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Reversed Weighted Hardy-type Inequalities with Negative Indices

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This research paper presents a comprehensive investigation of novel Hardy-type dynamic inequalities that incorporate two independent weight functions, denoted as $\mathtt{u}$ and $\mathtt{v}$. A distinctive feature of this work is its focus on time scales calculus with negative parameters, a generalization that unifies and extends discrete and continuous analysis. The basic methodology involves the application of the reverse Hölder's inequality and the Minkowski integral inequality to rigorously deduce all essential results. To illustrate the adaptability of our results, we provide explicit examples of the corresponding discrete and integral analogues for various time scales: when $\mathbb{T} = \mathbb{N}$ (the natural numbers, indicating discrete sequences), $\mathbb{T} = l^{\mathbb{N}_0}$ for $l > 1$ (a quantum time scale), and $\mathbb{T} = \mathbb{R}$ (the real numbers, signifying the classical continuous case). This paper situates its findings within a wider mathematical framework by demonstrating how they contain and extend certain cases of reverse Hardy-type dynamic inequalities previously formulated by distinguished scholars including Prokhorov, Kufner, Yang, Nguyen, and Benaissa. Consequently, this work presents a cohesive framework that broadens the theoretical terrain of Hardy-type inequalities.

Keywords: time scales, dynamic inequalities, reverse Hardy inequality, negative exponents, delta differentiation, Keller's rule, reverse time scale Hölder's inequality, integral time scale Minkowski's inequality.

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Introduction

In a pivotal paper of Hardy [1], he demonstrated the discrete classical inequality

$$\sum_{\mathfrak{n}=1}^{\infty}\left(\frac{1}{\mathfrak{n}}\sum_{\mathfrak{k}=1}^{\mathfrak{n}}\mathtt{b}^{\frac{1}{\mathtt{r}}}(\mathfrak{k})\right)^{\mathtt{r}} \leq \left(\frac{\mathtt{r}}{\mathtt{r}-1}\right)^{\mathtt{r}}\sum_{\mathfrak{n}=1}^{\infty}\mathtt{b}(\mathfrak{n}), \quad \mathtt{r}>1,$$

that provides an essential implementation for double series of Hilbert's inequality, a widely recognized concept at this time, where $\mathtt{b}(\mathfrak{n}) \geq 0$ for $\mathfrak{n} \geq 1$. Moreover, he [2] discovered the corresponding inequality which affirms that, for $\mathtt{p} > 1$ and $\mathfrak{h}(\mathfrak{x})$ is an integrable positive function that belongs to the weighted space $L^{\mathtt{p}}(\mathbb{R}^{+})$, the inequality

$$\|H_{\mathfrak{h}}\|_{L^{\mathtt{p}}(\mathbb{R}^{+})} \leq \frac{\mathtt{p}}{\mathtt{p}-1}\|\mathfrak{h}\|_{L^{\mathtt{p}}(\mathbb{R}^{+})} \tag{1}$$

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holds, where $H_{\mathfrak{h}}(\mathfrak{x}) = \frac{1}{\mathfrak{x}} \int_{\mathtt{a}}^{\mathfrak{x}} \mathfrak{h}(t)\, dt$ and $\|\mathfrak{h}\|_{L^{\mathtt{p}}(\mathbb{R}^{+})} = \left(\int_{\mathtt{a}}^{\infty} \mathfrak{h}^{\mathtt{p}}(\mathfrak{x})\, d\mathfrak{x}\right)^{1/p}$, with $\mathtt{a} > 0$ where the constant factor $\left(\frac{\mathtt{p}}{\mathtt{p}-1}\right)^{\mathtt{p}}$ is sharp one. Extending inequality (1) over the past few decades has led to the following generic mixed norm inequality

$$\|S_{\mathfrak{h}}\|_{L^{q}_{\mathtt{u}}[\mathtt{a},\mathtt{b})} \leq \mathbf{C}\, \|\mathfrak{h}\|_{L^{\mathtt{p}}_{\mathtt{v}}[\mathtt{a},\mathtt{b})}, \quad \text{where} \quad S_{\mathfrak{h}}(\mathfrak{x}) = \int_{\mathtt{a}}^{\mathfrak{x}} \mathfrak{h}(t)\Delta t, \tag{2}$$

that holds with $\mathfrak{h} \in L^{\mathtt{p}}_{\mathtt{v}}[\mathtt{a},\mathtt{b})$ and $S_{\mathfrak{h}} \in L^{q}_{\mathtt{u}}[\mathtt{a},\mathtt{b})$, where the norm of $\mathfrak{h}$ is defined as

$$\|\mathfrak{h}\|_{L^{\mathtt{p}}_{\mathtt{v}}[\mathtt{a},\mathtt{b})} = \left(\int_{\mathtt{a}}^{\mathtt{b}} \mathfrak{h}^{\mathtt{p}}(\mathfrak{x})\mathtt{v}(\mathfrak{x})d\mathfrak{x}\right)^{1/\mathtt{p}},$$

for any given positive measurable weights $\mathtt{u}$ and $\mathtt{v}$, defined on the open interval $(\mathtt{a},\mathtt{b})$ and $1 < \mathtt{p} \leq q < \infty$, for all values of $-\infty \leq \mathtt{a} < \mathtt{b} \leq \infty$. A large body of literature has addressed new generalizations and extensions of the Hardy inequality (2); the interested reader is referred to the relevant papers [3–5] and books [6–8]. Beesack et al. in [9] considered inequality (2), they obtained the sufficient condition for exponents $\mathtt{p}$ and q that are less than one, where $\mathtt{p} \neq 0, q \neq 0$, then for positive weights $\mathtt{u}$ and $\mathtt{v}$ defined on $(\mathtt{a},\mathtt{b})$, they concluded Hardy reverse inequality that has the form

$$\left(\int_{\mathtt{a}}^{\mathtt{b}} (\mathtt{v}(\mathfrak{x})\, \mathfrak{h}(\mathfrak{x}))^{\mathtt{p}}\, d\mathfrak{x}\right)^{\frac{1}{\mathtt{p}}} \leq \mathbf{C} \left(\int_{\mathtt{a}}^{\mathtt{b}} \left(\mathtt{u}(\mathfrak{x}) \int_{\mathtt{a}}^{\mathfrak{x}} \mathfrak{h}(t)\, dt\right)^{q} d\mathfrak{x}\right)^{\frac{1}{q}}, \tag{3}$$

that is valid if and only if

$$\inf_{s>0} \left(\int_{\mathtt{a}}^{s} \mathtt{u}^{q}(\mathfrak{x})\, d\mathfrak{x}\right)^{\frac{1}{q}} \left(\int_{\mathtt{a}}^{s} \mathtt{v}^{-\mathtt{p}'}(\mathfrak{x})\, d\mathfrak{x}\right)^{\frac{1}{\mathtt{p}'}} = \mathcal{B}_1 < \infty, \qquad \frac{1}{\mathtt{p}'} + \frac{1}{\mathtt{p}} = 1,$$

where $s > 0$. Moreover, they deduced the dual reverse integral Hardy inequality

$$\left(\int_{\mathtt{a}}^{\mathtt{b}} (\mathtt{v}(\mathfrak{x})\, \mathfrak{h}(\mathfrak{x}))^{\mathtt{p}}\, d\mathfrak{x}\right)^{\frac{1}{\mathtt{p}}} \leq \mathbf{C} \left(\int_{\mathtt{a}}^{\mathtt{b}} \left(\mathtt{u}(\mathfrak{x}) \int_{\mathfrak{x}}^{\mathtt{b}} \mathfrak{h}(t)\, dt\right)^{q} d\mathfrak{x}\right)^{\frac{1}{q}}, \tag{4}$$

which also asserts if and solely if $\inf\limits_{s>0} \left(\int_{s}^{\mathtt{b}} \mathtt{u}^{q}(\mathfrak{x})\, d\mathfrak{x}\right)^{\frac{1}{q}} \left(\int_{s}^{\mathtt{b}} \mathtt{v}^{-\mathtt{p}'}(\mathfrak{x})\, d\mathfrak{x}\right)^{\frac{1}{\mathtt{p}'}} = \mathcal{B}_2 < \infty$.

The authors in [10], expanded the reverse Hardy inequalities (3) and (4) with kernels. In specific, they showed that for the non-increasing $\mathbf{K}(\mathfrak{x}, y)$ with $q \leq \mathtt{p} < 0$, the resulting inequality

$$\|\mathfrak{h}\|_{L^{\mathtt{p}}_{\mathtt{v}}[0,\infty)} \leq \mathbf{C}\, \|S_{\mathfrak{h}}\|_{L^{q}_{\mathtt{u}}[0,\infty)}, \quad \text{where} \quad S_{\mathfrak{h}} = \left(\int_{0}^{\mathfrak{x}} \mathbf{K}(\mathfrak{x}, y)\mathfrak{h}(y)\, dy\right)^{q}, \tag{5}$$

that for every $\mathfrak{h} \in L^{\mathtt{p}}_{\mathtt{v}}[0,\infty)$, $S_{\mathfrak{h}} \in L^{q}_{\mathtt{u}}[0,\infty)$ where $\|\mathfrak{h}\|_{L^{\mathtt{p}}_{\mathtt{v}}[0,\infty)} = \left(\int_{0}^{\infty} \mathfrak{h}^{\mathtt{p}}(\mathfrak{x})\mathtt{v}(\mathfrak{x})d\mathfrak{x}\right)^{1/\mathtt{p}}$, then the inequality (5) holds if and only if $\inf\limits_{s} \mathfrak{J}_1(s) = \mathcal{B}_3 > 0$, where $\mathfrak{J}_1(s) = \left(\int_{0}^{s} \mathtt{v}^{1-\mathtt{p}'}(\mathfrak{x})\, \mathbf{K}^{p'}(\mathfrak{x}, y)dx\right)^{\frac{1}{\mathtt{p}'}} \left(\int_{0}^{s} \mathtt{u}(\mathfrak{x})\, d\mathfrak{x}\right)^{\frac{1}{q}}$ is not decreasing and the constant $\mathbf{C}$ satisfies the condition that $1/\mathbf{C} \geq (\mathtt{p}')^{1/\mathtt{p}'} (-\mathtt{p})^{1/q} \mathcal{B}_3$. On the other hand, they deduced the dual reverse integral Hardy inequality

$$\left(\int_{0}^{\infty} \mathtt{v}(\mathfrak{x})\, h^{\mathtt{p}}(\mathfrak{x})\, d\mathfrak{x}\right)^{\frac{1}{\mathtt{p}}} \leq \mathbf{C} \left(\int_{0}^{\infty} \mathtt{u}(\mathfrak{x}) \left(\int_{\mathfrak{x}}^{\infty} \mathbf{K}(y, \mathfrak{x})\mathfrak{h}(y)\, dy\right)^{q} d\mathfrak{x}\right)^{\frac{1}{q}},$$

which also holds for the non-decreasing $\mathbf{K}(\mathfrak{x},y)$ if and only if $\inf_s \mathfrak{J}_2(s)=\mathcal{B}_4>0$, where $\mathfrak{J}_2(s)=\left(\int_s^{\mathsf{b}}\mathsf{v}^{-\mathsf{p}'}(\mathfrak{x})\,d\mathfrak{x}\right)^{\frac{1}{\mathsf{p}'}}\left(\int_s^{\mathsf{b}}\mathsf{u}^{q}(\mathfrak{x})\,d\mathfrak{x}\right)^{\frac{1}{q}}$ is non-increasing and $\mathbf{C}$ here satisfies $1/\mathbf{C}\geq(\mathsf{p}')^{1/\mathsf{p}'}(-p)^{1/q}\mathcal{B}_4$.

