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Time-fractional parabolic equation with Zaremba-type boundary conditions: analysis and applications

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This paper investigates a time-fractional parabolic equation with Zaremba-type boundary conditions. The main objective of the present work is to construct reliable and efficient numerical approximations for such problems. To this end, stable finite difference schemes are developed within a consistent analytical framework. A key result is obtaining a coercive stability estimate for the first-order scheme, which guarantees its consistency and supports its practical use in computations. In addition, both first- and second-order schemes are implemented in the one-dimensional case using a modified Gaussian elimination approach. This implementation simplifies the solution process and improves computational reliability when handling the resulting systems. The behavior of the proposed methods is examined through several numerical experiments designed to reflect different parameter choices and settings. The results demonstrate that the schemes achieve the expected levels of accuracy, consistency, and efficiency. An accompanying error analysis explains the observed outcomes and supports the theoretical findings. The numerical results, presented in tables, show strong agreement with the theoretical predictions, thereby confirming the validity and effectiveness of the proposed approach. These conclusions highlight the practical applicability of the proposed numerical schemes for solving this fractional parabolic problem with mixed boundary conditions.

Keywords: Zaremba-type boundary conditions, Riemann–Liouville derivative, time-fractional parabolic equation, boundary value problems, positive operators, modified Gaussian elimination method, difference schemes, stability.

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Introduction

The analysis of fractional partial differential equations has gained considerable attention in recent decades due to their wide applicability in physics, engineering, and mathematical biology [1–3]. Among them, the time-fractional diffusion and parabolic equations [4, 5] are of particular interest because they provide realistic models for anomalous diffusion processes with memory effects. We may refer to [6, 7] and [8–10] for further studies.

The study of boundary value problems for parabolic equations with classical derivatives has a long history. Zaremba [11], following a suggestion by Wirtinger, introduced the mixed Dirichlet–Neumann boundary problem (now known as the Zaremba problem) for the Laplace equation. Modern studies have extended these ideas in various directions, including the analysis of singular interfaces [12].

In [13], fractional powers (FPs) of positive operators (POs) were investigated. The author examined the conditions under which the sum of coercively POs retains positivity and defined their FPs. This work has provided a theoretical foundation for the analysis of fractional operator powers in Banach and Hilbert spaces, which is fundamental for applications in differential and integral equations.

The paper [14] established the well-posedness results for a boundary value problem of fractional parabolic equations (FPEs) with mixed conditions. The authors derived coercive stability estimates

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for the solution associated with the mixed problem and presented stable difference schemes (DSs) for its approximate solution. They considered first and second order accurate DSs in time and first-order accuracy in space, applying a modified Gaussian elimination method for their numerical solution.

Despite these advances, study of time-FPEs with Zaremba-type boundary conditions remains limited. In particular, the development of stable numerical methods and stability estimates for such problems has not been fully addressed in the literature.

This paper is devoted to the study of the time-FPE with Zaremba-type (mixed Dirichlet–Neumann) boundary conditions:

$$\begin{gathered} D_t^{\mu} u(t,\mathrm{z}) - \frac{\partial}{\partial \mathrm{z}}\left(\xi(\mathrm{z})\, u_{\mathrm{z}}(t,\mathrm{z})\right) + \eta\, u(t,\mathrm{z}) = g(t,\mathrm{z}), \quad t \in (0,\mathsf{T}), \quad \mathrm{z} \in (0,L), \\ u_{\mathrm{z}}(t,0) = 0, \quad u(t,L) = 0, \quad t \in [0,\mathsf{T}], \\ u(\mathsf{T},\mathrm{z}) = 0, \quad \mathrm{z} \in [0,L]. \end{gathered} \tag{1}$$

Here, $D_t^{\mu} = D_{\mathsf{T}-}^{\mu}$ stands for the right-sided Riemann–Liouville fractional derivative (FD) of order $\mu \in (0,1)$ [15]. We assume that $\xi(\mathrm{z})$ and $g(t,\mathrm{z})$ are smooth functions for $t \in (0,\mathsf{T})$ and $\mathrm{z} \in (0,L)$, with $\xi(\mathrm{z}) \geq a > 0$ and $\eta > 0$. We construct stable DSs for the approximate solution of the boundary value problem (1) and derive coercive stability estimate for the first-order DS. Additionally, a modified Gaussian elimination method is employed to solve both the first and second order accurate DSs for FPEs.

