

## On an inverse boundary control of a system of integro-differential equations of fractional order in a locally convex space

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This paper investigates an inverse boundary optimal control problem for a nonlinear system of fractional-order integro-differential equations in a complete locally convex space. The state equation is supplemented with unknown boundary data, which are identified simultaneously with the optimal control. By introducing an adjoint final-value problem, a Pontryagin-type function is constructed, and a necessary optimality condition is derived in the form of a nonlinear integral equation for the control function. The original boundary value problem is transformed into an equivalent coupled system of nonlinear integral equations with respect to the state function, the boundary value, and the control. The solvability of this system is established by the Banach contraction principle in spaces whose topology is generated by a family of seminorms. Sufficient conditions guaranteeing the existence and uniqueness of the optimal control, the corresponding state trajectory, and the identified boundary value are obtained. In addition, the continuous dependence of the solution on the boundary data is proved. The proposed approach provides a constructive framework for studying inverse optimal control problems for fractional systems in locally convex spaces and can be extended to more general classes of nonlocal and infinite-dimensional dynamical systems.

**Keywords:** fractional order equation, locally convex algebra, equations with three unknown quantities, Pontryagin function, method of contraction mapping, existence, uniqueness, stability.

**2020 Mathematics Subject Classification:** 45J05, 34K29, 47G20.

### Introduction

Optimal control of dynamic systems is a pressing area of modern mathematics. In classical optimal control theory, the state function is typically studied in finite-dimensional spaces or Banach spaces. However, many models arising in analysis, operator theory, integro-differential equations, memory problems, and distributed parameter problems are naturally formulated in a more general class of spaces, namely, locally convex spaces.

On the other hand, fractional calculus has become one of the effective tools for modeling processes with memory and heredity. Therefore, it is natural to transfer optimal control problems for fractional systems to locally convex spaces.

The study of differential equations in locally convex spaces is a current area of research in the theory of differential equations. A large number of papers have been published in this area.

Optimal control theory for differential and integro-differential equations has become one of the fundamental areas of modern applied mathematics owing to its numerous applications in engineering,

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This research was funded by the Science Committee of the Ministry of Science and Higher Education of the Republic of Kazakhstan (Grant No. AP23488740, 2024–2026.)

Received: 10 February 2026; Accepted: 8 June 2026.

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mechanics, physics, biology, and economics. The foundations of optimal control theory for distributed parameter systems were established in the classical works of Butkovsky and Pustynnikov [1], while the general duality theorems and optimality conditions for nonconvex optimization problems were developed by Deumlich and Elster [2]. Methods of optimal control for distributed parameter systems were further investigated by Deineka in the context of thin compound plates [3]. Time-optimal control problems for heat transfer and evolutionary partial differential equations were studied in [4,5]. Optimal control of heat transfer processes with integral constraints was considered in [6]. Controllability of singularly perturbed discrete systems was analyzed in [7], whereas stability and null controllability of infinite systems of differential equations were investigated in [8]. Optimal control problems for impulsive functional-differential systems were addressed in [9]. Optimization problems involving multipoint and integral conditions were considered in [10,11]. Necessary optimality conditions for thermal control and tracking problems in nonlinear vector optimization were established in [12,13]. Convex analysis, infimal convolution, and duality methods for optimal control problems with second-order evolution differential inclusions were developed in [14].

During the last decades, considerable attention has been devoted to optimal control problems for systems governed by fractional-order differential equations because fractional derivatives provide an adequate mathematical description of memory and hereditary effects that cannot be captured by classical integer-order models. The mathematical foundations of fractional calculus are comprehensively presented in the monographs by Samko et al. [15], Podlubny [16], and Kilbas et al. [17]. Existence and uniqueness of solutions to nonlinear fractional differential equations were established in [18,19], while analytical properties and qualitative analysis of fractional differential equations were investigated in [20,21]. Applications of fractional calculus to viscoplasticity and damping models were presented in [22,23]. Fractional models in protein dynamics and continuum mechanics were developed in [24,25], whereas a broad collection of applications in physics is summarized in [26]. Boundary value problems for fractional differential equations with nonlocal conditions were investigated in [27]. A systematic treatment of partial differential equations of fractional order was presented by Pskhu [28], while inverse problems involving Hilfer-type fractional differential equations were studied in [29]. Recent investigations of loaded hyperbolic equations and heat equations involving fractional integro-differential operators are reported in [30,31].

Most existing results have been developed for finite-dimensional systems or for equations posed in Banach and Hilbert spaces. However, many models arising in functional analysis, operator theory, distributed parameter systems, and infinite-dimensional dynamics are naturally formulated in locally convex spaces (see, for example, [32–34]). Since such spaces are generated by families of seminorms rather than a single norm, many classical analytical techniques require substantial modification. Consequently, the development of constructive methods for inverse and optimal control problems in locally convex spaces remains an important and challenging topic.

Another significant feature of many practical applications is the presence of unknown boundary data that must be identified simultaneously with the optimal control. Such inverse boundary control problems arise in parameter identification, distributed systems, heat transfer, viscoelasticity, and other models involving hereditary effects. Nevertheless, the available literature contains relatively few results devoted to fractional integro-differential equations in locally convex spaces with simultaneous determination of the state, the control, and the unknown boundary value.