Since the previous inequalities were discovered, various papers have been published in the literature that dealt with contemporary proofs, generalizations, and extensions. Now, we review some of these results that can stimulate and clarify the content of this paper. Prokhorov in [11] focused on generalizing inequality (3) to the range of $-\infty<q\leq\mathsf{p}<0$ which has the form

$$\left(\int_{\mathsf{a}}^{\mathsf{b}}\mathfrak{h}^{\mathsf{p}}(\mathfrak{x})\,d\mathfrak{x}\right)^{\frac{1}{\mathsf{p}}}\leq\mathbf{C}\left(\int_{\mathsf{a}}^{\mathsf{b}}\mathsf{u}(\mathfrak{x})\left(\int_{\mathsf{a}}^{\mathfrak{x}}\mathsf{v}(t)\,\mathfrak{h}(t)\,dt\right)^{q}d\mathfrak{x}\right)^{\frac{1}{q}},$$

and it holds for the condition that $\sup_{\mathsf{a}<t<\mathsf{b}}\left(\int_{\mathsf{a}}^{\mathfrak{x}}\mathsf{v}^{\mathsf{p}'}(t)\,dt\right)^{\frac{-1}{\mathsf{p}'}}\left(\int_{\mathsf{a}}^{\mathfrak{x}}\mathsf{u}^{q}(t)\,dt\right)^{\frac{-1}{q}}=\mathcal{B}_5<\infty$, for the lowest suitable constant $\mathbf{C}$, likewise he generated the dual case. Through involving a constraint that $q/\mathsf{p}\geq1$, Kufner et al. [12] extended inequality (3) to get the best possible estimation that makes the next inequality

$$\left(\int_{\mathsf{a}}^{\mathsf{b}}\mathsf{v}(\mathfrak{x})\,\mathfrak{h}^{\mathsf{p}}(\mathfrak{x})\,d\mathfrak{x}\right)^{\frac{1}{\mathsf{p}}}\leq\mathbf{C}\left(\int_{\mathsf{a}}^{\mathsf{b}}\mathsf{u}(\mathfrak{x})\left(\int_{\mathsf{a}}^{\mathfrak{x}}\mathfrak{h}(t)\,dt\right)^{q}d\mathfrak{x}\right)^{\frac{1}{q}}\tag{6}$$

holds. They concluded that the best possible constant $\mathbf{C}$ for $-\infty<q\leq\mathsf{p}<0$, is given by

$$\mathcal{B}_6\leq\mathbf{C}<2^{-\frac{1}{q}}\left(\frac{\mathsf{p}+s-2}{\mathsf{p}-1}\right)^{\frac{1}{\mathsf{p}'}}\mathcal{B}_6,\quad s\in[\mathsf{p},1),$$

where $\mathcal{B}_6=\sup_{\mathsf{a}<\mathfrak{x}<\mathsf{b}}\left(\int_{\mathsf{a}}^{\mathfrak{x}}V^{\frac{\mathsf{p}-s}{\mathsf{p}}q}(t)\mathsf{u}(t)dt\right)^{\frac{-1}{q}}V^{\frac{1-s}{\mathsf{p}}}(\mathfrak{x})$, they also deduced analogy inequality of (6). In our paper, we will concentrate on the previous inequality (6) and its analogy.

Motivated by developments in the continuous setting, it is natural to investigate weighted discrete Hardy inequalities with negative indices, as demonstrated by Nguyen et al. [13], they proved that the following discrete inequality

$$\sum_{\mathfrak{n}=1}^{\infty}\left(\frac{1}{\mathfrak{n}}\sum_{k=1}^{\mathfrak{n}}\mathsf{b}^{r}(k)\right)^{\frac{1}{r}}\leq2^{1-\frac{1}{r}}\frac{r}{r-1}\sum_{\mathfrak{n}=1}^{\infty}\mathsf{b}(\mathfrak{n}),\quad r<-1$$

applies to any non-negative realistic convergent sequence $\{\mathsf{b}(\mathfrak{n})\}$ that for all $\mathfrak{n}\in\mathbb{N}$. Furthermore, the author in [14] deduced that for the case that p and r are less than one where r is not equal to zero, the inequality

$$\sum_{\mathfrak{n}=1}^{\mathbf{N}}\mathsf{a}_{\mathfrak{n}}\left(\frac{1}{A_{\mathfrak{n}}}\sum_{i=1}^{\mathfrak{n}}\mathsf{a}_i\mathsf{b}^{r}(i)\right)^{1/\mathsf{p}}\leq A_{\mathfrak{n}}^{1-1/r}\left(\sum_{i=1}^{\mathbf{N}}\mathsf{a}_iA^{-r}(i)\right)^{1/r}\times\sum_{\mathfrak{n}=1}^{\mathbf{N}}\left(1-\frac{\sum_{i=1}^{\mathfrak{n}-1}\mathsf{a}_iA^{-r}(i)}{\sum_{i=1}^{\mathbf{N}}\mathsf{a}_iA^{-r}(i)}\right)\mathsf{b}_{\mathfrak{n}}\mathsf{a}_{\mathfrak{n}}$$

holds for all $\mathbf{N}\in\mathbb{N}$, $\{\mathsf{b}_{\mathfrak{n}}\}$ is a sequence that is not negative and $\{\mathsf{a}_{\mathfrak{n}}\}$ is a sequence of real positive numbers.

Over the past few decades several authors [15–17] have focused on the development of inequality (3) by expanding and generalizing the continuous Hardy's inequality with negative indices, thereby deriving corresponding discrete versions. Due to the significance of this inequality in harmonic analysis and mathematics, we restrict our attention to the representative papers [18–20].

The time scale dynamic inequalities have recently attracted much attention [21–23], as they may serve as examples of discrete and integral inequalities. Time scale calculus consists of three primary instances: differential calculus for $\mathbb{T}=\mathbb{R}$, difference calculus if $\mathbb{T}=\mathbb{N}$, and quantum calculus for

$\mathbb{T} = l^{\mathbb{N}_0} = \left\{ l^t : t \in \mathbb{N}_0,\ l > 1 \right\}$. For the convenience of the reader, we include here several time scale inequalities that are closely related to Hardy-type inequalities. We refer the reader to the previous references [24–26]. For example in [27] authors derived that the condition

$$\sup_{\mathtt{a}<\mathfrak{r}<\mathtt{b}} \left(\int_{\mathfrak{r}}^{\mathtt{b}} \mathtt{u}(t)\,\Delta t \right)^{1/q} \left(\int_{\mathtt{a}}^{\sigma(\mathfrak{r})} \mathtt{v}^{1-\mathtt{p}^*}(t)\,\Delta t \right)^{1/\mathtt{p}^*} = \mathcal{B}_7 < \infty, \qquad \mathtt{p}^* = \frac{\mathtt{p}}{\mathtt{p}-1}$$

is the necessary and sufficient for the generalized dynamic inequality

$$\|R_{\mathfrak{h}}\|_{L^q_{\mathtt{u}}([\mathtt{a},\mathtt{b})_{\mathbb{T}})} \le \mathbf{C}\, \|\mathfrak{h}\|_{L^{\mathtt{p}}_{\mathtt{v}}([\mathtt{a},\mathtt{b})_{\mathbb{T}})}, \quad R_{\mathfrak{h}}(\mathfrak{r}) = \int_{\mathtt{a}}^{\sigma(\mathfrak{r})} \mathfrak{h}(t)\Delta t, \tag{7}$$

where $1 < \mathtt{p} \le q < \infty$ and the constant $\mathbf{C}$ in (7) satisfied the estimation

$$\mathcal{B}_7 \le \mathbf{C} \le \left(1 + \frac{q}{\mathtt{p}^*}\right)^{1/q} \left(1 + \frac{\mathtt{p}^*}{q}\right)^{1/\mathtt{p}^*} \mathcal{B}_7.$$

The natural follow-up question is: Is it possible to define new weight functions on an arbitrary time scale $\mathbb{T}$ that satisfy Hardy-type inequalities with negative indices? The purpose of this paper is to respond affirmatively to this inquiry. To be precise, I will extend inequality (6) and its analogue on time scales, while explaining the associated validity condition in the case that $-\infty < q \le \mathtt{p} < 0$ with the additional requirement that $q/\mathtt{p} \ge 1$. Moreover, we will generate discrete and integral characterizations employing these new characterizations. Our main results will be established using available tools on time scales including reverse Hölder inequality and the integral Minkowski inequality, while developing a novel approach. Following this introduction, Section 2 of the paper consists of a presentation of some fundamental concepts of time scale calculus and auxiliary results that are essential for instituting all main results. Ultimately, Section 3 concludes with some time scale variations of inequality (6). It is worth mentio-ning here that, as an exception case stemmed from our results, for $\mathbb{T} = \mathbb{R}$, $\mathbb{T} = \mathbb{N}$ and $\mathbb{T} = l^{\mathbb{N}_0}$, we will also discuss specific dynamic inequalities of Hardy-type that Prokhorov, Kufner, Yang, Nguyen, and Benaissa have captured in their literature previously.

1 Preliminaries and Basic Lemmas

Here we provide a high-level overview of the principles within time scale theory. Furthermore, we include supporting concepts that are required to validate our fundamental results.We recommend consulting two monographs authored by Bohner and Peterson [28–30] for a comprehensive examination of time scale calculus. To keep this work concise, we will just provide the core information that is required to prove our results. The time scale $\mathbb{T}$ is a closed non-empty arbitrary subset of real numbers $\mathbb{R}$.

The operator $\sigma : \mathbb{T} \to \mathbb{T}$ that identified as $\sigma(t) := \inf\{s \in \mathbb{T} : s > t\}$ is called the forward jump operator and that for all $t \in \mathbb{T}$. The graininess function μ is defined as $\mu(t) = \sigma(t) - t$, where $\mu : \mathbb{T} \to [0, \infty)$.

