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Numerical solution of the boundary value problems for the parabolic equation with involution

In this work, we study two boundary value problems for involutary parabolic equation with the first and second kind conditions. We propose absolute stable difference schemes for numerical solutions of these boundary value problems. Actually the stability estimates for solutions of difference schemes are proved. Later error analysis for the numerical solution of both difference schemes are illustrated by test examples.

Keywords: involution, parabolic, finite difference scheme, stability estimate, boundary value problem.

Introduction

It is well known that various models in physics can be reduced to a parabolic equation with delay and involution. Time delay and involutary parabolic equations with local and non local boundary conditions have been investigated by several researchers [1–17].

1 Finite differences for involutary parabolic equation with Dirichlet condition

We consider boundary value problem for parabolic equation with involution and Dirichlet condition as follows

$$\begin{cases} u_t(t,x) - (a(x)u_x(t,x))_x + \delta u(t,x) + q\left(-(a(x)u_t(-t,x))_x + \delta u(-t,x)\right) = f(t,x), \\ t \in I,\ x \in (0,l), \\ u(0,x) = \varphi(x),\ x \in [0,l], \\ u(t,0) = 0,\ u(t,l) = 0,\ t \in I. \end{cases} \tag{1}$$

Here and in future a, φ and f are given smooth functions and δ and q are known numbers such that $a(x) \geq a_0 > 0$, $\forall x \in (0,l)$, δ, $|q| < 1$, $I = (-\infty, \infty)$.

1.1 Stability of differential problem

Denote by $W_2^2(0,l)$, the Sobolev space of all functions $v(x)$ defined on $[0,l]$ equipped with norm

$$\|v\|_{W_2^2(0,l)} = \left(\int_0^l |v(x)|^2\,dx\right)^{\frac{1}{2}} + \left(\int_0^l \left|v''(x)\right|^2\,dx\right)^{\frac{1}{2}}.$$

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Theorem 1. Let $\varphi \in W_2^2(0,l)$ and $f(t,x)$ be continuously differentiable on $I\times[0,l]$. Then, for the solution of initial boundary value problem (1) the stability estimates

$$\sup_{t\in I}\|u(t,\cdot)\|_{L_2(0,l)} \le M(\delta)\left[\|\varphi\|_{L_2(0,l)} + \int\limits_{-\infty}^{\infty}\|f(s,\cdot)\|_{L_2(0,l)}\,ds\right],$$

$$\sup_{t\in I}\|u_t(t,\cdot)\|_{L_2(0,l)} + \sup_{t\in I}\|u_t(t,\cdot)\|_{W_2^2(0,l)} \le \tag{2}$$

$$\le M(\delta)\left[\|\varphi\|_{W_2^2(0,l)} + \|f(0,\cdot)\|_{L_2(0,l)} + \int\limits_{-\infty}^{\infty}\|f'(s,\cdot)\|_{L_2(0,l)}\,ds\right]$$

hold, where $M(\delta)$ does not depend on both functions φ and f.

Proof. One can write problem (1) in the abstract initial value problem

$$\begin{cases} u_t(t) + Au(t) + q\ Au(-t) = f(t),\ t\in I, \\ u(0)=\varphi. \end{cases} \tag{3}$$

Here $A=A^x$ is a self adjoint positive definite operator in $H=L_2(0,l)$ which is defined by formula

$$Au(x) = -\left(a(x)u_x(x)\right)_x + \delta u(x) \tag{4}$$

with the domain $D(A)=\left\{u\in W_2^2(0,l)\ \mid u(0)=0,\ u(l)=0\ \right\}$, $\varphi=\varphi(x)$ is given element of H and $f(t)=f(t,x)$ is a given abstract function. The proof of Theorem 1 is based on the stability of abstract problem (3) and positiveness and self-adjointness of the abstract operator A defined by (4).

