https://doi.org/10.31489/2026M2/136-148 *Research article*

Cauchy problem for an essentially loaded fractional diffusion equation

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In this paper, we solve the Cauchy problem for a loaded fractional diffusion equation in an infinite strip. The loaded term is defined as the trace of the fractional derivative of the desired solution on a continuous curve lying inside the domain. We consider all three cases of possible distribution of the order of differentiation in the loaded term (μ) and the order of the time-fractional derivative in the principal differential part of the equation (α). In the first case considered ($\alpha > \mu$), the problem under study is reduced to an integral equation. In the second case ($\alpha = \mu$), we obtain a functional equation. In the third case ($\alpha < \mu$), we are dealing with a differential equation. We show that the condition $\alpha > \mu$ ensures the unique solvability of the problem under consideration. In the case of an essentially loaded equation ($\alpha \le \mu$), the problem may lose both uniqueness and solvability. In particular, it is shown that if $\alpha < \mu$, then the problem under consideration ceases to be uniquely solvable, and the corresponding homogeneous problem has infinitely many nontrivial solutions. Moreover, in this case, the solvability requires additional conditions that narrow the set of admissible input data.

Keywords: loaded differential equation, Cauchy problem, essentially loaded equation, fractional diffusion equation, non-uniqueness, moving load, Wright function, Mittag-Leffler function.

2020 Mathematics Subject Classification: 45D05, 35K20.

Introduction

Consider the equation

$$\left(D_{0y}^{\alpha} - \frac{\partial^2}{\partial x^2}\right) u(x,y) = \lambda \left[D_{0y}^{\mu} u(x,y)\right]_{x=z(y)}, \tag{1}$$

where D_{0y}^{σ} denotes the Riemann–Liouville fractional derivative (integral) of order σ with respect to y with origin at the point $y = 0$ [1]; $\lambda \in \mathbb{R}$ and $z(y)$ is a given continuous function, $z : (0,T) \to \mathbb{R}$. Here we assume that $\alpha \in (0,1]$ and $\mu \in \mathbb{R}$.

Equation (1) belongs to the class of loaded differential equations [2–4]. Loaded equations are an important and actively developing section of the modern theory of differential equations [5, 6]. Boundary value problems for loaded equations are considered for equations of parabolic [7], hyperbolic [8–10] type, as well as integro-differential equations [11] and equations of mixed type [12]. Moreover, loaded equations arise in the theory of inverse problems [13], in control problems [14], in numerical methods [15], in modeling [16], etc.

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This research was funded by the Science Committee of the Ministry of Science and Higher Education of the Republic of Kazakhstan (Grant No. AP23488729, 2024–2026.)

Received: 19 September 2025; *Accepted:* 30 March 2026.

The main part of equation (1) is the fractional diffusion operator [17–19]. The loaded term is given in the form of a trace of the derivative of the desired solution on line $x = z(y)$. Loads of this type are usually called moving loads. Another peculiarity of this equation is that μ can be greater than α, that is, the order of differentiation in the loaded term can be greater than the order of the fractional derivative in the principal differential part of the equation. It turns out that this feature affects the unique solvability of problems for the equation (1). In particular, the uniqueness of solution may be violated, and the parameter λ may play the role of a spectral parameter.

This effect of an essentially loaded term was discovered in [20–22]. These works considered parabolic equations with loads differentiated with respect to spatial variables. It has been shown that in problems for equations with a load of this type, a spectrum appears with respect to the coefficient of the load. Among the works close to the present work, we also cite articles [23–25], in which various issues of solvability of problems for loaded heat and fractional diffusion equations were considered. In this paper, we consider the loaded term in the form of a fractional derivative with respect to the time variable. We solve the Cauchy problem for the equation (1) in all three cases of possible distribution of α and μ. We show that the condition $\alpha > \mu$ ensures the unique solvability of the problem under consideration (Theorem 1). In the case of essentially loaded equation ($\alpha \le \mu$), the problem may lose uniqueness or solvability. If $\alpha = \mu$, then the problem ceases to be solvable for $\lambda = 1$ (Theorem 2). When $\alpha < \mu$, the problem loses uniqueness of solution (Section 7). Moreover, in this case, the solvability requires additional conditions that narrow the set of admissible input data of the problem (Theorem 3, Remark 1).

