Bulletin of the Karaganda University. Mathematics Series, No. 1(121), 2026, pp. 228–237

https://doi.org/10.31489/2026M1/228-237 *Research article*

Integration of the negative order Nonlinear Schrödinger Equation with self-consistent source

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This paper focuses on the integrability properties of the negative-order nonlinear Schrödinger equation with a source. The source consists of the combination of the eigenfunctions of the corresponding spectral problem for the Dirac system which has not spectral singularities. The connection between the negative-order nonlinear Schrödinger equation with a self-consistent source and the Dirac system of equations is crucial, as it allows the complex dynamics of the original nonlinear model to be interpreted through the spectral theory of the Dirac operator. Building on this relationship, the evolution equations for the scattering data of the Dirac operator are derived, which is the central part in the inverse scattering transform (IST) framework. Due to the IST procedure, the rapidly decaying potential of the Dirac operator can be reconstructed from the derived differential equations for the scattering data, and this potential corresponds precisely to the solution of the problem under consideration. To illustrate the practical value of the theoretical results, the paper presents a detailed example demonstrating each stage of the method, from the formulation of the scattering data to the final reconstruction of the potential. This example clarifies the overall procedure and highlights the effectiveness of the approach in concrete applications.

Keywords: soliton solution, negative-order, nonlinear equations, nonlinear Schrödinger equation, self-consistent source, inverse scattering transform, eigenvalue, eigenfunction.

2020 Mathematics Subject Classification: 35P25, 35P30, 35Q51, 35Q53, 37K15.

Introduction

The nonlinear Schrödinger equation (NLSE) arises in several physical systems characterized by wave-like behavior interacting with nonlinear effects, resulting in distinct phenomena such as optical solitons in fiber optics, Bose–Einstein condensates in ultracold atomic gases, and wave dynamics in plasma physics [1, 2]. The NLSE provides a mathematical foundation to understand nonlinear wave dynamics, emphasizing the interplay between dispersion, which tends to spread wave packets, and nonlinearity, which can counteract dispersion through self-interaction. This framework helps elucidate the emergence of solitons, vortices, and other intricate wave phenomena observed in both natural and engineered systems. Well-known bright, dark, and gray solitons, as well as so-called optical rogue waves, whose experimental observations in optical systems are supported by numerical simulations based on probabilistic supercontinuum generation in highly nonlinear microstructured optical fibers, are modeled using the generalized NLSE [3]. The NLSE can be written in several forms depending on the context, but a common form for the NLSE in one spatial dimension is:

$$iu_t(x,t) = u_{xx}(x,t) \pm 2u^*(x,t)u^2(x,t).$$

The motivation for developing negative-order nonlinear equations originates from Peter J. Olver's [4] on recursion operators for symmetries of evolution equations, which was subsequently extended to derive negative-order analogs of evolution equations [5–7].

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Received: 27 November 2024; *Accepted:* 26 December 2025.

Note that in [8], the focusing NLSE was derived as the following

$$\begin{cases} \mu_x = |u|_t^2\,, \\ u_{xt} + 2\mu u + iu_x = 0, \quad x \in \mathbb{R}, \quad t \geq 0 \end{cases}$$

negative-order nonlinear Schrödinger equation (nNLSE) and subsequently solved using the IST.

The study of soliton equations with self-consistent sources has been particularly pursued in the work of Mel'nikov [9]. Subsequently, numerous works [10–12] have explored various methods for solving these types of equations. Mathematically, such systems of equations arise through a multiscaling limit of well-known integrable systems. Several of these systems have also independently appeared in various physical contexts [13, 14]. In this work we concern on applying the inverse scattering transform (IST) for integrating the nNLSE with a self-consistent source in the class of decreasing function.

