

A note on the space-dependent source identification problem for the two-dimensional neutron transport equation

A. Ashyralyev^{1,2,3}, A. Taskin^{4,*}

¹Bahcesehir University, Istanbul, Turkey;

²Institute of Mathematics and Mathematical Modeling, Almaty, Kazakhstan;

³Peoples' Friendship University of Russia (RUDN University), Moscow, Russian Federation;

⁴ENKA Schools, Istanbul, Turkey

(E-mail: allaberen.ashyralyev@bau.edu.tr, gafurtaskin@hotmail.com & gafur.taskin@enka.k12.tr)

The space-dependent source identification problem for the two-dimensional neutron transport equation with spatially variable coefficients and periodic boundary conditions is investigated. The neutron transport equation is a fundamental model for neutron propagation in a medium, accounting for both streaming and collision events, and is critical for nuclear reactor safety analysis. While the source identification problem has been extensively studied for parabolic, elliptic, and hyperbolic equations, recent research has extended these methods to transport equations as well. The investigation begins with the homogeneous problem with constant coefficients, where an explicit solution is derived using the method of characteristics. This solution serves as the basis for studying the variable-coefficient case. A function is introduced to quantify the deviation of variable coefficients from constant values. Under suitable assumptions, a stability estimate is established, showing that the solution operator is exponentially bounded in time. Building on these results, this study investigates the inverse problem of identifying space-dependent source terms. By analyzing the equation structure and applying the stability estimate for the homogeneous problem, the study derives conditions for stable recovery of source terms from measured data. We introduce a time interval parameter and a contraction condition to make sure the solution is unique and depends continuously on the data. The main result provides stability estimates, showing that small changes in measured data lead to small changes in the identified sources. These theoretical contributions enhance the understanding of inverse problems for transport equations and have potential applications in reactor physics and related fields.

Keywords: source identification problem, two-dimensional neutron transport equation, spatially variable coefficients, stability inequality, periodic boundary conditions, method of characteristics, inverse problem, space-dependent source term, homogeneous transport problem.

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Introduction

The neutron transport equation (NTE) describes the distribution of neutrons as a function of spatial position, angular direction, energy, and time, accounting for both streaming and collision processes. Fundamental studies have provided comprehensive analyses of the NTE's computational methods, numerical techniques, and mathematical theory [1–3], while recent research has addressed the latest theoretical and computational developments [4]. The theory of linear differential equations in Banach spaces [5] and the method of positive operators [6] provide key tools for such analyses. The time-dependent NTE is fundamental to safety analysis for nuclear reactor systems.

Inverse problems, notably those involving the identification of unknown source terms, are central to numerous scientific disciplines. The literature presents extensive conceptual models and applications [7–9], including recent progress in fractional Schrödinger-type equations [10]. Parabolic equations

*Corresponding author. E-mail: gafurtaskin@hotmail.com

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have been investigated numerically for reconstructing distributed sources [11] and for using stability estimates in delay problems [12]. Further studies address well-posedness [13] and source identification in reverse problems [14]. For elliptic partial differential equations, research has focused on stable difference schemes for mixed boundary conditions [15] and source identification in elliptic-telegraph equations [16]. Investigations of hyperbolic equations include identification in hyperbolic-parabolic systems [17], source identification for hyperbolic differential equations [18], and combined parabolic-hyperbolic formulations [19]. Recent studies have expanded the scope of the neutron transport equation (NTE) by analyzing fractional spaces generated by neutron transport operators [20], examining their structure in two-dimensional contexts [21], and addressing source identification problems for neutron transport equations [22]. Related research explores nonlocal problems and operator positivity, with particular attention to the positivity of differential operators generated by hyperbolic systems [23]. Building on these foundations, the present study investigates the simultaneous determination of two space-dependent source terms in the NTE. This problem also arises in heat conduction, where similar simultaneous identification challenges have been addressed for two-dimensional heat conduction equations [24].

This paper presents stability estimates for the space-dependent source identification problem in the two-dimensional neutron transport equation with spatially variable coefficients and periodic boundary conditions

$$\left\{ \begin{array}{l} \frac{\partial u(t, x_1, x_2)}{\partial t} - a(x_1, x_2) \frac{\partial u(t, x_1, x_2)}{\partial x_1} - b(x_1, x_2) \frac{\partial u(t, x_1, x_2)}{\partial x_2} + \delta u(t, x_1, x_2) \\ = p(x_1, x_2) + f(t, x_1, x_2), \\ t \in (0, T), \quad x_1, x_2 \in (0, 1), \\ u(t, 0, x_2) = u(t, 1, x_2), \quad t \in [0, T], \quad x_2 \in [0, 1], \\ u(t, x_1, 0) = u(t, x_1, 1), \quad t \in [0, T], \quad x_1 \in [0, 1], \\ u(T_1, x_1, x_2) = u(0, x_1, x_2) + \psi_1(x_1, x_2), \quad T_1 \in (0, T), \\ u(T_2, x_1, x_2) = \psi_2(x_1, x_2), \quad x_1, x_2 \in [0, 1], \quad T_1 < T_2, \quad T_2 \in (0, T]. \end{array} \right. \quad (1)$$

