

Fundamental relation on hyper BCI -algebras

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This paper investigates several structural aspects of hyper BCI -algebras and their associated equivalence relations. In particular, we study a notion related to families of hyper BCI -algebras, provide illustrative examples, and analyze the relation β defined on these algebraic systems. Special attention is devoted to its transitive closure, denoted by β^* , and its role in the general theory of hyper BCI -algebras. We prove that β^* is the smallest equivalence relation for which the corresponding quotient structure is a classical BCI -algebra. Thus, β^* can naturally be regarded as the fundamental relation associated with a hyper BCI -algebra. We also establish a number of conditions that are equivalent to the transitivity of the relation β , thereby providing further characterizations of the algebraic structures under consideration and extending existing results in the theory of hyper BCI -algebras. Several examples are included to illustrate the introduced concepts and the obtained results.

Keywords: algebraic hyperstructure, hypergroupoid, hyperoperation, BCI -algebra, BCK -algebra, hyper BCI -algebra, hyper BCK -algebra, ideal, fundamental relation.

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1 Introduction and preliminaries

BCI -algebras are an important class of non-classical algebraic structures that arise in the study of abstract algebra and logic. These algebras generalize groups and are defined using a binary operation that satisfies specific conditions known as BCI -conditions. The BCI system is linked to substructural logics, which weaken some of the classical logical laws like associativity or distributivity. In a BCI -algebra, the operation is not necessarily associative or commutative, leading to a more flexible structure, useful for formalizing certain logical calculi. BCI -algebras find applications in areas such as automata theory, information systems, and fuzzy logic, where the relaxation of strict algebraic rules allows for new forms of reasoning and computation.

BCK -algebras and BCI -algebras, which are shortly called B -algebras, were introduced by two Japanese mathematicians, Imai and Iseki, in [1]. B -algebra structure comes from two different sources. The first origin is based on set theory. In set theory, there are three basic operations including union, intersection and set difference, so that Boolean algebras are derived from these three operations. The operations of union and intersection create distributable lattices; by examining only one of the union and intersection, we reach upper or lower semi-lattices. Until Iseki, the sets difference was not considered as a system. Among the logical operations, there are systems that include only one implicative operation such as positive implication system, weak positive implication system, BCK system and BCI system. There is a close relationship between the concept of sets difference in set theory and implicative operation in a logical device, for example see [2–5].

Let H be a non-empty set and $\mathcal{P}^*(H)$ be the family of all non-empty subsets of H . A mapping

$$\begin{aligned} \circ : H \times H &\rightarrow \mathcal{P}^*(H) \\ (x, y) &\mapsto x \circ y \subseteq H \end{aligned}$$

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is referred to a *binary hyperoperation* on H . The pair $(H, \circ)$ is called a *hypergroupoid*. Considering such definition, if A_1 and A_2 are two non-empty subsets of H as well as $x \in H$, we define

$$A_1 \circ A_2 = \bigcup_{\substack{a_1 \in A_1 \\ a_2 \in A_2}} a_1 \circ a_2, \quad x \circ A_1 = \{x\} \circ A_1 \text{ and } A_1 \circ x = A_1 \circ \{x\}.$$

A hypergroupoid $(H, \circ)$ is called as a hypergroup if

1. For all $a, a', a'' \in H$, $a \circ (a' \circ a'') = (a \circ a') \circ a''$;
2. For all $a \in H$, $a \circ H = H = H \circ a$.

For details about algebraic hyperstructures we refer to [6], and many other books.

There are various axiom systems for *BCI*-algebra, see [7]. In this paper, we will present the following axiom system related to *BCI*-algebra and give definition related to hyper *BCI*-algebra. The reader may refer to [8–11].

Definition 1. Consider A as a non-empty set equipped with a binary operation $\circ$ and a constant 0 . The structure $(A; \circ, 0)$ of type $(2, 0)$ is referred to a *BCI*-algebra if it meets the following conditions:

- (BCI-1) $((a_1 \circ a_2) \circ (a_1 \circ a_3)) \circ (a_3 \circ a_2) = 0$;
- (BCI-2) $(a_1 \circ (a_1 \circ a_2)) \circ a_2 = 0$;
- (BCI-3) $a_1 \circ a_1 = 0$;
- (BCI-4) $a_1 \circ a_2 = 0$ and $a_2 \circ a_1 = 0$ imply that $a_1 = a_2$.