1 *Statement of the problem*

This paper is concerned with the following system of equations:

$$\begin{cases} \mu_x = |u|_t^2\,, \\ iu_{xt} + 2\mu u + iu_x = i\sum\limits_{k=1}^{2N} (\Phi_{k_1}^2 - \Phi^{*2}_{k_2}), \\ L\Phi_k = \xi_k \Phi_k, \ k = 1, 2, ..., 2N, \ x \in R, \end{cases} \tag{1}$$

for the complex-valued continuous function $u(x,t)$ and $\Phi_k = (\Phi_{k1}(x,t),\ \Phi_{k2}(x,t))^T,\ k = 1, 2, ..., N$ is the eigenvector functions of the operator $L(t) = i\begin{pmatrix} \frac{d}{dx} & -u(x,t) \\ -u^*(x,t) & -\frac{d}{dx} \end{pmatrix}$, corresponding to the eigenvalue ξ_k. (*) means the complex conjugate of the function. For definiteness, we will assume that the sum involved in the right-hand side of (1) first includes terms with $Im\xi_k > 0,\ k = 1, 2, ..., N$. It is also assumed that

$$\int_{-\infty}^{\infty} \Phi_{k1}\Phi_{k2} dx = A_k(t), \qquad k = 1, 2, ..., 2N, \tag{2}$$

with given continuous, non-zero functions $A_k(t)$, that satisfy the conditions $A_k(t) = A_n(t)$ for $\xi_k = -\xi_n$, under initial condition

$$u(x,0) = u_0(x), \quad x \in R, \tag{3}$$

where, the initial function $u_0(x)$ $(-\infty < x < \infty)$ has the following properties:

1) $\int_{-\infty}^{\infty} (1 + |x|)\ |u_0(x)|\, dx < \infty$,

2) the operator $L\,(0) = i\begin{pmatrix} \frac{\partial}{\partial x} & -u_0(x) \\ -u_0^*(x) & -\frac{\partial}{\partial x} \end{pmatrix}$ has no spectral singularities and has exactly $2N$ simple eigenvalues $\xi_1(0),\ \xi_2(0),\ ...,\ \xi_{2N}(0)$. Here the function $u_0^*(x)$ is the complex conjugate of the function $u_0(x)$.

Let the function $u = u(x,t)$ be a complex valued and sufficiently smooth function of x and t, $\mu = \mu(x,t)$ is the sufficiently smooth real function of x and t, for all $t \geq 0$ satisfies the requirement

$$\begin{gathered} \int\limits_{-\infty}^{\infty} \left((1 + |x|)(|u(x,t)| + |u_x(x,t)|)\right) dx < \infty, \\ \mu(x,t) \to c^2 \text{ as } x \to \pm\infty, \end{gathered} \tag{4}$$

respectively.

In this work, we obtain the evolution of scattering data for the system with a potential that is a solution to the considered Cauchy problem (1)–(4) for the nNLSE with a simple self-consistent source.

2 Scattering problem

In this section, we provide known brief facts from the Dirac spectral problem on the real axis [15].

$$\begin{cases} y_{1x} = -i\xi y_1 \ + u\left(x\right) y_2, \\ y_{2x} = \ i\xi y_2 - u^*\left(x\right) y_1, \end{cases} \tag{5}$$

where $u\left(x\right)$ is complex-valued potential satisfies the condition

$$\int_{-\infty}^{\infty} \left(1+|x|\right)\ |u(x)|\, dx < \infty \tag{6}$$

and $\xi \in C$ is spectral parameter.

The system (5) under the condition (6) posses "Jost solutions" $\varphi\left(x,\xi\right)$ and $\psi\left(x,\xi\right)$, for real ξ Jost solutions have the following asymptotes:

$$\left.\begin{array}{l} \varphi(x,\xi)e^{i\xi x} \to \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ \bar{\varphi}(x,\xi)e^{-i\xi x} \to \begin{pmatrix} 0 \\ -1 \end{pmatrix} \end{array}\right\}, \quad x \to -\infty, \qquad \left.\begin{array}{l} \psi(x,\xi)e^{-i\xi x} \to \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ \bar{\psi}(x,\xi)e^{i\xi x} \to \begin{pmatrix} 1 \\ 0 \end{pmatrix} \end{array}\right\}, \quad x \to +\infty.$$

