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Fractal completeness and compactness in first-order model theory

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This article develops a formal framework for fractal structures within classical first-order model theory. The notion of fractality is reformulated in purely logical terms by replacing metric self-similarity with logical self-similarity induced by elementary endomorphisms of structures. A hierarchy of morphisms is introduced, including endomorphisms, elementary endomorphisms, and endiks, which preserve the truth of formulas in one or both directions. Based on these morphisms, fractal subsets and fractal models are defined via finite families of elementary self-maps. On the syntactic level, fractality is expressed through finite systems of T-elementary syntactic endomorphisms generating a stabilization process called the fractal corridor (a sequence of theories generated by iterated application of syntactic endomorphisms). A compatibility condition between syntactic and semantic fractality is formulated and proved. Using a Henkin-type construction, syntactic operators are lifted to semantic self-maps of a canonical model, yielding fractal completeness. A corresponding compactness theorem is also established. All constructions are carried out within standard first-order logic.

Keywords: first-order logic, fractal model, elementary endomorphism, syntactic fractality, semantic fractality, fractal completeness, fractal compactness, model theory.

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Introduction

Fractal structures originally arose in geometry as self-similar sets constructed via finite systems of contractions. In classical model theory, no metric structure is available; therefore, if fractality is to be formulated in logical terms, self-similarity must be expressed through structure-preserving self-maps. Traditionally, fractal structures are investigated within the framework of geometry as self-similar sets formed through iterated function systems of contractions, as detailed in the classical works of [1–3] and [4], as well as in the context of hierarchical and tree-like structures [5]. However, in classical model theory, the metric structure is absent, necessitating a reformulation of the notion of self-similarity in logical terms. In this paper, this transition is achieved by replacing metric self-similarity with logical self-similarity induced by elementary endomorphisms of structures. This approach allows fractality to be viewed as an internal structural condition imposed on theories themselves, based on the fundamental principles of constructing first-order models as laid out by [6] and [7]. The guiding principle of this paper is the replacement of metric self-similarity by logical self-similarity. Instead of contractions, we consider elementary endomorphisms of structures [7, 8]. Using finite families of such maps, fractal subsets and fractal models are defined. On the syntactic side, fractality is encoded via finite systems of T-elementary syntactic endomorphisms generating a stabilization mechanism called the fractal corridor [9].

A compatibility condition between semantic and syntactic systems is introduced. Under this condition, fractal-proof coincides with truth in compatible fractal models. A Henkin-type construction

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[6, 10] yields a canonical fractal model, establishing fractal completeness. Compactness follows by a finitary reduction argument.

The paper is organized as follows. Section 1 introduces the morphisms underlying fractal self-similarity. Section 2 defines fractal subsets and models. Section 3 develops syntactic fractality and the fractal corridor. Section 4 establishes the agreement theorem. Sections 5 and 6 contain the completeness and compactness of the fractal.

1 Morphisms in fractal model theory

Let L be a first-order signature and M an L-structure. A crucial aspect of formalizing fractality is the introduction of a hierarchy of mappings: endomorphisms, elementary endomorphisms, and "endiks". The use of the latter ensures the preservation of formula truth in both directions, which is fully consistent with the modern theory of morphisms by [7]. The proposed method extends the research on the semantic properties of models and fragments of Jonsson theories presented in the works of [8] and [11]. Within this paradigm, a fractal model is defined as a structure that constitutes a finite union of elementary images of itself. Conceptually, this correlates with the Iterated Function Systems (IFS) of [3] and [4], but implemented at the level of logical semantics.

Definition 1. [6] A mapping $f : M \to M$ is an endomorphism if it preserves functions and relations in the forward direction.

Definition 2. [6, 7] An endomorphism f is elementary if for every formula $\varphi(\bar{x})$ and tuple $\bar{a}$, $M \models \varphi(\bar{a}) \Rightarrow M \models \varphi(f(\bar{a}))$.

Definition 3. An elementary endomorphism is an *endik* if $M \models \varphi(\bar{a}) \iff M \models \varphi(f(\bar{a}))$.

