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A risk model for insurance companies on time scales

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This article deals with the problems of constructing and analyzing a collective risk model for an insurance company when the time evolution is defined on a general time scale. The relevance of the study is determined by the need to describe premium accumulation and claim payments occurring at discrete or irregular time instants within a unified analytical framework. The characteristic features of the classical risk model and its extension to time scales are analyzed, and the need to investigate the behavior of the non-ruin probability under such a generalization is identified and justified. On the basis of the study, the authors construct an analogue of the classical model on time scales and derive a dynamic equation for the distribution of the number of claims. An integral equation on a time scale for the non-ruin probability is formulated. Conditions ensuring the correctness of the constructed model are established. It is proved that the non-ruin probability defined on a family of time scales converges pointwise to the corresponding probability in the classical continuous-time risk model as the graininess function tends to zero. It is shown that the proposed approach provides a rigorous justification of the transition from discrete to continuous risk models.

Keywords: time scales, risk process, non-ruin probability, graininess function, weak convergence, local asymptotic stability, dynamic equation, integral equation, claim number distribution.

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Introduction

In modeling the operation of an insurance company, the classical risk model is well known (see, e.g., [1, 2]). According to this model, premium income from policyholders accumulates linearly over time as ct, where $c > 0$ denotes the premium intensity. At random times $\tau_1, \tau_2, \dots, \tau_n, \dots$, insurance claims are paid out, with claim sizes described by a sequence of independent and identically distributed random variables $Y_1, Y_2, \dots, Y_n, \dots$. It is assumed that the claim arrival times form a Poisson process with intensity $\alpha > 0$, that is, the stochastic process N_t, representing the number of claims occurring in the interval $[0, t)$, is a Poisson process. The total capital (surplus) of the insurance company at time t is given by

$$U_t = u + ct - S_t,$$

where u denotes the initial capital of the company and S_t is the aggregate claims up to time t. Clearly,

$$S_t = \sum_{k=1}^{N_t} Y_k.$$

The main quantitative characteristic of this model is the function $\varphi(u)$, which represents the non-ruin probability, defined by

$$\varphi(u) = P\{U_t \geq 0 \text{ for all } t \geq 0\}. \tag{1}$$

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It is well known that this function satisfies the following integro-differential equation:

$$\varphi'(u) = \frac{\alpha}{c}\varphi(u) - \frac{\alpha}{c}\int_0^u \varphi(u-z)\,dF(z),$$

where $F(z)$ is the distribution function of the claim sizes Y_k.

However, in practical applications, premium income does not accumulate continuously according to the linear law ct, but rather occurs at discrete or irregular time instants. Such time structures can be naturally described within the framework of time scale calculus.

The theory of dynamic equations on time scales, introduced in the pioneering work of S. Hilger [3], provides a unified approach to the study of continuous and discrete dynamical systems. Comprehensive treatments of the theory can be found in the monographs [4–6]. Recent developments and applications of dynamic equations on time scales are discussed, for example, in [7–9].

In this paper, we construct and study a mathematical model of an insurance company operating on time scales. First, under suitable assumptions, we develop an analogue of the classical risk model in this setting. Next, we derive a dynamic equation on a time scale for the function $P_n(t)$, defined by

$$P_n(t) = P\{N_t = n\}. \tag{2}$$

At a subsequent stage, we derive an integral equation on a time scale for the non-ruin probability $\varphi(u)$.

The aim of this work is to show that the non-ruin probability $\varphi_\lambda(u)$, defined on a family of time scales $\mathbb{T}_\lambda$, converges pointwise to the function $\varphi(u)$ as the graininess function $\mu_\lambda(t)$ tends to zero. Here, $\varphi(u)$ denotes the probability of non-ruin in the classical continuous-time risk model.

Noteworthy results concerning dynamic equations on time scales with complex topological structures (for example, Cantor-type sets) were obtained in [10, 11]. It should be noted that quantum calculus (or q-calculus) can be viewed as a particular case of the theory of dynamic equations on time scales, corresponding to time scales of the form $\mathbb{T} = q^{\mathbb{N}_0}$ or their modifications. Recent investigations of differential equations in the framework of q-calculus include, for example, [12–14].

Various properties of solutions have been investigated in the context of transitions between differential and dynamic equations. In particular, optimal control problems for ordinary differential equations and dynamic equations on time scales have been studied in [15–17], which is important for applications. Related questions concerning qualitative properties and boundary-value problems for dynamic equations on time scales were also studied in [18–20].

Applications of dynamic equations on time scales arise in various fields. In [21–23], a population model of Beverton–Holt type is investigated. Economic applications include the Solow growth model studied in [24, 25]. Furthermore, in [26, 27], an Arrow–Pratt type model on time scales is considered in the context of optimal insurance decisions.

Despite the extensive development of both risk theory and time scale calculus, collective risk models describing the operation of insurance companies within the framework of time scales have not been sufficiently studied. The present work aims to address this gap.

The paper is organized as follows. In Section 1, we introduce basic notions related to time scales and present auxiliary results. In Section 2, we construct a mathematical model of an insurance company operating on a time scale. Section 3 contains the main result concerning the convergence of the non-ruin probability $\varphi_\lambda(u)$ for models defined on a family of time scales to the non-ruin probability $\varphi(u)$ of the classical risk model.

1 Preliminaries

1.1 Time scales and basic definitions

We begin by recalling some basic notions from the theory of time scales (see, e.g., [4, 5]). A *time scale* $\mathbb{T}$ is a nonempty closed subset of $\mathbb{R}$. For a given set $A \subset \mathbb{R}$, we define $A_{\mathbb{T}} := A \cap \mathbb{T}$.

The *forward jump operator* $\sigma : \mathbb{T} \to \mathbb{T}$ is defined by

$$\sigma(t) := \inf\{s \in \mathbb{T} : s > t\}.$$

Similarly, the *backward jump operator* $\rho : \mathbb{T} \to \mathbb{T}$ is given by

$$\rho(t) := \sup\{s \in \mathbb{T} : s < t\}.$$

A point $t \in \mathbb{T}$ is called *left-dense (LD)*, *left-scattered (LS)*, *right-dense (RD)*, or *right-scattered (RS)* if $\rho(t) = t$, $\rho(t) < t$, $\sigma(t) = t$, or $\sigma(t) > t$, respectively.

