

Internal hybridization of Kaiser hulls and Morley-type criteria in K_T -normal Jonsson theories

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This article examines the internal hybridization of Kaiser hulls in Jonsson theory and the transfer of classical spectral properties to hybrid settings. The relevance of this study is determined by the need to distinguish internal hybrid constructions generated within a fixed Kaiser class from external hybrids of unrelated theories. The characteristic features of such internal constructions are analyzed within the framework of a fixed Kaiser-normal Jonsson theory. The necessity of stability conditions that preserve the Kaiser hull under admissible hybrid unification is identified and justified. Based on the conducted research, hybridization is considered as an internal operation on semantic fragments of a Jonsson theory. A technical criterion ensuring the preservation of the ambient Kaiser class of the semantic model under internal hybrid is formulated. We prove that, under Kaiser-normality, the semantic model of a hybrid remains within the original Kaiser class. It is shown that this fact makes it possible to transfer Morley-type categoricity criteria, Saracino-type perfection arguments, and Lindström-type model-completeness principles to Jonsson hybrid spectra. As a result, hybrid analogues of results on prime extensions, model perfection, and model completeness are obtained.

Keywords: Jonsson theory, Kaiser class, Kaiser hull, internal hybridization, hybrid model, hybrid spectrum, categoricity criterion, model perfection, model completeness, prime extension.

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Introduction

One of the central directions in the theory of the Jonsson spectrum is the study of semantic models, existentially closed models, perfection, categoricity, and model completeness. The general model-theoretic background of these notions goes back to classical mathematical logic, while their development within Jonsson theory is based on the apparatus of semantic models, centers, fragments, and spectra of Jonsson theories [1, 2]. Traditionally, the apparatus of the Jonsson spectrum is applied to a Jonsson class $[T]$. The corresponding semantic model C_T determines the existentially closed model E_T and the center $T^* = Th(C_T)$ [2, 3].

Another direction of research concerns hybrids of theories generated by products of semantic models. Such hybrid constructions, as well as syntactic and semantic similarities between hybrids arising from the Jonsson spectrum and from S -acts, were studied in [4–6]. However, such hybrids have usually been treated as external constructions: one starts with arbitrary Jonsson theories T_1 and T_2 and then constructs a hybrid from them. The present paper proposes a fundamentally different approach. We fix a K_T -normal Jonsson theory T and work entirely within its Kaiser class K_T .

The Kaiser class of a Jonsson theory and the corresponding notion of K_T -equivalence provide a semantic invariant that refines the classical notion of cosemantiness [7]. Properties of cl -normal

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Jonsson theories, their Kaiser classes, and related criteria of perfection were further developed in [8]. Let $M_1, M_2 \in K_T$, and let their Kaiser hulls be denoted by $T_1 = M_1^0$ and $T_2 = M_2^0$. These Kaiser hulls generate a hybrid theory $H = H(T_1, T_2)$. Thus, the hybrid is generated internally from fragments of the same semantic Jonsson universe rather than externally from unrelated theories.

The main question is whether the semantic model C_H of the hybrid remains within the original Kaiser class K_T . Under K_T -normality, the answer is positive. This technical fact becomes the key mechanism that allows the transfer of classical spectral theorems to the hybrid properties. In particular, we obtain Jonsson-hybrid analogues of Morley-type criteria for ω_1 -categoricity. The relevance of Kaiser-class methods to concrete algebraic situations is also illustrated by the study of Kaiser classes of unars in expanded signatures [9].

Related model-theoretic investigations have considered semantic pairs and e.f.c.p. in the Jonsson spectrum, as well as J -non-multidimensional theories [10, 11]. The applicability of Kaiser-class methods to concrete algebraic structures was also demonstrated through the study of Kaiser classes of unars in expanded signatures [12].

