

Ternary Menger hyperalgebras: some algebraic properties

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In this article, we first establish an algebraic hyperstructure called a *ternary Menger hyperalgebra of rank n* , where n is a natural number. The algebraic hyperstructure can be regarded as a novel generalization of ternary semihypergroups. In particular, by setting the natural number n equal to 1, the algebraic hyperstructures of ternary Menger hyperalgebras of rank 1 and ternary semihypergroups are the same. And then, we extend some fundamental results on the ternary semihypergroup theory to study on ternary Menger hyperalgebras of rank n including subhyperalgebras and homomorphisms. Moreover, we investigate some interesting algebraic connections among Menger algebras of rank n , Menger hyperalgebras of rank n , ternary Menger algebras of rank n and ternary Menger hyperalgebras of rank n . In this section, we present that the algebraic hyperstructure of ternary Menger hyperalgebras of rank n can also be considered as an extension of the concepts Menger hyperalgebras of rank n and ternary Menger algebras of rank n . Finally, we use algebraic hyperstructures of ternary Menger hyperalgebras of rank n to construct the so-called diagonal ternary semihypergroups of the ternary Menger hyperalgebras of rank n .

Keywords: Menger algebras, Menger hyperalgebras, ternary Menger algebras, ternary Menger hyperalgebras, semigroups, semihypergroups, ternary semigroups, ternary semihypergroups, diagonal ternary semihypergroups.

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Introduction

A unary function, which is defined on a nonempty X (i.e. a mapping from X into X), is known as a transformation of the set X . Later, the idea of *multiplace functions*, which are also called functions of many variables, has been studied. In 1946, Menger [1] presented an algebraic property of the composition of multiplace functions, which is called *superassociative law*. This study has been credited as the first research work concerning on the study of *Menger algebras of rank n* , where n is any positive integer. A Menger algebra (M, o) of rank n is an $(n+1)$ -ary algebraic structure such that its $(n+1)$ -ary operation o satisfies the superassociative law, i.e.,

$$o(o(x, y_1, \dots, y_n), z_1, \dots, z_n) = o(x, o(y_1, z_1, \dots, z_n), \dots, o(y_n, z_1, \dots, z_n)) \quad (1)$$

for all $x, y_1, \dots, y_n, z_1, \dots, z_n \in M$. It is easy to see that the superassociative law is an extension of the associative law:

$$o(o(x, y), z) = o(x, o(y, z)) \quad (2)$$

for all $x, y, z \in M$, i.e., the superassociative law (1) is reduced to the associative law (2) if $n = 1$. Furthermore, the Menger algebra (M, o) of rank n is also reduced to a semigroup. As a result, Menger algebras of rank n can be regarded as a natural generalization of semigroups.

Based on the concept of Menger algebras of rank n , there is a number of published papers studied on this structure in various directions and fields, both theoretical and applied mathematics (e.g. [2, 3]). The theory of Menger algebras of rank n and its applications have been developed by Dudek and

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Trokhimenko who introduced the idea of principal v -congruences on Menger algebras of rank n , which is an extension of principal right (left) congruences on semigroups [4]. By using the perspective of commutativity, Dudek and Trokhimenko [5–7] investigated some related algebraic properties on Menger algebras of rank n . Moreover, Dudek and Trokhimenko [8] used some operations to characterize the set of multiplace functions. In 1997, Trokhimenko [9] introduced the so-called v -regular Menger algebras of rank n , which can be acted as a generalization of regular semigroups, and also investigated some interesting results on the structures. In 2003, Denecke [10] studied on partial Menger algebras of rank n of terms. Up to 2021, Denecke and Hounnon [11] discussed some properties on Menger algebras of rank n and clones of terms.

The idea of ternary semigroups was first known when Banach (c.f. [12]) demonstrated some examples of ternary semigroups which need not to be reduced to semigroups. In 1932, the algebraic theory of ternary semigroups was firstly introduced by Lehmer [13]. A ternary semigroup (M, \star) consists of a nonempty set M and a ternary operation \star defined on M satisfying the so-called the *ternary associative law*, i.e.,

$$\star(\star(u, v, x), y, z) = \star(u, \star(v, x, y), z) = \star(u, v, \star(x, y, z)) \quad (3)$$

for all $u, v, x, y, z \in M$. According to the significant remark which was demonstrated by Banach, the algebraic properties and applications of ternary semigroups have been extensively studied (c.f. [14]), including ternary algebras and Banach ternary algebras, see [15–17]. Moreover, Chronowski [18] studied some results on ternary linear algebras and topological ternary structures. In [19], Jin et al. investigated some algebraic properties on C^* -ternary algebras. In 1955, Los [12] showed that each ternary semigroup can be embedded into a semigroup. In [20], Santiago and Sri Bala presented a regularity condition of ternary semigroups and studied some properties of regular ternary semigroups. Recently, Nongmanee and Leeratanavalee [21] extended the well-known result, i.e., Cayley's theorem, to study on ternary semigroups.

We note here that Menger hyperalgebras of rank n can be reduced to ternary algebraic structures with its ternary operation satisfying the following identity:

$$o(o(x, y_1, y_2), z_1, z_2) = o(x, o(y_1, z_1, z_2), o(y_2, z_1, z_2)) \quad (4)$$

for all $x, y_1, y_2, z_1, z_2 \in M$ if $n = 2$. It is easy to see that the identities (3) and (4) are not the same. It means that Menger algebras of rank n cannot be a generalization of ternary semigroups. Based on the previous remark, Nongmanee and Leeratanavalee [22] initiated the idea of *ternary Menger algebras of rank n* as a new generalization of ternary semigroups in sense of Menger algebras in 2021. A ternary Menger algebra (M, \diamond) of rank n consists of a nonempty set M and a $(2n + 1)$ -ary operation which satisfies the so-called the *ternary superassociative law* given as follows:

$$\begin{aligned} & \diamond(\diamond(u, v_1, \dots, v_n, x_1, \dots, x_n), y_1, \dots, y_n, z_1, \dots, z_n) \\ &= \diamond(u, \diamond(v_1, x_1, \dots, x_n, y_1, \dots, y_n), \dots, \diamond(v_n, x_1, \dots, x_n, y_1, \dots, y_n), z_1, \dots, z_n) \\ &= \diamond(u, v_1, \dots, v_n, \diamond(x_1, y_1, \dots, y_n, z_1, \dots, z_n), \dots, \diamond(x_n, y_1, \dots, y_n, z_1, \dots, z_n)) \end{aligned} \quad (5)$$

for all $u, v_i, x_i, y_i, z_i \in M$, $i \in \{1, \dots, n\}$.

Ternary Menger algebras of rank n can be considered as a natural generalization of ternary semigroups by setting the natural number $n = 1$. Then, the ternary superassociative law (5) is reduced to the ternary associative law on ternary semigroups (3) and implies that ternary Menger algebras of rank 1 and ternary semigroups are the same. Nongmanee and Leeratanavalee [23] recently presented the regular condition on ternary Menger algebras of rank n and investigated some algebraic properties of v -regular ternary Menger algebras of rank n .

