https://doi.org/10.31489/2026M1/217-227 *Research article*

Numerical Scheme for Singularly Perturbed Differential Equations with Small Shifts Using Non-Polynomial Quartic Spline

K. Ragula[1,*], G.B.S.L. Soujanya[2]

[1] *Rajiv Gandhi University of Knowledge Technologies, Basar, India;*
[2] *Kakatiya University, Warangal, India*
(E-mail: rksm39@gmail.com, gbslsoujanya@gmail.com)

In this paper, a non-polynomial quartic spline technique with a fitting parameter is applied to solve a second-order singularly perturbed differential-difference equation (SPDDE) having small shifts. Taylor series expansion is employed for the delayed and advanced terms in the considered problem to produce a singularly perturbed differential equation (SPDE), and then a non-polynomial quartic spline technique is applied. To manage the layer structure, a fitting parameter is introduced in the proposed computational method; based on the step size, this parameter is evaluated using the theory of singular perturbation theory. Two model examples with left-end boundary layer behavior are considered to theory validate the theoretical finding. The convergence method is analyzed, and the solutions are reported in terms of maximum absolute error with quadratic convergence rate using the fitting parameter. For comparison, solutions without the fitting parameter are reported for test problems. The graphs depict the layer profile for the values of perturbation and shift parameters using the fitting factor and the oscillations without it. The proposed scheme gives uniformly convergent and valid results.

Keywords: singularly perturbed differential-difference equation, fitting parameter, non-polynomial spline, small shifts, boundary layer, truncation Error, maximum absolute error, convergence.

2020 Mathematics Subject Classification: 65L10, 65L11, 65L12.

Introduction

The differential equation in which the highest derivative is multiplied by a small parameter and having delay/advanced on the terms different from the highest derivative is known as a singularly perturbed differential-difference equation (SPDDE). A challenging and frequent task in the mathematical modeling of many physical, engineering, and biological problems is finding solutions for SPDDEs in the interval of boundary conditions. Such problems include the initial exit time problem in neuronal variability activation models [1] and oscillations of the human pupil light reflex with delayed and mixed responses [2]. A comprehensive overview of SPDDEs is given in [3]. The efficient approximation schemes SCEM and MMAE to solve SPDDE were introduced in [4]. The authors suggested a hybrid technique and a midpoint upwind strategy for inside and outside the boundary layer region on the Shishkin mesh in [5]. In [6], the researchers proposed a numerical method for solving SPDDEs with small and large delays using a non-polynomial spline. In [7], the authors developed two adaptive methods based on the r-refinement strategy to solve SPDDE with mixed parameters. In [8, 9], the authors suggested spline methods with fitting factor to solve the problem of SPDDEs. An exponentially fitted spline method was proposed to solve SPDDE with delay in the convection term in [10]. The authors of [11] came up with a way to solve the singularly perturbed boundary value problems with mixed shifts using non polynomial splines. In [12], the authors developed a uniform convergent computational technique for solving singularly perturbed delay reaction-diffusion equations. The authors

*Corresponding author. *E-mail: rksm39@gmail.com*
Received: 30 August 2024; *Accepted:* 18 December 2025.

of [13, 14] proposed numerical methods for solving SPDDE with mixed shifts using non-polynomial cubic splines. The fourth-order finite difference method (FDM) with a fitting parameter is suggested to solve SPDDE with mixed shifts in [15]. The mixed finite difference method (FDM) was proposed to solve SPDDE with mixed shifts in [16]. The parametric spline approach developed by the authors in [17] is used to solve differential-difference equations (DDEs) with mixed parameters having twin layers. In [18], the authors proposed a second-order computational approach to solving SPDDEs using Stormer's method.

1 Description of the Problem

Consider the following linear SPDDE with mixed shifts

$$\varepsilon u''(\vartheta) + p(\vartheta)\, u'(\vartheta - \delta) + q(\vartheta)\, u(\vartheta + \eta) + r(\vartheta)\, u(\vartheta) = s(\vartheta), \quad 0 < \vartheta < 1 \tag{1}$$

with boundary conditions

$$u(\vartheta) = \varphi(\vartheta), \ -\delta \le \vartheta \le 0; \quad u(\vartheta) = \psi(\vartheta), \ 1 \le \vartheta \le 1 + \eta, \tag{2}$$

where $0 < \varepsilon \ll 1$ and $p(\vartheta), \ q(\vartheta), \ r(\vartheta), \ s(\vartheta), \ \varphi(\vartheta)$ and $\psi(\vartheta)$ are sufficiently smooth functions on $(0,1)$ and $0 < \delta = o(\varepsilon), \ 0 < \eta = o(\varepsilon), \ \delta$ is the delay parameter, η is the advanced parameter.

The Taylor series expansions of $u'(\vartheta - \delta), \ u(\vartheta + \eta)$ about the point ϑ, we have

$$u'(\vartheta - \delta) = u'(\vartheta) - \delta u''(\vartheta) + o\left(\delta^2\right); \quad u(\vartheta + \eta) = u(\vartheta) + \eta u'(\vartheta) + \frac{\eta^2}{2} u''(\vartheta) + o\left(\eta^3\right). \tag{3}$$

Using Eq. (3) in Eq. (1), we get

$$\varepsilon u''(\vartheta) + a(\vartheta)\, u'(\vartheta) + b(\vartheta)\, u(\vartheta) = f(\vartheta), \quad 0 < \vartheta < 1 \tag{4}$$

with the boundary constraints $u(0) = \varphi_0 = \varphi(0)$ and $u(1) = \psi_1 = \psi(1)$,

where $a(\vartheta) = \left(\dfrac{p(\vartheta) + \eta q(\vartheta)}{1 - \frac{p(\vartheta)\delta}{\varepsilon} + \frac{q(\vartheta)\eta^2}{2\varepsilon}}\right)$, $b(\vartheta) = \left(\dfrac{q(\vartheta) + r(\vartheta)}{1 - \frac{p(\vartheta)\delta}{\varepsilon} + \frac{q(\vartheta)\eta^2}{2\varepsilon}}\right)$ and $f(\vartheta) = \left(\dfrac{s(\vartheta)}{1 - \frac{p(\vartheta)\delta}{\varepsilon} + \frac{q(\vartheta)\eta^2}{2\varepsilon}}\right)$.