If $\mathbb{T}$ has a left-scattered maximum M, then we define $\mathbb{T}^\kappa := \mathbb{T} \setminus \{M\}$; otherwise, we set $\mathbb{T}^\kappa := \mathbb{T}$.

The *graininess function* $\mu : \mathbb{T} \to [0, \infty)$ is defined by

$$\mu(t) := \sigma(t) - t,$$

and characterizes the local structure of the time scale.

A function $f : \mathbb{T} \to \mathbb{R}^d$ is said to be Δ-*differentiable* at a point $t \in \mathbb{T}^\kappa$ if the limit

$$f^\Delta(t) := \lim_{s \to t} \frac{f(\sigma(t)) - f(s)}{\sigma(t) - s}$$

exists in $\mathbb{R}^d$. In this case, $f^\Delta(t)$ is called the Δ-*derivative* of f at t.

If f is continuous at t and t is right-scattered, then f is Δ-differentiable at t and

$$f^\Delta(t) = \frac{f(\sigma(t)) - f(t)}{\mu(t)}.$$

If t is right-dense, then f is Δ-differentiable at t provided that the limit

$$\lim_{s \to t} \frac{f(t) - f(s)}{t - s}$$

exists, and in this case

$$f^\Delta(t) = \lim_{s \to t} \frac{f(t) - f(s)}{t - s}.$$

Analogously to the classical approach based on the Carathéodory construction, one can introduce the notion of the Lebesgue Δ-measure on a time scale $\mathbb{T}$, which is denoted by μ_Δ.

The following properties hold:

1) for any $t_0 \in \mathbb{T}^\kappa$, one has

$$\mu_\Delta(\{t_0\}) = \mu(t_0);$$

2) if $a, b \in \mathbb{T}$ and $a \le b$, then

$$\mu_\Delta([a, b)) = b - a, \quad \mu_\Delta((a, b)) = b - \sigma(a), \quad \mu_\Delta((a, b]) = \sigma(b) - \sigma(a), \quad \mu_\Delta([a, b]) = \sigma(b) - a.$$

The Lebesgue integral associated with the measure μ_Δ is called the *Lebesgue* Δ-*integral* on the time scale $\mathbb{T}$.

For a Δ-measurable set $E \subset \mathbb{T}$ and a function $f : E \to \mathbb{R}$, the corresponding integral of f over E is denoted by

$$\int_E f(t)\, \Delta t.$$

Consequently, all results of the general theory of Lebesgue integration, including theorems concerning limit processes, are valid for the Lebesgue Δ-integral on $\mathbb{T}$.

Let $A \subset \mathbb{T} \setminus \{\sup \mathbb{T}\}$ be a μ_Δ-measurable set. Consider a function f, defined Δ-almost everywhere on A, with values in $\mathbb{R}^d$.

Define the set

$$\widetilde{A} := A \cup \bigcup_{r \in A \cap RS} (r, \sigma(r)),$$

and introduce the function $\widetilde{f}$, which is an extension of f defined almost everywhere on $\widetilde{A}$, by

$$\widetilde{f}(t) = \begin{cases} f(t), & t \in A, \\ f(r), & t \in (r, \sigma(r)), \quad r \in A \cap RS. \end{cases}$$

Note that the function f is μ_Δ-measurable on A if and only if the function $\widetilde{f}$ is Lebesgue measurable on $\widetilde{A}$. The following result holds.

Theorem 1. [28] The function f is Δ-integrable on A if and only if the function $\widetilde{f}$ is Lebesgue integrable on $\widetilde{A}$, and

$$\int_A f(t)\,\Delta t = \int_{\widetilde{A}} \widetilde{f}(t)\,dt.$$

In what follows, the exponential function on a time scale, denoted by $e_P(t,s)$, will play an essential role. Specifically, the function $e_P(\cdot, s)$ is defined as the unique solution of the matrix initial value problem

$$x^\Delta(t) = P(t)x(t), \quad x(s) = E,$$

where $P(t)$ is a matrix-valued function and E denotes the identity matrix of size $d \times d$.

Throughout the paper, $|x|$ denotes the Euclidean norm of a vector $x \in \mathbb{R}^d$, and $\|A\|$ denotes the matrix norm of $A \in \mathbb{R}^{d\times d}$ induced by the Euclidean vector norm.

1.2 On the correspondence between solutions of dynamic equations on time scales and ordinary differential equations

For the subsequent analysis, we require several results concerning the relationship between the properties of solutions of dynamic equations on time scales and solutions of the corresponding ordinary differential equations.

We consider the system of ordinary differential equations on the semi-axis $t \geq 0$ given by

$$\frac{dx}{dt} = \dot{x} = f(x), \quad x(0) = x_0, \tag{3}$$

and the corresponding family of initial value problems for dynamic equations on time scales $\mathbb{T}_\lambda$, where $\lambda \in \Lambda \subset \mathbb{R}$ and $\lambda = 0$ is a limit point of the set Λ.

We denote

$$[0,T]_\lambda := [0,T] \cap \mathbb{T}_\lambda$$

and assume that the points 0 and T belong to all time scales $\mathbb{T}_\lambda$. The corresponding dynamic initial value problems are of the form

$$x_\lambda^\Delta = f(x), \quad x_\lambda(0) = x_0. \tag{4}$$

Here $x \in B_r$, $B_r := \{x \in \mathbb{R}^d : |x| \leq r\}$, $r > 0$, and $f : B_r \to \mathbb{R}^d$. Let $\mu_\lambda(t) : \mathbb{T}_\lambda \to [0, \infty)$ denote the graininess function of the time scale $\mathbb{T}_\lambda$. We set

$$\mu_\lambda := \sup_{t \in \mathbb{T}_\lambda} \mu_\lambda(t),$$

and assume that $\mu_\lambda \to 0$ as $\lambda \to 0$.

Assume further that the function f is continuously differentiable on B_r. That is, there exists a constant $C > 0$ such that

$$|f(x)| + \left\| \frac{\partial f(x)}{\partial x} \right\| \leq C \tag{5}$$

for all $x \in B_r$.

We will need a result on the preservation of exponential stability when passing from ordinary differential equations to dynamic equations on time scales. In our opinion, this result is also of independent interest.

We assume that the function $f(x)$ is defined for $x \in B_r$ and that condition (5) holds on this set. Moreover, we impose the following assumptions:

(a1) $\sup \mathbb{T}_\lambda = \infty$ for all $\lambda \in \Lambda$;

(a2) $\mu_\lambda \to 0$ as $\lambda \to 0$.