The purpose of this paper is to develop a constructive approach to solving an inverse boundary optimal control problem for a nonlinear system of fractional-order integro-differential equations in a complete locally convex space. By introducing an adjoint final-value problem, a Pontryagin-type function is constructed and a necessary optimality condition is obtained. The original boundary value problem is then reduced to a coupled system of nonlinear integral equations with respect to three unknown quantities: the state function, the control function, and the boundary value. The solvability

of the resulting system is established by applying the contraction mapping principle in locally convex spaces.

The main contributions of the paper can be summarized as follows. First, an equivalent integral formulation of the inverse boundary optimal control problem is obtained. Second, a nonlinear integral equation determining the optimal control is derived from the Pontryagin optimality condition. Third, sufficient conditions guaranteeing the existence and uniqueness of the control function, the corresponding state trajectory, and the unknown boundary value are established. Finally, the continuous dependence of the solution on the identified boundary data is proved. These results provide a rigorous theoretical basis for further investigations of inverse optimal control problems for fractional and nonlocal systems in locally convex spaces and locally convex algebras.

So, unlike the standard approach, in this paper we find the state function, boundary value, and control function simultaneously as the solution to a functional system consisting three nonlinear equations.

### 1 Preliminary information and statement of the problem

Let  $E$  be a linear space over the field  $F$  of real numbers. A seminorm on  $E$  is a non-negative function  $E : p \rightarrow R_+$  satisfying the homogeneity condition  $p(\lambda x) = |\lambda|p(x)$  for all  $\lambda \in F$  and the triangle inequality  $p(x + y) \leq p(x) + p(y)$  for all  $x, y \in E$ .

*Definition 1.* A sequence  $\{x_n\} \in E$  is said to converge to an element  $x \in E$  if for any  $\varepsilon > 0$  and  $p \in \mathcal{P}$  there exists a positive integer  $N$  such that  $p(x_n - x) < \varepsilon$  for all  $n \geq N$ .

*Definition 2.* A sequence  $\{x_n\} \in E$  is called a Cauchy sequence if for any  $\varepsilon > 0$  and  $p \in \mathcal{P}$  there exists a positive integer  $N$  such that  $p(x_n - x_m) < \varepsilon$  for all  $n, m \geq N$ .

*Definition 3.* A locally convex space  $E$  is called complete if every Cauchy sequence is convergent.

So, let the space  $E$  be a complete locally convex space whose topology is defined by a family of seminorms

$$\mathcal{P} = \{p_\alpha : \alpha \in \mathcal{A}\}.$$

The operations of addition and multiplication by a number are continuous in any locally convex topology of the space  $E$ .

Let  $\Omega = [0, T]$ ,  $0 < T$ , and let  $C(\Omega, E)$  denote the space of all continuous functions  $x : \Omega \rightarrow E$ . For each seminorm  $p_\alpha \in \mathcal{P}$  we define the induced seminorm on  $C(\Omega, E)$ :

$$\tilde{p}_\alpha(x) = \sup_{t \in \Omega} p_\alpha(x(t)).$$

Then  $C(\Omega, E)$  is a locally convex space with respect to the family of seminorms  $\{\tilde{p}_\alpha\}_{\alpha \in \mathcal{A}}$ .

Let  $U$  be a control space. For definiteness, we assume that  $U$  is a locally convex space with the family of seminorms  $\{q_\beta\}_{\beta \in \mathcal{B}}$ , and by  $C(\Omega, U)$  we denote the space of continuous control functions.

In addition, we suppose that the spaces  $E$  and  $U$  are topological algebra. For two elements  $x, y \in E$  is defined factor  $xy \in E$ , where the mapping  $xy : E \times E \rightarrow E$  is continuous. For each seminorm  $p_\alpha$  there exist  $p_\beta, p_\gamma$  and  $0 < C_\alpha$  that

$$p_\alpha(xy) \leq C_\alpha p_\beta(x) p_\gamma(y).$$

Let us consider the optimal control problem for the following nonlinear integro-differential equation of fractional order

$${}_CD_{0t}^\gamma x(t) = g(t, x(t)) \int_0^t f(s, x(s), u(s)) ds, \quad t \in \Omega \quad (1)$$

with boundary condition

$$a x(0) + b x(T) = x_r \in E, \quad (2)$$

where  $x(t)$  is the state function,  $x_r$  is the redefinition value,  $a, b \in R$ ,  $a \neq 0$ ,  $b \neq 0$ ,  $a \pm b \neq 0$ ,  $u(t)$  is the control function,  ${}_C D_{0t}^\gamma$  denotes the Gerasimov–Caputo fractional operator of order  $0 < \gamma < 1$ :

$${}_C D_{0t}^\gamma x(t) = I_{0t}^{1-\gamma} x'(t) = \frac{1}{\Gamma(1-\gamma)} \int_0^t \frac{x'(s) ds}{(t-s)^\gamma}, \quad t \in \Omega,$$

and the Riemann–Liouville integral of order  $0 < \gamma < 1$  is given by the expression

$$I_{0t}^\gamma x(t) = \frac{1}{\Gamma(\gamma)} \int_0^t \frac{x(s) ds}{(t-s)^{1-\gamma}}, \quad 0 < \gamma < 1,$$

$\Gamma(\gamma)$  is Gamma function:  $\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt$ ,  $\operatorname{Re} z > 0$ ,  $g \in C(\Omega \times E, E)$ ,  $f \in C(\Omega \times E \times U, E)$ .