The mapping $\mathfrak{h} : [\mathtt{a}, b] \to \mathbb{R}$ is interpreted as the right-dense continuous function ($rd-$continuous) if it is right continuous at every right-dense point and the left limit is finite and exists at left–dense point, then we can denote the space of all right dense continuous functions as $C_{rd}(\mathbb{T}, \mathbb{R})$.
If the derivative of any function $\mathfrak{h}$ exists then this function is called differentiable. The forward shift of the function $\mathfrak{h}$ is $\mathfrak{h}^\sigma$, where $\mathfrak{h}^\sigma(t) = \mathfrak{h} \circ \sigma(t)$ and the delta derivative of $\mathfrak{h}$ signifies $\mathfrak{h}^\Delta$, where $\mathfrak{h}^\sigma = \mathfrak{h} + \mu \mathfrak{h}^\Delta$. For every function $\Omega : \mathbb{T} \to \mathbb{R}$, the inscription $\Omega^\sigma(\mathtt{t})$ indicates $\Omega(\sigma(\mathtt{t}))$. For each positive value of ϵ there is a neighbourhood U of $\mathtt{t}$, where

$$|[\Omega(\sigma(\mathtt{t})) - \Omega(s)] - \Omega^\Delta(\mathtt{t})[\sigma(\mathtt{t}) - s]| \le \epsilon |\sigma(\mathtt{t}) - s|,$$

for all $s \in U$ for the existence number $\Omega^{\Delta}(\mathfrak{t})$. Here, we can say that $\Omega^{\Delta}(\mathfrak{t})$ is a delta derivative of Ω at $\mathfrak{t}$ and Ω is a delta differentiable at $\mathfrak{t}$. Recapture the product and quotient formulae for the derivatives of two delta differentiable functions φ and $\imath$, denoted as $\varphi\imath$ and $\varphi/\imath$, respectively

$$(\varphi\imath)^{\Delta} = \varphi^{\Delta}\imath + \varphi^{\sigma}\imath^{\Delta} = \varphi\imath^{\Delta} + \varphi^{\Delta}\imath^{\sigma}, \qquad \left(\frac{\varphi}{\imath}\right)^{\Delta} = \frac{\varphi^{\Delta}\imath - \varphi\imath^{\Delta}}{\imath\imath^{\sigma}}, \qquad \imath\imath^{\sigma} \neq 0.$$

Time scale integration for any delta differentiable function $\jmath : \mathbb{T} \to \mathbb{R}$ is defined as follows: the Cauchy integral of $\jmath^{\Delta}$ is defined as $\int_{\mathsf{a}}^{\mathsf{b}} \jmath^{\Delta}(t)\Delta t = \jmath(b) - \jmath(\mathsf{a})$, for a delta differentiable function $\jmath$, with $\mathsf{a}, b \in \mathbb{T}$, (for more details, see [30]). Observe that for time scale $\mathbb{T} = \mathbb{R}$, we acquire that $t = \sigma(t)$, $\jmath^{\Delta}(t) = \jmath'(t)$ and

$$\int_{\mathsf{a}}^{\mathsf{b}} \jmath(t)\Delta t = \int_{\mathsf{a}}^{\mathsf{b}} \jmath(t)dt,$$

if $\mathbb{T} = \mathbb{Z}$, then $\sigma(t) = 1 + t$, $\jmath^{\Delta}(t) = \Delta\jmath(t)$ and $\int_{\mathsf{a}}^{\mathsf{b}} \jmath(t)\Delta t = \sum_{t=\mathsf{a}}^{b-1} \jmath(t)$.

Further, by setting $\mathbb{T} = \ell^{\mathbb{N}}$, we have $\sigma(t) = \ell t$, $\mu(t) = (\ell - 1)t$ and $\int_{t_0}^{\infty} \jmath(t)\Delta t = \sum_{k=n_0}^{\infty} \jmath(\ell^k)\mu(\ell^k)$, where for every $t \in \mathbb{T}$, $\ell^{\mathbb{N}} = \{t = \ell^k,\ k \in \mathbb{N}_0,\ \ell > 1\}$, and $t_0 = \ell^{n_0}$.

The following two lemmas are mentioned in [30, 31].

Lemma 1. Suppose that η and φ are right-dense continuous mappings that defined on the interval $[0, \infty)_{\mathbb{T}}$. Then, integration by parts formulation is stated as

$$\int_0^{\infty} \eta(t)\varphi^{\Delta}(t)\Delta t = \eta(t)\varphi(t)|_0^{\infty} - \int_0^{\infty} \eta^{\Delta}(t)\varphi^{\sigma}(t)\Delta t. \tag{8}$$

Lemma 2. Assume that the function $\jmath : \mathbb{R} \to \mathbb{R}$ is continuously differentiable function and let $h : \mathbb{T} \to \mathbb{R}$ be a delta-differentiable mapping. Then, the composition of both functions $\jmath$ and h is a delta-differentiable mapping and for every $\zeta \in [t, \sigma(t)]$ the following equation

$$(\jmath h\,(t))^{\Delta} = \jmath'\,(h\,(\zeta))\,h^{\Delta}(t) \tag{9}$$

holds.

The two lemmas that follow are the derived reverse time scale Hölder's inequality and Minkowski's integral inequality, which are crucial to our results [31, 32].

Theorem 1. Suppose η, and φ are two Δ-integrable functions therefore

$$\int_{\mathsf{a}}^{\mathsf{b}} |\eta(t)\varphi(t)|\Delta t \leq \left[\int_{\mathsf{a}}^{\mathsf{b}} |\eta(t)|^{\varepsilon}\Delta t\right]^{\frac{1}{\varepsilon}} \left[\int_{\mathsf{a}}^{\mathsf{b}} |\varphi(t)|^{\varsigma}\Delta t\right]^{\frac{1}{\varsigma}},$$

for every a, and b belongs to $\mathbb{T}$, where $\varepsilon > 1$ and $\frac{1}{\varepsilon} + \frac{1}{\varsigma} = 1$. If we replace ε with negative value $\varepsilon < 0$, we capture the so-called reverse time scale Hölders inequality [33]

$$\int_{\mathsf{a}}^{\mathsf{b}} |\eta(t)\varphi(t)|\Delta t \geq \left[\int_{\mathsf{a}}^{\mathsf{b}} |\eta(t)|^{\varepsilon}\Delta t\right]^{\frac{1}{\varepsilon}} \left[\int_{\mathsf{a}}^{\mathsf{b}} |\varphi(t)|^{\varsigma}\Delta t\right]^{\frac{1}{\varsigma}}. \tag{10}$$

Now, we state Minkowski integral inequality [34, 35].

Theorem 2. Let κ, Θ be rd-continuous and non-negative functions identified on the interval $[\mathsf{a}, \infty)_{\mathbb{T}}$, where a belongs to the time scale $\mathbb{T}$. For $\mathfrak{k} \geq 1$, the inequality

$$\left[\int_{\mathsf{a}}^{\infty} \kappa(s)\left(\int_{\mathsf{a}}^{\sigma(s)} \Theta(t)\Delta t\right)^{\mathfrak{k}} \Delta s\right]^{1/\mathfrak{k}} \leq \int_{\mathsf{a}}^{\infty} \Theta(s)\left(\int_s^{\infty} \kappa(\mathfrak{r})\Delta\mathfrak{r}\right)^{1/\mathfrak{k}} \tag{11}$$

holds. If $0 < \mathfrak{k} \leq 1$, then the direction of inequality (11) is reversed.

We can consider the two next lemmas proven in [35] to be power rules of integration on the time scale $\mathbb{T}$.

Lemma 3. Consider a time scale $\mathbb{T}$, and let $\mathsf{a}, \mathfrak{x} \in \mathbb{T}$ with $\mathfrak{x} \geq a$. Then, the inequality

$$\left(\int_{\mathsf{a}}^{\sigma(\mathfrak{x})} \mathrm{g}(t)\Delta t\right)^{m} \leq m \int_{\mathsf{a}}^{\sigma(\mathfrak{x})} \mathrm{g}(t)\left(\int_{\mathsf{a}}^{\sigma(t)} \mathrm{g}(s)\Delta s\right)^{m-1} \Delta t$$

holds for $m \geq 1$; however, for $0 < m < 1$, the direction of the inequality will reverse.

Lemma 4. Assume that $\mathbb{T}$ is a time scale. Then, for $m \geq 1$, the inequality [36, 37]

$$\left(\int_{\mathfrak{x}}^{b} \mathrm{g}(t)\Delta t\right)^{m} \leq m \int_{\mathfrak{x}}^{b} \mathrm{g}(t)\left(\int_{t}^{b} \mathrm{g}(s)\Delta s\right)^{m-1} \Delta t$$

is valid for every b, where $\mathfrak{x}$ belongs to $\mathbb{T}$ with $\mathfrak{x} \leq b$. While for $0 < m < 1$, the inequality will reverse.

All results in this paper are considered to be based on continuous and non-negative functions.

2 Main Results and Applications

Now, we can explain and validate the main results. Here, it should be mentioned that the integrals in the statements of theorems that follow are assumed to exist. To demonstrate our initial result, we will adopt the reverse of Hölder inequality (10) and the integral inequality of Minkowski (11). In order to make items easier, we use the notations

$$\mathbb{E}_{k}(\mathfrak{x}, s) = \left(\Lambda^{\sigma}(\mathfrak{x})\right)^{\frac{1-s}{p}} \left(\int_{\mathsf{a}}^{\sigma(\mathfrak{x})} \mathfrak{u}(t)\Lambda^{\frac{q(p-s)}{p}}(t)\Delta t\right)^{-1/q}, \quad s \in [p, 1), \tag{12}$$

where the mapping $\Lambda(\mathfrak{x})$ is a non-negative mapping defined on $[\mathsf{a}, b]_{\mathbb{T}}$ and

$$\Lambda(\mathfrak{x}) := \int_{\mathsf{a}}^{\sigma(\mathfrak{x})} \mathrm{v}^{1-p'}(t)\Delta t, \tag{13}$$

where $p' := \frac{p}{p-1}$, further

$$\mathbb{E}_{s}(\mathfrak{x}, s) = \sup_{\mathfrak{x}\in(\mathsf{a},\ b)_{\mathbb{T}}} \left(\Lambda^{\sigma}(\mathfrak{x})\right)^{\frac{1-s}{p}} \left(\int_{\mathsf{a}}^{\sigma(\mathfrak{x})} \mathfrak{u}(t)\Lambda^{\frac{q(p-s)}{p}}(t)\Delta t\right)^{-1/q}, \quad s \in [p, 1), \tag{14}$$

then

$$\mathbb{E}_{s} := \sup_{\mathfrak{x}\in(\mathsf{a},\ b)_{\mathbb{T}}} \mathbb{E}_{k}. \tag{15}$$

Theorem 3. Suppose that $\mathbb{T}$ is a time scale and $-\infty < q \leq p < 0$, where $q/p \geq 1$, $\mathfrak{f} \in C_{rd}\left([\mathsf{a}, b]_{\mathbb{T}}, \mathbb{R}\right)$ is a non-negative function and assume that $\mathfrak{u}$, v are rd-continuous positive functions on the open interval $(\mathsf{a}, b)_{\mathbb{T}}$. Thus, the reverse Hardy inequality

$$\left(\int_{\mathsf{a}}^{b} \mathfrak{f}^{p}(\mathfrak{x})\,\mathrm{v}(\mathfrak{x})\,\Delta\mathfrak{x}\right)^{1/p} \leq \mathbf{C}\left(\int_{\mathsf{a}}^{b}\left(\int_{\mathsf{a}}^{\sigma(\mathfrak{x})} \mathfrak{f}(t)\,\Delta t\right)^{q} \mathfrak{u}(\mathfrak{x})\,\Delta\mathfrak{x}\right)^{1/q} \tag{16}$$

is valid for every function $\mathfrak{f}$, if and only if the following criterion applies,

$$\mathbb{E}_{s} < \infty. \tag{17}$$

Moreover, the value of $\mathbf{C}$ in (16) possesses the estimation

$$\mathbb{E}_s \leq \mathbf{C} < 2^{-\frac{1}{q}} \left(\frac{p+s-2}{p-1} \right)^{1/p'} \mathbb{E}_s,$$

where the definition of $\mathbb{E}_s$ is mentioned in (14).