1.2 Stability of difference problems

Let $[0,l]_h = \{x_n = nh,\ 0\le n\le M\}$ be grid space. Denote by $L_{2h}=L_2\,[0,l]_h$, Hilbert space of grid functions $\rho^h(x)=\{\rho^n\}_0^M$ defined on $[0,l]_h$ equipped with norm

$$\left\|\rho^h\right\|_{L_{2h}} = \left(\sum_{x\in[0,l]_h}\left|\rho^h(x)\right|^2\right)^{\frac{1}{2}}.$$

To the operator (4) we assign the difference operator by formula

$$A_h^x\rho^h(x) = -\left(a(x)\rho_{\bar{x}}^h(x)\right)_x + \delta\rho^h(x)$$

acting in the space of grid functions $\rho^h(x)=\{\rho^n(x)\}_0^M$ and satisfying the conditions $\rho^0\ =0$, $\rho^M\ =0$, where

$$\rho_{\bar{x}}^i = \frac{\rho^i-\rho^{i-1}}{h},\ 1\le i\le M,\ \ \rho_x^k = \frac{\rho^{k+1}-\rho^k}{h},\ 0\le k\le M-1.$$

In the first step of discreatization we get the following Dirichlet problem

$$\begin{cases} u_t^h(t,x) + A_h^x u^h(t,x) + qA_h^x u^h(-t,x) = f^h(t,x),\ t\in I, \\ u^h(0,x)=\varphi^h(x), x\in\ [0,l]_h\,. \end{cases}$$

In the second step of discreatization one can construct the first order of accuracy difference scheme

$$\begin{cases} \frac{u_k^h(x)-u_{k-1}^h(x)}{\tau} + A_h^x u_k^h(x) + qA_h^x u_{-k}^h(x) = f_k^h(x), \quad f_k^h(x) = f^h(t_k, x), \\ t_k = k\tau, \ k \in Z, \ x \in \ [0,l]_h \,, \\ u_0^h(x) = \varphi^h(x), \ x \in \ [0,l]_h \end{cases} \tag{5}$$

and the second order of accuracy difference scheme

$$\begin{cases} \frac{u_k^h(x)-u_{k-1}^h(x)}{\tau} + \frac{1}{2}\left(A_h^x u_k^h(x) + qA_h^x u_{-k}^h(x)\right) + \frac{1}{2}\left(A_h^x u_{k-1}^h(x) + qA_h^x u_{-k+1}^h(x)\right) = \\ = f_{k+\frac{1}{2}}^h(x) = f^h(t_{k+\frac{1}{2}}, x), \ t_{k+\frac{1}{2}} = \left(k+\frac{1}{2}\right)\tau, \\ t_k = k\tau, k \in Z, \quad x \in \ [0,l]_h \,, \\ u_0^h(x) = \varphi^h(x), \quad x \in \ [0,l]_h \,. \end{cases} \tag{6}$$

Theorem 2. Let τ and h be sufficiently small positive numbers. Then, for the solution $\left\{u_k^h(x)\right\}_{-\infty}^{\infty}$of difference schemes (5) and (6) the stability estimates

$$\begin{gathered} \sup_{k\in Z} \left\|u_k^h\right\|_{L_{2h}} \le M(\delta)\left[\left\|\varphi^h\right\|_{L_{2h}} + \sum_{k=-\infty}^{\infty}\left\|f_k^h\right\|_{L_{2h}}\right], \\ \sup_{k\in Z}\left\|\frac{u_k^h-u_{k-1}^h}{\tau}\right\|_{L_{2h}} + \sup_{k\in Z}\left\|u_k^h\right\|_{W_{2h}^2} \le \\ \le M(\delta)\left[\left\|\varphi^h\right\|_{W_{2h}^2} + \left\|f_0^h\right\|_{L_{2h}} + \sum_{k=-\infty}^{\infty}\left\|\frac{f_k^h-f_{k-1}^h}{\tau}\right\|_{L_{2h}}\right] \end{gathered} \tag{7}$$

are valid, where $M(\delta)$ does not depend on τ, h φ^h and f_k^h.

Proof. Difference schemes (5) and (6) can be rewritten as the following abstract difference schemes

$$\begin{cases} \frac{u_k^h-u_{k-1}^h}{\tau} + A_h u_k^h + qA_h u_{-k}^h = f_k^h \,, \quad k \in Z, \\ u_0^h = \varphi^h \end{cases} \tag{8}$$

and

$$\begin{cases} \frac{u_k^h-u_{k-1}^h}{\tau} + \frac{1}{2}\left(A_h u_k^h + qA_h u_{-k}^h\right) + \frac{1}{2}\left(A_h u_{k-1}^h + qA_h^x u_{-k+1}^h\right) = f_{k+\frac{1}{2}}^h \,, \\ k \in Z, \ u_0^h = \varphi^h, \end{cases} \tag{9}$$

correspondingly. So, the proof of Theorem 2 is based on the stability of the difference schemes (8) and (9) on the positive definiteness and self-adjointness of the operator A^h in the Hilbert space L_{2h}.

