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On the Equivalence of Elementary Surfaces with Respect to the Motion Group of Pseudo-Euclidean Space

R.A. Gafforov[1,*], S. Kerbal[2], S.S. Juraboyev[1]

[1] *Fergana State University, Fergana, Uzbekistan;*
[2] *Sultan Qaboos University, Muscat, Oman*
(E-mail: gafforov.rahmatjon@mail.ru, skerbal@squ.edu.om, saidaxbor.juraboyev@mail.ru)

This paper investigates the conditions for the equivalence of regular surfaces with respect to the action of a certain subgroup of linear transformations. This subgroup is pseudo-orthogonal and preserves a metric structure defined by a matrix with specific sign properties. The study focuses on elementary surfaces, which are considered as mappings from the square of the parameter domain $(0,1)\times(0,1)$ into an n-dimensional real vector space. The regularity of a surface is determined by the non-vanishing determinant of a special matrix composed of its partial derivatives. The paper also introduces the concept of surface equivalence. The main theorem establishes necessary and sufficient conditions for the equivalence of regular surfaces under the action of the pseudo-orthogonal group. These conditions are expressed through equalities between products of matrices constructed from the partial derivatives of the surfaces and the pseudo-orthogonal matrix. The obtained results provide a theoretical foundation for understanding the relationships between regular surfaces under the action of the pseudo-orthogonal group and contribute to the further study of their geometric properties and transformations.

Keywords: pseudo-orthogonal group, G-equivalent, equivalence of the surfaces, semidirect product of the groups, symplectic group, special pseudo-orthogonal group, action of the surfaces, Euclidean space.

2020 Mathematics Subject Classification: 53A05, 53A07, 53A15, 53A55.

Introduction

In differential geometry, one of the central tasks is to establish convenient criteria for determining the equivalence of elementary surfaces. This involves identifying properties that help us assess whether two surfaces are geometrically identical, even if they are positioned differently in space.

A powerful approach to solving this problem relies on the theory of differential invariants. Differential invariants are properties of surfaces that remain unchanged under specific geometric transformations. By utilizing these invariants, we can formulate equivalence criteria, which simplifies the process of classifying surfaces and identifying their similarities, regardless of their position or orientation in space.

Let $\mathbb{R}^n$ denote an n-dimensional linear space over the field of real numbers $\mathbb{R}$, and let $GL(n,\mathbb{R})$ be the group of all invertible linear transformations of the space $\mathbb{R}^n$. Elements of $\mathbb{R}^n$ are represented as n-dimensional column vectors $\overrightarrow{x} = \{\overrightarrow{x}_j\}_{j=1}^n$, while the transformations $g \in GL(n,\mathbb{R})$ are represented as $n\times n$ matrices $(g_{ij})_{i,j=1}^n$, where $x_i, g_{ij} \in \mathbb{R}$ for $i,j = 1,\dots,n$. The action of $g \in GL(n,\mathbb{R})$ on the vector $\overrightarrow{x} = \{\overrightarrow{x}_j\}_{j=1}^n \in \mathbb{R}^n$ is given by matrix-vector multiplication, denoted as $g\overrightarrow{x}$.

An infinitely differentiable mapping $x : (0,1)\times(0,1) \to \mathbb{R}^n$ is called an elementary surface. If G is a subgroup of $GL(n,\mathbb{R})$, then two elementary surfaces $\overrightarrow{y}(s,t)$ and $\overrightarrow{x}(s,t)$ are said to be G-equivalent if $\overrightarrow{y}(s,t) = g\overrightarrow{x}(s,t)$ for some $g \in G$ and for all $(s,t) \in (0,1)\times(0,1)$.

In this paper, we reformulate the problem of G-equivalence of elementary surfaces for the pseudo-orthogonal group $O(n,p,\mathbb{R})$ using the language of differential algebra. This reformulation facilitates

*Corresponding author. *E-mail: gafforov.rahmatjon@mail.ru*
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the application of an algebraic approach to solve the problem. Such an approach has previously been employed to derive the necessary and sufficient conditions for surface equivalence in the context of actions by the general linear, special linear [1], orthogonal, pseudo-orthogonal [2], symplectic [3], and special pseudo-orthogonal groups [4–6].