We also assume that (3) and (4) admit the trivial solution, i.e., $f(0) = 0$.

Definition 1. A function $\beta : [0, r_0) \times [0, \infty) \to [0, \infty)$ is said to belong to the class $\mathcal{K}$ if the following conditions hold:

1) the function $\beta(r, t)$ is continuous with respect to the variables $r \in [0, r_0)$ and $t \geq 0$;
2) for every $t \geq 0$, the function $\beta(\cdot, t)$ is strictly increasing with $\beta(0, t) = 0$, and for every $r \in [0, r_0)$, the function $\beta(r, \cdot)$ is strictly decreasing and tends to zero as $t \to \infty$.

Definition 2. Systems (3) and (4) are called *locally asymptotically stable (LAS)* if there exists a function $\beta \in \mathcal{K}$ such that

$$|x(t, t_0, x_0)| \leq \beta(|x_0|,\, t - t_0), \qquad \beta(|x_0|,\, t - t_0) \to 0 \quad \text{as } t \to \infty,$$

and

$$|x_\lambda(t, t_0, x_0)| \leq \beta(|x_0|,\, t - t_0), \qquad \beta(|x_0|,\, t - t_0) \to 0 \quad \text{as } t \to \infty, \quad t,\, t_0 \in \mathbb{T}_\lambda.$$

Here $x(t, t_0, x_0)$ and $x_\lambda(t, t_0, x_0)$ denote the solutions of the corresponding Cauchy problems with the initial condition

$$x(t_0, t_0, x_0) = x_\lambda(t_0, t_0, x_0) = x_0.$$

The given notion, as well as the more general concept of input-to-state stability (ISS), was introduced by E. Sontag [29] to characterize the robust stabilization of nonlinear systems. In this context, the dependence of the function β on $|x_0|$ describes the rate at which solutions converge to zero.

For example, for an exponentially stable system, one can take

$$\beta(|x_0|, t - t_0) = |x_0| e^{-\alpha(t - t_0)}, \quad \alpha > 0.$$

For the nonlinear equation $\dot{x} = -x^3$, we have

$$\beta(|x_0|, t - t_0) = \frac{|x_0|}{\sqrt{2x_0^2(t - t_0) + 1}}.$$

In the general case, ISS is characterized in terms of Lyapunov functions (see, e.g., [30, 31]).

Concerning the relationship between the local asymptotic stability of systems (3) and (4), the following theorem holds.

Theorem 2. Suppose that the following conditions hold:

(A1) System (3) is locally asymptotically stable for $x_0 \in D$.

(A2) There exists $\varepsilon > 0$ such that the trivial solution $x \equiv 0$ of (3) is exponentially stable in B_ε, i.e., there exist constants $L > 0$ and $\gamma > 0$ such that

$$|x(t, t_0, x_0)| \leq L e^{-\gamma(t - t_0)} |x_0|, \qquad t \geq t_0, \quad |x_0| \leq \varepsilon.$$

A3) For $|x| \le r$, $x \neq 0$, the inequality

$$\big(f(x), x\big) < 0$$

holds, where $(\cdot,\cdot)$ denotes the scalar product in $\mathbb{R}^d$.

Moreover, conditions (a1) and (a2) are satisfied.

Then there exists $\lambda_0 \in \Lambda$ such that for all $\lambda \le \lambda_0$ system (4) is also locally asymptotically stable for $x_0 \in D$.

Proof. Without loss of generality, we set $t_0 = 0 \in \mathbb{T}_\lambda$ for all $\lambda \in \Lambda$.

From condition (A3) it follows that $\frac{d}{dt}|x(t,x_0)|^2 = 2\big(x(t,x_0), f(x(t,x_0))\big) \le 0$. Hence, the function $|x(t,x_0)|$ is non-increasing and therefore $|x(t,x_0)| \le |x_0|$ for $t \ge 0$.

Moreover, by condition (A3) and the compactness of the set $A := \{x \in \mathbb{R}^d : \varepsilon \le |x| \le r\}$, there exists a constant $\alpha > 0$ such that $\big(f(x), x\big) \le -\alpha$ for all $x \in A$.

Now consider a solution $x_\lambda(t,x_0)$ of system (4) with $x_0 \in A$ on the interval $[0, t^*_\lambda)_{\mathbb{T}_\lambda}$, where t^*_λ denotes the first exit time of the solution from the set A.

For $t \in [0, t^*_\lambda)_{\mathbb{T}_\lambda}$, we compute the Δ-derivative

$$\begin{aligned}\frac{d^\Delta}{dt}\big|x_\lambda(t,x_0)\big|^2 &= \left(\int_0^1 2\big(x_\lambda(t,x_0) + h\mu_\lambda(t) f(x_\lambda(t,x_0))\big)dh,\ f(x_\lambda(t,x_0))\right)\\ &= 2\big(x_\lambda(t,x_0), f(x_\lambda(t,x_0))\big) + \mu_\lambda(t)\,\big|f(x_\lambda(t,x_0))\big|^2 \le -2\alpha + \mu_\lambda \sup_{x\in A}|f(x)|^2.\end{aligned}$$

Choose $\lambda_1 \in \Lambda$ such that

$$-2\alpha + \mu_\lambda \sup_{x\in A}|f(x)|^2 \le 0 \quad \text{for all} \quad \lambda \le \lambda_1.$$

Then $|x_\lambda(t,x_0)| \le r$ for all $t \in \mathbb{T}_\lambda$, $\lambda \le \lambda_1$. Therefore, for all $t \ge 0$ with $t \in \mathbb{T}_\lambda$, the solutions of systems (3) and (4) remain in the ball B_r, provided that $|x_0| \le r$ and $\lambda \le \lambda_1$.

From condition (A1) it follows that there exists a function $\beta \in \mathcal{K}$ such that

$$|x(t,x_0)| \le \beta(|x_0|, t) \le \beta(r,t) \to 0, \qquad t \to \infty.$$

Hence, there exists $T_1 > 0$ such that

$$|x(t,x_0)| < \frac{\varepsilon}{4}, \qquad t \ge T_1.$$

We choose $\lambda_2 \in \Lambda$, $\lambda_2 \le \lambda_1$, such that the interval $[T_1, T_1+1]$ contains a point $T_{1,\lambda} \in \mathbb{T}_\lambda$ for all $\lambda \le \lambda_2$.

