https://doi.org/10.31489/2026M1/55-66 *Research article*

Local derivation on the Schrödinger Lie algebra in $(n+1)$-dimensional space-time

A.K. Alauadinov[1,2], B.B. Yusupov[1,3,*]

[1] *V.I. Romanovskiy Institute of Mathematics Uzbekistan Academy of Sciences, Tashkent, Uzbekistan;*
[2] *Karakalpak State University, Nukus, Uzbekistan;*
[3] *Urgench State University, Urgench, Uzbekistan*
(E-mail: amir_t85@mail.ru, baxtiyor_yusupov_93@mail.ru)

This paper investigates local derivations on the Schrödinger Lie algebra $\mathfrak{s}_n$, the Lie algebra of the $(n+1)$-dimensional space-time Schrödinger group. As a finite-dimensional Lie algebra that is neither semisimple nor solvable, the Schrödinger algebra plays an important role in mathematical physics, particularly as the symmetry algebra of the free Schrödinger equation. While local derivations are well understood for semisimple, solvable, and certain infinite-dimensional Lie algebras, much less is known for non-semisimple algebras. We prove that for all integers $n \geq 3$, every local derivation on $\mathfrak{s}_n$ is a derivation. Our approach uses the explicit structure of the Schrödinger algebra together with a detailed description of its derivation algebra. First, we reduce the problem to derivations that act trivially on the semisimple part, and then we perform a coefficient-wise analysis in a fixed basis. This shows that every local derivation is an ordinary derivation. Moreover, such derivations decompose in the usual way into inner derivations and the known outer derivations. Our result extends earlier low-dimensional cases and shows a uniform rigidity phenomenon for all higher-dimensional Schrödinger algebras.

Keywords: Lie algebras, semisimple Lie algebra, solvable Lie algebra, nilpotent Lie algebra, Schrödinger algebras, inner derivations, derivations, local derivations.

2020 Mathematics Subject Classification: 17A32, 17B30, 17B10.

Introduction

Local derivations are useful tools in studying the structure of rings and algebras, where there are still many related unsolved problems. R.V. Kadison, D.R. Larson, and A.R. Sourour first introduced the notion of local derivations on algebras in their remarkable paper [1,2]. Since then, many researchers have been studying local derivations of different types of algebras (e.g., see [3–5]). In [6] the authors proved that every local derivation on a finite-dimensional semisimple Lie algebra $\mathcal{L}$ over an algebraically closed field of characteristic zero is a derivation.

In [4], local derivations on solvable Lie algebras are studied. It is shown that within this class, there exist solvable Lie algebras admitting local derivations that are not derivations, as well as solvable Lie algebras for which every local derivation is a derivation. Moreover, it is proved that every local derivation on a finite-dimensional solvable Lie algebra with a model nilradical and a complementary space of maximal dimension is a derivation. In [5], the authors proved that every local derivations on solvable Lie algebras whose nilradical has maximal rank is a derivation. In [3], the authors proved that every local derivation on the conformal Galilei algebra is a derivation.

We note that the aforementioned algebras are finite-dimensional algebras. In the infinite-dimensional case, the authors of [7–9] proved that every local derivation on some class of locally simple Lie algebras, generalized Witt algebras, Witt algebras, and Witt algebras over a field of prime characteristic is a derivation.

[*]Corresponding author. *E-mail: baxtiyor_yusupov_93@mail.ru*
Received: 30 October 2024; *Accepted:* 27 December 2025.

The Schrödinger Lie group is the symmetry group of the free-particle Schrödinger equation (see [10]). The Lie algebra $\mathfrak{s}_n$ in $(n+1)$-dimensional space-time of the Schrödinger Lie group is called the Schrödinger algebra, see [11, 12]. The Schrödinger algebra $\mathfrak{s}_n$ is a non-semisimple Lie algebra and plays an important role in mathematical physics. Recently there was a series of papers on studying the structure and representation theory of the Schrödinger algebra $\mathfrak{s}_1$ in the case of $(1+1)$-dimensional space-time, see [13–15].

In this paper, we generalize our previous result to all integers $n > 2$. In [16], we proved that for $n = 1, 2$, every local derivation on the Schrödinger algebra $\mathfrak{s}_n$ (in $(n+1)$-dimensional space–time) is a derivation. Hence, the same result holds for all $n \in \mathbb{N}$.

1 *Preliminaries*

In this section, we first recall the definition of $\mathfrak{s}_n$ from [11] in a different form. We know that the general linear Lie algebra $\mathfrak{gl}_{2n}$ has the natural representation on $\mathbb{C}^{2n}$ by left matrix multiplication. Let $\{e_1, e_2, \cdots, e_{2n}\}$ be the standard basis of $\mathbb{C}^{2n}$.

The Heisenberg Lie algebra $\mathfrak{h}_n = \mathbb{C}^{2n} \oplus \mathbb{C}z$ is the Lie algebra with Lie bracket given by

$$[e_i, e_{n+i}] = z, \quad [z, \mathfrak{h}_n] = 0, \quad 1 \leq i \leq n.$$

Recall that the Schrödinger Lie algebra $\mathfrak{s}_n$ is the semidirect product Lie algebra

$$\mathfrak{s}_n = (\mathfrak{sl}_2 \oplus \mathfrak{so}_n) \ltimes \mathfrak{h}_n,$$

where $\mathfrak{sl}_2$ is embedded in $\mathfrak{gl}_{2n}$ by the mapping

$$\begin{pmatrix} a & b \\ c & -a \end{pmatrix} \mapsto \begin{pmatrix} aI_n & bI_n \\ cI_n & -aI_n \end{pmatrix}$$

and $\mathfrak{so}_n$ is embedded in $\mathfrak{gl}_{2n}$ by

$$A \in \mathfrak{so}_n \mapsto \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix}.$$

Here I_n is the $n \times n$ identity matrix, $\mathfrak{sl}_2 \oplus \mathfrak{so}_n$ acts on $\mathfrak{h}_n$ by matrix multiplication, and $[z, \mathfrak{s}_n] = 0$.

