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Boundedness of the pseudo-differential operators generated by 1D-Dunkl operator

B. Bekbolat[1,2,*], N. Tokmagambetov[2]

[1]*SDU University, Kaskelen, Kazakhstan;*
[2]*Institute of Mathematics and Mathematical Modeling, Almaty, Kazakhstan*
(E-mail: bayan.bekbolat@sdu.edu.kz, tokmagambetov@math.kz)

This article is devoted to the study of pseudo-differential operators generated by Dunkl operators, focusing primarily on their boundedness properties. We establish that, under a set of suitable assumptions on the symbols and the underlying function spaces, these operators are bounded on specific Banach spaces. In addition, we define the composition of pseudo-differential operators generated by Dunkl operators and rigorously prove that this composition also inherits boundedness properties under appropriate conditions. The analysis is carried out using techniques based on the Dunkl transform, which provides a powerful tool for handling operators associated with reflection groups and allows for the derivation of precise estimates. Beyond the theoretical development, we illustrate an application of the obtained results, demonstrating how these boundedness properties can be employed to address complex problems in mathematical physics and harmonic analysis. Overall, the work contributes to a deeper understanding of Dunkl analysis and the structure of pseudo-differential operators in this context. The results presented not only consolidate existing knowledge but also open new perspectives for further investigations in the field, providing a solid foundation for future research on Dunkl operators and their applications in both theoretical and applied analysis.

Keywords: Dunkl analysis, Dunkl operator, Dunkl kernel, Dunkl transform, inverse Dunkl transform, pseudo-differential operators, composition of pseudo-differential operators, boundedness results.

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Introduction

Pseudo-differential operators T_a (Definition 3), generated by the Dunkl operator were introduced by A. Dachraoui in 2001 in [1]. In his paper, the author defined two classes of symbols, S_0^m and S^m for $m \in \mathbb{R}$, with $S^m \subset S_0^m$, and introduced Sobolev-type spaces $W_\alpha^{s,p}(\mathbb{R}, d\mu_\alpha)$, where $s \in \mathbb{R}$, $p \in [1, +\infty]$, and $\alpha \geq -1/2$ (definitions are provided below). He proved that the pseudo-differential operator T_a, generated by the Dunkl operator with symbol $a \in S^m$, is continuous from $W_\alpha^{\frac{m}{2},1}(\mathbb{R}, d\mu_\alpha)$ to $W_\alpha^{0,\infty}(\mathbb{R}, d\mu_\alpha)$, and from $W_\alpha^{\frac{m}{2},p}(\mathbb{R}, d\mu_\alpha)$ to $W_\alpha^{0,p}(\mathbb{R}, d\mu_\alpha)$ for $p \geq 1$.

Definition 1. Let m be a real number. The function $a : \mathbb{R} \times \mathbb{C} \to \mathbb{C}$ is called a symbol in the class S_0^m, if it satisfies

- for a fixed x in $\mathbb{R}$, the function $\lambda \mapsto a(x, \lambda)$ is a smooth function on $\mathbb{R}$;
- for a fixed λ in $\mathbb{R}$, the function $x \mapsto a(x, \lambda)$ is a smooth function on $\mathbb{R}$;
- for all $k, n \in \mathbb{N}$, there exists $C_{k,n,m} > 0$, such that

$$\left| \frac{\partial^k}{\partial x^k} \frac{\partial^n}{\partial \lambda^n} a(x, \lambda) \right| \leq C_{k,n,m} (1 + |\lambda|^2)^{\frac{m-n}{2}},$$

for all $x \in \mathbb{R}$ and $\lambda \in \mathbb{R}$.

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[*] Corresponding author. *E-mail: bekbolat@math.kz*

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Definition 2. Let m be a real number. The function $a : \mathbb{R} \times \mathbb{C} \to \mathbb{C}$ is called a symbol in the class S^m, if it satisfies

- for a fixed x in $\mathbb{R}$, the function $\lambda \mapsto a(x, \lambda)$ is a smooth function on $\mathbb{R}$;
- for a fixed λ in $\mathbb{R}$, the function $x \mapsto a(x, \lambda)$ is a smooth function on $\mathbb{R}$;
- for all $k, \ell, n \in \mathbb{N}$, there exists $C_{k,\ell,n,m} > 0$, such that

$$\left|(1 + |x|^2)^{\ell} \frac{\partial^k}{\partial x^k} \frac{\partial^n}{\partial \lambda^n} a(x, \lambda)\right| \leq C_{k,\ell,n,m}(1 + |\lambda|^2)^{\frac{m-n}{2}},$$

for all $x \in \mathbb{R}$ and $\lambda \in \mathbb{R}$.

Definition 3. Let $a \in S_0^m$ and $\alpha \geq -1/2$. The pseudo-differential operator associated with a symbol a is defined on $\mathcal{S}(\mathbb{R})$ by

$$T_a f(x) = \int_{\mathbb{R}} E_\alpha(x, \lambda) a(x, \lambda) \mathcal{F}_\alpha[f](\lambda) d\mu_\alpha(\lambda),$$

where E_α is the Dunkl kernel defined by

$$E_\alpha(x, \lambda) = j_\alpha(x\lambda) + i \frac{x\lambda}{2(\alpha + 1)} j_{\alpha+1}(x\lambda), \tag{1}$$

j_α is the normalized Bessel function of the first kind, $\mathcal{F}_\alpha[f]$ is the Dunkl transform given by

$$\mathcal{F}_\alpha[f](\lambda) = \int_{\mathbb{R}} E_\alpha(-x, \lambda) f(x) d\mu_\alpha(x), \tag{2}$$

and

$$d\mu_\alpha(x) = \frac{|x|^{2\alpha+1}}{2^{\alpha+1}\Gamma(\alpha + 1)} dx, \tag{3}$$

Γ is a Gamma function.

Definition 4. The space $W_\alpha^{s,p}(\mathbb{R}, d\mu_\alpha)$, where s is real number, and $1 \leq p \leq +\infty$, is defined as the closure of the space of C^∞-functions on $\mathbb{R}$ with compact supports, with respect to the norms

$$\|f\|_{W_\alpha^{s,p}} := \|(1 + \lambda^2)^{s/2} \mathcal{F}_\alpha[f]\|_{p,\alpha}, \quad \text{if} \quad 1 \leq p < +\infty,$$

and

$$\|f\|_{W_\alpha^{s,\infty}} := \sup_{\lambda \in \mathbb{R}} (1 + \lambda^2)^{s/2} |\mathcal{F}_\alpha[f](\lambda)| \quad \text{if} \quad p = +\infty,$$

where

$$\|f\|_{p,\alpha} = \sqrt[p]{\int_{\mathbb{R}} |f(x)|^p d\mu_\alpha(x)}.$$

Then L^2 and L^p-boundedness of the pseudo-differential operators T_a associated with the Dunkl operators were studied by the authors C. Abdelkefi, B. Amri, and M. Sifi [2] for classes of symbols $S_{1,0}^0$, or simply S^0, which contains symbols with property

$$\left|\partial_\lambda^n \partial_x^k a(x, \lambda)\right| \leq \frac{C_{k,n}}{(1 + |\lambda|)^n},$$

for all $k, n \in \mathbb{N}$ and $x, \lambda \in \mathbb{R}$.

After, B. Amri, S. Mustapha, and M. Sifi [3] have extended L^2-theorem of Calderón–Vaillancourt to the pseudo-differential operator T_a associated with the Dunkl operator on $\mathbb{R}$.