Next, we need the following proposition.

Proposition 1. Let $t_0 \in \mathbb{T}_\lambda$, and let $x(t)$ and $x_\lambda(t)$ be the solutions of systems (3) and (4), respectively, satisfying the initial condition $x(t_0) = x_\lambda(t_0)$. Assume that

$$x(t) \in B_r \quad \text{for } t \in [t_0, t_0+T], \qquad x_\lambda(t) \in B_r \quad \text{for } t \in [t_0, t_0+T]_{\mathbb{T}_\lambda}.$$

Then, for all $t \in [t_0, t_0+T]_{\mathbb{T}_\lambda}$, the estimate

$$|x(t) - x_\lambda(t)| \le \mu_\lambda\, K(T, C_r) \max_{s\in[t_0,t_0+T]} |f(x(s))| \tag{6}$$

holds, where $C_r := \max_{x\in B_r}\left\{|f(x)|,\ \left\|\frac{\partial f(x)}{\partial x}\right\|\right\}$.

Proof. The proof follows from the proof of Lemma 2.1 in [32, p. 2102], provided that we observe that the constant C_1 in inequality (2.12) of [32, p. 2103] can be chosen as $C_1^2 = \max_{t\in[t_0,t_0+T]} \left|\frac{\partial f(x(t))}{\partial x} f(x(t))\right|$, which completes the proof of Proposition 1. □

We now proceed with the proof of the theorem. Since $f(0) = 0$, it follows that

$$|f(x)| \leq \sup_{x\in B_r} \left\|\frac{\partial f(x)}{\partial x}\right\| |x|. \tag{7}$$

Then, from assumption (A3), inequalities (6) and (7), we obtain

$$|x(t) - x_\lambda(t)| \leq \mu_\lambda\, K(T, C_r)\, |x(t_0)|.$$

Hence, for $t \in [0, T_{1,\lambda}]_{\mathbb{T}_\lambda}$,

$$|x_\lambda(t)| \leq |x(t)| + |x(t) - x_\lambda(t)| \leq \beta(|x_0|, t) + \mu_\lambda K|x_0|. \tag{8}$$

In particular,

$$|x_\lambda(T_{1,\lambda})| \leq \beta(|x_0|, T_{1,\lambda}) + \mu_\lambda K|x_0| \leq \frac{\varepsilon}{4} + \mu_\lambda K|x_0|.$$

Now choose $\lambda_3 \leq \lambda_2$ such that for all $\lambda \in (0, \lambda_3]$,

$$\mu_\lambda K r \leq \frac{\varepsilon}{4}. \tag{9}$$

Then, from (8) and (9), we obtain

$$|x_\lambda(t)| \leq \beta(|x_0|, t) + \mu_\lambda K|x_0| \quad \text{and} \quad |x_\lambda(T_{1,\lambda})| \leq \beta(|x_0|, T_{1,\lambda}) + \frac{\varepsilon}{4} \leq \frac{\varepsilon}{2}.$$

Next, by condition (A2), we obtain

$$|x(t, x_0)| \leq \frac{|x(t_0)|}{8}, \qquad \text{for } t - t_0 \geq T := \frac{1}{\gamma}\ln(8L).$$

Let $x_T(t)$ be the solution of system (3) such that $x_T(T_{1,\lambda}) = x_\lambda(T_{1,\lambda})$.

Consider the interval $[T_{1,\lambda}, T_{2,\lambda}]_{\mathbb{T}_\lambda}$, where $T_{2,\lambda} \in [T_{1,\lambda} + T,\ T_{1,\lambda} + T + 1]$.

From the condition $\mu_\lambda \to 0$ as $\lambda \to 0$, it follows that for all sufficiently small $\lambda \leq \lambda_4 \leq \lambda_3$ such a point $T_{2,\lambda}$ exists. Therefore, for $t \in [T_{1,\lambda}, T_{2,\lambda}]$, we obtain

$$|x_T(t)| \leq |x_\lambda(T_{1,\lambda})| \leq \beta(|x_0|, T_{1,\lambda}) + \mu_\lambda K|x_0|.$$

Now choose $\lambda_5 \leq \lambda_4$ such that for all $\lambda \leq \lambda_5$

$$\mu_\lambda K(T + 1, C_r) \leq \frac{1}{8}.$$

Then

$$|x_T(t)| \leq \beta(|x_0|, T_{1,\lambda}) + \frac{|x_0|}{8}.$$

Hence,

$$|x_\lambda(t)| \leq |x_T(T_{1,\lambda})| + \frac{1}{8}|x_\lambda(T_{1,\lambda})| = \frac{9}{8}|x_\lambda(T_{1,\lambda})| \leq \frac{9}{8}\big(\beta(|x_0|, T_{1,\lambda}) + |x_0|\big), \tag{10}$$

and

$$|x_\lambda(T_{2,\lambda})| \le \frac{1}{4}\big|x_\lambda(T_{1,\lambda})\big| \le \frac{1}{4}\big(\beta(|x_0|, T_{1,\lambda}) + |x_0|\big). \tag{11}$$

Next, consider the interval $[T_{2,\lambda}, T_{3,\lambda}]$, where $T_{3,\lambda} \in \mathbb{T}_\lambda$ and $T_{3,\lambda} \in [T_{1,\lambda} + 2T,\ T_{1,\lambda} + 2T + 1]$.

Let $x_{2T}(t)$ be the solution of system (3) such that $x_{2T}(T_{2,\lambda}) = x_\lambda(T_{2,\lambda})$. Then, from (10), we obtain

$$|x_\lambda(t)| \le \frac{9}{8}|x_\lambda(T_{2,\lambda})| \le \frac{9}{8} \cdot \frac{1}{4}\big(\beta(|x_0|, T_{1,\lambda}) + |x_0|\big),$$

and from (11) we get

$$|x_\lambda(T_{3,\lambda})| \le \frac{1}{4}|x_\lambda(T_{2,\lambda})| \le \left(\frac{1}{4}\right)^2 \big(\beta(|x_0|, T_{1,\lambda}) + |x_0|\big). \tag{12}$$

Proceeding inductively, on the interval $[T_{k,\lambda}, T_{k+1,\lambda}]_{\mathbb{T}_\lambda}$, we similarly obtain

$$|x_\lambda(t)| \le \frac{9}{8}\left(\frac{1}{4}\right)^{k-1} \big(\beta(|x_0|, T_{1,\lambda}) + |x_0|\big), \quad |x_\lambda(T_{k+1,\lambda})| \le \left(\frac{1}{4}\right)^{k} \big(\beta(|x_0|, T_{1,\lambda}) + |x_0|\big). \tag{13}$$

This completes the proof of the theorem. □

Remark 1. It follows from the proof of Theorem 2 (see estimates (12) and (13)) that if the trivial solution of system (3) is globally exponentially stable, then for sufficiently small λ the trivial solution of system (4) is also globally exponentially stable.