1 *Problem statement*

For a positive σ, the Riemann–Liouville fractional derivative of order σ with origin at $y = 0$ is defined by

$$D^{\sigma}_{0y} g(y) := \frac{\partial^n}{\partial y^n} D^{\sigma-n}_{0y} g(y) \qquad (n-1 < \sigma \le n, \quad n \in \mathbb{N}),$$

where

$$D^{-\gamma}_{0y} g(y) := \frac{1}{\Gamma(\gamma)} \int_0^y g(t)(y-t)^{\gamma-1} dt \qquad (\gamma > 0) \tag{2}$$

is the Riemann–Liouville fractional integral of order γ; it is also assumed that $D^0_{0y} g(y) := g(y)$.

In what follows we use the notations

$$\Omega := \{(x,y) : \, x \in \mathbb{R}, \; y \in (0,T)\} = \mathbb{R} \times (0,T)$$

and

$$\Omega_0 := \{(x,y) : \, x \in \mathbb{R}, \; y \in [0,T)\} = \mathbb{R} \times [0,T).$$

As usual, $AC[0,T)$ stands for the space of absolutely continuous functions on the segment $[0,c]$ for any $c \in (0,T)$.

A function $u(x,y)$ is called a *regular solution* of the equation (1) in Ω if $y^{1-\varepsilon} u(x,y) \in C(\Omega_0)$ for some $\varepsilon > 0$; $D^{\alpha-1}_{0y} u(x,y)$ belongs to $AC[0,T)$ as a function of y for every fixed $x \in \mathbb{R}$; $u(x,y)$ is twice continuously differentiable with respect to $x \in \mathbb{R}$ for every $y \in (0,T)$; and $u(x,y)$ satisfies the equation (1) for all $(x,y) \in \Omega$.

The Cauchy problem for the equation (1) is formulated as follows: *find a regular solution of the equation* (1) *in* Ω *satisfying the initial condition*

$$\lim_{y \to 0} D^{\alpha-1}_{0y} u(x,y) = \tau(x), \qquad x \in \mathbb{R}. \tag{3}$$

2 Auxiliary statements

Consider the function

$$w_\delta(x,y) := \frac{y^{\delta-1}}{2}\phi\left(-\beta,\delta;-\frac{|x|}{y^\beta}\right), \tag{4}$$

where

$$\phi\,(a,b;\,z) := \sum_{k=0}^{\infty}\frac{z^k}{k!\,\Gamma(ak+b)} \qquad (a>-1)$$

is the Wright function [26, 27]. Also, here and in what follows, β means $\frac{\alpha}{2}$, i.e.

$$\beta := \frac{\alpha}{2}. \tag{5}$$

It is known that [27]

$$\phi(-\beta,\delta;-t) = C\,t^{\frac{1-2\delta}{2(1-\beta)}}\exp\left(-\beta^{\frac{\beta}{1-\beta}}(1-\beta)t^{\frac{1}{1-\beta}}\right)\left[1+O\left(t^{-\frac{1}{1-\beta}}\right)\right] \quad (t\to\infty, \quad 0<\beta<1), \tag{6}$$

$$\frac{d}{dx}\,\phi(-\beta,\,\delta;\,x) = \phi(-\beta,\,\delta-\beta;\,x), \tag{7}$$

$$D_{0y}^{\nu}\left[y^{\delta-1}\phi\left(-\beta,\,\delta;\,-\frac{c}{y^\beta}\right)\right] = y^{\delta-\nu-1}\phi\left(-\beta,\,\delta-\nu;\,-\frac{c}{y^\beta}\right) \quad (c>0), \tag{8}$$

and

$$\int_0^\infty \phi\,(-\beta,\,\delta;\,-x)\,dx = \frac{1}{\Gamma(\beta+\delta)}. \tag{9}$$

The asymptotic expansion (6) gives that

$$\left|y^{\delta-1}\phi\left(-\beta,\,\delta;\,-\frac{x}{y^\beta}\right)\right| \le C\,x^{-\theta}y^{\beta\theta+\delta-1}, \tag{10}$$

where

$$\theta \ge \begin{cases} 0, & (-\delta)\notin\mathbb{N}\cup\{0\}, \\ -1, & (-\delta)\in\mathbb{N}\cup\{0\}. \end{cases}$$

Here and in what follows, the letter C denotes positive constants, which are assumed to be different in different cases. When necessary, the parameters on which they may depend will be indicated in parentheses: $C = C(\alpha,\beta,...)$.