Next, we will introduce a basis of $\mathfrak{s}_n$. Let

$$h = \begin{pmatrix} I_n & 0 \\ 0 & -I_n \end{pmatrix}, \quad e = \begin{pmatrix} 0 & I_n \\ 0 & 0 \end{pmatrix}, \quad f = \begin{pmatrix} 0 & 0 \\ I_n & 0 \end{pmatrix},$$
$$s_{ij} = \begin{pmatrix} e_{ij} - e_{ji} & 0 \\ 0 & e_{ij} - e_{ji} \end{pmatrix}, \quad 1 \leq i < j \leq n,$$
$$u_k = e_k, v_k = e_{n+k}, \quad 1 \leq k \leq n,$$

where $e_{i,j}$ $(1 \leq i, j \leq n)$ the $n \times n$ matrix with zeros everywhere except a 1 on position (i, j).

The Schrödinger algebra $\mathfrak{s}_n$ is a Lie algebra with a $\mathbb{C}$-basis

$$\{e, f, h, z, u_i, v_i, s_{jk}(= -s_{kj}) \mid 1 \leqslant i \leqslant n,\ 1 \leqslant j < k \leqslant n\}$$

equipped with the following non-trivial commutation relations:

$$[h, e] = 2e, \quad [h, f] = -2f, \quad [e, f] = h,$$
$$[u_i, v_i] = z, \quad [h, u_i] = u_i, \quad [h, v_i] = -v_i,$$
$$[e, v_i] = u_i, \quad [f, u_i] = v_i,$$
$$[s_{kl}, u_i] = \delta_{li} u_k - \delta_{ki} u_l, \quad [s_{kl}, v_i] = \delta_{li} v_k - \delta_{ki} v_l,$$
$$[s_{ij}, s_{kl}] = \delta_{kj} s_{il} + \delta_{il} s_{jk} + \delta_{lj} s_{ki} + \delta_{ki} s_{lj},$$

where δ_{ij} is the Kronecker Delta defined as 1 for $i=j$ and as 0 otherwise.

We fix an order on the basis as follows:

$$\{e, f, h, z, u_i, v_i, s_{jk}(=-s_{kj}) \mid 1 \leq i \leq n,\ 1 \leq j < k \leq n\}.$$

The Schrödinger algebra $\mathfrak{s}_n$ is a finite-dimensional Lie algebra that is neither semisimple nor solvable. It can be realized as the semidirect product

$$\mathfrak{s}_n = (\mathfrak{sl}_2 \oplus \mathfrak{so}_n) \ltimes \mathfrak{h}_n,$$

where $\mathfrak{sl}_2 = \mathrm{Span}_{\mathbb{C}}\{e, f, h\}$ is the 3-dimensional simple Lie algebra, $\mathfrak{so}_n = \mathrm{Span}_{\mathbb{C}}\{s_{kl} \mid 1 \leq k < l \leq n\}$ is the orthogonal Lie algebra, and $\mathfrak{h}_n = \mathrm{Span}_{\mathbb{C}}\{z, u_i, v_i \mid 1 \leq i \leq n\}$ is the Heisenberg Lie algebra.

A derivation on a Lie algebra $\mathcal{L}$ is a linear map $D : \mathcal{L} \to \mathcal{L}$ which satisfies the Leibniz rule:

$$D([x, y]) = [D(x), y] + [x, D(y)], \quad \text{for any} \quad x, y \in \mathcal{L}.$$

For any element $y \in \mathcal{L}$ the operator of right multiplication $\mathrm{ad}_y : \mathcal{L} \to \mathcal{L}$, defined as $\mathrm{ad}_y(x) = [y, x]$ is a derivation, and derivations of this form are called inner derivations. The set of all inner derivations of $\mathcal{L}$, denoted by $\mathrm{Inn}(\mathcal{L})$, is an ideal in $\mathrm{Der}(\mathcal{L})$.

Definition 1. A linear operator Δ is called a local derivation if for any $x \in \mathcal{L}$, there exists a derivation $D_x : \mathcal{L} \to \mathcal{L}$ (depending on x) such that $\Delta(x) = D_x(x)$. The set of all local derivations on $\mathcal{L}$ we denote by $\mathrm{LocDer}(\mathcal{L})$.

We use the following definition given in [17].

Definition 2. The following derivations are outer derivations of $\mathfrak{s}_n$.

- When $n \geq 2$, the derivation $\sigma : \mathfrak{s}_n \to \mathfrak{s}_n$ is given by

$$\sigma(e) = \sigma(f) = \sigma(h) = \sigma(s_{kl}) = 0,\ \ \sigma(z) = z,\ \ \sigma(u_i) = \frac{1}{2}u_i,\ \ \sigma(v_i) = \frac{1}{2}v_i,$$

  for all $1 \leq i \leq n,\ 1 \leq k < l \leq n$.
- When $n = 2$, the derivation $\tau : \mathfrak{s}_2 \to \mathfrak{s}_2$ is given by

$$\tau(e) = \tau(f) = \tau(h) = \tau(z) = \tau(u_i) = \tau(v_i) = 0,\ \ \tau(s_{12}) = z,\ \ i = 1, 2.$$

- When $n = 1$, the derivation $\sigma_1 : \mathfrak{s}_1 \to \mathfrak{s}_1$ is given by

$$\sigma_1(e) = \sigma_1(f) = \sigma_1(h) = 0,\ \ \sigma_1(z) = z,\ \ \sigma_1(u_1) = \frac{1}{2}u_1,\ \ \sigma_1(v_1) = \frac{1}{2}v_1.$$

The following theorem is proved in [17].

