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Integral representations for a class of triple confluent hypergeometric functions and their applications in boundary value problems

A. Hasanov[1,3], T.G. Ergashev[1,2,3], A.R. Ryskan[4,5,*]

[1] *Institute of Mathematics of Uzbekistan Academy of Sciences, Tashkent, Uzbekistan;*
[2] *"TIIAME" National Research University, Tashkent, Uzbekistan;*
[3] *Ghent University, Ghent, Belgium;*
[4] *Narxoz University, Almaty, Kazakhstan;*
[5] *UC Davis, Davis, USA*
(E-mail: anvarhasanov@yahoo.com, ergashev.tukhtasin@gmail.com, ryskan.a727@gmail.com)

Hypergeometric functions are divided into complete and confluent functions. Srivastava and Karlsson were the first to propose a method for constructing the complete set of triple Gaussian hypergeometric series and compiled a table containing definitions and regions of convergence for 205 distinct complete series in three variables. Subsequently, several authors obtained various integral representations and transformation formulas for the functions introduced by Srivastava and Karlsson. More recently, Ergashev identified 395 hypergeometric series of three variables that represent confluent forms of the known 205 complete hypergeometric series. In the present study, new Euler-type integral representations are derived for certain Gaussian hypergeometric functions of three variables. The main results are obtained using properties of the gamma and beta functions. New integral representations are established for 14 functions from the list of confluent hypergeometric functions of three variables. All derived integrals can be regarded as generalized Euler type representations of the classical Gaussian hypergeometric functions of one and two variables. In addition, it is demonstrated how one of these confluent functions, together with its integral representation, can be applied to construct solutions of the three-dimensional singular Helmholtz equation.

Keywords: Gaussian hypergeometric function, Appell functions, Humbert functions, Srivastava–Karlsson hypergeometric functions, triple confluent hypergeometric series, integral representation, singular Helmholtz equation, application of hypergeometric functions.

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Introduction

The great interest in the theory of hypergeometric functions (including functions of one, two or more variables) is primarily due to the fact that hypergeometric functions allow us to find solutions to various applied problems related to thermal conductivity and dynamic processes, electromagnetic oscillations, aerodynamics, quantum mechanics and potential theory. These functions, which relate to higher and special functions [1–3], are often called special functions of mathematical physics.

It is known that hypergeometric function $F(a, b; c; z)$ was studied by Leonhard Euler, but the first complete and systematic interpretation of it was given by Carl Friedrich Gauss in 1813. In the Gaussian hypergeometric function $F(a, b; c; z)$ there are two numerator parameters a, b, and one denominator parameter c. A natural generalization of this function is to introduce an arbitrary number of parameters for both the numerator and the denominator. The resulting function, denoted as ${}_pF_q$, is called the generalized Gauss function or generalized hypergeometric function (for more information, see [4, p. 19]). In 1880, Appell introduced four series F_1 to F_4, each of which is an analogue of the Gauss function $F(a, b; c; z)$. Horn [5] introduced the following ten hypergeometric functions in two variables, denoting

*Corresponding author. *E-mail: ryskan.a727@gmail.com*
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them as G_1, G_2, G_3, H_1, ..., H_7; he thus completed the set of all possible second-order (complete) hypergeometric functions in two variables [4, p. 24]. Humbert defined seven confluent forms for the four Appell functions [6], and he denoted these confluent hypergeometric functions in two variables by Φ_1, Φ_2, Φ_3, Ψ_1, Ψ_2, Ξ_1, Ξ_2. In addition, there exist 13 confluent forms of the Horn functions, which are denoted by Γ_1, Γ_2, H_1, ..., H_{11} [5] (see, also [7]). A significant contribution to the further development of the theory of hypergeometric series in two variables was made by Horn, who proposed a general definition and classification of double hypergeometric series. He studied the convergence properties of hypergeometric series in two variables and identified systems of partial differential equations to which these series correspond. Horn investigated hypergeometric series of the second order. He found that among them there are series that are expressed through one variable, or are products of two hypergeometric series, each of which depends on one variable. In addition, according to his conclusions, there are 14 complete and 20 confluent different convergent series of the second order. Definitions and convergence conditions for all 34 hypergeometric series in two variables are also given in [7].

Lauricella [8, p. 114] further generalized the four Applell series F_1, F_2, F_3, F_4 to series in n variables and defined his multiple hypergeometric series denoted by $F_A^{(n)}$, $F_B^{(n)}$, $F_C^{(n)}$ and $F_D^{(n)}$; in this work he introduced 14 complete hypergeometric series in three variables of the second order . He denoted his triple hypergeometric series by the symbols F_1, F_2, ..., F_{14} of which F_1, F_2, F_5 and F_9 correspond, respectively, to the three-variable Lauricella series $F_A^{(3)}$, $F_B^{(3)}$, $F_C^{(3)}$ and $F_D^{(3)}$. The remaining series F_3, F_4, F_6, F_7,F_8, F_{10},..., F_{14} of Lauricella's set apparently fell into oblivion. Saran [9] initiated a systematic study of these ten triple hypergeometric series of Lauricella's set. Sahai and Verma [10, 11] proved new recursion and infinite summation formulas for the triple Lauricella functions. Currently, Bezrodnykh [12–14] has obtained interesting results in the study of Lauricella functions.

In further study of Lauricella's 14 hypergeometric series in three variables, Srivastava [15, 16] discovered three additional complete triple hypergeometric series of the second order. These series, labeled H_A, H_B, and H_C, were not part of Lauricella's set and had not been previously reported in the literature. At present, the properties of various generalizations of the Srivastava's triple hypergeometric functions are being studied [17], integral representations are established for them, and they are applied to solving fractional differential equations [18]. A (p, q)-extensions of H_A, H_B, and H_C are defined and investigated in [19–21], respectively. Applications of the hypergeometric structures to the theory of Feynman integrals are found recently in [22].

An extended presentation of results on hypergeometric functions of three variables, as well as references to the original sources, is presented in the monograph by Srivastava and Karlsson [4], which is considered a classic work in this area. This monograph contains an extensive bibliography, including all relevant publications up to 1985. In particular, the authors compiled a table of 205 different complete triple hypergeometric Gauss series, accompanied by references to their sources, if known. Realizing the importance of integral representations of multiple hypergeometric functions for solving applied problems, Hasanov and Ruzhansky [23] developed Euler-type integral representations for all 205 complete triple hypergeometric functions. Authors of the paper [24] established several new more interesting integral representations of the Euler type and Laplace type for ten Gauss hypergeometric functions of three variables. Later, a systems of partial differential equations that the indicated 205 functions satisfy are constructed and all their linearly independent solutions near the origin are found, in those cases where such solutions exist [25].

However, comparatively less attention has been paid to the study of confluent hypergeometric functions of three variables. In the work of Jain [26], individual functions representing confluent forms of complete hypergeometric functions of three variables were investigated. In his paper [27] Ergashev identified 395 degenerate hypergeometric functions of three variables, denoting them as E_1, ..., E_{395}. He thus probably completed the classification of all possible second-order confluent hypergeometric functions for three variables. The study also includes an analysis of systems of partial differential equations associated with these 395 functions. In addition, particular solutions of some systems of differential equations near the origin were found, if such solutions exist.

Here we present integral representations for the functions E_1, ..., E_{14} defined in [27]. We also demonstrate the application of the confluent hypergeometric function E_2 by constructing particular solutions of the three-dimensional Helmholtz equation with singular coefficients.

This paper uses standard definitions and notations, including the Pochhammer symbol $(\lambda)_n$, the beta function $B(x, y)$, the gamma function $\Gamma(z)$, the Gauss hypergeometric function and its generalization ${}_pF_q$ [7], the Appell functions [28], the Humbert functions [6], the Horn functions [5] in two variables, and the complete [4] and confluent [27] hypergeometric functions of three variables.

1 *Preliminaries*

A function

$$F(a,b;c;z) \equiv F\left[\begin{array}{l} a,b; \\ c; \end{array} z\right] := \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k} \frac{z^k}{k!}, |z| < 1, \;\; c \neq 0, -1, -2, ... \tag{1}$$

is known as the Gaussian hypergeometric function.