1 Preliminaries

Let $O(n,p,\mathbb{R})$ be the pseudo-orthogonal subgroup of $GL(n,\mathbb{R})$, i.e.,

$$O(n,p,\mathbb{R}) = \{g \in GL(n,\mathbb{R}) : g^T e_p g = e_p\},$$

where g^T is the transpose of the matrix g, and e_p is an identity matrix in $GL(n,\mathbb{R})$ given by $e_p = (e^p_{ij})^n_{i,j=1}$, where

$$e^p_{ij} = \begin{cases} 1 & \text{for } i = 1, 2, \ldots, p, \\ -1 & \text{for } i = p+1,\, p+2, \ldots,\, n, \\ 0 & \text{for } i \neq j,\, i, j \in \{1, 2, \ldots, n\} \end{cases}$$

for some $p \in \{1, \ldots, n-1\}$.

For each elementary surface $\overrightarrow{x}(s,t) = (x_j(s,t))^n_{j=1}$, we define $M_s(\overrightarrow{x})$ as the $n \times n$ matrix $(m_{ij}(s,t))^n_{i,j=1}$, where the i-th column has coordinates $m_{ij}(s,t) = \frac{\partial^{i-1} x_j(s,t)}{\partial s^{i-1}}$, with $i, j = 1, \ldots, n$, and we set $\frac{\partial^0 x_j(s,t)}{\partial s^0} = x_j(s,t)$ for all $j = 1, \ldots, n,\ \ s, t \in (0,1)$.

Let $M_{ss}(\overrightarrow{x})$ be the matrix $\left\{\frac{\partial^i x_j(s,t)}{\partial s^i}\right\}^n_{i,j=1}$, $M_{sss}(\overrightarrow{x})$ be the matrix $\left\{\frac{\partial^{i+1} x_j(s,t)}{\partial s^{i+1}}\right\}^n_{i,j=1}$, and $M_{st}(\overrightarrow{x})$ be the matrix $\left\{\frac{\partial^i x_j(s,t)}{\partial s^{i-1}\partial t}\right\}^n_{i,j=1}$, while $M_{sst}(\overrightarrow{x})$ is the matrix $\left\{\frac{\partial^{i+1} x_j(s,t)}{\partial s^i \partial t}\right\}^n_{i,j=1}$.

Below, we consider only regular surfaces, i.e., elementary surfaces $\overrightarrow{x}(s,t)$ such that the determinant $\det M_s(\overrightarrow{x})(s,t) \neq 0$ for all $s, t \in (0,1)$.

Let $Aff(\mathbb{R}^n)$ be the group of all affine transformations of the n-dimensional linear space $\mathbb{R}^n$. Each affine transformation in $Aff(\mathbb{R}^n)$ is a composition of a non-degenerate linear transformation $g \in GL(n,\mathbb{R})$ and a translation by an element $\overrightarrow{u} = (u_i)^n_{i=1} \in \mathbb{R}^n$, i.e., any affine transformation $(\overrightarrow{u}, g) \in Aff(\mathbb{R}^n)$ acts in $\mathbb{R}^n$ according to the following rule:

$$(\overrightarrow{u}, g)(\overrightarrow{x}) = g\overrightarrow{x} + \overrightarrow{u},$$

where $\overrightarrow{x}, \overrightarrow{u} \in \mathbb{R}^n$ and $g \in GL(n,\mathbb{R})$.

