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On certain identities of generalized derivations of semirings with involution

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MA-semirings form a proper subclass of inverse semirings that properly contains both the class of rings and the class of distributive lattices with the least element. In this paper, we study generalized derivations satisfying certain algebraic identities of MA-semirings with involution. The main objective of this research is to investigate identities involving three, two, one generalized derivation in MA-semirings with involution, ensuring commutativity. Hermitian and skew-Hermitian elements are primarily used to formulate the basic tools for the development of this paper and these notions are the fundamental units of the second kind involution. Involution of the second kind plays a key role not only for proving the main results (see Theorems 1, 3, 5) but also it enables us to observe more results from their proofs (see Theorems 2, 4, 6). Since every derivation is a generalized derivation, the results obtained naturally extend a variety of results on derivations. Moreover, several well-established results on derivations of MA-semirings and rings under the similar environment can be concluded as special cases.

Keywords: semirings, MA-semirings, prime semirings, Hermitian elements, skew-Hermitian elements, involution, second kind involution, derivations, generalized derivations.

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Introduction

The theory of semirings has tremendous and direct applications in the sciences. For instance, idempotent analysis based on additive inverse semirings has interesting applications in quantum physics (see [1,2]), and the same algebraic structure is used to develop the formal languages [3,4] and automata theory [4–6]. One can find the applications of semirings in other fields of science and mathematics such as theoretical computer sciences and engineering, parallel computational systems, optimization theory, combinatorics, functional analysis, topology, graph theory, Euclidean geometry, and mathematical modeling of quantum physics (see [7–9]). Moreover, semirings have some notable applications in cryptography (see [10, 11]). B^*-algebras as well as C^*-algebras are well-known examples of rings with involution (see [12–14]) in the canvass of functional analysis, which is indeed a primitive source of motivation for ring theorists. For the ring's theoretical background, we would like to refer to [15–17].

The class of MA-semirings [18] has a significant potential to accommodate the study of derivations and generalized derivations satisfying different identities on semirings with involution [19–21] and without involution [22–24] for exploring commuting conditions and other features. In the present paper, we generalize a few results of [25] in the framework MA-semirings with second-kind involution.

In the next section, we include some necessary preliminaries for the sake of completeness and examples for exploring the features of this paper.

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1 Preliminaries and Examples

An additive commutative inverse semiring $(S,+,.)$ with absorbing zero 0 and center $Z(S)$ is said to be an MA-semiring [18] if $w+w' \in Z(S)$ for all $w \in S$, where w' denotes the pseudo inverse of w, which is indeed unique (see [8, 9]). Throughout the paragraph, by S we mean an MA-semiring. A mapping $\varrho : S \longrightarrow S$ is a derivation if $\varrho(w+v) = \varrho(w)+\varrho(v)$ and $\varrho(wv) = \varrho(w)v + w\varrho(v)$. An additive mapping $F_\varrho : S \longrightarrow S$ is said to be a generalized derivation associated with a derivation ϱ, if $F_\varrho(uv) = F_\varrho(u)v + u\varrho(v)$. The commutator and anti-commutator of $w, v \in S$ are respectively defined as $[w,v] = wv+v'w$ and $w \circ v = wv+vw$. Involution is an additive mapping $* : S \longrightarrow S$ satisfying $(w^*)^* = w$ and $(vw)^* = w^*v^*$, for all $v, w \in S$. $\mathbb{H}(S) = \{w \in S : w^* = w\}$ and $\mathbb{K}(S) = \{w \in S : w^* = w'\}$ respectively represents the sets of Hermitian and skew Hermitian elements of S. Involution of the second kind was introduced in [26] in the framework of MA-semirings. If $Z(S) \nsubseteq \mathbb{H}(S)$, then involution is of second kind otherwise it is of first kind.

Example 1. [27] Let $(\mathbb{Z},+,.)$ be the ring of integers and $I(\mathbb{Z})$ be the collection of all ideals of $\mathbb{Z}$. Consider the set $S = M_2(\mathbb{Z}) \times I(\mathbb{Z})$ and let $u = (A_1, I), v = (A_2, J) \in S$. Define addition $\oplus$ and multiplication $\odot$ by $u \oplus v = (A_1 + A_2, I + J)$ and $u \odot v = (A_1A_2, IJ)$. Then $(S, \oplus, \odot)$ is an example of a proper MA-semiring. Furthermore, define a mapping $* : S \longrightarrow S$ by $(A, I)^* = (A^T, I)$, where A^T is the transpose of A. Then $*$ defines an involution on S. We further see that $Z(S) \subseteq \mathbb{H}(S)$, therefore $*$ is an involution of first kind.

Example 2. [27] Let $\mathbb{Z}$ be the set of integers, $\mathbb{Z}_0^+$ be the set of all non-negative integers and $R = \mathbb{Z}\times\mathbb{Z}_0^+$. Define addition $\oplus$ and multiplication $\odot$ by $(u_1, v_1) \oplus (u_2, v_2) = (u_1 + u_2, v_1 \vee v_2)$ and $(u_1, v_1) \odot (u_2, v_2) = (u_1.u_2, v_1.v_2)$, where $v_1 \vee v_2 = max\{v_1, v_2\}$. Then the triplet $(R, \oplus, \odot)$ forms an MA-semiring which is not a ring. One can observe that

$$M_R = \left\{ \begin{bmatrix} w & v & u & x \\ 0 & w & 0 & u \\ 0 & 0 & w & v' \\ 0 & 0 & 0 & w \end{bmatrix} : u, v, w, x \in R \right\}$$

(where v' is the pseudo inverse of v) is an MA-semiring under matrix addition and multiplication. Next, we define a mapping $* : M_R \longrightarrow M_R$ by

$$\begin{bmatrix} w & v & u & x \\ 0 & w & 0 & u \\ 0 & 0 & w & v' \\ 0 & 0 & 0 & w \end{bmatrix}^* = \begin{bmatrix} w & v & u & x' \\ 0 & w & 0 & u \\ 0 & 0 & w & v' \\ 0 & 0 & 0 & w \end{bmatrix}.$$

The mapping $*$ defines a second kind involution on M_R.