Definition 2. A subset K of a *BCI*-algebra $(A; \circ, 0)$ is called a subalgebra of A if it meets the following criteria:

- (1) $0 \in K$,
- (2) $(K; \circ, 0)$ itself is a *BCI*-algebra.

A and $\{0\}$ are two subalgebras of A , which called the trivial subalgebra of A .

Definition 3. Consider a_1, a_2, a_3 as three element of a *BCI*-algebra $(A; \circ, 0)$. Then A is associative if the following identity holds:

$$(a_1 \circ a_2) \circ a_3 = a_1 \circ (a_2 \circ a_3).$$

Theorem 1. Suppose that $(A; \circ, 0)$ is a *BCI*-algebra. Then the following conditions are equivalent:

- (1) A is associative;
- (2) $0 \circ a = a$ for every $a \in A$;
- (3) $a_1 \circ a_2 = a_2 \circ a_1$ for every $a_1, a_2 \in A$.

It is well-established that while commutative *BCK*-algebra form a variety, but *BCK*-algebras in general do not. This observation prompts the search for specific classes of *BCK*-algebras such that each of them forms a variety. Consequently, in 1977, H. Yutani introduced the concept of quasi-commutative *BCK*-algebra [12]. Later, in 1980, K. Iseki also extended this concept to *BCI*-algebra [13]. For a *BCI*-algebra A , we inductively define a polynomial $Q_{m,n}(a, b)$ involving two variables a, b in A as follows:

$$\begin{aligned} Q_{0,0}(a, b) &= a * (a * b), \\ Q_{m+1,n}(a, b) &= Q_{m,n}(a, b) * (a * b), \\ Q_{m,n+1}(a, b) &= Q_{m,n}(a, b) * (b * a), \end{aligned}$$

where m, n are any natural numbers. In fact, $Q_{m,n}(a, b)$ can be achieved from right $*$ multiplying $Q_{0,0}(a, b)$ by $a * b$ m times and next by $b * a$ n times, that is,

$$Q_{m,n}(a, b) = (Q_{0,0}(a, b) * (a * b)^m) * (b * a)^n,$$

in other words, we have $Q_{m,n}(a, b) = (a * (a * b)^{m+1}) * (b * a)^n$.

Definition 4. A *BCI*-algebra A is referred to quasi-commutative of type $(i, j; m, n)$ if there exist two pairs of natural numbers i, j and m, n such that the following identity holds:

$$Q_{i,j}(a, b) = Q_{m,n}(b, a)$$

or equivalently,

$$(a * (a * b)^{i+1}) * (b * a)^j = (b * (b * a)^{m+1}) * (a * b)^n.$$

Correspondingly, we can explain a quasi-commutative *BCK*-algebra. If A is a quasi-commutative *BCI*-algebra of type $(i, j; m, n)$, the *BCK*-part of A is naturally a quasi-commutative *BCK*-algebra of the same type. Exchanging a and b in the identity $Q_{i,j}(a, b) = Q_{m,n}(b, a)$, we get $Q_{m,n}(a, b) = Q_{i,j}(b, a)$.

Definition 5. Suppose that $(A; \circ, 0)$ is a *BCI*-algebra and I is a subset of A and I is also non-empty. Then I is referred to an ideal of A , if for all $a, b \in A$, it meets the following conditions:

- (1) $0 \in I$;
- (2) $b \in I$ and $a \circ b \in I$ imply $a \in I$.

$\{0\}$ and A are obviously two ideals of A , which are called the trivial ideals of A . An ideal I is said to be proper if $I \subsetneq A$.