Theorem 1. The derivations of the Schrödinger algebra $\mathfrak{s}_n$ are given by

$$\mathrm{Der}(\mathfrak{s}_n) = \begin{cases} \mathrm{Inn}(\mathfrak{s}_1) \oplus \mathbb{C}\sigma_1, & n = 1, \\ \mathrm{Inn}(\mathfrak{s}_2) \oplus \mathbb{C}\sigma \oplus \mathbb{C}\tau, & n = 2, \\ \mathrm{Inn}(\mathfrak{s}_n) \oplus \mathbb{C}\sigma, & n > 2, \end{cases}$$

where σ_1, τ, σ are given by Definition 2.

For any $x \in \mathfrak{s}_n,\ n \geq 3$, there exists

$$a = a_e e + a_f f + a_h h + a_z z + \sum_{i=1}^{n} a_{u_i} u_i + \sum_{i=1}^{n} a_{v_i} v_i + \sum_{1 \leq k < l \leq n} a_{s_{k,l}} s_{k,l}$$

and $\lambda \in \mathbb{C}$ such that, by Theorem 1, we can write

$$D(x) = [a, x] + \lambda\sigma(x).$$

Now consider

$$\begin{aligned}
D(e) &= [a, e] + \lambda\sigma(e) \\
&= \left[a_e e + a_f f + a_h h + a_z z + \sum_{i=1}^{n} a_{u_i} u_i + \sum_{i=1}^{n} a_{v_i} v_i + \sum_{1\le p<q\le n} a_{s_{p,q}} s_{p,q}, e\right] \\
&= 2a_h e - a_f h - \sum_{i=1}^{n} a_{v_i} u_i, \\
D(f) &= [a, f] + \lambda\sigma(f) = -2a_h f + a_e h - \sum_{i=1}^{n} a_{u_i} v_i, \\
D(h) &= [a, h] + \lambda\sigma(h) = -2a_e e + 2a_f f - \sum_{i=1}^{n} a_{u_i} u_i + \sum_{i=1}^{n} a_{v_i} v_i, \\
D(u_i) &= [a, u_i] + \lambda\sigma(u_i) = a_f v_i + \left(a_h + \frac{\lambda}{2}\right) u_i - a_{v_i} z + \sum_{1\le p<i} a_{s_{p,i}} u_p - \sum_{i<q\le n} a_{s_{i,q}} u_q, \\
D(v_i) &= [a, v_i] + \lambda\sigma(v_i) = \left(-a_h + \frac{\lambda}{2}\right) v_i + a_e u_i + a_{u_i} z + \sum_{1\le p<i} a_{s_{p,i}}, v_p - \sum_{i<q\le n} a_{s_{i,q}} v_q, \\
D(s_{k,l}) &= [a, s_{k,l}] + \lambda\sigma(s_{k,l}) = -a_{u_l} u_k + a_{u_k} u_l - a_{v_l} v_k + a_{v_k} v_l + \\
&+ \sum_{1\le p<k,\ p\ne l} a_{s_{p,k}} s_{p,l} + \sum_{l<q\le n,\ q\ne k} a_{s_{l,q}} s_{q,k} + \\
&+ \sum_{1\le p<l,\ p\ne k} a_{s_{p,l}} s_{k,p} + \sum_{k<q\le n,\ l\ne q} a_{s_{k,q}} s_{l,q}.
\end{aligned}$$

2 Main results

In this section, we will prove that every local derivation on the Schrödinger algebra $\mathfrak{s}_n$ is a derivation.

Theorem 2. Every local derivation on the Schrödinger algebra $\mathfrak{s}_n,\ n \ge 3$ is a derivation.

To obtain this result, we first prove several lemmas.

Lemma 1. Let Δ be a local derivation on $\mathfrak{s}_n$ and $D \in \mathrm{Der}(\mathfrak{s}_n)$. Define $\Delta' = \Delta - D$. Then

$$\Delta'(x) \in \mathfrak{h}_n \quad \text{for all } x \in \mathfrak{s}_n. \tag{1}$$

Proof. By Theorem 1, every derivation $D \in \mathrm{Der}(\mathfrak{s}_n)$ can be written in the form

$$D(y) = [a, y] + \lambda\,\sigma(y)$$

for some $a \in \mathfrak{s}_n$ and $\lambda \in \mathbb{C}$, where σ is the outer derivation from Definition 2. Since Δ is a local derivation, for each $x \in \mathfrak{s}_n$ there exists a derivation $D_x \in \mathrm{Der}(\mathfrak{s}_n)$ of the same type such that

$$\Delta(x) = D_x(x) = [a(x), x] + \lambda(x)\,\sigma(x).$$

We use the theorem of Ayupov and Kudaybergenov [6], which asserts that any local derivation on a finite-dimensional semisimple Lie algebra is a derivation. Since $\mathfrak{sl}_2$ and $\mathfrak{so}_n$ are semisimple, their restrictions $\Delta|_{\mathfrak{sl}_2}$ and $\Delta|_{\mathfrak{so}_n}$ are derivations, and hence such a D exists.