Example 3. [28] Let $(R,+,.)$ be a ring and $I(R)$ be the collection of all ideals of R. Consider the set $S = R \times I(R)$ and let $u = (r_1, I), v = (r_2, J) \in S$. Define addition $\oplus$ and multiplication $\odot$ by $u \oplus v = (r_1 + r_2, I + J)$ and $u \odot v = (r_1r_2, IJ)$. Then $(S, \oplus, \odot)$ forms an MA-semiring which is not a ring.

Throughout the sequel by a semiring S, we mean an MA-semiring S unless mentioned otherwise. Furthermore, we take $h_z, \tilde{h}_z \in \mathbb{H}(S) \cap Z(S)$ and $k_z \in \mathbb{K}(S) \cap Z(S)$, for the sake of convenience.

Following results are indeed useful to establish the main results of this paper.

Lemma 1. [18] Let S be a semiring and ϱ be a derivation of S. Then for all $u, v, w \in S,\ z \in Z(S)$, we have

(i) $[w, wu] = w[w, u]$,
(ii) $[w, uv] = [w, u]v + u[w, v]$,
(iii) $[wu, v] = w[u, v] + [w, v]u$,
(iv) $(wu)' = w'u = wu'$,
(v) $[w, u] + [u, w] = u(w + w') = w(u + u')$,
(vi) $[w, u]' = [w, u'] = [w', u] = [u, w]$,
(vii) $w \circ (u + v) = w \circ u + w \circ v$,
(viii) $\varrho(w') = (\varrho(w))'$,
(ix) $(w')^* = ((w)^*)'$,
(x) $[w, uz] = z[w, u] = [w, u]z$,
(xi) $[w, w] = [w, w]'$,
(xii) $w + u = 0 \ \Rightarrow w = u'$, however the converse may not hold in general.

For more such identities, one can see [28–30].

Lemma 2. [31] Let S be a semiprime semiring with involution $*$ of second kind. Then $\mathbb{K}(S) \cap Z(S) \neq \{0\}$ and hence $\mathbb{H}(S) \cap Z(S) \neq \{0\}$.

The following lemma is readily discernible from the definitions of Hermitian and the skew Hermitian elements of a semiring with second kind involution.

Lemma 3. [27] If S is a semiring with second kind involution $*$, then for any $k \in \mathbb{K}(S)$ and $h \in \mathbb{H}(S)$, we have

(i) $k^2 \in \mathbb{H}(S)$,
(ii) $hh_z \in \mathbb{H}(S)$,
(iii) $kk_z \in \mathbb{H}(S)$,
(iv) $hk_z \in \mathbb{K}(S)$.

Lemma 4. [27] Let ϱ be a derivation of a 2-torsion free prime semiring S with involution $*$ of second kind. If $\varrho(h_z) = 0$, then $\varrho(k_z) = 0$.

Lemma 5. [27] Let ϱ be a derivation of a 2-torsion free prime semiring S with involution $*$ of second kind. If $\varrho(h_z) = 0$, then $\varrho(z) = 0$, for all $z \in Z(S)$.

Following lemma is a special case of Theorem 2.2 of [31].

Lemma 6. [31] Let ϱ be a nonzero derivation of a prime semiring S such that $[\varrho(w), w] = 0$, for all $w \in S$. Then S is commutative.

Lemma 7. Let F_σ be a generalized derivation associated with a nonzero derivation σ of a prime semiring S. If $[F_\sigma(w), w] = 0$, for all $w \in S$, then S is commutative.

Proof. The hypothesis states that

$$[F_\sigma(w), w] = 0. \tag{1}$$

Linearizing (1) and again using (1), we get

$$[F_\sigma(w), s] + [F_\sigma(s), w] = 0. \tag{2}$$

Substituting ws for w in (2), we find

$$[F_\sigma(w), s]s + [w\sigma(s), s] + [F_\sigma(s), w]s + w[F_\sigma(s), s] = 0.$$

Using (1) and (2) in the last expression, we get

$$[w\sigma(s), s] = 0. \tag{3}$$

In (3) substituting rw for w, and using Lemma 1, we obtain

$$0 = [rw\sigma(s), s] = r[w\sigma(s), s] + [r, s]w\sigma(s),$$

and using (3) again, we obtain

$$[r,s]w\sigma(s)=0. \tag{4}$$

Multiplying (4) by s from the right, we get

$$[r,s]w\sigma(s)s=0. \tag{5}$$

In (4) writing ws' for w, we obtain

$$[r,s]ws'\sigma(s)=0. \tag{6}$$

Adding (5) and (6) and then substituting $\sigma(s)$ for r, we get $[\sigma(s),s]S[\sigma(s),s]=0$. As S is prime, we can write $[\sigma(s),s]=0$ for all $s\in S$. By Lemma 6, S is commutative. □

2 Main Results

In this section, we present the key results of this research article. We investigate several identities involving generalized derivations working in pairs and triplets with a key role of involution of the second kind. Through out this section by a prime semiring we mean a prime MA-semiring unless mentioned otherwise.

Following result describes an identity involving three generalized derivations, which leads to the commutativity of a semiring and this result is an extended version of Theorem 2.2 of [25].

Theorem 1. For a 2-torsion free prime semiring S with involution $*$ of second kind, let F_σ,G_ϱ and D_δ be generalized derivations respectively associated with derivations σ,ϱ and δ such that $\sigma\neq 0$ and $\varrho\neq 0$. If

$$[F_\sigma(w)G_\varrho(w^*)+D_\delta(ww^*),t]=0, \tag{7}$$

for all $t,w\in S$, then S is commutative.