In 1934, Marty [24] initiated the concept of algebraic hyperstructures when he defined the so-called hypergroups associated with hyperoperations. Such a concept is a new perspective on the classical

algebraic theory. That is, on the classical algebras, a composition of two elements (or n elements) of a base set is again an element, but on algebraic hyperstructures, a composition of two elements (or n elements) is a nonempty set. Since then, algebraic properties on hyperstructures have been investigated by many authors and there are many books written on this topic. Corsini and Leoreanu [25] point out the application of hyperstructures to graphs, hypergraphs, cryptography, codes, automata theory, fuzzy set theory, rough set theory and geometry (see also [26,27]). *Semihypergroups* and *ternary semihypergroups* (sometimes called *hypersemigroups* and *ternary hypersemigroups*, respectively) are also algebraic hyperstructures that can be considered as generalizations of semigroups and ternary semigroups, respectively. The idea of semihypergroups was initiated by Bonansinga and Corsini [28,29]. There are currently many articles studying semihypergroups and ternary semihypergroups by many mathematicians in different contexts. In 2021, Kumduang and Leeratanavalee [30] combined the concepts of Menger algebras of rank n and hyperstructures to create a generalization of semihypergroups called a *Menger hyperalgebra of rank n* . In their work, they also investigated some fundamental results on the hyperstructures, including the Cayley's theorem and the isomorphism theorem. Recently, Nongmanee and Leeratanavalee [31] studied the algebraic connection between Menger hyperalgebras of rank n and Menger algebras of rank n through regularity.

Inspired by the previous structural and hyperstructural studies, in this article we will start with the following facts: (i) each semihypergroup can be formed into a ternary semihypergroup, while some ternary semihypergroups do not necessarily reduce to an ordinary semihypergroup; (ii) Menger hyperalgebras can be regarded as a generalization of semihypergroups, but it cannot be regarded as a generalization of ternary semihypergroups. Therefore, a potential research question arises as to what is the algebraic structural generalization of ternary semihypergroups in terms of the arbitrary arity of a hyperoperation that satisfies the ternary superassociative law? Thus, in Section 2, the concept of *ternary Menger hyperalgebras of rank n* is introduced by combining the concepts of ternary Menger algebras of rank n and algebraic hyperstructures. In particular, we will examine their basic results and study the algebraic connections between ternary Menger hyperalgebras of rank n and other structures. In Section 3, we use the hyperstructure of ternary Menger algebras of rank n to construct what are called *diagonal ternary semihypergroups*. Moreover, we present some algebraic characteristics of the hyperstructure. In the last section, we will conclude the article by showing the relationship between structures and hyperstructures.

1 Preliminary background

To obtain the main results, we first need to recall the initial definitions and results of Menger algebras of rank n , ternary Menger algebras of rank n and Menger hyperalgebras of rank n .

Let (M, o) be a Menger algebra of rank n together with an $(n + 1)$ -ary operation o given by $(x, y_1, \dots, y_n) \mapsto o(x, y_1, \dots, y_n)$. Then, (M, \diamond) forms a ternary Menger algebra of rank n under a $(2n + 1)$ -ary operation \diamond defined by $(x, y_1, \dots, y_n, z_1, \dots, z_n) \mapsto o(o(x, y_1, \dots, y_n), z_1, \dots, z_n)$.

The ternary Menger algebra (M, \diamond) of rank n formed by the Menger algebra (M, o) of rank n is called a *ternary Menger algebra of rank n induced by a Menger algebra of rank n* . There are also some examples of ternary Menger algebras of rank n that cannot be reduced to Menger algebras of rank n , which are given in [22, 23].

Let M be a nonempty set and an operation $g : M^{n+1} \rightarrow P^*(M)$, where $P^*(M)$ is the family of all nonempty subsets of M , be defined by: for any $A, B_i \in P^*(M)$, $i \in \{1, \dots, n\}$,

$$g(A, B_1, \dots, B_n) = \bigcup_{a \in A, b_i \in B_i, i \in \{1, \dots, n\}} g(a, b_1, \dots, b_n).$$

Then g is called a $(2n + 1)$ -ary hyperoperation. In [30], an $(n + 1)$ -ary hypergroupoid (M, g) is a pair of a nonempty set M and an $(n + 1)$ -ary hyperoperation g defined on M . If the $(n + 1)$ -ary hyperoperation

g satisfies the superassociative law given as in (1), i.e.,

$$\bigcup_{a \in g(x, y_1, \dots, y_n)} g(a, z_1, \dots, z_n) = \bigcup_{b_i \in g(y_i, z_1, \dots, z_n), i \in \{1, \dots, n\}} g(x, b_1, \dots, b_n),$$

for all $a, x, y_1, \dots, y_n, z_1, \dots, z_n \in M$, then the $(n + 1)$ -ary hypergroupoid (M, g) is called a *Menger hyperalgebra of rank n* . The first observation about Menger hyperalgebras of rank n is the fact that Menger hyperalgebras of rank n immediately reduce to semihypergroups by setting $n = 1$. This means that it is a structural generalization of all semihypergroups. The following theorem shows a closed connection between Menger algebras of rank n and Menger hyperalgebras of rank n .

Theorem 1. [30] Let (M, o) be a Menger algebra of rank n . Then, (M, g) forms a Menger hyperalgebra of rank n under an $(n + 1)$ -ary hyperoperation g defined by $g(x, y_1, \dots, y_n) = \{o(x, y_1, \dots, y_n)\}$ for all $x, y_1, \dots, y_n \in M$.

Example 1. Some examples of Menger hyperalgebras of rank n have already been presented in [30].

(i) The set of all real numbers \mathbb{R} together with an $(n + 1)$ -ary hyperoperation g defined by

$$g(x, y_1, \dots, y_n) = \left\{ x + \frac{y_1 + \dots + y_n}{n} \right\} \quad \text{for all } x, y_1, \dots, y_n \in \mathbb{R},$$

where $+$ is the usual addition on \mathbb{R} , is a Menger hyperalgebra of rank n .

(ii) Define an $(n + 1)$ -ary hyperoperation g on the unit interval $[0, 1]$ as follows:

$$g(x, y_1, \dots, y_n) = \left[0, \frac{x \times y_1 \times \dots \times y_n}{n+1} \right] \quad \text{for all } x, y_1, \dots, y_n \in [0, 1],$$

where \times is the usual multiplication on $[0, 1]$. Then, $([0, 1], g)$ forms a Menger hyperalgebra of rank n .

(iii) Let \mathbb{N} be the set of all nonnegative integers. We define an $(n + 1)$ -ary hyperoperation g on \mathbb{N} by

$$g(x, y_1, \dots, y_n) = \{m \in \mathbb{N} \mid m \geq \max\{x, y_1, \dots, y_n\}\} \quad \text{for all } x, y_1, \dots, y_n \in \mathbb{N}.$$

Then, (\mathbb{N}, g) is a Menger hyperalgebra of rank n .

To better understand the Menger hyperalgebras of rank n , we refer the reader to [30].

2 Ternary Menger hyperalgebras of rank n

In this section, we will first create what is called *ternary Menger hyperalgebras of rank n* , where n is a positive integer. The hyperstructure is a new appropriate generalization of the whole ternary semihypergroup. Then, we will examine some of its interesting algebraic properties, including subhyperalgebras and homomorphisms. The algebraic connections among Menger algebras of rank n , Menger hyperalgebras of rank n , ternary Menger algebras of rank n and ternary Menger hyperalgebras of rank n have also been studied.