2 Finite differences for involutary parabolic equation with Neumann condition

Let us take boundary value problem for parabolic equation with involution and Neumann condition as follows

$$\begin{cases} u_t(t,x) - (a(x)u_x(t,x))_x + \delta u(t,x) + q\left(-\left(a(x)u_t(-t,x)\right)_x + \delta u(-t,x)\right) = f(t,x), \\ t \in I,\ x \in (0,l), \\ u(0,x) = \varphi(x),\ x \in [0,l], \\ u_x(t,0) = 0,\ u_x(t,l) = 0,\ t \in I. \end{cases} \tag{10}$$

Theorem 3. Let $\varphi \in W_2^2(0,l)$ and $f(t,x)$ be continuously differentiable on $I \times [0,l]$. Then, for the solution of initial boundary value problem (10) the stability estimates (2) hold.

Proof. One can write problem (10) in the abstract initial value problem (3), where $A = A^x$ is a self adjoint positive definite operator in $H = L_2(0,l)$ which is defined by formula (4) with the domain $D(A) = \left\{u \in W_2^2(0,l) \ \mid u_x(0) = 0,\ u_x(l) = 0\right\}$. So, the proof of Theorem 1 is based on the stability of abstract problem (3) and positiveness and self-adjointness of the abstract operator A defined by (4).

To the operator (4) we assign the difference operator by formula

$$A_h^x \rho^h(x) = -\left(a(x)\rho_{\bar{x}}^h(x)\right)_x + \delta\rho^h(x),$$

acting in the space of grid functions $\rho^h(x) = \{\rho^n(x)\}_0^M$ and satisfying the conditions $\rho^2 = 4\rho^1 - 3\rho^0$, $\rho^{M-2} = 4\rho^{M-1} - 3\rho^M$, where

$$\rho_{\bar{x}}^i = \frac{\rho^i - \rho^{i-1}}{h},\ 1 \le i \le M,\ \rho_x^k = \frac{\rho^{k+1} - \rho^k}{h},\ 0 \le k \le M-1.$$

After discreatization one can construct the following difference schemes

$$\begin{cases} \frac{u_k^h(x) - u_{k-1}^h(x)}{\tau} + A_h^x u_k^h(x) + qA_h^x u_{-k}^h(x) = f_k^h(x), \quad f_k^h(x) = f^h(t_k,x), \\ t_k = k\tau,\ k \in Z,\ x \in \ [0,l]_h, \\ u_0^h(x) = \varphi^h(x),\ x \in \ [0,l]_h \end{cases} \tag{11}$$

and

$$\begin{cases} \frac{u_k^h(x) - u_{k-1}^h(x)}{\tau} + \frac{1}{2}\left(A_h^x u_k^h(x) + qA_h^x u_{-k}^h(x)\right) + \frac{1}{2}\left(A_h^x u_{k-1}^h(x) + qA_h^x u_{-k+1}^h(x)\right) = \\ = f_{k+\frac{1}{2}}^h(x) = f^h(t_{k+\frac{1}{2}},x),\ t_{k+\frac{1}{2}} = \left(k + \frac{1}{2}\right)\tau, \\ t_k = k\tau,\ k \in Z,\ x \in \ [0,l]_h, \\ u_0^h(x) = \varphi^h(x), \quad x \in \ [0,l]_h. \end{cases} \tag{12}$$

Theorem 4. Let τ and h be sufficiently small positive numbers. Then, for the solution $\left\{u_k^h(x)\right\}_{-\infty}^{\infty}$ of difference schemes (11) and (12) the stability estimates (7) are valid.

Proof. Difference schemes (11) and (12) can be rewritten as the abstract difference schemes (8) and (9), correspondingly. So, the proof of Theorem 2 is based on the stability of the difference schemes (8) and (9) on the positive definiteness and self-adjointness of the operator A^h in the Hilbert space L_{2h}.

3 Numerical implementation

In this section we will consider test examples with the first and second kind boundary conditions.