The formulas (7), (8), and (9) yield that

$$D_{0y}^{\nu}w_\delta(x,y) = w_{\delta-\nu}(x,y), \qquad \lim_{\varepsilon\to0+}\left[\frac{\partial}{\partial x}w_\delta(x-s,y)\right]_{s=x-\varepsilon}^{s=x+\varepsilon} = \frac{y^{\delta-\beta-1}}{\Gamma(\delta-\beta)},$$

$$\left(D_{0y}^{\alpha}-\frac{\partial^2}{\partial x^2}\right)w_\delta(x-s,y) = 0, \quad \left(D_{0y}^{\alpha}-\frac{\partial^2}{\partial s^2}\right)w_\delta(x-s,y) = 0 \quad (x\ne s),$$

and

$$\int_{-\infty}^{\infty} w_\delta(x-s,y)ds = \frac{y^{\delta+\beta-1}}{\Gamma(\mu+\beta)}. \tag{11}$$

In what follows, we will need the solution of the Cauchy problem for the fractional diffusion equation in a particular case. Now we recall it.

From now on, by $\mathbf{T}_{\alpha,\omega}$ we will denote the set of continuous functions that grow no faster than $\exp\left(\omega x^{\frac{2}{2-\alpha}}\right)$ as $|x|\to\infty$ for given α and ω, i.e.

$$\mathbf{T}_{\alpha,\omega} := \left\{ g(x)\in C(\mathbb{R}) : \lim_{|x|\to\infty} g(x)\exp\left(-\omega|x|^{\frac{2}{2-\alpha}}\right)=0\right\}.$$

As usual, $L_{\mathrm{loc}}(J)$ denotes a set of locally integrable functions on J, that is the set of functions that are integrable on any compact subset of J. In particular,

$$L_{\mathrm{loc}}[0,T) := \{g(x)\in L(0,c),\ \forall c\in(0,T)\}.$$

Lemma 1. Let $v(y)\in L_{\mathrm{loc}}[0,T)$ and $\tau(x)\in\mathbf{T}_{\alpha,\omega}$ for some $\omega<(1-\beta)\left(\frac{\beta}{T}\right)^{\frac{\beta}{1-\beta}}$. Then the function

$$u(x,y) = D_{0y}^{-\alpha}v(y) + \int_{-\infty}^{\infty}\tau(s)\, w_\beta(x-s,y)\, ds \tag{12}$$

is a regular solution of the equation

$$\left(D_{0y}^{\alpha} - \frac{\partial^2}{\partial x^2}\right)u(x,y) = v(y), \tag{13}$$

and satisfies the condition (3).

Moreover, the problem (13) and (3) has at most one solution in the class of functions satisfying the condition

$$\sup_{y\in(0,T)} y^{1-\alpha}u(x,y)\in\mathbf{T}_{\alpha,\rho} \tag{14}$$

for some $\rho>0$.

Proof. It follows from [28, Theorem 2] that a regular solution of problem (13) and (3) has the form

$$u(x,y) = \int_{-\infty}^{\infty}\int_0^y v(t)\, w_\beta(x-s,y)\, dt\, ds + \int_{-\infty}^{\infty}\tau(s)\, w_\beta(x-s,y)\, ds. \tag{15}$$

Given (2), (5), and (11), the first term on the right-hand side of (15) can be written as follows:

$$\int_{-\infty}^{\infty}\int_0^y v(t)\, w_\beta(x-s,y)\, dt\, ds = \int_0^y v(t)\int_{-\infty}^{\infty} w_\beta(x-s,y)\, ds\, dt = \int_0^y v(t)\frac{(y-s)^{\alpha-1}}{\Gamma(\alpha)}\, dt = D_{0y}^{-\alpha}v(y).$$

This gives (12) and proves, in particular, the uniqueness of the problem.