Let M be a nonempty set and f be a $(2n + 1)$ -ary hyperoperation from M into $P^*(M)$. For each $X, Y_i, Z_i \in P^*(M)$, $i \in \{1, \dots, n\}$, we define

$$f(X, Y_1, \dots, Y_n, Z_1, \dots, Z_n) = \bigcup_{x \in X, y_i \in Y_i, z_i \in Z_i} f(x, y_1, \dots, y_n, z_1, \dots, z_n).$$

For convenience, throughout this article, the sequence of elements y_1, \dots, y_n is replaced by the symbol \bar{y} and the composition $f(x, y_1, \dots, y_n, z_1, \dots, z_n)$ under the $(2n + 1)$ -ary hyperoperation f is denoted by the abbreviated notation $x[y_1 \dots y_n z_1 \dots z_n]$ or $x[\bar{y}\bar{z}]$. In the case where $z_1 = \dots = z_n = z$, we write $x[\bar{y}z^n]$ instead of $f(x, y_1, \dots, y_n, z_1, \dots, z_n)$.

Moreover, for any Menger hyperalgebra (M, g) of rank n , we also write $x[\bar{y}]$ instead of $g(x, y_1, \dots, y_n)$. Also, in the case where $y_1 = \dots = y_n = y$, the notation $x[y^n]$ is written instead of $g(x, y_1, \dots, y_n)$.

Definition 1. A $(2n + 1)$ -ary hypergroupoid (M, f) is a $(2n + 1)$ -ary algebraic hyperstructure consisting of a nonempty set M and a $(2n + 1)$ -ary hyperoperation f defined on M .

Definition 2. A $(2n + 1)$ -ary hypergroupoid (M, f) is called a *ternary Menger hyperalgebra of rank n* if its $(2n + 1)$ -ary hyperoperation f satisfies the ternary superassociative law given in (5), i.e.,

$$u[\bar{v}\bar{x}][\bar{y}\bar{z}] = u[v_1[\bar{x}\bar{y}] \dots v_n[\bar{x}\bar{y}]\bar{z}] = u[\bar{v}x_1[\bar{y}\bar{z}] \dots x_n[\bar{y}\bar{z}]]$$

for all $u, \bar{v}, \bar{x}, \bar{y}, \bar{z} \in M$.

For algebraic hyperstructures, the ternary superassociative law given in (5) means that:

$$\bigcup_{a \in u[\bar{v}\bar{x}]} a[\bar{y}\bar{z}] = \bigcup_{b_i \in v_i[\bar{x}\bar{y}], i \in \{1, \dots, n\}} u[\bar{b}\bar{z}] = \bigcup_{c_i \in x_i[\bar{y}\bar{z}], i \in \{1, \dots, n\}} u[\bar{v}\bar{c}]$$

for all $u, \bar{v}, \bar{x}, \bar{y}, \bar{z} \in M$.

By Definition 2, we see that the algebraic hyperstructure of ternary Menger hyperalgebras of rank n can be observed as a new generalization of ternary semihypergroups. This concept differs from the well-known concepts such as: n -ary semihypergroups and ordered n -ary semihypergroups. In particular, in case where $n = 1$, the ternary superassociative law immediately reduces to the ternary associative law presented in (3). Moreover, a ternary Menger hyperalgebra of rank n is immediately reduced to a ternary semihypergroup. However, the n -ary semihypergroup is reduced to a ternary semihypergroup by setting $n = 3$.

For the first result, we show the closed algebraic connection between the algebraic structures of ternary Menger algebras of rank n and the algebraic hyperstructures of ternary Menger hyperalgebras of rank n .

Theorem 2. Any ternary Menger algebra of rank n can form a ternary Menger hyperalgebra of rank n .

Proof. Let (M, \diamond) be a ternary Menger algebra of rank n . On the base set M , we define a $(2n + 1)$ -ary hyperoperation f by

$$x[\bar{y}\bar{z}] = \{\diamond(x, \bar{y}, \bar{z})\} \quad \text{for all } x, \bar{y}, \bar{z} \in M. \tag{6}$$

Indeed, for each $u, \bar{v}, \bar{x}, \bar{y}, \bar{z} \in M$, we have

$$\begin{aligned} u[\bar{v}\bar{x}][\bar{y}\bar{z}] &= \diamond(u, \bar{v}, \bar{x})[\bar{y}\bar{z}] \\ &= \{\diamond(\diamond(u, \bar{v}, \bar{x}), \bar{y}, \bar{z})\} \\ &= \{\diamond(u, \diamond(v_1, \bar{x}, \bar{y}), \dots, \diamond(v_n, \bar{x}, \bar{y}), \bar{z})\} \\ &= u[\{\diamond(v_1, \bar{x}, \bar{y})\} \dots \{\diamond(v_n, \bar{x}, \bar{y})\} \bar{z}] \\ &= u[v_1[\bar{x}\bar{y}] \dots v_n[\bar{x}\bar{y}]\bar{z}], \\ u[\bar{v}x_1[\bar{y}\bar{z}] \dots x_n[\bar{y}\bar{z}]] &= u[\bar{v} \diamond(x_1, \bar{y}, \bar{z}) \dots \diamond(x_n, \bar{y}, \bar{z})] \\ &= \{\diamond(u, \bar{v}, \diamond(x_1, \bar{y}, \bar{z}) \dots \diamond(x_n, \bar{y}, \bar{z}))\} \\ &= \{\diamond(\diamond(u, \bar{v}, \bar{x}), \bar{y}, \bar{z})\} \\ &= \diamond(u, \bar{v}, \bar{x})[\bar{y}\bar{z}] \\ &= u[\bar{v}\bar{x}][\bar{y}\bar{z}]. \end{aligned}$$

It follows that f satisfies the ternary superassociative law. Thus, (M, f) forms a ternary Menger hyperalgebra of rank n . □

We call (M, f) , where f is defined in (6) of Theorem 2, a *ternary Menger hyperalgebra of rank n induced by a ternary Menger algebra of rank n* or a *trivial ternary Menger hyperalgebra of rank n* . To better understand the nature of the ternary Menger hyperalgebras of rank n , the following example is needed.

Example 2. (i) Let M be a nonempty set. Define a $(2n + 1)$ -ary hyperoperation $f : M^{2n+1} \longrightarrow P^*(M)$ by

$$x[\bar{y}\bar{z}] = \{x\} \quad \text{for all } x, \bar{y}, \bar{z} \in M.$$

Therefore, (M, f) forms a ternary Menger hyperalgebra of rank n .

(ii) Let \mathbb{Z} be the set of all integers. A $(2n + 1)$ -ary hypergroupoid $(\mathbb{Z} \times \mathbb{Z}, f)$, together with a $(2n + 1)$ -ary hyperoperation f defined by:

$$(a, b)[(u_1, v_1) \dots (u_n, v_n)(x_1, y_1) \dots (x_n, y_n)] = \{(a, y_n)\} \quad \text{for all } a, b, \bar{u}, \bar{v}, \bar{x}, \bar{y} \in \mathbb{Z}$$

forms a ternary Menger hyperalgebra of rank n .

(iii) Define a $(2n + 1)$ -ary hyperoperation f on the set \mathbb{R} of all real numbers as follows:

$$x[\bar{y}\bar{z}] = \left\{x + \frac{y_1 + \dots + y_n + z_1 + \dots + z_n}{n}\right\} \quad \text{for all } x, \bar{y}, \bar{z} \in \mathbb{R}.$$

Therefore, (\mathbb{R}, f) is a ternary Menger hyperalgebra of rank n .

(iv) On the set \mathbb{R}_+ of all positive real numbers, we define a $(2n + 1)$ -ary hyperoperation f as follows:

$$x[\bar{y}\bar{z}] = \left\{x \times \sqrt[n]{y_1 \times \dots \times y_n \times z_1 \times \dots \times z_n}\right\} \quad \text{for all } x, \bar{y}, \bar{z} \in \mathbb{R}_+$$

Then, (\mathbb{R}_+, f) is a ternary Menger hyperalgebra of rank n .