Proposition 1. If f is an endik, then $f[M]$ is an elementary substructure of M and f is an isomorphism onto its image.

Example 1. In $(\mathbb{Q}, <)$, the maps $f_0(x) = x/2,\ f_1(x) = (x + 1)/2$ are elementary endomorphisms.

Example 2. (An endomorphism which is not elementary) Let $M = (Z, +)$ and define $f(x) = 2x$. Then f is an endomorphism of M, but not elementary.

Consider the formula

$$\psi(x) = \exists y(y + y = x).$$

Then $M \models \neg\psi(1)$, while $M \models \psi(f(1))$, since $f(1) = 2 = 1 + 1$. Thus, truth is not preserved under f, so f is not elementary.

Example 3. (Elementary endomorphism with proper image) Let V be an infinite-dimensional vector space over a field K, considered in the language of K-vector spaces. Fix a basis $(e_i)_{i\in\mathbb{N}}$ and define the shift

$$s(e_i) = e_{i+1},$$

extended linearly. Then s is an injective endomorphism with proper image. Since the theory of vector spaces over K admits quantifier elimination [6], s is elementary.

Example 4. (Endiks in the pure equality language) Let $M = (U; =)$ be an infinite structure in the language consisting only of equality. Every injective self-map $f : U \to U$ is an endik, since all formulas reduce to Boolean combinations of equalities among variables.

These examples show that [7, 8]

$$\text{endomorphism} \subsetneq \text{elementary endomorphism} \subsetneq \text{endik}.$$

2 Fractal subsets and fractal models

Let $n \neq 1$.

Definition 4. A subset $X \subseteq M$ is *fractal* if

$$X = \bigcup_{i=1}^{n} f_i[X]$$

for a finite family $\{f_1, \ldots, f_n\}$ of elementary endomorphisms of M where $n \geq 1$ [3,4].
If $n = 1$, the map f_1 must not be an automorphism.

Definition 5. A structure M is called *fractal* if

$$M = \bigcup_{i=1}^{n} f_i[M]$$

for a finite family $\{f_1, \ldots, f_n\}$ of elementary endomorphisms of M where $n \geq 1$ [3,4].
If $n = 1$, the map f_1 must not be an automorphism.

Remark 1. Degenerate one-map case. In Definitions 4 and 5, if the family consists of a single map ($n = 1$), this map must not be an automorphism. Otherwise, the equalities

$$X = f[X] \quad \text{or} \quad M = f[M]$$

become tautological and do not express genuine fractal self-similarity.

Proposition 2. If each f_i is an endik, then each image $f_i[M]$ is an elementary substructure of M [6,7].

3 Syntactic fractality and the fractal corridor

Let T be a first-order L-theory.
Convention. All syntactic constructions are considered relative to a fixed finite family

$$\Psi = \{\sigma_1, \ldots, \sigma_m\}.$$

If $m = 1$, the operator σ_1 must be non-trivial, i.e. it must not act as an identity or as a purely automorphic symmetry preserving all sentences tautologically.

At the syntactic level, fractality generates a sequence of theories called the “fractal corridor”. Unlike geometric fractals, where the key parameters are measured and set dimension [2,3,5], stabilization in logical systems is considered on fragments of limited quantifier depth. The analysis of formula complexity and its fragments is a standard in modern model theory, allowing the application of finitary proof methods to infinite syntactic chains [9,12].

Let Form_L denote the set of all L-formulas.

Definition 6. A mapping $\sigma : L \to L$ is called a syntactic endomorphism if it induces a map on terms and on quantifier-free formulas such that:

1. For every function symbol f, $\sigma(f(t_1, \ldots, t_n)) = f(\sigma(t_1), \ldots, \sigma(t_n))$.
2. For every predicate symbol P, $\sigma(P(t_1, \ldots, t_n)) = P(\sigma(t_1), \ldots, \sigma(t_n))$.
3. Boolean structure is preserved: $\sigma(\neg\varphi) = \neg\sigma(\varphi)$, $\quad \sigma(\varphi \wedge \psi) = \sigma(\varphi) \wedge \sigma(\psi)$, and similarly for other connectives.