The multiplication operation in $Aff(\mathbb{R}^n)$ is defined by

$$(\overrightarrow{u}, g)(\overrightarrow{v}, h) = (\overrightarrow{u} + g\overrightarrow{v}, gh),$$

where $\overrightarrow{u}, \overrightarrow{v} \in \mathbb{R}^n$ and $g, h \in GL(n,\mathbb{R})$. This implies that $Aff(\mathbb{R}^n)$ is a semidirect product of the groups $\mathbb{R}^n$ and $GL(n,\mathbb{R})$, written as

$$Aff(\mathbb{R}^n) = \mathbb{R}^n \lhd GL(n,\mathbb{R}).$$

If G is a subgroup of $GL(n,\mathbb{R})$, then the set

$$\mathbb{R}^n \lhd G = \{(\overrightarrow{u}, g) \in \mathbb{R}^n \lhd GL(n,\mathbb{R}) : g \in G\}$$

is a subgroup of $\mathbb{R}^n \lhd GL(n,\mathbb{R})$, and is also referred to as the semidirect product of the groups $\mathbb{R}^n$ and G.

It is well-known (see, for example, [7, Chapter XVII, §2]) that the group $\mathbb{R}^n \lhd O(n,\mathbb{R})$ coincides with the group of all motions in Euclidean space $(\mathbb{R}^n, (.,.))$, i.e., the group of all bijections U from $\mathbb{R}^n$ to $\mathbb{R}^n$ such that $(Ux, Uy) = (x, y)$ for all $x, y \in \mathbb{R}^n$. Similarly, the group $\mathbb{R}^n \lhd O(n,p,\mathbb{R})$ is the group of all motions in pseudo-Euclidean space $(\mathbb{R}^n, [.,.]_p)$, i.e., the group of all bijections V from $\mathbb{R}^n$ to $\mathbb{R}^n$ such that

$$[Vx, Vy]_p = [x, y]_p \ \text{ for all } \ x, y \in \mathbb{R}^n,$$

(see, for example, [8, Chapter III, §1]).

Let G be a subgroup of $\mathbb{R}^n \lhd GL(n,\mathbb{R})$. Two regular surfaces $\overrightarrow{x}(s,t)$ and $\overrightarrow{y}(s,t)$, defined in $\mathbb{R}^n$, are said to be G-equivalent if there exists $(\overrightarrow{u}, g) \in \mathbb{R}^n$ such that:

$$\overrightarrow{y}(s,t) = g\overrightarrow{x}(s,t) + \overrightarrow{u} \ \text{ for all } \ (s,t) \in (0,1).$$

The following statement reduces the problem of $\mathbb{R}^n \lhd G$-equivalence of regular surfaces $\overrightarrow{x}(s,t)$ and $\overrightarrow{y}(s,t)$ to the problem of G-equivalence of regular surfaces $\overrightarrow{x}'_s(s,t)$ and $\overrightarrow{y}'_s(s,t)$.

Statement 1. Two regular surfaces $\overrightarrow{x}(s,t)$ and $\overrightarrow{y}(s,t)$, defined in $\mathbb{R}^n$, are $\mathbb{R}^n \lhd G$-equivalent if and only if the regular surfaces $\overrightarrow{x}'_s(s,t)$ and $\overrightarrow{y}'_s(s,t)$ are G-equivalent.

Proof. If two regular surfaces $\overrightarrow{x}(s,t)$ and $\overrightarrow{y}(s,t)$ are $\mathbb{R}^n \lhd G$-equivalent, then there exist $\overrightarrow{u} = \{u_i\}_{i=1}^n \in \mathbb{R}^n$ and $g = \{g_{ij}\}_{i,j=1}^n \in G$ such that

$$\overrightarrow{y}(s,t) = g\overrightarrow{x}(s,t) + \overrightarrow{u} \ \text{ for all } \ (s,t) \in (0,1).$$

For each $i = 1, \ldots, n$, we have

$$\overrightarrow{y}_i(s,t) = \sum_{j=1}^n g_{ij}\overrightarrow{x}_j(s,t) + u_i.$$

Taking the derivatives with respect to s, we get

$$\overrightarrow{y}'_i(s,t) = \sum_{j=1}^n g_{ij}\overrightarrow{x}'_j(s,t). \tag{1}$$

Thus, for all $i = 1, \ldots, n$, we have

$$\overrightarrow{y}'_s(s,t) = g\overrightarrow{x}'_s(s,t), \quad (s,t) \in (0,1),$$

which implies that the regular surfaces $\overrightarrow{x}'_s(s,t)$ and $\overrightarrow{y}'_s(s,t)$ are G-equivalent.