2 Quartic Non-Polynomial Spline Approach

The interval $[0,1]$ partitioned into $\mathcal{N}$ sub intervals of equal length with constant step size h. Let $0 = \vartheta_0 < \vartheta_1 < \ldots < \vartheta_{\mathcal{N}} = 1$ be the $\mathcal{N}$ grid points. Then we have $h = \frac{1}{\mathcal{N}}$ and $\vartheta_i = ih, i = 0, 1, 2, \ldots, \mathcal{N}$.

In [19], the authors explained about the non-polynomial quartic spline and defined in $[\vartheta_i, \vartheta_{i+1}]$, $i = 0, 1, \ldots, \mathcal{N} - 1$ the spline is of the form

$$\mathcal{G}_i(\vartheta) = p_i (\vartheta - \vartheta_i)^2 + q_i (\vartheta - \vartheta_i) + r_i \sin \tau (\vartheta - \vartheta_i) + s_i \cos \tau (\vartheta - \vartheta_i) + t_i, \tag{5}$$

where $p_i, \ q_i, \ r_i, \ s_i$ and t_i are unknown constants and the function $\mathcal{G}_i(\vartheta)$ interpolates $u(\vartheta_i)$ at the points ϑ_i by depending on arbitrary parameter τ and reducing to quartic spline in $[0,1]$ as $\tau \to 0$.

To examine the coefficients $p_i, \ q_i, \ r_i, \ s_i$ and t_i in Eq. (5) in terms of $u_i, \ u_{i+1}, \ \mathcal{M}_i$, $\mathcal{M}_{i+1}, \mathcal{F}_i$ and $\mathcal{F}_{i+1}$, we define

$$\mathcal{G}_i(\vartheta_i) = u_i, \quad \mathcal{G}_i(\vartheta_{i+1}) = u_{i+1}, \quad \mathcal{G}_i''(\vartheta_i) = \mathcal{M}_i, \quad \mathcal{G}_i''(\vartheta_{i+1}) = \mathcal{M}_{i+1}, \quad \mathcal{G}_i^{(4)}(\vartheta_i) = \frac{1}{2}(\mathcal{F}_i + \mathcal{F}_{i+1}).$$

By using the conditions, we calculate the coefficients in Eq.(5) as

$$\begin{cases} p_i = \frac{\mathcal{M}_i}{2} + \frac{\mathcal{F}_i+\mathcal{F}_{i+1}}{4\tau^2}, \\ q_i = \frac{1}{h}(\ u_{i+1} - u_i) - (\frac{1}{h\tau^2} + \frac{h}{2})\mathcal{M}_i + \frac{1}{h\tau^2}\ \mathcal{M}_{i+1} - \frac{h}{4\tau^2}\left(\mathcal{F}_i + \mathcal{F}_{i+1}\right), \\ r_i = \frac{1}{\tau^2 \sin\omega}\left(\mathcal{M}_i - \mathcal{M}_{i+1}\right) + \frac{1-\cos\omega}{2\tau^4 \sin\omega}\left(\mathcal{F}_i + \mathcal{F}_{i+1}\right), \\ s_i = \frac{\mathcal{F}_i\ +\ \mathcal{F}_{i+1}}{2\tau^4}, \\ t_i = u_i - \frac{1}{2\tau^4}\left(\mathcal{F}_i + \mathcal{F}_{i+1}\right), \end{cases}$$

where $\tau h = \omega$.

Using continuity of first and third derivatives, $\mathcal{G}_{i-1}{}^{(m)}(\vartheta_i) = \mathcal{G}_i{}^{(m)}(\vartheta_i)$, $m = 1, 3$ then, we get relations

$$\frac{4h\tau^3 \sin\omega}{2(1-\cos\omega) - h\tau\sin\omega}\left[\frac{u_{i+1} - 2u_i + u_{i-1}}{h^2}\right] + (2\mathcal{F}_i + \mathcal{F}_{i+1} + \mathcal{F}_{i-1})$$

$$= \frac{2\tau\left(2h\tau\cos\omega + h^2\tau^2\sin\omega - 2\sin\omega\right)}{h\left[2(1-\cos\omega) - h\tau\sin\omega\right]}\mathcal{M}_{i-1} + \frac{2\tau\left(4\sin\omega + h^2\tau^2\sin\omega - 2h\tau(\cos\omega + 1)\right)}{h\left[2(1-\cos\omega) - h\tau\sin\omega\right]}\mathcal{M}_i \quad (6)$$

$$+ \frac{4\tau(h\tau - \sin\omega)}{h\left[2(1-\cos\omega) - h\tau\sin\omega\right]}\mathcal{M}_{i+1},$$

$$(2\mathcal{F}_i + \mathcal{F}_{i+1} + \mathcal{F}_{i-1}) = \frac{2\tau^2\cos\omega}{(1-\cos\omega)}\mathcal{M}_{i-1} + \frac{2\tau^2(\cos\omega + 1)}{(1-\cos\omega)}\mathcal{M}_i + \frac{2\tau^2}{(1-\cos\omega)}\mathcal{M}_{i+1}. \quad (7)$$

Substituting Eq. (7) in Eq. (6), we get the consistent relation

$$u_{i-1} - 2u_i + u_{i+1} = h^2\left[\alpha(\mathcal{M}_{i-1} + \mathcal{M}_{i+1}) + 2\beta\mathcal{M}_i\right], \quad i = 1, 2, \ldots, \mathcal{N} - 1, \quad (8)$$

where

$$\alpha = \frac{\omega^2 - 2(1-\cos\omega)}{2\omega^2(1-\cos\omega)}, \quad \beta = \frac{4(1-\cos\omega) + \omega^2(1 - 3\cos\omega)}{4\omega^2(1-\cos\omega)}.$$

If $h \to 0$, then $\omega = h\tau \to 0$. Thus, using L'Hospital's rule, we have $(\beta, \alpha) \to \left(\frac{5}{12}, \frac{1}{12}\right)$.