Definition 7. A syntactic endomorphism σ is called elementary if it extends to all formulas of L by the rules

$$\sigma(\forall x\,\varphi) = \forall x\,\sigma(\varphi), \quad \sigma(\exists x\,\varphi) = \exists x\,\sigma(\varphi),$$

and therefore acts on the whole set of sentences of L.

Definition 8. Let T be a theory in L. An elementary syntactic endomorphism σ is called T-elementary if for every sentence φ

$$T \vdash \varphi \quad \Longleftrightarrow \quad T \vdash \sigma(\varphi).$$

This condition is the syntactic analogue of endik symmetry [7, 8].

Definition 9. Let T be a theory in a first-order language L, and let

$$\Psi = \{\sigma_1, \dots, \sigma_m\}$$

be a finite family of T-elementary syntactic endomorphisms. Define the fractal corridor recursively by

$$T[0] = T, \quad T[k+1] = \mathrm{Th}\Big(\bigcup_{j=1}^{m} \sigma_j(T[k])\Big), \quad k \geq 0.$$

The sequence T[k] is called the fractal corridor generated by Ψ.

A theory T is called syntactically fractal with respect to Ψ if there exists $n \geq 1$ such that $T[n] = T$.

Thus, every sentence of T is obtained as the image of some sentence of T under one of the operators in Ψ.

Definition 10. A sentence φ is called *fractal-proofed* from T relative to Ψ if there exists $k \in N$ such that $\varphi \in T[k]$.

Lemma 1. (Monotonicity dichotomy). Let $\Psi = \{\sigma_1, \dots, \sigma_m\}$ be a finite family of syntactic endomorphisms and define $\{T[k]\}$ by

$$T[0] = T, \quad T[k+1] = \mathrm{Th}\Big(\bigcup_{j=1}^{m} \sigma_j(T[k])\Big).$$

(1) If the identity operator belongs to Ψ, then

$$T[k] \subseteq T[k+1] \quad \text{for all } k.$$

(2) If the identity operator does not belong to Ψ, monotonicity may fail; however, each $T[k]$ is a theory.

Proof. (1) Assume $id \in \Psi$. Let $\phi \in T[k]$. Then

$$\phi = id(\phi) \in \bigcup_{j=1}^{m} \sigma_j(T[k]).$$

Hence

$$\phi \in \mathrm{Th}\Big(\bigcup_{j=1}^{m} \sigma_j(T[k])\Big) = T[k+1],$$

so $T[k] \subseteq T[k+1]$.

(2) For the general case, by definition

$$T[k+1] = \mathrm{Th}\Big(\bigcup_{j=1}^{m} \sigma_j(T[k])\Big),$$

so $T[k+1]$ is closed deductively. Since $T[0] = T$ is a theory, by induction it follows that every $T[k]$ is a theory. □

This completes the syntactic framework. In the next section, we establish the agreement theorem connecting syntactic fractality with semantic fractal models.

Lemma 2. Let σ be a T-elementary syntactic endomorphism. If $T \vdash \varphi$, then $T \vdash \sigma(\varphi)$.

Proof. If $T \vdash \varphi$, then $T \vdash \forall \bar{x}\, (\varphi \leftrightarrow \top)$. Since σ respects T-equivalence, $T \vdash \forall \bar{x}\, (\sigma(\varphi) \leftrightarrow \sigma(\top))$. But $\sigma(\top) = \top$, hence $T \vdash \sigma(\varphi)$. □

Proposition 3. For every $k \in \mathbb{N}$, the set $T[k]$ is a theory.

Proof. This follows immediately from the definition of $T[k+1]$ as a deductive closure. □

Recall that the corridor levels are theories defined by

$$T[0] = T, \qquad T[k+1] = \mathrm{Th}\Big(\bigcup_{j=1}^{m} \sigma_j(T[k])\Big).$$

We define the limit set of sentences

$$T[\infty] := \bigcup_{k \in \mathbb{N}} T[k].$$

Lemma 3. Assume that each $T[k]$ is consistent. Then $T[\infty]$ is consistent.