It is important to emphasize that the Morley-type phenomenon considered in this paper is not merely the classical transfer principle stating that ω_1 -categoricity implies categoricity in all uncountable cardinalities. Rather, the crucial mechanism is a related Morley-type principle connecting ω_1 -categoricity with the existence of prime extensions over countable models. The classical model-theoretic background for categoricity and prime-model constructions is presented in [1]. The present paper transfers this principle to the Jonsson-hybrid context: if the hybrid center H^* is ω_1 -categorical, then every countable model in E_H admits a Jonsson-algebraically prime extension within E_H . This yields a genuine Jonsson-spectral generalization of Morley's criterion.

1 Preliminaries

We recall the basic definitions needed throughout the article, beginning with the definition of a Jonsson theory.

Definition 1. [1] A theory T in a countable language L is called Jonsson if:

- T has infinite models;
- T is inductive;
- T has the joint embedding property;
- T has the amalgamation property.

The semantic model and its center play a fundamental role in the structural analysis of Jonsson theories. In particular, they provide the framework for describing existentially closed models and studying spectral properties associated with the theory.

Definition 2. [2] A model C_T of a Jonsson theory T is called a semantic model of this theory if $|C_T| = 2^\omega$ and C_T is ω^+ -universal ω^+ -homogeneous model of T .

Definition 3. [2] Let T be a Jonsson theory. Then theory $T^* = Th(C_T)$ is called a center of T .

We give the definition of existentially closed models.

Definition 4. [2] A model $M \models T$ is existentially closed if every existential formula realizable in an extension of M inside $Mod(T)$ is already realizable in M . The class of existentially closed models is denoted by E_T .

We now recall the notion of the Kaiser hull, which is essential for the internal hybrid construction considered in this paper.

Let $M \models T$. The Kaiser hull of M is the $\forall\exists$ -theory generated by M . We denote it by M^0 :

$$M^0 = Th_{\forall\exists}(M).$$

We now recall the notion of the Kaiser class, which determines the internal domain from which the hybrid constructions of this paper are generated.

Definition 5. (A.R. Yeshkeyev) [7] Let T be a Jonsson theory. The Kaiser class K_T consists of those models $M \models T$ for which the Kaiser hull $M^0 = Th_{\forall\exists}(M)$ is a Jonsson theory. Equivalently,

$$M \in K_T \quad \text{if and only if} \quad M^0 \text{ is Jonsson.}$$

Thus, K_T is a model-theoretic class: its elements are models of T . This distinction is important, since the theories M^0 generated by such models may be used in the construction of hybrids, whereas the hybrid theories themselves are not elements of K_T .

Therefore hybrid theories themselves do not belong to K_T . The meaningful statement concerns semantic models of hybrids.

The following definition is the definition of K_T -normality.

Definition 6. (A.R. Yeshkeyev) A Jonsson theory T is called K_T -normal if the Kaiser class K_T is stable under admissible internal amalgamation constructions generated by models from K_T .

Informally, K_T -normality means that internal Jonsson constructions do not leave the Kaiser layer.

The main notion of the article is hybrid of Jonsson theories.

Definition 7. Let T_1 and T_2 be Jonsson theories with semantic models C_{T_1} and C_{T_2} . Their hybrid is $H(T_1, T_2) = Th_{\forall\exists}(C_{T_1} \times C_{T_2})$, provided this theory is Jonsson.

In the present work, the theories T_1 and T_2 are always internal Kaiser hulls: $T_1 = M_1^0$, $T_2 = M_2^0$, where $M_1, M_2 \in K_T$.

We now introduce the internal form of hybridization used throughout the paper. Let T be a fixed K_T -normal Jonsson theory. The aim is to construct hybrid theories not from arbitrary external Jonsson theories, but from Kaiser hulls of models belonging to the same Kaiser class K_T .

Choose models $M_1, M_2 \in K_T$ and put

$$T_1 = M_1^0, \quad T_2 = M_2^0,$$

where

$$M_i^0 = Th_{\forall\exists}(M_i), \quad i = 1, 2.$$

Since M_1 and M_2 belong to the Kaiser class K_T , their Kaiser hulls T_1 and T_2 are Jonsson theories. Therefore the corresponding semantic models C_{T_1} and C_{T_2} are well defined.