Conversely, suppose that

$$\overrightarrow{y}'_s(s,t) = g\overrightarrow{x}'_s(s,t) \ \text{ for all } \ (s,t) \in (0,1),$$

for some $g = \{g_{ij}\}_{i,j=1}^n \in G$. From equation (1), we deduce that

$$\overrightarrow{y}_i(s,t) = \sum_{j=1}^n g_{ij}\overrightarrow{x}_j(s,t) + u_i,$$

where $u_i(s,t)$ is a constant (independent of s and t). To find $u_i(s,t)$, we define $\overrightarrow{u}_i(s,t) = \overrightarrow{y}_i(s,t) - \sum_{j=1}^n g_{ij}\overrightarrow{x}_j(s,t)$, and compute the derivative with respect to s:

$$\overrightarrow{u}'_i(s,t) = 0 \ \text{ for all } \ (s,t) \in (0,1).$$

Since $\overrightarrow{u}'_i(s,t) = 0$, we conclude that $\overrightarrow{u}_i(s,t) = \overrightarrow{u}_i(0)$, a constant for each i. Therefore, we define $\overrightarrow{u} = \{u_i(0)\}_{i=1}^n \in \mathbb{R}^n$, and obtain the relation

$$\overrightarrow{y}(s,t) = g\overrightarrow{x}(s,t) + \overrightarrow{u} \ \text{ for all } \ (s,t) \in (0,1).$$

This shows that the two regular surfaces $\overrightarrow{x}(s,t)$ and $\overrightarrow{y}(s,t)$ are $\mathbb{R}^n \lhd G$-equivalent.

Thus, we have proven that two regular surfaces $\overrightarrow{x}(s,t)$ and $\overrightarrow{y}(s,t)$ are $\mathbb{R}^n \lhd G$-equivalent if and only if $\overrightarrow{x}'_s(s,t)$ and $\overrightarrow{y}'_s(s,t)$ are G-equivalent. $\square$

2 Equivalence of Elementary Surfaces

The following theorem provides necessary and sufficient conditions for the $\mathbb{R}^n \lhd O(n,p,\mathbb{R})$-equivalence of regular surfaces $\overrightarrow{x}(s,t)$ and $\overrightarrow{y}(s,t)$.

Theorem 1. Two regular surfaces $\overrightarrow{x}(s,t)$ and $\overrightarrow{y}(s,t)$ are $\mathbb{R}^n \lhd O(n,p,\mathbb{R})$-equivalent if and only if for any $(s,t) \in (0,1)$, the following equalities hold:
(a) $M_{ss}^{-1}(\overrightarrow{x}(s,t))M_{sss}(\overrightarrow{x}(s,t)) = M_{ss}^{-1}(\overrightarrow{y}(s,t))M_{sss}(\overrightarrow{y}(s,t))$;
(b) $M_{st}^{-1}(\overrightarrow{x}(s,t))M_{sst}(\overrightarrow{x}(s,t)) = M_{st}^{-1}(\overrightarrow{y}(s,t))M_{sst}(\overrightarrow{y}(s,t))$;
(c) $M_{ss}^{T}(\overrightarrow{x}(s,t))e_pM_{ss}(\overrightarrow{x}(s,t)) = M_{ss}^{T}(\overrightarrow{x}(s,t))e_pM_{ss}(\overrightarrow{y}(s,t))$.