Here, $u(t, x_1, x_2)$ and $p(x_1, x_2)$ denote unknown functions, whereas $f(t, x_1, x_2)$, $\psi_1(x_1, x_2)$, and $\psi_2(x_1, x_2)$ are sufficiently smooth functions. All compatibility conditions are assumed to be satisfied. $a(x_1, x_2)$ and $b(x_1, x_2)$ specify the directional components of the transport velocity. The parameter $\delta > 0$ is an absorption coefficient that models a sink or loss term. The problem is defined on the space-time domain $(t, x_1, x_2) \in (0, T) \times (0, 1)^2$, with periodic boundary conditions in both spatial directions.

1 Preliminaries

The Banach space $C([0, T], E) = C(E)$ is introduced to support the results. For all abstract continuous functions $\phi(t)$ with the values in E , which has the maximum norm, maps the interval $[0, T]$ into the set E ,

$$\|\phi\|_{C(E)} = \max_{0 \leq t \leq T} \|\phi(t)\|_E.$$

Let $\Omega = [0, 1] \times [0, 1]$ be the spatial domain, and define $E = C(\Omega)$ as the space of continuous functions $\psi(x_1, x_2)$ defined on Ω equipped with the norm

$$\|\psi\|_{C(\Omega)} = \max_{(x_1, x_2) \in \Omega} |\psi(x_1, x_2)|,$$

and let $C^{(1)}(\Omega)$ denote the space of continuously differentiable functions $\psi(x_1, x_2)$ on Ω with the norm

$$\|\psi\|_{C^{(1)}(\Omega)} = \|\psi\|_{C(\Omega)} + \max_{(x_1, x_2) \in \Omega} |\psi_{x_1}(x_1, x_2)| + \max_{(x_1, x_2) \in \Omega} |\psi_{x_2}(x_1, x_2)|.$$

Positive definite operator $A^{(x_1, x_2)}$ defined by the formula

$$A^{(x_1, x_2)}v = -\left(a(x_1, x_2)\frac{\partial v(x_1, x_2)}{\partial x_1} + b(x_1, x_2)\frac{\partial v(x_1, x_2)}{\partial x_2} - \delta v(x_1, x_2)\right) \quad (2)$$

which acts on the domain

$$D(A^{(x_1, x_2)}) = \{v \in C^{(1)}(\Omega) : v(0, x_2) = v(1, x_2), v(x_1, 0) = v(x_1, 1), \forall x_1, x_2 \in [0, 1]\}.$$

The following section presents a fundamental property of the homogeneous neutron transport equation, which is essential for stability analysis. It can be established that $v(t, x_1, x_2)$ is the solution of the following problem

$$\begin{cases} \frac{\partial v(t, x_1, x_2)}{\partial t} - a(x_1, x_2)\frac{\partial v(t, x_1, x_2)}{\partial x_1} - b(x_1, x_2)\frac{\partial v(t, x_1, x_2)}{\partial x_2} + \delta v(t, x_1, x_2) = 0, \\ t \in (0, T), x_1, x_2 \in [0, 1], \\ v(0, x_1, x_2) = \phi(x_1, x_2), x_1, x_2 \in [0, 1], \\ v(t, 0, x_2) = v(t, 1, x_2), t \in [0, T], x_2 \in [0, 1], \\ v(t, x_1, 0) = v(t, x_1, 1), t \in [0, T], x_1 \in [0, 1]. \end{cases} \quad (3)$$

Here, $v(t, x_1, x_2)$ is the unknown function, $\phi(x_1, x_2)$, $a(x_1, x_2)$, and $b(x_1, x_2)$ are periodic and smooth functions. $\delta > 0$ is a constant. By applying the method of characteristics to equation (3), one can demonstrate that for constant coefficients $a_0 = a(x_0, y_0)$ and $b_0 = b(x_0, y_0)$ with $\delta > 0$, equation (3) has the unique solution

$$v(t, x_1, x_2) = e^{-\delta t} \phi(x_1 + a_0 t, x_2 + b_0 t). \quad (4)$$

We can analyze this solution by examining the semigroup generated by the differential operator $A^{(x_0, y_0)}$,

$$e^{-tA^{(x_0, y_0)}} \phi(x_1, x_2) = e^{-\delta t} \phi(x_1 + a_0 t, x_2 + b_0 t). \quad (5)$$