The Gaussian hypergeometric series $F(a, b; c; z)$ includes two numerator parameters a and b, and one denominator parameter c. Its natural generalization is the introduction of an arbitrary number of parameters in both the numerator and the denominator. The resulting series

$${}_pF_q\left[\begin{array}{l} (a_p); \\ (b_q); \end{array} z\right] \equiv {}_pF_q\left[(a_p);(b_q);z\right] := \sum_{k=0}^{\infty} \frac{\prod_{j=1}^{p} (a_j)_k}{\prod_{j=1}^{q} (b_j)_k} \frac{z^k}{k!}, \;\; |z| < 1,$$

is known as the generalized Gauss series [7, p. 182], or simply, the generalized hypergeometric series. Here p and q are positive integers or zero (interpreting an empty product as 1), and we assume that the variable z, the numerator parameters a_1,..., a_p, and the denominator parameters b_1, ..., b_q take on complex values, provided that $b_j \neq 0, -1, -2, \ldots; \; j = 1, \ldots, q$. In general (that is, except for certain integer values of the parameters for which the series terminates or is undefined) ${}_pF_q$ converges for all finite z if $p \leq q$, converges for $|z| < 1$ if $p = q + 1$, and diverges for all $z \neq 0$ if $p > q + 1$.

Gauss' series (1) in the present notation is ${}_2F_1(a, b; c; z) \equiv F(a, b; c; z)$.

The double Appell hypergeometric functions are defined as following [28]:

$$F_1\left(a,b,b';c;x,y\right) = \sum_{m,n=0}^{\infty} \frac{(a)_{m+n}(b)_m\left(b'\right)_n}{(c)_{m+n} m!n!} x^m y^n, \quad \max\{|x|,|y|\} < 1, \tag{2}$$

$$F_2\left(a,b,b';c,c';x,y\right) = \sum_{m,n=0}^{\infty} \frac{(a)_{m+n}(b)_m\left(b'\right)_n}{(c)_m\left(c'\right)_n m!n!} x^m y^n, \quad |x|+|y| < 1, \tag{3}$$

$$F_3\left(a,a',b,b';c;x,y\right) = \sum_{m,n=0}^{\infty} \frac{(a)_m\left(a'\right)_n(b)_m\left(b'\right)_n}{(c)_{m+n} m!n!} x^m y^n, \quad \max\{|x|,|y|\} < 1, \tag{4}$$

$$F_4\left(a,b;c,c';x,y\right) = \sum_{m,n=0}^{\infty} \frac{(a)_{m+n}(b)_{m+n}}{(c)_m\left(c'\right)_n m!n!} x^m y^n, \quad \sqrt{|x|}+\sqrt{|y|} < 1, \tag{5}$$

here, in all definitions (2)–(5), as usual, the denominator parameters c and c' are neither zero nor a negative integer.

Seven confluent forms of the four Appell series were introduced by Humbert [6], who denoted these confluent hypergeometric series of two variables as follows:

$$\Phi_1(\alpha,\beta;\gamma;x,y)=\sum_{m,n=0}^{\infty}\frac{(\alpha)_{m+n}(\beta)_m}{(\gamma)_{m+n}m!n!}x^m y^n,\ \ |x|<1, \tag{6}$$

$$\Phi_2\left(\beta,\beta';\gamma;x,y\right)=\sum_{m,n=0}^{\infty}\frac{(\beta)_m(\beta')_n}{(\gamma)_{m+n}m!n!}x^m y^n, \tag{7}$$

$$\Phi_3(\beta;\gamma;x,y)=\sum_{m,n=0}^{\infty}\frac{(\beta)_m}{(\gamma)_{m+n}\,m!n!}x^m y^n, \tag{8}$$

$$\Psi_1\left(\alpha,\beta;\gamma,\gamma';x,y\right)=\sum_{m,n=0}^{\infty}\frac{(\alpha)_{m+n}(\beta)_m}{(\gamma)_m(\gamma')_n m!n!}x^m y^n,\ \ |x|<1, \tag{9}$$

$$\Psi_2\left(\alpha;\gamma,\gamma';x,y\right)=\sum_{m,n=0}^{\infty}\frac{(\alpha)_{m+n}}{(\gamma)_m(\gamma')_n\,m!n!}x^m y^n, \tag{10}$$

$$\Xi_1\left(\alpha,\alpha',\beta;\gamma;x,y\right)=\sum_{m,n=0}^{\infty}\frac{(\alpha)_m(\alpha')_n(\beta)_m}{(\gamma)_{m+n}\,m!n!}x^m y^n,\ \ |x|<1, \tag{11}$$

$$\Xi_2(\alpha,\beta;\gamma;x,y)=\sum_{m,n=0}^{\infty}\frac{(\alpha)_m(\beta)_m}{(\gamma)_{m+n}\,m!n!}x^m y^n,\ \ |x|<1, \tag{12}$$

where the denominator parameters γ and γ' are neither zero nor a negative integer. The hypergeometric functions defined in (6)–(12) are called *Humbert functions*.

In this paper we will establish integral representations for the following functions:

$$\mathrm{E}_1(a_1,a_2,a_3,a_4,a_5;c;x,y,z)=\sum_{m,n,p=0}^{\infty}\frac{(a_1)_m(a_2)_m(a_3)_n(a_4)_n(a_5)_p}{(c)_{m+n+p}}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!},\ \ \frac{1}{|x|}+\frac{1}{|y|}>1, \tag{13}$$

$$\mathrm{E}_2(a_1,a_2,a_3,a_4;c;x,y,z)=\sum_{m,n,p=0}^{\infty}\frac{(a_1)_m(a_2)_m(a_3)_n(a_4)_n}{(c)_{m+n+p}}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!},\ \ \frac{1}{|x|}+\frac{1}{|y|}>1, \tag{14}$$

$$\mathrm{E}_3(a_1,a_2,a_3,a_4;c;x,y,z)=\sum_{m,n,p=0}^{\infty}\frac{(a_1)_m(a_2)_m(a_3)_n(a_4)_p}{(c)_{m+n+p}}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!},\ \ |x|<1, \tag{15}$$

$$\mathrm{E}_4(a_1,a_2,a_3;c;x,y,z)=\sum_{m,n,p=0}^{\infty}\frac{(a_1)_m(a_2)_m(a_3)_n}{(c)_{m+n+p}}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!},\ \ |x|<1, \tag{16}$$

$$\mathrm{E}_5(a_1,a_2,a_3;c;x,y,z)=\sum_{m,n,p=0}^{\infty}\frac{(a_1)_m(a_2)_n(a_3)_p}{(c)_{m+n+p}}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}, \tag{17}$$

$$\mathrm{E}_6(a,b;c;x,y,z)=\sum_{m,n,p=0}^{\infty}\frac{(a)_m(b)_n}{(c)_{m+n+p}}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}, \tag{18}$$

$$\mathrm{E}_7(a_1,a_2,a_3,a_4;c;x,y,z)=\sum_{m,n,p=0}^{\infty}\frac{(a_1)_{m+n}(a_2)_m(a_3)_n(a_4)_p}{(c)_{m+n+p}}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!},\ \ |x|<1,\ |y|<1, \tag{19}$$

$$\mathrm{E}_8\left(a_1, a_2, a_3, a_4; c; x, y, z\right)=\sum_{m,n,p=0}^{\infty} \frac{(a_1)_{m+n}(a_2)_m(a_3)_p(a_4)_p}{(c)_{m+n+p}} \frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}, \quad \frac{1}{|x|}+\frac{1}{|z|}>1, \tag{20}$$

$$\mathrm{E}_9\left(a_1, a_2, a_3; c; x, y, z\right)=\sum_{m,n,p=0}^{\infty} \frac{(a_1)_{m+n}(a_2)_m(a_3)_n}{(c)_{m+n+p}} \frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}, \quad |x|<1,\ |y|<1, \tag{21}$$

$$\mathrm{E}_{10}\left(a_1, a_2, a_3; c; x, y, z\right)=\sum_{m,n,p=0}^{\infty} \frac{(a_1)_{m+n}(a_2)_m(a_3)_p}{(c)_{m+n+p}} \frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}, \quad |x|<1, \tag{22}$$