We choose a derivation $D \in \mathrm{Der}(\mathfrak{s}_n)$ satisfying

$$D|_{\mathfrak{sl}_2} = \Delta|_{\mathfrak{sl}_2}, \quad D|_{\mathfrak{so}_n} = \Delta|_{\mathfrak{so}_n}.$$

For arbitrary $x \in \mathfrak{s}_n$ we have

$$\Delta'(x) = \Delta(x) - D(x) = [a(x) - a, x] + (\lambda(x) - \lambda)\,\sigma(x). \tag{2}$$

Using (2) and the fact that $a(s) - a \in \mathfrak{h}_n \oplus \mathbb{C}z$, we obtain $\Delta'(s) = 0$ for all $s \in \mathfrak{sl}_2 \cup \mathfrak{so}_n$.

Consequently, $a(x) - a \in \mathfrak{h}_n \oplus \mathbb{C}z$ for any $x \in \mathfrak{s}_n$. Because $\mathfrak{h}_n$ is an ideal of $\mathfrak{s}_n$ and $[\mathbb{C}z, \mathfrak{s}_n] = 0$, we have $[a(x) - a, x] \in \mathfrak{h}_n$. Moreover, by Definition 2, $\sigma(x)$ maps $\mathfrak{s}_n$ into $\mathfrak{h}_n$. Therefore, both terms in (2) belong to $\mathfrak{h}_n$, and hence

$$\Delta'(x) \in \mathfrak{h}_n \quad \text{for all} \quad x \in \mathfrak{s}_n.$$

□

For each $x \in \mathfrak{s}_n$, there exist an element $a = a(x) \in \mathfrak{s}_n$ of the form

$$a = a_e e + a_f f + a_h h + a_z z + \sum_{i=1}^{n} a_{u_i} u_i + \sum_{i=1}^{n} a_{v_i} v_i + \sum_{1 \le k < l \le n} a_{s_{k,l}}\, s_{k,l},$$

and a scalar $\lambda = \lambda(x) \in \mathbb{C}$ such that, by Theorem 1,

$$\Delta'(x) = [a, x] + \lambda\,\sigma(x).$$

Here $a_e, a_f, a_h, a_z, a_{u_i}, a_{v_i}, a_{s_{k,l}}$, and λ are complex numbers depending on x.

By applying (1) to $x = h$ and $x = z$, we get

$$\begin{aligned}\Delta'(h) &= -\sum_{i=1}^{n} a_{u_i}^{(h)} u_i + \sum_{i=1}^{n} a_{v_i}^{(h)} v_i,\\ \Delta'(z) &= \lambda^{(z)} z.\end{aligned}$$

Let $x_0 = \sum\limits_{i=1}^{n} a_{u_i}^{(h)} u_i + \sum\limits_{i=1}^{n} a_{v_i}^{(h)} v_i$. Consider the following statement

$$\Delta'' = \Delta' - \mathrm{ad}(x_0) - \lambda^{(h)}\sigma.$$

Then Δ'' is a local derivation. By direct verification we have

$$\Delta''(x) \in \mathfrak{h}_n, \quad \text{for all} \quad x \in \mathfrak{s}_n, \tag{3}$$

and

$$\Delta''(h) = \Delta''(z) = 0.$$

Considering (3), we find the values of the operator Δ'' in the basis elements:

$$\begin{aligned}
\Delta''(f) &= -\sum_{i=1}^{n} a_{u_i}^{(f)} v_i,\\
\Delta''(e) &= -\sum_{i=1}^{n} a_{v_i}^{(e)} u_i,\\
\Delta''(u_i) &= a_f^{(u_i)} v_i + \left(a_h^{(u_i)} + \frac{\lambda^{(u_i)}}{2}\right) u_i - a_{v_i}^{(u_i)} z + \sum_{1 \le k < i} a_{s_{k,i}}^{(u_i)} u_k - \sum_{i < l \le n} a_{s_{i,l}}^{(u_i)} u_l,\\
\Delta''(v_i) &= \left(\frac{\lambda^{(v_i)}}{2} - a_h^{(v_i)}\right) v_i + a_e^{(v_i)} u_i + a_{u_i}^{(v_i)} z + \sum_{1 \le k < i} a_{s_{k,i}}^{(v_i)} v_k - \sum_{i < l \le n} a_{s_{i,l}}^{(v_i)} v_l,\\
\Delta''(s_{k,l}) &= -a_{u_l}^{(s_{k,l})} u_k + a_{u_k}^{(s_{k,l})} u_l - a_{v_l}^{(s_{k,l})} v_k + a_{v_k}^{(s_{k,l})} v_l.
\end{aligned} \tag{4}$$

We take an element $b = b_e e + b_f f + b_h h + b_z z + \sum\limits_{i=1}^{n} b_{u_i} u_i + \sum\limits_{i=1}^{n} b_{v_i} v_i + \sum\limits_{1\le k<l\le n} b_{s_{k,l}} s_{k,l}$ and $\mu \in \mathbb{C}$, where $b \in \mathfrak{s}_n$, $b_e, b_f, b_h, b_z, b_{u_i}, b_{v_i}, b_{s_{k,l}}, \mu$ are complex numbers depending on b.

Lemma 2. Coefficients $a_f^{(u_i)}$ and $a_e^{(v_i)}$, $(1 \le i \le n)$ in the formula (4) are equal to zero.