Proof. First of all and to facilitate, we reform inequality (16) to take the equivalent form

$$\int_{\mathsf{a}}^{b} \left(\int_{\mathsf{a}}^{\sigma(\mathfrak{x})} \mathfrak{g}^{\frac{1}{p}}(t)\, \varrho(t) \Delta t \right)^{q} \mathfrak{u}(\mathfrak{x})\, \Delta \mathfrak{x} \leq \mathbf{C}^{-q} \left(\int_{\mathsf{a}}^{b} \mathfrak{g}(\mathfrak{x})\, \Delta \mathfrak{x} \right)^{q/p}, \tag{18}$$

and that by setting $\mathfrak{f}(\mathfrak{x}) = \mathfrak{g}^{\frac{1}{p}}(\mathfrak{x})\, w(\mathfrak{x})$ and $\varrho(\mathfrak{x}) = \mathrm{v}^{-1/p}(\mathfrak{x})$. Then, equation (13) can be expressed to

$$\Lambda(\mathfrak{x}) := \int_{\mathsf{a}}^{\sigma(\mathfrak{x})} \varrho^{p'}(t) \Delta t, \quad p' := \frac{p}{p-1}. \tag{19}$$

Now, we prove the sufficiency of condition (17), so we first assume that $\mathbb{E}_s < \infty$, this implies that

$$0 < \Lambda(\mathfrak{x}) := \int_{\mathsf{a}}^{\sigma(\mathfrak{x})} \varrho^{p'}(t) \Delta t < \infty,$$

for every $t \in (\mathsf{a}, b)_{\mathbb{T}}$. Invoking the reverse of Hölder inequality (10) with parameters p and p' on the left-hand side of the inequality (18), we obtain that

$$\begin{aligned}
&\int_{\mathsf{a}}^{b} \left(\int_{\mathsf{a}}^{\sigma(\mathfrak{x})} \mathfrak{g}^{\frac{1}{p}}(t)\, \varrho(t) \mathfrak{u}(\mathfrak{x})\, \Delta t \right)^{q} \Delta \mathfrak{x} \\
&= \int_{\mathsf{a}}^{b} \left(\int_{\mathsf{a}}^{\sigma(\mathfrak{x})} \mathfrak{g}^{\frac{1}{p}}(t)\, \Lambda^{\frac{1-s}{p}}(t)\, \Lambda^{\frac{s-1}{p}}(t)\, \varrho(t) \Delta t \right)^{q} \mathfrak{u}(\mathfrak{x})\, \Delta \mathfrak{x} \\
&\leq \int_{\mathsf{a}}^{b} \left(\int_{\mathsf{a}}^{\sigma(\mathfrak{x})} \mathfrak{g}(t)\, \Lambda^{1-s}(t)\, \Delta t \right)^{q/p} \mathfrak{u}(\mathfrak{x}) \times \left(\int_{\mathsf{a}}^{\sigma(\mathfrak{x})} \Lambda^{\frac{p'(s-1)}{p}}(t)\, \varrho^{p'}(t)\, \Delta t \right)^{q/p'} \Delta \mathfrak{x},
\end{aligned} \tag{20}$$

from the definition of $\Lambda(t)$ that mentioned in (19), we display that

$$\Lambda^{\Delta}(t) \leq \varrho^{p'}(\sigma(t)) \leq \varrho^{p'}(t). \tag{21}$$

Substituting from (21) into (20) and by recalling chain rule (9), we exhibit that

$$\begin{aligned}
&\int_{\mathsf{a}}^{b} \left(\int_{\mathsf{a}}^{\sigma(\mathfrak{x})} \mathfrak{g}^{\frac{1}{p}}(t)\, \varrho(t) \Delta t \right)^{q} \mathfrak{u}(\mathfrak{x})\, \Delta \mathfrak{x} \\
&\leq \int_{\mathsf{a}}^{b} \left(\int_{\mathsf{a}}^{\sigma(\mathfrak{x})} \mathfrak{g}(t)\, \Lambda^{1-s}(t)\, \Delta t \right)^{q/p} \mathfrak{u}(\mathfrak{x}) \times \left(\int_{\mathsf{a}}^{\sigma(\mathfrak{x})} \Lambda^{\frac{p'(s-1)}{p}}(t)\, \Lambda^{\Delta}(t) \Delta t \right)^{q/p'} \Delta \mathfrak{x},
\end{aligned}$$

now by applying the time scale concept of differentiation and the fact that $\Lambda(a) = 0$, so the previous inequality reduces to

$$\int_{\mathfrak{a}}^{b}\left(\int_{\mathfrak{a}}^{\sigma(\mathfrak{x})}\mathfrak{g}^{\frac{1}{p}}(t)\,\varrho(t)\Delta t\right)^{q}\mathfrak{u}(\mathfrak{x})\,\Delta\mathfrak{x}$$

$$\leq\int_{\mathfrak{a}}^{b}\left(\int_{\mathfrak{a}}^{\sigma(\mathfrak{x})}\mathfrak{g}(t)\,\Lambda^{1-s}(t)\,\Delta t\right)^{q/p}\times\left(\frac{\Lambda^{\frac{(s-1)p'}{p}+1}(\sigma(\mathfrak{x}))}{\frac{(s-1)p'}{p}+1}\right)^{q/p'}\mathfrak{u}(\mathfrak{x})\,\Delta\mathfrak{x}$$

$$=\int_{\mathfrak{a}}^{b}\left(\int_{\mathfrak{a}}^{\sigma(\mathfrak{x})}\mathfrak{g}(t)\,\Lambda^{1-s}(t)\,\Delta t\right)^{q/p}\times\left(\frac{p-1}{s+p-2}\Lambda^{\frac{p+s-2}{-1+p}}(\sigma(\mathfrak{x}))\right)^{q/p'}\Delta\mathfrak{x}$$

$$=\left(\frac{-1+p}{p+s-2}\right)^{q/p'}\int_{\mathfrak{a}}^{b}\mathfrak{u}(\mathfrak{x})\left(\int_{\mathfrak{a}}^{\sigma(\mathfrak{x})}\mathfrak{g}(t)\,\Lambda^{1-s}(t)\,\Delta t\right)^{q/p}\Lambda^{\frac{p+s-2}{-1+p}\cdot\frac{q}{p'}}(\sigma(\mathfrak{x}))\,\mathfrak{u}(\mathfrak{x})\,\Delta\mathfrak{x},$$

by simple computations and by interacting the reality that $\mathfrak{x}\leq\sigma(\mathfrak{x})$, we indicate the following

$$\int_{\mathfrak{a}}^{b}\left(\int_{\mathfrak{a}}^{\sigma(\mathfrak{x})}\mathfrak{g}^{\frac{1}{p}}(t)\,\varrho(t)\Delta t\right)^{q}\mathfrak{u}(\mathfrak{x})\,\Delta\mathfrak{x}\leq\left(\frac{-1+p}{p+s-2}\right)^{q/p'}$$

$$\times\left[\left(\int_{\mathfrak{a}}^{b}\left[\left(\int_{\mathfrak{a}}^{\sigma(\mathfrak{x})}\mathfrak{g}(t)\,\Lambda^{1-s}(t)\,\Delta t\right)^{\frac{q}{p}\cdot\frac{p}{q}}\Lambda^{\frac{p+s-2}{-1+p}\cdot\frac{q}{p'}\cdot\frac{p}{q}}(\sigma(\mathfrak{x}))\,\mathfrak{u}^{p/q}(\mathfrak{x})\right]^{q/p}\Delta\mathfrak{x}\right)^{p/q}\right]^{q/p}$$

$$\leq\left(\frac{-1+p}{p+s-2}\right)^{q/p'}\times\left[\left(\int_{\mathfrak{a}}^{b}\left[\left(\int_{\mathfrak{a}}^{\sigma(\mathfrak{x})}\mathfrak{g}(t)\,\Lambda^{1-s}(t)\,\Delta t\right)\Lambda^{\frac{s+p-2}{p-1}\cdot\frac{p}{p'}}(\mathfrak{x})\,\mathfrak{u}^{p/q}(\mathfrak{x})\right]^{q/p}\Delta\mathfrak{x}\right)^{p/q}\right]^{q/p}. \quad (22)$$

For $q/p\geq 1$, and by employing Minkowski integral time scale inequality (11) on the inequality (22), we determine I, where

$$I=\int_{\mathfrak{a}}^{b}\left(\int_{\mathfrak{a}}^{\sigma(\mathfrak{x})}\mathfrak{g}^{\frac{1}{p}}(t)\,\varrho(t)\Delta t\right)^{q}\mathfrak{u}(\mathfrak{x})\,\Delta\mathfrak{x}$$

$$\leq\left(\frac{-1+p}{p+s-2}\right)^{q/p'}\left[\left(\int_{\mathfrak{a}}^{b}\left(\int_{t}^{b}\Lambda^{\frac{s+p-2}{p-1}\cdot\frac{q}{p'}}(\mathfrak{x})\,\mathfrak{u}(\mathfrak{x})\,\Delta\mathfrak{x}\right)\mathfrak{g}^{q/p}(t)\,\Lambda^{\frac{q(1-s)}{p}}(t)\,\Delta t\right)^{p/q}\right]^{q/p}$$

$$=\left(\frac{-1+p}{p+s-2}\right)^{q/p'}\left[\int_{\mathfrak{a}}^{b}\left(\int_{t}^{b}\Lambda^{\frac{s+p-2}{p}\cdot q}(\mathfrak{x})\,\mathfrak{u}(\mathfrak{x})\,\Delta\mathfrak{x}\right)^{p/q}\mathfrak{g}(t)\,\Lambda^{1-s}(t)\,\Delta t\right]^{q/p}. \quad (23)$$

Setting

$$\mathcal{B}(t)=\left(\int_{t}^{b}\Lambda^{\frac{s+p-2}{p}\cdot q}(\mathfrak{x})\,\mathfrak{u}(\mathfrak{x})\,\Delta\mathfrak{x}\right)^{p/q}\Lambda^{1-s}(t), \quad (24)$$

in inequality (23), we indicate the following inequality

$$I\leq\left(\frac{-1+p}{p+s-2}\right)^{q/p'}\left(\int_{\mathfrak{a}}^{b}\mathfrak{g}(t)\,\Delta t\right)^{q/p}\mathcal{B}^{q/p},$$

where we denote that $\mathcal{B}=\sup_{t\in(\mathfrak{a},\,b)_{\mathbb{T}}}\mathcal{B}(t)$.