Theorem 1 (Calderón–Vaillancourt). Assume that $0 \leq \rho < 1$ and $a \in S^0_{\rho,\rho}$, which is $a \in C^\infty(\mathbb{R} \times \mathbb{R})$ and satisfies

$$\left|\partial_x^k \partial_\lambda^n a(x,\lambda)\right| \leq C_{n,k}(1+|\lambda|)^{\rho(k-n)},$$

for all $n, k \in \mathbb{N}$ and all $x, \lambda \in \mathbb{R}$. Then T_a can be extended to a bounded operator on $L^2(\mathbb{R}, d\mu_\alpha)$.

In [3], the L^p-boundedness of the operator T_a with symbols in $S^0_{1,\delta}$, $0 \leq \delta < 1$, was established. A symbol a is said to belong to the class $S^0_{1,\delta}$, $0 \leq \delta < 1$, if $a \in C^\infty(\mathbb{R} \times \mathbb{R})$ and satisfies

$$\left|\partial_x^k \partial_\lambda^n a(x,\lambda)\right| \leq \frac{C_{n,k}}{(1+|\lambda|)^{n-\delta k}},$$

for all $n, k \in \mathbb{N}$ and all $x, \lambda \in \mathbb{R}$.

Theorem 2. Let $a \in S^0_{1,\delta}$, $0 \leq \delta < 1$. Then T_a can be extended to a bounded operator from $L^p(\mathbb{R}, d\mu_\alpha)$ into itself for all $1 < p < +\infty$.

In this paper, we establish several boundedness results for pseudo-differential operators generated by the Dunkl operators on the space $L(\mathbb{R}, d\mu_\alpha)$ (Definition 6) under certain assumptions. The techniques used in this paper are adapted from [4] and [5].

For a comprehensive overview of recent developments in this area, the reader is referred to the work [6].

1 Preliminaries

In this section, we recall some basic definitions from Dunkl analysis. The Dunkl operator

$$D_\alpha : C^1(\mathbb{R}) \to C(\mathbb{R}), \quad \alpha \geq -\frac{1}{2},$$

associated with the reflection group $\mathbb{Z}_2$ on $\mathbb{R}$, is defined by

$$D_\alpha f(x) := \frac{d}{dx} f(x) + \left(\alpha + \frac{1}{2}\right) \frac{f(x) - f(-x)}{x}.$$

Following the definition of the Dunkl operator, we note that this operator was firstly introduced by C.F. Dunkl [7].

Note that if $\alpha = -1/2$, then the Dunkl operator D_α is a first order differential operator and operator is well defined on other important function spaces, some of them listed below.

Lemma 1. [8, Lemma 2.2, p. 6] If $f \in C^m(\mathbb{R})$ with $m \geq 1$, then we have $D_\alpha f \in C^{m-1}(\mathbb{R})$.

Lemma 2. [9, Proposition 3.4, p. 28] The Dunkl operators leaves invariant

$$C^\infty(\mathbb{R}), \quad C_c^\infty(\mathbb{R}) \quad \text{and} \quad \mathcal{S}(\mathbb{R}).$$

As we frequently work with the Schwartz space, let us recall its definition.

Definition 5 (Schwartz space $\mathcal{S}(\mathbb{R})$). The Schwartz space $\mathcal{S}(\mathbb{R})$ is the topological vector space of functions $f : \mathbb{R} \longrightarrow \mathbb{R}$ such that $f \in C^\infty(\mathbb{R})$ and

$$x^k \frac{d^n}{dx^n} f(x) \to 0 \quad \text{as} \quad |x| \to \infty$$

for all $n, k \in \mathbb{N}$.

Let $\alpha \geq -1/2$ and $\lambda \in \mathbb{R}$. The equation

$$D_\alpha f(x) = i\lambda f(x)$$

with initial condition $f(0) = 1$ has a unique solution defined by (1). In the literature the function $E_\alpha(x, \lambda)$ is called the Dunkl kernel. The Dunkl kernel has following properties:

- for all $\lambda \in \mathbb{C}$, the function $x \longmapsto E_\alpha(x, \lambda)$ is a C^∞-function on $\mathbb{R}$;
- for all $x \in \mathbb{R}$, the function $\lambda \longmapsto E_\alpha(x, \lambda)$ is an entire function on $\mathbb{C}$;
- $E_\alpha(x, \lambda) = E_\alpha(\lambda, x)$;
- $E_\alpha(\xi x, \lambda) = E_\alpha(x, \xi\lambda), \quad \xi \in \mathbb{C}$;
- $\overline{E_\alpha(x, \lambda)} = E_\alpha(-x, \overline{\lambda}) = E_\alpha(x, -\overline{\lambda})$;
- $|E_\alpha(x, \lambda)| \leq 1, \quad \lambda \in \mathbb{R}$.

Note that the Dunkl kernel is the exponential function $\exp(ix\lambda)$ when $\alpha = -1/2$. The Dunkl kernel leads to the Dunkl transform. Before introducing the Dunkl transform, we need to consider the L^p space with the $d\mu_\alpha$ measure (defined below).

The space $L^p(\mathbb{R}, d\mu_\alpha)$, $1 \leq p \leq +\infty$, is the space of measurable functions f on $\mathbb{R}$, for which norms defined as

$$\|f\|_{p,\alpha} = \sqrt[p]{\int_{\mathbb{R}} |f(x)|^p d\mu_\alpha(x)} < +\infty, \quad \text{if} \quad 1 \leq p < +\infty,$$

and

$$\|f\|_\infty = \operatorname*{ess\,sup}_{x \in \mathbb{R}} |f(x)| < +\infty, \quad \text{if} \quad p = +\infty,$$

where $d\mu_\alpha$ is defined by (3).

Using property of the Dunkl kernel, we obtain

$$\|\mathcal{F}_\alpha[f]\|_\infty \leq \|f\|_{1,\alpha},$$

where the Dunkl transform $\mathcal{F}_\alpha$ is defined by (2).

The inverse Dunkl transform is defined by

$$\mathcal{F}_\alpha^{-1}[f](\lambda) = \mathcal{F}_\alpha[f](-\lambda) = \int_{\mathbb{R}} E_\alpha(x, \lambda) f(x) d\mu_\alpha(x), \quad \lambda \in \mathbb{R}.$$

If $\alpha = -1/2$, then we obtain the classical Fourier transform and inverse Fourier transform. The Dunkl transform has following important properties:

Theorem 3. (1) (Plancherel theorem) [10, Theorem 4.26, p. 160] The Dunkl transform has a unique extension to an isometric isomorphism of $L^2(\mathbb{R}, d\mu_\alpha)$, i.e.,

$$\|\mathcal{F}_\alpha[f]\|_{2,\alpha} = \|f\|_{2,\alpha},$$

for all $f \in L^2(\mathbb{R}, d\mu_\alpha)$.

(2) [10, Corollary 4.22, p. 159] The Dunkl transform is a homeomorphism of $\mathcal{S}(\mathbb{R})$.

