

Singular discrete Sturm–Liouville problems with transmission conditions

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This paper examines singular discrete Sturm–Liouville equations subject to transmission conditions, focusing on their fundamental spectral and geometric properties. In mathematical physics and operator theory, understanding the asymptotic behavior of solutions at singular boundaries is highly essential. While the classical spectral theory for continuous systems has been comprehensively analyzed, the structural nuances of their discrete counterparts containing interior discontinuities require a more detailed investigation. For the discrete equations under consideration, Weyl’s famous alternative classification is successfully established, dividing the underlying equations into limit-circle and limit-point cases based on the behavior of their solutions. By incorporating transmission conditions at an interior point, this study bridges a critical gap between discrete calculus and boundary value problems with transmission interactions. We construct the corresponding Hilbert space framework and thoroughly investigate the core properties of the corresponding Weyl function. Furthermore, the geometric characteristics of Weyl circles are systematically derived. Ultimately, the generalized spectral characteristics established herein offer comprehensive insights into discrete systems with transmission conditions, contributing significantly to the ongoing advancement of the spectral analysis of discontinuous difference operators.

Keywords: difference equations, discrete Sturm–Liouville problem, boundary conditions, Weyl theory and its generalizations, limit circle case, limit point case, transmission conditions, singular case.

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Introduction

In differential equation theory, the Sturm–Liouville problem is one of the most studied problems. The significance of the topic is increased by the appearance of such issues in the solution of partial differential equations using the technique of separating their variables [1–3]. Such problems have been studied under a variety of boundary condition types and situations [4–6]. Similar to impulsive boundary conditions, these conditions have been extensively studied in the literature (see [7–9] and its references).

For many mathematicians today, discrete equations are one of the most important topics to study. Discrete equations come from the approximate solution of differential equations through discretization. Nonetheless, they often manifest independently as mathematical encodings of various real-life events. Using algorithmic techniques, computers can solve discrete equations with ease. Spectral theory of difference equations has been addressed in [10–12]. In [13], the authors investigated the spectral properties of the following second-order difference equation

$$\begin{aligned} & a(\zeta - 1)z(\zeta - 1) + b(\zeta)z(\zeta) + a(\zeta)z(\zeta + 1) \\ & = (2 \cosh y)z(\zeta), \quad \zeta \in \mathbb{N} \setminus \{k - 1, k, k + 1\} \quad (\mathbb{N} := \{1, 2, 3, \dots\}), \end{aligned}$$

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$$z(0) = 0, \quad \begin{cases} z(k+1) = \theta_1 z(k-1), \\ \Delta z(k+1) = \theta_2 \nabla z(k-1), \end{cases} \quad \theta_1, \theta_2 \in \mathbb{R},$$

where $\{a(\zeta)\}_{\zeta \in \mathbb{N}}, \{b(\zeta)\}_{\zeta \in \mathbb{N}}$ are real sequences. Ergün and Guseinov [14] studied the following difference equation

$$\begin{aligned} & \Delta^2 z(\zeta - 1) + q(\zeta) z(\zeta) \\ & = \lambda \rho(\zeta) z(\zeta), \quad \zeta \in \mathbb{Z} \setminus \{0, 1\}, \quad (\mathbb{Z} := \{0, \pm 1, \pm 2, \pm 3, \dots\}), \\ & z(-1) = z(1), \quad \Delta z(-1) = e^{2i\delta} \Delta z(1), \end{aligned}$$

where $z = z(\zeta)$ with $\zeta \in \mathbb{Z}$ is a desired solution, $q(\zeta) \in \mathbb{R}$ for $\zeta \in \mathbb{Z} \setminus \{0, 1\}$, $\delta \in [0, \frac{\pi}{2})$, and

$$\rho(\zeta) = \begin{cases} e^{2i\delta} & \text{if } n \leq -1, \\ e^{-2i\delta} & \text{if } n \geq 2. \end{cases}$$

They gave simple conditions for the discreteness of its spectrum. In [15], Bairamov et al. considered a discrete Schrödinger operator defined as

$$\begin{aligned} & a(\zeta - 1) z(\zeta - 1) + b(\zeta) z(\zeta) + a(\zeta) z(\zeta + 1) \\ & = \lambda z(\zeta), \quad \zeta \in \mathbb{N} \setminus \{m_0 - 1, m_0, m_0 + 1\}, \end{aligned}$$

$$\begin{aligned} & z(m_0 + 1) = \gamma_1 z(m_0 - 1), \\ & z(m_0 + 2) - z(m_0 + 1) = \gamma_2 (z(m_0 - 1) - z(m_0 - 2)), \end{aligned}$$

where $\gamma_1 \gamma_2 \neq 0, \gamma_1, \gamma_2 \in \mathbb{R}$; λ is a spectral parameter, $\{a(\zeta)\}_{\zeta \in \mathbb{N} \cup \{0\}}$ and $\{b(\zeta)\}_{\zeta \in \mathbb{N} \cup \{0\}}$ are real sequences satisfying the condition

$$\sum_{\zeta \in \mathbb{N}} \zeta (|1 - a(\zeta)| + |b(\zeta)|) < \infty.$$

By using the scattering solutions, they investigated the properties of the scattering function of this problem.

In this study, we will transfer the famous classification given by Weyl [16] in 1910 for classical Sturm–Liouville problems to discrete Sturm–Liouville problems with transmission conditions. It has been studied in [10, 12] for discrete Sturm–Liouville problems in the singular case without transmission conditions.