Proof. Fix i with $1 \le i \le n$ and set $x = e + u_i$. By the definition of a local derivation, there exist $b = b(x) \in \mathfrak{s}_n$ and $\mu = \mu(x) \in \mathbb{C}$ such that

$$\Delta''(x) = [b, x] + \mu\, \sigma(x).$$

Hence

$$\begin{aligned}\Delta''(x) &= \Delta''(e + u_i) = [b, e + u_i] + \mu\, \sigma(e + u_i)\\ &= \Big[b_e e + b_f f + b_h h + b_z z + \sum_{j=1}^{n} b_{u_j} u_j + \sum_{j=1}^{n} b_{v_j} v_j + \sum_{1\le k<l\le n} b_{s_{k,l}} s_{k,l},\ e + u_i\Big] + \mu\, \sigma(e + u_i)\\ &= -b_f h + b_f v_i + *e + \sum_{j=1}^{n} *u_j + *z.\end{aligned}$$

On the other hand, based on (4), we calculate the following equality:

$$\Delta''(x) = \Delta''(e + u_i) = \Delta''(e) + \Delta''(u_i) = a_f^{(u_i)} v_i + \sum_{j=1}^{n} *u_j + *z.$$

Comparing the coefficients at the basis elements h and v_i, we get $b_f = 0,\ b_f = a_f^{(u_i)}$, which implies

$$a_f^{(u_i)} = 0.$$

Now, consider the element $x = f + v_i$ for fixed $1 \le i \le n$,

$$\Delta''(x) = \Delta''(f + v_i) = [b, f + v_i] + \mu\sigma(f + v_i) = b_e h + b_e u_i + *f + \sum_{j=1}^{n} *v_j + *z.$$

On the other hand,

$$\Delta''(x) = \Delta''(f + v_i) = \Delta''(f) + \Delta''(v_i) = a_e^{(v_i)} u_i + \sum_{j=1}^{n} *v_j + *z.$$

Comparing the coefficients at the basis elements h and u_i, we get $b_e = 0,\ b_e = a_e^{(v_i)}$, which implies

$$a_e^{(v_i)} = 0.$$

□

Lemma 3. Coefficients $a_{v_i}^{(u_i)}$ and $a_{u_i}^{(v_i)}$, $1 \le i \le n$ in the formula (4) are equal to zero.

Proof. Fix i with $1 \le i \le n$ and set $x = h + u_i$. Then

$$\begin{aligned}\Delta''(x) &= [b, h + u_i] + \mu\, \sigma(h + u_i)\\ &= 2b_f f + \sum_{j=1}^{n} b_{v_j} v_j + b_f v_i - b_{v_i} z + *e + \sum_{j=1}^{n} *u_j.\end{aligned}$$

On the other hand,

$$\Delta''(x) = \Delta''(h + u_i) = \Delta''(h) + \Delta''(u_i) = -a^{(u_i)}_{v_i} z + \sum_{j=1}^{n} * u_j.$$

Comparing the coefficients at the basis elements $f,\ z$ and v_i, we get $b_f = b_{v_i} = 0,\ b_{v_i} = a^{(u_i)}_{v_i}$, which implies

$$a^{(u_i)}_{v_i} = 0.$$

Fix i with $1 \le i \le n$ and set $x = h + v_i$. Then

$$\begin{aligned}\Delta''(x) &= \Delta''(h + v_i) = [b, h + v_i] + \mu\sigma(h + v_i)\\ &= -2b_e e - \sum_{j=1}^{n} b_{u_j} u_j + b_e u_i + b_{u_i} z + *f + \sum_{j=1}^{n} * v_j.\end{aligned}$$

On the other hand,

$$\Delta''(x) = \Delta''(h + v_i) = \Delta''(h) + \Delta''(v_i) = a^{(v_i)}_{u_i} z + \sum_{j=1}^{n} * v_j.$$

Comparing the coefficients at the basis elements $e,\ z$ and u_i, we get $b_e = b_{u_i} = 0,\ b_{u_i} = a^{(v_i)}_{u_i}$, which implies

$$a^{(v_i)}_{u_i} = 0.$$ □

Lemma 4. $\Delta''(f) = 0$ and $a^{(v_i)}_{h} - \frac{\lambda^{(v_i)}}{2} = 0$ in the formula (4).

Proof. Take an element $x = f - \frac{1}{2}z + v_i$ $(1 \le i \le n)$. Then

$$\begin{aligned}\Delta''(x) &= \left[b,\ f - \tfrac{1}{2}z + v_i\right] + \mu\,\sigma\left(f - \tfrac{1}{2}z + v_i\right)\\ &= -2b_h f + b_e h - \sum_{j=1}^{n} b_{u_j} v_j - \frac{\mu}{2} z - b_h v_i + b_e u_i + b_{u_i} z\\ &\quad + \sum_{1 \le k < i} b_{s_{k,i}} v_k - \sum_{i < l \le n} b_{s_{i,l}} v_l + \frac{\mu}{2} v_i.\end{aligned} \tag{5}$$

On the other hand,

$$\begin{aligned}\Delta''(x) &= \Delta''(f) - \Delta''(\frac{z}{2}) + \Delta''(v_i) = -\sum_{i=1}^{n} a^{(f)}_{u_i} v_i - a^{(v_i)}_{h} v_i\\ &\quad + \sum_{1 \le k < i} a^{(v_i)}_{s_{k,i}} v_k - \sum_{i < l \le n} a^{(v_i)}_{s_{i,l}} v_l + \frac{\lambda^{(v_i)}}{2} v_i.\end{aligned} \tag{6}$$

Comparing the coefficients at the basis elements $f,\ z$ and v_i, (5) and (6), we get

$$\begin{cases} -2b_h &= 0,\\ -\frac{\mu}{2} + b_{u_i} &= 0,\\ -b_{u_i} - b_h + \frac{\mu}{2} &= -a^{(f)}_{u_i} - a^{v_i}_{h} + \frac{\lambda^{v_i}}{2}, \end{cases}$$

which implies

$$a_{u_i}^{(f)} = -a_h^{(v_i)} + \frac{\lambda^{(v_i)}}{2}. \tag{7}$$

Take an element $x = f - \frac{1}{2}z - v_i$. Then

$$\begin{aligned}\Delta''(x) = \left[b, f - \frac{1}{2}z - v_i\right] + \mu\sigma(f - \frac{1}{2}z - v_i) = -2b_h f + b_e h - \sum_{j=1}^{n} b_{u_j} v_j \\ - \frac{\mu}{2}z + b_h v_i - b_e u_i - b_{u_i} z - \sum_{1\le k<i} b_{s_{k,i}} v_k + \sum_{i<l\le n} b_{s_{i,l}} v_l - \frac{\mu}{2} v_i.\end{aligned} \tag{8}$$