Let's consider the following quality functional

$$J[u] = (x_r - \eta)^2 + \mu \int_0^T \varepsilon(t) (u(t))^2 dt, \quad (3)$$

where  $0 < \eta$ ,  $\mu = \text{const}$ ,  $0 < \varepsilon(t) \in C(\Omega)$ ,  $x_r$  will be defined from the following internal condition

$$x(t_1) = x_r, \quad 0 < t_1 < T. \quad (4)$$

In the present paper we use  $p_\alpha$  is a fixed seminorm of the locally convex algebra  $E$  and  $q_\beta$  is a fixed seminorm of the locally convex algebra  $U$ .

*Statement of the problem.* It is required to find the control function  $u^*(t) \in C(\Omega, U)$ , the redefinition value  $x_r^* \in E$ , and the corresponding state function  $x^*(t) \in C(\Omega, E)$  that minimize the functional (3) under the condition that  $(x^*(t), x_r^*, u^*(t))$  satisfies the equation (1) and the boundary condition (2) and (4).

## 2 Reduction of the problem (1), (2) to an integral equation

Let  $x(t) \in C(\Omega, E)$  be a solution to problem (1), (2). Apply the operator  $I_{0t}^\gamma$  to equation (1):

$$I_{0t}^\gamma {}_C D_{0t}^\gamma x(t) = I_{0t}^\gamma g(t, x(t)) \int_0^t f(s, x(s), u(s)) ds$$

and use the definition of the Gerasimov–Caputo fractional operator

$$I_{0t}^\gamma {}_C D_{0t}^\gamma (x(t)) = I_{0t}^\gamma I_{0t}^{1-\gamma} (x'(t)) = I_{0t}^1 (x'(t)) = \int_0^t x'(s) ds = x(t) - x(0),$$

$$I_{0t}^\gamma \eta_0(t, x(t)) = \frac{1}{\Gamma(\gamma)} \int_0^t (t-s)^{\gamma-1} \eta_0(s, x(s)) ds.$$

Then, applying these two formulas to the last fractional integro-differential equation, we obtain

$$x(t) = x(0) + \frac{1}{\Gamma(\gamma)} \int_0^t (t-s)^{\gamma-1} g(s, x(s)) \int_0^s f(\theta, x(\theta), u(\theta)) d\theta ds. \quad (5)$$

We consider

$$x(T) = x(0) + \frac{1}{\Gamma(\gamma)} \int_0^T (T-s)^{\gamma-1} g(s, x(s)) \int_0^s f(\theta, x(\theta), u(\theta)) d\theta ds.$$

Here we use the boundary condition (2) and the last equality, then we get that

$$x(0) = \frac{x_r}{a+b} - \frac{1}{\Gamma(\gamma)} \frac{b}{a+b} \int_0^T (T-s)^{\gamma-1} g(s, x(s)) \int_0^s f(\theta, x(\theta), u(\theta)) d\theta ds. \quad (6)$$

Now, we substitute the found value (6) of the state function at the initial point into the integral equation (5):

$$x(t) = \mathbf{K}_1(t; x, x_r, u) \equiv \frac{x_r}{a+b} + \int_0^T V(t, s) g(s, x(s)) \int_0^s f(\theta, x(\theta), u(\theta)) d\theta ds, \quad (7)$$

where

$$V(t, s) = \frac{1}{\Gamma(\gamma)} \begin{cases} (t-s)^{\gamma-1} - \frac{b}{a+b}(T-s)^{\gamma-1}, & 0 \leq s < t, \\ (T-s)^{\gamma-1}, & t \leq s \leq T. \end{cases}$$

Equation (7) is a nonlinear Fredholm integral equation of the second kind and satisfies problem (1), (2). We differentiate equation (7) with respect to  $t$ :

$$x'(t) = \int_0^t V'(t, s) g(s, x(s)) \int_0^s f(\theta, x(\theta), u(\theta)) d\theta ds, \quad (8)$$

where

$$\begin{aligned} V'(t, s) &= \left( \frac{1}{\Gamma(\gamma)} \begin{cases} (t-s)^{\gamma-1} - \frac{b}{a+b}(T-s)^{\gamma-1}, & 0 \leq s < t, \\ (T-s)^{\gamma-1}, & t \leq s \leq T, \end{cases} \right)'_t = \\ &= \frac{1}{\Gamma(\gamma)} \left( (t-s)^{\gamma-1} \right)'_t = \frac{\gamma-1}{\Gamma(\gamma)} (t-s)^{\gamma-2}. \end{aligned}$$

### 3 Relation for redefinition value

We suppose that  $a+b \neq 1$ . Then we apply the intermediate condition (4) to the equation (7):

$$x(t_1) = \frac{x_r}{a+b} + \int_0^T V(t_1, s) g(s, x(s)) \int_0^s f(\theta, x(\theta), u(\theta)) d\theta ds = x_r.$$

From here, by a simple transformation, we find that

$$x_r = \mathbf{K}_2(x, x_r, u) \equiv \frac{a+b}{a+b-1} \int_0^T V(t_1, s) g(s, x(s)) \int_0^s f(\theta, x(\theta), u(\theta)) d\theta ds. \quad (9)$$

The formula (9), like (7), is a relation between unknown functions  $(x(t), x_r, u(t))$ .