1 Preliminaries

The following statements are established and are used throughout this article.

Let K be a Banach space, and let $A : \mathcal{D}(A) \subset \mathsf{K} \to \mathsf{K}$ be a linear and unbounded operator that is densely defined in K. A is said to be strongly PO in K if its spectrum $\eta(A)$ lies entirely within a sector of angle Θ, with $0 < \Theta < \pi/2$, symmetric w.r.t. the real axis. On the edges of this sector,

$$S_1(\Theta) = \{\rho e^{i\Theta} : 0 \leq \rho < \infty\}, \quad S_2(\Theta) = \{\rho e^{-i\Theta} : 0 \leq \rho < \infty\},$$

and outside the sector, the resolvent $(\lambda I - A)^{-1}$ satisfies the estimate [16]

$$\|(\lambda I - A)^{-1}\| \leq \frac{M}{1 + |\lambda|}.$$

The infimum of all such angles is referred to as the spectral angle $\Theta(A, \mathsf{K})$ of A.

In what follows, M stands for a positive constant, which may take different values in each occurrence. When it is necessary to indicate the dependence of this constant on certain parameters, we write $M(\alpha, \beta, \gamma, \dots)$.

A is a PO in K. For $0 < \gamma < 1$, we define the fractional space $\mathsf{K}_\gamma = \mathsf{K}_\gamma(\mathsf{K}, A)$ as the set of all $u \in \mathsf{K}$ for which the norm

$$\|u\|_{\mathsf{K}_\gamma} = \sup_{\lambda > 0} \lambda^{\gamma} \|(A + \lambda I)^{-1} u\|_{\mathsf{K}} + \|u\|_{\mathsf{K}}$$

is finite.

Theorem 1. [13] If A and B are two commuting POs with $\Theta(A, \mathsf{K}) + \Theta(B, \mathsf{K}) < \pi$, the bounded operator $(A + B)^{-1}$ defined in K exists. In addition, for every $\gamma \in (0,1)$ and every $g \in \mathsf{K}$, the problem $Au + Bu = g$ admits a unique solution $u = u(g)$ and the next estimates hold:

$$\begin{gathered} \|Au\|_{\mathsf{K}_\gamma(\mathsf{K},B)} + \|Bu\|_{\mathsf{K}_\gamma(\mathsf{K},B)} + \|Bu\|_{\mathsf{K}_\gamma(\mathsf{K},A)} \leq M(\gamma) \|g\|_{\mathsf{K}_\gamma(\mathsf{K},B)}, \\ \|Au\|_{\mathsf{K}_\gamma(\mathsf{K},A)} + \|Bu\|_{\mathsf{K}_\gamma(\mathsf{K},A)} + \|Au\|_{\mathsf{K}_\gamma(\mathsf{K},B)} \leq M(\gamma) \|g\|_{\mathsf{K}_\gamma(\mathsf{K},A)}. \end{gathered}$$

Theorem 2. [17] Suppose A is a PO with $\Theta(A,\mathsf{K}) < \pi$. Then, for $\gamma \le \frac{1}{2}$, the FP A^γ is also a PO with $\Theta(A^\gamma,\mathsf{K}) < \frac{\pi}{2}$.