Proof. Let the surfaces $\overrightarrow{x}(s,t)$ and $\overrightarrow{y}(s,t)$ be $\mathbb{R}^n \lhd O(n,p,\mathbb{R})$-equivalent, i.e., there exists an element $g \in \mathbb{R}^n \lhd O(n,p,\mathbb{R})$ such that the following holds:

$$\overrightarrow{y}(s,t) = g\overrightarrow{x}(s,t) + \overrightarrow{u}.$$

Consequently, by the definition of the matrices $M_{ss}(\overrightarrow{x})$, we have

$$M_{ss}(\overrightarrow{y}) = gM_{ss}(\overrightarrow{x}).$$

We will now show that from this equality, the validity of equations (a), (b), and (c) follows. Indeed:

$$M_{ss}^{-1}(\overrightarrow{y}(s,t))M_{sss}(\overrightarrow{y}(s,t)) = (gM_{ss}(\overrightarrow{x}(s,t)))^{-1}(gM_{sss}(\overrightarrow{x}(s,t))).$$

This simplifies to

$$(M_{ss}(\overrightarrow{x}(s,t)))^{-1}(g^{-1}g)(M_{sss}(\overrightarrow{x}(s,t))) = M_{ss}^{-1}(\overrightarrow{x}(s,t))M_{sss}(\overrightarrow{x}(s,t)),$$

which shows that (a) holds.

Similarly,

$$M_{st}^{-1}(\overrightarrow{y}(s,t))M_{sst}(\overrightarrow{y}(s,t)) = (gM_{st}(\overrightarrow{x}(s,t)))^{-1}(gM_{sst}(\overrightarrow{x}(s,t))).$$

This simplifies to

$$(M_{st}(\overrightarrow{x}(s,t)))^{-1}(g^{-1}g)(M_{sst}(\overrightarrow{x}(s,t))) = M_{st}^{-1}(\overrightarrow{x}(s,t))M_{sst}(\overrightarrow{x}(s,t)),$$

which shows that (b) holds.

Finally, for equation (c), we have

$$M_{ss}^{T}(\overrightarrow{y}(s,t))e_pM_{ss}(\overrightarrow{y}(s,t)) = (gM_{ss}(\overrightarrow{x}(s,t)))^{T}e_p(gM_{ss}(\overrightarrow{x}(s,t))).$$

This simplifies to

$$(M_{ss}(\overrightarrow{x}(s,t)))^{T}(g^{T}e_pg)(M_{ss}(\overrightarrow{x}(s,t))) = M_{ss}^{T}(\overrightarrow{x}(s,t))e_pM_{ss}(\overrightarrow{x}(s,t)),$$

which shows that (c) holds.

Conversely, assume that the relations (a), (b), and (c) hold for the surfaces $\overrightarrow{x}(s,t)$ and $\overrightarrow{y}(s,t)$. Since $A(s,t) = A_s$ is invertible, assume that equalities (a) and (b) are valid. Differentiating the equality $A_s^{-1}A_s = A_sA_s^{-1} = E$, we get

$$(A^{-1})_s = -(A_s)^{-1}A_{ss}(A_s)^{-1}.$$

$$M_{ss}(\overrightarrow{x}(s,t)) \cdot M_{ss}^{-1}(\overrightarrow{x}(s,t)) = E,$$

$$(M_{ss}(\overrightarrow{x}(s,t)) \cdot M_{ss}^{-1}(\overrightarrow{x}(s,t)))_s = 0,$$

$$M_{sss}(\overrightarrow{x}(s,t)) \cdot M_{ss}^{-1}(\overrightarrow{x}(s,t)) + M_{ss}(\overrightarrow{x}(s,t)) \cdot M_{sss}^{-1}(\overrightarrow{x}(s,t)) = 0,$$

$$M_{ss}(\overrightarrow{x}(s,t)) \cdot M_{sss}^{-1}(\overrightarrow{x}(s,t)) = -M_{sss}(\overrightarrow{x}(s,t)) \cdot M_{ss}^{-1}(\overrightarrow{x}(s,t)),$$

$$M_{sss}^{-1}(\overrightarrow{x}(s,t)) = -M_{ss}^{-1}(\overrightarrow{x}(s,t)) \cdot M_{sss}(\overrightarrow{x}(s,t)) \cdot M_{ss}^{-1}(\overrightarrow{x}(s,t)),$$