1 Set up

In this paper, we consider the discrete Sturm–Liouville equation with transmission conditions

$$\Gamma(z) := \frac{1}{w(\zeta)} \nabla [p(\zeta) \Delta z(\zeta)] + q(\zeta) z(\zeta) = \lambda z(\zeta), \quad \zeta \in J, \quad (1)$$

$$\begin{aligned} & p(\eta + 1) z(\eta + 1) - dp(\eta - 1) z(\eta - 1) = 0, \\ & z(\eta + 2) - \frac{1}{d} z(\eta) = 0, \end{aligned} \quad (2)$$

where Δ is the forward difference operator ($\Delta y(\zeta) = y(\zeta + 1) - y(\zeta)$), and ∇ is the backward difference operator ($\nabla y(\zeta) = y(\zeta) - y(\zeta - 1)$), and $-\infty \leq a < \eta - 1 < \eta < \eta + 1 < b \leq \infty$, $a, \eta, b \in \mathbb{Z}$, and $J = [a, b] \cap \mathbb{Z} \setminus \{\eta, \eta + 1\}$, λ is a parameter, $p(\zeta)$, $q(\zeta)$, and $w(\zeta)$ are real numbers such that $w(\zeta) > 0$ for $\zeta \in J$, and $p(\zeta) > 0$ for $\zeta \in J$, $q(\zeta)$ is a finite at $\eta, \eta + 1$, and $p(\zeta)$ is a positive at $\eta, \eta + 1$, and $0 \neq d \in \mathbb{R}$.

Recently, the spectral properties of the problem (1)–(2) were investigated in the references [17, 18] in the regular and singular cases.

Theorem 1. [18] Let ζ_0 be a fixed point in $[a, b] \cap \mathbb{Z}$. Then Problem (1), (2) has a unique solution $(z(\zeta))_{\zeta \in [a, b] \cap \mathbb{Z}}$, such that

$$z(\zeta_0) = k_1, \quad \Delta z(\zeta_0) = k_2,$$

where $k_1, k_2 \in \mathbb{C}$.

Theorem 2. [18] Let Ξ and Θ be solutions of the equation $\Gamma z = \lambda z$. Then the Wronskian defined as

$$W[\Xi, \Theta](\zeta) := p(\zeta - 1) [\Xi(\zeta) \Delta \Theta(\zeta - 1) - \Theta(\zeta) \Delta \Xi(\zeta - 1)]$$

is constant on each of the sets $[a, \eta - 1] \cap \mathbb{Z}$ and $[\eta + 2, b] \cap \mathbb{Z}$:

$$W[\Xi, \Theta](\zeta) = \begin{cases} W^-, & \zeta \in [a, \eta - 1] \cap \mathbb{Z}, \\ W^+, & \zeta \in [\eta + 2, b] \cap \mathbb{Z}. \end{cases}$$

Furthermore,

$$W^- = W^+,$$

and

$$W[\Xi, \Theta](\eta) = -\frac{1}{d} W^+.$$

Definition 1. Problem (1)–(2) is singular when the interval $[a, b]$ is infinite length.

Denote by $l_w^2(J)$ the Hilbert space with the inner product

$$(f, g) := \sum_{\zeta=a}^{\eta-1} f(\zeta) \overline{g(\zeta)} w(\zeta) + \sum_{\zeta=\eta+2}^b f(\zeta) \overline{g(\zeta)} w(\zeta),$$

where $f, g \in l_w^2(J)$.

For any Ξ and Θ in $l_w^2(J)$, we have

$$\begin{aligned} & (\Gamma(\Xi), \Theta) - (\Xi, \Gamma(\Theta)) \\ &= p(a-1) [\Xi(a-1) \overline{\Theta(a)} - \Xi(a) \overline{\Theta(a-1)}] \\ & \quad - p(\eta-1) [\Xi(\eta-1) \overline{\Theta(\eta)} - \Xi(\eta) \overline{\Theta(\eta-1)}] \\ & \quad + p(\eta+1) [\Xi(\eta+1) \overline{\Theta(\eta+2)} - \Xi(\eta+2) \overline{\Theta(\eta+1)}] \\ & \quad - p(b) [\Xi(b) \overline{\Theta(b+1)} - \Xi(b+1) \overline{\Theta(b)}] \\ &= [\Xi, \Theta](a) - [\Xi, \Theta](\eta) + [\Xi, \Theta](\eta+2) - [\Xi, \Theta](b+1), \end{aligned}$$

where

$$[\Xi, \Theta](\zeta) := W[\Xi, \overline{\Theta}](\zeta) = p(\zeta-1) [\Xi(\zeta-1) \overline{\Theta(\zeta)} - \Xi(\zeta) \overline{\Theta(\zeta-1)}].$$

2 The nested circles at b

Let us consider $J_1 = \{[c, b'] \cap \mathbb{Z}\} \setminus \{\eta, \eta + 1\}$, where $a < c < \eta - 1 < \eta < \eta + 1 < b' < b$ and let χ_1 and χ_2 be solutions of Problem (1)–(2) satisfying

$$\chi_1(c) = 0, \quad \chi_1(c-1) = -\frac{1}{p(c-1)},$$

$$\chi_2(c) = 1, \quad \chi_2(c-1) = 0.$$

It is clear that

$$\begin{aligned} & W[\chi_1, \chi_2](n) \\ &= -\frac{1}{d} W^-[\chi_1, \chi_2](c) p(c-1) [\chi_1(c-1) \chi_2(c) - \chi_1(c) \chi_2(c-1)] = \frac{1}{d}, \quad n \in J_1. \end{aligned}$$