Further, a direct verification shows that the conditions of the lemma guarantee that the function (12) is a solution of the equation (13) and satisfies the initial condition (3). □

Consider the operator H^δ, which acts on the function $g(x)\in\mathbf{T}_{\alpha,\omega}$ as follows:

$$\left(H^\delta g\right)(x,y) := \int_{-\infty}^{\infty} g(s)\, w_\delta(x-s,y)\, ds. \tag{16}$$

Lemma 2. Let $g(x)\in\mathbf{T}_{\alpha,\omega}$ for some $\omega<(1-\beta)\left(\frac{\beta}{T}\right)^{\frac{\beta}{1-\beta}}$. Then

$$\left(H^\delta g\right)(x,y)\in C^\infty(\Omega), \tag{17}$$

$$y^{1-\beta-\delta}\left(H^{\delta}g\right)(x,y)\in C(\Omega_0) \tag{18}$$

and

$$D_{0y}^{\sigma}\left(H^{\delta}g\right)=\left(H^{\delta+\sigma}g\right)(x,y) \tag{19}$$

for $\delta>-\beta$ if $\sigma\notin\mathbb{N}$, and for $\delta\in\mathbb{R}$ if $\sigma\in\mathbb{N}$.

Proof. The inclusion (17) follows from the formulas (6), (7) and (8).

Next, by (4) we can write

$$\left(H^{\delta}g\right)(x,y)=y^{\delta-1}\int_{-\infty}^{\infty}g(s)\,\phi\left(-\beta,\delta;\,-\frac{|x-s|}{y^{\beta}}\right)ds=y^{\delta-1}\int_{-\infty}^{\infty}g(x+s)\,\phi\left(-\beta,\delta;\,-\frac{|s|}{y^{\beta}}\right)ds=$$

$$=y^{\delta+\beta-1}\int_{-\infty}^{\infty}g(x+y^{\beta}s)\,\phi\left(-\beta,\delta;\,-|s|\right)ds.$$

Given (6), this proves (18).

Taking into account (6) and (10), by (8) we get (19). □

3 Reduction to an integro-differential equation

Let $u(x,y)$ be a regular solution of the problem (1) and (3) that satisfies (14) and let $v(y)$ be the loaded term in (1), i.e.

$$v(y)=\left[D_{0y}^{\mu}u(x,y)\right]_{x=z(y)}.$$

We will assume that $\tau(x)$ and $v(y)$ satisfy the conditions imposed in Lemma 1. Moreover, in the case $\mu>\alpha$ we will assume that $D_{0y}^{\mu-\alpha-m}v(y)\in AC^{m}[0,T)$, where

$$m=\lceil\mu-\alpha\rceil:=\min\{n\in\mathbb{N}:\,\mu-\alpha\leq n\} \tag{20}$$

is the floor of the number $\mu-\alpha$.

Under the above assumptions, taking into account (12) and (16), we can write

$$u(x,y)=\lambda\,D_{0y}^{-\alpha}v(y)+\left(H^{\beta}\tau\right)(x,y). \tag{21}$$

Acting with D_{0y}^{μ} on both sides of (21) and substituting $x=z(y)$, using (19), we obtain

$$v(y)=\lambda\,D_{0y}^{\mu-\alpha}v(y)+f_{\tau,z}(y), \tag{22}$$

where

$$f_{\tau,z}(y):=\left(H^{\beta-\mu}\tau\right)(z(y),y). \tag{23}$$

Thus, the question of the solvability of the problem (1), (3) is reduced to the question of the solvability of the equation (22). This equation is integral if $\mu<\alpha$, functional if $\mu=\alpha$, and differential if $\mu>\alpha$. Below we will consider all three of these cases.

4 *The case of integral equation* ($\mu < \alpha$)

Let $\mu < \alpha$. In this case, (22) is an integral equation. In accordance with (18) and (23) we have that $f_{\tau,z}(y) \in L_{\text{loc}}[0,T)$. Thus, we can conclude that the equation (22) has a unique solution $v(y) \in L_{\text{loc}}[0,T)$, and this solution can be written as (see, e.g., [1, 29])

$$v(y) = f_{\tau,z}(y) + \lambda \int_0^y f_{\tau,z}(t)\,(y-t)^{\alpha-\mu-1} E_{\alpha-\mu,\alpha-\mu}\left(\lambda (y-t)^{\alpha-\mu}\right) dt, \tag{24}$$

where

$$E_{\xi,\eta}\,(z) := \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\xi k + \eta)}$$

is the Mittag-Leffler function.

Now we can formulate a theorem on the solvability of the problem (1), (3) in the case $\mu < \alpha$.