Note that here and below the function $\bar{\varphi}$ is not complex conjugate to φ. On the continuous spectrum of the problem (5)-(6) pairs of vector functions $\left\{\psi, \bar{\psi}\right\}$ are linearly independent, and for them holds the following relation

$$\varphi(x,\xi) = a(\xi)\bar{\psi}(x,\xi) + b(\xi)\psi(x,\xi), \tag{7}$$

where

$$a(\xi) = W\left\{\varphi, \psi\right\} \equiv \varphi_1(x,\xi)\psi_2(x,\xi) - \varphi_2(x,\xi)\psi_1(x,\xi) \tag{8}$$

$$\bar{\psi}(x,\xi) = \begin{pmatrix} \psi_2^*(x,\xi^*) \\ -\psi_1^*(x,\xi^*) \end{pmatrix}, \quad \bar{\varphi}(x,\xi) = \begin{pmatrix} \varphi_2^*(x,\xi^*) \\ -\varphi_1^*(x,\xi^*) \end{pmatrix}. \tag{9}$$

The quantity $r^+\left(\xi\right) = \frac{b(\xi)}{a(\xi)}$ is known as reflection coefficient. Furthermore [4], $r^+(\xi)$ uniquely determines $a(\xi)$. The function $a(\xi)$ admits analytic continuation into the upper half-plane $Im\,\xi > 0$. The function $a(\xi)$ can only have a finite number of zeros $\xi_k,\ k = 1,\ 2,\ ...,\ N$ in the half-plane $Im\,\xi > 0$. The zeros $\xi_k,\ k = 1,\ 2,\ ...,\ N$, of the functions $a(\xi)$ correspond to the eigenvalues $\xi_k,\ k = 1,\ 2,\ ...,\ N$, of the operator L in the upper half-plane.

In general, the operator L may have multiple eigenvalues and spectral singularities that lie on the continuous spectrum. The continuous spectrum of the operator fills the real axis.

In this paper, we assume that the operator L has no spectral singularities and all its eigenvalues ξ_k are simple, so that since the quantities ξ_k are the zeros of $a(\xi)$, it follows from relation (8) that

$$\varphi(x,\xi_k) = C_k\psi(x,\xi_k), \quad k = 1, 2, ..., N,$$

where C_k do not depend on x.

Definition. The set $\left\{r^+\left(\xi\right),\ C_k,\ \xi_k,\ k = 1, 2, ..., N\right\}$ is called scattering data for the system of equations (5). The set of scattering data uniquely determines the potential $u\left(x\right)$.

The potential function $u(x)$ is determined by the equality

$$u\left(x\right) = -2K\left(x,x\right).$$

Here $K\left(x,y\right)$ is a solution of the integral equation Gelfand–Levitan–Marchenko

$$K\left(x,y\right) - F^*\left(x+y\right) + \int_x^{+\infty}\int_x^{+\infty} K\left(x,y\right) F\left(z+s\right) F^*\left(s+y\right) ds dz = 0,$$

where $F\left(x\right) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \frac{b(\xi)}{a(\xi)} e^{i\xi x} d\xi - i\sum_{j=1}^{N} C_j e^{i\xi_j x}$.

3 Evolution of the scattering data

For further calculations and to obtain the main result, we provide the necessary notes.

Lemma 1. If the vector functions $Y(x,\zeta) = (y_1, y_2)^T$ and $Z(x,\eta) = (z_1, z_2)^T$ are solutions of the equations $LY = \zeta Y$ and $LZ = \eta Z$, respectively, then their components satisfy the equalities

$$\frac{d}{dx}(y_1 z_2 - y_2 z_1) = -i(\zeta - \eta)(y_1 z_2 + y_2 z_1),$$

$$\frac{d}{dx}(y_1 z_1^* + y_2 z_2^*) = -i(\zeta - \eta^*)(y_1 z_1^* - y_2 z_2^*).$$

This Lemma 1 can be proved by direct verification.