Taking the maximum norm on $E = C(\Omega)$ with $\Omega = [0, 1]^2$ and from (4), we obtain the estimate

$$\|e^{-tA^{(x_0, y_0)}}\|_{E \rightarrow E} \leq e^{-\delta t}.$$

Consequently, the operator $I - e^{-TA^{(x_0, y_0)}}$ is invertible and its inverse obeys

$$\|(I - e^{-TA^{(x_0, y_0)}})^{-1}\|_{E \rightarrow E} \leq \frac{1}{1 - e^{-T\delta}}.$$

2 Stability analysis

Theorem 1. Let $\delta > 0$, and let $a(x_1, x_2)$ and $b(x_1, x_2)$ be smooth periodic functions on Ω . Set $a_0 = a(x_0, y_0) > 0$ and $b_0 = b(x_0, y_0) > 0$, and define

$$\alpha(x_1, x_2) = \frac{a(x_1, x_2) - a_0}{a_0} + \frac{b(x_1, x_2) - b_0}{b_0}. \quad (6)$$

Assume that

$$1 + \alpha(x_1, x_2) \neq 0, \quad (x_1, x_2) \in \Omega.$$

Introduce the constants

$$M_1 = \max_{(x_1, x_2) \in \Omega} \frac{1}{|1 + \alpha(x_1, x_2)|}, \quad M_2 = \max_{(x_1, x_2) \in \Omega} |1 + \alpha(x_1, x_2)|,$$

and

$$L = \max_{\substack{(x_1, x_2) \in \Omega \\ 0 \leq s < t \leq T}} \left| \delta \alpha(x_1 + a_0(t-s), x_2 + b_0(t-s)) + \frac{d}{ds} \alpha(x_1 + a_0(t-s), x_2 + b_0(t-s)) \right|.$$

Then, for $0 \leq t \leq T$, the semigroup $e^{-tA^{(x_1, x_2)}}$ satisfies the estimate

$$\left\| e^{-tA^{(x_1, x_2)}} \right\|_{C(\Omega) \rightarrow C(\Omega)} \leq M e^{-\delta t}, \quad (7)$$

where $M = M_1 M_2 e^{M_1 L T}$.

Proof. Now, we investigate $v(t, x_1, x_2)$, defined by equation (3) with $a_0, b_0, \delta > 0$, and periodic boundary conditions. We have that

$$\frac{\partial v(t, x_1, x_2)}{\partial t} = a_0 \frac{\partial v(t, x_1, x_2)}{\partial x_1} + b_0 \frac{\partial v(t, x_1, x_2)}{\partial x_2} - \delta v(t, x_1, x_2) + R(t, x_1, x_2), \quad (8)$$

where

$$R(t, x_1, x_2) = (a(x_1, x_2) - a_0) \frac{\partial v(t, x_1, x_2)}{\partial x_1} + (b(x_1, x_2) - b_0) \frac{\partial v(t, x_1, x_2)}{\partial x_2}. \quad (9)$$

We can rewrite (8) in abstract form

$$\frac{dv(t)}{dt} = -A^{(x_0, y_0)} v(t) + R(t), \quad v(0) = \phi, \quad (10)$$

where $R(t) = R(t, x_1, x_2)$. Therefore, we have the following formula (see [25])

$$v(t) = e^{-tA^{(x_0, y_0)}} \phi + \int_0^t e^{-(t-s)A^{(x_0, y_0)}} R(s) ds \quad (11)$$

for the solution of (10). Substituting (5) and (9) into (11), we obtain

$$\begin{aligned} v(t, x_1, x_2) &= e^{-\delta t} \phi(x_1 + a_0 t, x_2 + b_0 t) \\ &+ \int_0^t e^{-\delta(t-s)} R(s, x_1 + a_0(t-s), x_2 + b_0(t-s)) ds = G_0(t, x_1, x_2) + G_1(t, x_1, x_2), \end{aligned} \quad (12)$$

where

$$G_0(t, x_1, x_2) = e^{-\delta t} \phi(x_1 + a_0 t, x_2 + b_0 t),$$

$$\begin{aligned} G_1(t, x_1, x_2) &= \int_0^t e^{-\delta(t-s)} \left[(a(x_1 + a_0(t-s), x_2 + b_0(t-s)) - a_0) \frac{\partial v(s, x_1 + a_0(t-s), x_2 + b_0(t-s))}{\partial(x_1 + a_0(t-s))} \right. \\ &\quad \left. + (b(x_1 + a_0(t-s), x_2 + b_0(t-s)) - b_0) \frac{\partial v(s, x_1 + a_0(t-s), x_2 + b_0(t-s))}{\partial(x_2 + b_0(t-s))} \right] ds. \end{aligned}$$