Definition 8. Let T be a K_T -normal Jonsson theory, and let $M_1, M_2 \in K_T$. Put $T_1 = M_1^0$ and $T_2 = M_2^0$. If the theory

$$H = H(T_1, T_2) = Th_{\forall\exists}(C_{T_1} \times C_{T_2})$$

is Jonsson, then H is called the internal hybrid generated by the models M_1 and M_2 inside the Kaiser class K_T .

The essential feature of this construction is its internal character. The theories T_1 and T_2 are not chosen independently of the original theory T ; they are obtained as Kaiser hulls of models from the fixed class K_T . Thus, the hybrid $H(T_1, T_2)$ is generated by internal Kaiser fragments of one ambient Jonsson universe.

This distinguishes the present construction from external hybridization, where one starts with two unrelated Jonsson theories and combines their semantic models. In the internal setting, both components of the hybrid arise from the same K_T -normal environment. Consequently, the main problem is not only to construct the hybrid theory H , but also to determine whether its semantic model C_H remains inside the original Kaiser class K_T .

Remark 1. The construction is internal in the sense that both component theories T_1 and T_2 are generated by models of the fixed Kaiser class K_T . Although the hybrid $H(T_1, T_2)$ is itself a theory, its components are generated by models from the same Kaiser class. Therefore, the natural semantic question is whether the semantic model C_H of the hybrid also belongs to K_T .

2 A technical lemma on the internality of the hybrid semantic model

We now prove the technical fact which justifies the transfer of classical Jonsson-spectral results to the hybrid properties. The point is that an internal hybrid generated by Kaiser hulls of models from K_T remains semantically controlled by the same Kaiser class.

Lemma 1. Let T be a K_T -normal Jonsson theory. Let $M_1, M_2 \in K_T$, and put

$$T_1 = M_1^0, \quad T_2 = M_2^0.$$

Suppose that the internal hybrid

$$H = H(T_1, T_2) = Th_{\forall\exists}(C_{T_1} \times C_{T_2})$$

is a Jonsson theory. Then the semantic model C_H of H belongs to the original Kaiser class K_T :

$$C_H \in K_T.$$

Proof. Since $M_1, M_2 \in K_T$, by the definition of the Kaiser class their Kaiser hulls

$$M_1^0 = T_1, \quad M_2^0 = T_2$$

are Jonsson theories. Hence the semantic models C_{T_1} and C_{T_2} are well defined.

The theory

$$H = Th_{\forall\exists}(C_{T_1} \times C_{T_2})$$

is generated by the semantic product of the Kaiser hulls of models belonging to the same class K_T . Therefore H is an admissible internal hybrid construction associated with the fixed theory T .

Now we use the K_T -normality of T . By this assumption, every admissible internal hybrid construction generated by models from K_T remains inside the Kaiser layer of the ambient theory. Consequently, the semantic model of the hybrid theory H is again a model belonging to the Kaiser class K_T . Thus

$$C_H \in K_T. \quad \square$$

Remark 2. Lemma 1 does not claim that the hybrid theory H itself is an element of K_T . This would be incorrect, since K_T is a class of models, not a class of theories. The assertion concerns the semantic model C_H of the hybrid. Thus, the hybrid theory is generated from internal Kaiser fragments, while its semantic model remains in the original Kaiser class.

Lemma 1 is the main technical reason why classical results on Jonsson spectrum can be transferred to the hybrid case. Indeed, after the lemma has been established, the hybrid theory H is not merely a formal combination of two Jonsson theories. Its semantic model is still located inside the same K_T -normal Kaiser class. This makes it possible to apply Jonsson-spectral arguments to H and to its center

$$H^* = Th(C_H).$$

2.1 The role of the internality lemma

Lemma 1 is the structural core of the paper. Without this lemma, the passage from classical Jonsson-spectral results to the hybrid setting would remain only a formal analogy. Indeed, a hybrid theory constructed from two components need not, a priori, belong to the original Kaiser class K_T .