3 Numerical Algorithm

At each ϑ_i, Eq. (4) can be written as

$$\varepsilon u_i'' = -a(\vartheta_i)\, u_i' - b(\vartheta_i)\, u_i + f(\vartheta_i).$$

Using $\mathcal{G}_i''(\vartheta_i) = \mathcal{M}_i = u_i''$ in above equation, we get

$$\varepsilon\mathcal{M}_j = -a_j(\vartheta_i)\, u_i' - b_j(\vartheta_i)\, u_i + f_j(\vartheta_i) \quad \text{for} \quad j = i,\ i \pm 1. \quad (9)$$

Using Eq. (9) in Eq. (8) and then using u_j', for $j = i-1,\ i,\ i+1$

$$u_i' \approx \frac{1}{2h}(u_{i+1} - u_{i-1}),\ u_{i+1}' \approx \frac{1}{2h}(-4u_i + u_{i-1} + 3u_{i+1})$$

and

$$u_{i-1}' \approx \frac{1}{2h}(+4u_i - 3u_{i-1} - u_{i+1}),$$

$$\frac{\varepsilon}{h^2}(u_{i+1} - 2u_i + u_{i-1}) = -\alpha a_{i-1}\frac{(-u_{i+1} + 4u_i - 3u_{i-1})}{2h} - 2\beta a_i\frac{(u_{i+1} - u_{i-1})}{2h}$$

$$-\alpha a_{i+1}\frac{(u_{i-1} - 4u_i + 3u_{i+1})}{2h} - \alpha b_{i-1}u_{i-1} - 2\beta b_i u_i - \alpha\, u_{i+1} + (\alpha(f_{i-1} + f_{i+1}) + 2\beta f_i).$$

To control the oscillations and increase the accuracy of the solution, we introduce the fitting parameter $\sigma\ (\rho)$ in the proposed approach then, we have

$$\frac{\varepsilon\sigma\left(\rho\right)}{h^{2}}\left(u_{i+1}-2u_{i}+u_{i-1}\right)=-\alpha a_{i-1}\frac{\left(4u_{i}-u_{i+1}-3u_{i-1}\right)}{2h}-2\beta a_{i}\frac{\left(u_{i+1}-u_{i-1}\right)}{2h}$$

$$-\alpha a_{i+1}\frac{\left(u_{i-1}-4u_{\ i}+3u_{i+1}\right)}{2h}-\alpha b_{i-1}u_{i-1}-2\beta b_{i}u_{i}-\alpha b_{i+1}u_{i+1}+\left(\alpha f_{i-1}+2\beta f_{i}+\ \alpha f_{i+1}\right). \quad (10)$$

Eq. (10) can be written as

$$E_{i}u_{i-1}+F_{i}u_{i}+G_{i}u_{i+1}=H_{i},\quad i=1,2,\ldots,\mathcal{N}-1, \quad (11)$$

where

$$E_{i}=\varepsilon\sigma-h\frac{3\alpha a_{i-1}}{2}-h\beta a_{i}+h\frac{\alpha a_{i+1}}{2}+h^{2}\alpha b_{i-1},$$

$$F_{i}=-2\sigma\varepsilon+2\alpha h a_{i-1}-2\alpha h a_{i+1}+2h^{2}\beta b_{i},$$

$$G_{i}=\varepsilon\sigma-h\frac{\alpha a_{i-1}}{2}+h\beta a_{i}+h\frac{3\alpha a_{i+1}}{2}+h^{2}\alpha b_{i+1},$$

$$H_{i}=h^{2}\left(\alpha(f_{i-1}+f_{i+1})\ +2\beta f_{i}\right).$$

With the help of Thomas algorithm and boundary conditions $u\left(0\right)=\varphi_{0}$, $u\left(1\right)=\psi_{1}$ Eq. (11) can be solved.

To calculate fitting parameter from singular perturbations theory, an approximation for the solution of Eq. (4)

$$u\left(\vartheta\right)=u_{0}\left(\vartheta\right)+\frac{a\left(0\right)}{a\left(\vartheta\right)}\left(\varphi_{0}-u_{0}\left(0\right)\right)\exp^{-\int_{0}^{\vartheta}\left(\frac{a\left(\vartheta\right)}{\varepsilon}\right)d\vartheta}+o\left(\varepsilon\right), \quad (12)$$

where $u_{0}\left(\vartheta\right)$ is the solution of

$$a\left(\vartheta\right)u_{0}'\left(\vartheta\right)+b\left(\vartheta\right)u_{0}\left(\vartheta\right)=f\left(\vartheta\right),\quad u_{0}\left(1\right)=\psi_{1}.$$

If we expand $a\left(\vartheta\right)$ and $b\left(\vartheta\right)$ about the point zero using Taylor's series, then Eq. (12) becomes

$$u\left(\vartheta\right)=u_{0}\left(\vartheta\right)+\left(\varphi_{0}-u_{0}\left(0\right)\right)\exp^{-\left(\frac{a\left(\vartheta\right)}{\varepsilon}\right)\vartheta}+o\left(\varepsilon\right). \quad (13)$$

From Eq. (13), we have

$$\lim_{h\to0}u\left(ih\right)=u_{0}\left(0\right)+\exp^{-a\left(\vartheta_{i}\right)\rho}\left(\varphi_{0}-u_{0}\left(0\right)\right).$$