Proof. If $T[\infty]$ were inconsistent, then some finite set $\Delta \subseteq T[\infty]$ would be inconsistent. Since Δ is finite, there exists k_0 such that $\Delta \subseteq T[k_0]$. Hence $T[k_0]$ is inconsistent, which contradicts the assumption. □

Remark 2. Global stabilization of the entire chain $\{T[k]\}$ does not need to be in first-order logic. The correct finitary substitute is stabilization on bounded syntactic fragments [9], which is captured by Theorem 1. Thus, Theorem 2 provides finitary control of the corridor, while $T[\infty]$ serves as the global limit object used in the Henkin construction.

Corollary 1. If each $T[k]$ is consistent, then $T[\infty]$ admits a Henkin extension $\widehat{T} \supseteq T[\infty]$ and therefore has a term model $M_{\widehat{T}}$.

Proof. By Lemma 3, $T[\infty]$ is consistent. The standard Henkin [6, 10] extension procedure applies to any consistent set of sentences. □

To formulate a strict stabilization statement, we work with the quantifier depth $\mathrm{qd}(\varphi)$ of formulas.

Definition 11. For $N \in \mathbb{N}$, let $\mathrm{Sent}_{\leq N}$ be the set of all L-sentences φ with $\mathrm{qd}(\varphi) \leq N$. Define the N-fragment of $T[k]$ by

$$T[k]{\leq N} := T[k] \cap \mathrm{Sent}_{\leq N}.$$

Lemma 4. For every $j \leq m$ and every sentence φ, $\mathrm{qd}(\sigma_j(\varphi)) = \mathrm{qd}(\varphi)$.

Proof. By Definition 6, each σ_j commutes with $\exists$ and $\forall$ and preserves logical form. Hence quantifier nesting is preserved. □

Theorem 1. (Finite stabilization on bounded depth). For every $N \in \mathbb{N}$ there exists $k(N) \in \mathbb{N}$ such that

$$T[k(N)]{\leq N} = T[k(N)+1]{\leq N}.$$

Proof. (By induction on N).

Base $N = 0$. Sentences of quantifier depth 0 are Boolean combinations of atomic sentences. By Definition 6, each σ_j preserves Boolean structure and does not introduce quantifiers, hence acts inside Sent≤ 0. Therefore the sequence $T[k]\leq 0$ is an ascending chain of subsets of $\mathrm{Sent}_{\leq 0}$. Since at each step we apply only finitely many operators to already obtained sentences (and do not increase depth), there is a stage $k(0)$ after which no new depth-0 sentences appear. Hence stabilization holds.

Inductive step. Assume the claim holds for N. Consider depth $N + 1$. Any sentence φ of depth $\leq N + 1$ is built from:

- Boolean combinations of sentences of depth $\leq N + 1$, and
- quantified sentences $\exists x\, \psi(x)$ or $\forall x\, \psi(x)$, where $\mathrm{qd}(\psi) \leq N$.

By Lemma 4, each σ_j preserves quantifier depth and commutes with quantifiers:

$$\sigma_j(\exists x\, \psi) = \exists x\, \sigma_j(\psi), \qquad \sigma_j(\forall x\, \psi) = \forall x\, \sigma_j(\psi).$$

Hence, to generate new sentences of depth $\leq N + 1$ at stage $k + 1$, it suffices to generate new inner formulas of depth $\leq N$ at stage k. By the inductive hypothesis, there exists $k(N)$ such that $T[k(N)]\leq N = T[k(N) + 1]\leq N$. From that stage onward, applying σ_j cannot produce new depth$\leq N$ components, hence cannot produce genuinely new quantified sentences of depth $\leq N+1$, nor new Boolean combinations thereof. Therefore stabilization holds at depth $N+1$ for some $k(N+1)$. □

4 Agreement between syntactic and semantic fractality

Let T be a first-order theory and $M \models T$.

Let

$$\Phi = \{f_1, \ldots, f_n\}$$

be a finite family of elementary endomorphisms of M [6, 7], and

$$\Psi = \{\sigma_1, \ldots, \sigma_m\}$$

a finite family of T-elementary syntactic endomorphisms.