Recalling inequality (18), we conclude that

$$\mathbf{C}^{-q} \leq \left(\frac{-1+p}{p+s-2}\right)^{q/p'} \mathcal{B}^{q/p}. \tag{25}$$

Now, by utilizing the power of q/p, we can raise both sides of inequality (24) to the power of q/p, then we derive that

$$\begin{aligned}\mathcal{B}^{q/p} &= \Lambda^{\frac{q(1-s)}{p}}(t)\left(\int_t^b \Lambda^{q\frac{s+p-2}{p}}(\mathfrak{x})\,\mathfrak{u}(\mathfrak{x})\,\Delta\mathfrak{x}\right)\\ &= \Lambda^{\frac{q(1-s)}{p}}(t)\left(\int_t^b \Lambda^{\frac{2q(s-1)}{p}}(\mathfrak{x})\,\Lambda^{\frac{p-s}{p}\cdot q}(\mathfrak{x})\,\mathfrak{u}(\mathfrak{x})\,\Delta\mathfrak{x}\right)\\ &= \Lambda^{\frac{q(1-s)}{p}}(t)\int_t^b \Lambda^{\frac{2q(s-1)}{p}}(\mathfrak{x})\left(\int_t^{\mathfrak{x}} \Lambda^{\frac{p-s}{p}\cdot q}(\tau)\,\mathfrak{u}(\tau)\,\Delta\tau\right)^{\Delta}\Delta\mathfrak{x}.\end{aligned} \tag{26}$$

Putting $L = \int_t^b \Lambda^{\frac{2(s-1)}{p}\cdot q}(\mathfrak{x})\left(\int_t^{\mathfrak{x}} \Lambda^{\frac{p-s}{p}\cdot q}(\tau)\,\mathfrak{u}(\tau)\,\Delta\tau\right)^{\Delta}\Delta\mathfrak{x}$. Then, by applying integration by parts formulation (8) with $\eta(\mathfrak{x}) = \Lambda^{\frac{2(s-1)}{p}\cdot q}(\mathfrak{x})$ and $\varphi^{\Delta}(\mathfrak{x}) = \left(\int_t^{\mathfrak{x}} \Lambda^{\frac{p-s}{p}\cdot q}(\tau)\,u(\tau)\,\Delta\tau\right)^{\Delta}$, we have that

$$\begin{aligned}L &= \eta(\mathfrak{x})\varphi(\mathfrak{x})\big|_t^b - \int_t^b \eta^{\Delta}(\mathfrak{x})\varphi^{\sigma}(\mathfrak{x})\Delta\mathfrak{x}\\ &= \Lambda^{\frac{2(s-1)}{p}\cdot q}(\mathfrak{x})\left(\int_t^{\mathfrak{x}} \Lambda^{\frac{p-s}{p}\cdot q}(\tau)\,\mathfrak{u}(\tau)\,\Delta\tau\Big|_t^b\right) - \int_t^b \left(\Lambda^{\frac{2(s-1)}{p}\cdot q}(\mathfrak{x})\right)^{\Delta}\left(\int_t^{\sigma(\mathfrak{x})} \Lambda^{\frac{p-s}{p}\cdot q}(\tau)\,\mathfrak{u}(\tau)\,\Delta\tau\right)\Delta\mathfrak{x}.\end{aligned} \tag{27}$$

From (27), setting

$$J_1 = \Lambda^{\frac{2(s-1)}{p}\cdot q}(\mathfrak{x})\left(\int_t^{\mathfrak{x}} \Lambda^{\frac{p-s}{p}\cdot q}(\tau)\,\mathfrak{u}(\tau)\,\Delta\tau\Big|_t^b\right) \tag{28}$$

and

$$J_2 = -\int_t^b \left(\int_t^{\sigma(\mathfrak{x})} \Lambda^{\frac{p-s}{p}\cdot q}(\tau)\,\mathfrak{u}(\tau)\,\Delta\tau\right)\left(\Lambda^{\frac{2(s-1)}{p}\cdot q}(\mathfrak{x})\right)^{\Delta}\Delta\mathfrak{x}, \tag{29}$$

by employing equations (12) and (15), the equation (28) can take the form

$$\begin{aligned}J_1 &= \lim_{\mathfrak{x}\to b-} \Lambda^{\frac{2q(s-1)}{p}}(\mathfrak{x})\int_t^{\mathfrak{x}} \Lambda^{\frac{p-s}{p}\cdot q}(\tau)\,\mathfrak{u}(\tau)\,\Delta\tau\\ &\leq \lim_{\mathfrak{x}\to b-} \Lambda^{\frac{2q(s-1)}{p}}(\mathfrak{x})\int_{\mathtt{a}}^{\mathfrak{x}} \Lambda^{\frac{p-s}{p}\cdot q}(\tau)\,\mathfrak{u}(\tau)\,\Delta\tau\\ &= \lim_{\mathfrak{x}\to b-} \Lambda^{q\left(\frac{s-1}{p}+\frac{-1+s}{p}\right)}(\mathfrak{x})\left[\left(\left(\int_{\mathtt{a}}^{\mathfrak{x}} \Lambda^{\frac{p-s}{p}\cdot q}(\tau)\,\mathfrak{u}(\tau)\,\Delta\tau\right)^{-1/q}\right]^{-q}\\ &= \lim_{\mathfrak{x}\to b-} \Lambda^{\frac{s-1}{p}\cdot q}(\mathfrak{x})\,\mathbb{E}_k^{-q}(\mathfrak{x},s) \leq \mathbb{E}_s^{-q}\lim_{\mathfrak{x}\to b-} \Lambda^{\frac{s-1}{p}\cdot q}(\mathfrak{x}) \leq \mathbb{E}_s^{-q}\lim_{\mathfrak{x}\to b-} \Lambda^{\frac{s-1}{p}\cdot q}(\sigma(\mathfrak{x})),\end{aligned} \tag{30}$$

that for $q < 0$. In a similar manner, for $t \geq \mathtt{a}$ and by recalling equation (12), where $\mathbb{E}_k(\mathfrak{x}, s) \leq \mathbb{E}_s$, we conclude that equation (29) becomes

$$\begin{aligned} J_2 &= -\int_t^b \Lambda^{\frac{q(s-1)}{p}} (\sigma(\mathfrak{x})) \left(\int_t^{\sigma(\mathfrak{x})} \Lambda^{\frac{q(p-s)}{p}} (\tau)\, \mathfrak{u}(\tau)\, \Delta\tau \right) \times \Lambda^{\frac{q(1-s)}{p}} (\sigma(\mathfrak{x})) \left(\Lambda^{\frac{2q(s-1)}{p}} (\mathfrak{x}) \right)^{\Delta} \Delta\mathfrak{x} \\ &\leq -\int_t^b \Lambda^{\frac{q(s-1)}{p}} (\sigma(\mathfrak{x})) \left(\int_{\mathsf{a}}^{\sigma(\mathfrak{x})} \Lambda^{\frac{q(p-s)}{p}} (\tau)\, \mathfrak{u}(\tau)\, \Delta\tau \right) \times \Lambda^{\frac{q(1-s)}{p}} (\sigma(\mathfrak{x})) \left(\Lambda^{\frac{2q(s-1)}{p}} (\mathfrak{x}) \right)^{\Delta} \Delta\mathfrak{x} \\ &\leq -\mathbb{E}_s^{-q}(\mathfrak{x}, s) \int_t^b \Lambda^{\frac{q(1-s)}{p}} (\sigma(\mathfrak{x})) \left(\Lambda^{\frac{2q(s-1)}{p}} (\mathfrak{x}) \right)^{\Delta} \Delta\mathfrak{x}. \end{aligned} \tag{31}$$

Now, by implementing the chain rule (9) where $(\mathfrak{f}(\mathsf{g}(t)))^{\Delta} = \mathfrak{f}'(\mathsf{g}(c))\mathsf{g}^{\Delta}(t)$, with $c \in [\mathfrak{x}, \sigma(\mathfrak{x})]$, we attain that $(\mathbf{U}^{-q}(\mathfrak{x}))^{\Delta} = -q\mathbf{U}^{-q-1}(c)\mathbf{U}^{\Delta}(\mathfrak{x})$. Now, for $c \leq \sigma(\mathfrak{x})$, the non-increasing mapping $\mathbf{U}$ reaches that $\mathbf{U}^{\Delta}(t) \leq 0$ and $\mathbf{U}(c) \geq \mathbf{U}^{\sigma}(\mathfrak{x})$, it follows that

$$\left(\mathbf{U}^{-q}(\mathfrak{x})\right)^{\Delta} \leq -q\mathbf{U}^{-q-1}(\sigma(\mathfrak{x}))\, \mathbf{U}^{\Delta}(\mathfrak{x}). \tag{32}$$

Then, combining relations (31) and (32), we get the inequality

$$\left(\Lambda^{\frac{2q(s-1)}{p}} (\mathfrak{x}) \right)^{\Delta} \leq \frac{2q(s-1)}{p} \Lambda^{\frac{2q(s-1)}{p}-1} (\sigma(\mathfrak{x}))\, \Lambda^{\Delta}(\mathfrak{x}), \tag{33}$$

substituting from (33) into (31) and for $q/p \geq 1$, we conclude that

$$\begin{aligned} J_2 &\leq -\mathbb{E}_s^{-q}(\mathfrak{x}, s) \int_t^b \Lambda^{\frac{1-s}{p}\cdot q} (\sigma(\mathfrak{x})) \left(\frac{2(s-1)q}{p} \Lambda^{-1+\frac{2(s-1)q}{p}} (\sigma(\mathfrak{x}))\, \Lambda^{\Delta}(\mathfrak{x}) \right) \Delta\mathfrak{x} \\ &= -\frac{2q(s-1)}{p} \mathbb{E}_s^{-q}(\mathfrak{x}, s) \int_t^b \Lambda^{\frac{(s-1)q}{p}-1} (\sigma(\mathfrak{x}))\, \Lambda^{\Delta}(\mathfrak{x})\, \Delta\mathfrak{x} \\ &= -\frac{2q(s-1)}{p} \mathbb{E}_s^{-q}(\mathfrak{x}, s) \left(\frac{\Lambda^{\frac{q(s-1)}{p}} (\sigma(\mathfrak{x}))}{\frac{q(s-1)}{p}} \Bigg|_t^b \right) \\ &= -2\mathbb{E}_s^{-q}(\mathfrak{x}, s) \left[\lim_{\mathfrak{x}\to b-} \Lambda^{\frac{s-1}{p}\cdot q} (\sigma(\mathfrak{x})) - \left(\Lambda^{\frac{s-1}{p}\cdot q} (\sigma(t)) \right) \right]. \end{aligned} \tag{34}$$