$$M_{ss}(\overrightarrow{y}(s,t)) = gM_{ss}(\overrightarrow{x}(s,t)),$$

$$(M_{ss}(\overrightarrow{y}(s,t))M_{ss}^{-1}(\overrightarrow{x}(s,t)))_s = 0.$$

Therefore, the equalities (a) and (b) can be rewritten as

$$\text{(a')}\ (M_{ss}(\overrightarrow{y}(s,t)) \cdot M_{ss}^{-1}(\overrightarrow{x}(s,t)))_s = 0,$$

$$\text{(b')}\ (M_{ss}(\overrightarrow{y}(s,t)) \cdot M_{ss}^{-1}(\overrightarrow{x}(s,t)))_t = 0.$$

These qualities imply that

$$M_{ss}(\overrightarrow{y}(s,t)) \cdot M_{ss}^{-1}(\overrightarrow{x}(s,t)) = g = (g_{ij})_{i,j=1}^n \in GL(n, \mathbb{R}),$$

where $s, t \in (0,1)$. Consequently, we obtain

$$M_{ss}(\overrightarrow{y}(s,t)) = gM_{ss}(\overrightarrow{x}(s,t)),$$

and in particular,

$$\overrightarrow{y}'_s(s,t) = g\overrightarrow{x}'_s(s,t) \quad \text{for all } (s,t) \in (0,1).$$

Thus, for $g = \{g_{ij}\}_{i,j=1}^n$, $\overrightarrow{y}(s,t) = \{y_i(t)\}_{i=1}^n$, and $\overrightarrow{x}(s,t) = \{x_i(t)\}_{i=1}^n$, we have

$$\overrightarrow{y}'_i(s,t) = \sum_{j=1}^n g_{ij}\overrightarrow{x}'_j(s,t).$$

Let $\overrightarrow{u}_i = \overrightarrow{y}_i(s,t) - \sum_{j=1}^n g_{ij}\overrightarrow{x}_j(s,t)$, which yields

$$\overrightarrow{u} = \{u_i\}_{i=1}^n \in \mathbb{R}^n,$$

and for the element $(\overrightarrow{u}, g) \in \mathbb{R}^n \lhd O(n,p,\mathbb{R})$, we have the equality

$$\overrightarrow{y}(s,t) = g\overrightarrow{x}(s,t) + \overrightarrow{u} \quad \text{for all} \ \ (s,t) \in (0,1).$$

Furthermore, due to equation (c), we have

$$g^T e_p g = (M_{ss}(\overrightarrow{y}(s,t)) \cdot M_{ss}^{-1}(\overrightarrow{x}(s,t)))^T e_p (M_{ss}(\overrightarrow{y}(s,t)) \cdot M_{ss}^{-1}(\overrightarrow{x}(s,t))) = e_p,$$

which implies that

$$g^T e_p g = e_p.$$

Thus, $g \in O(n,p,\mathbb{R})$. Theorem 1 is proved. □

Now consider the problem of the equivalence of k-dimensional surfaces. An infinitely differentiable mapping $x : [0,1]^k \to \mathbb{R}^n$, where $1 \leq k < n$, is called a parameterized k-dimensional surface in the finite-dimensional vector space $\mathbb{R}^n$.

Definition 1. [9] Two k-dimensional surfaces $\overrightarrow{x}(t_1,\dots,t_k)$ and $\overrightarrow{y}(t_1,\dots,t_k)$ are said to be G-equivalent if there exists an element $g\in G$ such that

$$\overrightarrow{y}(t_1,\dots,t_k)=g\overrightarrow{x}(t_1,\dots,t_k)$$

for all $(t_1,\dots,t_k)\in(0,1)^k$.

Definition 2. [3] A function f of $\overrightarrow{x}(t_1,\dots,t_k)$ and its finite number of partial derivatives is called G-invariant if its values coincide for G-equivalent surfaces.