(3) (Inverse Dunkl transform) [10, Theorem 4.20, p. 159] Let $f \in L^1(\mathbb{R}, d\mu_\alpha)$ and $\mathcal{F}_\alpha[f] \in L^1(\mathbb{R}, d\mu_\alpha)$, then we have

$$f(x) = \mathcal{F}_\alpha^{-1}[\mathcal{F}_\alpha[f]](x) \quad \text{a.e.}$$

We have the following product formula for the function $j_\alpha(x\lambda)$ with $\alpha > -\frac{1}{2}$ and parameter $\lambda \in \mathbb{C}$ [11, Formula 1.II.23, p. 13]:

$$j_\alpha(x\lambda) j_\alpha(y\lambda) = \int_0^{+\infty} j_\alpha(z\lambda) k_\alpha(x, y, z) z^{2\alpha+1} dz,$$

for $x, y > 0$, where

$$k_\alpha(x,y,z) = 2^{2\alpha-1}\frac{\Gamma(\alpha+1)}{\Gamma(\alpha+\frac{1}{2})\Gamma(\frac{1}{2})}\frac{\Delta(x,y,z)^{2\alpha-1}}{(xyz)^{2\alpha}} \cdot 1_{[|x-y|,x+y]}(z).$$

Here 1_A is the indicator function of A and

$$\Delta(x,y,z) := \frac{1}{4}\sqrt{(x+y+z)(x+y-z)(x-y+z)(y+z-x)}$$

denotes the area of the triangle with sides $x, y, z > 0$. The function $k_\alpha(x,y,z)$ satisfies the following properties [11, p. 13-14]:

- for all $z > 0$, we have $k_\alpha(x,y,z) \geq 0$;
- for $x, y > 0$, we have
$$\int_0^{+\infty} k_\alpha(x,y,z) z^{2\alpha+1} dz = 1;$$
- for all $x, y, z > 0$, we have
$$k_\alpha(x,y,z) = k_\alpha(y,x,z) \quad \text{and} \quad k_\alpha(x,y,z) = k_\alpha(x,z,y).$$

For our convenience, we fix some notations. For all $x, y, z \in \mathbb{R}$, we put

$$b_{x,y,z} := \begin{cases} \frac{x^2+y^2-z^2}{2xy}, & \text{if} \quad x, y \neq 0, \\ 0, & \text{otherwise,} \end{cases}$$

and

$$\rho(x,y,z) := \frac{1}{2}(1 - b_{x,y,z} + b_{z,x,y} + b_{z,y,x}).$$

Theorem 4. [12, Theorem 2.4, p. 5] (1) Let $\alpha > -\frac{1}{2}$ and $\lambda \in \mathbb{C}$. The Dunkl kernel satisfies the following product formula:

$$E_\alpha(x,\lambda)E_\alpha(y,\lambda) = \int_{\mathbb{R}} E_\alpha(z,\lambda) d\nu_{x,y}(z)$$

for $x, y \in \mathbb{R}$, where

$$d\nu_{x,y}(z) := \begin{cases} W_\alpha(x,y,z)|z|^{2\alpha+1}\, dz, & \text{if} \quad x, y \neq 0, \\ d\delta_x(z), & \text{if} \quad y = 0, \\ d\delta_y(z), & \text{if} \quad x = 0, \end{cases}$$

with kernel

$$W_\alpha(x,y,z) = k_\alpha(|x|,|y|,|z|)\rho(x,y,z).$$

(2) The measures $\nu_{x,y}$ have the following properties:

- $\operatorname{supp}\nu_{x,y} = [-|x|-|y|, -||x|-|y||] \cup [||x|-|y||, |x|+|y|]$ for $x, y \neq 0$;
- $\|\nu_{x,y}\| := \int_{\mathbb{R}} W_\alpha(x,y,z)|z|^{2\alpha+1} dz \leq 4$ for all $x, y \in \mathbb{R}$.

Remark 1. In Theorem 4, δ_x is the Dirac measure. So, we have

- if $y = 0$, then
$$E_\alpha(x,\lambda) = E_\alpha(x,\lambda)E_\alpha(0,\lambda) = \int_{\mathbb{R}} E_\alpha(z,\lambda) d\delta_x(z) = E_\alpha(x,\lambda),$$
- if $x = 0$, then
$$E_\alpha(y,\lambda) = E_\alpha(0,\lambda)E_\alpha(y,\lambda) = \int_{\mathbb{R}} E_\alpha(z,\lambda) d\delta_y(z) = E_\alpha(y,\lambda).$$

Remark 2. Let $x, y \neq 0$. Then from

$$E_\alpha(x,\lambda)E_\alpha(y,\lambda) = \int_{\mathbb{R}} E_\alpha(z,\lambda)W_\alpha(x,y,z)|z|^{2\alpha+1}dz$$
$$= 2^{\alpha+1}\Gamma(\alpha+1)\int_{\mathbb{R}} E_\alpha(z,\lambda)W_\alpha(x,y,z)d\mu_\alpha(z),$$

we obtain

$$W_\alpha(x,y,z) = \frac{1}{2^{\alpha+1}\Gamma(\alpha+1)}\int_{\mathbb{R}} E_\alpha(-z,\lambda)E_\alpha(x,\lambda)E_\alpha(y,\lambda)d\mu_\alpha(\lambda). \tag{4}$$

Lemma 3. Let $x, y, z \in \mathbb{R}$. Then

$$W_\alpha(x,-y,z) = W_\alpha(x,-z,y).$$

Furthermore, we have

$$W_\alpha(x,-y,z)|z|^{2\alpha+1}dzd\mu_\alpha(y) = W_\alpha(x,-z,y)|y|^{2\alpha+1}dyd\mu_\alpha(z).$$

Proof. For any $x, y, z \in \mathbb{R}$ a short calculation gives us the following equalities:

$$b_{x,-y,z} = \frac{x^2+(-y)^2-z^2}{2x(-y)} = -\frac{x^2+y^2-z^2}{2xy},$$
$$b_{z,x,-y} = \frac{z^2+x^2-(-y)^2}{2zx} = \frac{z^2+x^2-y^2}{2zx},$$
$$b_{z,-y,x} = \frac{z^2+(-y)^2-x^2}{2z(-y)} = -\frac{z^2+y^2-x^2}{2zy},$$

and

$$\begin{aligned}\rho(x,-y,z) &= \frac{1}{2}(1-b_{x,-y,z}+b_{z,x,-y}+b_{z,-y,x})\\ &= \frac{1}{2}\left(1+\frac{x^2+y^2-z^2}{2xy}+\frac{z^2+x^2-y^2}{2zx}-\frac{z^2+y^2-x^2}{2zy}\right)\\ &= \frac{1}{2}\left(1+\frac{x^2+z^2-y^2}{2zx}+\frac{y^2+x^2-z^2}{2xy}-\frac{y^2+z^2-x^2}{2zy}\right)\\ &= \frac{1}{2}(1-b_{x,-z,y}+b_{y,x,-z}+b_{y,-z,x})\\ &= \rho(x,-z,y).\end{aligned}$$

Then using property of the function $k_\alpha(x,y,z)$, we obtain

$$\begin{aligned}W_\alpha(x,-y,z) &= k_\alpha(|x|,|-y|,|z|)\rho(x,-y,z)\\ &= k_\alpha(|x|,|-z|,|y|)\rho(x,-z,y)\\ &= W_\alpha(x,-z,y).\end{aligned}$$

Thus, we have

$$W_\alpha(x,-y,z)|z|^{2\alpha+1}dzd\mu_\alpha(y) = W_\alpha(x,-z,y)\frac{|y|^{2\alpha+1}}{2^{\alpha+1}\Gamma(\alpha+1)}dy|z|^{2\alpha+1}dz$$
$$= W_\alpha(x,-z,y)|y|^{2\alpha+1}dyd\mu_\alpha(z).$$

□

For all $x, y \in \mathbb{R}$ and f, continuous function on $\mathbb{R}$, we define

$$\tau_x f(y) := \int_{\mathbb{R}} f(z) d\nu_{x,y}(z).$$

The operators $\tau_x, x \in \mathbb{R}$ are called Dunkl translation operators on real line.