Example 3. Let \mathbb{N} be the set of all natural numbers. Define a $(2n + 1)$ -ary hyperoperation f by

$$x[\bar{y}\bar{z}] = \{m \in \mathbb{N} \mid m \geq \max\{x, \bar{y}, \bar{z}\}\} \quad \text{for all } x, \bar{y}, \bar{z} \in \mathbb{N}.$$

Then, (\mathbb{N}, f) is a ternary Menger hyperalgebra of rank n .

Example 4. Let M be the unit interval $[0, 1]$. Define a $(2n + 1)$ -ary hyperoperation f by

$$x[\bar{y}\bar{z}] = [m, 1] \quad \text{for all } x, \bar{y}, \bar{z} \in M,$$

where $m = \min\{x, \bar{y}, \bar{z}\}$. Hence, the $(2n + 1)$ -ary hypergroupoid (M, f) forms a ternary Menger hyperalgebra of rank n .

Example 5. On the unit interval $[0, 1]$, we define a $(2n + 1)$ -ary hyperoperation f by

$$x[\bar{y}\bar{z}] = \left[0, \frac{x \times y_1 \times \dots \times y_n \times z_1 \times \dots \times z_n}{2n+1}\right] \quad \text{for all } x, \bar{y}, \bar{z} \in [0, 1],$$

where \times is the usual multiplication on $[0, 1]$. Then the $(2n + 1)$ -ary hypergroupoid (M, f) forms a ternary Menger hyperalgebra of rank n .

To obtain our main result, some special elements of ternary Menger hyperalgebras of rank n are defined in a manner similar to semihypergroup theory and ternary semihypergroup theory.

Definition 3. Let (M, f) be a ternary Menger hyperalgebra of rank n . An element $e \in M$ is said to be

(i) a (scalar) left identity element if

$$x \in e[e^n x^n] \quad (\{x\} = e[e^n x^n]) \quad \text{for all } x \in M;$$

(ii) a (scalar) right identity element if

$$x \in x[e^n e^n] \quad (\{x\} = x[e^n e^n]) \quad \text{for all } x \in M;$$

(iii) a (scalar) lateral identity element if

$$x \in e[x^n e^n] \quad (\{x\} = e[x^n e^n]) \quad \text{for all } x \in M;$$

(iv) an (scalar) identity element if

$$x \in e[e^n x^n] \cap e[x^n e^n] \cap x[e^n e^n] (\{x\} = e[e^n x^n] = e[x^n e^n] = x[e^n e^n]) \quad \text{for all } x \in M.$$

Next, we provide basic methods, namely subhyperalgebras and homomorphic images, to construct new ternary Menger hyperalgebras of rank n by using the given ternary Menger hyperalgebras of rank n .

Definition 4. Let (M, f) be a ternary Menger hyperalgebra of rank n and S be a nonempty subset of M . A $(2n + 1)$ -ary algebraic hyperstructure (S, f) is called a *ternary Menger subhyperalgebra of rank n* of (M, f) if S is closed under the $(2n + 1)$ -ary hyperoperation f , i.e., if $x, \bar{y}, \bar{z} \in S$, then $x[\bar{y}\bar{z}] \subseteq S$. If (S, f) is a ternary Menger subhyperalgebra of rank n of (M, f) , then we write $(S, f) \leq (M, f)$.

- Example 6.* (i) Consider the ternary Menger hyperalgebra (M, f) of rank n in Example 4. Let $S_p = [p, 1]$, where $m \leq p \leq 1$. Then, $(S_p, f) \leq (M, f)$.
 (ii) Consider the ternary Menger hyperalgebra (M, f) of rank n in Example 5. Let $S_q = [0, q]$, where $0 \leq q \leq 1$. Then $(S_q, f) \leq (M, f)$.

Using the concept of ternary Menger subhyperalgebras of rank n , we will examine some basic results about the $(2n + 1)$ -ary algebraic hyperstructures.

Theorem 3. Let $(S_1, f), (S_2, f)$ and (S_3, f) be ternary Menger subhyperalgebras of rank n of a ternary Menger hyperalgebra (M, f) of rank n . Then, the following assertions hold:

- (i) if $(S_1, f) \leq (S_2, f)$ and $(S_2, f) \leq (S_3, f)$, then $(S_1, f) \leq (S_3, f)$;
- (ii) if $S_1 \subseteq S_2 \subseteq S_3$, $(S_1, f) \leq (S_3, f)$ and $(S_2, f) \leq (S_3, f)$, then $(S_1, f) \leq (S_2, f)$.

Proof. The proof is straightforward. □

Theorem 4. Let I be an indexed set and let (S_i, f) , where $i \in I$, be ternary Menger subhyperalgebras of rank n of a ternary Menger hyperalgebra (M, f) of rank n . Then, $(\bigcap_{i \in I} S_i, f) \leq (M, f)$.

Proof. The proof is straightforward. □

Let (M, f) be a ternary Menger hyperalgebra of rank n and S be a nonempty subset of M . Then, there is at least one ternary Menger hyperalgebra of rank n containing the subset S . That is, (M, f) is a ternary Menger hyperalgebra of rank n containing S . By Theorem 4, we immediately obtain that the intersection of all ternary Menger subhyperalgebras of rank n of (M, f) which contain the subset S forms a ternary Menger subhyperalgebra of rank n of (M, f) . We denote it by (\hat{S}, f) . Using the previous fact, the following algebraic property is examined.

Proposition 1. Let (M, f) be a ternary Menger hyperalgebra of rank n and S be a nonempty subset of M . Then,

- (i) $S \subseteq \hat{S}$;
- (ii) if (T, f) is a ternary Menger subhyperalgebra of rank n of (M, f) which contains the subset S , then $(\hat{S}, f) \leq (T, f)$.

Proof. The proof is straightforward. □

The next purpose of this section is to demonstrate other basic results of ternary Menger hyperalgebras of rank n related to the concepts of homomorphism and ternary Menger subhyperalgebra of rank n . In order to obtain the result, it is necessary to introduce the concept of several types of homomorphism in ternary Menger hyperalgebras of rank n .

Definition 5. Let (M, f) and (N, g) be ternary Menger hyperalgebras of rank n . A mapping $\phi : (M, f) \rightarrow (N, g)$ is called:

- (i) a *homomorphism* if for each $x, \bar{y}, \bar{z} \in M$,

$$\phi(x[\bar{y}\bar{z}]) \subseteq \phi(x)[\phi(y_1) \dots \phi(y_n)\phi(z_1) \dots \phi(z_n)];$$

(ii) a *strong homomorphism* if for each $x, \bar{y}, \bar{z} \in M$,

$$\phi(x[\bar{y}\bar{z}]) = \phi(x)[\phi(y_1) \dots \phi(y_n)\phi(z_1) \dots \phi(z_n)].$$

By Definition 5, if the homomorphism ϕ is injective, it is called a *monomorphism* (or an *embedding*). Moreover, if the strong homomorphism ϕ is both injective and surjective, it is called an *isomorphism* from (M, f) onto (N, g) . In this particular situation, we say that (M, f) is isomorphic to (N, g) and denote by $(M, f) \cong (N, g)$.

Theorem 5. Let (M, f) and (N, g) be ternary Menger hyperalgebras of rank n and let $\phi : (M, f) \rightarrow (N, g)$ be a homomorphism. Then,

(i) if $(S, f) \leq (M, f)$ then $(\phi(S), g) \leq (N, g)$;

(ii) if $(T, g) \leq (N, g)$ and $\phi^{-1}(T) \neq \emptyset$ then $(\phi^{-1}(T), f) \leq (M, f)$,

where $\phi(S) = \{t \in N \mid \phi(s) = t \text{ for some } s \in S\}$ and $\phi^{-1}(T) = \{s \in M \mid \phi(s) = t \text{ for some } t \in T\}$.