The purpose of this section is to formulate a precise compatibility condition under which syntactic fractality corresponds to semantic fractality.

We first formalize the connection between Φ and Ψ.

Definition 12. The pair (Φ, Ψ) is called compatible if for every $j \leq m$ there exists $i \leq n$ such that for every formula $\varphi(\bar{x})$ and every tuple $\bar{a} \in M$,

$$M \models \sigma_j(\varphi)(\bar{a}) \quad \Longleftrightarrow \quad M \models \varphi(f_i(\bar{a})).$$

Thus each syntactic operator corresponds to evaluation along a semantic endomorphism [7, 13].

Let the fractal corridor $T[k]$ be defined relative to Ψ.

Lemma 5. Assume (Φ, Ψ) is compatible and $M \models T$. Then for every k and every sentence $\varphi \in T[k]$, $M \models \varphi$.

Proof. We argue by induction on k. For $k = 0$, the claim holds since $M \models T$.

Assume the claim holds for k. Let $\varphi \in T[k + 1]$ be a *sentence*. By Definition 10,

$$T[k + 1] = \mathrm{Th}\Big(\bigcup_{j=1}^{m} \sigma_j(T[k])\Big),$$

hence φ is a logical consequence of finitely many sentences from $\bigcup_{j=1}^{m} \sigma_j(T[k])$.

By the induction hypothesis, $M \models \psi$ for each sentence $\psi \in T[k]$. By compatibility of the syntactic and semantic operators used in the paper, truth is preserved along the corresponding images, and therefore $M \models \sigma_j(\psi)$ for all such ψ and all $j \leq m$. Consequently, $M \models \varphi$. □

We now obtain the main result of this section.

Theorem 2. (Agreement Theorem). Let T be syntactically fractal with respect to Ψ, and let M be fractal with respect to Φ. Assume (Φ, Ψ) is compatible. Then:

1. Every fractal-proofed sentence from T is true in M.
2. The stabilized corridor $T[\infty]$ is contained in $\mathrm{Th}(M)$.

Proof. (1) Follows immediately from Lemma 5.

(2) By definition,

$$T[\infty] = \bigcup_{k \in N} T[k].$$

By Lemma 5, every sentence in each $T[k]$ is true in M. Hence every sentence in $T[\infty]$ is true in M, and therefore $T[\infty] \subseteq Th(M)$. □

Under compatibility, the syntactic corridor describes iterated evaluation along the finite system of elementary endomorphisms Φ. Stabilization therefore, corresponds to semantic closure under logical self-similarity [8, 11].

This establishes the precise bridge between syntactic and semantic fractality.

5 *Fractal completeness*

In this section we establish a completeness theorem corresponding to the fractal proof mechanism generated by the corridor $T[k]$. The proof is based on a Henkin-type construction in which the finite syntactic system Ψ induces a finite semantic system of elementary self-maps.

The proof of the fractal completeness theorem is based on a modified Henkin construction, detailed by [6, 10]. Within this framework, syntactic operators are canonically lifted to semantic mappings of the term model. This enables the establishment of an equivalence between syntactic fractal consistency and the existence of a corresponding fractal model. The resulting fractal compactness theorem confirms that the developed framework preserves the key properties of classical first-order logic [6, 7], adapting them to the requirements of the theory of self-similar structures [3, 4].

Throughout this section T is a first-order L-theory syntactically fractal with respect to a fixed finite family

$$\Psi = \{\sigma_1, \ldots, \sigma_m\}$$

of T-elementary syntactic endomorphisms.

We begin with the basic preservation property.

Lemma 6. If T is consistent, then each level $T[k]$ is consistent.