Applying integration by parts to $G_1(t, x_1, x_2)$, we get

$$\begin{aligned}
 G_1(t, x_1, x_2) = & - \left\{ \frac{a(x_1, x_2) - a_0}{a_0} + \frac{b(x_1, x_2) - b_0}{b_0} \right\} v(t, x_1, x_2) \\
 & + \left\{ \frac{a(x_1 + a_0 t, x_2 + b_0 t) - a_0}{a_0} + \frac{b(x_1 + a_0 t, x_2 + b_0 t) - b_0}{b_0} \right\} e^{-\delta t} v(0, x_1 + a_0 t, x_2 + b_0 t) \\
 & + \delta \int_0^t \left\{ \frac{a(x_1 + a_0(t-s), x_2 + b_0(t-s)) - a_0}{a_0} + \frac{b(x_1 + a_0(t-s), x_2 + b_0(t-s)) - b_0}{b_0} \right\} \\
 & \quad \times e^{-\delta(t-s)} v(s, x_1 + a_0(t-s), x_2 + b_0(t-s)) ds \\
 & + \int_0^t \frac{d}{ds} \left\{ \frac{a(x_1 + a_0(t-s), x_2 + b_0(t-s)) - a_0}{a_0} + \frac{b(x_1 + a_0(t-s), x_2 + b_0(t-s)) - b_0}{b_0} \right\} \\
 & \quad \times e^{-\delta(t-s)} v(s, x_1 + a_0(t-s), x_2 + b_0(t-s)) ds.
 \end{aligned}$$

Multiplying (12) by the factor $e^{\delta t}$ gives

$$\begin{aligned}
 e^{\delta t} v(t, x_1, x_2) = & \phi(x_1 + a_0 t, x_2 + b_0 t) \\
 & - \left\{ \frac{a(x_1, x_2) - a_0}{a_0} + \frac{b(x_1, x_2) - b_0}{b_0} \right\} e^{\delta t} v(t, x_1, x_2) \\
 & + \left\{ \frac{a(x_1 + a_0 t, x_2 + b_0 t) - a_0}{a_0} + \frac{b(x_1 + a_0 t, x_2 + b_0 t) - b_0}{b_0} \right\} \phi(x_1 + a_0 t, x_2 + b_0 t) \\
 & + \delta \int_0^t e^{\delta s} \left\{ \frac{a(x_1 + a_0(t-s), x_2 + b_0(t-s)) - a_0}{a_0} + \frac{b(x_1 + a_0(t-s), x_2 + b_0(t-s)) - b_0}{b_0} \right\} \\
 & \quad \times v(s, x_1 + a_0(t-s), x_2 + b_0(t-s)) ds \\
 & + \int_0^t e^{\delta s} \frac{d}{ds} \left\{ \frac{a(x_1 + a_0(t-s), x_2 + b_0(t-s)) - a_0}{a_0} + \frac{b(x_1 + a_0(t-s), x_2 + b_0(t-s)) - b_0}{b_0} \right\} \\
 & \quad \times v(s, x_1 + a_0(t-s), x_2 + b_0(t-s)) ds.
 \end{aligned}$$

Applying the substitution

$$z(t, x_1, x_2) = e^{\delta t} v(t, x_1, x_2), \quad (13)$$

and using condition (6), we obtain

$$\begin{aligned}
 z(t, x_1, x_2) = & \phi(x_1 + a_0 t, x_2 + b_0 t) - \alpha(x_1, x_2) z(t, x_1, x_2) \\
 & + \alpha(x_1 + a_0 t, x_2 + b_0 t) \phi(x_1 + a_0 t, x_2 + b_0 t) \\
 & + \delta \int_0^t \alpha(x_1 + a_0(t-s), x_2 + b_0(t-s)) z(s, x_1 + a_0(t-s), x_2 + b_0(t-s)) ds \\
 & + \int_0^t \frac{d}{ds} \alpha(x_1 + a_0(t-s), x_2 + b_0(t-s)) z(s, x_1 + a_0(t-s), x_2 + b_0(t-s)) ds.
 \end{aligned}$$