Proof. (i) Assume that $(S, f) \leq (M, f)$. Let $x, \bar{y}, \bar{z} \in \phi(S)$. Then, there exist $a, \bar{b}, \bar{c} \in S$ such that $\phi(a) = x, \phi(b_i) = y_i$ and $\phi(c_i) = z_i$, where $i \in \{1, \dots, n\}$.

By our assumption, we get $a[\bar{b}\bar{c}] \in S$ and hence, $\phi(a[\bar{b}\bar{c}]) \in \phi(S)$. Since ϕ is a homomorphism from (M, f) into (N, g) , we have

$$x[\bar{y}\bar{z}] = \phi(a)[\phi(b_1) \dots \phi(b_n)\phi(c_1) \dots \phi(c_n)] = \phi(a[\bar{b}\bar{c}]) \in \phi(S).$$

It follows that $(\phi(S), g)$ is a ternary Menger subhyperalgebra of rank n of (N, g) .

(ii) The proof is similar to the argument of the assertion (i). □

We next conclude this section by examining the algebraic connections among Menger algebras of rank n , Menger hyperalgebras of rank n , ternary Menger algebras of rank n and ternary Menger hyperalgebras of rank n .

Theorem 6. Any Menger hyperalgebra of rank n can form a ternary Menger hyperalgebra of rank n .

Proof. Let (M, g) be a Menger hyperalgebra of rank n . We define a $(2n + 1)$ -ary hyperoperation f on M by

$$x[\bar{y}\bar{z}] = x[\bar{y}][\bar{z}] \quad \text{for all } x, \bar{y}, \bar{z} \in M. \tag{7}$$

Let $u, \bar{v}, \bar{x}, \bar{y}, \bar{z} \in M$. Using the definition of f and the superassociativity of g , we obtain

$$\begin{aligned} u[\bar{v}\bar{x}][\bar{y}\bar{z}] &= u[\bar{v}][\bar{x}][\bar{y}][\bar{z}] \\ &= u[v_1[\bar{x}] \dots v_n[\bar{x}]][\bar{y}][\bar{z}] \\ &= u[v_1[\bar{x}][\bar{y}] \dots v_n[\bar{x}][\bar{y}]][\bar{z}] \\ &= u[v_1[\bar{x}\bar{y}] \dots v_n[\bar{x}\bar{y}]][\bar{z}] \\ &= u[v_1[\bar{x}\bar{y}] \dots v_n[\bar{x}\bar{y}]]\bar{z}, \\ u[\bar{v}x_1[\bar{y}\bar{z}] \dots x_n[\bar{y}\bar{z}]] &= u[\bar{v}][x_1[\bar{y}\bar{z}] \dots x_n[\bar{y}\bar{z}]] \\ &= u[\bar{v}][x_1[\bar{y}][\bar{z}] \dots x_n[\bar{y}][\bar{z}]] \\ &= u[\bar{v}][x_1[\bar{y}] \dots x_n[\bar{y}]][\bar{z}] \\ &= u[\bar{v}][\bar{x}][\bar{y}][\bar{z}] \\ &= u[\bar{v}\bar{x}][\bar{y}\bar{z}]. \end{aligned}$$

It implies that

$$u[\bar{v}\bar{x}][\bar{y}\bar{z}] = u[v_1[\bar{x}\bar{y}] \dots v_n[\bar{x}\bar{y}]]\bar{z} = u[\bar{v}x_1[\bar{y}\bar{z}] \dots x_n[\bar{y}\bar{z}]] \quad \text{for all } u, \bar{v}, \bar{x}, \bar{y}, \bar{z} \in M.$$

Thus, the $(2n + 1)$ -ary hyperoperation f satisfies the ternary superassociative law. Thus, (M, f) forms a ternary Menger hyperalgebra of rank n . \square

However, there are some ternary Menger hyperalgebras of rank n that do not need to be reduced to Menger hyperalgebras of rank n . To illustrate the previous statement, the following related examples are given.

Example 7. (i) Let \mathbb{Z} be the set of all integers. Define a ternary hyperoperation \cdot on \mathbb{Z} by

$$\cdot(x, y, z) = \{x - y + z\} \quad \text{for all } x, y, z \in \mathbb{Z},$$

where $-$ and $+$ are the usual subtraction and the usual addition, respectively. Then, (\mathbb{Z}, \cdot) forms a ternary semihypergroup that is not reduced to a semihypergroup. Next, we define a $(2n + 1)$ -ary hyperoperation f on \mathbb{Z} as follows:

$$x[\bar{y}\bar{z}] = \{\cdot(x, y_1, z_1)\} \quad \text{for all } x, \bar{y}, \bar{z} \in \mathbb{Z}.$$

Using the ternary associativity of \cdot on \mathbb{Z} , the $(2n + 1)$ -ary hyperoperation f also satisfies the ternary superassociative law. Thus, (\mathbb{Z}, f) is a ternary Menger hyperalgebra of rank n that is not reduced to a Menger hyperalgebra of rank n .

(ii) Let $M = \{1, -1, i, -i\}$, where i is the imaginary unit, i.e., $i^2 = -1$. Define a $(2n + 1)$ -ary hyperoperation f_p , where $p \in \{1, \dots, n\}$, on M by

$$x[\bar{y}\bar{z}] = \{x \times y_p \times z_p\} \quad \text{for all } x, y, z \in M,$$

where \times is the usual multiplication. Then, (M, f_p) forms a ternary Menger hyperalgebra of rank n for all $p \in \{1, \dots, n\}$. It also does not reduce to a Menger hyperalgebra of rank n .

Proposition 2. Every ternary Menger algebra of rank n induced by a Menger algebra of rank n can be formed into a ternary Menger hyperalgebra of rank n induced by a Menger hyperalgebra of rank n .

Proof. Let (M, \diamond) be a ternary Menger algebra of rank n induced by a Menger algebra (M, o) of rank n , where a $(2n + 1)$ -ary operation \diamond is defined by

$$\diamond(x, \bar{y}, \bar{z}) = o(o(x, \bar{y}), \bar{z}) \quad \text{for all } x, \bar{y}, \bar{z} \in M.$$

By Theorem 1, we obtain that (M, g) is a Menger hyperalgebra of rank n under an $(n + 1)$ -ary hyperoperation g defined by

$$x[\bar{y}] = \{o(x, \bar{y})\} \quad \text{for all } x, \bar{y} \in M.$$

By Theorem 2, (M, f) forms a ternary Menger hyperalgebra of rank n together with a $(2n + 1)$ -ary hyperoperation f defined as in (6), i.e.,

$$x[\bar{y}\bar{z}] = \{\diamond(x, \bar{y}, \bar{z})\} \quad \text{for all } x, \bar{y}, \bar{z} \in M.$$

Then, for each $x, \bar{y}, \bar{z} \in M$, we have

$$x[\bar{y}\bar{z}] = \{\diamond(x, \bar{y}, \bar{z})\} = \{o(o(x, \bar{y}), \bar{z})\} = \bigcup_{a \in \{o(x, \bar{y})\}} \{o(a, \bar{z})\} = \bigcup_{a \in x[\bar{y}]} a[\bar{z}] = x[\bar{y}][\bar{z}].$$

So, we get $x[\bar{y}\bar{z}] = x[\bar{y}][\bar{z}]$ for all $x, \bar{y}, \bar{z} \in M$. It means that (M, f) forms a ternary Menger hyperalgebra of rank n induced by the Menger hyperalgebra (M, g) of rank n . \square

Proposition 3. Let e be a scalar identity element of a Menger hyperalgebra (M, g) of rank n , i.e., $\{x\} = x[e^n] = e[x^n]$ for all $x \in M$. Then, the element is again a scalar identity element of a ternary Menger hyperalgebra (M, f) of rank n induced by (M, g) .