These limit values used in Eq. (10), we obtain the $\sigma\left(\rho\right)$

$$\sigma\left(\rho\right)=\rho\left(\beta+\alpha\right)a_{i}\coth\left(\frac{a_{i}\rho}{2}\right),\ \text{where}\ \ \rho=\frac{h}{\varepsilon}.$$

4 Convergence Analysis

The local truncation error estimate for the computational scheme of Eq. (11) is

$$T\left(h\right)=h^{2}\left[1-2\left(\beta+\alpha\right)\right]\varepsilon u_{i}''+$$

$$h^{4}\left[\alpha\left(b_{i}''u_{i}+a_{i}''u_{i}'+2b_{i}'u_{i}'-2a_{i}'u_{i}''+b_{i}u_{i}''+\frac{a_{i}}{3}u_{i}'''-f_{i}''\right)+\frac{\beta a_{i}}{3}u_{i}'''\right]+o\left(h^{6}\right).$$

Hence, the truncation error of order four as $(\alpha,\beta)\to\left(\frac{1}{12},\frac{5}{12}\right)$.

Using the boundary conditions in Eq. (2), the matrix form of Eq. (11) is

$$(\mathbb{Q}+\mathbb{R})\,U+\widetilde{M}+\mathbb{T}\,(h)=O, \tag{14}$$

where

$$\mathbb{Q}=\begin{bmatrix} -2\sigma\varepsilon & \sigma\varepsilon & 0 & 0 & ... & 0\\ \sigma\varepsilon & -2\sigma\varepsilon & \sigma\varepsilon & .. & ... & 0\\ 0 & .. & .. & .. & ... & 0\\ .. & .. & .. & .. & .. & ..\\ .. & .. & .. & .. & .. & ..\\ 0 & .. & .. & .. & \sigma\varepsilon & -2\sigma\varepsilon \end{bmatrix},$$

$$\mathbb{R}=\begin{bmatrix} v_1 & w_1 & 0 & 0 & .... & 0\\ x_2 & v_2 & w_2 & 0 & .... & 0\\ 0 & x_3 & v_3 & w_3 & .... & 0\\ .. & .. & .. & .. & .. & ..\\ .. & .. & .. & .. & .. & ..\\ 0 & .. & .. & 0 & x_{\mathcal{N}-1} & v_{\mathcal{N}-1} \end{bmatrix},$$

with

$$x_i=-\frac{3\alpha h a_{i-1}}{2}-\beta h a_i+\frac{\alpha h a_{i+1}}{2}+h^2\alpha b_{i-1},\quad v_i=2\alpha h a_{i-1}-2\alpha h a_{i+1}+2h^2\beta b_i,$$

$$w_i=-\frac{\alpha h a_{i-1}}{2}+\beta h a_i+\frac{3\alpha h a_{i+1}}{2}+h^2\alpha b_{i+1},\ \forall\, i=1,2,...,\mathcal{N}-1,$$

and

$$\widetilde{M}\ =\ [m_1+(\sigma\varepsilon+x_1)\,\varphi_0, m_2, m_3\ ,\dots, m_{\mathcal{N}-2}, m_{\mathcal{N}-1}+(\varepsilon\sigma+w_{\mathcal{N}-1})\,\psi_1]^T\ ,$$

with $m_i=h^2\left(\alpha(f_{i-1}+f_{i+1})+2\beta f_i\right)$ for $i=1,2,\dots,\mathcal{N}-1$, $\mathbb{T}\,(h)=O\left(h^4\right)$; $U=[U_1,U_2,...,U_{\mathcal{N}-1}]^T$, $\mathbb{T}\,(h)=[\bar{t}_1,\bar{t}_2,...,\bar{t}_{\mathcal{N}-1}]^T$, $O=[0,0,...,0]^T$ are corresponding vectors of Eq. (14).

Let $u=[u_1,u_2,...,u_{\mathcal{N}-1}]^T\cong U$ which satisfies the equation

$$(\mathbb{Q}+\mathbb{R})\,u+\widetilde{M}=0. \tag{15}$$

If $e_i=u_i-U_i$, $i=1,2,\dots,\mathcal{N}-1$ denote discretization error, then $\widetilde{E}=[\bar{e}_1,\bar{e}_2,...,\bar{e}_{\mathcal{N}-1}]^T=u-U$.

Subtracting Eq. (14) from Eq. (15), we obtain

$$(\mathbb{Q}+\mathbb{R})\,E=\mathbb{T}(h). \tag{16}$$

Let $|a_i|\le P_1$, $|b_i|\le P_2$ so that if $R_{i,j}$ is the $(i,j)^{th}$ element of matrix $\mathbb{R}$, then

$$|R_{i,i+1}|=|w_i|\le\varepsilon+(h\alpha+h\beta)\,P_1+h^2\alpha P_2,\quad i=1,2,\dots,\mathcal{N}-2,$$

$$|R_{i,\ i-1}|=|x_i|\le\varepsilon+(h\alpha+h\beta)\,P_1+h^2\alpha P_2,\quad i=2,3\dots,\mathcal{N}-1.$$

Thus, for relatively small h ($h\to 0$), we observe that

$$|R_{i,i+1}|<\varepsilon,\quad \forall\, i=1,2,\dots,\mathcal{N}-2,$$
$$|R_{i,i+1}|<\varepsilon,\quad \forall\, i=2,3,\dots,\mathcal{N}-1.$$

Hence $(\mathbb{Q}+\mathbb{R})$ is irreducible [20].