Hence from (30) and (34) and by substituting into (26), we obtain that

$$\begin{aligned} B^{q/p} &\leq \mathbb{E}_s^{-q}(\mathfrak{x}, s)\, \Lambda^{\frac{1-s}{p}\cdot q}(t) \left[\lim_{\mathfrak{x}\to b-} \Lambda^{\frac{s-1}{p}\cdot q} (\sigma(\mathfrak{x})) - 2\lim_{\mathfrak{x}\to b-} \Lambda^{\frac{s-1}{p}\cdot q} (\sigma(\mathfrak{x})) + 2\Lambda^{\frac{s-1}{p}\cdot q} (\sigma(t)) \right] \\ &= \mathbb{E}_s^{-q}(\mathfrak{x}, s)\, \Lambda^{\frac{1-s}{p}\cdot q}(t) \left[-\lim_{\mathfrak{x}\to b-} \Lambda^{\frac{s-1}{p}\cdot q} (\sigma(\mathfrak{x})) + 2\Lambda^{\frac{s-1}{p}\cdot q} (\sigma(t)) \right] \\ &\leq \mathbb{E}_s^{-q}(\mathfrak{x}, s)\, \Lambda^{\frac{1-s}{p}\cdot q}(\sigma(t)) \left[-\lim_{\mathfrak{x}\to b-} \Lambda^{\frac{s-1}{p}\cdot q} (\sigma(\mathfrak{x})) + 2\Lambda^{\frac{s-1}{p}\cdot q} (\sigma(t)) \right] \leq 2\mathbb{E}_s^{-q}(\mathfrak{x}, s). \end{aligned}$$

Combining with (25), we get that $\mathbf{C}^{-q} \leq 2\left(\frac{-1+p}{p+s-2}\right)^{q/p'} \mathbb{E}_s^{-q}(\mathfrak{x}, s)$. Therefore

$$\mathbf{C} \leq 2^{-1/q} \left(\frac{-1+p}{p+s-2} \right)^{-1/p'} \mathbb{E}_s(\mathfrak{x}, s),$$

which is the desired sufficient condition. Now, we will show the necessity of condition (17). Just consider that the inequality (18) is valid for every rd-function that is non-negative defined on $[\mathsf{a}, b]_{\mathbb{T}}$, then we will prove that $\mathbb{E}_s < \infty$. Firstly, let $s \in (p, 1)$ and for fixed value $t \in (\mathtt{a}, b)$, we formulate the function

$$\mathfrak{g}(\mathfrak{x}) = \begin{cases} \Lambda^{-s}(\mathfrak{x})\varrho^{p'}(\mathfrak{x}), & \mathfrak{x} \in (\mathtt{a}, t)_{\mathbb{T}}, \\ 0, & \mathfrak{x} \in [t, b)_{\mathbb{T}}, \end{cases} \tag{35}$$

therefore, we obtain that

$$\left(\int_{\mathtt{a}}^{b} \mathfrak{g}(\mathfrak{x})\, \Delta\mathfrak{x}\right)^{q/p} = \left(\int_{\mathtt{a}}^{\sigma(t)} \Lambda^{-s}(\mathfrak{x})\varrho^{p'}(\mathfrak{x})\Delta\mathfrak{x}\right)^{q/p},$$

recalling the definition of $\Lambda(\mathfrak{x})$ in (19) and the fact that $\Lambda(\mathtt{a}) = 0$, we get that

$$\begin{aligned} \left(\int_{\mathtt{a}}^{b} \mathfrak{g}(\mathfrak{x})\, \Delta\mathfrak{x}\right)^{q/p} &\leq \left(\int_{\mathtt{a}}^{\sigma(t)} \Lambda^{-s}(\mathfrak{x})\Lambda^{\Delta}(\mathfrak{x})\Delta\mathfrak{x}\right)^{q/p} \\ &= \left(\frac{\Lambda^{-s+1}(\mathfrak{x})}{-s+1}\right)^{q/p}\Bigg|_{\mathtt{a}}^{\sigma(t)} \leq \left(\frac{1}{-s+1}\right)^{q/p} \Lambda^{\frac{q(-s+1)}{p}}(t), \end{aligned} \tag{36}$$

on the other hand, by raising both sides of (35) to the power of $1/p$, and by multiplying both sides with the function $\varrho(\mathfrak{x})$, we have the equation

$$\mathfrak{g}^{1/p}(\mathfrak{x})\varrho(\mathfrak{x}) = \begin{cases} \Lambda^{-\frac{s}{p}}(\mathfrak{x})\varrho^{\frac{p'}{p}}(\mathfrak{x}), & \mathfrak{x} \in (\mathtt{a}, t)_{\mathbb{T}}, \\ 0, & \mathfrak{x} \in [t, b)_{\mathbb{T}}. \end{cases} \tag{37}$$

By utilizing (37) into (18), we determine the following

$$\begin{aligned} &\int_{\mathtt{a}}^{b} \left(\int_{\mathtt{a}}^{\sigma(\mathfrak{x})} \mathfrak{g}^{\frac{1}{p}}(t)\, \varrho(t)\Delta t\right)^{q} \mathfrak{u}(\mathfrak{x})\, \Delta\mathfrak{x} = \int_{\mathtt{a}}^{b} \left(\int_{\mathtt{a}}^{\sigma(\mathfrak{x})} \Lambda^{\frac{-s}{p}}(\tau)\varrho^{\frac{p'}{p}}(\tau)\Delta\tau\right)^{q} \mathfrak{u}(\mathfrak{x})\, \Delta\mathfrak{x} \\ &\geq \int_{\mathtt{a}}^{\sigma(t)} \left(\int_{\mathtt{a}}^{\sigma(\mathfrak{x})} \Lambda^{\frac{-s}{p}}(\tau)\Lambda^{\Delta}(\tau)\Delta\tau\right)^{q} \mathfrak{u}(\mathfrak{x})\, \Delta\mathfrak{x} \\ &= \int_{\mathtt{a}}^{\sigma(t)} \left(\left(\frac{\Lambda^{\frac{-s}{p}+1}(\tau)}{\frac{-s}{p}+1}\right)^{q}\Bigg|_{\mathtt{a}}^{\sigma(\mathfrak{x})}\right) \mathfrak{u}(\mathfrak{x})\, \Delta\mathfrak{x} \geq \left(\frac{p}{-s+p}\right)^{q} \int_{\mathtt{a}}^{\sigma(t)} \Lambda^{\frac{q(-s+p)}{p}} \mathfrak{u}(\mathfrak{x})\, \Delta\mathfrak{x}. \end{aligned} \tag{38}$$

Combining (36) and (38) implies that

$$\left(\frac{p}{-s+p}\right)^{q} \int_{\mathtt{a}}^{\sigma(t)} \Lambda^{q\left(\frac{p-s}{p}\right)}(\mathfrak{x})\, \mathfrak{u}(\mathfrak{x})\, \Delta\mathfrak{x} \leq \mathbf{C}^{-q} \left(\frac{1}{-s+1}\right)^{q/p} \Lambda^{q\left(\frac{-s+1}{p}\right)}(t),$$

therefore

$$\left(\frac{p}{-s+p}\right)^{q} (-s+1)^{q/p}\, \Lambda^{q\left(\frac{-1+s}{p}\right)}(t) \int_{\mathtt{a}}^{\sigma(t)} \Lambda^{q\left(\frac{-s+p}{p}\right)}(\mathfrak{x})\, \mathfrak{u}(\mathfrak{x})\, \Delta\mathfrak{x} \leq \mathbf{C}^{-q}. \tag{39}$$

Raising both sides of (39) to the power of $-1/q$, yields the resultant inequality

$$\left(\frac{-s+p}{p}\right)(-s+1)^{-1/p}\, \Lambda^{\left(\frac{-s+1}{p}\right)}(\mathfrak{x}) \left(\int_{\mathtt{a}}^{\sigma(t)} \mathfrak{u}(\mathfrak{x})\, \Lambda^{q\left(\frac{-s+p}{p}\right)}(\mathfrak{x})\, \Delta\mathfrak{x}\right)^{-1/q} \leq \mathbf{C},$$

that is identical to the inequality

$$\left(\frac{-s+p}{p}\right)(-s+1)^{-1/p}\, \mathbb{E}_k(t, s) \leq \mathbf{C}.$$

Secondly, for $s = p$ we take the mapping $\mathfrak{g}(\mathfrak{x})$ as the form

$$\mathfrak{g}(\mathfrak{x}) = \begin{cases} \varrho^{p'}(\mathfrak{x}), & \mathfrak{x} \in (\mathsf{a}, t)_{\mathbb{T}}, \\ 0, & \mathfrak{x} \in [t, b)_{\mathbb{T}}, \end{cases} \tag{40}$$

consequently and from (19), we get that

$$\left(\int_{\mathsf{a}}^{b} \mathfrak{g}(\mathfrak{x}) \Delta\mathfrak{x}\right)^{q/p} = \left(\int_{\mathsf{a}}^{\sigma(t)} \varrho^{p'}(\mathfrak{x}) \Delta\mathfrak{x}\right)^{q/p} = \Lambda^{\frac{q}{p}}(\sigma(t)). \tag{41}$$

By substituting from (40) and (41) into (18) we obtain that

$$\int_{\mathsf{a}}^{b} \left(\int_{\mathsf{a}}^{\sigma(t)} \varrho^{p'}(\mathfrak{x}) \mathfrak{u}(\mathfrak{x}) \Delta\mathfrak{x}\right)^{q} \Delta\mathfrak{x} \leq \mathbf{C}^{-q} \Lambda^{\frac{q}{p}}(\sigma(t)),$$

then

$$\Lambda^{q}(\sigma(t)) \int_{\mathsf{a}}^{t} \mathfrak{u}(\mathfrak{x}) \Delta\mathfrak{x} \leq \Lambda^{q}(\sigma(t)) \int_{\mathsf{a}}^{b} \mathfrak{u}(\mathfrak{x}) \Delta\mathfrak{x} \leq \mathbf{C}^{-q} \Lambda^{\frac{q}{p}}(\sigma(t)).$$

Finally, we get that

$$\Lambda^{-\frac{1}{p'}}(\sigma(t)) \left(\int_{\mathsf{a}}^{\sigma(t)} \mathfrak{u}(\mathfrak{x}) \Delta\mathfrak{x}\right)^{-1/q} \leq \mathbf{C} < \infty.$$

Thus, $C < \infty$ implies the boundedness of $\mathbb{E}_k(\mathfrak{x}, s)$ in (12). This concludes the theorem's proof. □

Consequently, by employing a similar technique we can easily conclude the duality of (Theorem 3) to obtain the following theorem.