3.1 Test example for first kind boundary condition

Test example 1. Consider boundary value problem for parabolic equation with involution and Dirichlet condition

$$\begin{cases} u_t(t,x) - ((2+\cos(x))u_x(t,x))_x + \ u(t,x) - ((2+\cos(x))u_x(-t,x))_x + \ u(-t,x) = f(t,x), \\ f(t,x) = \cos t \sin x, \ t \in \ (-\pi, \pi)\,, \ x \in (0,\pi), \\ u(0,x) = 0, \ x \in [0,\pi], \\ u(t,0) = 0, \ u(t,l) = 0, \ t \in [-\pi,\pi]. \end{cases} \tag{13}$$

Here and in future we define sets of grid points as follows

$$[-\pi,\pi]_\tau \times [0, \ \pi]_h = \left\{(t_k, x_i) : t_k = k\tau, \ -N \le k \le N, \ x_i = ih, \ 0 \le k \le M, \ N\tau = \pi, \ Mh = \pi\right\}.$$

By using Taylor decomposition in two points

$$u(t_k) - u(t_{k-1}) = \tau u'(t_k) + o(\tau^2), \tag{14}$$

$$u(t_k) - u(t_{k-1}) = \frac{\tau}{2} u'(t_k) + \frac{\tau}{2} u'(t_{k-1}) + o(\tau^3), \tag{15}$$

$$u''(x_n) = \frac{u(x_{n+1}) - 2u(x_n) + u(x_{n-1})}{h^2} + o(h^2), \tag{16}$$

we present the first order of accuracy difference scheme in t

$$\begin{cases} \frac{u_k^n - u_{k-1}^n}{\tau} - (2+\cos(x_n))\frac{u_k^{n+1} - 2u_k^n + u_k^{n-1}}{h^2} + \sin(x_n)\frac{u_k^{n+1} - u_k^{n-1}}{2h} + u_k^n - \\ - \ (2+\cos(x_n))\frac{u_{-k}^{n+1} - 2u_{-k}^n + u_{-k}^{n-1}}{h^2} + \sin(x_n) \ \frac{u_{-k}^{n+1} - u_{-k}^{n-1}}{2h} + u_{-k}^n = f_k^n, \ f_k^n \ = f\ (t_k, x_n), \\ t_k = k\tau, \ -N+1 \le k \le N, \\ u_k^0 \ = 0, \ u_k^M \ = 0, \ k = 0, \pm 1, \pm 2, ..., \pm N, \\ u_0^n \ = 0, \ x_n = nh, \ n = 0, 1, ..., M \end{cases} \tag{17}$$

and the second order of accuracy difference scheme in t

$$\begin{cases} \frac{u_k^n-u_{k-1}^n}{\tau}-\frac{(2+\cos(x_n))}{2}\left(\frac{u_k^{n+1}-2u_k^n+u_k^{n-1}}{h^2}+\frac{u_{-k}^{n+1}-2u_{-k}^n+u_{-k}^{n-1}}{h^2}\right)+\frac{u_k^n+u_{-k}^n}{2}+ \\ +\frac{\sin(x_n)}{2}\left(\frac{u_k^{n+1}-u_k^{n-1}}{2h}+\frac{u_{-k}^{n+1}-u_{-k}^{n-1}}{2h}\right)- \\ -\frac{(2+\cos(x_n))}{2}\left(\frac{u_{k-1}^{n+1}-2u_{k-1}^n+u_k^{n-1}}{h^2}+\frac{u_{-k+1}^{n+1}-2u_{-k+1}^n+u_{-k+1}^{n-1}}{h^2}\right)+ \\ +\frac{\sin(x_n)}{2}\left(\frac{u_{k-1}^{n+1}-u_{k-1}^{n-1}}{2h}+\frac{u_{-k+1}^{n+1}-u_{-k+1}^{n-1}}{2h}\right)+\frac{u_{k-1}^n+u_{-k+1}^n}{2}= \\ = f^h(t_{k+\frac{1}{2}},x),\ t_{k+\frac{1}{2}}=\left(k+\frac{1}{2}\right)\tau,\ t_k=k\tau,\ \ -N+1\le k\le N, \\ u_k^0=0, u_k^M=0,\ k=0,\pm1,\pm2,...,\pm N, \\ u_0^n=0,\ x_n=nh,\ n=0,1,...,M. \end{cases} \tag{18}$$

Later, system of equations (17) and (18) can be rewritten in the matrix form as follows

$$\begin{cases} A_nU_{n-1}+B_nU_n+C_nU_{n+1}=R\varphi_n,\ n=1,..,M-1, \\ U_0=\overrightarrow{O},\ U_M=\overrightarrow{O}. \end{cases} \tag{19}$$