Proposition 1. [13, Proposition 2, p. 20] The operators $\tau_x, x \in \mathbb{R}$ have the following properties:

- for all $x \in \mathbb{R}$ and $f \in L^p(\mathbb{R}, d\mu_\alpha)$, $p \in [1, +\infty]$, we have
$$\|\tau_x f\|_{p,\alpha} \leq 4\|f\|_{p,\alpha};$$
- for all $\lambda, x \in \mathbb{R}$ and $f \in L^1(\mathbb{R}, d\mu_\alpha)$, we obtain
$$\mathcal{F}_\alpha[\tau_x f](\lambda) = E_\alpha(x, \lambda)\mathcal{F}_\alpha[f](\lambda).$$

For two continuous functions f and g on $\mathbb{R}$ with compact supports, we define the convolution product $*_\alpha$ by

$$(f *_\alpha g)(x) := \int_{\mathbb{R}} \tau_x f(-y) g(y) d\mu_\alpha(y), \quad x \in \mathbb{R},$$

where $\tau_x, x \in \mathbb{R}$ is the Dunkl translation operators on $\mathbb{R}$.

Remark 3. Note that $*_{-\frac{1}{2}}$ is the standard convolution $*$.

Proposition 2. [13, Proposition 3, p. 21] (i) Let $p, q, r \in [1, \infty]$ and satisfy $\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}$. Then the map $(f, g) \mapsto f *_\alpha g$ can be extended to a continuous map from $L^p(\mathbb{R}, d\mu_\alpha) \times L^q(\mathbb{R}, d\mu_\alpha)$ to $L^r(\mathbb{R}, d\mu_\alpha)$, and

$$\|f *_\alpha g\|_{r,\alpha} \leq 4\|f\|_{p,\alpha}\|g\|_{q,\alpha}.$$

(ii) For any $f \in L^1(\mathbb{R}, d\mu_\alpha)$ and $g \in L^2(\mathbb{R}, d\mu_\alpha)$, we have

$$\mathcal{F}_\alpha[f *_\alpha g](\lambda) = \mathcal{F}_\alpha[f](\lambda)\mathcal{F}_\alpha[g](\lambda), \quad \lambda \in \mathbb{R}.$$

Now we define our $L(\mathbb{R}, d\mu_\alpha)$ space, as following:

Definition 6. Let us define the space $L(\mathbb{R}, d\mu_\alpha)$, as following:

$$L(\mathbb{R}, d\mu_\alpha) := \{f \in L^1(\mathbb{R}, d\mu_\alpha) \, : \, \mathcal{F}_\alpha[f] \in L^1(\mathbb{R}, d\mu_\alpha)\}$$

with the norm

$$\|f\|_L := \|\mathcal{F}_\alpha[f]\|_{1,\alpha} = \int_{\mathbb{R}} |\mathcal{F}_\alpha[f](\lambda)| \, d\mu_\alpha(\lambda).$$

Remark 4. The space $L(\mathbb{R}, d\mu_\alpha)$ is a subspace of $L^1(\mathbb{R}, d\mu_\alpha)$ and the norm $\|\cdot\|_L$ is defined equivalently via the Dunkl transform. Additionally, the space $(L(\mathbb{R}, d\mu_\alpha), \|\cdot\|_L)$ is Banach space.

2 *Main results*

In this section, we obtain some boundedness results for pseudo-differential operators and the composition of pseudo-differential operators generated by the Dunkl operator on the space $L(\mathbb{R}, d\mu_\alpha)$.

Assumption 1. We assume the symbol $a \in S^m_{\rho,\delta}(\mathbb{R} \times \mathbb{R})$ is defined as:

$$a(x, \lambda) = \int_{\mathbb{R}} E_\alpha(x, \xi) V(\xi, \lambda) d\mu_\alpha(\xi),$$

where $V(\xi, \lambda)$ is a complex-valued measurable function on $\mathbb{R} \times \mathbb{R}$, such that

$$|V(\xi, \lambda)| \leq K(\xi),$$

for all $\xi, \lambda \in \mathbb{R}$, and $K \in L^1(\mathbb{R}, d\mu_\alpha)$ is a continuous function.

Remark 5. The integral (1) exists, because

$$|a(x,\lambda)| \leq \int_{\mathbb{R}} |E_\alpha(x,\xi)V(\xi,\lambda)| d\mu_\alpha(\xi) \leq \int_{\mathbb{R}} |K(\xi)| d\mu_\alpha(\xi) < +\infty.$$

Theorem 5. Let $f \in \mathcal{S}(\mathbb{R})$. Then the pseudo-differential operator

$$T_a f(x) = \int_{\mathbb{R}} E_\alpha(x,\lambda) a(x,\lambda) \mathcal{F}_\alpha[f](\lambda) d\mu_\alpha(\lambda)$$

is a bounded linear operator under Assumption 1 on $L(\mathbb{R}, d\mu_\alpha)$, i.e.,

$$\|T_a f\|_L \leq 4\|K\|_{1,\alpha}\|f\|_L.$$

Remark 6. In the statements of theorems, the operators are formulated as bounded operators on the space $L(\mathbb{R}, d\mu_\alpha)$. However, in the proofs, the corresponding estimates are first established for functions from $\mathcal{S}(\mathbb{R})$. This is sufficient, since the Schwartz space $\mathcal{S}(\mathbb{R})$ is dense in $L(\mathbb{R}, d\mu_\alpha)$, and the operators under consideration are continuous with respect to the $\|\cdot\|_L$ norm. Therefore, by a standard density argument, each operator admits a unique continuous extension from $\mathcal{S}(\mathbb{R})$ to the whole space $L(\mathbb{R}, d\mu_\alpha)$, and all estimates obtained for Schwartz functions remain valid for arbitrary functions in $L(\mathbb{R}, d\mu_\alpha)$.

Proof. Let $f \in \mathcal{S}(\mathbb{R})$. Then by the definition of the pseudo-differential operator we obtain

$$\begin{aligned}
&\int_{\mathbb{R}} E_\alpha(x,\lambda) a(x,\lambda) \mathcal{F}_\alpha[f](\lambda) d\mu_\alpha(\lambda) \\
&= \int_{\mathbb{R}} E_\alpha(x,\lambda) \left(\int_{\mathbb{R}} E_\alpha(x,\xi) V(\xi,\lambda) d\mu_\alpha(\xi) \right) \mathcal{F}_\alpha[f](\lambda) d\mu_\alpha(\lambda) \\
&= \int_{\mathbb{R}} \int_{\mathbb{R}} E_\alpha(x,\lambda) E_\alpha(x,\xi) V(\xi,\lambda) \mathcal{F}_\alpha[f](\lambda) d\mu_\alpha(\xi) d\mu_\alpha(\lambda) \\
&= \int_{\mathbb{R}} \int_{\mathbb{R}} \left(\int_{\mathbb{R}} E_\alpha(x,\eta) d\nu_{\lambda,\xi}(\eta) \right) V(\xi,\lambda) \mathcal{F}_\alpha[f](\lambda) d\mu_\alpha(\xi) d\mu_\alpha(\lambda) \\
&= \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}} E_\alpha(x,\eta) V(\xi,\lambda) \mathcal{F}_\alpha[f](\lambda) W_\alpha(\lambda,\xi,\eta) |\eta|^{2\alpha+1} d\eta d\mu_\alpha(\xi) d\mu_\alpha(\lambda) \\
&= \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}} E_\alpha(x,\eta) V(\xi,\lambda) \mathcal{F}_\alpha[f](\lambda) W_\alpha(-\lambda,\eta,\xi) |\xi|^{2\alpha+1} d\xi d\mu_\alpha(\eta) d\mu_\alpha(\lambda) \\
&= \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}} E_\alpha(x,\eta) V(\xi,\lambda) \mathcal{F}_\alpha[f](\lambda) d\nu_{-\lambda,\eta}(\xi) d\mu_\alpha(\eta) d\mu_\alpha(\lambda) \\
&= \int_{\mathbb{R}} E_\alpha(x,\eta) \left(\int_{\mathbb{R}} \int_{\mathbb{R}} V(\xi,\lambda) \mathcal{F}_\alpha[f](\lambda) d\nu_{-\lambda,\eta}(\xi) d\mu_\alpha(\lambda) \right) d\mu_\alpha(\eta),
\end{aligned}$$

using above assumption and Fubini's theorem. After applying Dunkl transform $\mathcal{F}_\alpha$ to the both sides of the equation, we have

$$\begin{aligned}
\mathcal{F}_\alpha[T_a f](\eta) &= \int_{\mathbb{R}} \int_{\mathbb{R}} V(\xi,\lambda) \mathcal{F}_\alpha[f](\lambda) d\nu_{-\lambda,\eta}(\xi) d\mu_\alpha(\lambda) \\
&= \int_{\mathbb{R}} \int_{\mathbb{R}} V(\xi,\lambda) \mathcal{F}_\alpha[f](\lambda) W_\alpha(-\lambda,\eta,\xi) |\xi|^{2\alpha+1} d\xi d\mu_\alpha(\lambda).
\end{aligned}$$