On the other hand,

$$\begin{aligned}\Delta''(x) = \Delta''(f) - \Delta''(\frac{z}{2}) - \Delta''(v_i) = -\sum_{j=1}^{n} a_{u_j}^{(f)} v_j + a_h^{(v_i)} v_i \\ - \sum_{1\le k<i} a_{s_{k,i}}^{(v_i)} v_k + \sum_{i<l\le n} a_{s_{i,l}}^{(v_i)} v_l - \frac{\lambda^{(v_i)}}{2} v_i.\end{aligned} \tag{9}$$

Comparing the coefficients at the basis elements $f,\ z$ and v_i, (8) and (9), we get

$$\begin{cases} -2b_h & = 0, \\ -\frac{\mu}{2} - b_{u_i} & = 0, \\ -b_{u_i} + b_h - \frac{\mu}{2} & = -a_{u_i}^{(f)} + a_h^{v_i} - \frac{\lambda^{v_i}}{2}, \end{cases}$$

which implies

$$a_{u_i}^{(f)} = a_h^{(v_i)} - \frac{\lambda^{(v_i)}}{2}. \tag{10}$$

Comparing (7) and (10), we obtain that

$$a_{u_i}^{(f)} = 0, \quad a_h^{(v_i)} = \frac{\lambda^{(v_i)}}{2}.$$

So, $\Delta''(f) = 0$ follows from equality (4). Thus, the coefficients satisfy the relation

$$a_h^{(v_i)} - \frac{\lambda^{(v_i)}}{2} = 0.$$

□

Lemma 5. $\Delta''(e) = 0$ and $a_h^{(u_i)} + \frac{\lambda^{(u_i)}}{2} = 0$ in the formula (4).

Proof. Take an element $x = e + \frac{1}{2}z + u_i$. Then

$$\begin{aligned}\Delta''(x) &= [b, e + \frac{1}{2}z + u_i] + \mu\sigma(e + \frac{1}{2}z + u_i) \\ &= 2b_h e - b_f h - \sum_{j=1}^{n} b_{v_j} u_j + \frac{\mu}{2} z \\ &+ b_f v_i + b_h u_i - b_{v_i} z + \sum_{1\le k<i} b_{s_{k,i}} u_k - \sum_{i<l\le n} b_{s_{i,l}} u_l + \frac{\mu}{2} u_i.\end{aligned} \tag{11}$$

On the other hand,

$$\begin{aligned}\Delta''(x) = \Delta''(e) + \Delta''(\frac{1}{2}z) + \Delta''(u_i) = -\sum_{j=1}^{n} a_{v_j}^{(e)} u_j + a_h^{(u_i)} u_i \\ + \sum_{1\le k<i} a_{s_{k,i}}^{(u_i)} u_k - \sum_{i<l\le n} a_{s_{i,l}}^{(u_i)} u_l + \frac{\lambda^{(u_i)}}{2} u_i.\end{aligned} \tag{12}$$

Comparing the coefficients at the basis elements $e,\ z$ and u_i, (11) and (12), we obtain that

$$\begin{cases} 2b_h & = 0, \\ \frac{\mu}{2} - b_{v_i} & = 0, \\ -b_{v_i} + b_h + \frac{\mu}{2} & = -a_{v_i}^{(e)} + a_h^{u_i} + \frac{\lambda^{u_i}}{2}, \end{cases}$$

which implies

$$a_{v_i}^{(e)} = a_h^{(u_i)} + \frac{\lambda^{(u_i)}}{2}. \tag{13}$$

Next, we take an element $x = e + \frac{1}{2}z - u_i$, then

$$\begin{aligned}\Delta''(x) = [b, e + \frac{1}{2}z - u_i] + \mu\sigma(e + \frac{1}{2}z - u_i) = 2b_h e - b_f h - \sum_{j=1}^{n} b_{v_j} u_j + \frac{\mu}{2} z \\ - b_f v_i - b_h u_i + b_{v_i} z - \sum_{1\le k<i} b_{s_{k,i}} u_k + \sum_{i<l\le n} b_{s_{i,l}} u_l - \frac{\mu}{2} u_i.\end{aligned} \tag{14}$$

On the other hand,

$$\begin{aligned}\Delta''(x) = \Delta''(e) + \Delta''(\frac{z}{2}) - \Delta''(u_i) = -\sum_{j=1}^{n} a_{v_j}^{(e)} u_j - a_h^{(u_i)} u_i \\ - \sum_{1\le k<i} a_{s_{k,i}}^{(u_i)} u_k + \sum_{i<l\le n} a_{s_{i,l}}^{(u_i)} u_l - \frac{\lambda^{(u_i)}}{2} u_i.\end{aligned} \tag{15}$$

Comparing the coefficients at the basis elements $e,\ z$ and u_i, (14) and (15), we get

$$\begin{cases} 2b_h & = 0, \\ \frac{\mu}{2} + b_{v_i} & = 0, \\ -b_{v_i} - b_h - \frac{\mu}{2} & = -a_{v_i}^{(e)} - a_h^{u_i} - \frac{\lambda^{u_i}}{2}, \end{cases}$$

which implies

$$a_{v_i}^{(e)} = -a_h^{(u_i)} - \frac{\lambda^{(u_i)}}{2}. \tag{16}$$

Comparing (13) and (16), we obtain that

$$a_{v_i}^{(e)} = 0, \quad a_h^{(u_i)} = -\frac{\lambda^{(u_i)}}{2}.$$

Thus, $\Delta''(e) = 0$ follows from equality (4). We have the following connection

$$a_h^{(u_i)} + \frac{\lambda^{(u_i)}}{2} = 0$$

between the coefficients. □

Lemma 6. $\Delta''(\mathfrak{so}_n) = \{0\}$ and $\Delta''(\mathfrak{h}_n) = \{0\}$.