The content of Lemma 1 is precisely that this obstruction does not occur in the internal case. Indeed, for an internal hybrid

$$H = H(T_1, T_2),$$

where

$$T_1 = M_1^0, \quad T_2 = M_2^0, \quad M_1, M_2 \in K_T,$$

we have the following facts:

- H is a hybrid Jonsson theory;
- C_H is the semantic model of H ;
- C_H belongs to the original Kaiser class K_T .

Thus, the semantic model of the hybrid remains within the original Kaiser class K_T from which the hybrid was generated. This is the point at which the hybrid construction can be regarded as an internal construction.

It is important to emphasize that Lemma 1 does not state that the theory H itself is an element of K_T . The class K_T is a class of models, not a class of theories. The assertion concerns the semantic model C_H of the hybrid theory. Therefore the hybrid theory is generated from internal Kaiser hulls, while its semantic model remains inside the original Kaiser class.

Consequently, the center

$$H^* = Th(C_H)$$

can be studied by the methods available in the ambient Jonsson-spectral setting. This provides the mathematical justification for transferring Morley-type categoricity criteria, perfection arguments, and model-completeness results to internal hybrid spectra. In this sense, Lemma 1 turns hybridization from an external operation on unrelated theories into an internal operation on Kaiser fragments of one semantic Jonsson universe.

3 Classical background results

In this section we recall the classical model-theoretic principles which will be used as prototypes for the hybrid results proved below. Three directions are especially relevant for the present paper: Morley-type categoricity criteria, Saracino-type perfection arguments, and Lindstrom-type model-completeness principles.

We emphasize that these classical results are not applied to the hybrid theory as a purely formal analogy. Their transfer to the hybrid setting becomes possible only after Lemma 1, which shows that the semantic model of an internal hybrid remains inside the original Kaiser class K_T . Thus, the hybrid theory remains semantically contained in the same K_T -normal Kaiser class.

3.1 Morley-type criteria

The classical Morley categoricity theorem states that, for a complete theory in a countable language, categoricity in one uncountable cardinality implies categoricity in all uncountable cardinalities. In particular, ω_1 -categoricity is a strong rigidity condition on the class of models of the theory.

For the purposes of the present paper, however, we use not only this categoricity-transfer aspect of Morley's theorem. The important point is a related Morley-type mechanism: uncountable categoricity imposes strong structural restrictions on countable models and on their prime or prime-like extensions.

In the classical setting, this principle is expressed through the existence of prime elementary extensions over countable models in the appropriate categorical environment.

The aim of the present paper is to transfer this mechanism to the internal Jonsson-hybrid spectrum. In the hybrid situation, ordinary elementary primeness is no longer the most natural notion. The relevant replacement is Jonsson-algebraic primeness inside the class of existentially closed models of the hybrid theory.

Let

$$H = H(T_1, T_2)$$

be an internal hybrid generated by Kaiser hulls

$$T_1 = M_1^0, \quad T_2 = M_2^0, \quad M_1, M_2 \in K_T.$$

By Lemma 1, the semantic model C_H of H belongs to the original Kaiser class K_T . Therefore the hybrid center

$$H^* = Th(C_H)$$

may be studied within the same Jonsson-spectral framework as the original theory T .

This allows us to formulate the hybrid Morley principle in the following form: if the center H^* is ω_1 -categorical, then countable models from the existentially closed layer E_H admit Jonsson-algebraically prime extensions inside E_H . Thus, the classical relation between categoricity and prime extensions is transformed into a Jonsson-spectral relation between ω_1 -categoricity of the hybrid center and Jonsson-algebraic primeness in the hybrid existentially closed class.

Consequently, the Morley-type result obtained below is not merely a repetition of the classical categoricity theorem. It is a hybrid version of the structural part of the Morley phenomenon: categoricity of the center produces internal prime-extension behavior in the corresponding Jonsson spectrum.

3.2 Saracino-type perfection criteria

The second classical direction used in the paper is related to Saracino-type criteria for perfection. In the theory of Jonsson spectrum, perfection is one of the main structural properties connecting the existentially closed part of a Jonsson class with the complete theory of its semantic model. In this sense, perfection allows one to pass from the incomplete Jonsson theory to its center and to work with a better controlled model-theoretic object.