Therefore,

$$\begin{aligned} (1 + \alpha(x_1, x_2))z(t, x_1, x_2) &= \phi(x_1 + a_0t, x_2 + b_0t)(1 + \alpha(x_1 + a_0t, x_2 + b_0t)) \\ &+ \delta \int_0^t \alpha(x_1 + a_0(t-s), x_2 + b_0(t-s))z(s, x_1 + a_0(t-s), x_2 + b_0(t-s)) ds \\ &+ \int_0^t \frac{d}{ds} \alpha(x_1 + a_0(t-s), x_2 + b_0(t-s))z(s, x_1 + a_0(t-s), x_2 + b_0(t-s)) ds. \end{aligned}$$

Equivalently, dividing by $1 + \alpha(x_1, x_2)$, we have

$$\begin{aligned} z(t, x_1, x_2) &= \frac{1}{1 + \alpha(x_1, x_2)} \left\{ \phi(x_1 + a_0t, x_2 + b_0t)(1 + \alpha(x_1 + a_0t, x_2 + b_0t)) \right. \\ &+ \delta \int_0^t \alpha(x_1 + a_0(t-s), x_2 + b_0(t-s))z(s, x_1 + a_0(t-s), x_2 + b_0(t-s)) ds \\ &\left. + \int_0^t \frac{d}{ds} \alpha(x_1 + a_0(t-s), x_2 + b_0(t-s))z(s, x_1 + a_0(t-s), x_2 + b_0(t-s)) ds \right\}. \end{aligned}$$

Combining the integrals, we get

$$\begin{aligned} z(t, x_1, x_2) &= \frac{1}{1 + \alpha(x_1, x_2)} \left\{ \phi(x_1 + a_0t, x_2 + b_0t)(1 + \alpha(x_1 + a_0t, x_2 + b_0t)) \right. \\ &+ \int_0^t \left(\delta \alpha(x_1 + a_0(t-s), x_2 + b_0(t-s)) + \frac{d}{ds} \alpha(x_1 + a_0(t-s), x_2 + b_0(t-s)) \right) \\ &\left. \times z(s, x_1 + a_0(t-s), x_2 + b_0(t-s)) ds \right\}. \end{aligned} \quad (14)$$

Using the triangle inequality and applying the maximum norm $\|\cdot\|_{C(\Omega)}$ to (14), we obtain

$$\|z(t)\|_{C(\Omega)} \leq M_1 M_2 \|\phi\|_{C(\Omega)} + M_1 L \int_0^t \|z(s)\|_{C(\Omega)} ds.$$

Applying Grönwall's inequality, we get

$$\|z(t)\|_{C(\Omega)} \leq M \|\phi\|_{C(\Omega)}, \quad 0 \leq t \leq T.$$

From this inequality and (13), it follows that

$$\|v(t)\|_{C(\Omega)} \leq M e^{-\delta t} \|\phi\|_{C(\Omega)}.$$

Therefore,

$$\left\| e^{-tA(x_1, x_2)} \right\|_{C(\Omega) \rightarrow C(\Omega)} \leq M e^{-\delta t}.$$

This establishes the result as stated in Theorem 1. □

Remark 1. The operator $I - e^{-TA(x_1, x_2)}$ is invertible for $\delta > \frac{1}{T} \ln M$, and the following estimate holds

$$\|(I - e^{-TA(x_1, x_2)})^{-1}\|_{C(\Omega) \rightarrow C(\Omega)} \leq \frac{1}{1 - M e^{-T\delta}}. \quad (15)$$

3 The main theorem

This section establishes stability estimates for the space-dependent source identification problem associated with the two-dimensional neutron transport equation with spatially variable coefficients and periodic boundary conditions, as given in (1).

Theorem 2. Assume that the conditions of Theorem 1 are satisfied. The following estimates hold

$$\|u\|_{C(C(\Omega))} + \left\| \left(A^{(x_1, x_2)} \right)^{-1} p \right\|_{C(\Omega)} \leq M_3 \left(\|\psi_1\|_{C(\Omega)} + \|\psi_2\|_{C(\Omega)} + \|f\|_{C(C(\Omega))} \right),$$

and

$$\begin{aligned} \|u_t\|_{C(C(\Omega))} + \|u\|_{C(C^{(1)}(\Omega))} + \|p\|_{C(\Omega)} &\leq M_3 \left(\|\psi_1\|_{C^{(1)}(\Omega)} + \|\psi_2\|_{C^{(1)}(\Omega)} \right. \\ &\quad \left. + \min \left\{ \|f\|_{C(C^{(1)}(\Omega))}, \|f(0)\|_{C(\Omega)} + \|f'\|_{C(C(\Omega))} \right\} \right). \end{aligned}$$

Here, M_3 depends only on T_1, T_2, δ, M .