2 Construction of the mathematical model

Let $\mathbb{T}$ be a time scale satisfying $\sup \mathbb{T} = +\infty$ and $\mu(t) \le 1$ for all $t \in \mathbb{T}$. In this section, we construct a mathematical model of an insurance company whose dynamics evolve on the time scale $\mathbb{T}$.

We assume that the insurance company starts its operation at time $t = 0$ (with the initial moment included, $0 \in \mathbb{T}$) with an initial capital $u \ge 0$. Premiums are assumed to be received continuously according to a linear law ct, where $c > 0$ denotes the premium intensity. Insurance claims occur at random time instants $\tau_1, \tau_2, \ldots, \tau_n, \ldots$. Let $\overline{\tau}_1, \overline{\tau}_2, \ldots, \overline{\tau}_n, \ldots$ denote the points of the time scale $\mathbb{T}$ immediately to the right of $\tau_1, \tau_2, \ldots, \tau_n, \ldots$, at which insurance payments (claims) are made.

The corresponding claim amounts $Y_1, Y_2, \ldots, Y_n, \ldots$ form a sequence of independent and identically distributed random variables defined on a probability space $(\Omega, \mathcal{F}, \mathcal{P})$. Obviously, $Y_n \ge 0$ almost surely.

Let $F(y)$ denote the distribution function of Y_n, with $F(0) = 0$. Moreover, the first two moments are assumed to exist: $\mathbb{E}Y_n = \mu$ and $\mathrm{Var}(Y_n) = \sigma^2$. We further assume that the claim sizes $\{Y_k\}$ are independent of the claim arrival times $\{\tau_k\}$. Let $N(t)$ denote, as above, the number of insurance claims occurring on the interval $[0, t)$.

With respect to the claim arrival process, we impose the following assumptions:
1. The numbers of claims occurring on disjoint time intervals are independent random variables.
2. (a) If t is a right-scattered point of the time scale $\mathbb{T}$, then the probability that exactly one insurance claim occurs at time t equals $\alpha\,\mu(t)$, where $\alpha \in (0,1)$ is the claim intensity. The probability that more than one claim occurs at such a point is equal to zero.
   (b) If t is a right-dense point of $\mathbb{T}$, then the probability that at least one insurance claim occurs on the interval $[t, t+h)_{\mathbb{T}}$ is equal to $\alpha h + o(h)$ as $h \to 0$, while the probability that more than one claim occurs on $[t, t+h)_{\mathbb{T}}$ is $o(h)$ as $h \to 0$. Here $h \in V_t := \{\beta \geq 0 : t + \beta \in \mathbb{T}\}$. Moreover, the distribution of the number of claims on $[t, t+h)_{\mathbb{T}}$ depends only on h.

Recall that we have previously introduced the risk process as a random process equal to the total capital (surplus) of the insurance company on the interval $[0, t)_{\mathbb{T}}$, namely,

$$U_t = u + ct - S_t,$$

where $S_t = \sum_{k=1}^{N_t} Y_k$. Here it is assumed that $\sum_{k=1}^{0} Y_k = 0$.

Recall also that the non-ruin probability in this risk model is defined by formula (1).

Similarly to the classical risk model, we now derive an equation for the function $P_n(t)$ defined by (2). We begin with the function $P_0(t)$, which is the probability that no insurance claims occur on the interval $[0, t)_{\mathbb{T}}$.

Let t be a right-scattered point of the time scale $\mathbb{T}$. Then the event

$$A = \{\text{no insurance claims occur on } [0, \sigma(t))_{\mathbb{T}}\}$$

can be represented as the intersection of the two events

$$A_1 = \{\text{no insurance claims occur on } [0, t)_{\mathbb{T}}\} \quad \text{and} \quad A_2 = \{\text{no claim occurs at time } t\}.$$

By assumption 1 on the claim arrival process, the events A_1 and A_2 are independent.

Therefore,

$$P(A) = P(A_1) \cdot P(A_2).$$

By the definition of $P_0(t)$, it follows that $P_0(\sigma(t)) = P_0(t)\big(1 - \alpha\,\mu(t)\big)$, or, equivalently,

$$\frac{P_0(\sigma(t)) - P_0(t)}{\mu(t)} = -\alpha\,P_0(t).$$

Thus, in this case we obtain

$$P_0^{\Delta}(t) = -\alpha\,P_0(t).$$

If the point t is right-dense, then the same arguments yield

$$P_0(t+h) = P_0(t)\big(1 - \alpha h + o(h)\big), \qquad h \to 0.$$

Hence, $P_0^{\Delta}(t) = -\alpha\,P_0(t)$. Therefore, the function $P_0(t)$ satisfies the dynamic equation

$$P_0^{\Delta}(t) = -\alpha\,P_0(t), \tag{14}$$

with the obvious initial condition $P_0(0) = 1$.

The solution of this Cauchy problem is given by the exponential function on time scales, $e_{-\alpha}(t, 0) = e_{-\alpha}(t)$. Consequently,

$$P_0(t) = e_{-\alpha}(t). \tag{15}$$

Example 1. Let $\mathbb{T} = \mathbb{R}$. Then the exponential function on time scales coincides with the classical exponential, i.e., $e_{-\alpha}(t) = e^{-\alpha t}$, $t \in \mathbb{R}$.

Example 2. Let $\mathbb{T} = h\mathbb{Z}$ with $h > 0$. Then the exponential function on this time scale is given by

$$e_{-\alpha}(t) = (1 - \alpha h)^{\frac{t}{h}}, \qquad t \in \mathbb{T}. \tag{16}$$

Let now $n \geq 1$. Assume that t is a right-scattered point of the time scale $\mathbb{T}$. Then the event

$$A = \{\text{exactly } n \text{ insurance claims occur on } [0, \sigma(t))_{\mathbb{T}}\}$$

can be represented as the union of two mutually exclusive events $A = A_1 \cup A_2$.