Proof. Assume $T[k]$ is consistent. Suppose for contradiction that $T[k+1]$ is inconsistent. Then there exist sentences $\varphi_1, \ldots, \varphi_r \in T[k+1]$ such that

$$\vdash (\varphi_1 \wedge \cdots \wedge \varphi_r) \to \bot.$$

By definition of $T[k+1]$, for each $\ell \leq r$ there exist $j(\ell) \leq m$ and $\psi_\ell \in T[k]$ such that

$$\varphi_\ell = \sigma_{j(\ell)}(\psi_\ell).$$

Consider the conjunction $\psi := \psi_1 \wedge \cdots \wedge \psi_r$. Since each σ_j preserves conjunctions,

$$\sigma_{j(\ell)}(\psi) = \sigma_{j(\ell)}(\psi_1) \wedge \cdots \wedge \sigma_{j(\ell)}(\psi_r).$$

Using the inconsistency of $\varphi_1, \ldots, \varphi_r$ and T-elementarity of the operators, we obtain a contradiction to consistency of $T[k]$. Hence $T[k+1]$ is consistent. The base case $T[0] = T$ is consistent by assumption. □

Let L_H be the Henkin expansion of L: for every formula $\exists x\, \varphi(x, \bar{y})$ we add a new function symbol $c_\varphi(\bar{y})$ intended to witness existence.

Let T_H be the corresponding Henkin extension of $T[\infty]$ obtained by adding all Henkin axioms.

$$\forall \bar{y}\, \big(\exists x\, \varphi(x, \bar{y}) \to \varphi(c_\varphi(\bar{y}), \bar{y})\big).$$

Lemma 7. If $T[\infty]$ is consistent, then the Henkin extension T_H is consistent.

Proof. Standard Henkin construction [6, 10]. □

Let Tm be the set of L_H-terms. Define an equivalence relation:

$$t \equiv s \quad \Longleftrightarrow \quad T_H \vdash t = s.$$

Let M be the term model whose universe is $\mathrm{Tm}/\equiv$, with interpretation given by

$$F^M([t_1], \ldots, [t_n]) := [F(t_1, \ldots, t_n)].$$

By the standard Henkin argument, $M \models T_H$, hence $M \models T[\infty]$, and in particular $M \models T$.

We now lift the syntactic operators to semantic self-maps of the term model. For each $\sigma_j \in \Psi$ define a mapping $f_j : M \to M$ by $f_j([t]) := [\sigma_j(t)]$.

Lemma 8. (Well-definedness). Each f_j is well-defined.

Proof. Assume $[t] = [s]$, i.e. $T_H \vdash t = s$. Then $T_H \vdash (t = s) \leftrightarrow (s = s)$. Since σ_j is T-elementary and preserves equality, it respects provable equivalence, hence $T_H \vdash \sigma_j(t) = \sigma_j(s)$, so $[\sigma_j(t)] = [\sigma_j(s)]$. □

Lemma 9. Each f_j is an elementary endomorphism of M.

Proof. Let $\varphi(\bar{x})$ be any L-formula and $\bar{t}$ a tuple of terms. By the Truth Lemma for the term model [6],

$$M \models \varphi([\bar{t}]) \quad \Longleftrightarrow \quad T_H \vdash \varphi(\bar{t}).$$

Applying σ_j and using preservation of logical form,

$$T_H \vdash \varphi(\bar{t}) \;\Rightarrow\; T_H \vdash \sigma_j(\varphi(\bar{t})) = \sigma_j(\varphi)(\sigma_j(\bar{t})).$$

Hence

$$M \models \sigma_j(\varphi)(f_j([\bar{t}])).$$

This is precisely the elementarity condition for f_j. □

We now show that the resulting family $\Phi = \{f_1, \ldots, f_m\}$ yields a fractal covering of M.

Proposition 4. The term model M is a fractal model with respect to $\Phi = \{f_1, \ldots, f_m\}$, and the pair (Φ, Ψ) is compatible.

Proof. Compatibility follows from the construction:

$$M \models \sigma_j(\varphi)(\bar{a}) \quad \Longleftrightarrow \quad M \models \varphi(f_j(\bar{a})).$$

To show covering, let $[t] \in M$. By construction of the term model, every term is obtained by applying finitely many syntactic operators from Ψ to initial terms. Hence, there exists $j \leq m$ and a term s such that $t = \sigma_j(s)$. Therefore

$$[t] = [\sigma_j(s)] = f_j([s]) \in f_j[M].$$

Thus

$$M = \bigcup_{j=1}^{m} f_j[M],$$

and M is fractal. □

We can now state the completeness theorem.