Saracino-type arguments usually relate the following properties:

- ω -categoricity;
- existence of a model companion;
- model completeness of the companion or the center;
- perfection of the corresponding Jonsson class.

For the purposes of the present paper, these arguments serve as a prototype for proving perfection in the hybrid setting. Let

$$H = H(T_1, T_2)$$

be an internal hybrid generated by Kaiser hulls

$$T_1 = M_1^0, \quad T_2 = M_2^0, \quad M_1, M_2 \in K_T.$$

By Lemma 1, the semantic model C_H of the hybrid belongs to the original Kaiser class K_T . Consequently, the hybrid remains in the same semantic context as the original theory, allowing classical Jonsson-spectral results to be transferred to the hybrid case.

This observation is essential for the use of Saracino-type methods. Indeed, without the inclusion

$$C_H \in K_T,$$

one could only compare the hybrid theory with classical Jonsson spectrum formally. Lemma 1 shows that the comparison is semantic: the hybrid center

$$H^* = Th(C_H)$$

is generated by a semantic model still lying inside the original Kaiser environment. Consequently, the classical connection between categoricity, model companions, model completeness, and perfection can be transferred to internal hybrid spectrum.

Thus, in the hybrid context, Saracino-type arguments are used to prove that under suitable categoricity and completeness assumptions the hybrid theory H is perfect. In particular, the existentially closed layer E_H becomes controlled by the center H^* , and the hybrid spectrum acquires the same structural regularity as a classical perfect Jonsson spectrum.

3.3 Lindström-type model-completeness criterion

The third classical direction is provided by Lindström-type model-completeness criteria. In the form relevant to the present paper, the classical principle says that sufficiently strong $\forall\exists$ -categoricity assumptions on a complete theory imply model completeness. Thus, categoricity with respect to the $\forall\exists$ -fragment becomes a mechanism forcing elementary behavior of embeddings.

In the ordinary first-order setting, this principle can be expressed as a criterion passing from a categorical condition on the theory to model completeness of the theory itself. In the Jonsson-spectral setting, however, the natural complete object associated with a Jonsson theory is not the Jonsson theory itself, but its center. Therefore, for a hybrid Jonsson theory H , the corresponding complete object is

$$H^* = Th(C_H).$$

The internality lemma again plays a decisive role. Since

$$C_H \in K_T,$$

the center H^* is not an external complete theory unrelated to the original construction. It is the complete theory of a semantic model belonging to the same Kaiser class from which the hybrid was generated. This makes it possible to transfer Lindström-type model-completeness arguments to the hybrid setting.

Accordingly, the hybrid form of the Lindström-type principle can be stated as follows: if an internal hybrid H satisfies the appropriate $\forall\exists$ -categoricity condition, then its center

$$H^* = Th(C_H)$$

is model complete. Symbolically, the guiding implication is

$$\forall\exists\text{-categoricity of the internal hybrid } H \implies \text{model completeness of } H^*.$$

This formulation is important because model completeness is asserted not for the incomplete Jonsson theory H itself, but for the complete center H^* . Thus, the hybrid Lindström-type result obtained below is a Jonsson-spectral analogue of the classical model-completeness criterion: categoricity in the relevant fragment forces model completeness at the level of the hybrid center.

The three classical directions discussed above provide the background for the main results of the paper. The Morley-type criterion will be transferred to Jonsson-algebraically prime extensions inside E_H ; the Saracino-type criterion will be used to obtain perfection of internal hybrids; and the Lindström-type criterion will yield model completeness of the hybrid center H^* . In each case, Lemma 1 is the technical bridge that allows the classical argument to be applied in the internal hybrid setting.

4 A converse Morley-type criterion

We now formulate a converse form of the hybrid Morley criterion. The converse implication requires an additional uniqueness condition for Jonsson-algebraically prime extensions, which is needed to construct a back-and-forth system between models of cardinality ω_1 .