Proof. Let k, l $(k \neq l)$ fixed numbers in the set $\{1, 2, ..., n\}$. Next, set $x = u_k + s_{k,l}$ (if $l < k$, then $s_{k,l} = -s_{l,k}$). Then

$$\begin{aligned} \Delta''(x) &= [b, u_k + s_{k,l}] + \mu\sigma(u_k + s_{k,l}) \\ &= b_f v_k + b_h u_k - b_{v_k} z + \sum_{1\le p<k} b_{s_{p,k}} u_p \\ &- \sum_{k<q\le n} b_{s_{k,q}} u_q + \frac{\mu}{2} u_k - b_{u_l} u_k + b_{u_k} u_l - b_{v_l} v_k + b_{v_k} v_l \\ &+ \sum_{1\le p<k,\ p\neq l} b_{s_{p,k}} s_{p,l} + \sum_{l<q\le n,\ q\neq k} b_{s_{l,q}} s_{q,k} \\ &+ \sum_{1\le p<l,\ p\neq k} b_{s_{p,l}} s_{k,p} + \sum_{k<q\le n,\ l\neq q} b_{s_{k,q}} s_{l,q}. \end{aligned} \tag{17}$$

On the other hand,

$$\begin{aligned} \Delta''(x) = \Delta''(u_k) + \Delta(s_{k,l}) &= \sum_{1\le j<k} a^{(u_k)}_{s_{j,k}} u_j - \sum_{k<j\le n} a^{(u_i)}_{s_{k,j}} u_j \\ &- a^{(s_{k,l})}_{u_l} u_k + a^{(s_{k,l})}_{u_k} u_l - a^{(s_{k,l})}_{v_l} v_k + a^{(s_{k,l})}_{v_k} v_l. \end{aligned} \tag{18}$$

Comparing the coefficients at the basis elements z and v_l, (17) and (18), we obtain that

$$\begin{cases} b_{v_k} &= 0, \\ b_{v_k} &= a^{(s_{k,l})}_{v_k}, \end{cases}$$

which implies

$$a^{(s_{k,l})}_{v_k} = 0.$$

Similarly, from equality:

$$\begin{aligned} &\Delta''(u_l + s_{l,k}) = \Delta''(u_k) + \Delta''(s_{l,k}) \text{ we obtain } a^{(s_{k,l})}_{v_l} = 0; \\ &\Delta''(v_k + s_{k,l}) = \Delta''(v_k) + \Delta''(s_{k,l}) \text{ we obtain } a^{(s_{k,l})}_{u_k} = 0; \\ &\Delta''(v_l + s_{l,k}) = \Delta''(v_k) + \Delta''(s_{l,k}) \text{ we obtain } a^{(s_{k,l})}_{u_l} = 0. \end{aligned}$$

If we substitute the above four results into (4), we get $\Delta''(s_{l,k}) = 0,\ 1 \le l < k \le n$.
Then equation (18) can be rewritten as

$$\Delta''(x) = \Delta''(u_k) + \Delta(s_{k,l}) = \sum_{1\le j<k} a^{(u_k)}_{s_{j,k}} u_j - \sum_{k<j\le n} a^{(u_k)}_{s_{k,j}} u_j. \tag{19}$$

Comparing the coefficients at the basis elements u_j $(j \neq k, j \neq l)$ and $s_{j,l}$, (17) and (19), we obtain that

- if $j < k$, then $b_{s_{j,k}} = 0$ and $b_{s_{j,k}} = a^{(u_k)}_{s_{j,k}}$;
- if $j > k$, then $b_{s_{k,j}} = 0$ and $b_{s_{k,j}} = a^{(u_k)}_{s_{k,j}}$;

we get

$$a_{s_{k,j}}^{(u_k)} = 0, \quad (j \neq k). \tag{20}$$

Take a number i such that $i \neq k$ and $i \neq l$.

$$\begin{aligned} \Delta''(u_k + s_{k,i}) &= \Delta''(u_k) + \Delta''(s_{k,i}) \ \Rightarrow \ a_{s_{k,i}}^{(u_k)} = 0, \\ \Delta''(v_k + s_{k,j}) &= \Delta''(v_k) + \Delta''(s_{k,j}) \ \Rightarrow \ a_{s_{k,j}}^{(v_k)} = 0, \quad (j \neq k), \\ \Delta''(v_k + s_{k,i}) &= \Delta''(v_k) + \Delta''(s_{k,i}) \ \Rightarrow \ a_{s_{k,i}}^{(v_k)} = 0. \end{aligned} \tag{21}$$

Thus, according to (4), (20) and (21), we have

$$\Delta''(\mathfrak{so}_n) = \{0\} \quad \text{and} \quad \Delta''(\mathfrak{h}_n) = \{0\}.$$ □

Now we are in position to prove Theorem 2.
Proof of Theorem 2. From (2) and Lemmas 2–6 we obtain

$$\Delta'' = 0. \tag{22}$$

Together (2) and (22) give

$$\Delta' = \mathrm{ad}(x_0) + \lambda^{(h)}\sigma. \tag{23}$$

Together (1) and (23) give

$$\Delta = D + \mathrm{ad}(x_0) + \lambda^{(h)}\sigma.$$

Hence, any local derivation of the algebra $\mathfrak{s}_n$ $(n \geq 3)$ is a derivation. □

Conclusion

We study local derivations on the Schrödinger algebra $\mathfrak{s}_n$ in $(n+1)$-dimensional space-time of Schrödinger Lie groups for any integer n. We prove that every local derivations on the Schrödinger algebra $\mathfrak{s}_n$ in $(n+1)$-dimensional space-time are derivations.