Theorem 3. [18] Assume A is the operator in $\mathsf{K} = \mathcal{C}[0,\mathsf{T}]$ given by $(A\,\omega)(t) = -\omega'(t)$, whose domain is $\mathcal{D}(A) = \{\, \omega \in \mathcal{C}[0,\mathsf{T}] : \omega' \in \mathcal{C}[0,\mathsf{T}],\ \omega(\mathsf{T}) = 0 \,\}$. This implies that A is a PO on the space $\mathsf{K} = \mathcal{C}[0,\mathsf{T}]$. Moreover, for every $g \in \mathcal{D}(A)$ and every $\gamma \in (0,1)$, the identity

$$A^\gamma g(t) = D^\gamma_{\mathsf{T}-} g(t)$$

holds.

Theorem 4. Suppose A and B are POs with $\Theta(A,\mathsf{K}) < \pi$ and $\Theta(B,\mathsf{K}) < \frac{\pi}{2}$. Consequently, for every $\gamma \le \frac{1}{2}$, the operator $(D^\gamma + B)^{-1}$ exists as a bounded operator on K. Additionally, for each $g \in \mathsf{K}$, there is a unique solution $u = u(g)$ of the equation $D^\gamma u + Bu = g$, and the next estimate holds:

$$\|D^\gamma u\|_{\mathsf{K}_\gamma(\mathsf{K},B)} + \|Bu\|_{\mathsf{K}_\gamma(\mathsf{K},B)} \le M(\gamma)\, \|g\|_{\mathsf{K}_\gamma(\mathsf{K},B)}.$$

B^{z}, the differential operator of order two, is defined by

$$B^{\mathrm{z}} u(\mathrm{z}) = -\frac{\partial}{\partial \mathrm{z}}(\xi(\mathrm{z}) u_{\mathrm{z}}(\mathrm{z})) + \eta u(\mathrm{z}) \tag{2}$$

whose domain is $\mathcal{D}(B^{\mathrm{z}}) = \{u \ :\ u, u', u'' \in \mathcal{C}[0,L],\ u'(0) = 0,\ u(L) = 0\}$.

For $\gamma \in (0,1]$, let $C^\gamma[0,L]$ be the Banach space of all continuous functions $\psi(\mathrm{z})$ on $[0,L]$ satisfying a Hölder condition, with the norm

$$\|\psi\|_{\mathcal{C}^\gamma[0,L]} = \|\psi\|_{\mathcal{C}[0,L]} + \sup_{\mathrm{z}_1 \ne \mathrm{z}_2} \frac{|\psi(\mathrm{z}_1) - \psi(\mathrm{z}_2)|}{|\mathrm{z}_1 - \mathrm{z}_2|^\gamma},$$

where $\mathcal{C}[0,L]$ is the Banach space consisting of continuous functions $\psi(\mathrm{z})$ on $[0,L]$, endowed with the norm $\|\psi\|_{\mathcal{C}[0,L]} = \max_{\mathrm{z}\in[0,L]} |\psi(\mathrm{z})|$.

The positivity of B^{z} in $\mathcal{C}[0,L]$ has been proved. Furthermore, for any $\gamma \in (0,1/2)$, the norms in the spaces $\mathsf{K}_\gamma(\mathsf{K},B)$ and $\mathcal{C}^{2\gamma}[0,L]$ are equivalent.

Theorem 5. The norms in the space $\mathsf{K}_\gamma(\mathcal{C}[0,L], B^{\mathrm{z}})$ and the Hölder space $\mathcal{C}^{2\gamma}[0,L]$ are equivalent if $\gamma \in (0,1/2)$.

The proof of Theorem 5 relies on the following estimates for the Green function G^{z} of the operator B^{z} defined in (2):

$$|G^{\mathrm{z}}(\mathrm{z},\mathrm{z}_0;\lambda)| \le \frac{M(\eta,a)}{\sqrt{\eta+\lambda}} \begin{cases} e^{-\frac{1}{2}\sqrt{\frac{\eta+\lambda}{a}}(\mathrm{z}-\mathrm{z}_0)}, & 0 \le \mathrm{z}_0 \le \mathrm{z}, \\ e^{-\frac{1}{2}\sqrt{\frac{\eta+\lambda}{a}}(\mathrm{z}_0-\mathrm{z})}, & \mathrm{z} \le \mathrm{z}_0 \le L, \end{cases}$$