Proof. Linearizing (7) and using (7) again, we obtain

$$[F_\sigma(w)G_\varrho(s^*)+F_\sigma(s)G_\varrho(w^*)+D_\delta(ws^*)+D_\delta(sw^*),t]=0. \tag{8}$$

Writing $s\tilde{h}_z$ for s in (8) and hence after the rearrangement of terms, we obtain

$$[F_\sigma(w)G_\varrho(s^*)+F_\sigma(s)G_\varrho(w^*)+D_\delta(ws^*)+D_\delta(sw^*),t]\tilde{h}_z$$
$$+[F_\sigma(w)s^*\varrho(\tilde{h}_z)+s\sigma(\tilde{h}_z)G_\varrho(w^*)+(ws^*+sw^*)\delta(\tilde{h}_z),t]=0.$$

Using (8) again, we obtain

$$[F_\sigma(w)s^*\varrho(\tilde{h}_z)+s\sigma(\tilde{h}_z)G_\varrho(w^*)+(ws^*)\delta(\tilde{h}_z)+(sw^*)\delta(\tilde{h}_z),t]=0. \tag{9}$$

Substituting sk_z for s in (9), we obtain

$$[(F_\sigma(w)s^*\varrho(\tilde{h}_z))'+s\sigma(\tilde{h}_z)G_\varrho(w^*)+(ws^*)'\delta(\tilde{h}_z)+sw^*\delta(\tilde{h}_z),t]Sk_z=\{0\}.$$

Because of the primeness of S, we have

$$[(F_\sigma(w)s^*\varrho(\tilde{h}_z))'+s\sigma(\tilde{h}_z)G_\varrho(w^*)+(ws^*)'\delta(\tilde{h}_z)+sw^*\delta(\tilde{h}_z),t]=0.$$

Using assertion (xii) of Lemma 1, we get

$$[F_\sigma(w)s^*\varrho(\tilde{h}_z)+ws^*\delta(\tilde{h}_z),t]=[s\sigma(\tilde{h}_z)G_\varrho(w^*)+sw^*\delta(\tilde{h}_z),t]. \tag{10}$$

In view of the 2-torsion freeness of S, using (10) in (9), we find

$$[s(\sigma(\tilde{h}_z)G_\varrho(w^*) + w^*\delta(\tilde{h}_z)), t] = 0. \tag{11}$$

Writing rs for s in (11) and again using (11), we can write

$$[r,t]S(\sigma(\tilde{h}_z)G_\varrho(w^*) + w^*\delta(\tilde{h}_z)) = \{0\}$$

and by the primeness of S, we have either S is commutative or $\sigma(\tilde{h}_z)G_\varrho(w^*) + w^*\delta(\tilde{h}_z) = 0$, which further implies

$$\sigma(\tilde{h}_z)G_\varrho(w) + w\delta(\tilde{h}_z) = 0. \tag{12}$$

Writing wh_z for w in (12), we get $(\sigma(\tilde{h}_z)G_\varrho(w) + w\delta(\tilde{h}_z))h_z + \sigma(\tilde{h}_z)w\varrho(h_z) = 0$ and by the use of (12) again, we further get $\sigma(\tilde{h}_z)S\varrho(h_z) = \{0\}$. Due to the primeness of S, we have either $\sigma(\tilde{h}_z) = 0$, for all $\tilde{h}_z \in \mathbb{H}(S) \cap Z(S)$ or $\varrho(h_z) = 0$, for all $h_z \in \mathbb{H}(S) \cap Z(S)$. Assume that $\sigma(\tilde{h}_z) = 0$, then from (12), we obtain $w\delta(\tilde{h}_z) = 0$, which further implies $\delta(\tilde{h}_z) = 0$. Now since $\sigma(\tilde{h}_z) = 0 = \delta(\tilde{h}_z)$, from (9) we obtain $[F_\sigma(w)s^*\varrho(\tilde{h}_z), t] = 0$ and therefore

$$[F_\sigma(w)s\varrho(\tilde{h}_z), t] = 0. \tag{13}$$

Substituting wp for w in (13), we obtain $[F_\sigma(w)ps\varrho(\tilde{h}_z), t] + [w\sigma(p)s\varrho(\tilde{h}_z), t] = 0$ and using (13) again, we get

$$[w\sigma(p)s\varrho(\tilde{h}_z), t] = 0. \tag{14}$$

On replacement of w by rw in (14) and making use of (14) again, we obtain

$$[r,t]S\sigma(p)s\varrho(\tilde{h}_z) = \{0\}$$

and by the primeness of S, we have either S is commutative or $\sigma(p)S\varrho(\tilde{h}_z) = \{0\}$. From the second possibility, since $\sigma \neq 0$, we have $\varrho(\tilde{h}_z) = 0$. Hence we conclude that $\sigma(h_z) = \varrho(h_z) = \delta(h_z) = 0$, for all $h_z \in \mathbb{H}(S) \cap Z(S)$. By Lemma 4, we have $\sigma(k_z) = \varrho(k_z) = \delta(k_z) = 0$, for all $k_z \in \mathbb{K}(S) \cap Z(S)$ and by Lemma 5, we have $\sigma(z) = \varrho(z) = \delta(z) = 0$, for all $z \in Z(S)$. Substituting sk_z for s in (8) and using the assumption that $\sigma(k_z) = \varrho(k_z) = \delta(k_z) = 0$, we get

$$[(F_\sigma(w)G_\varrho(s^*))' + F_\sigma(s)G_\varrho(w^*) + D_\delta(ws^*))' + D_\delta(sw^*), t]k_z = 0.$$

and by the primeness, we have

$$[(F_\sigma(w)G_\varrho(s^*))' + F_\sigma(s)G_\varrho(w^*) + D_\delta(ws^*))' + D_\delta(sw^*), t] = 0.$$

As $s + t = 0$ implies $s = t'$ for all $s, t \in S$, therefore from the last identity, we can write

$$[F_\sigma(s)G_\varrho(w^*) + D_\delta(sw^*), t] = [F_\sigma(w)G_\varrho(s^*) + D_\delta(ws^*), t]. \tag{15}$$

In view of the 2-torsion freeness of S, using (15) in (8), and then substituting w^* for w, we have

$$[F_\sigma(s)G_\varrho(w) + D_\delta(sw), t] = 0. \tag{16}$$

Replacing w by wt in (16), we get $[F_\sigma(s)G_\varrho(w)t + F_\sigma(s)w\varrho(t) + D_\delta(sw)t + sw\delta(t), t] = 0$, which further implies by using Lemma 1 that $[F_\sigma(s)G_\varrho(w) + D_\delta(sw), t]t + [F_\sigma(s)w\varrho(t) + sw\delta(t), t] = 0$ and using (16) again, we obtain

$$[F_\sigma(s)w\varrho(t) + sw\delta(t), t] = 0. \tag{17}$$

In (17) writing sp in place of s, we find

$$[F_\sigma(s)pw\varrho(t) + s\sigma(p)w\varrho(t) + spw\delta(t), t] = 0. \tag{18}$$