Author Contributions

All authors contributed equally to this work.

Conflict of Interest

The authors declare no conflict of interest.

References

1 Kadison, R.V. (1990). Local derivations. *Journal of Algebra, 130*(2), 494–509. https://doi.org/10.1016/0021-8693(90)90095-6

2 Larson, D.R., & Sourour, A.R. (1990). Local derivations and local automorphisms of $B(X)$. *Proceedings of Symposia in Pure Mathematics, 51*, 187–194.

3 Alauadinov, A.K., & Yusupov, B.B. (2024). Local derivations of conformal Galilei algebra. *Communications in Algebra, 52*(6), 2489–2508. https://doi.org/10.1080/00927872.2023.2301539

4 Ayupov, Sh., & Khudoyberdiyev, A. (2021). Local derivations on solvable Lie algebras. *Linear and Multilinear Algebra, 69*(7), 1286–1301. https://doi.org/10.1080/03081087.2019.1626336

5 Kudaybergenov, K.K., Omirov, B.A., & Kurbanbaev, T.K. (2022). Local derivations on solvable Lie algebras of maximal rank. *Communications in Algebra, 50*(9), 1–11. https://doi.org/10.1080/00927872.2022.2045494

6 Ayupov, Sh.A., & Kudaybergenov, K.K. (2016). Local derivations on finite-dimensional Lie algebras. *Linear Algebra and its Applications, 493*, 381–398. https://doi.org/10.1016/j.laa.2015.11.034

7 Ayupov, Sh.A., Kudaybergenov, K.K., & Yusupov, B.B. (2024). Local and 2-Local Derivations of Locally Simple Lie Algebras. *Journal of Mathematical Sciences, 278*(4), 613–622. https://doi.org/10.1007/s10958-024-06943-2

8 Chen, Y., Zhao, K., & Zhao, Y. (2022). Local derivations on Witt algebras. *Linear and multilinear algebra, 70*(6), 1159–1172. https://doi.org/10.1080/03081087.2020.1754750

9 Yao, Y.-F. (2020). Local derivations on the Witt algebra in prime characteristic. *Linear and multilinear algebra, 70*(15), 1–15. https://doi.org/10.1080/03081087.2020.1819189

10 Perroud, M. (1997). Projective representations of the Schrödinger group. *Helvetica Physica Acta, 50*, 233–252.

11 Dobrev, V.K., Doebner, H.D., & Mrugalla, C. (1997). Lowest weight representations of the Schrödinger algebra and generalized heat equations. *Reports on Mathematical Physics, 39*(2), 201–218. https://doi.org/10.1016/S0034-4877(97)88001-9

12 Liu, G., Li, Y., & Wang, K. (2021). Irreducible weight modules over the Schrödinger Lie algebra in (n+1) dimensional space-time. *Journal of Algebra, 575*, 1–13. https://doi.org/10.1016/j.jalgebra.2021.01.034

13 Aizawa, N., & Dobrev, V. (2010). Intertwining operator realization of non-relativistic holography. *Nuclear Physics B, 828*(3), 581–593. https://doi.org/10.1016/j.nuclphysb.2009.10.019

14 Yang Y., & Tang, X. (2018). Derivation of the Schrödinger algebra and their applications. *Journal of Applied Mathematics and Computing, 58*, 567–576. https://doi.org/10.1007/s12190-017-1157-5

15 Zhang, X., & Cheng, Y. (2015). Simple Schrödinger modules which are locally finite over the positive part. *Journal of Pure and Applied Algebra, 219*(7), 2799–2815. https://doi.org/10.1016/j.jpaa.2014.09.029

16 Alauadinov, A.K., & Yusupov, B.B. (2025). Local derivations of the Schrödinger algebras. *Algebra Colloquium, 32*(3), 467–480. https://doi.org/10.1142/S1005386725000343

17 Wu, Q., & Tang, X. (2022). Derivations and Biderivations of the Schrodinger algebra in $(n+1)$-dimensional space-time. *Linear and Multilinear Algebra, 71*(7), 1073–1097. https://doi.org/10.1080/03081087.2022.2050884

Author Information[*]

Amir Kadirbergenovich Alauadinov — PhD in Physical and Mathematical Sciences, Associate Professor, Department of Mathematics, Karakalpak State University, 1 Ch. Abdirov St., Nukus, 230113, Uzbekistan; e-mail: *amir_t85@mail.ru*; https://orcid.org/0000-0002-1492-4939

Bakhtiyor Bakhrambek ugli Yusupov (*corresponding author*) — PhD in Physical and Mathematical Sciences, Senior Researcher, V.I. Romanovskiy Institute of Mathematics Uzbekistan Academy of Sciences, 9 University St., Tashkent, 100174, Uzbekistan; e-mail: *baxtiyor_yusupov_93@mail.ru*; https://orcid.org/0000-0001-9400-5811

[*]Authors' names are presented in the following order: first name, middle name (if any), last name.