Let $\mathbb{S}_i$ be sum of i^{th} row elements of the matrix $(\mathbb{Q}+\mathbb{R})$, then

$$\mathbb{S}_i=-\sigma\varepsilon+h\frac{3\alpha a_{i-1}}{2}+\beta h a_i-h\frac{\alpha a_{i+1}}{2}+h^2\left(\alpha b_{i+1}+2\beta b_i\right)\ \text{for}\ \ i=1,$$

$$\mathbb{S}_i = h^2\left(\alpha(b_{i-1}+b_{i+1})+2\beta b_i\right) \text{ for } i=2,3,\ldots,\mathcal{N}-2,$$

$$\mathbb{S}_i = -\sigma\varepsilon + h\frac{\alpha a_{i-1}}{2} - \beta h a_i - h\frac{3\alpha a_{i+1}}{2} + h^2\left(\alpha b_{i-1}+2\beta b_i\right) \text{ for } i=\mathcal{N}-1.$$

Let

$$P_{1*} = \min_{1\ \le i \le\ \mathcal{N}-1} |a_i|, \quad {P_1}^* = \max_{1\ \le i\ \le \mathcal{N}} |a_i|\,, \quad P_{2*} = \min_{1\ \le i\ \le \mathcal{N}-1} |b_i|, \quad {P_2}^* = \max_{1\ \le i\ \le \mathcal{N}} |b_i|,$$

then $0 \le P_{1*} \le P_1 \le {P_1}^*,\ 0 \le P_{2*} \le P_2 \le {P_2}^*$.

If h sufficiently small h $(h \to 0)$, then $(\mathbb{Q}+\mathbb{R})$ is monotone [20]. Hence, $(\mathbb{Q}+\mathbb{R})^{-1}$ exists and $(\mathbb{Q}+\mathbb{R})^{-1} \ge 0$. Therefore, from Eq. (16), we have

$$||E|| \le\ ||\mathbb{Q}+\mathbb{R}||^{-1}||\mathbb{T}||. \tag{17}$$

Let $(\mathbb{Q}+\mathbb{R})^{-1}_{i,k}$ be the $(i,k)^{th}$ element of $(\mathbb{Q}+\mathbb{R})^{-1}$ and define

$$||\mathbb{Q}+\mathbb{R}||^{-1} = \max_{1\le\ i \le\ \mathcal{N}-1} \sum_{k=1}^{\mathcal{N}-1} (\mathbb{Q}+\mathbb{R})^{-1}_{i,k}\,, \text{ and } \quad ||\mathbb{T}(h)|| = \max_{1\le\ i\le\ \mathcal{N}-1} |\mathbb{T}_i|.$$

Since $(\mathbb{Q}+\mathbb{R})^{-1}_{i,k} \ge 0$ and $\sum_{k=1}^{\mathcal{N}-1}(\mathbb{Q}+\mathbb{R})^{-1}_{i,k}\,,\ \mathbb{S}_k = 1$ for $i = 1,2,...,\mathcal{N}-1$, hence,

$$(\mathbb{Q}+\mathbb{R})^{-1}_{i,k} \le \frac{1}{\mathbb{S}_i} < \frac{1}{h^2 P_2}, \quad i=1, \tag{18a}$$

$$(\mathbb{Q}+\mathbb{R})^{-1}_{i,k} \le \frac{1}{\mathbb{S}_i} < \frac{1}{h^2 P_2}, \quad i=\mathcal{N}-1. \tag{18b}$$

Furthermore,

$$\sum_{k=1}^{\mathcal{N}-1} (\mathbb{Q}+\mathbb{R})^{-1}_{i,k} \le \frac{1}{\min\limits_{2\le i\le \mathcal{N}-2} \mathbb{S}_i} < \frac{1}{h^2 P_2}. \tag{18c}$$

From Eq. (17) and Eq. (18a)–(18c), we get

$$||E|| \le O\ (h^2).$$

This illustrates the quadratic rate of convergence for Eq. (11), as $(\alpha,\beta) \to \left(\frac{1}{12},\frac{5}{12}\right)$.

5 Numerical Illustrations

To examine the quality and robustness of the suggested technique, we solved two different test problems and reported the numerical results in the form of maximum absolute errors (MAEs) with and without fitting parameter and the computed rates of convergence (ROC) in the tables. The MAEs are calculated with the double mesh principle because the exact solutions to test problems are unknown.

$$E_{\mathcal{N}} = \max_{0\le i\le \mathcal{N}} \left|{u_{2i}}^{2\mathcal{N}} - {u_i}^{\mathcal{N}}\right|,$$

where ${u_i}^{\mathcal{N}}$ and ${u_{2i}}^{2\mathcal{N}}$ are the numerical solutions of the problem for $\mathcal{N}$ and $2\mathcal{N}$ mesh points respectively. Further, formula was used to determine the numerical rate of convergence (ROC).

$$R_{\mathcal{N}} = \log_2\left|\frac{E_{\mathcal{N}}}{E_{2\mathcal{N}}}\right|.$$

Example 1.

$$\varepsilon u''(\vartheta) + (1+\vartheta)\, u'(\vartheta-\delta) + \exp(-2\vartheta)\, u(\vartheta+\eta) - 2\exp(-\vartheta)\, u(\vartheta) = 0,$$

with boundary constraints $u(\vartheta) = 1;\ -\delta \le \vartheta \le 0,\ u(1) = 0.$

Example 2.

$$\varepsilon u''(\vartheta) + (1+\vartheta)\, u'(\vartheta-\delta) + \sin(2\vartheta)\, u(\vartheta+\eta) - \exp(-\vartheta)\, u(\vartheta) = \sin(2\vartheta) + 3\exp(-\vartheta),$$

with boundary constraints $u(\vartheta) = -1;\ -\delta \le \vartheta \le 0,\ u(1) = 1.$

Table 1

MAEs and ROCs of Example 1 (with fitting factor)