Now consider the following regular problem:

$$\Gamma(z) = \lambda z(\zeta), \quad \zeta \in J_1, \quad (3)$$

$$z(c-1) = 0, \quad (4)$$

$$z(b'+1) + kz(b') = 0, \quad k \in \mathbb{R}, \quad (5)$$

$$\begin{aligned} p(\eta+1)z(\eta+1) - dp(\eta-1)z(\eta-1) &= 0, \\ z(\eta+2) - \frac{1}{d}z(\eta) &= 0. \end{aligned} \quad (6)$$

For $\text{Im } \lambda \neq 0$, let $\varphi_{b'} = \chi_1 + m_{b'}\chi_2$ be the solution of Problem (3)–(6). The function $m_{b'}$ is called the Titchmarsh–Weyl function of Problem (3)–(6). Here $\varphi_{b'}$ and χ_2 must be linearly independent.

Proposition 1. Let Ξ be a solution of $\Gamma u = \lambda u$ and Θ is a solution of $\Gamma u = \mu u$, both satisfying (4) and (6). Then, for $\zeta \in J_1$,

$$\begin{aligned} & (\lambda - \mu) \left(\sum_{\gamma=a}^{\eta-1} \Xi(\gamma) \Theta(\gamma) w(\gamma) + \sum_{\gamma=\eta+2}^{\zeta} \Xi(\gamma) \Theta(\gamma) w(\gamma) \right) \\ &= p(\zeta) [\Xi(\zeta+1) \Theta(\zeta) - \Xi(\zeta) \Theta(\zeta+1)]. \end{aligned}$$

Proof. From (3), we obtain

$$\lambda w(\gamma) \Xi(\gamma) - q(\gamma) \Xi(\gamma) = \nabla [p(\gamma) \Delta \Xi(\gamma)]$$

and

$$\mu w(\gamma) \Theta(\gamma) - q(\gamma) \Theta(\gamma) = \nabla [p(\gamma) \Delta \Theta(\gamma)].$$

Multiplying these equations, respectively, by $\Theta(\gamma)$, $-\Xi(\gamma)$, adding gives

$$(\lambda - \mu) \Xi(\gamma) \Theta(\gamma) = \nabla [p(\gamma) \Delta \Xi(\gamma)] \Theta(\gamma) - \nabla [p(\gamma) \Delta \Theta(\gamma)] \Xi(\gamma),$$

i.e.,

$$\begin{aligned} & (\lambda - \mu) \Xi(\gamma) \Theta(\gamma) = p(\gamma) [\Xi(\gamma+1) \Theta(\gamma) - \Xi(\gamma) \Theta(\gamma+1)] \\ & - p(\gamma-1) [\Xi(\gamma) \Theta(\gamma-1) - \Xi(\gamma-1) \Theta(\gamma)]. \end{aligned} \quad (7)$$

Thus, we find

$$\begin{aligned}
 & (\lambda - \mu) \left(\sum_{\gamma=a}^{\eta-1} \Xi(\gamma) \Theta(\gamma) w(\gamma) + \sum_{\gamma=\eta+2}^{\zeta} \Xi(\gamma) \Theta(\gamma) w(\gamma) \right) \\
 &= \sum_{\gamma=a}^{\eta-1} p(\gamma) [\Xi(\gamma+1) \Theta(\gamma) - \Xi(\gamma) \Theta(\gamma+1)] \\
 &\quad - \sum_{\gamma=a}^{\eta-1} p(\gamma-1) [\Xi(\gamma) \Theta(\gamma-1) - \Xi(\gamma-1) \Theta(\gamma)] \\
 &\quad + \sum_{\gamma=\eta+2}^{\zeta} p(\gamma) [\Xi(\gamma+1) \Theta(\gamma) - \Xi(\gamma) \Theta(\gamma+1)] \\
 &\quad - \sum_{\gamma=\eta+2}^{\zeta} p(\gamma-1) [\Xi(\gamma) \Theta(\gamma-1) - \Xi(\gamma-1) \Theta(\gamma)].
 \end{aligned}$$

By virtue of (4) and (6), we see that

$$\begin{aligned}
 & (\lambda - \mu) \left(\sum_{\gamma=a}^{\eta-1} \Xi(\gamma) \Theta(\gamma) w(\gamma) + \sum_{\gamma=\eta+2}^{\zeta} \Xi(\gamma) \Theta(\gamma) w(\gamma) \right) \\
 &= p(\zeta) [\Xi(\zeta+1) \Theta(\zeta) - \Xi(\zeta) \Theta(\zeta+1)].
 \end{aligned}$$

□

Theorem 3. Let $\text{Im } \lambda \neq 0$. $\varphi_{b'}$ satisfies conditions (4)–(6) if and only if $m_{b'}$ lies on a circle $C_{b'}$ in the complex plane. The equation of $C_{b'}$ is

$$W[\varphi_{b'}, \overline{\varphi_{b'}}] (b' + 1) = 0, \quad (8)$$

with radius

$$r_{b'} = |W[\chi_2, \overline{\chi_2}] (b' + 1)|^{-1},$$

and center

$$\widetilde{m}_{b'} = -\frac{W[\chi_1, \overline{\chi_2}] (b' + 1)}{W[\chi_2, \overline{\chi_2}] (b' + 1)}.$$