Definition 6. Let a_1, a_2, a_3 be any elements in a non-empty set H_B , $\circ$ be a binary hyperoperation on H_B as well as 0 be a constant. Then $(H_B, \circ, 0)$ is a hyper *BCK*-algebra if

- (HK-1) $(a_1 \circ a_3) \circ (a_2 \circ a_3) \ll a_1 \circ a_2$;
- (HK-2) $(a_1 \circ a_2) \circ a_3 = (a_1 \circ a_3) \circ a_2$;
- (HK-3) $a_1 \circ H_B \ll \{a_1\}$;
- (HK-4) $a_1 \ll a_2$ and $a_2 \ll a_1$ imply that $a_1 = a_2$;

where $a_1 \ll a_2$ is defined by $0 \in a_1 \circ a_2$. Let H_1, H_2 be two subsets of H_B , then we have $H_1 \ll H_2$ if and only if for all $h_1 \in H_1$ there exist $h_2 \in H_2$ such that $h_1 \ll h_2$.

Remark 1. In a hyper *BCK*-algebra $(H_B, \circ)$, the condition (HK-3) is equivalent to the following condition:

$$a_1 \circ a_2 \ll \{a_1\}, \text{ for all } a_1, a_2 \in H_B.$$

Definition 7. A hyper *BCI*-algebra is a structure $(H_B, \circ, 0)$ such that H_B is a non-empty set equipped with a hyperoperation $\circ : H_B \times H_B \rightarrow \mathcal{P}^*(H_B)$ and a constant 0 satisfying the following axioms:

- (HI-1) $(a_1 \circ a_3) \circ (a_2 \circ a_3) \ll a_1 \circ a_2$;
- (HI-2) $(a_1 \circ a_2) \circ a_3 = (a_1 \circ a_3) \circ a_2$;
- (HI-3) $a \ll a$;
- (HI-4) $a_1 \ll a_2$ and $a_2 \ll a_1$ imply $a_1 = a_2$;
- (HI-5) $0 \circ (0 \circ a_1) \ll a_1$

for all $a_1, a_2, a_3 \in H_B$.

Definition 8. Suppose that H_B is a hyper *BCI*-algebra. Then we say H_B is ordered if for any $a_1, a_2, a_3 \in H_B$ it satisfies the following condition:

$$a_1 \ll a_2 \text{ and } a_2 \ll a_3 \text{ imply } a_1 \ll a_3.$$

Definition 9. Suppose that $(H, \circ)$ is a hypergroupoid that includes 0 . We say H is a quasi-hyper *BCI*-algebra if it satisfies the following conditions:

- (QHCI-1) $(a_1 \circ a_3) \circ (a_2 \circ a_3) \ll a_1 \circ a_2$;
- (QHCI-2) $(a_1 \circ a_2) \circ a_3 = (a_1 \circ a_3) \circ a_2$;
- (QHCI-3) $a_1 \ll a_1$;
- (QHCI-4) $a_1 \ll a_2$ and $a_2 \ll a_1$ imply that $a_1 = a_2$;

- (QHCI-5) $a_1 \ll a_1 \circ 0$;
 (QHCI-6) $a_1 \ll 0 \Rightarrow a_1 = 0$.

Proposition 1. Every hyper BCK-algebra is a hyper BCI-algebra, but the reverse is not necessarily true.

Proposition 2. Assume that $(H_B, \circ)$ is a hyper BCI-algebra. Then the following are hold:

- (1) $a_1 \ll 0$ implies $a_1 = 0$,
- (2) $0 \in a_1 \circ (a_1 \circ 0)$,
- (3) $a_1 \ll a_1 \circ 0$,
- (4) $0 \circ (a_1 \circ a_2) \ll a_2 \circ a_1$,
- (5) $H_1 \ll H_1$,
- (6) $H_1 \subseteq H_2$ implies $H_1 \ll H_2$,
- (7) $H_1 \ll \{0\}$ implies $H_1 = \{0\}$,
- (8) $a_1 \circ 0 \ll \{a_2\}$ implies $a_1 \ll a_2$,
- (9) $a_2 \ll a_3$ implies $a_1 \circ a_3 \ll a_1 \circ a_2$,
- (10) $a_1 \circ a_2 = \{0\}$ implies $(a_1 \circ a_3) \circ (a_2 \circ a_3) = \{0\}$ and $a_1 \circ a_3 \ll a_2 \circ a_3$,
- (11) $H_1 \circ \{0\} = \{0\}$ implies $H_1 = \{0\}$,

for all $a_1, a_2, a_3 \in H_B$ and all non-empty subsets, H_1, H_2 of H_B .