Proof. Let (M, g) be a Menger hyperalgebra of rank n consisting of a scalar identity element e . By Theorem 6, we immediately obtain that (M, f) forms a ternary Menger hyperalgebra of rank n under a $(2n + 1)$ -ary hyperoperation f defined as (7), i.e., $x[\bar{y}\bar{z}] = x[\bar{y}][\bar{z}]$ for all x, \bar{y}, \bar{z} .

In fact, for each $x \in M$ we have $x[e^n e^n] = x[e^n][e^n] = x[e^n] = \{x\}$, $e[x^n e^n] = e[x^n][e^n] = x[e^n] = \{x\}$ and $e[e^n x^n] = e[e^n][x^n] = e[x^n] = \{x\}$.

Hence, the element e is again a scalar identity element of (M, f) . □

3 Diagonal ternary semihypergroups

In this section, we use the $(2n + 1)$ -ary algebraic hyperstructures of ternary Menger hyperalgebras of rank n to construct new ternary semihypergroups called *diagonal ternary semihypergroups*. We then study the algebraic connections between ternary Menger hyperalgebras of rank n and its diagonal ternary semihypergroups.

Lemma 1. Let (M, f) be a ternary Menger hyperalgebra of rank n . Then, (M, \bullet_f) forms a ternary semihypergroup with respect to a ternary hyperoperation \bullet_f which is defined via a $(2n + 1)$ -ary hyperoperation f on M by

$$\bullet_f(x, y, z) = x[y^n z^n] \quad \text{for all } x, y, z \in M. \tag{8}$$

Proof. Let (M, f) be a ternary Menger hyperalgebra of rank n . Define a ternary hyperoperation \bullet_f via a $(2n + 1)$ -ary hyperoperation f on M as (8). Next, we show that \bullet_f satisfies the ternary associative law.

Using the ternary superassociativity of f on M , we obtain that

$$\begin{aligned} \bullet_f(\bullet_f(u, v, x), y, z) &= u[v^n x^n][y^n z^n] \\ &= u[(v[x^n y^n])^n z^n] \\ &= \bullet_f(u, \bullet_f(v, x, y), z), \\ \bullet_f(u, v, \bullet_f(x, y, z)) &= u[v^n (x[y^n z^n])^n] \\ &= u[v^n x^n][y^n z^n] \\ &= \bullet_f(\bullet_f(u, v, x), y, z) \end{aligned}$$

for all $u, v, x, y, z \in M$. It immediately implies that \bullet_f satisfies the ternary associative law. Thus, (M, \bullet_f) forms a ternary semihypergroup. □

As in Lemma 1, the important point is to know that each ternary Menger hyperalgebra (M, f) of rank n can be used to form a ternary semihypergroup (M, \bullet_f) with a ternary hyperoperation \bullet_f defined as (8). In this article, (M, \bullet_f) is called a *diagonal ternary semihypergroup of a ternary Menger hyperalgebra (M, f) of rank n* .

According to the definition of the ternary hyperoperation \bullet_f , we immediately get the following result.

Proposition 4. Let (M, f) be a ternary Menger hyperalgebra of rank n . Then, a diagonal ternary semihypergroup of a ternary Menger hyperalgebra (M, f) of rank n is unique.

Proof. The proof is straightforward. □

Remark 1. Non-isomorphic ternary Menger hyperalgebras of rank n can have the same diagonal ternary semihypergroup.

The following example is provided to demonstrate that the statement in Remark 1 is true.

Example 8. Let $M = \{a, b, c, d, e\}$. Define a ternary hyperoperation \cdot on M by

$$\cdot(x, y, z) = \{x, y, z\} \quad \text{for all } x, y, z \in M.$$

It is easily to show that (M, \cdot) forms a ternary semihypergroup. Next, we define $(2n + 1)$ -ary hyperoperations f_1 and f_2 on M as follows:

$$f_1(x, \bar{y}, \bar{z}) = \cdot(x, y_1, y_1) \text{ and } f_2(x, \bar{y}, \bar{z}) = \cdot(x, y_2, y_2) \text{ for all } x, \bar{y}, \bar{z} \in M.$$

Then, we can see that (M, f_1) and (M, f_2) are ternary Menger hyperalgebras of rank n . Suppose that (M, f_1) and (M, f_2) are isomorphic via an isomorphism ϕ . Consider, $\phi(f_1(a, a, b^{n-1}, a^n)) = \phi(\cdot(a, a, a)) = \phi(a)$ and $f_2(\phi(a), \phi(a), (\phi(b))^{n-1}, (\phi(a))^n) = \cdot(\phi(a), \phi(b), \phi(b)) = \{\phi(a), \phi(b)\}$.

It implies that $\phi(f_1(a, a, b^{n-1}, a^n)) \neq f_2(\phi(a), \phi(a), (\phi(b))^{n-1}, (\phi(a))^n)$. This means that ϕ is not an isomorphism. Thus, (M, f_1) and (M, f_2) are non-isomorphic ternary Menger hyperalgebras of rank n . We also obtain that (M, \bullet_{f_1}) and (M, \bullet_{f_2}) are diagonal ternary semihypergroups of (M, f_1) and (M, f_2) , respectively, where \bullet_{f_1} and \bullet_{f_2} are defined by

$$\bullet_{f_1}(x, y, z) = x[y^n z^n] \text{ and } \bullet_{f_2}(x, y, z) = x[y^n z^n] \quad \text{for all } x, y, z \in M.$$

Furthermore, we can also see that (M, \bullet_{f_1}) and (M, \bullet_{f_2}) are the same.

Proposition 5. Let (M, f) be a ternary Menger hyperalgebra of rank n containing a scalar right (resp. left, lateral) identity element e . If there exists $m \in M$ satisfying the property

$$e[e^n m^n] = \{e\} \text{ (resp. } m[e^n e^n] = \{e\}, e[m^n e^n] = \{e\}),$$

then m is also a scalar right (resp. left, lateral) identity element.

Proof. Let e be a scalar right identity element of a ternary Menger hyperalgebra (M, f) of rank n . Then, $x[e^n e^n] = \{x\}$ for all $x \in M$. Assume that $m \in M$ satisfies the property $e[e^n m^n] = \{e\}$. Consider, for each $x \in M$,

$$\begin{aligned} x[m^n m^n] &= x[(m[e^n e^n])^n m^n] \\ &= x[m^n (e[e^n m^n])^n] \\ &= x[e^n e^n][m^n (e[e^n m^n])^n] \\ &= x[e^n e^n][m^n e^n] \\ &= x[(e[e^n m^n])^n e^n] \\ &= x[e^n e^n] \\ &= \{x\}. \end{aligned}$$

This means that m is also a scalar right identity element. Similar to the previous argument, we can prove the rest. □

Corollary 1. Let (M, f) be a ternary Menger hyperalgebra of rank n containing a scalar identity element e . If there exists $m \in M$ satisfying the property

$$\{e\} = m[e^n e^n] = e[m^n e^n] = e[e^n m^n],$$

then m is also a scalar identity element.