Proof. We can rewrite problem (1) in the following abstract form:

$$\begin{cases} u'(t) + Au(t) = p + f(t), & t \in (0, T), \\ u(T_1) = u(0) + \psi_1, & T_1 \in (0, T), \\ u(T_2) = \psi_2, & T_2 \in (0, T], \end{cases} \quad (16)$$

in a Banach space $E = C(\Omega)$ with a positive operator $A = A^{(x_1, x_2)}$ defined by (2). The proof of Theorem 2 is based on the positivity of the operator A in $E = C(\Omega)$ and the following abstract theorem. $\square$

Theorem 3. Let $\psi_1, \psi_2 \in D(A)$ and $f \in C(D(A)) \cap C^{(1)}(E)$. The following estimates are valid for the solution of problem (1)

$$\|u(t)\|_E + \|A^{-1}p\|_E \leq M_4 (\|\psi_1\|_E + \|\psi_2\|_E + \|f\|_{C(E)}),$$

and

$$\begin{aligned} \|u_t\|_{C(E)} + \|u\|_{C(E)} + \|p\|_E &\leq M_4 \left(\|\psi_1\|_{D(A)} + \|\psi_2\|_{D(A)} \right. \\ &\quad \left. + \min \{ \|f\|_{C(D(A))}, \|f(0)\|_E + \|f'\|_{C(E)} \} \right). \end{aligned} \quad (17)$$

Here, M_4 depends only on T_1, T_2, δ, M .

Proof. Applying the formula (16), we obtain

$$e^{-(t-s)A} \frac{du(s)}{ds} + e^{-(t-s)A} Au(s) = e^{-(t-s)A} (p + f(s))$$

for $0 \leq s \leq t$. Then, taking the integral with respect to s from 0 to t , we get

$$u(t) = e^{-tA} u(0) + \int_0^t e^{-(t-s)A} (p + f(s)) ds. \quad (18)$$

Evaluating (18) at $t = T_1$ and using $u(T_1) = u(0) + \psi_1$ yields

$$u(0) + \psi_1 = e^{-T_1 A} u(0) + \int_0^{T_1} e^{-(T_1-s)A} (p + f(s)) ds,$$

which can be rearranged as

$$(I - e^{-T_1 A})u(0) = -\psi_1 + \int_0^{T_1} e^{-(T_1-s)A} f(s) ds + (I - e^{-T_1 A})A^{-1}p.$$

Since $I - e^{-T_1 A}$ is invertible, we obtain

$$u(0) = -(I - e^{-T_1 A})^{-1}\psi_1 + (I - e^{-T_1 A})^{-1} \int_0^{T_1} e^{-(T_1-s)A} f(s) ds + A^{-1}p. \quad (19)$$

Now evaluating (18) at $t = T_2$ and using $u(T_2) = \psi_2$ yields

$$\psi_2 = e^{-T_2 A}u(0) + \int_0^{T_2} e^{-(T_2-s)A} f(s) ds + (I - e^{-T_2 A})A^{-1}p. \quad (20)$$

Substituting (19) into (20) gives

$$\begin{aligned} \psi_2 = & -e^{-T_2 A}(I - e^{-T_1 A})^{-1}\psi_1 + e^{-T_2 A}(I - e^{-T_1 A})^{-1} \int_0^{T_1} e^{-(T_1-s)A} f(s) ds \\ & + \int_0^{T_2} e^{-(T_2-s)A} f(s) ds + A^{-1}p. \end{aligned}$$

Solving for $A^{-1}p$, we obtain

$$\begin{aligned} A^{-1}p = & \psi_2 + e^{-T_2 A}(I - e^{-T_1 A})^{-1}\psi_1 - e^{-T_2 A}(I - e^{-T_1 A})^{-1} \int_0^{T_1} e^{-(T_1-s)A} f(s) ds \\ & - \int_0^{T_2} e^{-(T_2-s)A} f(s) ds. \end{aligned} \quad (21)$$

After taking the norm in E and applying the triangle inequality, we obtain

$$\|A^{-1}p\|_E \leq M_5 \left(\|\psi_1\|_E + \|\psi_2\|_E + \|f\|_{C(E)} \right), \quad (22)$$

where M_5 is defined by

$$M_5 = \max \left\{ 1, \frac{Me^{-\delta T_2}}{1 - Me^{-\delta T_1}}, \frac{M(2 - e^{-\delta T_1})}{\delta} \right\}.$$