It easy to show that the vector functions

$$h_n(x) = \frac{\left.\frac{d}{d\xi}(\varphi - C_n\psi)\right|_{\xi = \xi_n}}{\dot{a}(\xi_n)}, \quad n = 1, 2, ..., N, \tag{10}$$

are solutions to the system of equations $Lh_n = \xi_n h_n$. According to the equality (8), which can be rewritten in the $Im\ \xi > 0$, and using the equality (10), we obtain the following asymptotes

$$\begin{aligned} h_n &\sim -C_n \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{i\xi_n x} \quad at \quad x \to -\infty, \\ h_n &\sim \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-i\xi_n x} \quad at \quad x \to \infty. \end{aligned} \tag{11}$$

In particular,

$$W\{\varphi_n, h_n\} \equiv \varphi_{n1} h_{n2} - \varphi_{n2} h_{n1} = -C_n, \tag{12}$$

where $\varphi_n \equiv \varphi(x, \xi_n)$, $n = 1, 2, ..., N$.

Now, let's consider the nNLSE with the source

$$\begin{cases} \mu_x = |u|_t^2, \\ iu_{xt} + 2\mu u + iu_x = iG, \end{cases} \tag{13}$$

where $G = G(x,t)$ is a sufficiently smooth function and for any nonnegative value t, satisfies the condition

$$G(x,t) = o(1) \ \text{ at } \ x \to \pm\infty.$$

Equation (13) is considered under the initial condition (3).

Lemma 2. If the potential $u(x,t)$, in the system of equations (5), is a solution to equation (13) in the class of functions (4), then the scattering data of the system of equations (5) varies as follows:

$$\frac{\partial r^+}{\partial t} = \frac{i}{\xi}\left(c^2 + \xi\right) r^+ + \frac{1}{2\xi a^2(\xi)} \int_{-\infty}^{\infty} \left(G\varphi_2^2 + G^*\varphi_1^2\right) dx, \quad (Im\xi = 0),$$

$$\frac{dC_n}{dt} = \left(i\left(1 + \frac{c^2}{\xi_n}\right) + \frac{1}{2\xi_n}\int_{-\infty}^{\infty}(G^* h_{n1}\psi_{n1} + G h_{n2}\psi_{n2})\, dx\right) C_n,$$

$$\frac{d\xi_n}{dt} = \frac{i}{4\xi_n} \frac{\int_{-\infty}^{\infty}(G^*\varphi_{n1}^2 + G\varphi_{n2}^2) dx}{\int_{-\infty}^{\infty} \varphi_{n1}\varphi_{n2} dx}, \quad n = 1, 2, ..., N.$$

Proof. The system of equation (13) can be represented as a Lax equation:

$$L_t + [L, A] = R, \tag{14}$$

where $[L,\ A]=\ LA-AL$ and

$$L(t)=i\left(\begin{array}{cc}\frac{d}{dx} & -u(x,t)\\ -u^{*}(x,t) & -\frac{d}{dx}\end{array}\right),$$

$$A=\frac{i}{2\xi}\left(\begin{array}{cc}-\mu-\xi & -iu-u_t\\ iu^{*}-u_t^{*} & \mu+\xi\end{array}\right),$$

$$R=\left(\begin{array}{cc}0 & -\frac{G}{2\xi}\\ \frac{G^{*}}{2\xi} & 0\end{array}\right).$$

Let $\varphi\left(x,\xi,t\right)$ is the Jost solution of the equation $L\varphi=\xi\varphi$. Differentiating this equality with respect to t, we obtain

$$L_t\varphi+L\varphi_t=\xi\varphi_t,$$

substituting L_t from (14), we get

$$\left(L-\xi\right)\left(\varphi_t-A\varphi\right)=R\varphi. \tag{15}$$

We will look for a solution (15) in the form

$$\varphi_t-A\varphi=\alpha\left(x\right)\psi+\beta\left(x\right)\varphi.$$

Using the approach described in [5], we obtain the following equality

$$\varphi_t-A\varphi=-\frac{1}{a\left(\xi\right)}\int_{-\infty}^{x}\hat{\varphi}^{T}R\varphi dx\cdot\psi+\left(\frac{1}{a\left(\xi\right)}\int_{-\infty}^{x}\hat{\psi}^{T}R\varphi dx+\frac{i\left(c^2+\xi\right)}{2\xi}\right)\varphi. \tag{16}$$