Proof. From (5), we obtain

$$k = -\frac{\varphi_{b'}(b' + 1)}{\varphi_{b'}(b')}.$$

Since $k \in \mathbb{R}$, we have

$$\begin{aligned}
 0 = k - \overline{k} &= -\frac{\varphi_{b'}(b' + 1)}{\varphi_{b'}(b')} + \frac{\overline{\varphi_{b'}(b' + 1)}}{\overline{\varphi_{b'}(b')}} \\
 &= \frac{\varphi_{b'}(b') \overline{\varphi_{b'}(b' + 1)} - \varphi_{b'}(b' + 1) \overline{\varphi_{b'}(b')}}{|\varphi_{b'}(b')|^2},
 \end{aligned}$$

i.e.,

$$\varphi_{b'}(b') \overline{\varphi_{b'}(b' + 1)} - \varphi_{b'}(b' + 1) \overline{\varphi_{b'}(b')} = 0.$$

Hence,

$$W[\varphi_{b'}, \overline{\varphi_{b'}}] (b' + 1) = 0.$$

$\varphi_{b'} (b' + 1) + k\varphi_{b'} (b') = 0$ is equivalent to

$$m_{b'} = -\frac{\chi_1 (b' + 1) + k\chi_1 (b')}{\chi_2 (b' + 1) + k\chi_2 (b')},$$

which describes a circle in the complex plane. By (8), we find

$$\begin{aligned} |m_{b'}|^2 W[\chi_2, \overline{\chi_2}] (b' + 1) + m_{b'} W[\chi_2, \overline{\chi_1}] (b' + 1) \\ \overline{m_{b'}} W[\chi_1, \overline{\chi_2}] (b' + 1) + W[\chi_1, \overline{\chi_1}] (b' + 1) = 0. \end{aligned} \quad (9)$$

Let

$$m_{b'} = u + iv, \quad W[\chi_2, \overline{\chi_2}] (b' + 1) = 2iA,$$

and

$$-W[\chi_2, \overline{\chi_1}] (b' + 1) = B + iC.$$

Then we obtain

$$W[\chi_1, \overline{\chi_2}] (b' + 1) = B - iC,$$

and

$$W[\chi_1, \overline{\chi_1}] (b' + 1) = 2iD.$$

Thus, by (9), we conclude that

$$(2iA) (u^2 + v^2) - (u + iv) (B + iC) + (u - iv) (B - iC) + 2iD = 0.$$

Hence,

$$\left(u - \frac{C}{2A}\right)^2 + \left(v - \frac{B}{2A}\right)^2 = \frac{B^2 + C^2 - 4AD}{4A^2},$$

which implies

$$\widetilde{m_{b'}} = \frac{C + iB}{2A} = -\frac{B - iC}{2iA} = -\frac{W[\chi_1, \overline{\chi_2}] (b' + 1)}{W[\chi_2, \overline{\chi_2}] (b' + 1)},$$

and

$$r_{b'} = \sqrt{\frac{B^2 + C^2 - 4AD}{4A^2}} = |W[\chi_2, \overline{\chi_2}] (b' + 1)|^{-1}.$$

□

Theorem 4. $m_{b'}$ is inside or on $C_{b'}$ if and only if

$$\sum_{\zeta=c}^{\eta-1} |\varphi_{b'} (\zeta)|^2 w (\zeta) + \sum_{\zeta=\eta+2}^{b'} |\varphi_{b'} (\zeta)|^2 w (\zeta) \leq -\frac{\operatorname{Im} m_{b'}}{\operatorname{Im} \lambda}.$$

Furthermore, we have

$$\widetilde{m_{b'}} = \frac{1 - 2i \operatorname{Im} \lambda \left[\sum_{\zeta=c}^{\eta-1} \chi_1 (\zeta) \overline{\chi_2} (\zeta) w (\zeta) + \sum_{\zeta=\eta+2}^{b'} \chi_1 (\zeta) \overline{\chi_2} (\zeta) w (\zeta) \right]}{2i \operatorname{Im} \lambda \left[\sum_{\zeta=c}^{\eta-1} |\chi_2 (\zeta)|^2 w (\zeta) + \sum_{\zeta=\eta+2}^{b'} |\chi_2 (\zeta)|^2 w (\zeta) \right]}, \quad (10)$$

and

$$r_{b'} = \left| 2i \operatorname{Im} \lambda \left[\sum_{\zeta=c}^{\eta-1} |\chi_2(\zeta)|^2 w(\zeta) + \sum_{\zeta=\eta+2}^{b'} |\chi_2(\zeta)|^2 w(\zeta) \right] \right|^{-1}. \quad (11)$$

Proof. From (8) and (9), we see that $m_{b'}$ is inside or on $C_{b'}$ if and only if

$$\frac{W[\varphi_{b'}, \overline{\varphi_{b'}}](b'+1)}{W[\chi_2, \overline{\chi_2}](b'+1)} \leq 0. \quad (12)$$

Let Ξ and Θ be solutions of (3), (6). From (7), we find

$$\begin{aligned} & W[\Xi, \overline{\Theta}](b'+1) - W[\Xi, \overline{\Theta}](c) \\ &= 2i \operatorname{Im} \lambda \left[\sum_{\zeta=c}^{\eta-1} \Xi(\zeta) \overline{\Theta}(\zeta) w(\zeta) + \sum_{\zeta=\eta+2}^{b'} \Xi(\zeta) \overline{\Theta}(\zeta) w(\zeta) \right]. \end{aligned}$$