#### 4 Integral equations for the control function

Let  $u^*(t) \in U$  be the optimal control function, where  $u^*(t) + \Delta u^*(t) \in C(\Omega, U)$ . Consider the problem of finding the control function  $u^*(t) \in U$  and the corresponding state function  $x^*(t)$  that ensure the minimum of the functional (3). To obtain the optimality conditions using the functional (3) and the equation (8), we introduce the Pontryagin function

$$H(u(t), x(t)) = \psi(t) \int_0^t V'(t, s)g(s, x(s)) \int_0^s f(\theta, x(\theta), u(\theta)) d\theta ds - \mu\varepsilon(t)u^2(t), \quad (10)$$

where the function  $\psi(t)$  belongs to the dual space  $E'$  and is determined from the following equation:

$${}_CD_{0t}^\gamma \psi(t) = -\psi(t) \quad (11)$$

with final point condition

$$\psi(T) = -2[x_r - \eta]. \quad (12)$$

As a result of solving the fractional differential equation (11), we obtain the function

$$\psi(t) = \psi(0) - \frac{1}{\Gamma(\gamma)} \int_0^t (t-s)^{\gamma-1} \psi(s) ds.$$

We consider

$$\psi(T) = \psi(0) - \frac{1}{\Gamma(\gamma)} \int_0^T (T-s)^{\gamma-1} \psi(s) ds.$$

To determine the constant of integration  $\psi(0)$ , we use the condition specified at the end point (12):

$$\psi(0) - \frac{1}{\Gamma(\gamma)} \int_0^T (T-s)^{\gamma-1} \psi(s) ds = -2[x_r - \eta].$$

Hence, we obtain that

$$\psi(0) = -2x_r + 2\eta + \frac{1}{\Gamma(\gamma)} \int_0^T (T-s)^{\gamma-1} \psi(s) ds. \quad (13)$$

Therefore, by virtue of (13), we have

$$\psi(t) = -2x_r + 2\eta + \frac{1}{\Gamma(\gamma)} \int_0^T V_0(t, s) \psi(s) ds, \quad (14)$$

where

$$V_0(t, s) = \frac{1}{\Gamma(\gamma)} \begin{cases} (T-s)^{\gamma-1} - (t-s)^{\gamma-1}, & 0 \leq s < t, \\ -(T-s)^{\gamma-1}, & t \leq s \leq T. \end{cases}$$

Equation (14) is an integral equation of the second kind and has a unique solution  $\psi(t, x_r) \in E'$ , depending on  $x_r$ .

By differentiating the expression (10) with respect to the control function, we obtain an integral equation with respect to the control function

$$H_u(u(t), x(t)) = -2\mu\varepsilon(t)u(t) +$$

$$+ \psi(t, x_r) \int_0^t V'(t, s) g(s, x(s)) \int_0^s f_u(\theta, x(\theta), u(\theta)) d\theta ds. \quad (15)$$

We rewrite the equation (15) as an operator equation with respect to the control function

$$\begin{aligned} u(t) &= \mathbf{K}_3(t; x, x_r, u) \equiv \\ &\equiv a_0(t, x_r) \int_0^t V'(t, s) g(s, x(s)) \int_0^s f_u(\theta, x(\theta), u(\theta)) d\theta ds, \end{aligned} \quad (16)$$

where

$$a_0(t, x_r) = \frac{\psi(t, x_r)}{\mu\varepsilon(t)}.$$

The equation (16) is the third relation between the unknown functions  $(x(t), x_r, u(t))$ .

### 5 Key results

Let  $C(\Omega, E) \times E \times C(\Omega, U)$  is a complete locally convex space with topological algebra. For each seminorm  $p_{\alpha,0,\beta} \in \mathcal{P}$  we define the family of seminorm on  $C(\Omega, E) \times E \times C(\Omega, U)$ :

$$\tilde{p}_{\alpha,0,\beta}(x) = \sup_{t \in \Omega} p_{\alpha}(x(t)) + p_0(x(0)) + \sup_{t \in \Omega} q_{\beta}(u(t)).$$

Consider the right-hand side of three operators (7), (9) and (16):  $\mathbf{K}_j(t; x, x_r, u)$ ,  $j = 1, 2, 3$ , as a system of equations.