Hence, taking integral from both sides, we obtain

$$\begin{aligned}
&\int_{\mathbb{R}} |\mathcal{F}_\alpha[T_a f](\eta)| d\mu_\alpha(\eta)\\
&\leq \int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}} |V(\xi,\lambda)\mathcal{F}_\alpha[f](\lambda)W_\alpha(-\lambda,\eta,\xi)||\xi|^{2\alpha+1} d\xi d\mu_\alpha(\lambda) d\mu_\alpha(\eta)\\
&\leq \int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}} K(\xi)|\mathcal{F}_\alpha[f](\lambda)W_\alpha(-\lambda,\eta,\xi)||\xi|^{2\alpha+1} d\xi d\mu_\alpha(\lambda) d\mu_\alpha(\eta)\\
\\
&\quad = \int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}} K(\xi)|\mathcal{F}_\alpha[f](\lambda)W_\alpha(\lambda,\xi,\eta)||\eta|^{2\alpha+1} d\eta d\mu_\alpha(\xi) d\mu_\alpha(\lambda)\\
&\quad \leq 4\int_{\mathbb{R}}\int_{\mathbb{R}} K(\xi)|\mathcal{F}_\alpha[f](\lambda)| d\mu_\alpha(\xi) d\mu_\alpha(\lambda)\\
&\quad \leq 4\|K\|_{1,\alpha}\int_{\mathbb{R}} |\mathcal{F}_\alpha[f](\lambda)| d\mu_\alpha(\lambda).
\end{aligned}$$

This completes proof of the theorem. □

Let $f, g \in \mathcal{S}(\mathbb{R})$. The composition of two pseudo-differential operators

$$T_a g(x) = \int_{\mathbb{R}}\int_{\mathbb{R}} E_\alpha(x,\lambda)E_\alpha(-y,\lambda)a(x,\lambda)g(y) d\mu_\alpha(y) d\mu_\alpha(\lambda)$$

and

$$T_b f(y) = \int_{\mathbb{R}}\int_{\mathbb{R}} E_\alpha(y,\xi)E_\alpha(-z,\xi)b(y,\xi)f(z) d\mu_\alpha(z) d\mu_\alpha(\xi),$$

with the symbols $a(x,\lambda)$ and $b(y,\xi)$ respectively, is

$$\begin{aligned}
T_a(T_b f)(x) &= \int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}} E_\alpha(x,\lambda)E_\alpha(-y,\lambda)a(x,\lambda)E_\alpha(y,\xi)E_\alpha(-z,\xi)b(y,\xi)f(z)\\
&\times d\mu_\alpha(z) d\mu_\alpha(\xi) d\mu_\alpha(y) d\mu_\alpha(\lambda)\\
&= \int_{\mathbb{R}}\int_{\mathbb{R}} E_\alpha(x,\xi)E_\alpha(-z,\xi)c(x,\xi)f(z) d\mu_\alpha(z) d\mu_\alpha(\xi)\\
&= \int_{\mathbb{R}} E_\alpha(x,\xi)c(x,\xi)\mathcal{F}_\alpha[f](\xi) d\mu_\alpha(\xi),
\end{aligned}$$

where

$$c(x,\xi) = \frac{1}{E_\alpha(x,\xi)}\int_{\mathbb{R}}\int_{\mathbb{R}} E_\alpha(x,\lambda)E_\alpha(-y,\lambda)E_\alpha(y,\xi)a(x,\lambda)b(y,\xi) d\mu_\alpha(y) d\mu_\alpha(\lambda).$$

Thus,

$$T_c f(x) = T_a(T_b f)(x) = \int_{\mathbb{R}} E_\alpha(x,\xi)c(x,\xi)\mathcal{F}_\alpha[f](\xi) d\mu_\alpha(\xi)$$

is a pseudo-differential operator with symbol

$$c(x,\xi) = \frac{1}{E_\alpha(x,\xi)}\int_{\mathbb{R}}\int_{\mathbb{R}} E_\alpha(x,\lambda)E_\alpha(-y,\lambda)E_\alpha(y,\xi)a(x,\lambda)b(y,\xi) d\mu_\alpha(y) d\mu_\alpha(\lambda).$$

Now, let us discuss the existence of such an integral under Assumption 1. Let us have

$$a(x,\lambda) = \int_{\mathbb{R}} E_\alpha(x,\eta)V_a(\eta,\lambda) d\mu_\alpha(\eta) \tag{5}$$

and

$$b(y,\xi)=\int_{\mathbb{R}} E_\alpha(y,\sigma)V_b(\sigma,\xi)d\mu_\alpha(\sigma), \tag{6}$$

where $V_a(\eta,\lambda)$ and $V_b(\sigma,\xi)$ are complex-valued measurable functions on $\mathbb{R}\times\mathbb{R}$, such that

$$|V_a(\eta,\lambda)|\le K_a(\eta) \quad \text{and} \quad |V_b(\sigma,\xi)|\le K_b(\sigma)$$

for all $\eta,\lambda,\sigma,\xi\in\mathbb{R}$, and $K_a, K_b\in L^1(\mathbb{R},d\mu_\alpha)$ are continuous functions. Then by using integral expressions (5) and (6) of $a(x,\lambda)$ and $b(y,\xi)$ respectively, we obtain

$$\begin{aligned}
c(x,\xi) &= \frac{1}{E_\alpha(x,\xi)}\int_{\mathbb{R}}\int_{\mathbb{R}} E_\alpha(x,\lambda)E_\alpha(-y,\lambda)E_\alpha(y,\xi)a(x,\lambda)b(y,\xi)d\mu_\alpha(y)d\mu_\alpha(\lambda)\\
&= \frac{1}{E_\alpha(x,\xi)}\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}} E_\alpha(x,\lambda)E_\alpha(-y,\lambda)E_\alpha(y,\xi)E_\alpha(x,\eta)E_\alpha(y,\sigma)\\
&\times V_a(\eta,\lambda)V_b(\sigma,\xi)d\mu_\alpha(\sigma)d\mu_\alpha(\eta)d\mu_\alpha(y)d\mu_\alpha(\lambda)\\
&= \frac{1}{E_\alpha(x,\xi)}\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}} E_\alpha(x,\lambda)\left(\int_{\mathbb{R}} E_\alpha(-y,\lambda)E_\alpha(y,\xi)E_\alpha(y,\sigma)d\mu_\alpha(y)\right)\\
&\times E_\alpha(x,\eta)V_a(\eta,\lambda)V_b(\sigma,\xi)d\mu_\alpha(\sigma)d\mu_\alpha(\eta)d\mu_\alpha(\lambda)\\
&= \frac{1}{E_\alpha(x,\xi)}\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}} E_\alpha(x,\lambda)W_\alpha(\xi,\sigma,\lambda)E_\alpha(x,\eta)V_a(\eta,\lambda)V_b(\sigma,\xi)\,d\mu_\alpha(\sigma)d\mu_\alpha(\eta)d\mu_\alpha(\lambda).
\end{aligned}$$

After taking absolute value from both sides of the equation, as following:

$$\begin{aligned}
|c(x,\xi)| &\le \int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}} |W_\alpha(\xi,\sigma,\lambda)V_a(\eta,\lambda)V_b(\sigma,\xi)|d\mu_\alpha(\sigma)d\mu_\alpha(\eta)d\mu_\alpha(\lambda)\\
&\le 4\int_{\mathbb{R}}\int_{\mathbb{R}} K_a(\eta)K_b(\sigma)d\mu_\alpha(\sigma)d\mu_\alpha(\eta)\\
&\le 4\|K_a\|_{1,\alpha}\|K_b\|_{1,\alpha},
\end{aligned}$$

we can see that the $c(x,\xi)$ is a bounded function.