$$\mathrm{E}_{11}\left(a, b; c; x, y, z\right)=\sum_{m,n,p=0}^{\infty} \frac{(a)_{m+n}(b)_m}{(c)_{m+n+p}} \frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}, \quad |x|<1, \tag{23}$$

$$\mathrm{E}_{12}\left(a_1, a_2, a_3; c; x, y, z\right)=\sum_{m,n,p=0}^{\infty} \frac{(a_1)_{m+n}(a_2)_{n+p}(a_3)_m}{(c)_{m+n+p}} \frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}, \quad |x|<1,\ |y|<1, \tag{24}$$

$$\mathrm{E}_{13}\left(a, b; c; x, y, z\right)=\sum_{m,n,p=0}^{\infty} \frac{(a)_{m+n}(b)_{n+p}}{(c)_{m+n+p}} \frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}, \quad |y|<1, \tag{25}$$

$$\mathrm{E}_{14}\left(a_1, a_2, a_3; c; x, y, z\right)=\sum_{m,n,p=0}^{\infty} \frac{(a_1)_{m+n+p}(a_2)_m(a_3)_n}{(c)_{m+n+p}} \frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}, \quad |x|<1,\ |y|<1, \tag{26}$$

New integral transforms for the two-variable analogous of the confluent hypergeometric functions (13)–(26) are found in [29].

2 *Single integral Representations*

Theorem 1. If $\mathrm{Re}(\alpha)>0$ and $\mathrm{Re}(\beta)>0$, then each of the following integral representation for E_1–E_8 holds true:

$$\mathrm{E}_1\left(\alpha, a_2, \beta, a_4, a_5; c; x, y, z\right)=k_1\int_0^1 \xi^{\alpha-1}\left(1-\xi\right)^{\beta-1}\mathrm{E}_7\left(\alpha+\beta, a_2, a_4, a_5; c; x\xi, y-y\xi, z\right)d\xi, \tag{27}$$

$$\mathrm{E}_2\left(\alpha, a_2, \beta, a_4, a_5; c; x, y, z\right)=k_1\int_0^1 \xi^{\alpha-1}\left(1-\xi\right)^{\beta-1}\mathrm{E}_9\left(\alpha+\beta, a_2, a_4; c; x\xi, y-y\xi, z\right)d\xi, \tag{28}$$

$$\mathrm{E}_3\left(\alpha, a_2, \beta, a_4; c; x, y, z\right)=k_1\int_0^1 \xi^{\alpha-1}\left(1-\xi\right)^{\beta-1}\mathrm{E}_{10}\left(\alpha+\beta, a_2, a_4; c; x\xi, y-y\xi, z\right)d\xi, \tag{29}$$

$$\mathrm{E}_4\left(a_1, \alpha, \beta; c; x, y, z\right)=k_1\int_0^1 \xi^{\alpha-1}\left(1-\xi\right)^{\beta-1}\mathrm{E}_{11}\left(\alpha+\beta, a_1; c; x\xi, y-y\xi, z\right)d\xi, \tag{30}$$

$$\mathrm{E}_5\left(\alpha, \beta, a; c; x, y, z\right)=k_1\int_0^1 \xi^{\alpha-1}\left(1-\xi\right)^{\beta-1}\Xi_2\left(\alpha+\beta, a; c; x\xi+y-y\xi, z\right)d\xi, \tag{31}$$

$$\mathrm{E}_6\left(\alpha, \beta; c; x, y, z\right)=k_1\int_0^1 \xi^{\alpha-1}\left(1-\xi\right)^{\beta-1}\Phi_3\left(\alpha+\beta; c; x\xi+y-y\xi, z\right)d\xi, \tag{32}$$

$$\mathrm{E}_7\left(a, \alpha, \beta, b; c; x, y, z\right)=k_1\int_0^1 \xi^{\alpha-1}\left(1-\xi\right)^{\beta-1}\Xi_1\left(a, b, \alpha+\beta; c; x\xi+y-y\xi, z\right)d\xi, \tag{33}$$

$$\mathrm{E}_8\left(a_1,\alpha,\beta,a_4;c;x,y,z\right)=k_1\int_0^1\xi^{\alpha-1}\left(1-\xi\right)^{\beta-1}\mathrm{E}_{12}\left(\alpha+\beta,a_1,a_4;c;z-z\xi,x\xi,y\right)d\xi, \tag{34}$$

where $k_1=\dfrac{\Gamma\left(\alpha+\beta\right)}{\Gamma\left(\alpha\right)\Gamma\left(\beta\right)}$.

Proof. Using the definition of the Beta function

$$\mathrm{B}(\alpha,\beta)=\int_0^1\xi^{\alpha-1}(1-\xi)^{\beta-1}d\xi,\;\; Re\alpha>0,\;\; Re\beta>0,$$

it is easy to establish the relation

$$\frac{(\alpha)_m(\beta)_n}{(\alpha+\beta)_{m+n}}=\frac{\Gamma\left(\alpha+\beta\right)}{\Gamma\left(\alpha\right)\Gamma\left(\beta\right)}\mathrm{B}(\alpha+m,\beta+n),\;\; m,n=0,1,2,\ldots,$$

i.e.,

$$\frac{(\alpha)_m(\beta)_n}{(\alpha+\beta)_{m+n}}=k_1\int_0^1\xi^{\alpha-1+m}(1-\xi)^{\beta-1+n}d\xi,\;\; Re\alpha>0,\;\; Re\beta>0,\;\; m,n=0,1,2,\ldots, \tag{35}$$

where $(\lambda)_\nu=\Gamma(\lambda+\nu)/\Gamma(\lambda)$ is a Pochhammer symbol.

Applying the relation (35), we obtain the integral representations (27)–(30).

The proof of the representations (31)–(34) can be distinguished from the others by using the well-known property of double power series

$$\sum_{m,n=0}^{\infty}f(m+n)\frac{x^m}{m!}\frac{y^n}{n!}=\sum_{k=0}^{\infty}f(k)\frac{(x+y)^k}{k!}. \tag{36}$$

To give an example, by virtue of the definition of E_5, we have

$$\mathrm{E}_5\left(\alpha,\beta,a;c;x,y,z\right)=\sum_{m,n,p=0}^{\infty}\frac{(\alpha+\beta)_{m+n}(a)_p}{(c)_{m+n+p}}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}\int_0^1\xi^{\alpha-1+m}(1-\xi)^{\beta-1+n}d\xi$$

$$=\int_0^1\xi^{\alpha-1}(1-\xi)^{\beta-1}\sum_{m,n,p=0}^{\infty}\frac{(\alpha+\beta)_{m+n}(a)_p}{(c)_{m+n+p}}\frac{(x\xi)^m}{m!}\frac{(y-y\xi)^n}{n!}\frac{z^p}{p!}d\xi.$$

Then using the property (36), we obtain

$$\mathrm{E}_5\left(\alpha,\beta,a;c;x,y,z\right)=\int_0^1\xi^{\alpha-1}(1-\xi)^{\beta-1}\sum_{k,p=0}^{\infty}\frac{(\alpha+\beta)_k(a)_p}{(c)_{k+p}}\frac{(x\xi+y-y\eta)^k}{k!}\frac{z^p}{p!}d\xi,$$

which asserts the validity of the representation (31). The Theorem 1 is proven. □