*Theorem 1.* Assume that the following conditions are satisfied:

- 1)  $a + b \neq 1$ ;
- 2) the functions  $g, f, f_u$  satisfy the inclusions:

$$g \in C(\Omega \times E, E), f \in C(\Omega \times E \times U, E), f_u \in C(\Omega \times E \times U, E)$$

and  $\tilde{p}_{\alpha}(g) < \infty$ ,  $\tilde{p}_{\alpha}(f) < \infty$ ,  $\tilde{p}_{\alpha}(f_u) < \infty$ ;

- 3) for each seminorm  $p_{\alpha} \in \mathcal{P}$  there exist constants  $0 < L_1, L_{\kappa i}$ ,  $\kappa = 2, 3$ ,  $i = 1, 2$  such that the following inequalities hold:

$$p_{\alpha}(g(t, x_1) - g(t, x_2)) \leq L_1 p_{\alpha}(x_1 - x_2),$$

$$p_{\alpha,\beta}(f(t, x_1, u_1) - f(t, x_2, u_2)) \leq L_{21} p_{\alpha}(x_1 - x_2) + L_{22} q_{\beta}(u_1 - u_2),$$

$$p_{\alpha,\beta}(f_u(t, x_1, u_1) - f_u(t, x_2, u_2)) \leq L_{31} p_{\alpha}(x_1 - x_2) + L_{32} q_{\beta}(u_1 - u_2),$$

where  $q_{\beta}$  is a fixed seminorm in  $U$ ;

- 4) for each seminorm  $p_{\alpha,0,\beta}$ , the following condition is satisfied

$$\tilde{\rho}_{\alpha,0,\beta}(x(t), x_r, u(t)) = \tilde{\rho}_{\alpha}(x(t)) + \tilde{\rho}_0(x_r) + \tilde{\rho}_{\beta}(u(t)) = \chi_1 + \chi_2 + \chi_3 < 1,$$

where  $\chi_1, \chi_2, \chi_3$  are determined from (33), (21), (22), (24), (25) and (27)–(29) below.

Then the system of three equations (7), (9) and (16) has a unique set of solutions

$$\left\{ x^*(t) \in C(\Omega, E), \quad x_r^* \in E, \quad u^*(t) \in C(\Omega, U) \right\}.$$

*Proof.* We fix  $\alpha \in \mathcal{A}$ ,  $\beta \in \mathcal{B}$ . Then, taking into account the conditions of the theorem, there the following estimate holds:

$$\begin{aligned} \tilde{p}_\alpha(\mathbf{K}_1 x(t)) &\leq C_1 p_0 \left( \frac{x_r}{a+b} \right) \sup_{t \in \Omega} \int_0^T p_\alpha(V(t, s)) p_\alpha(g(s, x(s))) \times \\ &\quad \times \int_0^s p_\alpha(f(\theta, x(\theta), u(\theta))) d\theta ds \leq \\ &\leq C_1 p_0 \left( \frac{x_r}{a+b} \right) \sup_{t \in \Omega} \int_0^T p_\alpha(V(t, s)) s ds p_\alpha(g(t, x(t))) p_\alpha(f(t, x(t), u(t))) = \delta_1 < \infty, \end{aligned} \quad (17)$$

where  $C_1, \delta_1$  are positive constants.

Similarly, we obtain the following estimate:

$$\begin{aligned} p_0(\mathbf{K}_2 x_r) &\leq C_2 p_0 \left( \frac{a+b}{a+b-1} \right) \int_0^T p_\alpha(V(t_1, s)) p_\alpha(g(s, x(s))) \times \\ &\quad \times \int_0^s p_\alpha(f(\theta, x(\theta), u(\theta))) d\theta ds \leq \\ &\leq C_2 b_0 \int_0^T p_\alpha(V(t_1, s)) s ds p_\alpha(g(t, x(t))) p_\alpha(f(t, x(t), u(t))) = \delta_2 < \infty, \end{aligned} \quad (18)$$

where  $C_2, \delta_2$  are positive constants,  $b_0 = p_0 \left( \frac{a+b}{a+b-1} \right)$ .

We also have that the estimate is valid

$$\begin{aligned} \tilde{q}_\beta(\mathbf{K}_3 u(t)) &\leq C_3 \sup_{t \in \Omega} p_0(a_0(t, x_r)) \int_0^t p_\alpha(V'(t, s)) p_\alpha(g(s, x(s))) \times \\ &\quad \times \int_0^s p_\alpha(f(\theta, x(\theta), u(\theta))) d\theta ds \leq \\ &\leq C_3 \sup_{t \in \Omega} p_\alpha(a_0(t, x_r)) \int_0^t p_\alpha(V'(t, s)) s ds p_\alpha(g(t, x(t))) p_\alpha(f(t, x(t), u(t))) = \delta_3 < \infty, \end{aligned} \quad (19)$$

where  $C_3, \delta_3$  are positive constants.