In (17) replacing w by pw, we get

$$[F_\sigma(s)pw\varrho(t) + spw\delta(t), t] = 0. \tag{19}$$

Using (19) in (18), we get

$$[s\sigma(p)w\varrho(t), t] = 0. \tag{20}$$

From (20), we have $s\sigma(p)w\varrho(t)t + t's\sigma(p)w\varrho(t) = 0$ and by Lemma 1, we can further write $s\sigma(p)w\varrho(t)t = ts\sigma(p)w\varrho(t) = 0$. In (20), replacing s by sr, we get $[sr\sigma(p)w\varrho(t), t] = 0$, and by Lemma 1, we can write $[s,t]r\sigma(p)w\varrho(t) = 0$, and therefore

$$[s,t]S\sigma(p)w\varrho(t) = \{0\}.$$

We consider the subsets $S_1 = \{t \in S : [s,t] = 0, \text{ for all } s \in S\}$ and $S_2 = \{t \in S : \sigma(p)S\varrho(t) = \{0\}, \text{ for all } p \in S\}$. We see that $S = S_1 \cup S_2$. Our claim is that either $S = S_1$ or $S = S_2$. For this we prove that either $S_1 \subseteq S_2$ or $S_2 \subseteq S_1$. On the contrary, let $t_1 \in S_1 \setminus S_2$ and $t_2 \in S_2 \setminus S_1$. Then $t_1 + t_2 \in S_1 + S_2 \subseteq S_1 \cup S_2 = S$. If $t_1 + t_2 \in S_1$, then $0 = [r, t_1 + t_2] = [r, t_1] + [r, t_2] = [r, t_2]$, which implies $t_2 \in S_1$, a contradiction. On the other hand if $t_1 + t_2 \in S_2$, then $\{0\} = \sigma(p)S\varrho(t_1 + t_2) = \sigma(p)S\varrho(t_1) + \sigma(p)S\varrho(t_2) = \sigma(p)S\varrho(t_1)$, therefore $t_1 \in S_2$, a contradiction. Hence we have either $S_1 \subseteq S_2$ or $S_2 \subseteq S_1$ and therefore we respectively have $S_1 = S$ or $S_2 = S$. Firstly if $S_1 = S$, then S is commutative. Secondly if $S_2 = S$, then $\sigma(p)S\varrho(t) = \{0\}$ for all $p, t \in S$, then by the primeness of S, we have either $\sigma = 0$ or $\varrho = 0$ which contradicts the hypothesis. This completes the proof. □

From the proof of Theorem 1, one can obtain the following result.

Theorem 2. Let F_σ, G_ϱ and D_δ be generalized derivations respectively associated with the nonzero derivations σ, ϱ and a derivation δ of a 2-torsion free prime semiring S with involution $*$ of second kind. If

$$[F_\sigma(w)G_\varrho(s) + D_\delta(ws), t] = 0$$

for all $t, w, s \in S$, then S is commutative.

If $D_\delta = 0$, then we can obtain the following result from Theorem 1.

Corollary 1. For a 2-torsion free prime semiring S with involution $*$ of second kind, let F_σ and G_ϱ be generalized derivations respectively associated with derivations σ and ϱ such that $\sigma \neq 0$ and $\varrho \neq 0$. If

$$[F_\sigma(w)G_\varrho(w^*), t] = 0,$$

for all $t, w \in S$, then S is commutative.

A generalized version of Theorem 2.4 of [25] is given in the following theorem.

Theorem 3. Let S be a 2-torsion free prime semiring with involution $*$ of second kind. Let F_σ be a nonzero generalized derivation associated with a derivation σ that satisfies one of the statements below:

1. $F_\sigma[w, w^*] + [(w^*), \sigma(w)] = 0$,
2. $[F_\sigma(w), w^*] + \sigma[(w^*), w] = 0$

for all $w \in S$. Then S is commutative.

Proof. 1. The hypothesis states that

$$F_\sigma[w, w^*] + [(w^*), \sigma(w)] = 0, \tag{21}$$

for all $w \in S$. Linearizing (21) and using (21) again, we get

$$F_\sigma[w, s^*] + F_\sigma[s, w^*] + [(w^*), \sigma(s)] + [(s^*), \sigma(w)] = 0. \tag{22}$$

If $\sigma = 0$, then from (22) and then replacing s by s^*, we have

$$F_\sigma[w,s] + F_\sigma[s^*,w^*] = 0. \tag{23}$$

Writing sk_z for s in (23), we obtain $(F_\sigma[w,s] + F_\sigma[s^*,w^*]')Sk_z = \{0\}$ and since S is prime, we have $F_\sigma[w,s] + F_\sigma[s^*,w^*]' = 0$ and since $u + v = 0$ implies $u = v'$ for all $u, v \in S$, therefore we can write

$$F_\sigma[w,s] = F_\sigma[s^*,w^*]. \tag{24}$$

Using (24) in (23) and then using 2-torsion freeness of S, we get

$$F_\sigma[w,s] = 0. \tag{25}$$

In (25) substituting wp for w and using Lemma 1, we get $F_\sigma([w,s]p + w[p,s]) = 0$, which further implies that $F_\sigma([w,s])p + [w,s]\sigma(p) + F_\sigma(w)[p,s] + w\sigma[p,s] = 0$ and using (25), we obtain

$$F_\sigma(w)[p,s] = 0. \tag{26}$$

In (26) replacing p by rp and using (26) again, we obtain $F_\sigma(w)S[p,s] = \{0\}$. As $F_\sigma \neq 0$, by the primeness of S is commutative.