Theorem 2. If $\mathrm{Re}\left(\beta\right)>\mathrm{Re}\left(\alpha\right)>0$, then the following integral representations are valid:

$$\mathrm{E}_7\left(\alpha,a_2,a_3,a_4;\beta;x,y,z\right)=k_2\int_0^1\frac{\xi^{\alpha-1}\left(1-\xi\right)^{\beta-\alpha-1}}{\left(1-x\xi\right)^{a_2}\left(1-y\xi\right)^{a_3}}{}_1F_1\left(a_4,\beta-\alpha;z-z\xi\right)d\xi, \tag{37}$$

$$\mathrm{E}_8\left(\alpha,a_2,a_3,a_4;\beta;x,y,z\right)=k_2\int_0^1\frac{\xi^{\alpha-1}\left(1-\xi\right)^{\beta-\alpha-1}}{\left(1-x\xi\right)^{a_2}}e^{y\xi}{}_2F_1\left(a_3,a_4;\beta-\alpha;z-z\xi\right)d\xi, \tag{38}$$

$$\mathrm{E}_9\left(\alpha,a_2,a_3;\beta;x,y,z\right)=k_2\int_0^1\frac{\xi^{\alpha-1}\left(1-\xi\right)^{\beta-\alpha-1}}{\left(1-x\xi\right)^{a_2}\left(1-y\xi\right)^{a_3}}{}_0F_1\left(-;\beta-\alpha;z-z\xi\right)d\xi, \tag{39}$$

$$\mathrm{E}_{10}\left(\alpha, a_2, a_3; \beta; x, y, z\right) = k_2 \int_0^1 \frac{\xi^{\alpha-1}\left(1-\xi\right)^{\beta-\alpha-1}}{\left(1-x\xi\right)^{a_2}} e^{y\xi} {}_1F_1\left(a_3; \beta-\alpha; z-z\xi\right) d\xi, \tag{40}$$

$$\mathrm{E}_{11}\left(\alpha, b; \beta; x, y, z\right) = k_2 \int_0^1 \frac{\xi^{\alpha-1}\left(1-\xi\right)^{\beta-\alpha-1}}{\left(1-x\xi\right)^{b}} e^{y\xi} {}_0F_1\left(-; \beta-\alpha; z-z\xi\right) d\xi, \tag{41}$$

$$\mathrm{E}_{12}\left(\alpha, a_2, a_3; \beta; x, y, z\right) = k_2 \int_0^1 \frac{\xi^{\alpha-1}\left(1-\xi\right)^{\beta-\alpha-1}}{\left(1-x\xi\right)^{a_3}\left(1-y\xi\right)^{a_2}} {}_1F_1\left(a_2; \beta-\alpha; \frac{z\left(1-\xi\right)}{1-y\xi}\right) d\xi, \tag{42}$$

$$\mathrm{E}_{13}\left(\alpha, b; \beta; x, y, z\right) = k_2 \int_0^1 \frac{\xi^{\alpha-1}\left(1-\xi\right)^{\beta-\alpha-1}}{\left(1-y\xi\right)^{b}} e^{x\xi} {}_1F_1\left(b; \beta-\alpha; \frac{z\left(1-\xi\right)}{1-y\xi}\right) d\xi, \tag{43}$$

$$\mathrm{E}_{14}\left(\alpha, a_2, a_3; \beta; x, y, z\right) = k_2 \int_0^1 \frac{\xi^{\alpha-1}\left(1-\xi\right)^{\beta-\alpha-1}}{\left(1-x\xi\right)^{a_2}\left(1-y\xi\right)^{a_3}} e^{z\xi} d\xi, \tag{44}$$

$$\mathrm{E}_{14}\left(a_1, \alpha, a_3; \beta; x, y, z\right) = k_2 \int_0^1 \frac{\xi^{\alpha-1}\left(1-\xi\right)^{\beta-\alpha-1}}{\left(1-x\xi\right)^{a_1}} \Phi_1\left(a_1, a_3; \beta-\alpha; \frac{y\left(1-\xi\right)}{1-x\xi}, \frac{z\left(1-\xi\right)}{1-x\xi}\right) d\xi, \tag{45}$$

where $k_2 = \dfrac{\Gamma\left(\beta\right)}{\Gamma\left(\alpha\right)\Gamma\left(\beta-\alpha\right)}$.

Proof. To prove integral representations (37)–(45), Euler's formula for Gauss function [7, p. 59]

$$F(a, b; c; x) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int_0^1 \frac{\xi^{a-1}(1-\xi)^{c-a-1}}{(1-x\xi)^b} d\xi, \quad \mathrm{Re}(c) > \mathrm{Re}(a) > 0 \tag{46}$$

and the integral representation formula for Appell function

$$F_1(a, b_1, b_2; c; x, y) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int_0^1 \frac{\xi^{a-1}(1-\xi)^{c-a-1}}{(1-x\xi)^{b_1}(1-y\xi)^{b_2}} d\xi, \quad \mathrm{Re}(c) > \mathrm{Re}(a) > 0$$

are used.

Let's consider the last equality (45) in Theorem 2. One can represent the confluent hypergeometric function E_{14} in the form

$$\mathrm{E}_{14}\left(a_1, \alpha, a_3; \beta; x, y, z\right) = \sum_{n,p=0}^{\infty} \frac{\left(a_1\right)_{n+p}\left(a_3\right)_n}{\left(\beta\right)_{n+p}} F\left(\alpha, a_1+n+p; \beta+n+p; x\right) \frac{y^n}{n!}\frac{z^p}{p!}.$$

Then, using the Euler's formula (46) and the definition of the Humbert function Φ_1 we obtain the integral representation (45). The remaining integral representations in the Theorem 2 are proved similarly. The Theorem 2 is proven. □

3 Double integral Representations

Theorem 3. If $\mathrm{Re}\left(\alpha\right) > 0$, $\mathrm{Re}\left(\beta\right) > 0$ and $\mathrm{Re}\left(\gamma\right) > 0$, then the following integral representations are valid:

$$\begin{aligned}\mathrm{E}_1\left(a_1, a_2, a_3, a_4, a_5; \alpha+\beta+\gamma; x, y, z\right) =& k_3 \int_0^1 \int_0^1 \xi^{\alpha-1}\left(1-\xi\right)^{\beta+\gamma-1}\eta^{\beta-1}\left(1-\eta\right)^{\gamma-1} \times \\ & \times {}_2F_1\left(a_1, a_2; \alpha; x\xi\right) {}_2F_1\left(a_3, a_4; \beta; y\eta\left(1-\xi\right)\right) {}_1F_1\left(a_5; \gamma; z\left(1-\xi\right)\left(1-\eta\right)\right) d\xi d\eta,\end{aligned} \tag{47}$$

$$\mathrm{E}_1(\alpha, a_2, \beta, a_4, \gamma; \alpha+\beta+\gamma; x, y, z) = k_3 \int_0^1 \int_0^1 \xi^{\alpha-1}(1-\xi)^{\beta+\gamma-1} \eta^{\beta-1}(1-\eta)^{\gamma-1} \times$$
$$\times (1-x\xi)^{-a_2}(1-y\eta+y\xi\eta)^{-a_4} e^{z(1-\xi)(1-\eta)} d\xi d\eta, \tag{48}$$

$$\mathrm{E}_1(\alpha, \beta, \gamma, a_4, a_5; c; x, y, z) = k_3 \int_0^1 \int_0^1 \xi^{\alpha-1}(1-\xi)^{\beta+\gamma-1} \eta^{\beta-1}(1-\eta)^{\gamma-1} \times$$
$$\times \mathrm{E}_{15}(\alpha+\beta+\gamma, a_4, a_5; c; x\xi(1-\xi)\eta, y(1-\xi)(1-\eta), z) d\xi d\eta, \tag{49}$$

$$\mathrm{E}_1(\alpha, a_2, \beta, a_4, \gamma; c; x, y, z) = k_3 \int_0^1 \int_0^1 \xi^{\alpha-1}(1-\xi)^{\beta-1} \eta^{\alpha+\beta-1}(1-\eta)^{\gamma-1} \times$$
$$\times \mathrm{E}_{14}(a_2+\gamma, a_2, a_4; c; x\xi\eta, y(1-\xi)\eta, z(1-\eta)) d\xi d\eta, \tag{50}$$

$$\mathrm{E}_1(\alpha, a_2, \beta, a_4, \gamma; a_2+\gamma; x, y, z) = k_3 \int_0^1 \int_0^1 \xi^{\alpha-1}(1-\xi)^{\beta-1} \eta^{\alpha+\beta-1}(1-\eta)^{\gamma-1} \times$$
$$\times (1-x\xi\eta)^{-a_2}(1-y\eta+y\xi\eta)^{-a_4} e^{z(1-\eta)} d\xi d\eta, \tag{51}$$