Definition 10. Suppose that I is a non-empty subset of a hyper BCI-algebra H_B . Then I is said to be a hyper BCI-ideal of H_B if:

- (HI-1) $0 \in I$;
 (HI-2) $a_1 \circ a_2 \ll I$ and $a_2 \in I$ imply $a_1 \in I$ for all $a_1, a_2 \in H_B$.

Every hyper BCI-ideal of a hyper BCI-algebra H is a hyper subalgebra of H , but the reverse may not hold.

2 The fundamental relation on hyper BCI-algebras

Assume that $(H_B, \circ, 0)$ is a hyper BCI-algebra and R be an equivalence relation on H , $H_1, H_2 \subseteq H$ and $H_1, H_2 \neq \emptyset$ we define:

- $H_1 R H_2 \iff \exists h_1 \in H_1, h_2 \in H_2$ such that $h_1 R h_2$;
 $H_1 \bar{R} H_2 \iff \forall h_1 \in H_1, \exists h_2 \in H_2$ such that $h_1 R h_2$ and $\forall h_2 \in H_2, \exists h_1 \in H_1$ such that $h_1 R h_2$;
 $H_1 \bar{\bar{R}} H_2 \iff \forall h_1 \in H_1, \forall h_2 \in H_2$ we have $h_1 R h_2$.

Definition 11. Assume that $(H_B, \circ, 0)$ is a hyper BCI-algebra and R is an equivalence relation on H . Then R is referred to a regular relation if for all a of H , we have

$$x R y \implies (x \circ a) \bar{R} (y \circ a) \quad \text{and} \quad ((a \circ x) \bar{R} (a \circ y) \quad \text{respectively}).$$

Definition 12. Suppose that $(H_B, \circ, 0)$ is a hyper BCI-algebra and R is an equivalence relation on H_B . Then R is referred to a strongly regular relation if for all a of H , we have:

$$x R y \implies (x \circ a) \bar{\bar{R}} (y \circ a) \quad \text{and} \quad ((a \circ x) \bar{\bar{R}} (a \circ y) \quad \text{respectively}).$$

Let $(H_B; \circ, 0)$ be a BCI-algebra and U_H be the set of all finite combination of elements of H and define β on H as follows:

Definition 13. If $(H_B; \circ, 0)$ is a hyper BCI-algebra then we define $\beta_1 = \{(x, x) : x \in H_B\}$ as the diagonal relation on H_B .

For any integer $n > 1$, β_n is defined as

$$x \beta_n y \iff \exists (u_1, \dots, u_n) \in H_B^n, \quad \exists u \in U_{(u_1, \dots, u_n)} \quad \text{such that} \quad \{a, b\} \subseteq u.$$

It is evident that, for all $n \geq 1$, the relations β_n are symmetric and β is both reflexive and symmetric. Let β^* denote the transitive closure of β .

Theorem 2. Suppose that $(H_B; \circ, 0)$ is a hyper BCI -algebra. Then, β^* is strongly regular on H_B .

Proof. If $x\beta^*y$, then by definition of β^* , there exist $n \in \mathbb{N}$, $i = 1, 2, \dots, n$ $(z_0, \dots, z_n) \in H^{n+1}$ and $\exists k_i \in \mathbb{N}$ such that $z_0 = x$, $z_n = y$ and $x = z_0\beta_{k_1}z_1, z_1\beta_{k_2}z_2, \dots, z_{n-1}\beta_{k_n}z_n = y$ by definition of relation β , we have

$$z_0\beta_{k_1}z_1 \implies \exists(u_1^1, \dots, u_{k_1}^1) \in H_B^{k_1}, \exists u^1 \in U_{(u_1^1, \dots, u_{k_1}^1)} \text{ such that } \{z_0, z_1\} \subseteq u^1;$$

$$z_1\beta_{k_2}z_2 \implies \exists(u_1^2, \dots, u_{k_2}^2) \in H_B^{k_2}, \exists u^2 \in U_{(u_1^2, \dots, u_{k_2}^2)} \text{ such that } \{z_1, z_2\} \subseteq u^2;$$