Theorem 3. (Fractal Completeness). Let T be syntactically fractal with respect to Ψ. For every sentence φ, if

$$M \models \varphi \quad \text{for every fractal model } M \models T \text{ compatible with } \Psi,$$

then φ is fractal-proofed from T.

Equivalently,

$$\varphi \notin \bigcup_k T[k] \quad \Longrightarrow \quad \exists M \models T \text{ fractal and compatible, such that } M \not\models \varphi.$$

Proof. Assume φ is not fractal-proofed from T. Then $T[\infty] \cup \{\neg\varphi\}$ is consistent. By Lemma 7 we obtain a Henkin extension and hence a term model $M \models T[\infty] \cup \{\neg\varphi\}$. By Proposition 4 this model is fractal and compatible. Thus $M \not\models \varphi$. □

6 *Fractal compactness*

In this section we derive a compactness theorem corresponding to the fractal proof system introduced above. The argument follows the classical scheme but uses the fractal-proof mechanism instead of ordinary derivability.

Throughout this section T is syntactically fractal with respect to a fixed finite family Ψ.

Definition 13. A set of sentences Σ is called finitely fractal-consistent relative to T and Ψ if for every finite subset $\Sigma_0 \subseteq \Sigma$,

$$\perp \notin \bigcup_k (T \cup \Sigma_0)[k],$$

where the corridor is defined relative to Ψ.

Theorem 4. (Fractal Compactness).

Let Σ be a set of sentences. If Σ is finitely fractal-consistent relative to T and Ψ, then there exists a fractal model $M \models T \cup \Sigma$ compatible with Ψ.

Proof. Assume Σ is finitely fractal-consistent. Suppose for contradiction that $T \cup \Sigma$ is fractal-inconsistent, i.e. $\perp \in \bigcup_k (T \cup \Sigma)[k]$. Then there exists k such that

$$\perp \in (T \cup \Sigma)[k].$$

Since the corridor construction is finitary at each step and each level depends only on finitely many applications of operators from Ψ, there exists a finite subset $\Sigma_0 \subseteq \Sigma$ such that

$$\perp \in \bigcup_k (T \cup \Sigma_0)[k].$$

This contradicts finite fractal-consistency. Hence $T \cup \Sigma$ is fractal-consistent. By Theorem 3 (Fractal Completeness), there exists a fractal model $M \models T \cup \Sigma$. Compatibility follows from the construction of Section 5. □

We therefore obtain:

Corollary 2. Fractal-proof over T satisfies compactness:

$$\varphi \text{ is fractal-proofed from } T \iff \text{there exists a finite } T_0 \subseteq T \text{ such that } \varphi \text{ is fractal-proofed from } T_0.$$

This completes the proof of fractal compactness.

Conclusion

We have introduced a formal framework for fractality within classical first-order model theory. The central idea consists in replacing metric self-similarity by logical self-similarity generated by finite systems of elementary self-maps. On the semantic side, fractal models are defined as finite unions of elementary images of themselves. On the syntactic side, fractality is expressed through finite systems of T-elementary syntactic endomorphisms generating the fractal corridor.

A precise compatibility condition between the semantic system Φ and the syntactic system Ψ establishes a bridge between model-theoretic self-embeddings and syntactic stabilization. Under this condition, fractal-proof coincides with truth in compatible fractal models, yielding a soundness theorem. Using a Henkin-type construction, syntactic operators are lifted canonically to elementary endomorphisms of the term model, producing a fractal model and establishing fractal completeness. Compactness follows by a finitary reduction argument inside the corridor mechanism.

All constructions remain entirely within standard first-order logic. No extension of the language or modification of classical proof theory is required. Fractality appears as an internal structural condition imposed on theories and models rather than as a new logical system.

This completes the development of fractal completeness and compactness within first-order model theory.

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Author Contributions

All authors contributed equally to this work.

Conflict of Interest

The authors declare no conflict of interest.

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