$$\mathrm{E}_2(a_1, a_2, a_3, a_4, a_5; \alpha+\beta+\gamma; x, y, z) = k_3 \int_0^1 \int_0^1 \xi^{\alpha-1}(1-\xi)^{\beta+\gamma-1} \eta^{\beta-1}(1-\eta)^{\gamma-1} \times$$
$$\times {}_2F_1(a_1, a_2; \alpha; x\xi)\, {}_2F_1(a_3, a_4; \beta; y\eta(1-\xi))\, {}_0F_1(-; \gamma; z(1-\xi)(1-\eta)) d\xi d\eta, \tag{52}$$

$$\mathrm{E}_2(\alpha, a_2, c_2, a_4, a_5; \alpha+\beta+\gamma; x, y, z) = k_3 \int_0^1 \int_0^1 \xi^{\alpha-1}(1-\xi)^{\beta+\gamma-1} \eta^{\beta-1}(1-\eta)^{\gamma-1} \times$$
$$\times (1-x\xi)^{-a_2}(1-y\eta+y\xi\eta)^{-a_4}\, {}_0F_1(-; \gamma; z(1-\xi)(1-\eta)) d\xi d\eta, \tag{53}$$

$$\mathrm{E}_2(\alpha, \beta, \gamma, a_4, a_5; c; x, y, z) = k_3 \int_0^1 \int_0^1 \xi^{\alpha-1}(1-\xi)^{\beta+\gamma-1} \eta^{\beta-1}(1-\eta)^{\gamma-1} \times$$
$$\times \mathrm{E}_{17}(\alpha+\beta+\gamma, a_4; c; x\xi(1-\xi)\eta, y(1-\xi)(1-\eta), z) d\xi d\eta, \tag{54}$$

$$\mathrm{E}_3(a_1, a_2, a_3, a_4; \alpha+\beta+\gamma; x, y, z) = k_3 \int_0^1 \int_0^1 \xi^{\alpha-1}(1-\xi)^{\beta+\gamma-1} \eta^{\beta-1}(1-\eta)^{\gamma-1} \times$$
$$\times {}_2F_1(a_1, a_2; \alpha; x\xi)\, {}_1F_1(a_3; \beta; y\eta(1-\xi))\, {}_1F_1(a_4; \gamma; z(1-\xi)(1-\eta)) d\xi d\eta, \tag{55}$$

$$\mathrm{E}_3(\alpha, a_2, \beta, a_4; \alpha+\beta+\gamma; x, y, z) =$$
$$= k_3 \int_0^1 \int_0^1 \xi^{\alpha-1}(1-\xi)^{\beta+\gamma-1} \eta^{\beta-1}(1-\eta)^{\gamma-1}(1-x\xi)^{-a_2} e^{(1-\xi)(y\eta+z-z\eta)} d\xi d\eta, \tag{56}$$

$$\mathrm{E}_3(\alpha, a_2, a_3, a_4; c; x, y, z) = k_3 \int_0^1 \int_0^1 \xi^{\alpha-1}(1-\xi)^{\beta-1} \eta^{\alpha+\beta-1}(1-\eta)^{\gamma-1} \times$$
$$\times \Phi_1(\alpha+\beta+\gamma, a_2; c; y\eta(1-\xi)+z(1-\eta), x\xi\eta) d\eta d\xi, \tag{57}$$

$$\mathrm{E}_3(\alpha, \beta, \gamma, a_4; c; x, y, z) = k_3 \int_0^1 \int_0^1 \xi^{\alpha-1}(1-\xi)^{\beta+\gamma-1} \eta^{\beta-1}(1-\eta)^{\gamma-1} \times$$
$$\times \mathrm{E}_{18}(\alpha+\beta+\gamma, b; c; x\xi(1-\xi)\eta, y(1-\xi)(1-\eta), z) d\xi d\eta, \tag{58}$$

$$\mathrm{E}_4\left(a_1,a_2,a_3;\alpha+\beta+\gamma;x,y,z\right)=k_3\int_0^1\int_0^1\xi^{\alpha-1}\left(1-\xi\right)^{\beta+\gamma-1}\eta^{\beta-1}\left(1-\eta\right)^{\gamma-1}\times$$
$$\times{}_2F_1\left(a_1,a_2;\alpha;x\xi\right){}_1F_1\left(a_3;\beta;y\eta\left(1-\xi\right)\right){}_0F_1\left(-;\gamma;z\left(1-\xi\right)\left(1-\eta\right)\right)d\xi d\eta, \tag{59}$$

$$\mathrm{E}_4\left(\alpha,\beta,\gamma;c;x,y,z\right)=k_3\int_0^1\int_0^1\xi^{\alpha-1}\left(1-\xi\right)^{\beta+\gamma-1}\eta^{\beta-1}\left(1-\eta\right)^{\gamma-1}\times$$
$$\times\mathrm{E}_{19}\left(\alpha+\beta+\gamma;c;x\xi\left(1-\xi\right)\eta,y\left(1-\xi\right)\left(1-\eta\right),z\right)d\xi d\eta, \tag{60}$$

$$\mathrm{E}_5\left(a_1,a_2,a_3;\alpha+\beta+\gamma;x,y,z\right)=k_3\int_0^1\int_0^1\xi^{\alpha-1}\left(1-\xi\right)^{\beta+\gamma-1}\eta^{\beta-1}\left(1-\eta\right)^{\gamma-1}\times$$
$$\times{}_1F_1\left(a_1;\alpha;x\xi\right){}_1F_1\left(a_2;\beta;y\eta\left(1-\xi\right)\right){}_1F_1\left(a_3;\gamma;z\left(1-\xi\right)\left(1-\eta\right)\right)d\xi d\eta, \tag{61}$$

$$\mathrm{E}_6\left(a,b;\alpha+\beta+\gamma;x,y,z\right)=k_3\int_0^1\int_0^1\xi^{\alpha-1}\left(1-\xi\right)^{\beta+\gamma-1}\eta^{\beta-1}\left(1-\eta\right)^{\gamma-1}\times$$
$$\times{}_1F_1\left(a;\alpha;x\xi\right){}_1F_1\left(b;\beta;y\eta\left(1-\xi\right)\right){}_0F_1\left(-;\gamma;z\left(1-\xi\right)\left(1-\eta\right)\right)d\xi d\eta, \tag{62}$$

$$\mathrm{E}_7\left(a_1,a_2,a_3,a_4;\alpha+\beta+\gamma;x,y,z\right)=k_3\int_0^1\int_0^1\xi^{\alpha-1}\left(1-\xi\right)^{\beta+\gamma-1}\eta^{\beta-1}\left(1-\eta\right)^{\gamma-1}\times$$
$$\times F_2\left(a_1;a_2,a_3;\alpha,\beta;x\xi,y\eta\left(1-\xi\right)\right){}_1F_1\left(a_4;\gamma;z\left(1-\xi\right)\left(1-\eta\right)\right)d\xi d\eta, \tag{63}$$

$$\mathrm{E}_7\left(\beta,a_2,a_3,a_4;\alpha+\beta+\gamma;x,y,z\right)=k_3\int_0^1\int_0^1\frac{\xi^{\alpha-1}\left(1-\xi\right)^{\beta+\gamma-1}\eta^{\beta-1}\left(1-\eta\right)^{\gamma-1}}{\left(1-y\eta+y\xi\eta\right)^{a_3}}\times$$
$$\times F_1\left(a_2;\beta-\gamma,a_3;\alpha;x\xi,\frac{x\xi}{1-y\eta+y\xi\eta}\right){}_1F_1\left(a_4;\gamma;z\left(1-\xi\right)\left(1-\eta\right)\right)d\xi d\eta, \tag{64}$$