Proposition 6. Let (M, f) be a ternary Menger hyperalgebra of rank n containing a scalar right (left) identity element e . If there exists $m \in M$ satisfying the property

$$e[m^n m^n] = \{e\} \text{ (} m[m^n e^n] = \{e\}),$$

then m is also a scalar right (left) identity element.

Proof. Let (M, f) be a ternary Menger hyperalgebra of rank n containing a scalar right identity element e . So, we get $x[e^n e^n] = \{x\}$ for all $x \in M$. Assume that $m \in M$ satisfies the property $e[m^n m^n] = \{e\}$. Indeed, for each $x \in M$, we have

$$x[m^n m^n] = x[e^n e^n][m^n m^n] = x[e^n (e[m^n m^n])^n] = x[e^n e^n] = \{x\}.$$

This yields that m is a scalar right identity element. Analogously, we can show the rest. □

According to Proposition 5, Corollary 1, and Proposition 6, we show that a scalar (right, left, or lateral) identity element of a ternary Menger hyperalgebra of rank n need not be unique. In Example 7 (ii), we see that 1 and -1 are two distinct scalar identity elements of the ternary Menger hyperalgebra (M, f_p) .

Proposition 7. Let (M, f) be a ternary Menger hyperalgebra of rank n containing a scalar left identity element e , and for each $x, y \in M$ there is $m \in M$ satisfying the property that $\{e\} = m[x^n y^n]$. Then, a diagonal ternary semihypergroup (M, \bullet_f) of (M, f) is left cancellative, i.e.,

$$\bullet_f(x, y, u) = \bullet_f(x, y, v) \implies u = v \quad \text{for all } u, v, x, y \in M.$$

Proof. Assume that $\bullet_f(x, y, u) = \bullet_f(x, y, v)$ for all $u, v, x, y \in M$. By the definition of \bullet_f , we have $x[y^n u^n] = x[y^n v^n]$. Consider,

$$\begin{aligned} \{u\} &= e[e^n u^n] \\ &= e[(m[x^n y^n])^n u^n] \\ &= e[m^n (x[y^n u^n])^n] \\ &= e[m^n (x[y^n v^n])^n] \\ &= e[(m[x^n y^n])^n v^n] \\ &= e[e^n v^n] \\ &= \{v\}. \end{aligned}$$

Hence, $\{u\} = \{v\}$ and then, $u = v$. It follows that (M, \bullet_f) is left cancellative. □

Proposition 8. Let (M, \bullet_f) be a diagonal ternary semihypergroup of a ternary Menger hyperalgebra (M, f) of rank n . Then,

$$\bullet_f(x, y, z)[\bar{u}\bar{v}] = \bullet_f(x, y, z[\bar{u}\bar{v}]) \quad \text{for all } \bar{u}, \bar{v}, x, y, z \in M.$$

Proof. The proposition is true by the ternary superassociativity of f on M . □

Previously, we presented interesting results showing several algebraic connections among ternary Menger hyperalgebras of rank n and their diagonal ternary semihypergroups. In fact, if (M, f) is a ternary Menger hyperalgebra of rank n , then its diagonal ternary semihypergroup can be constructed immediately. Conversely, we found that if a ternary semihypergroup satisfies certain significant conditions, it can act as a diagonal ternary semihypergroup of some ternary Menger hyperalgebras of rank n .

Theorem 7. Let (M, \circ) be a ternary semihypergroup. If there exists an n -ary operation h defined on M that satisfies the following properties:

- (i) $h(x^n) = x$ for all $x \in M$;
- (ii) $\circ(h(\bar{x}), y, z) = h(\circ(x_1, y, z), \dots, \circ(x_n, y, z))$ for all $\bar{x}, y, z \in M$,

then (M, \circ) forms a diagonal ternary semihypergroup of some ternary Menger hyperalgebras of rank n .

Proof. Assume that there exists an n -ary operation h defined on M that satisfies the above properties (i) and (ii). We define a $(2n + 1)$ -ary hyperoperation on M by

$$x[\bar{y}\bar{z}] = \circ(x, h(\bar{y}), h(\bar{z})) \quad \text{for all } x, \bar{y}, \bar{z} \in M. \tag{9}$$

In fact, for every $u, \bar{v}, \bar{x}, \bar{y}, \bar{z} \in M$, we have

$$\begin{aligned} u[\bar{v}\bar{x}][\bar{y}\bar{z}] &= \circ(\circ(u, h(\bar{v}), h(\bar{x})), h(\bar{y}), h(\bar{z})) \\ &= \circ(u, \circ(h(\bar{v}), h(\bar{x}), h(\bar{y})), h(\bar{z})) \\ &= \circ(u, h(\circ(v_1, h(\bar{x}), h(\bar{y})), \dots, \circ(v_n, h(\bar{x}), h(\bar{y}))), h(\bar{z})) \\ &= \circ(u, h(v_1[\bar{x}\bar{y}], \dots, v_n[\bar{x}\bar{y}]), h(\bar{z})) \\ &= u[v_1[\bar{x}\bar{y}] \dots v_n[\bar{x}\bar{y}]\bar{z}], \\ u[\bar{v}x_1[\bar{y}\bar{z}] \dots x_n[\bar{y}\bar{z}]] &= \circ(u, h(\bar{v}), h(x_1[\bar{y}\bar{z}], \dots, x_n[\bar{y}\bar{z}])) \\ &= \circ(u, h(\bar{v}), h(\circ(x_1, h(\bar{y}), h(\bar{z})), \dots, \circ(x_n, h(\bar{y}), h(\bar{z})))) \\ &= \circ(u, h(\bar{v}), \circ(h(\bar{x}), h(\bar{y}), h(\bar{z}))) \\ &= \circ(\circ(u, h(\bar{v}), h(\bar{x})), h(\bar{y}), h(\bar{z})) \\ &= u[\bar{v}\bar{x}][\bar{y}\bar{z}]. \end{aligned}$$

This implies that the $(2n + 1)$ -ary hyperoperation f satisfies the ternary superassociative law.

Finally, we assume that (M, \bullet_f) is a diagonal ternary semihypergroup of ternary Menger hyperalgebra (M, f) of rank n . Then, we obtain $\bullet_f(x, y, z) = x[y^n z^n]$ for all $x, y, z \in M$.

In fact, for each $x, y, z \in M$, we obtain

$$\circ(x, y, z) = \circ(x, h(y^n), h(z^n)) = x[y^n z^n] = \bullet_f(x, y, z).$$

So, the ternary hyperoperation \circ and \bullet_f are the same, which means that (M, \circ) forms a diagonal ternary semihypergroup of the ternary Menger hyperalgebra (M, f) of rank n . □

Theorem 8. Let (M, \circ) be a ternary semihypergroup containing a scalar left identity element. Then, (M, \circ) is a diagonal ternary semihypergroup of the ternary Menger hyperalgebra (M, f) of rank n containing a scalar left identity element if and only if there exists an n -ary operation h defined on M satisfying the given properties (i) and (ii) in Theorem 7.

Proof. (\implies) Firstly, we assume that (M, \circ) is a diagonal ternary semihypergroup of the ternary Menger hyperalgebra (M, f) of rank n containing a scalar left identity element e .