Corollary 1. Let T_a and T_b are pseudo-differential operators with symbols a and b, respectively. Then under Assumption 1 their composition is a pseudo-differential operator $T_a\circ T_b$, which is continuous linear map on $\mathcal{S}(\mathbb{R})$.

Corollary 2. Let $f\in\mathcal{S}(\mathbb{R})$. Then under Assumption 1 the composition of pseudo-differential operators T_a and T_b is a bounded linear operator on $L(\mathbb{R},d\mu_\alpha)$, i.e.,

$$\|T_a(T_bf)\|_L\le\frac{16}{2^{\alpha+1}\Gamma(\alpha+1)}\|K_a\|_{1,\alpha}\|K_b\|_{1,\alpha}\|f\|_L.$$

Proof. Let $f\in\mathcal{S}(\mathbb{R})$. Then we have

$$\begin{aligned}
T_a(T_bf)(x) &= \int_{\mathbb{R}} E_\alpha(x,\lambda)a(x,\lambda)\mathcal{F}_\alpha[T_bf](\lambda)d\mu_\alpha(\lambda)\\
&= \int_{\mathbb{R}} E_\alpha(x,\lambda)\left(\int_{\mathbb{R}} E_\alpha(x,\xi)V_a(\xi,\lambda)d\mu_\alpha(\xi)\right)\mathcal{F}_\alpha[T_bf](\lambda)d\mu_\alpha(\lambda)
\end{aligned}$$

and

$$\begin{aligned}
&\mathcal{F}_\alpha[T_a(T_bf)](\eta)\\
&= \int_{\mathbb{R}} E_\alpha(-x,\eta)T_a(T_bf)(x)d\mu_\alpha(x)\\
&= \int_{\mathbb{R}} E_\alpha(-x,\eta)\left(\int_{\mathbb{R}} E_\alpha(x,\lambda)\left(\int_{\mathbb{R}} E_\alpha(x,\xi)V_a(\xi,\lambda)d\mu_\alpha(\xi)\right)\mathcal{F}_\alpha[T_bf](\lambda)d\mu_\alpha(\lambda)\right)d\mu_\alpha(x)
\end{aligned}$$

$$
\begin{aligned}
&= \int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}} E_\alpha(-x,\eta)E_\alpha(x,\lambda)E_\alpha(x,\xi)V_a(\xi,\lambda)\mathcal{F}_\alpha[T_bf](\lambda)d\mu_\alpha(\xi)d\mu_\alpha(\lambda)d\mu_\alpha(x)\\
&= \int_{\mathbb{R}}\int_{\mathbb{R}} \left(\int_{\mathbb{R}} E_\alpha(-x,\eta)E_\alpha(x,\lambda)E_\alpha(x,\xi)d\mu_\alpha(x)\right) V_a(\xi,\lambda)\mathcal{F}_\alpha[T_bf](\lambda)d\mu_\alpha(\xi)d\mu_\alpha(\lambda)\\
&= \frac{1}{2^{\alpha+1}\Gamma(\alpha+1)}\int_{\mathbb{R}}\int_{\mathbb{R}} W_\alpha(\lambda,\xi,\eta)V_a(\xi,\lambda)\mathcal{F}_\alpha[T_bf](\lambda)d\mu_\alpha(\xi)d\mu_\alpha(\lambda),
\end{aligned}
$$

where we have used (4). Then taking absolute value and integrating, we have

$$
\begin{aligned}
&\int_{\mathbb{R}} |\mathcal{F}_\alpha[T_a(T_bf)](\eta)|\,d\mu_\alpha(\eta)\\
&\qquad\le \frac{1}{2^{\alpha+1}\Gamma(\alpha+1)}\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}} |W_\alpha(\lambda,\xi,\eta)V_a(\xi,\lambda)\mathcal{F}_\alpha[T_bf](\lambda)|\,d\mu_\alpha(\xi)d\mu_\alpha(\lambda)d\mu_\alpha(\eta)
\end{aligned}
$$

$$
\begin{aligned}
&\le \frac{4}{2^{\alpha+1}\Gamma(\alpha+1)}\int_{\mathbb{R}}\int_{\mathbb{R}} K_a(\xi)\,|\mathcal{F}_\alpha[T_bf](\lambda)|\,d\mu_\alpha(\xi)d\mu_\alpha(\lambda)\\
&\qquad\le \frac{4}{2^{\alpha+1}\Gamma(\alpha+1)}\|K_a\|_{1,\alpha}\int_{\mathbb{R}} |\mathcal{F}_\alpha[T_bf](\lambda)|\,d\mu_\alpha(\lambda)\\
&\qquad\qquad\le \frac{16}{2^{\alpha+1}\Gamma(\alpha+1)}\|K_a\|_{1,\alpha}\|K_b\|_{1,\alpha}\int_{\mathbb{R}} |\mathcal{F}_\alpha[f](\lambda)|\,d\mu_\alpha(\lambda).
\end{aligned}
$$

□

Assumption 2. We assume the symbol $a \in S^m_{\rho,\delta}(\mathbb{R}\times\mathbb{R})$ is defined by

$$
a(x,\lambda) = \int_{\mathbb{R}} E_\alpha(x,\xi)V(\xi,\lambda)d\mu_\alpha(\xi)
$$

satisfies

$$
a(x,\lambda) = \int_{\mathbb{R}} E_\alpha(x,\xi)V_1(\xi)V_2(\lambda)d\mu_\alpha(\xi) = V_2(\lambda)\int_{\mathbb{R}} E_\alpha(x,\xi)V_1(\xi)d\mu_\alpha(\xi),
$$

where $V_1 \in L^1(\mathbb{R}, d\mu_\alpha)$ is a continuous function.