$\varepsilon \downarrow \mathcal{N} \rightarrow$	2^4	2^5	2^6	2^7	2^8	2^9
$\eta = \delta = 0.5\varepsilon$						
2^{-1}	5.6311e-04	1.3401e-04	3.3083e-05	8.2448e-06	2.0596e-06	5.1483e-07
	2.0710	2.1199	2.0045	2.0011	2.0001	
2^{-2}	9.6942e-04	2.2072e-04	5.3696e-05	1.3362e-05	3.3344e-06	8.3321e-07
	2.1349	2.0393	2.0104	2.0066	2.0006	
2^{-3}	2.3099e-03	4.2972e-04	9.8534e-05	2.4310e-05	6.0432e-06	1.5087e-06
	2.4263	2.1247	2.0313	2.0190	2.0020	
2^{-4}	5.9356e-03	1.0579e-03	2.0386e-04	4.7300e-05	1.1613e-05	2.8940e-06
	2.4881	2.3755	2.1076	2.0260	2.0046	
2^{-5}	1.1620e-02	2.8395e-03	5.0813e-04	9.9643e-05	2.3249e-05	5.7093e-06
	2.0329	2.4823	2.3503	2.0995	2.0257	
2^{-6}	1.4231e-02	5.8252e-03	1.3925e-03	2.4936e-04	4.9330e-05	1.1542e-05
	1.2886	2.0646	2.4813	2.3376	2.0955	
2^{-7}	1.4493e-02	7.3470e-03	2.9211e-03	6.9013e-04	1.2358e-04	2.4555e-05
	0.9801	1.3306	2.0815	2.4814	2.3313	
2^{-8}	1.4524e-02	7.5002e-03	3.7407e-03	1.4634e-03	3.4363e-04	6.1524e-05
	0.9534	1.0036	1.3539	2.0903	2.4816	
2^{-9}	1.4538e-02	7.5104e-03	3.8238e-03	1.8884e-03	7.3247e-04	1.7147e-04
	0.9528	0.9738	1.0178	1.36623	2.0948	

Table 2

MAEs of Example 1 (without fitting factor)

$\varepsilon \downarrow \mathcal{N} \rightarrow$	2^4	2^5	2^6	2^7	2^8	2^9
$\eta = \delta = 0.5\varepsilon$						
2^{-1}	2.5261e-03	6.2615e-04	1.5621e-04	3.9031e-05	9.7564e-06	2.4391e-06
2^{-2}	8.1260e-03	1.9974e-03	4.9626e-04	1.2401e-04	3.0985e-05	7.7463e-06
2^{-3}	3.2720e-02	7.0941e-03	1.7178e-03	4.2932e-04	1.0711e-04	2.6766e-05
2^{-4}	1.1969e-01	2.9731e-02	6.5329e-03	1.5862e-03	3.9373e-04	9.8484e-05
2^{-5}	3.3077e-01	1.1540e-01	2.8201e-02	6.2431e-03	1.5181e-03	3.7697e-04
2^{-6}	6.6691e-01	3.3194e-01	1.1324e-01	2.7433e-02	6.0967e-03	1.4837e-03
2^{-7}	1.0199e+00	6.9033e-01	3.3256e-01	1.1217e-01	2.7049e-02	6.0235e-03
2^{-8}	1.2902e+00	1.0854e+00	7.0056e-01	3.3292e-01	1.1164e-01	2.6858e-02
2^{-9}	1.4613e+00	1.3962e+00	1.1125e+00	7.0578e-01	3.3312e-01	1.1137e-01

Table 3

MAEs and ROCs of Example 2 (with fitting factor)

$\varepsilon \downarrow \mathcal{N} \rightarrow$	2^4	2^5	2^6	2^7	2^8	2^9
$\eta = \delta = 0.5\varepsilon$						
2^{-1}	5.8352e-03	1.8204e-03	5.1499e-04	1.3599e-04	3.4639e-05	8.7050e-06
	1.6805	1.8216	1.9210	1.9730	1.9924	
2^{-2}	1.0479e-02	3.6556e-03	1.1655e-03	3.5046e-04	1.0087e-04	2.7900e-05
	1.5193	1.6491	1.7336	1.7967	1.8541	
2^{-3}	1.5717e-02	5.9555e-03	2.0005e-03	6.2793e-04	1.8872e-04	5.5003e-05
	1.4000	1.5742	1.6716	1.7343	1.9153	
2^{-4}	2.0188e-02	8.6036e-03	3.1435e-03	1.0417e-03	3.2435e-04	9.6961e-05
	1.2304	1.4525	1.5934	1.6833	1.7420	
2^{-5}	2.2262e-02	1.0756e-02	4.4448e-03	1.6094e-03	5.3115e-04	1.6496e-04
	1.0494	1.2749	1.4655	1.5993	1.6870	
2^{-6}	2.2569e-02	1.1731e-02	5.5199e-03	2.2582e-03	8.1433e-04	2.6826e-04
	0.9440	1.1740	1.2894	1.4714	1.6019	
2^{-7}	2.2572e-02	1.1895e-02	6.0190e-03	2.7959e-03	1.1381e-03	4.0962e-04
	0.9241	0.9827	1.1062	1.2966	1.4742	
2^{-8}	2.2569e-02	1.1900e-02	6.1045e-03	3.0472e-03	1.4070e-03	5.7135e-04
	0.9233	0.9630	1.0023	1.1148	1.3001	
2^{-9}	2.2568e-02	1.1899e-02	6.1074e-03	3.0909e-03	1.5330e-03	7.0573e-04
	0.9234	0.9622	0.9825	1.0116	1.1191	

Table 4

MAEs of Example 2 (without fitting factor)