Theorem 1. Let $\mu < \alpha$, $z(y) \in C[0,T)$ and $\tau(x) \in \mathbf{T}_{\alpha,\omega}$ for some $\omega < (1-\beta)\left(\frac{\beta}{T}\right)^{\frac{\beta}{1-\beta}}$. Then there exists a unique regular solution of the problem (1), (3) in the class of functions satisfying the condition (14). The solution has the form

$$u(x,y) = \lambda \int_0^y f_{\tau,z}(t)\,(y-t)^{\alpha-1} E_{\alpha-\mu,\alpha}\left(\lambda (y-t)^{\alpha-\mu}\right) dt + \left(H^\beta \tau\right)(x,y), \tag{25}$$

where H^β and $f_{\tau,z}(x)$ are defined by (16) and (23), respectively.

Proof. As shown in Section 3, if $u(x,y)$ is a regular solution of the problem (1), (3), then it has the form (21), where $v(x)$ should be found from (22). It was obtained above that in the case $\mu < \alpha$, which we are now considering, $v(x)$ is given by the formula (24). Taking this into account, we transform the first term on the right-hand side of the equation (21). Using (2) and the formulas (see, e.g., [1, 30, 31])

$$D_{0y}^{\gamma} y^{\eta-1} E_{\xi,\eta}\left(c\, y^{\xi}\right) = y^{\eta-\gamma-1} E_{\xi,\eta-\gamma}\left(c\, y^{\xi}\right) \qquad (\gamma \in \mathbb{R},\ \eta > 0), \tag{26}$$

$$E_{\xi,\eta}(z) = \frac{1}{\Gamma(\eta)} + z\, E_{\xi,\eta+\xi}(z) \tag{27}$$

and

$$D_{0y}^{-\gamma} \int_0^y g(t) h(y-t)\, dt = \int_0^y \left(D_{0y}^{-\gamma} g\right)(t) h(y-t)\, dt = \int_0^y g(t) \left(D_{0y}^{-\gamma} h\right)(y-t)\, dt \qquad (\gamma > 0), \tag{28}$$

we get

$$D_{0y}^{-\alpha} v(x) = D_{0y}^{-\alpha}\left[f_{\tau,z}(y) + \lambda \int_0^y f_{\tau,z}(t)\,(y-t)^{\alpha-\mu-1} E_{\alpha-\mu,\alpha-\mu}\left(\lambda (y-t)^{\alpha-\mu}\right) dt\right] =$$

$$= D_{0y}^{-\alpha} f_{\tau,z}(y) + \lambda \int_0^y f_{\tau,z}(t)\,(y-t)^{2\alpha-\mu-1} E_{2\alpha-\mu,\alpha-\mu}\left(\lambda (y-t)^{\alpha-\mu}\right) dt =$$

$$= \int_0^y f_{\tau,z}(t)\left[\frac{(y-t)^{\alpha-1}}{\Gamma(\alpha)} + \lambda\,(y-t)^{\alpha-\mu-1} E_{\alpha-\mu,\alpha-\mu}\left(\lambda (y-t)^{\alpha-\mu}\right)\right] dt =$$

$$= \int_0^y f_{\tau,z}(t)\,(y-t)^{\alpha-1} E_{\alpha-\mu,\alpha}\left(\lambda (y-t)^{\alpha-\mu}\right) dt.$$

This equality and (21) prove (25). In particular, the representation (25) yields the uniqueness of the problem (1), (3). Indeed, for the corresponding homogeneous problem ($\tau(x) \equiv 0$), we have $f_{\tau,z}(y) \equiv 0$.

Therefore, the difference between two different solutions with the same initial function $\tau(x)$ must be equal to zero. Due to the linearity of the problem under consideration, this is equivalent to the uniqueness of its solution.

The fact that the function (25) is a regular solution to the problem under consideration can be easily verified by direct computation. □

5 *The case of functional equation ($\mu = \alpha$)*

Now let us consider the case when $\mu = \alpha$. In this case the equation (22) takes the form

$$v(y) = \lambda\, v(y) + f_{\tau,z}(y). \tag{29}$$

It is easy to see that if $\lambda = 1$, then the equation (29) has a solution only for $f_{\tau,z}(y) \equiv 0$, and, moreover, then any function defined on the interval $(0, T)$ is a solution of this equation. Therefore, the solvability condition for (29) is the inequality $\lambda \neq 1$. If $\lambda \neq 1$, then

$$v(y) = \frac{f_{\tau,z}(y)}{1-\lambda}. \tag{30}$$

Now the solution of the problem (1), (3) can be found from (21). It should be noted that the function $v(y)$ must be integrable, therefore the function $f_{\tau,z}(y)$ must also be integrable.