Theorem 6. Let $f \in \mathcal{S}(\mathbb{R})$. Then the pseudo-differential operator T_a with symbol $a(x,\lambda)$, which satisfies Assumption 2, has a representation

$$
T_af(x) = 2^{\alpha+1}\Gamma(\alpha+1)\mathcal{F}_\alpha^{-1}(V_1 *_\alpha V_2\mathcal{F}_\alpha[f])(x)
$$

and satisfies following inequality

$$
\|T_af\|_L \le 2^{\alpha+3}\Gamma(\alpha+1)\|V_1\|_{1,\alpha}\|V_2\mathcal{F}_\alpha[f]\|_{1,\alpha}.
$$

Proof. By using Assumption 2, we have

$$
\begin{aligned}
&\int_{\mathbb{R}} E_\alpha(x,\lambda)a(x,\lambda)\mathcal{F}_\alpha[f](\lambda)d\mu_\alpha(\lambda)\\
&= \int_{\mathbb{R}} E_\alpha(x,\lambda)\left(V_2(\lambda)\int_{\mathbb{R}} E_\alpha(x,\xi)V_1(\xi)d\mu_\alpha(\xi)\right)\mathcal{F}_\alpha[f](\lambda)d\mu_\alpha(\lambda)\\
&= \int_{\mathbb{R}}\int_{\mathbb{R}} E_\alpha(x,\lambda)E_\alpha(x,\xi)V_2(\lambda)V_1(\xi)\mathcal{F}_\alpha[f](\lambda)d\mu_\alpha(\xi)d\mu_\alpha(\lambda)
\end{aligned}
$$

$$= \int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}} E_\alpha(x,\eta)V_2(\lambda)V_1(\xi)\mathcal{F}_\alpha[f](\lambda)d\nu_{\lambda,\xi}(\eta)d\mu_\alpha(\xi)d\mu_\alpha(\lambda)$$
$$= \int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}} E_\alpha(x,\eta)V_2(\lambda)V_1(\xi)\mathcal{F}_\alpha[f](\lambda)W_\alpha(\lambda,\xi,\eta)|\eta|^{2\alpha+1}d\eta d\mu_\alpha(\xi)d\mu_\alpha(\lambda)$$
$$= \int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}} E_\alpha(x,\eta)V_2(\lambda)V_1(\xi)\mathcal{F}_\alpha[f](\lambda)W_\alpha(-\lambda,\eta,\xi)|\xi|^{2\alpha+1}d\xi d\mu_\alpha(\eta)d\mu_\alpha(\lambda)$$

$$= 2^{\alpha+1}\Gamma(\alpha+1)\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}} E_\alpha(x,\eta)V_2(\lambda)V_1(\xi)\mathcal{F}_\alpha[f](\lambda)d\nu_{-\lambda,\eta}(\xi)d\mu_\alpha(\eta)d\mu_\alpha(\lambda)$$
$$= 2^{\alpha+1}\Gamma(\alpha+1)\int_{\mathbb{R}} E_\alpha(x,\eta)\left(\int_{\mathbb{R}}\int_{\mathbb{R}} V_2(\lambda)V_1(\xi)\mathcal{F}_\alpha[f](\lambda)d\nu_{-\lambda,\eta}(\xi)d\mu_\alpha(\lambda)\right)d\mu_\alpha(\eta)$$
$$= 2^{\alpha+1}\Gamma(\alpha+1)\int_{\mathbb{R}} E_\alpha(x,\eta)\left(\int_{\mathbb{R}} \tau_\eta V_1(-\lambda)V_2(\lambda)\mathcal{F}_\alpha[f](\lambda)d\mu_\alpha(\lambda)\right)d\mu_\alpha(\eta)$$
$$= 2^{\alpha+1}\Gamma(\alpha+1)\int_{\mathbb{R}} E_\alpha(x,\eta)(V_1 *_\alpha V_2\mathcal{F}_\alpha[f])(\eta)d\mu_\alpha(\eta)$$
$$= 2^{\alpha+1}\Gamma(\alpha+1)\mathcal{F}_\alpha^{-1}(V_1 *_\alpha V_2\mathcal{F}_\alpha[f])(x).$$

Thus, applying the Dunkl transform, we obtain

$$\mathcal{F}_\alpha[T_a f](\eta) = 2^{\alpha+1}\Gamma(\alpha+1)(V_1 *_\alpha V_2\mathcal{F}_\alpha[f])(\eta). \tag{7}$$

By taking the integral of both sides of the above equation, we are able to calculate

$$\int_{\mathbb{R}} |\mathcal{F}_\alpha[T_a f](\eta)|\, d\mu_\alpha(\eta) = 2^{\alpha+1}\Gamma(\alpha+1)\int_{\mathbb{R}} |(V_1 *_\alpha V_2\mathcal{F}_\alpha[f])(\eta)|\, d\mu_\alpha(\eta)$$

and

$$\|\mathcal{F}_\alpha[T_a f]\|_{1,\alpha} = 2^{\alpha+1}\Gamma(\alpha+1)\|V_1 *_\alpha V_2\mathcal{F}_\alpha[f]\|_{1,\alpha} \le 2^{\alpha+3}\Gamma(\alpha+1)\|V_1\|_{1,\alpha}\|V_2\mathcal{F}_\alpha[f]\|_{1,\alpha},$$

where we have used the Proposition 2. Furthermore, by using the Definition of the Sobolev-type space, it can be written as

$$\|T_a f\|_L \le 2^{\alpha+3}\Gamma(\alpha+1)\|V_1\|_{1,\alpha}\|V_2\mathcal{F}_\alpha[f]\|_{1,\alpha}.$$

□

Assumption 3. We assume the symbol $a \in S^m_{\rho,\delta}(\mathbb{R}\times\mathbb{R})$ is defined by

$$a(x,\lambda) = \int_{\mathbb{R}} E_\alpha(x,\xi)V(\xi,\lambda)d\mu_\alpha(\xi),$$

and satisfies

$$a(x,\lambda) = \int_{\mathbb{R}} E_\alpha(x,\xi)V_1(\xi)V_2(\lambda)d\mu_\alpha(\xi) = V_2(\lambda)\int_{\mathbb{R}} E_\alpha(x,\xi)V_1(\xi)d\mu_\alpha(\xi),$$

where $V_1 \in L^1(\mathbb{R}, d\mu_\alpha)$ is a continuous function and $V_2(\lambda) = A$ is a constant. So we have

$$a(x,\lambda) = A\int_{\mathbb{R}} E_\alpha(x,\xi)V_1(\xi)d\mu_\alpha(\xi).$$

Theorem 7. Let $f \in \mathcal{S}(\mathbb{R})$. Then the composition of the pseudo-differential operators T_a and T_b with symbols a and b, which satisfy Assumption 3, has a representation

$$T_a(T_b f)(x) = \left(2^{\alpha+1}\Gamma(\alpha+1)\right)^2 A \cdot \mathcal{F}_\alpha^{-1}[V_1 *_\alpha (W_1 *_\alpha B \cdot \mathcal{F}_\alpha[f])](x)$$

and satisfies the following inequality

$$\|T_a(T_b f)\|_L \leq 16 \left(2^{\alpha+1}\Gamma(\alpha+1)\right)^2 AB \|V_1\|_{1,\alpha} \|W_1\|_{1,\alpha} \|f\|_L.$$

Proof. Let $f \in \mathcal{S}(\mathbb{R})$. Then we have

$$\begin{aligned}
&T_a(T_b f)(x)\\
&= \int_{\mathbb{R}} E_\alpha(x,\lambda) a(x,\lambda) \mathcal{F}_\alpha[T_b f](\lambda) d\mu_\alpha(\lambda)\\
&= \int_{\mathbb{R}} E_\alpha(x,\lambda) \left(\int_{\mathbb{R}} E_\alpha(x,\xi) V_a(\xi,\lambda) d\mu_\alpha(\xi) \right) \mathcal{F}_\alpha[T_b f](\lambda) d\mu_\alpha(\lambda)\\
&= A \int_{\mathbb{R}} \int_{\mathbb{R}} E_\alpha(x,\lambda) E_\alpha(x,\xi) V_1(\xi) \mathcal{F}_\alpha[T_b f](\lambda) d\mu_\alpha(\xi) d\mu_\alpha(\lambda)\\
&= A \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}} E_\alpha(x,\eta) V_1(\xi) \mathcal{F}_\alpha[T_b f](\lambda) d\nu_{\lambda,\xi}(\eta) d\mu_\alpha(\xi) d\mu_\alpha(\lambda)\\
&= 2^{\alpha+1}\Gamma(\alpha+1) A \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}} E_\alpha(x,\eta) V_1(\xi) \mathcal{F}_\alpha[T_b f](\lambda) d\nu_{-\lambda,\eta}(\xi) d\mu_\alpha(\eta) d\mu_\alpha(\lambda)\\
&= 2^{\alpha+1}\Gamma(\alpha+1) A \int_{\mathbb{R}} E_\alpha(x,\eta) \left(\int_{\mathbb{R}} \int_{\mathbb{R}} V_1(\xi) \mathcal{F}_\alpha[T_b f](\lambda) d\nu_{-\lambda,\eta}(\xi) d\mu_\alpha(\lambda) \right) d\mu_\alpha(\eta).
\end{aligned}$$