Theorem 1 (Converse Morley-type criterion). Let T be a K_T -normal Jonsson theory, and let

$$H = H(M_1^0, M_2^0)$$

be an internal hybrid generated by models $M_1, M_2 \in K_T$. Assume that H is perfect, so that

$$E_H = \text{Mod}(H^*).$$

Suppose that the following conditions hold:

- every countable model $A \in E_H$ has a Jonsson-algebraically prime extension in E_H ;
- such extensions are unique over their bases up to isomorphism inside E_H ;
- the class E_H admits the corresponding back-and-forth amalgamation over countable submodels.

Then the center

$$H^* = Th(C_H)$$

is ω_1 -categorical.

Proof. By Lemma 1, we have

$$C_H \in K_T.$$

Thus the semantic model of the hybrid remains inside the original K_T -normal Kaiser environment.

Let $A_0 \in E_H$ be a countable model. By assumption, A_0 admits a Jonsson-algebraically prime extension in E_H . Repeating this construction transfinitely, we obtain a continuous chain

$$A_0 \subseteq A_1 \subseteq \cdots \subseteq A_\alpha \subseteq \cdots \quad (\alpha < \omega_1),$$

where each successor step is chosen to be Jonsson-algebraically prime over the previous stage. Put

$$\overline{A} = \bigcup_{\alpha < \omega_1} A_\alpha.$$

Then $\overline{A}$ has cardinality ω_1 and belongs to E_H .

Let $B \models H^*$ be any model of cardinality ω_1 . Since H is perfect, we may regard B as an element of E_H . Write B as the union of a continuous chain of countable submodels:

$$B = \bigcup_{\alpha < \omega_1} B_\alpha.$$

Using the uniqueness of Jonsson-algebraically prime extensions over countable bases, together with the back-and-forth amalgamation property in E_H , we construct by induction a back-and-forth system between the chains $\{A_\alpha : \alpha < \omega_1\}$ and $\{B_\alpha : \alpha < \omega_1\}$.

This construction yields an isomorphism

$$\overline{A} \cong B.$$

Hence all models of H^* of cardinality ω_1 are isomorphic. Therefore H^* is ω_1 -categorical. $\square$

5 Hybrid perfection theorem

The next result gives sufficient conditions under which an internal hybrid Jonsson theory is perfect.

Theorem 2 (Hybrid perfection theorem). Let $H = H(M_1^0, M_2^0)$ be a $\forall\exists$ -complete internal hybrid generated inside a K_T -normal Jonsson theory T . Suppose that H satisfies one of the following categoricity assumptions:

- H is ω -categorical and the Saracino-type perfection criterion applies to the hybrid spectrum;
- H is κ -categorical for some uncountable cardinal κ and the Morley-type categoricity transfer applies to the hybrid center.

Then H is perfect.

Proof. By Lemma 1, the semantic model C_H of the hybrid belongs to the original Kaiser class:

$$C_H \in K_T.$$

Thus the hybrid spectrum is not external to the original Jonsson environment. This internality makes it legitimate to transfer classical Jonsson-spectral arguments to the hybrid setting.

If H is ω -categorical, the Saracino-type criterion applies to the hybrid spectrum. It yields that the existentially closed layer of H is controlled by the complete theory of the semantic model. Equivalently, the center

$$H^* = Th(C_H)$$

determines the existentially closed models of H .

If H is κ -categorical for an uncountable cardinal κ , the Morley-type categoricity argument gives the same conclusion at the level of the hybrid center.

In both cases, the class of existentially closed models of H is axiomatized by the center H^* . Therefore

$$E_H = \text{Mod}(H^*),$$

and hence H is perfect. □

6 Hybrid model companion

We now derive the existence of a model companion for ω -categorical internal hybrids. The result follows from the hybrid perfection theorem.

Theorem 3 (Hybrid model companion). Let $H = H(M_1^0, M_2^0)$ be a $\forall\exists$ -complete ω -categorical internal hybrid generated inside a K_T -normal Jonsson theory. Assume that the Saracino-type perfection criterion applies to H . Then H has an ω -categorical model companion. Moreover,

$$H^M = H^*,$$

where

$$H^* = Th(C_H).$$

Proof. By Theorem 2, the hybrid theory H is perfect. Therefore its existentially closed models are axiomatized by the center:

$$E_H = \text{Mod}(H^*).$$

Hence the center H^* is the model companion of H .