Theorem 4. Assume that $\mathbb{T}$ is a time scale with $-\infty < q \leq p < 0$, $\mathfrak{f} \in \mathbf{C}_{rd}([\mathsf{a}, b]_{\mathbb{T}}, \mathbb{R})$ is a non-negative mapping and suppose that $\mathfrak{u}$, $\mathtt{v}$ are rd-continuous positive functions on $(\mathsf{a}, b)_{\mathbb{T}}$. Then the inequality

$$\left(\int_{\mathsf{a}}^{b} \mathfrak{f}^{p}(\mathfrak{x}) \mathtt{v}(\mathfrak{x}) \Delta\mathfrak{x}\right)^{1/p} \leq \mathbf{C} \left(\int_{\mathsf{a}}^{b} \left(\int_{\mathfrak{x}}^{b} \mathfrak{f}(t) \mathfrak{u}(\mathfrak{x}) \Delta t\right)^{q} \Delta\mathfrak{x}\right)^{1/q} \tag{42}$$

holds for each mapping $\mathfrak{f}$, if and only if

$$\mathbb{E}_r := \sup_{\mathfrak{x} \in (a,\, b)_{\mathbb{T}}} \Lambda^{\frac{1-s}{p}}(\mathfrak{x}) \left(\int_{\mathfrak{x}}^{b} \Lambda^{q(\frac{p-s}{p})}(t) \mathfrak{u}(t) \Delta t\right)^{-1/q} < \infty.$$

Moreover, the estimation $\mathbf{C}$ in (42) satisfies $\mathbb{E}_r \leq \mathbf{C} < 2^{-\frac{1}{q}} \left(\frac{p+s-2}{p-1}\right)^{1/p'} \mathbb{E}_r$, where $\Lambda(\mathfrak{x}) := \int_{\mathfrak{x}}^{b} \mathtt{v}^{1-p'}(t) dt$.

Now, we mention some suitable applications which closely related to Theorem 3 and 4. If we consider $\mathbb{T}$ equals to the set of real numbers $\mathbb{R}$ in Theorem 3, this leads to the following weighted Hardy inequality that is continuous due to Prokhorov [11] and Kufner et al. [12].

Remark 1. If $\mathbb{T} = \mathbb{R}$ for $-\infty < q \leq p < 0$ the inequality

$$\left(\int_{\mathsf{a}}^{b} \mathfrak{f}^{p}(\mathfrak{x}) \mathtt{v}(\mathfrak{x}) d\mathfrak{x}\right)^{1/p} \leq \mathbf{C} \left(\int_{\mathsf{a}}^{b} \mathfrak{u}(\mathfrak{x}) \left(\int_{\mathsf{a}}^{\mathfrak{x}} \mathfrak{f}(t) dt\right)^{q} d\mathfrak{x}\right)^{1/q}, \tag{43}$$

that holds, if and only if the next condition

$$\sup_{\mathfrak{x}\in(\mathtt{a},\, b)} \Lambda^{\frac{1-s}{p}}(\mathfrak{x}) \left(\int_{\mathtt{a}}^{\mathfrak{x}} \mathfrak{u}\,(t)\,\Lambda^{\frac{p-s}{p}q}(t)dt\right)^{-1/q} < \infty, \quad s \in [p, 1)$$

satisfies, where $\Lambda(\mathfrak{x}) := \int_{\mathtt{a}}^{\mathfrak{x}} \mathtt{v}^{1-p'}(t)dt,\ \ p' := \frac{p}{p-1}$.

By using an identical manner, we can put $\mathbb{T} = \mathbb{N}$ in Theorem 3 to obtain the discrete characterization of (43), that can be considered as the discrete result as mentioned in [13] and [14].

Remark 2. For $\mathbb{T} = \mathbb{N}$ and $-\infty < q \leq p < 0$, the inequality

$$\left(\sum_{\mathfrak{n}=0}^{\infty} \mathfrak{f}^p(\mathfrak{n})\mathtt{v}(\mathfrak{n})\right)^{1/p} \leq \mathbf{C} \left(\sum_{\mathfrak{n}=0}^{\infty} \mathfrak{u}(\mathfrak{n}) \left(\sum_{\mathbb{k}=0}^{\mathfrak{n}} \mathfrak{f}(\mathbb{k})\right)^q\right)^{1/q}$$

will be contented if and only if $\sup_{\mathfrak{n}} \left(\sum_{\mathfrak{n}=0}^{\infty} \left(\sum_{\mathbb{k}=0}^{\mathfrak{n}} \mathfrak{u}\,(\mathbb{k})\,\Lambda^{\frac{p-s}{p}q}(\mathbb{k})\right)^{-1/q} \Lambda^{\frac{1-s}{p}}(\mathfrak{n})\right) < \infty$, where $\mathfrak{u}\,(\mathfrak{n})$, $\mathtt{v}(\mathfrak{n})$, $\Lambda(\mathfrak{n}) = \sum_{\mathbb{k}=0}^{\mathfrak{n}-1} \mathtt{v}^{1-p'}(\mathbb{k})$ and $\mathfrak{f}(\mathfrak{n})$ are non-negative sequences.

If we consider $\mathbb{T} = l^{\mathbb{N}_0}$, we catch the next inequality.

Remark 3. If $-\infty < q \leq p < 0$ with $\mathbb{T} = l^{\mathbb{N}_0}$ then the inequality

$$\left(\sum_{n=0}^{\infty} \mathtt{v}(l^n)\eta(l^n)\mathfrak{f}^p(l^n)\right)^{1/p} \leq \mathbf{C} \left(\sum_{n=0}^{\infty} \mathfrak{u}(l^n)\eta(l^n) \left(\sum_{\mathbb{k}=0}^{n} \mathfrak{f}(l^{\mathbb{k}})\eta(l^{\mathbb{k}})\right)^q\right)^{1/q}$$

holds if and only if for the sequences $\mathfrak{f}(n)$, $\mathfrak{u}\,(n)$ and $\mathtt{v}(n)$ that are not negative and that for

$$\sup_n\,(l-1) \left(\sum_{n=0}^{\infty} \left(\sum_{\mathbb{k}=0}^{n} \mathfrak{u}(l^{\mathbb{k}})\eta(l^{\mathbb{k}})\Lambda^{\frac{p-s}{p}q}(l^{\mathbb{k}})\right)^{-1/q} \Lambda^{\frac{1-s}{p}}(l^n)\right) < \infty,$$

for $\Lambda(l^n) = (l-1)\sum_{\mathbb{k}=0}^{n-1} \mathtt{v}^{1-p'}(l^{\mathbb{k}})$, and $\eta(l^{\mathbb{k}}) = (l-1)\,l^{\mathbb{k}}$.

In the follow-up, we provide further deductions that demonstrate the useful implications of our main results. Through various substitutions, we can capture the following resultant reverse dynamic inequalities of Hardy-type that have negative parameters p and q.

Corollary 1. Suppose that $\mathbb{T}$ is a time scale, $-\infty < q \leq p < 0$, and the mapping $\mathfrak{f} \in C_{rd}\,([0,\infty)_{\mathbb{T}}, \mathbb{R})$. Consequently, there is a constant $\mathbf{C}$ in the form that Hardy inequality

$$\left(\int_0^{\infty} \mathfrak{f}^p\,(\mathfrak{x})\,\mathfrak{x}^{\beta}\Delta\mathfrak{x}\right)^{1/p} \leq \mathbf{C} \left(\int_0^{\infty} \left(\int_0^{\sigma(\mathfrak{x})} \mathfrak{f}\,(t)\,\Delta t\right)^q \mathfrak{x}^{\alpha}\Delta\mathfrak{x}\right)^{1/q} \tag{44}$$

fulfils for every $\mathfrak{f}$, if and only if $\sup_{\mathfrak{x}\in(0,\ b)_{\mathbb{T}}} \Lambda^{\frac{-s+1}{p}}(\sigma(\mathfrak{x})) \left(\int_0^{\sigma(\mathfrak{x})} \Lambda^{\frac{-s+p}{p}q}(t)t^{\alpha}\Delta t\right)^{-1/q}$, is convergent for $\Lambda(\mathfrak{x}) := \int_0^{\mathfrak{x}} t^{\beta(1-p')}\Delta t$ and α, β have positive values.

Proof. If we put $\mathfrak{u}\,(\mathfrak{x}) = \mathfrak{x}^{\alpha}$ and $\mathtt{v}(\mathfrak{x}) = \mathfrak{x}^{\beta}$ in Theorem 3, we gain the desired result. This concludes the proof. □

Remark 4. In inequality (44), if we set $\mathbb{T}=\mathbb{R}$, our inventory includes the continuous inequality

$$\left(\int_0^\infty \mathfrak{x}^{\beta}\mathfrak{f}^{p}\left(\mathfrak{x}\right)d\mathfrak{x}\right)^{1/p}\leq \mathbf{C}\left(\int_0^\infty\left(\int_0^{\mathfrak{x}}\mathfrak{f}\left(t\right)dt\right)^{q}\mathfrak{x}^{\alpha}d\mathfrak{x}\right)^{1/q},$$

due to [19], which valid for every function $\mathfrak{f}$, if and only if

$$\sup_{\mathfrak{x}\in(\mathtt{a},\ b)_{\mathbb{T}}}\Lambda^{\frac{-s+1}{p}}(\mathfrak{x})\left(\int_{\mathtt{a}}^{\mathfrak{x}}\Lambda^{\frac{-s+p}{p}q}(t)t^{\alpha}dt\right)^{-1/q}<\infty,$$

for positive numbers α, β and for $\Lambda(\mathfrak{x}):=\int_{\mathtt{a}}^{\mathfrak{x}}t^{\beta(1-p')}dt$.

Corollary 2. If we take $\mathbb{T}=\mathbb{N}$ in inequality (44), we obtain the discrete inequality

$$\left(\sum_{n=0}^{\infty}n^{\alpha}\mathfrak{f}^{p}(n)\right)^{1/p}\leq \mathbf{C}\left(\sum_{n=0}^{\infty}n^{\beta}\left(\sum_{\flat=0}^{n}\mathfrak{f}(\flat)\right)^{q}\right)^{1/q},$$

which shall be fulfilled if and only if $\sup_n\left(\sum_{n=0}^{\infty}\left(\sum_{\flat=0}^{n}\flat^{\alpha}\Lambda^{\frac{p-s}{p}q}(\flat)\right)^{-1/q}\Lambda^{\frac{1-s}{p}}(n)\right)<\infty$, which is essentially new.