Using (7) and passing to the limit in (16) at $x\to\infty$, using the definition of the reflection coefficient for the scattering problem (5)-(6), we obtain

$$\frac{\partial r^{+}}{\partial t}=\frac{i}{\xi}\left(c^2+\xi\right)r^{+}+\frac{1}{2\xi a^{2}\left(\xi\right)}\int_{-\infty}^{\infty}\left(G\varphi_2^2+G^{*}\varphi_1^2\right)dx,\quad \left(Im\xi=0\right).$$

Differentiating the equality $\varphi_n=C_n\psi_n$ by t, we receive

$$\left.\frac{\partial\varphi}{\partial t}\right|_{\xi=\xi_n}+\left.\frac{\partial\varphi}{\partial\xi}\right|_{\xi=\xi_n}\frac{d\xi_n}{dt}=\frac{dC_n}{dt}\psi_n+C_n\left.\frac{\partial\psi}{\partial t}\right|_{\xi=\xi_n}+C_n\left.\frac{\partial\psi}{\partial\xi}\right|_{\xi=\xi_n}\frac{d\xi_n}{dt}.$$

Substituting, instead $\left.\frac{d}{d\xi}\left(\varphi-C_n\psi\right)\right|_{\xi=\xi_n}$, the expression $h_n(x)$ from (10), we obtain

$$\frac{\partial\varphi_n}{\partial t}=\frac{dC_n}{dt}\psi_n+C_n\frac{\partial\psi_n}{\partial t}-\dot{a}\left(\xi_n\right)h_n\frac{d\xi_n}{dt}, \tag{17}$$

where $\frac{\partial\varphi_n}{\partial t}\equiv\left.\frac{\partial\varphi}{\partial t}\right|_{\xi=\xi_n}$.

Similarly to the continuous spectrum, for the discrete spectrum, we obtain the equality

$$\frac{\partial\varphi_n}{\partial t}-A\varphi_n=\left(\frac{1}{C_n}\int\limits_{-\infty}^{x}\hat{\varphi}_n^{T}R\varphi_n dx\right)h_n+\left(-\frac{1}{C_n}\int\limits_{-\infty}^{x}\hat{h}_n^{T}R\varphi_n\,dx+\frac{i}{2\xi_n}\left(c^2+\xi_n\right)\right)\varphi_n,$$

which is an analog of equality (16) for the continuous spectrum. According to (17), the last equality can be rewritten as

$$\frac{dC_n}{dt}\psi_n+C_n\frac{\partial\psi_n}{\partial t}-\dot{a}\left(\xi_n\right)\frac{d\xi_n}{dt}h_n-C_nA\psi_n=$$

$$= \left(\frac{1}{C_n} \int\limits_{-\infty}^{x} \hat{\varphi}_n^T R \varphi_n dx \right) h_n + \left(-\frac{1}{C_n} \int\limits_{-\infty}^{x} \hat{h}_n^T R \varphi_n \, dx + \frac{i}{2\xi_n} \left(c^2 + \xi_n \right) \right) C_n \psi_n. \tag{18}$$

Using asymptotics (11) and passing to the limit as $x \to \infty$ in (18), we obtain

$$\frac{dC_n}{dt} = \frac{i}{\xi_n} \left(c^2 + \xi_n \right) C_n - \int_{-\infty}^{\infty} \hat{h}_n R \varphi_n dx,$$

$$\frac{d\xi_n}{dt} = -\frac{\int_{-\infty}^{\infty} \hat{\varphi}_n R \varphi_n \, dx}{C_n \dot{a} \left(\xi_n \right)},$$

where $\dot{a}\left(\xi_n\right) = -\frac{2i}{\xi_n} \int_{-\infty}^{\infty} \varphi_{n1} \, \varphi_{n2} dx$.

As a result, we have

$$\frac{dC_n}{dt} = \left(i \left(1 + \frac{c^2}{\xi_n} \right) + \frac{1}{2\xi_n} \int_{-\infty}^{\infty} \left(G^* h_{n1} \psi_{n1} + G h_{n2} \psi_{n2} \right) dx \right) C_{n,},$$

$$\frac{d\xi_n}{dt} = \frac{i}{4\xi_n} \frac{\int_{-\infty}^{\infty} (G^* \varphi_{n1}^2 + G \varphi_{n2}^2) dx}{\int_{-\infty}^{\infty} \varphi_{n1} \varphi_{n2} dx}, \quad n = 1, 2, ..., N.$$

Lemma 2 is proved.