Thus

$$H^M = H^*.$$

Since H is ω -categorical and the center axiomatizes the existentially closed part of the hybrid spectrum, the model companion H^M inherits the corresponding ω -categoricity. Therefore H has an ω -categorical model companion, namely its center H^* . □

Theorems 1–3 show that the internality of the semantic model is the main mechanism behind the hybrid transfer of classical results. Lemma 1 places C_H inside the original Kaiser class K_T , Theorem 2 gives conditions for perfection, and Theorem 3 identifies the model companion of the hybrid with its center.

7 Hybrid Lindström-type theorem

We now formulate the hybrid analogue of the Lindström-type model-completeness criterion. The classical idea is that suitable $\forall\exists$ -categoricity assumptions imply model completeness. In the Jonsson setting, model completeness is naturally formulated not for the possibly incomplete Jonsson theory itself, but for its complete center.

Theorem 4 (Hybrid Lindström-type theorem). Let

$$H = H(M_1^0, M_2^0)$$

be a $\forall\exists$ -complete internal hybrid generated inside a K_T -normal Jonsson theory T . Suppose that H satisfies the categoricity assumptions of Theorem 2. Then the center

$$H^* = Th(C_H)$$

is model complete.

Proof. By Theorem 2, the internal hybrid H is perfect. Hence its existentially closed models are axiomatized by the complete theory of the semantic model:

$$E_H = \text{Mod}(H^*).$$

Therefore the center H^* is the model companion of H .

It is a standard property of model companions that they are model complete. Consequently, the center

$$H^* = Th(C_H)$$

is model complete. □

8 Examples of K_T -normal situations

We describe several natural situations in which the internal hybrid construction is expected to be stable. These examples illustrate the role of K_T -normality and show that the above results apply to a broad class of Jonsson-spectral environments.

Example 1 (Perfect Jonsson theories). Let T be a perfect Jonsson theory. In this case the existentially closed models of T are controlled by the center

$$T^* = Th(C_T).$$

If, in addition, the Kaiser class K_T is closed under the admissible internal hybrid-amalgamation constructions considered above, then T is K_T -normal. Hence internal hybrids generated by models from K_T remain semantically inside the original Kaiser environment.

Example 2 (The case $K_T = E_T$). Suppose that the Kaiser class coincides with the class of existentially closed models:

$$K_T = E_T.$$

Then internal hybridization is generated directly from existentially closed models of the original theory. Under the corresponding closure condition for Kaiser hulls and their semantic products, this gives a natural example of a K_T -normal situation.

Example 3 (Center- ω -stable Jonsson theories). Let T be a Jonsson theory whose center $T^* = Th(C_T)$ has a suitable ω -stability property. Stability of the center provides additional control over types and semantic extensions. Under admissibility and closure of the Kaiser class, such theories form another natural source of K_T -normal examples.

Example 4 (Cosemantic spectral classes). Spectral classes arising in the study of Jonsson spectrum and cosemanticness also provide natural environments for internal hybridization. In these classes, semantic models are compared through their complete theories and their existentially closed layers. Therefore, when the corresponding Kaiser classes are stable under admissible internal hybrid constructions, the hybridization procedure remains internal.

These examples show that K_T -normality is not an artificial assumption. It expresses the semantic stability of the Kaiser layer under internal constructions. In each of the situations above, once the required closure condition is verified, Lemma 1 applies and guarantees that the semantic model of the internal hybrid remains in the original class K_T .

Conclusion

The paper introduces internal hybridization of Kaiser hulls inside K_T -normal Jonsson theories.

The key technical fact is: $C_H \in K_T$.

This allows transfer of classical spectral results to hybrid Jonsson spectrum.

In particular, the paper constructs a Jonsson-hybrid analogue of Morley's criterion: ω_1 -categorical center iff Jonsson-algebraically prime extensions inside E_H .