The event A_1 is the intersection of two independent events $A_1 = B_{11} \cap B_{12}$, where

$$B_{11} = \{\text{exactly } n \text{ insurance claims occur on } [0, t)_{\mathbb{T}}\}, \quad B_{12} = \{\text{no insurance claim occurs at time } t\}.$$

Similarly, the event A_2 is the intersection of two independent events $A_2 = B_{21} \cap B_{22}$, where

$$B_{21} = \{\text{exactly } n - 1 \text{ insurance claims occur on } [0, t)_{\mathbb{T}}\},$$

$$B_{22} = \{\text{exactly one insurance claim occurs at time } t\}.$$

Consequently,

$$P_n(\sigma(t)) = P(A) = P(B_{11})P(B_{12}) + P(B_{21})P(B_{22}) = P_n(t)\big(1 - \alpha\mu(t)\big) + P_{n-1}(t)\,\alpha\mu(t).$$

Hence,

$$P_n^{\Delta}(t) = -\alpha P_n(t) + \alpha P_{n-1}(t), \qquad n \geq 1. \tag{17}$$

If the point t is right-dense, then by assumption 2 on the claim arrival process, analogous arguments show that the same relation holds. Indeed, the event

$$A = \{\text{exactly } n \text{ insurance claims occur on } [0, t + h)_{\mathbb{T}}\},$$

with $h \in V_t$, can be represented as a union of mutually exclusive events $A = \bigcup_{i=0}^{n} A_i$, where each event A_i is the intersection of two independent events, $A_i = B_{i1} \cap B_{i2}$, with

$$B_{i1} = \{\text{exactly } i \text{ insurance claims occur on } [0, t)_{\mathbb{T}}\},$$

$$B_{i2} = \{\text{exactly } n - i \text{ insurance claims occur on } [t, t + h)_{\mathbb{T}}\}.$$

By the assumptions on the claim arrival process, we have $P(B_{i1}) = P_i(t)$, $P(B_{i2}) = P_{n-i}(h)$.

Therefore,

$$P_n(t + h) = P_n(t)P_0(h) + P_{n-1}(t)P_1(h) + \sum_{k=2}^{n} P_{n-k}(t)P_k(h). \tag{18}$$

However, by assumption 2b we have $\sum_{k=2}^{n} P_{n-k}(t)P_k(h) = o(h)$ as $h \to 0$. Hence, from (18) it follows that

$$\frac{P_n(t + h) - P_n(t)}{h} = -\alpha P_n(t) + \alpha P_{n-1}(t) + \frac{o(h)}{h}.$$

Passing to the limit as $h \to 0$, we obtain

$$P_n^{\Delta}(t) = -\alpha P_n(t) + \alpha P_{n-1}(t), \qquad n = 1, 2, \ldots. \tag{19}$$

Hence, taking into account (17) and (19), the function $P_n(t)$ satisfies the following infinite system of linear dynamic equations:

$$P_n^{\Delta}(t) = -\alpha\, P_n(t) + \alpha\, P_{n-1}(t), \qquad t \in \mathbb{T}, \quad n = 1, 2, \ldots \tag{20}$$

with the initial conditions

$$P_n(0) = 0, \qquad n = 1, 2, \ldots$$

The system (20) can be simplified by introducing the substitution

$$P_n(t) = e_{-\alpha}(t)\, Q_n(t). \tag{21}$$

According to the product rule for the Δ-derivative, we obtain

$$P_n^{\Delta}(t) = -\alpha\, e_{-\alpha}(t)\, Q_n(t) + e_{-\alpha}(\sigma(t))\, Q_n^{\Delta}(t).$$

Since the exponential function on time scales satisfies $e_{-\alpha}(\sigma(t)) = \big(1 - \alpha\, \mu(t)\big) e_{-\alpha}(t)$, from (21) and (20) we obtain

$$Q_n^{\Delta}(t) = \frac{\alpha}{1 - \alpha\, \mu(t)}\, Q_{n-1}(t).$$

Note that $Q_0(0) = 1$ and $Q_n(0) = 0$, $n \geq 1$. Therefore,

$$Q_n(t) = \int_0^t \frac{\alpha}{1 - \alpha\, \mu(s)}\, Q_{n-1}(s)\, \Delta s,$$

and hence

$$P_n(t) = e_{-\alpha}(t) \int_0^t \frac{\alpha}{1 - \alpha\, \mu(s)}\, Q_{n-1}(s)\, \Delta s. \tag{22}$$

Remark 2. Expression (22) takes a particularly simple form in the case $\mathbb{T} = \mathbb{R}$. Indeed, in this case the exponential function on time scales reduces to the classical exponential, $e_{-\alpha}(t) = e^{-\alpha t}$, and we have $Q_0(t) \equiv 1$, while $Q_n(t) = \int_0^t \alpha\, Q_{n-1}(s)\, ds$. Therefore,

$$P_n(t) = e^{-\alpha t} \frac{(\alpha t)^n}{n!}, \qquad n = 0, 1, 2, \ldots$$

3 Non-ruin probability

As noted above, the principal quantitative characteristic used to assess risk is the non-ruin probability function $\varphi(u)$, defined by $\varphi(u) = P\{U_t \geq 0,\ t \geq 0\}$.

In this section, similarly to the classical risk model, we derive an integral equation for this function and study the convergence of the family of non-ruin probability functions $\{\varphi_\lambda(u)\}$ for risk models defined on a family of time scales, under the assumption that the graininess function tends to zero.

To derive an equation for $\varphi(u)$, we employ an integral analogue of the law of total probability. First, we condition on the time $\overline{\tau}_1$ of the first insurance payment, and then on the corresponding claim size Y_1.