Here R is identity matrix. For solving (19) we apply modified Gauss elimination method by formula

$$U_n=\alpha_nU_{n+1}+\beta_n,\ n=M-1,...,1,0, \tag{20}$$

where α_0 is matrix with zero elements and vector β_0 with zero elements, matrices α_n and vectors β_n are defined recurrently by

$$\begin{aligned} &\alpha_n=\left(B_n+A_n\alpha_{n-1}\right)^{-1}A_n, \\ &\beta_n=\left(B_n+A_n\alpha_{n-1}\right)^{-1}\left(R\varphi_n-A_n\,\beta_{n-1}\right), \\ &n=1,...,M-1. \end{aligned} \tag{21}$$

Error is calculated by formula

$$Error(N,M)=\max_{k=0,\pm1,...,\pm N,i=1,...,M}\left|u_k^i-u(t_k,x_i)\right|, \tag{22}$$

where u_k^i and $u(t_k,x_i)$ $(k=0,\pm1,...,\pm N,\ i=1,...,M)$ are values of solution of difference scheme and differential problem at point correspondingly. Table 1 shows that if numbers N and M increase by factor 2 then the values of errors decreases by a factor of approximately $\frac{1}{2}$ for the first order difference scheme (17) and $\frac{1}{4}$ for the second order of accuracy difference scheme (18).

Table 1

Error analysis for test example (13) with Dirichlet condition

$N=M$	1st order difference scheme	2nd order difference scheme
10	0.531	$8{,}35\times10^{-2}$
20	0.351	$2{,}88\times10^{-2}$
40	0.208	$7{,}36\times10^{-3}$
80	0.115	$1{,}92\times10^{-3}$
160	$6{,}09\times10^{-2}$	$5{,}01\times10^{-4}$
320	$3{,}13\times10^{-2}$	$1{,}28\times10^{-4}$

3.2 Test example for second kind boundary condition

Test example 2. Consider boundary problem for involutary parabolic equation with Neumann condition

$$\begin{cases} u_t(t,x) - ((2+\cos(x))u_x(t,x))_x + \ u(t,x) - ((2+\cos(x))u_x(-t,x))_x + \ u(-t,x) = f(t,x), \\ f(t,x) = \cos t\cos x, t \in \ (-\pi,\pi)\,,\ x\in(0,\pi), \\ u(0,x)=0,\ x\in[0,\pi]\,, \\ u_x(t,0)=0,\ u_x(t,l)=0,\ t\in[-\pi,\pi]\,. \end{cases} \tag{23}$$

By using (14), (15), (16) and

$$u'(0) = \frac{-u(x_2)+4u(x_1)-3u(x_0)}{h^2} \ + o(h^2),$$

$$u'(\pi) = \frac{u(x_{M-2})-4u(x_{M-1})+3u(x_M)}{h^2} \ + o(h^2),$$

one can get the first order of accuracy difference scheme in t

$$\begin{cases} \frac{u_k^n \ -u_{k-1}^n}{\tau} - (2+\cos(x_n))\frac{u_k^{n+1}-2u_k^n+u_k^{n-1}}{h^2} + \sin(x_n)\frac{u_k^{n+1}-u_k^{n-1}}{2h} + u_k^n - \\ -(2+\cos(x_n))\frac{u_{-k}^{n+1}-2u_{-k}^n+u_{-k}^{n-1}}{h^2} + \sin(x_n)\ \frac{u_{-k}^{n+1}-u_{-k}^{n-1}}{2h} + u_{-k}^n = f_k^n, \\ f_k^n \ = f\ (t_k,x_n),\ t_k=k\tau,\ -N+1\le k\le N, \\ u_k^0 \ = u_k^1\ ,\ u_k^M \ = u_k^{M-1},\ k=0,\pm1,\pm2,...,\pm N, \\ u_0^n \ = 0,\ x_n=nh,\ n=0,1,...,M \end{cases} \tag{24}$$