Thus, we obtain

$$\begin{aligned} & W[\chi_2, \overline{\chi_2}](b'+1) \\ &= 2i \operatorname{Im} \lambda \left[\sum_{\zeta=c}^{\eta-1} |\chi_2(\zeta)|^2 w(\zeta) + \sum_{\zeta=\eta+2}^{b'} |\chi_2(\zeta)|^2 w(\zeta) \right] = 2iA, \end{aligned}$$

and

$$\begin{aligned} & W[\chi_1, \overline{\chi_2}](b'+1) \\ &= -1 + 2i \operatorname{Im} \lambda \left[\sum_{\zeta=c}^{\eta-1} \chi_1(\zeta) \overline{\chi_2}(\zeta) w(\zeta) + \sum_{\zeta=\eta+2}^{b'} \chi_1(\zeta) \overline{\chi_2}(\zeta) w(\zeta) \right] = B - iC. \end{aligned}$$

Therefore, we have

$$\begin{aligned} & W[\chi_2, \overline{\chi_1}](b'+1) \\ &= 1 + 2i \operatorname{Im} \lambda \left[\sum_{\zeta=c}^{\eta-1} \chi_2(\zeta) \overline{\chi_1}(\zeta) w(\zeta) + \sum_{\zeta=\eta+2}^{b'} \chi_2(\zeta) \overline{\chi_1}(\zeta) w(\zeta) \right] = -(B + iC), \end{aligned}$$

and

$$\begin{aligned} & W[\chi_1, \overline{\chi_1}](b'+1) \\ &= 2i \operatorname{Im} \lambda \left[\sum_{\zeta=c}^{\eta-1} |\chi_1(\zeta)|^2 w(\zeta) + \sum_{\zeta=\eta+2}^{b'} |\chi_1(\zeta)|^2 w(\zeta) \right] = 2iD. \end{aligned}$$

It follows from (9) that

$$\begin{aligned} & W[\varphi_{b'}, \overline{\varphi_{b'}}](b'+1) \\ &= m_{b'} - \overline{m_{b'}} + 2i \operatorname{Im} \lambda \left[\sum_{\zeta=c}^{\eta-1} |\varphi_{b'}(\zeta)|^2 w(\zeta) + \sum_{\zeta=\eta+2}^{b'} |\varphi_{b'}(\zeta)|^2 w(\zeta) \right]. \end{aligned}$$

By virtue of (12), we conclude that $m_{b'}$ is inside or on $C_{b'}$ if and only if

$$\frac{m_{b'} - \overline{m_{b'}} + 2i \operatorname{Im} \lambda \left[\sum_{\zeta=c}^{\eta-1} |\varphi_{b'}(\zeta)|^2 w(\zeta) + \sum_{\zeta=\eta+2}^{b'} |\varphi_{b'}(\zeta)|^2 w(\zeta) \right]}{2i \operatorname{Im} \lambda \left[\sum_{\zeta=c}^{\eta-1} |\chi_2(\zeta)|^2 w(\zeta) + \sum_{\zeta=\eta+2}^{b'} |\chi_2(\zeta)|^2 w(\zeta) \right]} \leq 0.$$

From the previous calculations, we obtain (10) and (11). □

Theorem 5. The $C_{b'}$ circles are nested.

Proof. Let $b'' < b' < b$. Then we obtain

$$\begin{aligned} & \sum_{\zeta=c}^{\eta-1} |\varphi_{b'}(\zeta)|^2 w(\zeta) + \sum_{\zeta=\eta+2}^{b''} |\varphi_{b'}(\zeta)|^2 w(\zeta) \\ & \leq \sum_{\zeta=c}^{\eta-1} |\varphi_{b'}(\zeta)|^2 w(\zeta) + \sum_{\zeta=\eta+2}^{b'} |\varphi_{b'}(\zeta)|^2 w(\zeta) \leq -\frac{\operatorname{Im} m_{b'}}{\operatorname{Im} \lambda}. \end{aligned}$$

From Theorem 4, we see that $m_{b'}$ is inside or on $C_{b''}$. Hence the $C_{b'}$ circles are nested. □

Corollary 1. As $b' \rightarrow b = +\infty$, the $C_{b'}$ circles contract to a limit-circle or a limit-point. Let $m_{b'} \rightarrow m_b$ and let $\varphi_b = \chi_1 + m_b \chi_2$. Then we obtain

$$\sum_{\zeta=c}^{\eta-1} |\chi_1(\zeta) + m_b \chi_2(\zeta)|^2 w(\zeta) + \sum_{\zeta=\eta+2}^b |\chi_1(\zeta) + m_b \chi_2(\zeta)|^2 w(\zeta) \leq -\frac{\operatorname{Im} m_b}{\operatorname{Im} \lambda},$$

i.e., $\varphi_b \in l^2(J_3)$, where $J_3 = \{[c, b] \cap \mathbb{Z}\} \setminus \{\eta, \eta+1\}$. Moreover, $m_{b'}$ approaches m_b independently of k . If we take $k = 0$, then we have

$$m_b = -\lim_{b' \rightarrow b} \frac{\chi_1(b'+1)}{\chi_2(b'+1)}.$$

i) The limit-circle (LC) case: Let the $C_{b'}$ circles converge to a circle. By (11), we get

$$\sum_{\zeta=c}^{\eta-1} |\chi_2(\zeta)|^2 w(\zeta) + \sum_{\zeta=\eta+2}^b |\chi_2(\zeta)|^2 w(\zeta) < \infty,$$

i.e., all solutions of (3) in $l^2(J_3)$.

ii) The limit-point (LP) case: Let the $C_{b'}$ circles converge to a point. From (11), we find

$$\sum_{\zeta=c}^{\eta-1} |\chi_2(\zeta)|^2 w(\zeta) + \sum_{\zeta=\eta+2}^b |\chi_2(\zeta)|^2 w(\zeta) = \infty,$$

i.e., φ_b is the only linearly independent solution in $l^2(J_3)$.