$$|G^{\mathrm{z}}_{\mathrm{z}}(\mathrm{z},\mathrm{z}_0;\lambda)| \le M(\eta,a) \begin{cases} e^{-\frac{1}{2}\sqrt{\frac{\eta+\lambda}{a}}(\mathrm{z}-\mathrm{z}_0)}, & 0 \le \mathrm{z}_0 \le \mathrm{z}, \\ e^{-\frac{1}{2}\sqrt{\frac{\eta+\lambda}{a}}(\mathrm{z}_0-\mathrm{z})}, & \mathrm{z} \le \mathrm{z}_0 \le L. \end{cases}$$

Theorem 6. For the solution $u(t,\mathrm{z})$ of problem (1), the following coercive SE holds:

$$\max_{0\le t\le \mathsf{T}} \|u_{\mathrm{zz}}(t,\cdot)\|_{\mathcal{C}^\gamma[0,L]} \le M(\gamma) \|g(t,\cdot)\|_{\mathcal{C}^\gamma[0,L]}$$

for $M(\gamma)$ that is independent of $g(t,\mathrm{z})$ when $t \in [0,\mathsf{T}]$, $\mathrm{z} \in [0,L]$ and $0 < \gamma < 1$.

The proof of Theorem 6 relies on the positivity of the differential operator B^{z} (see (2)), on Theorem 3 establishing the relationship between FDs and FPs of positive operators, on Theorem 2 concerning the spectral angle of FPs, and on Theorem 4 regarding the FPs of coercively positive sums of two operators.

2 Main results

Problem (1) is discretized in two stages as follows. Firstly, we introduce the discrete mesh

$$[0, L]_h = \{\mathrm{z}_n : \mathrm{z}_n = nh,\ h = L/M,\ 0 \le n \le M\}.$$

To the differential operator B^{z} defined in (2), we associate the difference operator B^{z}_h given by

$$B^{\mathrm{z}}_h u^h(\mathrm{z}) = -\frac{\partial}{\partial \mathrm{z}}(\xi(\mathrm{z})u^h_{\mathrm{z}}(\mathrm{z})) + \eta u^h(\mathrm{z}), \tag{3}$$

which acts in the space of grid functions $u^h(\mathrm{z})$ subject to the boundary conditions

$$D^h u^h(0) = 0, \quad u^h(L) = 0,$$

where $D^h u^h(0)$ denotes the approximation of u_{z} at $\mathrm{z} = 0$.

Using the operator B^{z}_h, we consider the boundary value problem

$$\begin{cases} D^\mu_t \omega^h(t, \mathrm{z}) + B^{\mathrm{z}}_h \omega^h(t, \mathrm{z}) = g^h(t, \mathrm{z}),\ t \in (0, \mathsf{T}),\ \mathrm{z} \in [0, L]_h, \\ \omega^h(\mathsf{T}, \mathrm{z}) = 0,\ \mathrm{z} \in [0, L]_h \end{cases}$$

which represents a finite system of ordinary fractional differential equations.

Secondly, applying the first order of approximation formula [18]

$$D^\mu_\tau u_k = -\frac{1}{\Gamma(1-\mu)} \sum_{r=k}^{N} \frac{\Gamma(r-k-\mu+1)}{\Gamma(r-k+1)} \frac{u_r - u_{r-1}}{\tau^\mu}, \quad 1 \le k \le N$$

for

$$D^\mu_\tau u(t_k) = -\frac{1}{\Gamma(1-\mu)} \int_{t_k}^{\mathsf{T}} (s - t_k)^{-\mu} u'(s)\, ds$$

and applying the first-order accurate stable DS for parabolic equations, the first-order accurate DS in time can be formulated as