We now consider the case, when $\sigma \neq 0$. In (22) for each $h_z \in \mathbb{H}(S) \cap Z(S)$, replacing s by sh_z, we get

$$(F_\sigma[w,s^*] + F_\sigma[s,w^*] + [(w^*),\sigma(s)] + [(s^*),\sigma(w)])h_z \\ + [w,s^*]\sigma(h_z) + [s,w^*]\sigma(h_z) + [(w^*),s\sigma(h_z)] = 0$$

and using (22) again, we get

$$[w,s^*]\sigma(h_z) + [s,w^*]\sigma(h_z) + [w^*,s\sigma(h_z)] = 0. \tag{27}$$

Replacing s by sk_z in (27), we obtain

$$([w,s^*]'\sigma(h_z) + [s,w^*]\sigma(h_z) + [w^*,sd(h_z)])Sk_z = \{0\}.$$

Due to the primeness of S, we $[w,s^*]'\sigma(h_z) + [s,w^*]\sigma(h_z) + [w^*,s\sigma(h_z)] = 0$ and using Lemma 1, we further get

$$[s,w^*]\sigma(h_z) + [w^*,s\sigma(h_z)] = [w,s^*]\sigma(h_z). \tag{28}$$

Using (28) in (27) and then using 2-torsion freeness of S, we obtain $[w,s^*]\sigma(h_z) = 0$ and replacing s by s^*, we get

$$[w,s]\sigma(h_z) = 0. \tag{29}$$

In (29) substituting sr for s and again using (29), we obtain $[w,s]S\sigma(h_z) = \{0\}$. By the primeness of S, either S is commutative, or $\sigma(h_z) = 0$, for all $h_z \in \mathbb{H}(S) \cap Z(S)$.

Assume that $\sigma(h_z) = 0$. By Lemma 4, we have $\sigma(k_z) = 0$ for all $k_z \in \mathbb{K}(S) \cap Z(S)$. For each $k_z \in \mathbb{K}(S) \cap Z(S)$, replacing s by sk_z in (22) and using the assumption that $\sigma(k_z) = 0$, we obtain

$$(F_\sigma[w,s^*]' + F_\sigma[s,w^*] + [(w^*),\sigma(s)] + [(s^*),\sigma(w)]')Sk_z = \{0\}.$$

Due to the primeness of S, we obtain

$$F_\sigma[w,s^*]' + F_\sigma[s,w^*] + [(w^*),\sigma(s)] + [(s^*),\sigma(w)]' = 0,$$

and using Lemma 1, we have

$$F_\sigma[w,s^*]+[(s^*),\sigma(w)]=F_\sigma[s,w^*]+[(w^*),\sigma(s)]. \tag{30}$$

Using (30) in (22) and using 2-torsion freeness of S, we get

$$F_\sigma[w,s^*]+[(s^*),\sigma(w)]=0$$

and making substitution of s by s^*, we get

$$F_\sigma[w,s]+[s,\sigma(w)]=0. \tag{31}$$

In (31) replacing w by ws, we obtain

$$F_\sigma([w,s]s)+[s,\sigma(ws)]=0. \tag{32}$$

Using semiring identities from Lemma 1 and rearranging the terms, we have

$$F_\sigma([w,s]s)+[s,\sigma(ws)]=(F_\sigma[w,s]+[s,\sigma(w)])s+[w,s]\sigma(s)+[s,w\sigma(s)]$$

and using (31) and rearranging terms, we obtain

$$\begin{aligned}F_\sigma([w,s]s)+[s,\sigma(ws)]&=[w,s]\sigma(s)+[s,w\sigma(s)]\\&=ws\sigma(s)+(s'+s)w\sigma(s)+w\sigma(s)s'.\end{aligned}$$

As S is a semiring, $s+s'\in Z(S)$ and therefore

$$\begin{aligned}F_\sigma([w,s]s)+[s,\sigma(ws)]&=ws\sigma(s)+w\sigma(s)(s'+s)+w\sigma(s)s'\\&=ws\sigma(s)+w\sigma(s)(s'+s+s').\end{aligned}$$

As $s'+s+s'=s'$ and $s+s'+s=s$, therefore

$$F_\sigma([w,s]s)+[s,\sigma(ws)]=ws\sigma(s)+w\sigma(s)s'=w[s,\sigma(s)].$$

Therefore from (32) we can write $w[s,\sigma(s)]=0$ and on replacement of w by $[s,\sigma(s)]w$, it further implies $[s,\sigma(s)]S[s,\sigma(s)]=\{0\}$. Due to the primeness of S the last relation gives $[s,\sigma(s)]=0, \forall s\in S$. Hence by Lemma 6, S is commutative.

2. We have

$$[F_\sigma(w),w^*]+\sigma[(w^*),w]=0, \tag{33}$$

for all $w\in S$. Linearizing (33) and again using (33), we get

$$[F_\sigma(w),s^*]+[F_\sigma(s),w^*]+\sigma[(w^*),s]+\sigma[(s^*),w]=0. \tag{34}$$

If $\sigma=0$, then from (34), we have $[F_\sigma(w),s^*]+[F_\sigma(s),w^*]=0$ and replacing s by s^*, we get

$$[F_\sigma(w),s]+[F_\sigma(s^*),w^*]=0. \tag{35}$$

Writing sk_z for s in (35), we obtain $([F_\sigma(w),s]+[F_\sigma(s^*),w^*]')Sk_z=\{0\}$ and by the primeness, we obtain $[F_\sigma(w),s]+[F_\sigma(s^*),w^*]'=0$. Making use of the last equation, we obtain

$$[F_\sigma(w),s]=[F_\sigma(s^*),w^*]. \tag{36}$$

Using (36) in (35) and then by the 2-torsion freeness of S, we have

$$[F_\sigma(w), s] = 0. \tag{37}$$

In (37) replacing w by wr and using (37) again, we get $F_\sigma(w)[r,s] = 0$ and therefore $F_\sigma(w)S[r,s] = \{0\}$. As $F_\sigma \neq 0$, by the primeness of S, we have $[r,t] = 0$ which implies that S is commutative.