and the second order of accuracy difference scheme in t

$$\begin{cases} \frac{u_k^n - u_{k-1}^n}{\tau} - \frac{(2+\cos(x_n))}{2}\left(\frac{u_k^{n+1}-2u_k^n+u_k^{n-1}}{h^2} + \frac{u_{-k}^{n+1}-2u_{-k}^n+u_{-k}^{n-1}}{h^2}\right) u_k^n + \\ + \frac{\sin(x_n)}{2}\left(\frac{u_k^{n+1}-u_k^{n-1}}{2h} + \frac{u_{-k}^{n+1}-u_{-k}^{n-1}}{2h}\right) + \frac{u_k^n+u_{-k}^n}{2} + \\ + \frac{(2+\cos(x_n))}{2}\left(\frac{u_{k-1}^{n+1}-2u_{k-1}^n+u_k^{n-1}}{h^2} + \frac{u_{-k+1}^{n+1}-2u_{-k+1}^n+u_{-k+1}^{n-1}}{h^2}\right) + \\ + \frac{(2-\sin(x_n))}{2}\left(\frac{u_{k-1}^{n+1}-u_{k-1}^{n-1}}{2h} + \frac{u_{-k+1}^{n+1}-u_{-k+1}^{n-1}}{2h}\right) + \frac{u_{k-1}^n+u_{-k+1}^n}{2} = \\ = f^h(t_{k+\frac{1}{2}}, x),\ t_{k+\frac{1}{2}} = \left(k+\frac{1}{2}\right)\tau,\ t_k = k\tau,\ -N+1 \le k \le N, \\ 3u_k^0 = 4u_k^1 - u_k^2, 3u_k^M = 4u_k^{M-1} - u_k^{M-2},\ k = 0, \pm 1, \pm 2, ..., \pm N, \\ u_0^n = 0,\ x_n = nh,\ n = 0, 1, ..., M. \end{cases} \tag{25}$$

System of equations (24) and (25) can be written in the following forms

$$\begin{cases} A_n U_{n-1} + B_n U_n + C_n U_{n+1} = R\varphi_n,\ n = 1, .., M-1, \\ U_0 = U_1,\ U_M = U_{M-1}, \end{cases} \tag{26}$$

and

$$\begin{cases} A_n U_{n-1} + B_n U_n + C_n U_{n+1} = R\varphi_n,\ n = 1, .., M-1, \\ 3U_0 = 4U_1 - U_2,\ 3U_M = 4U_{M-1} - U_{M-2}, \end{cases} \tag{27}$$

correspondingly. For solving (26) we use formula (20), where $\alpha_0 = R$ is identity matrix and vector β_0 has only zero elements, matrices α_n and vectors β_n are defined by (21). Errors are computed by formula (22). Let us move to (27). We seek solution (27) in the form (see [18, 19])

$$U_n = \alpha_n U_{n+1} + \beta_n U_{n+2} + \gamma_n,\ n = M-2, M-1, ..., 1, 0.$$

Here auxiliary matrices α_n, β_n and vector γ_n are calculated by formulas

$$\alpha_n = -D_n(A_n + C_n\beta_{n-1}),\ \beta_n = O,$$

$$\gamma_n = D_n\ (R\varphi_n - C_n\ \gamma_{n-1}),$$

$$D_n = (B_n + C_n\ \alpha_{n-1})^{-1},\ n = 0, ..., M-2,$$

$$\alpha_0 = \frac{4}{3}R,\ \beta_0 = -\frac{1}{3}R,\ \alpha_1 = \frac{8}{5}R,\ \beta_1 = -\frac{3}{5}R,\ \gamma_0 = \gamma_1 = \overrightarrow{0}.$$

At the same time formulas for unknown U_M and U_{M-1} are given in [19].

Table 2 shows that if numbers N and M increase by factor 2 then the values of errors decrease by a factor of approximately $\frac{1}{2}$ for difference scheme (24) and $\frac{1}{4}$ for difference scheme (25).

Table 2

Error analysis for test example (23) with Neuman condition

$N = M$	1st order difference scheme	2nd order difference scheme
10	0.619	$8{,}41\times10^{-2}$
20	0.414	$2{,}91\times10^{-2}$
40	0.253	$7{,}41\times10^{-3}$
80	0.144	$1{,}95\times10^{-3}$
160	$7{,}77\times10^{-2}$	$5{,}03\times10^{-4}$
320	$4{,}05\times10^{-2}$	$1{,}30\times10^{-4}$

References

1 Abbas, T. (2019). *Schrodinger type Ivolutoury Partial Differential Equation, Master Thesis.* Near East Universty, Nicosia.

2 Agirseven, D. (2012). Approximate Solutions of delay parabolic equations with the Dirichlet condition. *Abstract and Applied Analysis, 10*(1155), 682752.