Theorem 2. Let $\mu = \alpha$, $\lambda \neq 1$, $z(y) \in C[0,T)$, $\tau(x) \in \mathbf{T}_{\alpha,\omega}$ for some $\omega < (1-\beta)\left(\frac{\beta}{T}\right)^{\frac{\beta}{1-\beta}}$, and let $\tau(x)$ be locally Hölder continuous. Then there exists a unique regular solution of the problem (1), (3) in the class of functions satisfying the condition (14). The solution is of the form

$$u(x,y) = \frac{\lambda}{1-\lambda}\, D_{0y}^{-\alpha}\, f_{\tau,z}(y) + \left(H^{\beta}\tau\right)(x,y). \tag{31}$$

Proof. The representation (31) follows from (21) and (30). In fact, to complete the proof it remains to show that the conditions of the theorem guarantee the inclusion $f_{\tau,z}(y) \in L_{\mathrm{loc}}[0,T)$. Let us check this. By (16), we can write

$$\left(H^{-\beta}\tau\right)(x,y) = \int_{-\infty}^{\infty} [\tau(s) - \tau(x)]\; w_{-\beta}(x-s,y)\, ds + \tau(x) \int_{-\infty}^{\infty} w_{-\beta}(x-s,y)\, ds.$$

Taking into account (6), (9) and (10), we get

$$\left|\left(H^{-\beta}\tau\right)(x,y)\right| \leq \int_{-\infty}^{\infty} |(x-s)^{\varepsilon}\, w_{-\beta}(x-s,y)|\, ds \leq C y^{\beta\delta - 1} \tag{32}$$

for any $\delta \in (0,\varepsilon)$ and $|x| < r$, where $C = C(\omega,\delta,r)$ and ε is the Hölder exponent for $\tau(x)$. Next, in accordance with (5), (23) and (32), we obtain

$$f_{\tau,z}(y) = \left(H^{-\beta}\tau\right)(z(y),y) \in L_{\mathrm{loc}}[0,T).$$

This completes the proof. □

6 The case of differtial equation ($\mu > \alpha$)

Now it remains to consider the last case, when $\mu > \alpha$. In this case, the order of differentiation in the loaded term exceeds the order of differentiation in the principal part of the equation. Equations with such a load are called essentially loaded [20] (as well as equations with $\mu = \alpha$ discussed in the previous section).

Let $u(x,y)$ be a solution of the problem (1) and (3). Then, as shown above, $u(x,y)$ can be represented as (21), where $v(y)$ is a solution of the equation (22), which in the case under consideration is a differential equation. The conditions $\lambda \neq 0$ and $f_{\tau,z}(y) \in L_{\mathrm{loc}}[0,T)$ guarantee (see, e.g., [31, 32]) that every solution of (22) can be given by

$$v(y) = \sum_{k=0}^{m-1} c_k\, G_{\mu-\alpha-k}(y) - \frac{1}{\lambda}\int_0^y f_{\tau,z}(t)\, G_{\mu-\alpha}(y-t)\, dt \tag{33}$$

for some set of numbers c_k, $k = 0, 1, \dots, m-1$. Here m is defined by (20) and

$$G_\sigma(y) := y^{\sigma-1} E_{\mu-\alpha,\sigma}\left(-\frac{1}{\lambda} y^{\mu-\alpha}\right). \tag{34}$$

Substituting (33) into (21) yields

$$u(x,y) = \lambda\, D_{0y}^{-\alpha}\left[\sum_{k=0}^{m-1} c_k\, G_{\mu-\alpha-k}(y) - \frac{1}{\lambda}\int_0^y f_{\tau,z}(t)\, G_{\mu-\alpha}(y-t)\, dt\right] + \left(H^\beta \tau\right)(x,y).$$

Using (26), (28) and (34), we get that if $u(x,y)$ is a solution of the problem (1) and (3) then it has the form

$$u(x,y) = \lambda \sum_{k=0}^{m-1} c_k\, G_{\mu-k}(y) - \int_0^y f_{\tau,z}(t)\, G_\mu(y-t)\, dt + \left(H^\beta \tau\right)(x,y) \tag{35}$$

for some c_k, $k = 0, 1, \dots, m-1$, where $f_{\tau,z}(y)$ is defined by (23).