We now consider the case when $\sigma \neq 0$. In (34) substituting s^* for s, we find

$$[F_\sigma(w), s] + [F_\sigma(s^*), w^*] + \sigma[(w^*), s^*] + \sigma[s, w] = 0. \tag{38}$$

In (38) replacing s by sh_z for each $h_z \in \mathbb{H}(S) \cap Z(S)$, we obtain

$$[F_\sigma(w), s]h_z + [F_\sigma(s^*), w^*]h_z + [s^*\sigma(h_z), w^*] + \sigma[w^*, s^*]h_z \\ + [w^*, s^*]\sigma(h_z) + \sigma[s, w]h_z + [s, w]\sigma(h_z) = 0.$$

After the rearrangement of terms, we get

$$([F_\sigma(w), s] + [F_\sigma(s^*), w^*] + \sigma[w^*, s^*] + \sigma[s, w])h_z \\ + [s^*\sigma(h_z), w^*] + [w^*, s^*]\sigma(h_z) + [s, w]\sigma(h_z) = 0.$$

Using (38) again, we obtain

$$[s^*\sigma(h_z), w^*] + [w^*, s^*]\sigma(h_z) + [s, w]\sigma(h_z) = 0. \tag{39}$$

In (39), writing sk_z in place of s, we obtain

$$([s^*\sigma(h_z), w^*]' + [w^*, s^*]'\sigma(h_z) + [s, w]\sigma(h_z))Sk_z = \{0\}.$$

Using primeness, after the rearrangement of terms, we obtain

$$[s^*\sigma(h_z), w^*]' + [w^*, s^*]'\sigma(h_z) + [s, w]\sigma(h_z) = 0.$$

Since $u + v = 0$ implies $u = v'$, for all $u, v \in S$, therefore from the last identity, we can write

$$[s^*\sigma(h_z), w^*] + [w^*, s^*]\sigma(h_z) = [s, w]\sigma(h_z). \tag{40}$$

Using (40) in (39), we obtain $[s, w]\sigma(h_z) = 0$ and therefore $[s, w]Sd(h_z) = \{0\}$. Because of the primeness, we have either S is commutative or $\sigma(h_z) = 0, h_z \in \mathbb{H}(S) \cap Z(S)$. Assume that $\sigma(h_z) = 0$, then by Lemma 4, $\sigma(k_z) = 0$ for all $k_z \in \mathbb{K}(S) \cap Z(S)$. Substituting sk_z for s in (34), and then using the assumption that $\sigma(k_z) = 0$, we obtain

$$([F_\sigma(w), s^*]' + [F_\sigma(s), w^*] + \sigma[(w^*), s] + \sigma[(s^*), w]')Sk_z = \{0\}.$$

Due to the primeness of S, we have

$$[F_\sigma(w), s^*]' + [F_\sigma(s), w^*] + \sigma[(w^*), s] + \sigma[(s^*), w]' = 0,$$

which further implies

$$[F_\sigma(w), s^*] + \sigma[(s^*), w] = [F_\sigma(s), w^*] + \sigma[(w^*), s]. \tag{41}$$

Using (41) in (34) and the 2-torsion freeness of S, we obtain $[F_\sigma(w), s^*] + \sigma[(s^*), w] = 0$ and therefore

$$[F_\sigma(w), s] + \sigma[s, w] = 0. \tag{42}$$

In (42) taking $s = w$, we have $[F_\sigma(w), w] + \sigma[w, w] = 0$. As $[w, w] = [w, w]'$, therefore $[F_\sigma(w), w] + \sigma[w, w]' = 0$ and hence

$$[F_\sigma(w), w] = \sigma[w, w]. \tag{43}$$

Using (42) and (43), we obtain $[F_\sigma(w), w] = 0, \forall w \in S$. By Lemma 7, we conclude that S is commutative. □

From the proof of Theorem 3, one can obtain the following result.

Theorem 4. Let F_σ be nonzero generalized derivation associated with a derivation σ of a 2-torsion free prime semiring S with involution $*$ of second kind. If one of the following holds:
1. $F_\sigma[w, s] + [s, \sigma(w)] = 0$,
2. $[F_\sigma(w), s] + \sigma[(s), w] = 0$

for all $s, w \in S$, then S is commutative.

Following result is an extended form of Theorem 2.6 of [25].

Theorem 5. Let S be a 2-torsion free prime semiring S with involution $*$ of second kind. Then there is no nonzero generalized derivation F_σ satisfying one of the following statements:
1. $F_\sigma(w) \circ w^* + \sigma(w^* \circ w)' = 0$,
2. $F_\sigma(w \circ w^*) + \sigma(w^*) \circ w' = 0$

for all $w \in S$.

Proof. (1). Assume that F_σ is a nonzero generalized derivation satisfying

$$F_\sigma(w) \circ w^* + \sigma(w^* \circ w)' = 0. \tag{44}$$

Linearizing (44) and using (44) again, we get

$$F_\sigma(w) \circ s^* + F_\sigma(s) \circ w^* + \sigma(w^* \circ s)' + \sigma(s^* \circ w)' = 0. \tag{45}$$

Substituting sh_z for s in (45), we get

$$(F_\sigma(w) \circ s^*)h_z + (F_\sigma(s) \circ w^*)h_z + ((s\sigma(h_z)) \circ w^*) \\ + \sigma(w^* \circ s)'h_z + (w^* \circ s)'\sigma(h_z) + \sigma(s^* \circ w)'h_z + (s^* \circ w)'\sigma(h_z) = 0$$

and therefore

$$((F_\sigma(w) \circ s^*) + (F_\sigma(s) \circ w^*) + \sigma(w^* \circ s)' + \sigma(s^* \circ w)')h_z \\ + ((s\sigma(h_z)) \circ w^*) + (w^* \circ s)'\sigma(h_z) + (s^* \circ w)'\sigma(h_z) = 0.$$

Using (45), we obtain

$$((s\sigma(h_z)) \circ w^*) + (w^* \circ s)'\sigma(h_z) + (s^* \circ w)'\sigma(h_z) = 0.$$

From the last equation, we can write

$$s\sigma(h_z)w^* + w^*s\sigma(h_z) + w^*s'\sigma(h_z) + s'w^*\sigma(h_z) + (s^* \circ w)'\sigma(h_z) = 0$$

and therefore

$$s\sigma(h_z)w^* + w^*(s + s')\sigma(h_z) + s'w^*\sigma(h_z) + (s^* \circ w)'\sigma(h_z) = 0.$$