Now for two triples  $(x_1(t), x_{r,1}, u_1(t))$  and  $(x_2(t), x_{r,2}, u_2(t))$ , using the Lipschitz conditions, we obtain

$$\tilde{p}_\alpha(\mathbf{K}_1 x_1(t) - \mathbf{K}_1 x_2(t)) \leq C_4 \left\{ p_0 \left( \frac{x_{r,1} - x_{r,2}}{a+b} \right) + \sup_{t \in \Omega} \int_0^T p_\alpha(V(t, s)) p_\alpha(g(s, x_1(s))) \times \right.$$

$$\begin{aligned}
 & \times \int_0^s [L_{21}p_\alpha(x_1(\theta) - x_2(\theta)) + L_{22}q_\beta(u_1(\theta) - u_2(\theta))] d\theta ds + \\
 & + \sup_{t \in \Omega} \int_0^T p_\alpha(V(t, s)) L_1 p_\alpha(x_1(s) - x_2(s)) \int_0^s p_\alpha(f(\theta, x_2(\theta), u_2(\theta))) d\theta ds \Big\} \leq \\
 & \leq C_4 \left\{ p_0 \left( \frac{x_{r,1} - x_{r,2}}{a + b} \right) + \sup_{t \in \Omega} \int_0^T p_\alpha(V(t, s)) s ds p_\alpha(g(t, x_1(t))) \times \right. \\
 & \quad \times [L_{21}p_\alpha(x_1(t) - x_2(t)) + L_{22}q_\beta(u_1(t) - u_2(t))] + \\
 & \quad \left. + \sup_{t \in \Omega} \int_0^T p_\alpha(V(t, s)) s ds p_\alpha(f(t, x_2(t), u_2(t))) L_1 p_\alpha(x_1(t) - x_2(t)) \right\} \leq \\
 & \leq C_4 p_0 \left( \frac{x_{r,1} - x_{r,2}}{a + b} \right) + \chi_{11} \tilde{p}_\alpha(x_1(t) - x_2(t)) + \chi_{12} \tilde{q}_\beta(u_1(t) - u_2(t)), \tag{20}
 \end{aligned}$$

where  $C_4$  is a positive constant,

$$\chi_{11} = C_4 \sup_{t \in \Omega} \int_0^T p_\alpha(V(t, s)) s ds [L_{21}p_\alpha(g(t, x_1(t))) + L_1 p_\alpha(f(t, x_2(t), u_2(t)))], \tag{21}$$

$$\chi_{12} = C_4 L_{22} \sup_{t \in \Omega} \int_0^T p_\alpha(V(t, s)) s ds p_\alpha(g(t, x_1(t))). \tag{22}$$

Similarly, we obtain the estimate

$$\begin{aligned}
 p_0(x_{r,1} - x_{r,2}) & \leq C_5 \left\{ p_0 \left( \frac{a + b}{a + b - 1} \right) \int_0^T p_\alpha(V(t_1, s)) p_\alpha(g(s, x_1(s))) \times \right. \\
 & \quad \times \int_0^s [L_{21}p_\alpha(x_1(\theta) - x_2(\theta)) + L_{22}q_\beta(u_1(\theta) - u_2(\theta))] d\theta ds + \\
 & \quad \left. + p_0 \left( \frac{a + b}{a + b - 1} \right) \int_0^T p_\alpha(V(t_1, s)) L_1 p_\alpha(x_1(s) - x_2(s)) \int_0^s p_\alpha(f(\theta, x_2(\theta), u_2(\theta))) d\theta ds \right\} \leq \\
 & \leq C_5 \left\{ p_0 \left( \frac{a + b}{a + b - 1} \right) \int_0^T p_\alpha(V(t_1, s)) s ds p_\alpha(g(t, x_1(t))) \times \right. \\
 & \quad \times [L_{21}p_\alpha(x_1(t) - x_2(t)) + L_{22}q_\beta(u_1(t) - u_2(t))] + \\
 & \quad \left. + p_0 \left( \frac{a + b}{a + b - 1} \right) \int_0^T p_\alpha(V(t_1, s)) s ds p_\alpha(f(t, x_2(t), u_2(t))) L_1 p_\alpha(x_1(t) - x_2(t)) \right\} \leq \\
 & \leq \chi_{21} \tilde{p}_\alpha(x_1(t) - x_2(t)) + \chi_{22} \tilde{q}_\beta(u_1(t) - u_2(t)), \tag{23}
 \end{aligned}$$

where

$$\chi_{21} = C_5 b_0 \int_0^T p_\alpha(V(t_1, s)) s ds [L_{21} p_\alpha(g(t, x_1(t))) + L_1 p_\alpha(f(t, x_2(t), u_2(t)))] , \quad (24)$$

$$\chi_{22} = C_5 L_{22} b_0 \int_0^T p_\alpha(V(t_1, s)) s ds p_\alpha(g(t, x_1(t))) , \quad (25)$$

$C_5$  is a positive constant.