Theorem 6. For some value λ , the limit-circle case holds at $b = +\infty$. Then the same is true for all $\mu \in \mathbb{C}$.

Proof. The proof is similar to [12, Theorem 3.2.7]. □

3 The nested circles at a

Let us consider the set $J_4 = \{[a', c-1] \cap \mathbb{Z}\} \setminus \{\eta, \eta+1\}$, where $a < a' < \eta-1 < \eta < \eta+1 < c-1$, $c < b$. Then the problem of finding the solution of

$$\Gamma(z) = \lambda z \quad (13)$$

satisfying the following conditions:

$$z(a' - 1) + hz(a') = 0, \quad (14)$$

$$z(c) = 0, \quad (15)$$

$$\begin{aligned} p(\eta+1)z(\eta+1) - dp(\eta-1)z(\eta-1) &= 0, \\ z(\eta+2) - \frac{1}{d}z(\eta) &= 0, \end{aligned} \quad (16)$$

is a regular self-adjoint problem on J_4 . For $\operatorname{Im} \lambda \neq 0$, let $\varphi_{a'} = \chi_1 + m_{a'}\chi_2$ be the solution of Problem (13)–(16). Here $\varphi_{a'}$ and χ_2 must be linearly independent.

Theorem 7. Let $\operatorname{Im} \lambda \neq 0$. $\varphi_{a'}$ satisfies conditions (13)–(16) if and only if $m_{a'}$ lies on a circle $C_{a'}$ in the complex plane. The equation of $C_{a'}$ is

$$W[\varphi_{a'}, \overline{\varphi_{a'}}](a') = 0,$$

with radius

$$r_{a'} = |W[\chi_2, \overline{\chi_2}](a')|^{-1},$$

and center

$$\widetilde{m}_{a'} = -\frac{W[\chi_1, \overline{\chi_2}](a')}{W[\chi_2, \overline{\chi_2}](a')}.$$

Proof. The proof is similar to Theorem 3. □

Theorem 8. $m_{a'}$ is inside or on $C_{a'}$ if and only if

$$\sum_{\zeta=a'}^{\eta-1} |\varphi_{a'}(\zeta)|^2 w(\zeta) + \sum_{\zeta=\eta+2}^{c-1} |\varphi_{a'}(\zeta)|^2 w(\zeta) \leq -\frac{\operatorname{Im} m_{a'}}{\operatorname{Im} \lambda}.$$

Proof. $m_{a'}$ is inside or on $C_{a'}$ if and only if

$$\frac{W[\varphi_{a'}, \overline{\varphi_{a'}}](a')}{W[\varphi_{a'}, \overline{\varphi_{a'}}](a')} \leq 0. \quad (17)$$

Let Ξ and Θ be solutions of (13), (16). From (7), we obtain

$$\begin{aligned} & W[\Xi, \overline{\Theta}](a') \\ &= W[\Xi, \overline{\Theta}](c) - 2i \operatorname{Im} \lambda \left[\sum_{\zeta=a'}^{\eta-1} \Xi(\zeta) \overline{\Theta}(\zeta) w(\zeta) + \sum_{\zeta=\eta+2}^{c-1} \Xi(\zeta) \overline{\Theta}(\zeta) w(\zeta) \right]. \end{aligned}$$

Thus, we get

$$W[\chi_2, \overline{\chi_2}](a') = -2i \operatorname{Im} \lambda \left[\sum_{\zeta=a'}^{\eta-1} |\chi_2(\zeta)|^2 w(\zeta) + \sum_{\zeta=\eta+2}^{c-1} |\chi_2(\zeta)|^2 w(\zeta) \right],$$

$$\begin{aligned}
& W[\chi_1, \overline{\chi_2}](a') \\
&= -1 - 2i \operatorname{Im} \lambda \left[\sum_{\zeta=a'}^{\eta-1} \chi_1(\zeta) \overline{\chi_2}(\zeta) w(\zeta) + \sum_{\zeta=\eta+2}^{c-1} \chi_1(\zeta) \overline{\chi_2}(\zeta) w(\zeta) \right], \\
& W[\chi_2, \overline{\chi_1}](a') \\
&= -1 - 2i \operatorname{Im} \lambda \left[\sum_{\zeta=a'}^{\eta-1} \overline{\chi_1}(\zeta) \chi_2(\zeta) w(\zeta) + \sum_{\zeta=\eta+2}^{c-1} \overline{\chi_1}(\zeta) \chi_2(\zeta) w(\zeta) \right],
\end{aligned}$$

and

$$W[\chi_1, \overline{\chi_1}](a') = -2i \operatorname{Im} \lambda \left[\sum_{\zeta=a'}^{\eta-1} |\chi_1(\zeta)|^2 w(\zeta) + \sum_{\zeta=\eta+2}^{c-1} |\chi_1(\zeta)|^2 w(\zeta) \right].$$