From (21), we can obtain

$$\begin{aligned} p = & A\psi_2 + e^{-T_2 A}(I - e^{-T_1 A})^{-1}A\psi_1 - e^{-T_2 A}(I - e^{-T_1 A})^{-1} \int_0^{T_1} e^{-(T_1-s)A} Af(s) ds \\ & - \int_0^{T_2} e^{-(T_2-s)A} Af(s) ds. \end{aligned}$$

Taking norms in E and using the triangle inequality, we obtain

$$\begin{aligned} \|p\|_E \leq & \|A\psi_2\|_E + \|e^{-T_2 A}(I - e^{-T_1 A})^{-1}A\psi_1\|_E \\ & + \left\| e^{-T_2 A}(I - e^{-T_1 A})^{-1} \int_0^{T_1} e^{-(T_1-s)A} Af(s) ds \right\|_E + \left\| \int_0^{T_2} e^{-(T_2-s)A} Af(s) ds \right\|_E. \end{aligned}$$

By using the estimates (7) and (15), we obtain

$$\|p\|_E \leq \|\psi_2\|_{D(A)} + \frac{Me^{-\delta T_2}}{1 - Me^{-\delta T_1}} \|\psi_1\|_{D(A)} + \left(\frac{M^2 e^{-\delta T_2} (1 - e^{-\delta T_1})}{\delta (1 - Me^{-\delta T_1})} + \frac{M(1 - e^{-\delta T_2})}{\delta} \right) \|f\|_{C(D(A))}.$$

Therefore, it follows that

$$\|p\|_E \leq M_6 \left(\|\psi_1\|_{D(A)} + \|\psi_2\|_{D(A)} + \|f\|_{C(D(A))} \right). \quad (23)$$

Here, M_6 depends only on T_1, T_2, δ, M .

Now using (20) and substituting (19), we obtain

$$\begin{aligned} (I - e^{-T_2 A})u(0) &= - (I - e^{-T_1 A})^{-1} \psi_1 + (I - e^{-T_1 A})^{-1} \int_0^{T_1} e^{-(T_1-s)A} f(s) ds - \psi_2 \\ &\quad + \int_0^{T_2} e^{-(T_2-s)A} f(s) ds + (2I - e^{-T_2 A})A^{-1}p. \end{aligned}$$

Since $I - e^{-T_2 A}$ is invertible, we obtain

$$\begin{aligned} u(0) &= (I - e^{-T_2 A})^{-1} \left[- (I - e^{-T_1 A})^{-1} \psi_1 + (I - e^{-T_1 A})^{-1} \int_0^{T_1} e^{-(T_1-s)A} f(s) ds - \psi_2 \right. \\ &\quad \left. + \int_0^{T_2} e^{-(T_2-s)A} f(s) ds + (2I - e^{-T_2 A})A^{-1}p \right]. \end{aligned}$$

By applying the norm in E and using the triangle inequality, we obtain

$$\begin{aligned} \|u(0)\|_E &\leq \frac{1}{1 - Me^{-\delta T_2}} \left(\frac{1}{1 - Me^{-\delta T_1}} \|\psi_1\|_E + \frac{M(1 - e^{-\delta T_1})}{\delta(1 - Me^{-\delta T_1})} \|f\|_{C(E)} \right. \\ &\quad \left. + \|\psi_2\|_E + \frac{M(1 - e^{-\delta T_2})}{\delta} \|f\|_{C(E)} + (2 + Me^{-\delta T_2}) \|A^{-1}p\|_E \right). \end{aligned}$$

Therefore, it follows that

$$\|u(0)\|_E \leq M_7 \left(\|\psi_1\|_E + \|\psi_2\|_E + \|f\|_{C(E)} + \|A^{-1}p\|_E \right). \quad (24)$$

Here, M_7 depends only on T_1, T_2, δ, M .

Substituting the estimate for (22) into the inequality for (24) gives

$$\|u(0)\|_E \leq M_8 (\|\psi_1\|_E + \|\psi_2\|_E + \|f\|_{C(E)}),$$

where the constant M_8 is depending only on T_1, T_2, δ, M .