Next, let us prove the converse statement: if the conditions $\lambda \neq 0$ and $f_{\tau,z}(y) \in L_{\mathrm{loc}}[0,T)$ are met, then the function (35) is a solution of the problem (1) and (3) for any set of c_k, $k = 0, 1, \dots, m-1$.

Indeed, using Lemma 1 and the formulas (19), (26), (27), (28), (34) and assuming that $u(x,y)$ is given by (35), we can write

$$\left(D_{0y}^\alpha - \frac{\partial^2}{\partial x^2}\right) u(x,y) = \lambda \sum_{k=0}^{m-1} c_k\, G_{\mu-\alpha-k}(y) - \int_0^y f_{\tau,z}(t)\, G_{\mu-\alpha}(y-t)\, dt, \tag{36}$$

$$D_{0y}^{\alpha-1} u(x,y) = \lambda \sum_{k=0}^{m-1} c_k\, G_{\mu-\alpha-k+1}(y) - \int_0^y f_{\tau,z}(t)\, G_{\mu-\alpha+1}(y-t)\, dt + \left(H^{1-\beta}\tau\right)(x,y), \tag{37}$$

$$D_{0y}^\mu u(x,y) = \sum_{k=0}^{m-1} c_k\, G_{\mu-\alpha-k}(y) - f_{\tau,z}(y) - \frac{1}{\lambda}\int_0^y f_{\tau,z}(t)\, G_{\mu-\alpha}(y-t)\, dt + \left(H^{\beta-\mu}\tau\right)(x,y),$$

and, by (23),

$$\left[D_{0y}^\mu u(x,y)\right]_{x=z(y)} = \sum_{k=0}^{m-1} c_k\, G_{\mu-\alpha-k}(y) - \frac{1}{\lambda}\int_0^y f_{\tau,z}(t)\, G_{\mu-\alpha}(y-t)\, dt. \tag{38}$$

By (18), (36), (37) and (38), we get

$$\left(D_{0y}^{\alpha} - \frac{\partial^2}{\partial x^2}\right) u(x,y) - \lambda \left[D_{0y}^{\mu} u(x,y)\right]_{x=z(y)} = 0$$

and

$$\lim_{y \to 0} D_{0y}^{\alpha-1} u(x,y) = \tau(x).$$

This proves that $u(x,y)$, defined by (35), is really a solution of the problem (1) and (3).

Thus, the problem (1) and (3) is solvable for those functions $\tau(x)$ and $z(y)$ that guarantee the validity of inclusion $f_{\tau,z}(y) \in L_{\mathrm{loc}}[0,T)$ (as everywhere, $f_{\tau,z}(y)$ is defined by the equality (23)).

Let us formulate what was proved above in the following rigorous statement.

Theorem 3. Let $\lambda \neq 0$, $\mu > \alpha$, $z(y) \in C[0,T)$, $\tau(x) \in \mathbf{T}_{\alpha,\omega}$ for some $\omega < (1-\beta)\left(\frac{\beta}{T}\right)^{\frac{\beta}{1-\beta}}$, and let

$$f_{\tau,z}(y) \in L_{\mathrm{loc}}[0,T). \tag{39}$$

Then every regular solution of the problem (1) and (3) from the class of functions satisfying the condition (14) has the form (35) for some set of c_k, $k = 0, 1, \dots, m-1$.

Conversely, any function $u(x,y)$ defined as (35) is a regular solution of the problem (1) and (3) for any set of c_k, $k = 0, 1, \dots, m-1$.

Remark 1. The condition (39) is essential for the solvability of the problem (1) and (3). However, for fairly simple functions $\tau(x)$ and $z(y)$ this condition may not be met. For example, if we take $\tau(x) = const$, then by (4), (11), (16) and (23), we obtain

$$f_{\tau,z}(y) = const \int_{-\infty}^{\infty} w_{\beta-\mu}(z(y)-s,y)\,ds = const\frac{y^{\alpha-\mu-1}}{\Gamma(\alpha-\mu)}.$$

This means that $f_{\tau,z}(y) \notin L_{\mathrm{loc}}[0,T)$ for any $\mu > \alpha$ regardless of the choice of $z(y)$.