Since $\varphi(u) = P\{U(t) \geq 0,\ \forall\, t \geq 0\} = P(A)$, where the event A means that ruin does not occur for the insurance company with initial capital u, by the law of total probability, conditioning first on the time $\overline{\tau}_1$ of the first insurance payment and then on the corresponding claim size Y_1, we obtain

$$\begin{aligned}\varphi(u) = P(A) = \int_0^{\infty} P\{A \mid \overline{\tau}_1 = s\}\, d^{\Delta} F_{\overline{\tau}_1}(s) &= \int_0^{\infty} \int_0^{\infty} P\{A \mid \overline{\tau}_1 = s,\ Y_1 = z\}\, dF(z)\, d^{\Delta} F_{\overline{\tau}_1}(s) \\ &= \int_0^{\infty} \int_0^{u+cs} P\{A \mid \overline{\tau}_1 = s,\ Y_1 = z\}\, dF(z)\, d^{\Delta} F_{\overline{\tau}_1}(s),\end{aligned}$$

where $F_{\overline{\tau}_1}(s)$ denotes the distribution function of the first insurance payment time.

By the assumptions of the model on the independence of $\overline{\tau}_1$ and Y_1, it follows from the previous formula that

$$\varphi(u) = \int_0^{\infty} \int_0^{u+cs} \varphi(u+cs-z)\, dF(z)\, d^{\Delta} F_{\overline{\tau}_1}(s). \tag{23}$$

Since

$$F_{\overline{\tau}_1}(s) = P\{\overline{\tau}_1 < s\} = P\{N(s) \geq 1\} = 1 - P\{N(s) = 0\} = 1 - e_{-\alpha}(s),$$

according to (15), we obtain $d^{\Delta} F_{\overline{\tau}_1}(s) = \alpha\, e_{-\alpha}(s)\, \Delta s$. Substituting this expression into (23), we obtain

$$\varphi(u) = \alpha \int_0^{\infty} \int_0^{u+cs} \varphi(u+cs-z)\, dF(z)\, e_{-\alpha}(s)\, \Delta s. \tag{24}$$

Equation (24) is the non-ruin probability equation on a time scale. It is a linear dynamic integral equation defined on $\mathbb{T}$.

Theorem 3. If the distribution function $F(x)$ is continuous, then equation (24) has a unique solution in the class of functions continuous on $[0, \infty)$.

Proof. Clearly, $\varphi(u) \in [0, 1]$ for all $u \geq 0$. In the metric space $\mathcal{B} = C([0, \infty), \mathbb{R})$, we consider the closed unit ball $B_1(0) := \{\varphi \in \mathcal{B} : \max_{u \geq 0} |\varphi(u)| \leq 1\}$.

Define the mapping H by

$$H(\varphi) = \alpha \int_0^{\infty} \int_0^{u+cs} \varphi(u+cs-z)\, dF(z)\, e_{-\alpha}(s)\, \Delta s.$$

We show that $H(B_1(0)) \subset B_1(0)$ and that H is a contraction mapping.

We establish the continuity of $H(\varphi)(u)$. Suppose that $u \to u_0$ and $u \leq u_0$. We have

$$\begin{aligned}
\lim_{u \to u_0} H(\varphi)(u) &= \lim_{u \to u_0} \alpha \int_0^{\infty} \int_0^{u+cs} \varphi(u+cs-z)\, dF(z)\, e_{-\alpha}(s)\, \Delta s \\
&= \lim_{u \to u_0} \alpha \int_0^{\infty} \int_0^{u_0+cs} \varphi(u+cs-z)\, dF(z)\, e_{-\alpha}(s)\, \Delta s \\
&\quad + \lim_{u \to u_0} \alpha \int_0^{\infty} \left[\int_0^{u+cs} \varphi(u+cs-z)\, dF(z) - \int_0^{u_0+cs} \varphi(u+cs-z)\, dF(z) \right] e_{-\alpha}(s)\, \Delta s \\
&=: \lim_{u \to u_0} I_1(u) + \lim_{u \to u_0} I_2(u).
\end{aligned} \tag{25}$$

Since

$$\int_0^{\infty} \int_0^{u+cs} \varphi(u+cs-z)\, dF(z)\, e_{-\alpha}(s)\, \Delta s \leq \int_0^{\infty} \int_0^{\infty} dF(z)\, e_{-\alpha}(s)\, \Delta s < \infty,$$

the continuity of the first term in (25) follows from the Lebesgue dominated convergence theorem, i.e.,

$$\lim_{u \to u_0} I_1(u) = I_1(u_0).$$

For the second term in (25) we have

$$\begin{aligned}
&\left| \int_0^{\infty} \left[\int_0^{u+cs} \varphi(u+cs-z)\, dF(z) - \int_0^{u_0+cs} \varphi(u+cs-z)\, dF(z) \right] e_{-\alpha}(s)\, \Delta s \right| \\
&\leq \int_0^{\infty} \left| \int_{u+cs}^{u_0+cs} \varphi(u+cs-z)\, dF(z) \right| e_{-\alpha}(s)\, \Delta s \leq \int_0^{\infty} |F(u_0+cs) - F(u+cs)|\, e_{-\alpha}(s)\, \Delta s.
\end{aligned} \tag{26}$$

Since the distribution function $F(t)$ is continuous, we have $F(u_0 + cs) - F(u + cs) \to 0$ as $u \to u_0$. Hence, the convergence of expression (26) to zero again follows from the Lebesgue dominated convergence theorem.

The case $u \geq u_0$ can be treated analogously.

Next, we have

$$\sup_{u \geq 0} |H(\varphi)(u)| \leq \alpha \int_0^{\infty} \int_0^{\infty} dF(z)\, e_{-\alpha}(s)\, \Delta s \leq \alpha \leq 1.$$

Therefore, the mapping H maps $B_1(0)$ into itself.

We now prove that H is a contraction mapping. We have

$$\sup_{u \geq 0} |H(\varphi_1)(u) - H(\varphi_2)(u)| \leq \alpha \int_0^{\infty} \int_0^{\infty} \sup_{u \geq 0} |\varphi_1(u) - \varphi_2(u)|\; dF(z)\, e_{-\alpha}(s)\, \Delta s \leq \alpha \sup_{u \geq 0} |\varphi_1(u) - \varphi_2(u)|\,.$$

Since $\alpha \in (0,1)$, the contraction property is proved. Hence, by the Banach fixed-point theorem, equation (24) has a unique solution. The theorem is proved.

□

Let $\{\mathbb{T}_\lambda\}$ be a family of time scales such that $\sup \mathbb{T}_\lambda = \infty$ for all $\lambda \in \Lambda$, and $\sup_{t \in \mathbb{T}_\lambda} \mu_\lambda(t) = \mu_\lambda \to 0$ as $\lambda \to 0$. On each time scale $\mathbb{T}_\lambda$, we consider the risk model described above and denote by $\varphi_\lambda(u)$ the corresponding non-ruin probability. We assume that the parameters of the risk process c, α, and the distribution function $F(t)$ are the same for all time scales.

