this integer sequence, we present a Binet-like formula and various algebraic properties, such as generating functions (both ordinary and exponential), various partial and binomial sums, etc. In addition, we present several identities in connection with the existing balancing and Lucas-balancing numbers. In particular, for $k = 1$ it gives the classical balancing numbers (A001109) and for $k = 2$ it gives the sequence A029547.

Following this study, this sequence can be extended and applied in many directions, such as hypercomplex numbers, polynomials, quaternions, octonions, matrix theory, transforms, etc. We are currently working on hypercomplex/2^m-ions numbers using this new sequence.

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Author Contributions

K.P.: Conceptualization, Methodology, Formal Analysis, Supervision; **I.:** Investigation, Formal Analysis, Validation; **M.K.:** Conceptualization, Formal Analysis, Validation, Manuscript preparation; All authors contributed equally to this work and approved the final submission.

Conflict of Interest

The authors declare no conflict of interest.

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