Hence

$$\begin{aligned}
& W[\varphi_{a'}, \overline{\varphi_{a'}}](a') = m_{a'} - \overline{m_{a'}} \\
& -2i \operatorname{Im} \lambda \sum_{\zeta=a'}^{\eta-1} \left[\begin{array}{c} |m_{a'}|^2 |\chi_2(\zeta)|^2 + m_{a'} \overline{\chi_1}(\zeta) \chi_2(\zeta) \\ + \overline{m_{a'}} \chi_1(\zeta) \overline{\chi_2}(\zeta) + |\chi_1(\zeta)|^2 \end{array} \right] w(\zeta) \\
& -2i \operatorname{Im} \lambda \sum_{\zeta=\eta+2}^{c-1} \left[\begin{array}{c} |m_{a'}|^2 |\chi_2(\zeta)|^2 + m_{a'} \overline{\chi_1}(\zeta) \chi_2(\zeta) \\ + \overline{m_{a'}} \chi_1(\zeta) \overline{\chi_2}(\zeta) + |\chi_1(\zeta)|^2 \end{array} \right] w(\zeta) \\
&= 2i \operatorname{Im}(m_{a'}) - 2i \operatorname{Im} \lambda \sum_{\zeta=a'}^{\eta-1} |\varphi_{a'}(\zeta)|^2 w(\zeta) + \sum_{\zeta=\eta+2}^{c-1} |\varphi_{a'}(\zeta)|^2 w(\zeta).
\end{aligned}$$

By (17), we conclude that $m_{a'}$ is inside or on $C_{a'}$ if and only if

$$\frac{\operatorname{Im}(m_{a'}) - \operatorname{Im} \lambda \sum_{\zeta=a'}^{\eta-1} |\varphi_{a'}(\zeta)|^2 w(\zeta) + \sum_{\zeta=\eta+2}^{c-1} |\varphi_{a'}(\zeta)|^2 w(\zeta)}{-\operatorname{Im} \lambda \left[\sum_{\zeta=a'}^{\eta-1} |\chi_2(\zeta)|^2 w(\zeta) + \sum_{\zeta=\eta+2}^{c-1} |\chi_2(\zeta)|^2 w(\zeta) \right]} \leq 0.$$

□

From the previous calculations, we get

$$\widetilde{m_{a'}} = \frac{1 + 2i \operatorname{Im} \lambda \left[\sum_{\zeta=a'}^{\eta-1} \chi_1(\zeta) \overline{\chi_2}(\zeta) w(\zeta) + \sum_{\zeta=\eta+2}^{c-1} \chi_1(\zeta) \overline{\chi_2}(\zeta) w(\zeta) \right]}{-2i \operatorname{Im} \lambda \left[\sum_{\zeta=a'}^{\eta-1} |\chi_2(\zeta)|^2 w(\zeta) + \sum_{\zeta=\eta+2}^{c-1} |\chi_2(\zeta)|^2 w(\zeta) \right]}$$

and

$$r_{a'} = \left| 2i \operatorname{Im} \lambda \left[\sum_{\zeta=a'}^{\eta-1} |\chi_2(\zeta)|^2 w(\zeta) + \sum_{\zeta=\eta+2}^{c-1} |\chi_2(\zeta)|^2 w(\zeta) \right] \right|^{-1}.$$

Theorem 9. The $C_{a'}$ circles are nested.

Proof. The proof is similar to Theorem 5. □

Corollary 2. As $a' \rightarrow a = -\infty$, the $C_{a'}$ circles contract to a limit-circle or a limit-point. Let $m_{a'} \rightarrow m_a$ and let $\varphi_a = \chi_1 + m_a \chi_2$. Then we obtain

$$\sum_{\zeta=a'}^{\eta-1} |\chi_1(\zeta) + m_a \chi_2(\zeta)|^2 w(\zeta) + \sum_{\zeta=\eta+2}^{c-1} |\chi_1(\zeta) + m_a \chi_2(\zeta)|^2 w(\zeta) \leq \frac{\operatorname{Im} m_a}{\operatorname{Im} \lambda},$$

i.e., $\varphi_a \in l^2(J_5)$, where $J_5 = \{[a, c-1] \cap \mathbb{Z}\} \setminus \{\eta, \eta+1\}$. Moreover, $m_{a'}$ approaches m_a independently of k . If we take $k=0$, then we have

$$m_a = - \lim_{a' \rightarrow a} \frac{\chi_1(a'-1)}{\chi_2(a'-1)}.$$

i) The limit-circle (LC) case: Let the $C_{a'}$ circles converge to a circle. Then all solutions of (13) in $l^2(J_5)$.

ii) The limit-point (LP) case: Let the $C_{a'}$ circles converge to a point. Then φ_a is the only linearly independent solution in $l^2(J_5)$.

Theorem 10. For some value λ , the limit-circle case holds at a . Then the same is true for all $\mu \in \mathbb{C}$.

4 Existence of boundary conditions

Let λ_0 be a complex number (possibly real in LC cases, but with nonzero imaginary part in LP cases) and χ_2 be a solution of

$$\Gamma(z) = \lambda_0 z. \quad (18)$$

Let φ_b be a solution of Eq. (18) in $l^2(J_3)$ such that $W[\varphi_b, \chi_2] = -1$.

Theorem 11. Let $\Gamma(z) = f$, where $z, f \in l^2(J_3)$. Then

i)

$$\lim_{b' \rightarrow b} W[z, \chi_2] (b' + 1)$$

exists when $b = +\infty$ is LC case, and

ii)

$$\lim_{b' \rightarrow b} W[z, \varphi_b] (b' + 1)$$

exists in all limit cases.