Rewriting the term involving p in (18) leads to

$$u(t) = e^{-tA}u(0) + \int_0^t e^{-(t-s)A} f(s) ds + A^{-1}(I - e^{-tA})p. \quad (25)$$

By applying the norm in E and using the triangle inequality, we obtain

$$\|u(t)\|_E \leq \|e^{-tA}\|_{E \rightarrow E} \|u(0)\|_E + \int_0^t \|e^{-(t-s)A}\|_{E \rightarrow E} \|f(s)\|_E ds + \|A^{-1}(I - e^{-tA})p\|_E.$$

From the semigroup estimate (7), the inequality $\|I - e^{-tA}\|_E \leq 1 + Me^{-\delta t} \leq 1 + M$, and estimate

$$\int_0^t \|e^{-(t-s)A}\|_{E \rightarrow E} \|f(s)\|_E ds \leq \frac{M}{\delta} \|f\|_{C(E)}$$

it follows that

$$\|u(t)\|_E \leq M\|u(0)\|_E + \frac{M}{\delta} \|f\|_{C(E)} + (1 + M)\|A^{-1}p\|_E.$$

Substituting the estimate for (22) into the inequality for (24) yields

$$\|u(t)\|_E \leq M_9(\|\psi_1\|_E + \|\psi_2\|_E + \|f\|_{C(E)}). \quad (26)$$

Here, M_9 depends only on T_1, T_2, δ, M . Now we take derivative of formula (25)

$$u_t(t) = -e^{-tA}Au(0) + f(t) - \int_0^t e^{-(t-s)A}Af(s) ds + e^{-tA}p. \quad (27)$$

By applying the norm in E and using the triangle inequality, we obtain

$$\begin{aligned} \|u_t(t)\|_E &\leq \|e^{-tA}\|_{E \rightarrow E} \|Au(0)\|_E + \|A^{-1}\|_{E \rightarrow E} \|Af(t)\|_E + \|e^{-tA}\|_{E \rightarrow E} \|p\|_E \\ &\quad + \int_0^t \|e^{-(t-s)A}\|_{E \rightarrow E} \|Af(s)\|_E ds. \end{aligned} \quad (28)$$

It is easy to see that

$$\int_0^t \|e^{-(t-s)A}\|_{E \rightarrow E} \|Af(s)\|_E ds \leq \frac{M}{\delta} (1 - e^{-\delta t}) \|f\|_{C(D(A))}. \quad (29)$$

Substituting the semigroup estimate $\|e^{-tA}\|_{E \rightarrow E} \leq Me^{-\delta t}$ from (7) and (29) into (28) yields

$$\|u_t(t)\|_E \leq Me^{-\delta t} \|Au(0)\|_E + \|A^{-1}\|_{E \rightarrow E} \|Af(t)\|_E + Me^{-\delta t} \|p\|_E + \frac{M}{\delta} (1 - e^{-\delta t}) \|f\|_{C(D(A))}, \quad (30)$$

for all $t \in [0, T]$. Applying integration by parts to the integral term in (27), we obtain

$$\int_0^t e^{-(t-s)A}Af(s) ds = f(t) - e^{-tA}f(0) - \int_0^t e^{-(t-s)A}f'(s) ds,$$

and then equation (27) becomes

$$u_t(t) = e^{-tA}(p + f(0) - Au(0)) + \int_0^t e^{-(t-s)A}f'(s) ds.$$

By applying the norm in E and using the triangle inequality, we obtain

$$\|u_t(t)\|_E \leq Me^{-\delta t} \|Au(0)\|_E + Me^{-\delta t} \|f(0)\|_E + Me^{-\delta t} \|p\|_E + \frac{M}{\delta} (1 - e^{-\delta t}) \|f'\|_{C(E)}, \quad (31)$$

for all $t \in [0, T]$. Combining (30) and (31) with the estimates for $\|u\|_{C(E)}$ and $\|p\|_E$ in (23) and (26) and taking the maximum norm yields inequality (17).

This establishes the result as stated in Theorem 3. $\square$

Conclusion

This article investigated the space-dependent source identification problem for the 2D NTE with spatially variable coefficients and periodic boundary conditions. We first presented a fundamental formula for an explicit solution to the homogeneous problem with constant coefficients. Using this result, we then established a stability estimate for the homogeneous problem with variable coefficients. Finally, we employed these findings to derive stability estimates for the space-dependent source identification problem in the two-dimensional periodic transport equation with variable coefficients.

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Author contributions

All authors contributed equally to this work.

Conflict of interest

The authors declare no conflict of interest.

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*Author information**

Allaberen Ashyralyev — Doctor of Physical and Mathematical Sciences, Professor, Department of Mathematics, Bahcesehir University, Istanbul 34353, Turkey; e-mail: aallaberen@gmail.com; <https://orcid.org/0000-0002-4153-6624>

Abdulgafur Taskin (*corresponding author*) — Doctor of Mathematical Sciences, PhD, Department of Mathematics, ENKA Schools, Istanbul, Turkey; e-mail: gafurtaskin@hotmail.com; <https://orcid.org/0000-0001-7432-7450>

*Authors' names are presented in the following order: first name, middle name (if any), last name.