We continue the estimation process. Note that for the function  $a_0(t, x_r)$  is true the estimate

$$p_\alpha(a_0(t, x_{r,1}) - a_0(t, x_{r,2})) \leq L_0 p_0(x_{r,1} - x_{r,2}), \quad 0 < L_0 = \text{const.}$$

Now for the third unknown function we obtain the estimate

$$\begin{aligned} \tilde{q}_\beta(u_1(t) - u_2(t)) &\leq C_6 \left\{ 3p_\alpha(a_0(t, x_{r,1})) \int_0^t p_\alpha(V'(t, s)) p_\alpha(g(s, x_1(s))) \times \right. \\ &\quad \times \int_0^s [L_{31} p_\alpha(x_1(\theta) - x_2(\theta)) + L_{32} q_\beta(u_1(\theta) - u_2(\theta))] d\theta ds + \\ &\quad + p_\alpha(a_0(t, x_{r,1})) \sup_{t \in \Omega} \int_0^t p_\alpha(V'(t, s)) L_1 p_\alpha(x_1(s) - x_2(s)) \int_0^s p_\alpha(f_u(\theta, x_2(\theta), u_2(\theta))) d\theta ds + \\ &\quad \left. + L_0 p_0(x_{r,1} - x_{r,2}) \sup_{t \in \Omega} \int_0^t p_\alpha(V'(t, s)) p_\alpha(g(s, x_2(s))) \int_0^s p_\alpha(f_u(\theta, x_2(\theta), u_2(\theta))) d\theta ds \right\} \leq \\ &\leq C_6 \left\{ 3p_\alpha(a_0(t, x_{r,1})) \sup_{t \in \Omega} \int_0^t p_\alpha(V'(t, s)) s ds p_\alpha(g(t, x_1(t))) \times \right. \\ &\quad \times [L_{31} p_\alpha(x_1(t) - x_2(t)) + L_{32} q_\beta(u_1(t) - u_2(t))] + \\ &\quad + L_1 p_\alpha(a_0(t, x_{r,1})) \sup_{t \in \Omega} \int_0^t p_\alpha(V'(t, s)) s ds p_\alpha(f_u(t, x_2(t), u_2(t))) p_\alpha(x_1(t) - x_2(t)) + \\ &\quad \left. + L_0 p_0(x_{r,1} - x_{r,2}) \sup_{t \in \Omega} \int_0^t p_\alpha(V'(t, s)) s ds p_\alpha(g(t, x_2(t))) p_\alpha(f_u(t, x_2(t), u_2(t))) \right\} \leq \\ &\leq \chi_{31} \tilde{p}_\alpha(x_1(t) - x_2(t)) + \chi_{32} p_0(x_{r,1} - x_{r,2}) + \chi_{33} \tilde{q}_\beta(u_1(t) - u_2(t)), \end{aligned} \quad (26)$$

where  $\tilde{a}_0 = p_\alpha(a_0(t))$ ,

$$\chi_{31} = 3C_6 p_\alpha(a_0(t, x_{r,1})) \sup_{t \in \Omega} \int_0^t p_\alpha(V'(t, s)) s ds p_\alpha(g(t, x_1(t))), \quad (27)$$

$$\chi_{32} = C_6 L_0 \sup_{t \in \Omega} \int_0^t p_\alpha(V'(t, s)) s ds p_\alpha(g(t, x_1(t))) p_\alpha(f_u(t, x_2(t), u_2(t))), \quad (28)$$

$$\chi_{33} = C_6 L_1 p_\alpha (a_0(t, x_{r,1})) \sup_{t \in \Omega} \int_0^t p_\alpha (V'(t, s)) s ds p_\alpha (f_u(t, x_2(t), u_2(t))), \quad (29)$$

$C_6$  is a positive constant.

We rewrite the estimates (20), (23) and (26) as follows:

$$\begin{aligned} \tilde{p}_\alpha (\mathbf{K}_1 x_1(t) - \mathbf{K}_1 x_2(t)) &\leq \\ &\leq \chi_1 [\tilde{p}_\alpha (x_1(t) - x_2(t)) + p_0 (x_{r,1} - x_{r,2}) + \tilde{q}_\beta (u_1(t) - u_2(t))], \end{aligned} \quad (30)$$

$$p_0 (\mathbf{K}_2 x_{r,1} - \mathbf{K}_2 x_{r,2}) \leq \chi_2 [\tilde{p}_\alpha (x_1(t) - x_2(t)) + p_0 (x_{r,1} - x_{r,2}) + \tilde{q}_\beta (u_1(t) - u_2(t))], \quad (31)$$

$$\begin{aligned} \tilde{q}_\beta (\mathbf{K}_3 u_1(t) - \mathbf{K}_3 u_2(t)) &\leq \\ &\leq \chi_3 [\tilde{p}_\alpha (x_1(t) - x_2(t)) + p_0 (x_{r,1} - x_{r,2}) + \tilde{q}_\beta (u_1(t) - u_2(t))], \end{aligned} \quad (32)$$

where

$$\chi_1 = \max \{p_0 (a + b), \chi_{11}, \chi_{12}\}, \quad \chi_2 = \max \{\chi_{21}, \chi_{22}\}, \quad \chi_3 = \max \{\chi_{31}, \chi_{32}, \chi_{33}\}. \quad (33)$$