$\vdots$

$$z_{n-1}\beta_{k_n}z_n \implies \exists(u_1^n, \dots, u_{k_n}^n) \in H_B^{k_n}, \exists u^n \in U_{(u_1^n, \dots, u_{k_n}^n)} \text{ such that } \{z_{n-1}, z_n\} \subseteq u^n.$$

Now let $a \in H_B$ be an arbitrary element of H_B . It follows that $\forall j \in \{0, \dots, n-1\}$ $z_j \circ a, z_{j+1} \circ a \subseteq u^j \circ a$. Hence if j is such that $0 \leq j \leq s$, $\forall u \in z_j \circ a, \forall v \in z_{j+1} \circ a$, we obtain $u\beta v$. Thus, $\forall u \in z_1 \circ a, \forall v \in z_n \circ a$ we have $u\beta^*v$. This means that β^* is strongly regular relation. $\square$

Definition 14. Suppose that $(H_B; \circ, 0)$ is a hyper BCI -algebra, R is a regular relation on H_B and $R(a), R(b)$ are two elements of H_B/R , where H_B/R is the set of all equivalence classes of R (i.e. $H_B/R = \{R(h) : h \in H\}$). Then we define hyperoperation $\bar{\circ}$ on H_B/R as follows:

$$R(a)\bar{\circ}R(b) := \{R(c) : c \in a \circ b\}.$$

Lemma 1. $\bar{\circ}$ is well-defined.

Proof. This proof is similar to the proof at [14]. $\square$

Definition 15. Let R be a regular relation on a hyper BCI -algebra $(H_B; \circ, 0)$ and $R(a), R(b)$ be two elements of H_B/R . Then we define

$$R(a) \ll R(b) \text{ if and only if } R(0) \in R(a)\bar{\circ}R(b).$$

Theorem 3. Suppose that $(H_B; \circ, 0)$ is a hyper BCI -algebra, R is an equivalence relation on H_B and hyperoperation $\bar{\circ}$ be one to one. Then R is regular relation if and only if $(H_B/R; \bar{\circ}, R(0))$ is a hyper BCI -algebra.

Proof. Assume that $R(a)$ is an arbitrary element of $(R(x)\bar{\circ}R(z))\bar{\circ}(R(y)\bar{\circ}R(z))$. By Definition 15, we have

$$\exists r \in x \circ z \text{ and } \exists s \in y \circ z : R(a) \in R(r)\bar{\circ}R(s).$$

Thus there exists $a' \in r \circ s$ such that $R(a) = R(a')$. Since $a' \in r \circ s$, we have

$$a' \in r \circ s \subseteq (x \circ z) \circ (y \circ z) \ll x \circ y.$$

So there exist $a'' \in x \circ y$ such that $a' \ll a''$. This means that $0 \in a' \circ a''$. So we have

$$R(0) \in R(a')\bar{\circ}R(a'') \implies R(a') \ll R(a'').$$

So $(R(x)\bar{\circ}R(z))\bar{\circ}(R(y)\bar{\circ}R(z)) \ll R(x)\bar{\circ}R(y)$.

Now, we investigate condition H_2 for H/R . In this case we must prove that $(R(x)\bar{\circ}R(y))\bar{\circ}R(z) = (R(x)\bar{\circ}R(z))\bar{\circ}R(y)$. For this purpose at first let $R(w) \in (R(x)\bar{\circ}R(y))\bar{\circ}R(z)$. Thus there is $u \in x \circ y$ such that $R(w) \in R(u)\bar{\circ}R(z)$. Then there exist $v \in u \circ z$ such that $R(w) = R(v)$. We have $v \in u \circ z \subseteq (x \circ y) \circ z$. As we know $(x \circ y) \circ z = (x \circ z) \circ y$. So $\exists r' \in x \circ z : v \in r' \circ y$, thus $R(w) = R(v) \in R(r')\bar{\circ}R(y) \subseteq (R(x)\bar{\circ}R(z))\bar{\circ}R(y)$. Consequently, it implies that $(R(x)\bar{\circ}R(y))\bar{\circ}R(z) \subseteq (R(x)\bar{\circ}R(z))\bar{\circ}R(y)$. In the same way, we earn the other side of equality.