Proof. i) By assumption, we obtain

$$\nabla [p(\zeta) \Delta z(\zeta)] + q(\zeta) z(\zeta) = w(\zeta) f(\zeta)$$

and

$$\nabla [p(\zeta) \Delta \chi_2(\zeta)] + q(\zeta) \chi_2(\zeta) = \lambda_0 w(\zeta) \chi_2(\zeta).$$

Multiplying these equations, respectively, by $\chi_2(\zeta)$, $-z(\zeta)$, adding gives

$$\begin{aligned} & \nabla [p(\zeta) \Delta z(\zeta)] \chi_2(\zeta) - \nabla [p(\zeta) \Delta \chi_2(\zeta)] z(\zeta) \\ &= w(\zeta) f(\zeta) \chi_2(\zeta) - \lambda_0 w(\zeta) \chi_2(\zeta) z(\zeta), \end{aligned}$$

i.e.,

$$W[z, \chi_2](\zeta) - W[z, \chi_2](\zeta + 1)$$

$$= w(\zeta) f(\zeta) \chi_2(\zeta) - \lambda_0 w(\zeta) \chi_2(\zeta) z(\zeta). \quad (19)$$

From (19), we find

$$\begin{aligned} W[z, \chi_2](c) - W[z, \chi_2](\eta) + W[z, \chi_2](\eta + 2) - W[z, \chi_2](b' + 1) \\ = \sum_{\zeta=c}^{\eta-1} [w(\zeta) f(\zeta) \chi_2(\zeta) - \lambda_0 w(\zeta) \chi_2(\zeta) z(\zeta)] \\ + \sum_{\zeta=\eta+2}^{b'} [w(\zeta) f(\zeta) \chi_2(\zeta) - \lambda_0 w(\zeta) \chi_2(\zeta) z(\zeta)]. \end{aligned}$$

Hence,

$$\begin{aligned} [z, \chi_2](b' + 1) &= W[z, \chi_2](c) - W[z, \chi_2](\eta) + W[z, \chi_2](\eta + 2) \\ &\quad - \left[\sum_{\zeta=c}^{\eta-1} w(\zeta) f(\zeta) \chi_2(\zeta) + \sum_{\zeta=\eta+2}^{b'} w(\zeta) f(\zeta) \chi_2(\zeta) \right] \\ &\quad + \lambda_0 \left[\sum_{\zeta=c}^{\eta-1} w(\zeta) \chi_2(\zeta) z(\zeta) + \sum_{\zeta=\eta+2}^{b'} [w(\zeta) \chi_2(\zeta) z(\zeta)] \right]. \end{aligned}$$

In the LC case at $b = +\infty$, $\chi_2 \in l^2(J_3)$. Consequently, $\lim_{b' \rightarrow b} W[z, \chi_2](b' + 1)$ exists.

ii) Similarly, we have

$$\nabla [p(\zeta) \Delta z(\zeta)] + q(\zeta) z(\zeta) = w(\zeta) f(\zeta)$$

and

$$\nabla [p(\zeta) \Delta \varphi_b(\zeta)] + q(\zeta) \varphi_b(\zeta) = \lambda_0 w(\zeta) \varphi_b(\zeta).$$

Then we find

$$\begin{aligned} [z, \varphi_b](b' + 1) &= W[z, \varphi_b](c) - W[z, \varphi_b](\eta) + W[z, \varphi_b](\eta + 2) \\ &\quad - \left[\sum_{\zeta=c}^{\eta-1} w(\zeta) f(\zeta) \varphi_b(\zeta) + \sum_{\zeta=\eta+2}^{b'} w(\zeta) f(\zeta) \varphi_b(\zeta) \right] \\ &\quad + \lambda_0 \left[\sum_{\zeta=c}^{\eta-1} w(\zeta) \varphi_b(\zeta) z(\zeta) + \sum_{\zeta=\eta+2}^{b'} [w(\zeta) \varphi_b(\zeta) z(\zeta)] \right]. \end{aligned}$$

Since $\varphi_b \in l^2(J_3)$, we see that $\lim_{b' \rightarrow b} W[z, \varphi_b](b' + 1)$ exists in all limit cases. $\square$

Similarly, let φ_a be a solution of Eq. (18) in $l^2(J_5)$ such that $W[\varphi_a, \chi_2] = -1$.

Theorem 12. Let $\Gamma(z) = f$, where $z, f \in l^2(J_5)$. Then

i)

$$\lim_{a' \rightarrow a} W[z, \chi_2](a')$$

exists when $a = -\infty$ is LC case, and

ii)

$$\lim_{a' \rightarrow a} W[z, \varphi_a](a')$$

exists in all limit cases.

Proof. The proof is similar to Theorem 11. $\square$

Conclusion

In this paper, we have successfully investigated the spectral and geometric properties of singular discrete Sturm–Liouville equations subject to interior transmission conditions. The core achievement of this research is the rigorous establishment of Weyl’s alternative classification for this class of discontinuous difference equations. By classifying the underlying configurations into limit-circle and limit-point cases based on the behavior of their solutions, this work extends classical continuous operator theory into the discrete and discontinuous domains.

Furthermore, we have constructed a robust Hilbert space framework that accommodates the discrete dynamics alongside the jump conditions at the interior singular point. Through this structure, the analytical properties of the corresponding Weyl function were thoroughly explored, and the geometric behavior of the associated Weyl circles was systematically derived. These results provide a deeper understanding of how interior transmission conditions influence the overall spectral landscape of discrete systems.

The generalized framework and methodologies developed in this study open new avenues for further research. Future work may focus on extending these results to discrete non-selfadjoint operators. Ultimately, the insights gained herein contribute significantly to the ongoing advancement of the spectral analysis of discontinuous difference operators.

Author contributions

All authors contributed equally to this work.

Conflict of interest

The authors declare no conflict of interest.

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