By the assumption, we have $\{x\} = e[e^n x^n]$ for all $x \in M$. We define an n -ary operation h on M by

$$h(\bar{x}) = e[e^n \bar{x}] \quad \text{for all } \bar{x} \in M.$$

Next, let $x, \bar{x}, y, z \in M$. Then, we have

$$\begin{aligned} h(x^n) &= e[e^n x^n] \\ &= \{x\}, \\ \circ(h(\bar{x}), y, z) &= \circ(e[e^n \bar{x}], y, z) \\ &= e[e^n \bar{x}][y^n z^n] \\ &= e[e^n x_1[y^n z^n] \dots x_n[y^n z^n]] \\ &= h(x_1[y^n z^n], \dots, x_n[y^n z^n]) \\ &= h(\circ(x_1, y, z), \dots, \circ(x_n, y, z)). \end{aligned}$$

Therefore, the n -ary operation h satisfies the properties (i) and (ii) in Theorem 7.

(\Leftarrow) Assume that there exists an n -ary operation h defined on M satisfies the given properties (i) and (ii) in Theorem 7. By Theorem 7, (M, \circ) forms a diagonal ternary semihypergroup of a ternary Menger hyperalgebra (M, f) of rank n , where a $(2n + 1)$ -ary hyperoperation f is given as (9), i.e.,

$$x[\bar{y}\bar{z}] = \circ(x, h(\bar{y}), h(\bar{z})) \quad \text{for all } x, \bar{y}, \bar{z} \in M.$$

Now, let $e \in M$ be a scalar left identity element in a ternary semihypergroup (M, \circ) . Then, we obtain $\circ(e, e, x) = \{x\}$ for all $x \in M$. Since (M, \circ) is also a diagonal ternary semihypergroup of the ternary Menger hyperalgebra (M, f) of rank n , we obtain

$$e[e^n x^n] = \circ(e, e, x) = \{x\} \quad \text{for all } x \in M.$$

Then, the element $e \in M$ is again a scalar left identity element of the ternary Menger hyperalgebra (M, f) of rank n .

Therefore, the proof is completed. \square

To illustrate Theorem 7 and Theorem 8, the following example is required.

Example 9. Let \mathbb{N} be the set of all natural numbers. Then, (\mathbb{N}, \circ) forms a ternary semihypergroup under a ternary hyperoperation \circ defined by

$$\circ(x, y, z) = \{x\} \quad \text{for all } x, y, z \in \mathbb{N}.$$

Moreover, each element in \mathbb{N} is a scalar identity element. We define an n -ary operation h on \mathbb{N} by

$$h(\bar{x}) = \min\{x_1, \dots, x_n\} \quad \text{for all } \bar{x} \in \mathbb{N}.$$

In fact, for each $x, \bar{x}, y, z \in \mathbb{N}$,

$$\begin{aligned} h(x^n) &= \min\{x, \dots, x\} = x, \\ \circ(h(\bar{x}), y, z) &= h(\bar{x}) \\ &= \min\{x_1, \dots, x_n\} \\ &= \min\{\circ(x_1, y, z), \dots, \circ(x_n, y, z)\} \\ &= h(\circ(x_1, y, z), \dots, \circ(x_1, y, z)). \end{aligned}$$

This implies that the n -ary operation h satisfies the properties (i) and (ii) in Theorem 7. By Theorem 7, the ternary semihypergroup (M, \circ) forms as a diagonal ternary semihypergroup of a ternary Menger hyperalgebra (M, f) of rank n such that a $(2n + 1)$ -ary hyperoperation f is defined as (9).

Conclusion

In this article, we begin with the fact that the concept of Menger hyperalgebras of rank n , where n is a fixed integer, can be considered as a generalization of semihypergroups, but not a generalization of ternary semihypergroups. Based on the previous observation, an interesting research question arises: what is a natural generalization of ternary semihypergroup? Since this is an important question, in this article, we start by establishing the so-called *a ternary Menger hyperalgebra of rank n* using the concept of Menger hyperalgebras of rank n . In particular, it is a $(2n + 1)$ -ary algebraic hyperstructure that consists of a nonempty set together with a $(2n + 1)$ -ary hyperoperation satisfying the *ternary superassociative law*. In case $n = 1$, the ternary Menger hyperalgebras of rank n are immediately reduced to ternary semihypergroups. Thus, the concept of ternary Menger hyperalgebras of rank n can be considered as an appropriate generalization of ternary semihypergroups. Moreover, it differs from the former generalizations of ternary semihypergroups such as n -ary semihypergroups and ordered

n -ary semihypergroups. That is, an n -ary semihypergroup can be reduced to a ternary semihypergroup if $n = 3$.

In Section 2, we presented some basic algebraic results on ternary Menger hyperalgebras of rank n through the concepts of subhyperalgebra and homomorphism. Then, we also show the algebraic connections between Menger algebras of rank n , ternary Menger algebras of rank n , Menger hyperalgebras of rank n and ternary Menger hyperalgebras of rank n . We have also proven that any Menger hyperalgebra of rank n can be induced to a ternary Menger hyperalgebra of rank n , while there are some ternary Menger hyperalgebras of rank n that do not necessarily reduce to Menger hyperalgebras of rank n .

Based on the above results, we show some algebraic connections as shown in Figure 1.

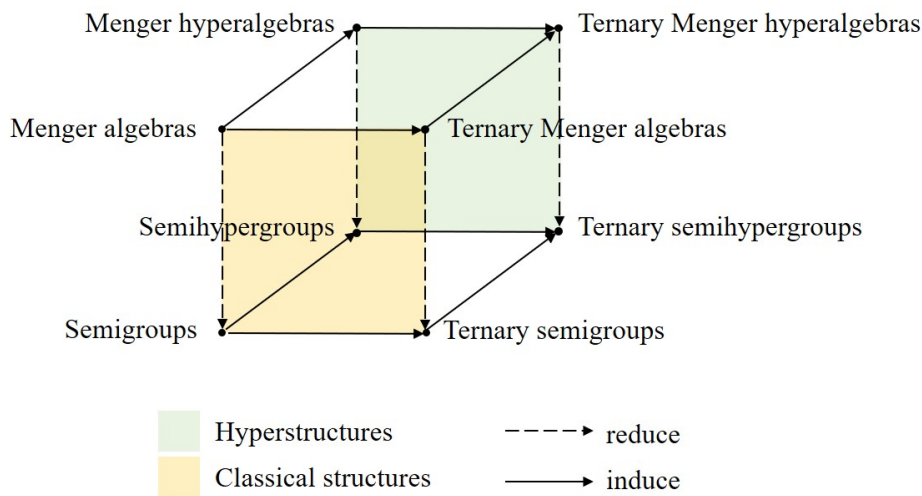


Figure 1. Algebraic connections among classical structures and hyperstructures

From Figure 1, we can briefly conclude that

- From a reductionist perspective, the concept of ternary Menger hyperalgebras of rank n can be regarded as a new generalization of ternary semihypergroups.
- From another perspective, the concept of ternary Menger hyperalgebras of rank n can also be considered as an extension of Menger hyperalgebras of rank n and ternary Menger algebras of rank n .

Finally in Section 3, we constructed the so-called *diagonal ternary semihypergroups* using base sets and $(2n + 1)$ -ary hyperoperation of ternary Menger hyperalgebras of rank n . Then, some interesting properties related to the connections between ternary Menger hyperalgebras of rank n and their diagonal ternary semihypergroups were examined. In the final, we characterize that some ternary semihypergroups that satisfy the special conditions can be diagonal ternary semihypergroups of some ternary Menger hyperalgebras of rank n .

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Author Contributions

All authors participated in the revision of the manuscript and approved the final submission. All authors contributed equally to this work.

Conflict of Interest

The authors declare no conflict of interest.

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