$$\mathrm{E}_7\left(\alpha,a_2,a_3,a_4;2\alpha+\beta;x,y,z\right)=k_4\int_0^1\int_0^1\frac{\xi^{\alpha-1}\left(1-\xi\right)^{\alpha+\beta-1}\eta^{\alpha-1}\left(1-\eta\right)^{\beta-1}}{\left(1-x\xi\right)^{a_2}\left(1-y\eta+y\xi\eta\right)^{a_3}}\times$$
$$\times{}_2F_1\left(a_2,a_3;\alpha;\frac{xy\xi\eta\left(1-\xi\right)}{\left(1-x\xi\right)\left(1-y\eta+y\xi\eta\right)}\right){}_1F_1\left(a_4;\beta;z\left(1-\xi\right)\left(1-\eta\right)\right)d\xi d\eta, \tag{65}$$

$$\mathrm{E}_7\left(a_1,a_2,a_3,a_4;\alpha+\beta+\gamma;x,y,z\right)=$$
$$=k_3\int_0^1\int_0^1\frac{\xi^{\alpha-1}\left(1-\xi\right)^{\beta+\gamma-1}\eta^{\beta-1}\left(1-\eta\right)^{\gamma-1}}{\left(1-x\xi-y\eta+y\xi\eta\right)^{a_1}}{}_1F_1\left(a_4;\gamma;z\left(1-\xi\right)\left(1-\eta\right)\right)\times$$
$$\times F_2\left(a_1,\alpha-a_2,\beta-a_3;\alpha,\beta;\frac{x\xi}{x\xi+y\eta-y\xi\eta-1},\frac{y}{x\xi+y\eta-y\xi\eta-1}\right)d\xi d\eta, \tag{66}$$

$$\mathrm{E}_7\left(a_1,\alpha,\beta,\gamma;c;x,y,z\right)=k_3\int_0^1\int_0^1\xi^{\alpha-1}\left(1-\xi\right)^{\beta-1}\eta^{\alpha+\beta-1}\left(1-\eta\right)^{\gamma-1}\times$$
$$\times\Phi_1\left(\alpha+\beta+\gamma,a_1;c;z-z\eta,x\xi\eta+y\eta-y\xi\eta\right)d\xi d\eta, \tag{67}$$

$$\mathrm{E}_7\left(a_1,\alpha,\beta,\gamma;c;x,y,z\right)=k_3\int_0^1\int_0^1\frac{\xi^{\alpha-1}\eta^{\alpha+\beta-1}\left(1-\xi\right)^{\beta-1}\left(1-\eta\right)^{\gamma-1}}{\left(1-z+z\eta\right)^{a_1}}\times$$
$$\times e^{x\xi\eta+y\eta-y\xi\eta}\Phi_1\left(c-\alpha-\beta-\gamma,a_1;c;\frac{z-z\eta}{z-z\eta-1},-x\xi\eta-y\eta+y\xi\eta\right)d\xi d\eta, \tag{68}$$

$$\mathrm{E}_8\left(a_1,a_2,a_3,a_4;\alpha+\beta+\gamma;x,y,z\right)=k_3\int_0^1\int_0^1\xi^{\alpha-1}\left(1-\xi\right)^{\beta+\gamma-1}\eta^{\beta-1}\left(1-\eta\right)^{\gamma-1}\times$$
$$\times\Psi_1\left(a_1,a_2;\beta,\alpha;y\eta-y\xi\eta,x\xi\right){}_2F_1\left(a_3,a_4;\gamma;z\left(1-\xi\right)\left(1-\eta\right)\right)d\xi d\eta, \tag{69}$$

$$\mathrm{E}_8\left(\alpha,a_2,a_3,a_4;c;x,y,z\right)=k_3\int_0^1\int_0^1\xi^{\alpha-1}\left(1-\xi\right)^{\beta-1}\eta^{\alpha+\beta-1}\left(1-\eta\right)^{\gamma-1}\times$$
$$\times\mathrm{E}_{21}\left(\alpha+\beta+\gamma,a_4;c;x\xi\eta^2,z,y\xi\eta\right)d\xi d\eta, \tag{70}$$

$$\mathrm{E}_9\left(a_1,a_2,a_3;\alpha+\beta+\gamma;x,y,z\right)=k_3\int_0^1\int_0^1\xi^{\alpha-1}\left(1-\xi\right)^{\beta+\gamma-1}\eta^{\beta-1}\left(1-\eta\right)^{\gamma-1}\times$$
$$\times F_2\left(a_1,a_2,a_3;\alpha,\beta;x\xi,y\eta-y\xi\eta\right){}_0F_1\left(-;\gamma;z\left(1-\xi\right)\left(1-\eta\right)\right)d\xi d\eta, \tag{71}$$

$$\mathrm{E}_9\left(a_1,a_2,a_3;\alpha+\beta+\gamma;x,y,z\right)=k_3\int_0^1\int_0^1\frac{\xi^{\alpha-1}\left(1-\xi\right)^{\beta+\gamma-1}\eta^{\beta-1}\left(1-\eta\right)^{\gamma-1}}{\left(1-y\eta+y\xi\eta\right)^{a_3}}\times$$
$$\times F_1\left(a_2;\beta-a_3,a_3;\alpha;x\xi,\frac{x\xi}{1-y\eta+y\xi\eta}\right){}_0F_1\left(-;\gamma;z\left(1-\xi\right)\left(1-\eta\right)\right)d\xi d\eta, \tag{72}$$

$$\mathrm{E}_{10}\left(a_1,a_2,a_3;\alpha+\beta+\gamma;x,y,z\right)=k_3\int_0^1\int_0^1\xi^{\alpha-1}\left(1-\xi\right)^{\beta+\gamma-1}\eta^{\beta-1}\left(1-\eta\right)^{\gamma-1}\times$$
$$\times\Psi_1\left(a_1,a_2;\alpha,\beta;x\xi,y\eta\left(1-\xi\right)\right){}_1F_1\left(a_3;\gamma;z\left(1-\xi\right)\left(1-\eta\right)\right)d\xi d\eta, \tag{73}$$

$$\mathrm{E}_{10}\left(a_1,a_2,a_3;\alpha+\beta+\gamma;x,y,z\right)=k_3\int_0^1\int_0^1\xi^{\alpha-1}\left(1-\xi\right)^{\beta+\gamma-1}\eta^{\beta-1}\left(1-\eta\right)^{\gamma-1}\times$$
$$\times\Psi_1\left(a_1,\alpha-a_2;\alpha,\beta;\frac{x\xi}{x\xi-1},\frac{y\eta\left(1-\xi\right)}{1-x\xi}\right){}_1F_1\left(a_3;\gamma;z\left(1-\xi\right)\left(1-\eta\right)\right)d\xi d\eta, \tag{74}$$

$$\mathrm{E}_{11}\left(a,b;\alpha+\beta+\gamma;x,y,z\right)=k_3\int_0^1\int_0^1\xi^{\alpha-1}\left(1-\xi\right)^{\beta+\gamma-1}\eta^{\beta-1}\left(1-\eta\right)^{\gamma-1}\times$$
$$\times\Psi_1\left(a,b;\alpha,\beta;x\xi,y\eta\left(1-\xi\right)\right){}_0F_1\left(-;\gamma;z\left(1-\xi\right)\left(1-\eta\right)\right)d\xi d\eta, \tag{75}$$

$$\mathrm{E}_{12}\left(a_1,a_2,a_3;\alpha+\beta+\gamma;x,y,z\right)=k_3\int_0^1\int_0^1\xi^{\alpha-1}\left(1-\xi\right)^{\beta+\gamma-1}\eta^{\beta-1}\left(1-\eta\right)^{\gamma-1}\times$$
$$\times\mathrm{E}_{62}\left(a_1,a_2,a_3;\alpha,\beta,\gamma;x\xi,y\eta\left(1-\xi\right),z\left(1-\xi\right)\left(1-\eta\right)\right)d\xi d\eta, \tag{76}$$

$$\mathrm{E}_{12}\left(\alpha,\beta,\gamma;c;x,y,z\right)=k_3\int\limits_0^1\int\limits_0^1\xi^{\alpha+\gamma-1}\left(1-\xi\right)^{\beta-1}\eta^{\alpha-1}\left(1-\eta\right)^{\gamma-1}\times$$
$$\times\mathrm{H}_6\left(\alpha+\beta+\gamma;c;x\xi^2\eta\left(1-\eta\right)+y\xi\left(1-\xi\right)\eta,z\left(1-\xi\right)\right)d\xi d\eta, \tag{77}$$