3 Arino, O., Hbid, M.L., & Ait Dads, E. (2006). *Delay Differential Equations and Applications.* Berlin: Springer.

4 Ashyralyev, A., & Sarsenbi, A.M. (2015). Well-posedness of an elliptic equation with involution. *Electronic Journal of Differential Equations, 2015*(284), 1–8.

5 Ashyralyev, A., & Sarsenbi, A.M. (2017). Well-posedness of a parabolic equation with the involution. *Numerical Functional Analysis and Optimization, 38*(10), 1295–1304.

6 Ashyralyev, A., & Sobolevskii, P.E. (2001). On the stability of the linear delay differential and difference equations. *Abstract and Applied Analysis, 6*(5), 267-297.

7 Bhalekar, S., & Patade, J. (2016). Analytical solutions of nonlinear equations with proportional delays. *Appl Comput Math., 15*, 331–445.

8 Bhrawy, A.H., Abdelkawy, M.A., & Mallawi, F. (2015). An accurate Chebyshev pseudospectral scheme formulti-dimensional parabolic problems with time delays. *Boundary Value Problems,* 2015:103.

9 Blasio, G.Di. (2003). Delay differential equations with unbounded operators on delay terms. *Nonlinear Analysis-Theory and Applications, 52*(2), 1–18.

10 Guidetti, D. (2012). Determining the source term in an abstract parabolic problem from a time integral of the solution. *Mediterr. J. Math., 9*(6), 11–633.

11 Nesbit, R. (1997). *Delay Differential Equations for Structured Populations, Structured Population Models in Marine.* Terrestrial and Freshwater Systems, Tuljapurkar & Caswell, ITP.

12 Poorkarimi, H., & Wiener, J. (1999). Bounded solutions of nonlinear parabolic equations with time delay 15-th Annual Conference of Applied Mathematics, Univ. of Central Oklahoma. *Electronic Journal of Differential Equations, 1999*, 87–91.

13 Shah, S.M., Poorkarimi, H., & Wiener, J. (1986). Bounded solutions of retarded nonlinear hyperbolice quations. *Bull. Allahabad Math. Soc., 1*, 1–14.

14 Sriram, K., & Gopinathan, M.S. (2004). A two variable delay model for the circadian rhythm of Neurospora crassa. *J Theor Biol., 231*, 1, 23–38.

15 Srividhyaa, J.A, & Gopinathan, M.S. (2006). Simple time delay model for eukaryotic cell cycle. *J Theor Biol., 241*, 3, 617–627.

16 Skubachevskii, A.L. (1994). On the problem of attainment of equilibrium for controlsystem with delay. *Doklady Akademii Nauk, 335*(2), 157–160.

17 Vlasov, V.V., & Rautian, N.A. (2010). *Spectral Analysis of Functional Differential Equations.* M.: MAKS Press.

18 Ashyralyyev, C., & Gonenc, A. (2021). Crank-Nicolson difference scheme for reverse parabolic nonlocal problem with integral and Neumann boundary conditions. *International Journal of Applied Mathematics, 34*(2), 273–282.

19 Ashyralyyev, C., Dural, A., & Sozen, Y. (2012). Finite difference method for the reverse parabolic problem with Neumann condition. *AIP Conf. Proc., 1470*, 102–105.

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Инволюциялы параболалық теңдеу үшін шеттік есептердің сандық шешімі

Мақалада бірінші және екінші типті шарттары бар эволюциялық параболалық теңдеудің екі шеттік есептерін зерттелген. Осы шеттік есептерді сандық түрде шешу үшін абсолютті тұрақты айырымдық схемалары ұсынылған. Айырымдық схемаларының шешімдерінің тұрақтылығын бағалау іс жүзінде дәлелденді. Екі айырымдық схемасының сандық шешімінің қателіктерін одан әрі талдау сынақ мысалдарымен келтірілген.

Кілт сөздер: инволюция, параболалық, ақырлы-айырымдық схемасы, тұрақтылықты бағалау, шеттік есеп.

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Численное решение краевых задач для параболического уравнения с инволюцией

В статье исследованы две краевые задачи для эвольвентного параболического уравнения с условиями первого и второго рода. Предложены абсолютно устойчивые разностные схемы для численного решения этих краевых задач. Фактически доказаны оценки устойчивости решений разностных схем. Дальнейший анализ погрешностей численного решения обеих разностных схем проиллюстрирован тестовыми примерами.

Ключевые слова: инволюция, парабола, конечно-разностная схема, оценка устойчивости, краевая задача.