Now, applying the Dunkl transform, we obtain

$$\begin{aligned}
\mathcal{F}_\alpha[T_a(T_b f)](\eta) &= 2^{\alpha+1}\Gamma(\alpha+1) A \int_{\mathbb{R}} \tau_\eta V_1(-\lambda) \mathcal{F}_\alpha[T_b f](\lambda) d\mu_\alpha(\lambda)\\
&= 2^{\alpha+1}\Gamma(\alpha+1) A (V_1 *_\alpha \mathcal{F}_\alpha[T_b f])(\eta).
\end{aligned}$$

Then by using (7), we obtain

$$\mathcal{F}_\alpha[T_a(T_b f)](\eta) = \left(2^{\alpha+1}\Gamma(\alpha+1)\right)^2 A (V_1 *_\alpha (W_1 *_\alpha B \mathcal{F}_\alpha[f]))(\eta),$$

so that

$$\int_{\mathbb{R}} |\mathcal{F}_\alpha[T_a(T_b f)](\eta)| \, d\mu_\alpha(\eta) = \left(2^{\alpha+1}\Gamma(\alpha+1)\right)^2 AB \int_{\mathbb{R}} |(V_1 *_\alpha (W_1 *_\alpha \mathcal{F}_\alpha[f]))(\eta)| \, d\mu_\alpha(\eta).$$

Thus, we have

$$\begin{aligned}
\|T_a(T_b f)\|_L = \left(2^{\alpha+1}\Gamma(\alpha+1)\right)^2 AB \|V_1 *_\alpha (W_1 *_\alpha \mathcal{F}_\alpha[f])\|_{1,\alpha}&\\
\leq 4 \left(2^{\alpha+1}\Gamma(\alpha+1)\right)^2 AB \|V_1\|_{1,\alpha} \|W_1 *_\alpha \mathcal{F}_\alpha[f]\|_{1,\alpha}&\\
\leq 16 \left(2^{\alpha+1}\Gamma(\alpha+1)\right)^2 AB \|V_1\|_{1,\alpha} \|W_1\|_{1,\alpha} \|\mathcal{F}_\alpha[f]\|_{1,\alpha}&\\
= 16 \left(2^{\alpha+1}\Gamma(\alpha+1)\right)^2 AB \|V_1\|_{1,\alpha} \|W_1\|_{1,\alpha} \|f\|_L.&
\end{aligned}$$

This completes proof of the theorem. □

3 Application

In this section, we present an application from the previous section. We also considered other applications of the Dunkl analysis to inverse source problems, as discussed in [14, 15].

Corollary 3. Let

$$a_k(x) = \int_{\mathbb{R}} E_\alpha(x,\xi) V_1^k(\xi) d\mu_\alpha(\xi),$$

where $V_1^k \in L^1(\mathbb{R}, d\mu_\alpha)$ is a continuous function for all k. Then the operator

$$\begin{cases} P_{n,\alpha} = \sum_{k=0}^{n} a_k(x) D_\alpha^k, \\ Dom(P_{n,\alpha}) = \mathcal{S}(\mathbb{R}) \end{cases}$$

is a continuous linear operator from $\mathcal{S}(\mathbb{R})$ to $L(\mathbb{R}, d\mu_\alpha)$. Moreover, we have

$$\|P_{n,\alpha} f\|_L \leq \sum_{k=0}^{n} 2^{\alpha+3} \Gamma(\alpha+1) \|V_1^k\|_{1,\alpha} \|V_2^k \mathcal{F}_\alpha[f]\|_{1,\alpha},$$

where $V_2^k(\lambda) = (i\lambda)^k$.

Remark 7. Here, the functions V_1^k and V_2^k are chosen to satisfy Assumption 2.

Proof. Let $f \in \mathcal{S}(\mathbb{R})$. Then

$$f(x) = \int_{\mathbb{R}} E_\alpha(x,\lambda) \mathcal{F}_\alpha[f](\lambda) d\mu_\alpha(\lambda)$$

and

$$\begin{aligned} P_{n,\alpha} f(x) &= \sum_{k=0}^{n} \int_{\mathbb{R}} a_k(x) D_\alpha^k E_\alpha(x,\lambda) \mathcal{F}_\alpha[f](\lambda) d\mu_\alpha(\lambda) \\ &= \sum_{k=0}^{n} \int_{\mathbb{R}} E_\alpha(x,\lambda) a_k(x) (i\lambda)^k \mathcal{F}_\alpha[f](\lambda) d\mu_\alpha(\lambda). \end{aligned}$$

Hence, symbol of the pseudo-differential operator $P_{n,\alpha}$ expressed by the form

$$a(x,\lambda) = \sum_{k=0}^{n} a_k(x,\lambda) = \sum_{k=0}^{n} a_k(x)(i\lambda)^k = \sum_{k=0}^{n} (i\lambda)^k \int_{\mathbb{R}} E_\alpha(x,\xi) V_1^k(\xi) d\mu_\alpha(\xi).$$

Then by applying Theorem 6, we obtain

$$\|P_{n,\alpha} f\|_L \leq \sum_{k=0}^{n} 2^{\alpha+3} \Gamma(\alpha+1) \|V_1^k\|_{1,\alpha} \|V_2^k \mathcal{F}_\alpha[f]\|_{1,\alpha},$$

where $V_2^k(\lambda) = (i\lambda)^k$. □

Conclusion

In this research paper, our aim is to obtain some boundedness results for pseudo-differential operator generated by the Dunkl operator. We obtain the following main results:

- Let $f \in \mathcal{S}(\mathbb{R})$. Then, under Assumption 1 the pseudo-differential operator
$$T_a f(x) = \int_{\mathbb{R}} E_\alpha(x,\lambda) a(x,\lambda) \mathcal{F}_\alpha[f](\lambda) d\mu_\alpha(\lambda)$$
is a bounded linear operator on $L(\mathbb{R}, d\mu_\alpha)$.
- Let $f \in \mathcal{S}(\mathbb{R})$. Then, under Assumption 1, the composition of pseudo-differential operators T_a and T_b is a bounded linear operator on $L(\mathbb{R}, d\mu_\alpha)$.

These results are obtained under the assumption that the symbol has an integral representation. Future improvements could focus on obtaining these results without this restriction.

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Author Contributions

All authors contributed equally to this work.

Conflict of Interest

The authors declare no conflict of interest.

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Author Information[*]

Bayan Bekbolat — PhD, Assistant Professor, SDU University, 1/1 Abylai Khan St., 040900, Kaskelen, Kazakhstan; e-mail: *bayan.bekbolat@sdu.edu.kz*; Senior Researcher, Institute of Mathematics and Mathematical Modeling, 28 Shevchenko St., Almaty, 050010, Kazakhstan; e-mail: *bekbolat@math.kz*; https://orcid.org/0000-0003-2347-0630

Niyaz Tokmagambetov — PhD, Leading Researcher, Institute of Mathematics and Mathematical Modeling, 28 Shevchenko St., Almaty, 050010, Kazakhstan; e-mail: *tokmagambetov@math.kz*; https://orcid.org/0000-0001-5725-9740

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[*] Authors' names are presented in the following order: first name, middle name (if any), last name.