Based on the conditions given for the function $A_k(t)$ in formula (2) and equalities (9), the right-hand side in equation (1) can be rewritten as

$$\sum_{k=1}^{2N} \left(\Phi_{k1}^2 - \Phi_{k2}^{*2} \right) = 2 \sum_{\substack{k=1 \\ Im\xi_k > 0}}^{N} \left(\Phi_{k1}^2 - \Phi_{k2}^{*2} \right).$$

Let us apply the results of Lemma 2 to the system of equations (1) assuming

$$G = 2 \sum_{\substack{k=1, \\ Im\xi_k > 0}}^{N} \left(\Phi_{k1}^2 - \Phi_{k2}^{*2} \right).$$

For ξ_k, $k \neq n$, according to Lemma 1, we have the following equality:

$$\left(\Phi_{k1}^2 - \Phi_{k2}^{*2} \right) \left(h_{n1} \psi_{n1} + h_{n2} \psi_{n2} \right) = -\frac{1}{2i} (\frac{1}{\xi_k + \xi_n} \frac{d}{dx} ((\Phi_{k1} h_{n1} + \Phi_{k2}^* h_{n2}) \times$$

$$\times (\Phi_{k1} \psi_{n1} + \Phi_{k2}^* \psi_{n2})) + \frac{1}{\xi_k - \xi_n} \frac{d}{dx} ((\Phi_{k1} h_{n2} - \Phi_{k2}^* h_{n1})(\Phi_{k1} \psi_{n2} - \Phi_{k2}^* \psi_{n1}))),$$

hence

$$\int_{-\infty}^{\infty} \left(\Phi_{k1}^2 - \Phi_{k2}^{*2} \right) \left(h_{n1} \psi_{n1} + h_{n2} \psi_{n2} \right) dx = 0.$$

If $\xi_k = \xi_n$, then

$$\left(\Phi_{n1}^2 - \Phi_{n2}^{*2} \right) \left(h_{n1} \psi_{n1} + h_{n2} \psi_{n2} \right) = -\frac{1}{4i\xi_n} \frac{d}{dx} ((\Phi_{n1} h_{n1} + \Phi_{n2}^* h_{n2}) \times$$

$$\times (\Phi_{n1} \psi_{n1} + \Phi_{n2}^* \psi_{n2})) + \Phi_{n1} \Phi_{n2}^* \left(\psi_{n1} h_{n2} - \psi_{n2} h_{n1} \right),$$

therefore, taking into account (2) and (12), we obtain

$$\int_{-\infty}^{\infty}\left(\Phi_{n1}^{2}-\Phi_{n2}^{*2}\right)\left(h_{n1}\psi_{n1}+h_{n2}\psi_{n2}\right)dx=\int_{-\infty}^{\infty}\Phi_{n1}\Phi_{n2}^{*}W\left\{\psi_{n},h_{n}\right\}dx=$$

$$=\frac{1}{C_{n}}\int_{-\infty}^{\infty}\Phi_{n1}\Phi_{n2}^{*}W\left\{\varphi_{n},h_{n}\right\}dx=-A_{n}\left(t\right).$$

Additionally, according to Lemma 2, we have

$$\frac{dC_{n}}{dt}=i\left(1+\frac{c^{2}}{\xi_{n}}\right)C_{n}-\frac{A_{n}}{\xi_{n}}C_{n}.$$

Similarly, it can be shown that

$$\int\limits_{-\infty}^{\infty}(G\varphi_{2}^{2}+G^{*}\varphi_{1}^{2})dx=0,$$

$$\int\limits_{-\infty}^{\infty}(G\varphi_{n2}^{2}+G^{*}\varphi_{n1}^{2})dx=0,$$

thus, we find

$$\frac{\partial r^{+}}{\partial t}=\frac{i}{\xi}\left(c^{2}+\xi\right)r^{+},\quad\left(Im\xi=0\right),$$

$$\frac{d\xi_{n}}{dt}=0.$$

Accordingly, we have proved the following theorem:

Theorem 1. If the set of functions $\left\{u\left(x,t\right),\mu\left(x,t\right),\Phi_{k}\left(x,t\right),\,k=1,2,\ldots,N\right\}$ is a solution to the problem (1)–(3) in the class of functions (4), then the scattering data of the system of equations (5) with potential $u\left(x,t\right)$ changes on t as follows:

$$\frac{\partial r^{+}}{\partial t}=\frac{i}{\xi}\left(c^{2}+\xi\right)r^{+},\quad\left(Im\xi=0\right),$$

$$\frac{dC_{n}}{dt}=i\left(1+\frac{c^{2}}{\xi_{n}}\right)C_{n}-\frac{A_{n}}{\xi_{n}}C_{n},$$

$$\frac{d\xi_{n}}{dt}=0,n=1,2,...,N.$$

The results allows us to apply the inverse scattering problem method to solve the Cauchy problem of the system of equations (1).

Consider the following example. Let the system of equations (1) be considered under the initial condition

$$u|_{t=0}=-\frac{2}{\mathrm{ch}2x},$$

which implies that the initial scattering data will be as follows

$$r^{+}\left(\xi,0\right)=0,\quad\xi_{1}\left(0\right)=i,\quad C_{1}\left(0\right)=2i.$$

According to the main theorem, we have

$$r^{+}\left(\xi,t\right)=0,\,\xi_{1}\left(t\right)=i,\quad C_{1}\left(t\right)=2i\exp\left\{\left(i+c^{2}\right)t-\int_{0}^{t}A_{1}\left(\tau\right)d\tau\right\},$$

from which, solving the inverse scattering problem, we obtain

$$K\left(x,y,t\right)=\frac{2\exp\left\{-x-y-it+\gamma\left(t\right)\right\}}{1+\exp\left\{-4x+2\gamma\left(t\right)\right\}}.$$

Applying the relation between potential and kernel $u\left(x\right)=-2K\left(x,x\right)$ yields that

$$u\left(x,t\right)=-\frac{2\exp\left\{-it\right\}}{\operatorname{ch}\left(2x-\gamma\left(t\right)\right)},$$

$$\mu\left(x,t\right)=\frac{2\left(A_1(t)-c^2\right)}{\operatorname{ch}^2\left(2x-\gamma\left(t\right)\right)}+c^2,$$

and consequently, using the integral representation of Levin for the vector function $\psi(x,\xi)$, we find

$$\psi_{11}\left(x,t\right)=\frac{\exp\left\{-it\right\}}{2\operatorname{ch}\left(2x-\gamma\left(t\right)\right)},$$

$$\psi_{12}\left(x,t\right)=\frac{\exp\left\{-x\right\}}{2}+\exp\left\{\frac{-x}{2}\right\}\operatorname{th}\left(2x-\gamma(t)\right).$$

Since, $\Phi_{11}\left(x\right)=d\psi_{11}$, $\Phi_{12}\left(x\right)=d\psi_{12}$, $d^2=4iA_1\exp\left\{it+\gamma(t)\right\}$ and from the normalization condition (2), we get

$$\Phi_{11}=\frac{\sqrt{iA_1(t)}}{\operatorname{ch}\left(2x-\gamma(t)\right)}\exp\left\{-x-\frac{it-\gamma(t)}{2}\right\},$$

$$\Phi_{12}=2\sqrt{iA_1(t)}\exp\left\{-x+\frac{it+\gamma(t)}{2}\right\}\left(1+\operatorname{th}\left(2x-\gamma(t)\right)\right),$$

where $\gamma\left(t\right)=c^2t-\int_0^t A_1(\tau)d\tau$.

Conclusion

This paper studies the integrability of the nNLSE with a self-consistent source using the inverse scattering transform (IST). The problem statement is presented in Section 1. Section 2 reviews the scattering theory for the Dirac system, including the definition of the scattering data and the IST technique. In Section 3, we obtain the evolution of scattering data for the system with a potential that is a solution to the considered Cauchy problem for the nNLSE with a simple self-consistent source.

Author Contributions

All authors contributed equally to this work.

Conflict of Interest

The authors declare no conflict of interest.

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