Let $\overrightarrow{x}(t_1,\dots,t_k)$ be a k-dimensional surface, and let $s\in\{1,\dots,k\}$ be a fixed index. For each surface $\overrightarrow{x}(t_1,\dots,t_k)$, the $(n\times n)$-matrix $M_{t_s}(\overrightarrow{x})$ has the form

$$M_{t_s}(\overrightarrow{x})=(\overrightarrow{x}^{\,0}_{t_s},\dots,\overrightarrow{x}^{\,n-1}_{t_s}),$$

where the i-th column consists of the coordinates

$$\frac{\partial^{i-1}x_j(t_1,\dots,t_k)}{\partial t_s^{i-1}},\quad j=1,\dots,n,\quad i=1,\dots,n.$$

Let $M_{t_st_s}(\overrightarrow{x})$ be the matrix

$$M_{t_st_s}(\overrightarrow{x})=\left\{\frac{\partial^{i}x_j(t_1,\dots,t_k)}{\partial t_s^{i}}\right\}_{i,j=1}^{n},$$

and let $M_{t_st_st_l}(\overrightarrow{x})$ denote the matrix

$$M_{t_st_st_l}(\overrightarrow{x})=\left(\frac{\partial^2x(t_1,\dots,t_k)}{\partial t_s\partial t_l},\dots,\frac{\partial^{n+1}x(t_1,\dots,t_k)}{\partial t_s^n\partial t_l}\right).$$

Throughout the rest of the text, only regular surfaces are considered, i.e., surfaces $\overrightarrow{x}(t_1,\dots,t_k)$ for which

$$\det M_{t_s}(\overrightarrow{x})(t_1,\dots,t_k)\neq 0$$

for all $(t_1,\dots,t_k)\in(0,1)^k$ and $s=1,\dots,k$.

The following theorem provides the necessary and sufficient conditions for $\mathbb{R}^n\lhd O(n,p,\mathbb{R})$-equivalence of two surfaces.

Theorem 2. Two surfaces $x(t_1,\dots,t_k)$ and $y(t_1,\dots,t_k)$ in $\mathbb{R}^n$ are $\mathbb{R}^n\lhd O(n,p,\mathbb{R})$-equivalent if and only if the following equalities hold

1. $M^{-1}_{t_st_s}(\overrightarrow{x})M_{t_st_st_l}(\overrightarrow{x})=M^{-1}_{t_st_s}(\overrightarrow{y})M_{t_st_st_l}(\overrightarrow{y})$,
2. $(M_{t_st_s}(\overrightarrow{x}))^Te_pM_{t_st_s}(\overrightarrow{x})=(M_{t_st_s}(\overrightarrow{y}))^Te_pM_{t_st_s}(\overrightarrow{y})$,

for all $l=1,\dots,k$.

The proof of this theorem follows a similar approach to the previously established theorem.

Remark. For two-dimensional regular surfaces, Theorem 1 under the action of the groups $G(n,\mathbb{C})$, $SL(n,\mathbb{C})$, $O(n,\mathbb{C})$, $SO(n,\mathbb{C})$, and $Sp(2n,\mathbb{C})$ was obtained in [2,3,10], while for m-dimensional regular surfaces ($m<n$) under the action of the same groups, it was obtained in [5]. For the semidirect product of the same groups, the result was obtained in [4]. For two-dimensional regular surfaces, Theorem 1 and Theorem 2 under the action of the group $SO(n,p,K)$ were obtained in [6], while for k-dimensional regular surfaces ($k<n$) under the action of $SO(p,q,K)$, they were obtained in [9]. Additionally, the above theorems for paths and curves were studied in [11], while the differential invariants of surfaces were examined in [12].

Conclusion

In the article, the problem of determining the equivalence of elementary and k-dimensional surfaces is considered. As the main result, the necessary and sufficient conditions for the equivalence of surfaces with respect to the action of the pseudo-orthogonal group are identified. The equivalence of surfaces under the action of this pseudo-orthogonal group is proven through the relations between special matrices constructed on the basis of the partial derivatives of the surfaces. The results presented in the article can be applied in the future for the classification and study of surfaces in differential geometry.

Author Contributions

All authors contributed equally to this work.

Conflict of Interest

The authors declare no conflict of interest.