$$\mathrm{E}_{13}\left(a,b;\alpha+\beta+\gamma;x,y,z\right)=k_3\int_0^1\int_0^1\xi^{\alpha-1}\left(1-\xi\right)^{\beta+\gamma-1}\eta^{\beta-1}\left(1-\eta\right)^{\gamma-1}\times$$
$$\times\mathrm{E}_{63}\left(a,b;\alpha,\beta,\gamma;x\xi,y\eta\left(1-\xi\right),z\left(1-\xi\right)\left(1-\eta\right)\right)d\xi d\eta, \tag{78}$$

$$\mathrm{E}_{14}\left(a_1,a_2,a_3;\alpha+\beta+\gamma;x,y,z\right)=k_3\int_0^1\int_0^1 \xi^{\alpha-1}\left(1-\xi\right)^{\beta+\gamma-1}\eta^{\beta-1}\left(1-\eta\right)^{\gamma-1}\times$$
$$\times\mathrm{E}_{64}\left(a_1,a_2,a_3;\alpha,\beta,\gamma;x\xi,y\eta\left(1-\xi\right),z\left(1-\xi\right)\left(1-\eta\right)\right)d\xi d\eta, \tag{79}$$

$$\mathrm{E}_1\left(\alpha,a_2,a_3,a_4,a_5;\alpha+\beta+\gamma+\delta;x,y,z\right)=k_5\int_0^1\int_0^1 \xi^{\alpha-1}\left(1-\xi\right)^{\beta-1}\eta^{\gamma-1}\left(1-\eta\right)^{\delta-1}\times$$
$$\times\,\Xi_1\left(\alpha+\beta,a_5,\gamma+\delta;\alpha+\beta+\gamma+\delta;x\xi\eta+y\left(1-\xi\right)\left(1-\eta\right),z\right)d\xi d\eta, \tag{80}$$

where

$$k_3=\frac{\Gamma\left(\alpha+\beta+\gamma\right)}{\Gamma\left(\alpha\right)\Gamma\left(\beta\right)\Gamma\left(\gamma\right)},\quad k_4=\frac{\Gamma\left(2\alpha+\beta\right)}{\Gamma^2\left(\alpha\right)\Gamma\left(\beta\right)},\quad k_5=\frac{\Gamma\left(\alpha+\beta\right)\Gamma\left(\gamma+\delta\right)}{\Gamma\left(\alpha\right)\Gamma\left(\beta\right)\Gamma\left(\gamma\right)\Gamma\left(\delta\right)},$$

confluent hypergeometric functions E_{15}, E_{17}, E_{18}, E_{19}, E_{21}, E_{62}, E_{63} and E_{64} are defined in [27].

Proof. The proofs of the representations (47)–(80) are similar to the proofs of the previous theorems. □

Theorem 4. If $\mathrm{Re}\left(\alpha\right)>0$, $\mathrm{Re}\left(\beta\right)>0$ and $\mathrm{Re}\left(\gamma\right)>\mathrm{Re}\left(\alpha\right)+\mathrm{Re}\left(\beta\right)$, then the following integral representations are valid:

$$\mathrm{E}_5\left(\alpha,\beta,a;\gamma;x,y,z\right)=k_6\int_0^1\int_0^1\frac{\xi^{\alpha-1}\left(1-\xi\right)^{\beta-1}\eta^{\alpha+\beta-1}\left(1-\eta\right)^{\gamma-\alpha-\beta-1}}{\left(1-y\eta-x\xi\eta+y\xi\eta\right)^a}\times$$
$$\times\,{}_0F_1\left(-;\gamma-\alpha-\beta;z-z\eta\right)d\eta d\xi, \tag{81}$$

$$\mathrm{E}_6\left(\alpha,\beta;\gamma;x,y,z\right)=k_6\int_0^1\int_0^1 \xi^{\alpha-1}\left(1-\xi\right)^{\beta-1}\eta^{\alpha+\beta-1}\left(1-\eta\right)^{\gamma-\alpha-\beta-1}\times$$
$$\times\,e^{x\xi\eta+y\eta-y\xi\eta}{}_0F_1\left(-;\gamma-\alpha-\beta;z-z\eta\right)d\xi d\eta, \tag{82}$$

where

$$k_6=\frac{\Gamma\left(\gamma\right)}{\Gamma\left(\alpha\right)\Gamma\left(\beta\right)\Gamma\left(\gamma-\alpha-\beta\right)}.$$

Proof. The proofs of the representations (81) and (82) are similar to the proofs of the previous theorems. □

4 Triple integral Representations

Theorem 5. The following integral representations are valid under certain restrictions on the numerical parameters:

$$\mathrm{E}_2\left(a_1,a_2,a_3,a_4,a_5;c;x,y,z\right)=\frac{\Gamma\left(a\right)}{\Gamma\left(a_1\right)\Gamma\left(a_2\right)\Gamma\left(a_3\right)\Gamma\left(a_4\right)}\times$$
$$\times\int_0^1\int_0^1\int_0^1 \xi^{a_1-1}\left(1-\xi\right)^{a_2+a_3-1}\eta^{a_2-1}\left(1-\eta\right)^{a_3-1}\zeta^{a_1+a_2+a_3-1}\left(1-\zeta\right)^{a_4-1}\times$$
$$\times\Xi_2\left(\frac{a}{2},\frac{a+1}{2};c;\,4x\xi\eta\left(1-\xi\right)\zeta^2+4y\left(1-\xi\right)\left(1-\eta\right)\zeta\left(1-\zeta\right),z\right)d\xi d\eta d\zeta, \tag{83}$$

$$a:=a_1+a_2+a_3+a_4,\ \ \mathrm{Re}\left(a_k\right)>0,\ \ k=1,2,3,4;$$

$$\mathrm{E}_3\left(a_1,a_2,a_3,a_4;c;x,y,z\right)=\frac{\Gamma\left(a\right)}{\Gamma\left(a_1\right)\Gamma\left(a_2\right)\Gamma\left(a_3\right)\Gamma\left(a_4\right)}\times$$
$$\times\int\limits_0^1\int\limits_0^1\int\limits_0^1\xi^{a_1-1}\left(1-\xi\right)^{a_2+a_3-1}\eta^{a_2-1}\left(1-\eta\right)^{a_3-1}\zeta^{a_1+a_2+a_3-1}\left(1-\zeta\right)^{a_4-1}\times$$
$$\times\mathrm{H}_6\left(a;c;x\xi\left(1-\xi\right)\eta\zeta^2,y\left(1-\xi\right)\left(1-\eta\right)\zeta+z\left(1-\zeta\right)\right)d\xi d\eta d\zeta,\tag{84}$$

$$a:=a_1+a_2+a_3+a_4,\ \ \mathrm{Re}\left(a_k\right)>0,\ \ k=1,2,3,4;$$

$$\mathrm{E}_7\left(a_1,a_2,a_3,a_4;c_1+c_2+c_3;x,y,z\right)=\frac{\Gamma\left(a_2+a_3\right)}{\Gamma\left(a_2\right)\Gamma\left(a_3\right)}\frac{\Gamma\left(c_1+c_2+c_3\right)}{\Gamma\left(c_1\right)\Gamma\left(c_2\right)\Gamma\left(c_3\right)}\times$$
$$\times\int\limits_0^1\int\limits_0^1\int\limits_0^1\xi^{c_1-1}\left(1-\xi\right)^{c_2+c_3-1}\eta^{c_2-1}\left(1-\eta\right)^{c_3-1}\zeta^{a_2-1}\left(1-\zeta\right)^{a_3-1}\times$$
$$\times F_4\left(a_1,a_2+a_3;c_1,c_2;x\xi\zeta,y\eta\left(1-\xi\right)\left(1-\zeta\right)\right){}_1F_1\left(a_4;c_3;z\left(1-\xi\right)\left(1-\eta\right)\right)d\xi d\eta d\zeta,\tag{85}$$

$$\mathrm{Re}\left(a_2\right)>0,\ \ \mathrm{Re}\left(a_3\right)>0,\ \ \mathrm{Re}\left(c_k\right)>0,\ \ k=1,2,3;$$