As $s + s' \in Z(S)$, therefore

$$s\sigma(h_z)w^* + (s' + s + s')w^*\sigma(h_z) + (s^* \circ w)'\sigma(h_z) = 0$$

and since $s' + s + s' = s'$, therefore

$$s\sigma(h_z)w^* + s'w^*\sigma(h_z) + (s^* \circ w)'\sigma(h_z) = 0,$$

which further implies

$$s[\sigma(h_z), w^*] + (s^* \circ w)'\sigma(h_z) = 0. \tag{46}$$

By Lemma 1, from (46), we can write

$$s[\sigma(h_z), w^*] = (s^* \circ w)\sigma(h_z). \tag{47}$$

In (46), replacing s by sk_z, we obtain $(s[\sigma(h_z), w^*] + (s^* \circ w)\sigma(h_z))k_z = 0$, which further implies $(s[\sigma(h_z), w^*] + (s^* \circ w)\sigma(h_z))Sk_z = \{0\}$. In view of Lemma 2, by the primeness of S, we can find

$$s[\sigma(h_z), w^*] + (s^* \circ w)\sigma(h_z) = 0. \tag{48}$$

Using (47) in (48), we obtain $2(s^* \circ w)\sigma(h_z) = 0$, and then by the 2-torsion freeness, we have $(s^* \circ w)\sigma(h_z) = 0$, and replacing s by s^*, we get

$$(s \circ w)\sigma(h_z) = 0. \tag{49}$$

Using Lemma 1 in (49), we can write

$$sw\sigma(h_z) = w' s\sigma(h_z). \tag{50}$$

In (49), substituting rw for w, we get

$$srw\sigma(h_z) + rws\sigma(h_z) = 0. \tag{51}$$

Using (50) in (51), we get $srw\sigma(h_z) + r'sw\sigma(h_z) = 0$ and so $[s, r]w\sigma(h_z) = 0$ i.e $[s, r]S\sigma(h_z) = \{0\}$. Due to the primeness, either S is commutative or $\sigma(h_z) = 0$ for all $h_z \in \mathbb{H}(S) \cap Z(S)$.

Assume that $\sigma(h_z) = 0$. By Lemma 4, $\sigma(k_z) = 0$ for all $k_z \in \mathbb{K}(S) \cap Z(S)$. In view of the fact that $\sigma(k_z) = 0$ for all $k_z \in \mathbb{K}(S) \cap Z(S)$, replacing s by $sk_z, k_z \in \mathbb{K}(S) \cap Z(S)$ in (45), we obtain

$$((F_\sigma(w) \circ s^*)' + F_\sigma(s) \circ w^* + \sigma(w^* \circ s)' + \sigma(s^* \circ w))Sk_z = \{0\}$$

and by the primeness, we obtain

$$(F_\sigma(w) \circ s^*)' + F_\sigma(s) \circ w^* + \sigma(w^* \circ s)' + \sigma(s^* \circ w) = 0,$$

which further implies

$$F_\sigma(s) \circ w^* + \sigma(w^* \circ s)' = F_\sigma(w) \circ s^* + \sigma(s^* \circ w)'. \tag{52}$$

As S is 2-torsion free, using (52) in (45), we get $F_\sigma(w) \circ s^* + \sigma(s^* \circ w)' = 0$ and making substitution of s by s^*, we get

$$F_\sigma(w) \circ s + \sigma(s \circ w)' = 0. \tag{53}$$

In (53), replacing s by h_z and using the assumption that $\sigma(h_z) = 0$, we get $(F_\sigma(w) + (\sigma(w))')Sh_z = \{0\}$. Because of the primeness, $F_\sigma(w) + (\sigma(w))' = 0$ for all $w \in S$ which implies that $F = \sigma$ and therefore from (53) becomes

$$\sigma(w) \circ s + \sigma(s \circ w)' = 0. \tag{54}$$

By the definition of Jordan product, we can write

$$\sigma(w) \circ s + \sigma(s \circ w)' = \sigma(w)s + s\sigma(w) + s'\sigma(w) + \sigma(s)w' + w'\sigma(s) + \sigma(w)s'.$$

Rearranging the terms, we have

$$\sigma(w) \circ s + \sigma(s \circ w))' = \sigma(w)(s + s') + (s + s')\sigma(w) + \sigma(s)w)' + w'\sigma(s)).$$

By the definition of MA-semiring $w + w' \in Z(S)$, therefore

$$\sigma(w) \circ s + (\sigma(s \circ w))' = (s + s' + s + s')\sigma(w) + \sigma(s)w' + w'\sigma(s).$$

By the definition of pseudo inverse, we have

$$\sigma(w) \circ s + \sigma(s \circ w))' = (s + s')\sigma(w) + \sigma(s)w' + w'\sigma(s).$$

Therefore (54) becomes

$$(s + s')\sigma(w) + (\sigma(s) \circ w') = 0. \tag{55}$$

As $s + s' = (s + s')'$, therefore from (55), we have $(s + s')'\sigma(w) + (\sigma(s) \circ w') = 0$, which further implies

$$(\sigma(s) \circ w') = (s + s')\sigma(w) \tag{56}$$

In view of the 2-torsion freeness of S, using (56) in (55), we obtain

$$\sigma(s) \circ w = \sigma(s)w + w\sigma(s) = 0. \tag{57}$$

From (57), we can write

$$\sigma(s)w = w'\sigma(s). \tag{58}$$

In (57) substituting wt for w, we obtain $\sigma(s)wt + wt\sigma(s) = 0$ and using (58), we further get $\sigma(s)wt + w'\sigma(s)t = 0$ that is $[\sigma(s), w]t = 0$. From the last relation, we can write $[\sigma(s), w]S[\sigma(s), w] = \{0\}$ and by the primeness of S, we have $[\sigma(s), w] = 0$ and so $\sigma(s) \in Z(S)$. Therefore, from (57) we have $2w\sigma(s) = 0$ and by 2-torsion freeness of S, we have $w\sigma(s) = 0$. Substituting wr for w in the last equation, we get $wS\sigma(s) = \{0\}$. As S is prime, we have $\sigma = 0$ and hence $F_\sigma = 0$, a contradiction. On the other hand if S is commutative, then $\sigma(w) \in Z(S)$. Repeating the same arguments as above we obtain $F_\sigma = 0$, a contradiction. This completes the proof.