We intend to prove condition H_3 for H_B/R . In other words, we need to prove that $R(x) \ll R(x)$ (i.e., $R(0) \in R(x) \bar{\circ} R(x)$). Since H_B is a hyper BCI -algebra, it follows that $0 \in x \circ x$. By using of the definition of hyperoperation $\bar{\circ}$ we have $R(0) \in R(x) \bar{\circ} R(x)$. Thus $R(x) \ll R(x)$.

For condition (H_4) , let $R(x) \ll R(y)$ and $R(y) \ll R(x)$, we want to prove that $R(x) = R(y)$. Thus we have, $R(0) \in R(x) \bar{\circ} R(y)$ and $R(0) \in R(y) \bar{\circ} R(x)$ so there exist $s \in x \circ y$ and $t \in y \circ x$ such that $R(0) = R(s)$ and $R(0) = R(t)$. Therefore we have $R(t) = R(s)$. This means that $R(x) \bar{\circ} R(y) = R(x) \bar{\circ} R(y)$, the hypothesis that $\bar{\circ}$ is one to one we conclude that $R(x) = R(y)$.

Finally we check condition H_5 . Let $a \in R(0) \bar{\circ} (R(0) \bar{\circ} R(x))$. So there is $s \in 0 \circ x$ and $h \in H_B$ such that $a = R(h) \in R(0) \bar{\circ} R(s)$. Therefore there exist $t \in 0 \circ s$ such that $R(h) = R(t)$. Since H_B is a hyper BCI -algebra, we have $0 \circ (0 \circ x) \ll x$. So $t \ll x$ and $R(t) \ll R(x)$. Consequently, $R(0) \bar{\circ} R(s) \ll R(x)$, which means that $R(0) \bar{\circ} (R(0) \bar{\circ} R(x)) \ll R(x)$.

Now, we want to show that R is a regular relation, first we have to show that

$$xRy \implies (x \circ a)R(y \circ a) \text{ and } (a \circ x)R(a \circ y).$$

For this purpose, let $x, y, a \in H_B$, xRy and $s \in x \circ a$, by the definition of hyperoperation $\bar{\circ}$ we have $R(s) \in R(x) \bar{\circ} R(a)$. Since $R(x) = R(y)$ so $R(s) \in R(x) \bar{\circ} R(a) = R(y) \bar{\circ} R(a)$ thus there exist $t \in y \circ a$ such that $R(s) = R(t)$. This means that $x \circ a \bar{R} y \circ a$. Therefore R is regular on the right. In the same way R is regular on the left. $\square$

Definition 16. Assume that $(H_B, \circ, 0)$ and $(H'_B, \circ', 0')$ are two hyper BCI -algebras. A function $f : H_B \rightarrow H'_B$ is referred to a homomorphism if the following conditions hold:

- (1) $f(x \circ y) = f(x) \circ' f(y)$;
- (2) $f(0) = 0'$ for every $(x, y) \in H_B^2$.

Theorem 4. Assume that $(H_B, \circ, 0)$ and $(H'_B, \circ', 0')$ are two hyper BCI -algebras and the function $f : H_B \rightarrow H'_B$ is a homomorphism. The equivalence R associated with f (i.e. xRy if and only if $f(x) = f(y)$) is regular as well as the function $g : f \rightarrow H_B/R$, $g(f(x)) = R(x)$ is an isomorphism.

Proof. It is similar to the proof of Theorem 3.9 in [14]. $\square$

Theorem 5. Suppose that $(H_B; \circ, 0)$ is a hyper BCI -algebra. Let R be a strongly regular relation on a hyper BCI -algebra $(H_B; \circ, 0)$. Then

- (1) H_B/R is a BCI -algebra (i.e. H_B/R is a fundamental BCI -algebra).
- (2) If H'_B is a BCI -algebra and $f : H_B \rightarrow H'_B$ is a homomorphism, then the equivalence γ associated with f is strongly regular.