The question arises: whether there exist functions $\tau(x)$ and $z(y)$ for which this condition is satisfied. Let us show that they do exist. Let $z(y) = y^{\beta}$ and $\tau(x) = |x|^{\varepsilon}$. Then, by (4), (16) and (23), we get

$$f_{\tau,z}(y) = \int_{-\infty}^{\infty} |s|^{\varepsilon} w_{\beta-\mu}(y^{\beta}-s,y)\,ds = \int_{-\infty}^{\infty} |y^{\beta}-s|^{\varepsilon} w_{\beta-\mu}(s,y)\,ds =$$

$$= y^{\beta-\mu+\beta\varepsilon-1} \int_{-\infty}^{\infty} |1-s|^{\varepsilon} w_{\beta-\mu}(s,1)\,ds.$$

This means that

$$|f_{\tau,z}(y)| \leq C y^{\beta-\mu+\beta\varepsilon-1}.$$

Thus, $f_{\tau,z}(y) \in L_{\mathrm{loc}}[0,T)$ for every $\varepsilon > \frac{\mu-\beta}{\beta}$.

7 Non-uniqueness of solution in the problem with an essential load

In the case $\mu > \alpha$, which is considered in the previous section, the problem (1) and (3) ceases to be uniquely solvable. Indeed, consider the function

$$u_0(x,y) = \sum_{k=0}^{m-1} c_k\, G_{\mu-k}(y), \tag{40}$$

where c_k, $k = 0, 1, \dots, m-1$ are constants, at least one of which is not equal to zero; and m is defined by (20). Using (26), (27) and (34), we can write

$$\left(D_{0y}^{\alpha} - \frac{\partial^2}{\partial x^2}\right) u_0(x,y) = \sum_{k=0}^{m-1} c_k\, G_{\mu-\alpha-k}(y), \qquad \Big[D_{0y}^{\mu} u_0(x,y)\Big]_{x=z(y)} = \frac{1}{\lambda} \sum_{k=0}^{m-1} c_k\, G_{\mu-\alpha-k}(y)$$

and

$$\lim_{y\to 0} D_{0y}^{\alpha-1} u_0(x,y) = \lim_{y\to 0} \sum_{k=0}^{m-1} c_k\, G_{\mu-\alpha-k+1}(y) = 0.$$

This gives that $u_0(x,y)$ is a regular solution of the equation (1) and satisfies the homogeneous initial value condition

$$\lim_{y\to 0} D_{0y}^{\alpha-1} u_0(x,y) = 0, \qquad x \in \mathbb{R}.$$

Due to the linearity, this means that solution of the problem (1) and (3) is not unique. If $u(x,y)$ is a regular solution of (1) and (3), then the function $u(x,y) + u_0(x,y)$ will also be a regular solution of this problem.

Conclusion

Thus, we considered the issue of solvability of the Cauchy problem (3) for the loaded equation (1) in all three cases of possible mutual distribution of α and μ ($\alpha > \mu$, $\alpha = \mu$ or $\alpha < \mu$).

It is shown that the condition $\alpha > \mu$ guarantees the unique solvability of the problem (1) and (3) (see Theorem 1). When $\alpha \le \mu$ the equation (1) becomes essentially loaded, and the problem under consideration may lose uniqueness or solvability. If $\alpha = \mu$, then the problem ceases to be solvable for $\lambda = 1$ (see Theorem 2). In the case $\alpha < \mu$, the problem (1) and (3) loses uniqueness of solution: the corresponding homogeneous problem has infinitely many non-trivial solutions (see Section 7) given in the form (40). Moreover, for the problem to be solvable in this case, it is necessary to impose an additional non-trivial condition (39) (see Theorem 3). This condition narrows the set of acceptable initial data, namely the initial value $\tau(x)$, as well as the function $z(y)$ that specifies the loaded term (see Remark 1).

In this last case, the questions arise: is it possible to achieve uniqueness of the solution by imposing additional conditions? And if so, what are these conditions? Also, can we equivalently reformulate the condition (39) in terms of $\tau(x)$ and $z(y)$? Further research is needed to answer these questions.

Author Contributions

All authors contributed equally to this work.

Conflict of Interest

The authors declare no conflict of interest.

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