$\varepsilon \downarrow \mathcal{N} \rightarrow$	2^4	2^5	2^6	2^7	2^8	2^9
$\eta = \delta = 0.5\varepsilon$						
2^{-1}	4.4262e-03	1.0866e-03	2.7065e-04	6.7609e-05	1.6897e-05	4.2241e-06
2^{-2}	1.3543e-02	3.2492e-03	8.1299e-04	2.0259e-04	5.0602e-05	1.2647e-05
2^{-3}	5.2911e-02	1.1506e-02	2.7860e-03	6.9285e-04	1.7304e-04	4.3238e-05
2^{-4}	1.9685e-01	4.8317e-02	1.0644e-02	2.5854e-03	6.4179e-04	1.6035e-04
2^{-5}	5.4951e-01	1.9032e-01	4.6290e-02	1.0263e-02	2.4963e-03	6.1989e-04
2^{-6}	1.1029e+00	5.5157e-01	1.8761e-01	4.5356e-02	1.0088e-02	2.4554e-03
2^{-7}	1.6521e+00	1.1505e+00	5.5337e-01	1.8642e-01	4.4912e-02	1.0005e-02
2^{-8}	2.0454e+00	1.8037e+00	1.1678e+00	5.5460e-01	1.8587e-01	4.4696e-02
2^{-9}	2.2859e+00	2.2936e+00	1.8556e+00	1.1768e+00	5.5530e-01	1.8561e-01

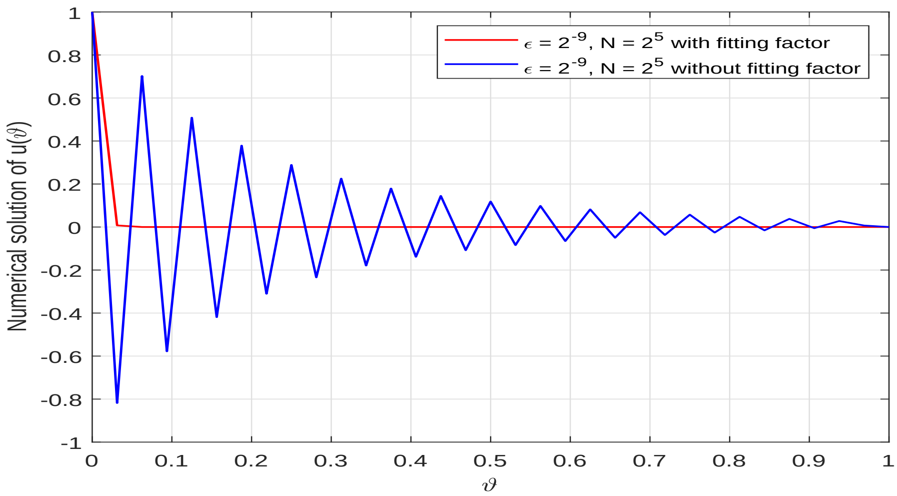

Figure 1. Solution profile of Ex.1 for $\eta = 0.5\varepsilon = \delta$ with and without fitting parameter

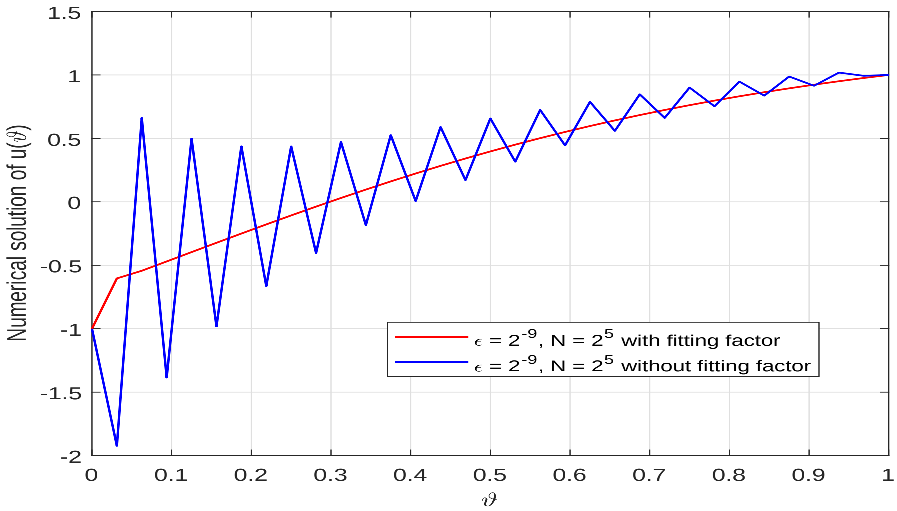

Figure 2. Solution profile of Ex.2 for $\eta = 0.5\varepsilon = \delta$ with and without fitting parameter

Conclusion

A simple and efficient computational non polynomial quartic spline technique is presented for solving second-order SPDDE with mixed parameters on the convection and reaction terms. To illustrate the accuracy and effectiveness of the approach, we solved two example problems for different values of ε, N with $\eta = \delta = 0.5\varepsilon$ and reported the numerical results in terms of maximum absolute errors (MAEs) and rate of convergence (ROC) with and without fitting factor. Using MATLAB, the results of the examples are listed in Tables 1, 2, 3, and 4 in terms of MAEs. Numerical solutions of Example 1 and Example 2 with fitting factors are shown in Table 1 and Table 3 in terms of MAEs and numerical ROCs, and Tables 2 and 4 show the numerical solutions of Examples 1 and 2 without fitting factor. The figures (1-2) depict graphs of the solutions of test problems with and without fitting factor respectively. As ε decreases for different values of $\mathcal{N}$, the solution can exhibit oscillatory behaviour. To manage these drawbacks in solutions of test problems, we introduced a fitting parameter so that we can control the layer structure shown in Figures 1-2. The maximum elapsed time is approximately 0.77 seconds for various values of $\mathcal{N}$ and ε for two test problems.

Author Contributions

All authors contributed equally to this work.

Conflict of Interest

The authors declare that they have no conflict of interest.