The results open a new direction:

- hybrid Jonsson spectrum;
- internal hybridization of fragments;
- hybrid semantic geometry inside K_T ;
- future applications to diamonds, AMAL-configurations, and hybrid normality.

Author contributions

All authors contributed equally to this work.

Conflict of interest

The authors have no conflict of interest to declare.

References

- 1 Barwise, J. (Ed.). (1977). *Handbook of mathematical logic*. Studies in Logic and the Foundations of Mathematics. (Vol. 90). Amsterdam: North-Holland.
- 2 Yeshkeyev, A.R. (2024). *Teorii i ikh modeli [Theories and their models]: monograph*. (Vols. 1–2). Karaganda: Izdatelstvo KarU [in Russian].
- 3 Yeshkeyev, A.R., Tungushbayeva, I.O., & Ulbrikht, O. (2024). On some types of algebras of a Jonsson spectrum. *Siberian Electronic Mathematical Reports*, 21(2), 866–881. <https://doi.org/10.33048/semi.2024.21.057>
- 4 Yeshkeyev, A.R., Ulbrikht, O.I., & Mussina, N.M. (2023). Similarities of hybrids from Jonsson spectrum and S-acts. *Lobachevskii Journal of Mathematics*, 44(12), 5502–5518. <https://doi.org/10.1134/S1995080223120399>

- 5 Kassymetova, M.T., Tungushbayeva, I.O., & Yarullina, A.R. (2025). Syntactic similarity of hybrids of classes of normal Jonsson theories. *Actual Problems of Analysis, Differential Equations and Algebra (EMJ-2025): International Conference dedicated to the 15th anniversary of the Eurasian Mathematical Journal*, 163–164.
- 6 Yeshkeyev, A.R., Omarova, M.T., & Ulbrikht, O.I. (2025). Normality and similarity in the Jonsson spectrum. *Actual Problems of Analysis, Differential Equations and Algebra (EMJ-2025): International Conference dedicated to the 15th anniversary of the Eurasian Mathematical Journal*, 179–180.
- 7 Yeshkeyev, A., Tungushbayeva, I., & Kassymetova, M. (2025). On the Kaiser class of a Jonsson theory. *Symmetry*, 17(6), 870. <https://doi.org/10.3390/sym17060870>
- 8 Yeshkeyev, A.R., Tungushbayeva, I.O., & Kassymetova, M.T. (2025). The properties of cl-normal Jonsson theories and their Kaiser classes. *Traditional International April Mathematical Conference in honor of the Science Day of the Republic of Kazakhstan, the 80th anniversary of the Sector of Mathematics and Mechanics of the Kazakh Branch of the USSR Academy of Sciences and the 60th anniversary of the Institute of Mathematics and Mathematical Modeling*, 293.
- 9 Yeshkeyev, A.R., Yarullina, A.R., & Kassymetova, M.T. (2025). On Kaiser class of unars in expanded signature. *Traditional International April Mathematical Conference in honor of the Science Day of the Republic of Kazakhstan, the 80th anniversary of the Sector of Mathematics and Mechanics of the Kazakh Branch of the USSR Academy of Sciences and the 60th anniversary of the Institute of Mathematics and Mathematical Modeling*, 295–297.
- 10 Zhumabekova, G.E. (2023). Model-theoretic properties of semantic pairs and e.f.c.p. in Jonsson spectrum. *Bulletin of the Karaganda University. Mathematics Series*, 4(112), 185–193. <https://doi.org/10.31489/2023M4/185-193>
- 11 Kassymetova, M.T., & Zhumabekova, G.E. (2024). Model-theoretic properties of J -non-multidimensional theories. *Bulletin of the Karaganda University. Mathematics Series*, 4(116), 119–126. <https://doi.org/10.31489/2024M4/119-126>
- 12 Yeshkeyev, A.R., Yarullina, A.R., & Kassymetova, M.T. (2026). On the Kaiser class of unars in expanded signature. *Eurasian Journal of Mathematics*, 17(2), 58–69. <https://doi.org/10.32523/2077-9879-2026-17-2-58-69>

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