Theorem 4. If the distribution function $F(t)$ is continuous, then

$$\varphi_\lambda(u) \to \varphi(u) \quad \text{for all } u \geq 0, \quad \text{as } \lambda \to 0,$$

where $\varphi(u)$ denotes the non-ruin probability in the classical risk model.

Proof. Clearly, $\int_0^{\infty} |\varphi_\lambda(x)|^2\, dF(x) \leq 1$. Thus, the family $\{\varphi_\lambda(u)\}$ is weakly compact in $L^2(\mathbb{R}, F(x))$ and, consequently, contains a weakly convergent subsequence $\{\varphi_{\lambda_n}(u)\}$ such that

$$\varphi_{\lambda_n} \xrightarrow{w} \psi \quad \text{in } L^2(\mathbb{R}, F(x)). \tag{27}$$

On the other hand, by (24), each function $\varphi_{\lambda_n}(u)$ satisfies the equation

$$\varphi_{\lambda_n}(u) = \alpha \int_0^{\infty} \int_0^{u+cs} \varphi_{\lambda_n}(u + cs - z)\, dF(z)\, e_{-\alpha}^{(\lambda_n)}(s)\, \Delta_{\lambda_n} s. \tag{28}$$

Here $e_{-\alpha}^{(\lambda_n)}(s)$ denotes the exponential function on the time scale $\mathbb{T}_{\lambda_n}$. From (27) we have

$$\int_0^{u+cs} \varphi_{\lambda_n}(u + cs - z)\, dF(z) \;\longrightarrow\; \int_0^{u+cs} \psi(u + cs - z)\, dF(z), \qquad \text{as } \lambda_n \to 0. \tag{29}$$

The functions $e_{-\alpha}^{(\lambda_n)}(s)$ are solutions of equation (14), which satisfies all the assumptions of Theorem 2. Moreover, the limiting equation corresponding to (14) has the form

$$\dot{x} = -\alpha x, \qquad x(0) = 1,$$

with the obvious solution.

According to Proposition 1, we have

$$e_{-\alpha}^{(\lambda_n)}(s) - e_{-\alpha}(s) \to 0, \quad \text{as } \lambda_n \to 0, \quad s \in \mathbb{T}_{\lambda_n}. \tag{30}$$

Moreover, analyzing the proof of Theorem 2, we note that the constant T in the present case is equal to $\frac{\ln 8}{\alpha}$.

Next, we have

$$\int_0^\infty e_{-\alpha}^{(\lambda_n)}(s)\,\Delta_{\lambda_n}s = \sum_{k=1}^\infty \int_{T_{k,\lambda}}^{T_{k+1,\lambda}} e_{-\alpha}^{(\lambda_n)}(s)\,\Delta_{\lambda_n}s, \tag{31}$$

where we used the decomposition of $[0,\infty)_{\mathbb{T}_{\lambda_n}}$ into the intervals $[T_{k,\lambda}, T_{k+1,\lambda}]_{\mathbb{T}_{\lambda_n}}$ introduced in the proof of Theorem 2 and the additivity of the Δ-integral.

Note that, according to the choice of the points $T_{k,\lambda}$, we have

$$T_{k+1,\lambda} - T_{k,\lambda} \le \max\{T, T_1\} + 1,$$

where T and T_1 are fixed constants. Therefore, from formula (31) we obtain

$$\sum_{k=1}^\infty \int_{T_{k,\lambda}}^{T_{k+1,\lambda}} e_{-\alpha}^{(\lambda_n)}(s)\,\Delta_{\lambda_n}s \le \sum_{k=1}^\infty \left(\frac{1}{4}\right)^{k-1} \left(e^{-\alpha T_1} + 1\right)(\max\{T, T_1\} + 1) < \infty. \tag{32}$$

Thus, from (29), (30), and (32), taking into account the Lebesgue dominated convergence theorem, it follows from (28) that

$$\varphi(u) = \alpha \int_0^\infty \int_0^{u+cs} \psi(u+cs-z)\,dF(z)\,e^{-\alpha s}\,ds. \tag{33}$$

By the uniqueness of the solution to equation (33), we conclude that $\psi(u) = \varphi(u)$. Consequently, $\varphi_\lambda(u) \to \varphi(u)$ for all $u \ge 0$ as $\lambda \to 0$.

□

Example 3. It is well known that in the classical risk model, under the assumption that the claim sizes are exponentially distributed,

$$F_{Y_k}(z) = \begin{cases} 1 - e^{-z/\mu}, & z \ge 0, \\ 0, & z < 0, \end{cases}$$

the probability of non-ruin can be found explicitly: $\varphi(u) = 1 - \frac{1}{1+\rho} e^{-\frac{\rho}{(1+\rho)\mu}u}$, where $\rho = \frac{c}{\alpha\mu} - 1$.

Now consider a time scale of the form $\mathbb{T} = \{kh \mid t = kh,\ k = 0, 1, 2, \dots\}$, for which the exponential function is given by (16). Accordingly, equation (24) takes the form

$$\varphi_h(u) = \frac{\alpha}{\mu} h \sum_{k=0}^\infty \int_0^{u+ckh} \varphi_h(u+ckh-z)\,e^{-z/\mu}\,dz\,(1-\alpha h)^k.$$

By Theorem 4, it follows that for every $u \ge 0$,

$$\varphi_h(u) \to 1 - \frac{1}{1+\rho} e^{-\frac{\rho}{(1+\rho)\mu}u}, \quad \text{as } h \to 0.$$

Conclusion

We developed a collective risk model for an insurance company on time scales, deriving a dynamic equation for the probabilities $P_n(t)$ and an integral equation for the non-ruin probability. The main result proves the pointwise convergence of the non-ruin probability $\varphi_\lambda(u)$ to the classical probability $\varphi(u)$ as the graininess function tends to zero. This rigorously justifies the classical risk model as a limiting case of models defined on arbitrary time scales. The approach extends time scale calculus to actuarial mathematics and opens perspectives for studying more general claim processes, stochastic perturbations, and optimal control problems on non-uniform time scales.

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Author Contributions

All authors contributed equally to this work.

Conflict of Interest

The authors declare no conflict of interest.

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