Proof. (1) Let x, y two elements of H_B . By assumption R is strongly regular relation on H_B . So for we have $x \circ y \bar{R} y \circ x$. Hence $|R(x) \circ R(y)| = 1$. This means that H_B/R is a BCI -algebra.

(2) To prove that γ is strongly regular relation, first we have to show that $x\gamma y \implies x \circ a \bar{\gamma} y \circ a$ and $a \circ x \bar{\gamma} a \circ y$. Let $a, x, y \in H_B$ and $x\gamma y$. It implies $\gamma(x) = \gamma(y)$ and $f(x) = f(y)$. Thus for every $s \in x \circ a$ and $t \in y \circ a$, we have $f(s), f(t) \in f(x \circ a) = f(x) \circ f(a) = f(y) \circ f(a) = f(y \circ a)$. As we know, H'_B is a BCI -algebra so cardinal of $f(x \circ a)$ and cardinal of $f(y \circ a)$ are equal to 1. It follows that $s \bar{R} t$. This means that $x \circ a \bar{R} y \circ a$. $\square$

Theorem 6. The relation β^* is the smallest equivalence relation such that the quotient H/β^* is a BCI -algebra.

Corollary 1. Relation β^* is the smallest strongly regular relation on a hyper BCI -algebra.

3 Transitivity condition for relation β

In this section, we want to identify the necessary and sufficient conditions for the transitivity of the relation β .

Definition 17. Consider X as a non-empty subset of H_B . If for every non-zero natural number n , for $i = 1, \dots, n$, $\forall k_i \in \mathbb{N}$, $\forall (u_{i1}, \dots, u_{ik_i}) \in H_B^{k_i}$ we have

$$U_{(u_{i1}, \dots, u_{ik_i})} \cap X \neq \emptyset \implies U_{(u_{i1}, \dots, u_{ik_i})} \subseteq X.$$

Then X is referred to a β -part of H_B .

Lemma 2. Assume that X is a non-empty subset of a hyper *BCI*-algebra of H_B . Then the following conditions are equivalent:

1. X is a β -part of H_B ;
2. if $x \in X$ and $x\beta y$, then $y \in X$;
3. if $x \in X$ and $x\beta^* y$, then $y \in X$.

Proof. (1 $\implies$ 2) If $(x, y) \in H_B$ is a pair such that $x \in X$ and $x\beta y$, then by the definition of relation β , there exist $n \in \mathbb{N}$, $(u_1, \dots, u_n) \in H^n$ such that $\{x, y\} \subseteq U_{(u_1, \dots, u_n)}$. By assumption X is a β -part and $U_{(u_1, \dots, u_n)} \cap X \neq \emptyset$, so we have $U_{(u_1, \dots, u_n)} \subseteq X$. This means that $y \in X$.

(2 $\implies$ 3) Assume that $(x, y) \in H^2$ such that $x \in X$ and $x\beta^* y$. By the definition of β^* , there exist $m \in \mathbb{N}$, $(u_0, \dots, u_m) \in H^{m+1}$ such that $u_0 = x$, $u_m = y$ and $x = u_0\beta u_1, u_1\beta u_2, \dots, u_{m-1}\beta u_m = y$. Since $x \in X$ and $x = u_0\beta u_1$ by applying (2), we obtain $u_1 \in X$. If we continue this process m times, we conclude that $y \in X$.

(3 $\implies$ 1) Now, we want to show that M is a β -part of H . So let $U_{(u_1, \dots, u_n)} \cap X \neq \emptyset$ and $x \in U_{(u_1, \dots, u_n)} \cap X$. For every $y \in U_{(u_1, \dots, u_n)}$ we have $x\beta y$. Thus, $x \in X$ and $x\beta^* y$. Eventually by applying (3) we conclude that $y \in X$. $\square$

At first we introduce several notations that they are applied to prove the next Theorem. Let H_B be a hyper *BCI*-algebra. For all $x \in H_B$, we set

1. $T_n(x) = \{(x_1, \dots, x_n) \in H^n \mid x \in U_{(x_1, \dots, x_n)}\}$;
2. $P_n(x) = \bigcup \{U_{(x_1, \dots, x_n)} \mid x \in U_{(x_1, \dots, x_n)}\}$;
3. $P_\sigma(x) = \bigcup_{n \geq 1} P_n(x)$.