Adding (30), (31) and (32) term by term, we obtain that

$$\begin{aligned} &\tilde{p}_\alpha (\mathbf{K}_1 x_1(t) - \mathbf{K}_1 x_2(t), \mathbf{K}_2 x_{r,1} - \mathbf{K}_2 x_{r,2}, \mathbf{K}_3 u_1(t) - \mathbf{K}_3 u_2(t)) = \\ &= \tilde{p}_\alpha (\mathbf{K}_1 x_1(t) - \mathbf{K}_1 x_2(t)) + p_0 (\mathbf{K}_2 x_{r,1} - \mathbf{K}_2 x_{r,2}) + \tilde{q}_\beta (\mathbf{K}_3 u_1(t) - \mathbf{K}_3 u_2(t)) \leq \\ &\leq \tilde{\rho}_\alpha \tilde{p}_\alpha (x_1(t) - x_2(t)) + \tilde{\rho}_0 p_0 (x_{r,1} - x_{r,2}) + \tilde{\rho}_\beta \tilde{q}_\beta (u_1(t) - u_2(t)) \leq \\ &\leq \tilde{\rho}_{\alpha,0,\beta} \tilde{p}_{\alpha,0,\beta} (x_1(t) - x_2(t), x_{r,1} - x_{r,2}, u_1(t) - u_2(t)), \end{aligned} \quad (34)$$

where

$$\tilde{\rho}_\alpha = \chi_1, \quad \tilde{\rho}_0 = \chi_2, \quad \tilde{\rho}_\beta = \chi_3, \quad \tilde{\rho}_{\alpha,0,\beta} = \chi_1 + \chi_2 + \chi_3.$$

Since  $\tilde{\rho}_{\alpha,0,\beta} < 1$ , it follows from the estimates (17)–(19), (34) that the operators given by the right-hand sides of equations (7), (9), and (16) are contracting with respect to seminorm of  $\tilde{\rho}_{\alpha,0,\beta}$ . Since the space  $C(\Omega, E) \times E \times C(\Omega, U)$  is complete, from the estimates (17)–(19), (20), (23) and (26) imply the existence and uniqueness of the control function  $u^*(t) \in C(\Omega, U)$ , redefinition value  $x_r^* \in E$  and the corresponding state function  $x^*(t) \in C(\Omega, E)$ . Denote the solution of the integral equation (16) by  $u^*(t) = h(t)$ . Then it is easy to establish the solvability of the differential equation (8). Theorem 1 is proved.  $\square$

*Theorem 2.* Let the conditions of Theorem 1 be satisfied. Then the state function  $x^*(t) \in E$  continuously depends on the boundary condition  $x_r^*$ .

*Proof.* In this case the equation (7) we rewrite as:

$$x(t) = \mathbf{K}_1(t; x, x_r, h) \equiv \frac{x_r}{a+b} + \int_0^T V(t, s) g(s, x(s)) \int_0^s f(\theta, x(\theta), h(\theta)) d\theta ds, \quad (35)$$

where  $h(t)$  is known function.

Now we show the continuous dependence of the solution to problem (1)–(2) on the right-hand side of the boundary condition (2). Let  $x_{r,1}, x_{r,2} \in E$  be two different numbers and the functions  $x_1(t), x_2(t) \in C(\Omega, E)$  be the corresponding solutions of problem (1)–(2). Then, similarly to the estimates (20) and (23) from equation (35) we obtain that

$$p_\alpha (x_1(t) - x_2(t)) \leq p_0 (x_{r,1} - x_{r,2}) + (\chi_1 + \chi_2 + \chi_3) p_\alpha (x_1(t) - x_2(t)). \quad (36)$$

Since  $\chi_1 + \chi_2 + \chi_3 < 1$ , it follows from the inequality (36) that

$$p_\alpha(x_1(t) - x_2(t)) \leq \frac{p_0(x_{r,1} - x_{r,2})}{1 - (\chi_1 + \chi_2 + \chi_3)}.$$

The right-hand side tends to zero as  $x_{r,1} \rightarrow x_{r,2}$ , which proves the continuous dependence of the solution on the boundary value. Theorem 2 is proved.  $\square$

### Conclusion

In this paper, an inverse boundary optimal control problem for a nonlinear system of fractional-order integro-differential equations in a complete locally convex space has been investigated. The proposed approach combines the theory of fractional differential equations with methods of optimal control and functional analysis in locally convex spaces.

The original boundary value problem has been transformed into an equivalent coupled system of nonlinear integral equations involving the state function, the unknown boundary value, and the control function. By introducing an adjoint final-value problem, a Pontryagin-type function has been constructed, and a necessary optimality condition has been derived in the form of an integral equation for the control. Under suitable boundedness, continuity, and Lipschitz assumptions on the nonlinear operators, the existence and uniqueness of the solution to the resulting system have been established by means of the contraction mapping principle. Furthermore, the continuous dependence of the state function on the identified boundary value has been proved.

The obtained results considerably extend the applicability of optimal control techniques to fractional dynamical systems formulated in locally convex spaces, where the absence of a global norm requires the use of families of seminorms and appropriate topological arguments. The proposed framework is constructive and can be applied to a broad class of inverse problems involving nonlocal conditions and fractional operators.

Future research will be directed toward nonlinear locally convex algebras, optimal control problems with general nonlocal boundary conditions, impulsive and delay fractional systems, and distributed parameter models in infinite-dimensional locally convex spaces. Another promising direction is the development of numerical algorithms based on the proposed theoretical framework and the extension of the results to more general classes of fractional evolution equations.

### Author contributions

All authors contributed equally to this work.

### Conflict of interest

The authors declare no conflict of interest.

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