$$\mathrm{E}_7\left(a_1,a_2,a_3,a_4;c;x,y,z\right)=\frac{\Gamma\left(c\right)}{\Gamma\left(a_2\right)\Gamma\left(a_3\right)\Gamma\left(a_4\right)\Gamma\left(c-a_2-a_3-a_4\right)}\times$$
$$\times\int\limits_0^1\int\limits_0^1\int\limits_0^1\xi^{a_2-1}\left(1-\xi\right)^{a_3-1}\eta^{a_4-1}(1-\eta)^{a_2+a_3-1}\zeta^{a_2+a_3+a_4-1}\left(1-\zeta\right)^{c-a_2-a_3-a_4-1}\times$$
$$\times\left(1-z\eta\zeta\right)^{-a_4}e^{x\xi\eta\zeta+y(1-\xi)(1-\eta)\zeta}d\zeta d\xi d\eta,\tag{86}$$

$$\mathrm{Re}\left(a_k\right)>0,\ k=2,3,4,\ \ \mathrm{Re}\left(c-a_2-a_3-a_4\right)>0;$$

$$\mathrm{E}_9\left(a_1,a_2,a_3;c_1+c_2+c_3;x,y,z\right)=\frac{\Gamma\left(a_2+a_3\right)}{\Gamma\left(a_2\right)\Gamma\left(a_3\right)}\frac{\Gamma\left(c_1+c_2+c_3\right)}{\Gamma\left(c_1\right)\Gamma\left(c_2\right)\Gamma\left(c_3\right)}\times$$
$$\times\int\limits_0^1\int\limits_0^1\int\limits_0^1\xi^{c_1-1}\left(1-\xi\right)^{c_2+c_3-1}\eta^{c_2-1}\left(1-\eta\right)^{c_3-1}\zeta^{a_2-1}\left(1-\zeta\right)^{a_3-1}\times$$
$$\times F_4\left(a_1,a_2+a_3;c_1,c_2;x\xi\zeta,y\eta\left(1-\xi\right)\left(1-\zeta\right)\right){}_0F_1\left(-;c_3;z\left(1-\xi\right)\left(1-\eta\right)\right)d\xi d\eta d\zeta,\tag{87}$$

$$\mathrm{Re}\left(a_2\right)>0,\ \ \mathrm{Re}\left(a_3\right)>0,\ \ \mathrm{Re}\left(c_k\right)>0,\ \ k=1,2,3;$$

$$\mathrm{E}_9\left(a_1,a_2,a_3;c_1+c_2+c_3;x,y,z\right)=\frac{\Gamma\left(a_1+a_2\right)}{\Gamma\left(a_1\right)\Gamma\left(a_2\right)}\frac{\Gamma\left(c_1+c_2+c_3\right)}{\Gamma\left(c_1\right)\Gamma\left(c_2\right)\Gamma\left(c_3\right)}\times$$
$$\times\int\limits_0^1\int\limits_0^1\int\limits_0^1\xi^{c_1-1}\left(1-\xi\right)^{c_2+c_3-1}\eta^{c_2-1}\left(1-\eta\right)^{c_3-1}\zeta^{a_1-1}\left(1-\zeta\right)^{a_2-1}\times$$
$$\times H_4\left(a_1+a_2,a_3;c_1,c_2;x\xi\zeta(1-\zeta),y\eta\zeta(1-\xi)\right){}_0F_1\left(-;c_3;z\left(1-\xi\right)\left(1-\eta\right)\right)d\xi d\eta d\zeta,\tag{88}$$

$$\mathrm{Re}\left(a_2\right)>0,\ \ \mathrm{Re}\left(a_3\right)>0,\ \ \mathrm{Re}\left(c_k\right)>0,\ \ k=1,2,3;$$

$$\mathrm{E}_{10}\left(a_1,a_2,a_3;c_1+c_2+c_3;x,y,z\right)=\frac{\Gamma\left(c_1+c_2+c_3\right)}{\Gamma\left(a_2\right)\Gamma\left(c_1-a_2\right)\Gamma\left(c_2\right)\Gamma\left(c_3\right)}\times$$
$$\times\int\limits_0^1\int\limits_0^1\int\limits_0^1\frac{\xi^{c_1-1}\left(1-\xi\right)^{c_2+c_3-1}\eta^{c_2-1}\left(1-\eta\right)^{c_3-1}\zeta^{a_2-1}\left(1-\zeta\right)^{c_1-a_2-1}}{\left(1-x\xi\zeta\right)^{a_1}}\times$$
$$\times{}_1F_1\left(a_1;c_2;\frac{y\eta\left(1-\xi\right)}{1-x\xi\zeta}\right){}_1F_1\left(a_3;c_3;z\left(1-\xi\right)\left(1-\eta\right)\right)d\xi d\eta d\zeta, \tag{89}$$

where

$$\mathrm{Re}\left(c_1-a_2\right)>0,\ \ \mathrm{Re}\left(c_k\right)>0,\ \ k=1,2,3.$$

Proof. The proofs of the representations (83)–(89) are similar to the proofs of the previous theorems. □

Similar integral formulas involving confluent hypergeometric functions of three variables are discussed in [30].

5 Application

The confluent hypergeometric function E_2 has important applications. In the recent paper [31] particular solutions, including the solution of the Dirichlet problem for the three-dimensional singular Helmholtz equation

$$u_{xx}+u_{yy}+u_{zz}+\frac{2\alpha}{x}u_x+\frac{2\beta}{y}u_y+\frac{2\gamma}{z}u_z-\lambda^2u=0,\quad 0<2\alpha,\,2\beta,\,2\gamma<1$$

in the infinite first octant $\Omega\equiv\{(x,y,z):x>0,\ y>0,\ z>0\}$ are expressed through a function E_2 (in the work [31] the function E_2 is denoted as A_2).

Another recent paper [32] derives important relations linking function E_1 with Appell function F_3, Humbert functions Φ_1, Φ_2, Φ_3, Ξ_1, Ξ_2, generalized hypergeometric function ${}_pF_q$ and Kampé de Fériet function $F^{p:q;k}_{l:m;n}$ (for details on Kampé de Fériet function see [33–35]).

Conclusion

As is known [27], the list of confluent hypergeometric functions of three variables was compiled recently, however, the confluent functions E_1–E_{14} investigated in this paper were first introduced by Jain [26] in 1966, who limited himself to composing systems of partial differential equations corresponding to these functions. Until now, the scientific community has not known any applications of the confluent hypergeometric functions E_1–E_{14}, except for the function E_2 discussed in the Application section.

Author Contributions

A. Hasanov served as the principal investigator of the research and supervised the research process. T.G. Ergashev collected and analyzed data and led manuscript preparation. A.R. Ryskan assisted in data collection and analysis. All authors participated in the revision of the manuscript and approved the final submission. All authors contributed equally to this work.

Conflict of Interest

The authors declare no conflict of interest.

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Author Information[*]

Anvar Hasanov — Doctor of Physical and Mathematical Sciences, Professor, Institute of Mathematics, Uzbekistan Academy of Sciences, 4 University St., Tashkent, 700174, Uzbekistan; e-mail: *anvarhasanov@yahoo.com*; https://orcid.org/0000-0002-9849-4103

Tukhtasin Gulamzhanovich Ergashev — Doctor of Physical and Mathematical Sciences, Professor, Department of Higher Mathematics, "Tashkent Institute of Irrigation and Agricultural Mechanization Engineer" National Research University, 39 Kari-Niyazi St., Tashkent, 100000, Uzbekistan; e-mail: *ergashev.tukhtasin@gmail.com*; https://orcid.org/0000-0003-3542-8309

Ainur Ryskankyzy Ryskan (*corresponding author*) — PhD, Associate Professor, School of Digital Technologies, Narxoz University, 55 Zhandosov St., Almaty, 050035, Kazakhstan; e-mail: *ryskan.a727@gmail.com*; https://orcid.org/0000-0002-8764-4751

[*]Authors' names are presented in the following order: first name, middle name (if any), last name.