It follows immediately the following from the previous notations and definitions:

Lemma 3. For every $x \in H_B$, $P_\sigma(x) = \{y \in H_B \mid x\beta y\}$.

Proof. For every $x, y \in H_B$, suppose that $x\beta y$ by definition of relation β , we have

$$\begin{aligned} x\beta y &\iff \exists n \in \mathbb{N}, \exists (u_1, \dots, u_n) \in H^n : \{x, y\} \subseteq U_{(u_1, \dots, u_n)} \\ &\iff \exists n \in \mathbb{N}, \exists (u_1, \dots, u_n) \in H^n : x \in U_{(u_1, \dots, u_n)} \\ &\iff \exists n \in \mathbb{N} : y \in P_n(x) \\ &\iff y \in P_\sigma(x). \end{aligned}$$

$\square$

Theorem 7. Let H be a hyper *BCI*-algebra. Then the following conditions are equivalent:

- (1) β is transitive;
- (2) $\beta^*(x) = P_\sigma(x)$ for all $x \in H$;
- (3) $P_\sigma(x)$ is a β -part of H .

Proof. (1 $\implies$ 2) Let $y \in \beta^*(x)$ by previous lemma, we have

$$\beta^*(x) = \beta^*(y) \iff x\beta^*y.$$

Since β is transitive, we have $x\beta y$. This means that $y \in P_\sigma(x)$.

(2 $\implies$ 3) By Lemma 2, if X is a non-empty subset of X , then M is a β -part of H_B if and only if it is a union of equivalence classes modulo β^* . As we know, every equivalence class modulo β^* is a β -part of H_B .

(3 $\implies$ 1) Now, we want to show that β is transitive. For this purpose, let $x\beta y$ and $y\beta z$, by the definition of β there exist $m, n \in \mathbb{N}$ and $(x_1, \dots, x_n) \in H^n$, $(y_1, \dots, y_m) \in H^m$ such that $\{x, y\} \subseteq U_{(x_1, \dots, x_n)}$ and $\{y, z\} \subseteq U_{(y_1, \dots, y_m)}$. Since $P_\sigma(x)$ is a β -part of H , we have

$$x \in U_{(x_1, \dots, x_n)} \cap P_\sigma(x) \quad \text{so} \quad y \in U_{(x_1, \dots, x_n)} \subseteq P_\sigma(x).$$

Thus $y \in P_\sigma(x)$ and $y \in U_{(y_1, \dots, y_m)}$ (i.e., $P_\sigma(x) \cap U_{(y_1, \dots, y_m)} \neq \emptyset$) since $P_\sigma(x)$ is a β -part of H , it implies that $z \in U_{(y_1, \dots, y_m)} \subseteq P_\sigma(x)$ so $z \in P_\sigma(x)$ and this means that there exist $n \in \mathbb{N}$ such that $z \in P_n(x)$. Therefore $x\beta z$ as well as β is transitive. $\square$

Conclusion

In this paper, we have studied structural properties of hyper *BCI*-algebras with respect to a naturally defined relation β . By considering families of hyper *BCI*-algebras, we provided several illustrative examples that clarify the behavior of the relation and its associated constructions. A central focus of this work was the investigation of the transitive closure β^* , which plays a crucial role in the structural analysis of these algebraic systems.

We showed that β^* is indeed the fundamental equivalence relation in the sense that it is the smallest equivalence relation on a hyper *BCI*-algebra whose corresponding quotient structure yields a classical *BCI*-algebra. This result highlights the importance of β^* in bridging hyper structures with their classical counterparts. Moreover, we demonstrated that β^* provides a natural framework for understanding the internal structure of hyper *BCI*-algebras.

Finally, we established several equivalent conditions for the transitivity of the relation β , offering additional characterizations of these algebraic structures. These results not only extend previous findings in the literature but also provide a foundation for further investigations in the theory of hyper *BCI*-algebras and related hyperstructures.

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Author contributions

All authors contributed equally to this work.

Conflict of interest

The authors declare no conflict of interest.

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