References

1 Bekbaev, U.D. (2005). On differential rational invariants of finite subgroups of affine group. *Bulletin of the Malaysian Mathematical Sciences Society, Series 2, 28*(1), 55–60.

2 Muminov, K.K. (2005). Ekvivalentnost putei i poverkhnostei dlia deistviia psevdoortogonalnoi gruppy [Equivalence of paths and surfaces for the action of a pseudo-orthogonal group]. *Uzbekskii matematicheskii zhurnal — Uzbek Mathematical Journal, 2,* 35–43 [in Russian].

3 Muminov, K.K. (1997). Ekvivalentnost poverkhnostei v kompleksnykh vektornykh prostranstvakh otnositelno $Sp(2, C)$ grupp [Equivalence of surfaces in complex vector spaces with respect to $Sp(2, C)$ groups]. *Uzbekskii matematicheskii zhurnal — Uzbek Mathematical Journal, 2*, 53–57 [in Russian].

4 Bekbaev, U.D., & Muminov, K.K. (2004). On differential rational invariants of the classical groups of motions of a vector space. *Methods of Functional Analysis and Topology, 10*(3), 26–30.

5 Muminov, K.K. (2010). Equivalence of multidimensional surfaces with to the acting of classical groups. *Uzbek Mathematical Journal 1,* 99–107.

6 Muminov, K.K., & Gafforov, R.A. (2024). Systems of matrix differential equations for surfaces. *Journal of Mathematical Sciences, 278*(4), 623–632. https://doi.org/10.1007/s10958-024-06944-1

7 Aleksandrov, P.S. (1988). *Kurs analiticheskoi geometrii i lineinoi algebry [Course of analytical geometry and linear algebra].* Moscow: Nauka [in Russian].

8 Rosenfeld, B.A. (1969). *Neevklidovy prostranstva [Non-Euclidean Spaces].* Moscow: Nauka [in Russian].

9 Gafforov, R.A. (2022). Equivalence of multidimensional surfaces under the action of the special pseudo-orthogonal group. *Bulletin of the Institute of Mathematics, 5*(4), 48–54.

10 Muminov, K.K., & Bekbaev, U.D. (1997). Ob ekvivalentnosti i invariantakh elementarnykh poverkhnostei otnositelno simplekticheskoi gruppy [On equivalence and invariants of elementary surfaces with respect to the symplectic group]. *Uzbekskii matematicheskii zhurnal — Uzbek Mathematical Journal, 4*, 26–30 [in Russian].

11 Muminov, K.K., & Chilin, V.I. (2015). *Ekvivalentnost krivykh v konechnomernykh vektornykh prostranstvakh [Equivalence of curves in finite-dimensional spaces].* Deutschland: LAMBERT Academic Publishing [in Russian].

12 Olver, P.J. (2009). Differential invariants of surfaces. *Differential Geometry and its Applications, 27*(2), 230–239. https://doi.org/10.1016/j.difgeo.2008.06.020

Author Information[*]

Rakhmatjon Abdukakhkhorovich Gafforov (*corresponding author*) — Doctor of Philosophy (PhD) in Physical and Mathematical Sciences, Associate Professor, Fergana State University, 19 Murabbiylar St., Fergana, 150100, Uzbekistan; e-mail: *gafforov.rahmatjon@mail.ru*

Sebti Kerbal — Doctor of Physical and Mathematical Sciences, Professor, Department of Mathematics, Sultan Qaboos University, P.O. Box 36, Al-Khoudh 123, Muscat, Oman, FracDiff Research Group DR/RG/03, Sultan Qaboos University, Muscat, Oman; e-mail: *skerbal@squ.edu.om*

Saidaxbor Solijonovich Juraboyev — Doctor of Physical and Mathematical Sciences, Teacher, Fergana State University, 19 Murabbiylar St., Fergana, 150100, Uzbekistan; e-mail: *saidaxbor.juraboyev@mail.ru*

[*]Authors' names are presented in the following order: first name, middle name (if any), last name.