(2). Assume that F_σ is a nonzero generalized derivation associated with a derivation σ such that

$$F_\sigma(w \circ w^*) + \sigma(w^*) \circ w' = 0, \tag{59}$$

for all $w \in S$. Linearizing (59) and again using (59), we obtain

$$F_\sigma(w \circ s^*) + F_\sigma(s \circ w^*) + \sigma(w^*) \circ s' + \sigma(s^*) \circ w' = 0. \tag{60}$$

For each $h_z \in \mathbb{H}(S) \cap Z(S)$, substituting sh_z for s in (60), we get

$$F_\sigma(w \circ (sh_z)^*) + F_\sigma((sh_z) \circ w^*) + \sigma(w^*) \circ (sh_z)' + \sigma((sh_z)^*) \circ w' = 0,$$

which implies

$$F_\sigma(w \circ (s^*h_z)) + F_\sigma((sh_z) \circ w^*) + \sigma(w^*) \circ (sh_z)' + \sigma((s^*h_z)) \circ w' = 0$$

and therefore, we can write

$$F_\sigma((w \circ s^*)h_z) + F_\sigma((s \circ w^*)h_z) + \sigma(w^*) \circ s')h_z + \sigma((s^*h_z)) \circ w' = 0.$$

As F_σ is a generalized derivation and σ is a derivation, therefore

$$\{F_\sigma(w \circ s^*)h_z + (w \circ s^*)\sigma(h_z)\} + \{F_\sigma(s \circ w^*)h_z + (s \circ w^*)\sigma(h_z)\}$$
$$+ \{(\sigma(w^*) \circ s')h_z\} + \{\sigma(s^*)h_z + s^*\sigma(h_z)\} \circ w' = 0,$$

which further implies

$$F_\sigma(w\circ s^*)h_z+(w\circ s^*)\sigma(h_z)+F_\sigma(s\circ w^*)h_z+(s\circ w^*)\sigma(h_z)\\ +(\sigma(w^*)\circ s')h_z+(\sigma(s^*)\circ w')h_z+(s^*\sigma(h_z))\circ w'=0.$$

Rearranging terms of the last expression, we obtain

$$\{F_\sigma(w\circ s^*)+F_\sigma(s\circ w^*)+(\sigma(w^*)\circ s')+(\sigma(s^*)\circ w')\}h_z\\ +(w\circ s^*)\sigma(h_z)+(s\circ w^*)\sigma(h_z)+(s^*\sigma(h_z))\circ w'=0.$$

Using (60), we obtain

$$(w\circ s^*)\sigma(h_z)+(s\circ w^*)\sigma(h_z)+(s^*\sigma(h_z))\circ w'=0. \tag{61}$$

Substituting sk_z for s in (61), we obtain

$$((w\circ s^*)'\sigma(h_z)+(s\circ w^*)\sigma(h_z)+(s^*\sigma(h_z))\circ w)Sk_z=\{0\}.$$

Since S is prime and $\mathbb{K}(S)\cap Z(S)\neq\{0\}$, we get

$$((w\circ s^*)'\sigma(h_z)+(s\circ w^*)\sigma(h_z)+(s^*\sigma(h_z))\circ w)=0$$

and therefore by Lemma 1, we obtain

$$(w\circ s^*)\sigma(h_z)+(s^*\sigma(h_z))\circ w'=(s\circ w^*)\sigma(h_z). \tag{62}$$

Using (62) in (61) and then using 2-torsion freeness, we get $(s\circ w^*)\sigma(h_z)=0$, which further gives

$$(s\circ w)\sigma(h_z)=0. \tag{63}$$

Equation (63) is the same as (49). Therefore, using similar arguments, we obtain that either $\sigma(h_z)=0$ or S is commutative. Firstly assume that $\sigma(h_z)=0$. By Lemma 4, we have $\sigma(k_z)=0$, for all $k_z\in\mathbb{K}(S)\cap Z(S)$ and by Lemma 5, $\sigma(z)=0$ for all $z\in Z(S)$. Substituting sk_z for s in (60), we obtain

$$(F_\sigma(w\circ s^*)'+F_\sigma(s\circ w^*)+\sigma(w^*)\circ s'+\sigma(s^*)\circ w)Sk_z=\{0\}.$$

Due to the primeness of S, we have

$$F_\sigma(w\circ s^*)'+F_\sigma(s\circ w^*)+\sigma(w^*)\circ s'+\sigma(s^*)\circ w=0.$$

By Lemma 1, since $u+v=0$ implies $u=v'$, therefore from the last identity, we can write

$$F_\sigma(w\circ s^*)+\sigma(s^*)\circ w'=F_\sigma(s\circ w^*)+\sigma(w^*)\circ s'. \tag{64}$$

Using (64) in (60) and the 2-torsion freeness of S, we obtain

$$F_\sigma(w\circ s)+\sigma(s)\circ w'=0. \tag{65}$$

In (65), replacing w by $z\in Z$ and using $\sigma(z)=0$, we obtain $(F_\sigma(s)+\sigma(s)')Sz=\{0\}$. Using the primeness of S, we have $F_\sigma(s)+\sigma(s)'=0$ and therefore $F_\sigma(s)=\sigma(s)$ and hence $F_\sigma=\sigma$. Therefore (65) becomes

$$\sigma(w\circ s)+\sigma(s)\circ w'=0. \tag{66}$$

Equation (66) is same as (54), therefore by the similar arguments, we can conclude that $F_\sigma=0$, a contradiction. On the other hand, if S is commutative, then by the similar arguments, we can again conclude that $F_\sigma=0$, a contradiction and this completes the proof. □

From the proof of Theorem 5, one can obtain the following result.

Theorem 6. Let S be a 2-torsion free prime semiring S with involution $*$ of second kind. Then there is no nonzero generalized derivation F_σ satisfying one of the following statements:

1. $F_\sigma(w) \circ s + \sigma(s \circ w)' = 0$,
2. $F_\sigma(w \circ s) + \sigma(s) \circ w' = 0$

for all $w, s \in S$.

Conclusion

The research work presented in this paper provides a motivation to investigate the results for semiprime MA-semirings with similar or different environment.

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Author Contributions

All authors contributed equally to this work.

Conflict of Interest

The authors declare no conflict of interest.

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