References

1 Stein, R.B. (1967). Some models of neuronal variability. *Biophysical Journal, 7*(1), 37–68. https://doi.org/10.1016/S0006-3495(67)86574-3

2 Longtin, A., & Milton, J.G. (1988). Complex oscillations in the human pupil light reflex with mixed and delayed feedback. *Mathematical Biosciences, 90*(1-2), 183–199. https://doi.org/10.1016/0025-5564(88)90064-8

3 Roos, H.-G., Stynes, M., & Tobiska, L. (2008). *Robust Numerical Methods for Singularly Perturbed Differential Equations: Convection-Diffusion-Reaction and Flow Problems.* Springer Berlin Heidelberg. https://doi.org/10.1007/978-3-540-34467-4

4 Priyadarshana, S., Sahu, S.R., & Mohapatra, J. (2022). Asymptotic and numerical methods for solving singularly perturbed differential difference equations with mixed shifts. *Iranian Journal of Numerical Analysis and Optimization, 1*(12), 55–72. https://doi.org/10.22067/ijnao.2021.731.1038

5 Kadalbajoo, M.K., & Ramesh, V.P. (2007). Hybrid method for numerical solution of singularly perturbed delay differential equations. *Applied Mathematics and Computation, 187*(2), 797–814. https://doi.org/10.1016/j.amc.2006.08.159

6 Lalu, M., Phaneendra, K., & Emineni, S.P. (2021). Numerical approach for differential-difference equations having layer behaviour with small or large delay using non-polynomial spline. *Soft Computing, 25*, 13709–13722. https://doi.org/10.1007/s00500-021-06032-5

7 Das, P., Rana, S., & Vigo-Aguiar, J. (2020). Higher order accurate approximations on equidistributed meshes for boundary layer originated mixed type reaction diffusion systems with multiple scale nature. *Applied numerical mathematics, 148*, 79–97. https://doi.org/10.1016/j.apnum.2019.08.028

8 Ragula, K., Soujanya, G.B.S.L., & Swarnakar, D. (2023). Computational Approach for a Singularly Perturbed Differential Equations With Mixed Shifts Using a Non-Polynomial Spline. *International Journal of Analysis and Applications, 21*, Article 5. https://doi.org/10.28924/2291-8639-21-2023-5.

9 Ranjan, R., & Prasad, H.S. (2021). A novel approach for the numerical approximation to the solution of singularly perturbed differential-difference equations with small shift. *Journal of Applied Mathematics and Computing, 65*, 403–427. https://doi.org/10.1007/s12190-020-01397-6

10 Adivi Sri Venkata, R.K., & Palli, M.M.K. (2017). A numerical approach for solving singularly perturbed convection delay problems via exponentially fitted spline method. *Calcolo, 54*, 943–961. https://doi.org/10.1007/s10092-017-0215-6

11 Prathap, T., & Rao, R.N. (2024). Fitted mesh methods based on non-polynomial splines for singularly perturbed boundary value problems with mixed shifts. *AIMS Mathematics, 9*(10), 26403–26434. https://doi.org/10.3934/math.20241285

12 Kiltu, G.G., Duressa, G.F., & Bullo, T.A. (2020). Numerical treatment of singularly perturbed delay reaction-diffusion equations. *International Journal of Engineering, Science and Technology, 12*(1), 15–24. https://doi.org/10.4314/ijest.v12i1.2

13 Debela, H.G., & Duressa, G.F. (2022). Robust Numerical Method for Singularly Perturbed Two-Parameter Differential-Difference Equations. *Applied Mathematics E-Notes, 22*(1), 200–209.

14 Tirfesa, B.B., Duressa, G.F., & Garoma, H. (2022). Non-polynomial cubic spline method for solving singularly perturbed delay reaction-diffusion equations. *Thai Journal of Mathematics, 20*(2), 679–692.

15 Ayalew, M., Kiltu, G.G., & Duressa, G.F. (2021). Fitted Numerical Scheme for Second-Order Singularly Perturbed Differential-Difference Equations with Mixed Shifts. *Abstract and Applied Analysis*, Article 4573847. https://doi.org/10.1155/2021/4573847

16 Sirisha, L., Phaneendra, K., & Reddy, Y.N. (2018). Mixed finite difference method for singularly perturbed differential difference equations with mixed shifts via domain decomposition. *Ain Shams Engineering Journal, 9*(4), 647–654. https://doi.org/10.1016/j.asej.2016.03.009

17 Soujanya, G., Ragula, K., & Phaneendra, K. (2023). A difference scheme using a parametric spline for differential difference equation with twin layers. *International Journal of Nonlinear Analysis and Applications, 14*(1), 2469–2479. https://doi.org/10.22075/ijnaa.2022.28237.3841

18 Joy, D., & Kumar, S.D. (2024). Computational techniques for singularly perturbed reaction-diffusion delay differential equations: a second-order approach. *Journal of mathematics and computer science, 35*, 304–318.

19 Justine, H., Chew, J., & Sulaiman, J. (2017). Quartic non-polynomial spline solution for solving two-point boundary value problems by using Conjugate Gradient Iterative Method. *Journal of Applied Mathematics and Computational Mechanics, 16*(1), 41–50. https://doi.org/10.17512/jamcm.2017.1.04

20 Varga, R.S. (1962). *Matrix iterative analysis.* Englewood Cliffs, N.J.: Prentice-Hall.

*Author Information**

Kumar Ragula (*corresponding author*) — Doctor of Mathematical Sciences, Assistant Professor, Department of Mathematics, Rajiv Gandhi University of Knowledge Technologies, Basar, 504107, India; e-mail: *rksm39@gmail.com*; https://orcid.org/0000-0003-3671-3590

G.B.S.L. Soujanya — Doctor of Mathematical Sciences, Assistant Professor, Department of Mathematics, Kakatiya University, Warangal, 506009, India; e-mail: *gbslsoujanya@gmail.com*; https://orcid.org/0000-0002-4522-7896

*Authors' names are presented in the following order: first name, middle name (if any), last name.