Corollary 3. Consider that $-\infty<q,\ p<0$, on the time scale $\mathbb{T}$ with $q=p$, $h\in C_{rd}\left([0,\infty)_{\mathbb{T}},\mathbb{R}^{+}\right)$ and the mappings $\mathfrak{u}$ and $\mathtt{v}$ are continuous positive right dense functions on an open interval $(0,\infty)_{\mathbb{T}}$. Further, for $r\neq 1$ suppose that

$$\Xi\left(\mathfrak{x}\right)=\begin{cases}\int_0^{\sigma(\mathfrak{x})}h\left(s\right)\Delta s, & r<1,\\ \int_{\mathfrak{x}}^{\infty}h\left(s\right)\Delta s, & r>1.\end{cases}$$

If r is less than 1, then the following inequality

$$\int_0^\infty \Xi^{p}\left(\mathfrak{x}\right)\mathfrak{u}\left(\mathfrak{x}\right)\Delta\mathfrak{x}\leq \mathbf{C}_1\int_0^\infty h^{p}\left(\mathfrak{x}\right)\mathtt{v}\left(\mathfrak{x}\right)\Delta\mathfrak{x} \tag{45}$$

holds. While the inequality

$$\int_0^\infty \Xi^{p}\left(\mathfrak{x}\right)\mathfrak{u}\left(\mathfrak{x}\right)\Delta\mathfrak{x}\leq \mathbf{C}_2\int_0^\infty h^{p}\left(\mathfrak{x}\right)\mathtt{v}\left(\mathfrak{x}\right)\Delta\mathfrak{x} \tag{46}$$

holds for r that is greater than 1.

Proof. When $r<1$, setting $q=p$, $\mathtt{a}=0$, and $b=\infty$ in Theorem 3 and by raising both sides to the power of p, we have the required result in (45). Suppose $r>1$, if we set $\mathtt{a}=0$, $b=\infty$ and $q=p$ in Theorem 4, we obtain the result in (46). This completes the proof. □

Remark 5. In inequality (45), as we consider $\mathbb{T}=\mathbb{R}$, $\mathfrak{u}\left(\mathfrak{x}\right)=\mathfrak{x}^{-r}$ and $\mathtt{v}\left(\mathfrak{x}\right)=\mathfrak{x}^{-r+p}$, we can establish the continuous inequality

$$\int_0^\infty\left(\int_0^{\mathfrak{x}}\mathfrak{f}\left(t\right)dt\right)^{p}\mathfrak{x}^{-r}d\mathfrak{x}\leq \mathbf{C}\int_0^\infty \mathfrak{f}^{p}\left(\mathfrak{x}\right)\mathfrak{x}^{-r+p}d\mathfrak{x},$$

due to [38,39], which holds for each $\mathfrak{f}$, if and only if $\sup_{\mathfrak{x}\in(0,\ \infty)_{\mathbb{T}}}\Lambda^{\frac{-s+1}{p}}(\mathfrak{x})\left(\int_0^{\mathfrak{x}}\mathfrak{x}^{-r}\Lambda^{-s+p}(t)\Delta t\right)^{-1/p}<\infty$, for $q=p$, $r<1$ and $\Lambda(\mathfrak{x}):=\int_0^{\mathfrak{x}}t^{-r+p}dt$.

Remark 6. For $\mathbb{T}=\mathbb{R}$, $\mathtt{u}(\mathfrak{x})=\dfrac{1}{\mathfrak{x}^r}$ and $\mathtt{v}(\mathfrak{x})=\dfrac{1}{\mathfrak{x}^{-p+r}}$ into inequality (46), we derive that

$$\int_0^\infty \left(\int_\mathfrak{x}^\infty \mathfrak{f}(t)\,dt\right)^p \frac{1}{\mathfrak{x}^r}d\mathfrak{x} \le \mathbf{C}\int_0^\infty \frac{1}{\mathfrak{x}^{-p+r}}\mathfrak{f}^p(\mathfrak{x})\,d\mathfrak{x},$$

which verifies for each function $\mathfrak{f}$, regarding to $\sup_{\mathfrak{x}\in(0,\ \infty)_\mathbb{T}}(\Lambda^{\frac{-s+1}{p}}(\mathfrak{x}))\left(\int_\mathtt{a}^\mathfrak{x}\frac{1}{\mathfrak{x}^r}\Lambda^{p-s}(t)\Delta t\right)^{-1/p}<\infty$, that for $q=p$, $\Lambda(\mathfrak{x}):=\int_\mathfrak{x}^\infty t^{-r+p}dt$ and $r>1$, as stated in [38, 39].

Example 1. If we consider that $\mathbb{T}=\mathbb{R}$, $\mathtt{v}(\mathfrak{x})=\mathfrak{x}^2$, $\mathtt{u}(\mathfrak{x})=1$ and $p=-1$, $q=-2$, in Theorem 3, where the integration on the interval $\mathtt{a}=1, b=\infty$, then we can directly simplify Theorem 3 to the resultant inequality becomes

$$\left(\int_1^\infty \mathfrak{f}^{-1}(\mathfrak{x})\,\mathfrak{x}^2 d\mathfrak{x}\right)^{-1} \le \mathbf{C}\left(\int_1^\infty \mathtt{u}(\mathfrak{x})\left(\int_1^\mathfrak{x}\mathfrak{f}(t)\,dt\right)^{-2}d\mathfrak{x}\right)^{-1/2},$$

holds for every non-negative function $\mathfrak{f}$ if and only if the following criterion is met:

$$\sup_{\mathfrak{x}\in(1,\infty)}\Lambda^{s-1}(\mathfrak{x})\left(\int_1^\mathfrak{x}\Lambda^{-2(s+1)}(t)dt\right)<\infty,$$

that for $\Lambda(t)=\int_1^\mathfrak{x}\mathtt{v}^{1-p'}(t)dt$.

Example 2. In Remark 4, assume that $\alpha=-3$, $\beta=-1$ and $p=-2$, $q=-2$, where the integration from $\mathtt{a}=0$ to $b=\infty$, then we can obtain the inequality

$$\left(\int_0^\infty \mathfrak{x}^{-1}\mathfrak{f}^{-2}(\mathfrak{x})\,d\mathfrak{x}\right)^{-1/2} \le \mathbf{C}\left(\int_0^\infty\left(\int_0^\mathfrak{x}\mathfrak{f}(t)\,dt\right)^{-2}\mathfrak{x}^{-3}d\mathfrak{x}\right)^{-1/2},$$

holds for every function $\mathfrak{f}$ if and only if

$$\sup_{\mathfrak{x}\in(0,\infty)}\Lambda^{\frac{s-1}{2}}(\mathfrak{x})\left(\int_0^\mathfrak{x}\Lambda^{\frac{s+2}{2}}(t)t^\alpha dt\right)^{1/2}<\infty,$$

that for $\Lambda(t)=\int_0^\mathfrak{x}\mathtt{v}^{1-p'}(t)dt$.

Example 3. In Remark 3, if we suppose that $l=2$, where $\mathbb{T}=l^{\mathbb{N}_0}$, then the inequality take the form

$$\left(\sum_{n=0}^\infty \mathtt{v}(2^n)\eta(2^n)\mathfrak{f}^p(2^n)\right)^{1/p} \le \mathbf{C}\left(\sum_{n=0}^\infty \mathtt{u}(2^n)\eta(2^n)\left(\sum_{\mathbb{k}=0}^n\mathfrak{f}(2^\mathbb{k})\eta(2^\mathbb{k})\right)^q\right)^{1/q},$$

and for $p=q=-1$, we capture the inequality

$$\left(\sum_{n=0}^\infty \mathtt{v}(2^n)\eta(2^n)\mathfrak{f}^{-1}(2^n)\right)^{-1} \le \mathbf{C}\left(\sum_{n=0}^\infty \mathtt{u}(2^n)\eta(2^n)\left(\sum_{\mathbb{k}=0}^n\mathfrak{f}(2^\mathbb{k})\eta(2^\mathbb{k})\right)^{-1}\right)^{-1},$$

that holds if and only if

$$\sup_n (2-1)\left(\sum_{n=0}^\infty\left(\sum_{\mathbb{k}=0}^n \mathtt{u}(2^\mathbb{k})\eta(2^\mathbb{k})\Lambda^{\frac{-1-s}{-1}-1}(2^\mathbb{k})\right)\Lambda^{\frac{1-s}{-1}}(2^n)\right)<\infty,$$

for the non-negative sequences $\mathfrak{f}(n)$, $\mathtt{u}(n)$ and $\mathtt{v}(n)$ and for $\Lambda(l^n)=(2-1)\sum_{\mathbb{k}=0}^{n-1}\mathtt{v}^{1-p'}(2^\mathbb{k})$, and $\eta(2^\mathbb{k})=(2-1)\,2^\mathbb{k}$.

Example 4. If we assume that $\mathfrak{u}(\mathfrak{x}) = \chi_{(0,1)}$, $\mathtt{v}(\mathfrak{x}) = \chi_{(0,1)} + \chi_{(1,\infty)}$ and $p = q = -1$, then for $\mathtt{a} = 0, b = \infty$. We can directly simplify Theorem 3 to the resultant inequality that established in [11]

$$\int_0^{\infty} \left(\mathtt{v}^{-1}(\mathfrak{x})\, \mathfrak{f}(\mathfrak{x}) d\mathfrak{x}\right)^{-1} \leq \mathbf{C} \left(\int_0^{\infty} \left(\int_0^{\mathfrak{x}} \mathfrak{f}(t) dt\right)^{-1} \mathfrak{u}(\mathfrak{x})\, d\mathfrak{x}\right)^{-1},$$

where $\mathbf{C}$ satisfies the following estimate $\sup_{\mathfrak{x}\in(0,\infty)} \Lambda^{-1+s}(\mathfrak{x}) \left(\int_0^{\mathfrak{x}} \Lambda^{-1-s}(t)\mathfrak{u}(t)\, dt\right) < \infty$, that for $\Lambda(t) = \int_{\mathtt{a}}^{\mathfrak{x}} \mathtt{v}^{1-p'}(t)dt$ on $\mathbb{T} = \mathbb{R}$.

Conclusion

This research formulates novel reverse Hardy-type inequalities with negative indices on arbitrary time scales, integrating two independent weight functions. Using the reverse Hölder and Minkowski integral inequalities within the time-scale framework, we derive the conditions under which these inequalities hold the obtained results, generalize and consolidate several classical discrete, continuous, and quantum analogies, building upon the foundational works of Prokhorov, Kufner, and others. The main scientific contribution lies in the comprehensive analysis of negative exponents and the formulation of a unified technique across many temporal dimensions. These results expand the theoretical framework of Hardy-type inequalities and suggest possible applications in harmonic analysis and dynamic equations. Future study might investigate multidimensional extensions, applications to partial dynamic equations, or similar outcomes in broader measure spaces.

Conflict of Interest

The authors declare no conflicts of interest.

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