$$-\frac{1}{\Gamma(1-\mu)} \sum_{r=k}^{N} \frac{\Gamma(r-k-\mu+1)}{\Gamma(r-k+1)} \frac{u^h_r(\mathrm{z}) - u^h_{r-1}(\mathrm{z})}{\tau^\mu} + B^{\mathrm{z}}_h u^h_k(\mathrm{z}) = g^h_k(\mathrm{z}),$$

$$g^h_k(\mathrm{z}) = g^h(t_k, \mathrm{z}), \quad t_k = k\tau, \quad \tau = \mathsf{T}/N, \quad 1 \le k \le N, \quad \mathrm{z} \in [0, L]_h, \tag{4}$$

$$u^h_N(\mathrm{z}) = 0, \quad \mathrm{z} \in [0, L]_h,$$

which represents the approximate solution of problem (1). In addition, employing the second-order approximation formula for $1 \le k \le N-2$,

$$D^\mu_\tau u_k = d\left\{ w_1 u_{k-1} + w_2 u_k + w_3 u_{k+1} + \sum_{m=k+2}^{N} [a(k,m)u_{m-2} + b(k,m)u_{m-1} + c(k,m)u_m] \right\}, \tag{5}$$

for $k = N - 1$,

$$\begin{aligned} D^{\mu}_{\tau} u_k &= d \left[-\frac{3^{1-\mu}}{2^{2-\mu}} \frac{1}{1-\mu} + \frac{3^{2-\mu}}{2^{2-\mu}} \frac{1}{(1-\mu)(2-\mu)} \right] u_{N-2} \\ &+ d \left[\frac{3^{1-\mu}}{2^{-\mu}} \frac{1}{1-\mu} - \frac{3^{2-\mu}}{2^{1-\mu}} \frac{1}{(1-\mu)(2-\mu)} \right] u_{N-1} \\ &+ d \left[-\frac{3^{2-\mu}}{2^{2-\mu}} \frac{1}{1-\mu} + \frac{3^{2-\mu}}{2^{2-\mu}} \frac{1}{(1-\mu)(2-\mu)} \right] u_N, \end{aligned}$$

for $k = N$,

$$\begin{aligned} D^{\mu}_{\tau} u_k &= d \left[-\frac{1}{2^{2-\mu}} \frac{1}{1-\mu} + \frac{1}{2^{2-\mu}} \frac{1}{(1-\mu)(2-\mu)} \right] u_{N-2} \\ &+ d \left[\frac{1}{2^{-\mu}} \frac{1}{1-\mu} - \frac{1}{2^{1-\mu}} \frac{1}{(1-\mu)(2-\mu)} \right] u_{N-1} \\ &+ d \left[-\frac{3}{2^{2-\mu}} \frac{1}{1-\mu} + \frac{1}{2^{2-\mu}} \frac{1}{(1-\mu)(2-\mu)} \right] u_N, \end{aligned}$$

for

$$D^{\mu}_{\tau} u(t_k - \tau/2) = -\frac{1}{\Gamma(1-\mu)} \int_{t_k-\tau/2}^{\mathsf{T}} (s - t_k + \tau/2)^{-\mu} u'(s)\, ds$$

and employing the Crank–Nicolson DS for parabolic equations, a second-order accurate DS in both t and z can be formulated as

$$\begin{gathered} D^{\mu}_{\tau} u^h_k(\mathrm{z}) + \frac{1}{2} B^{\mathrm{z}}_h \left(u^h_k(\mathrm{z}) + u^h_{k-1}(\mathrm{z}) \right) = g^h_k(\mathrm{z}), \\ g^h_k(\mathrm{z}) = g^h(t_k - \tau/2, \mathrm{z}), \quad t_k = k\tau, \quad \tau = \mathsf{T}/N, \quad 1 \le k \le N, \quad \mathrm{z} \in [0, L]_h, \\ u^h_N(\mathrm{z}) = 0, \quad \mathrm{z} \in [0, L]_h \end{gathered} \tag{6}$$

which represents an approximation for the solution of problem (1).

In (5), it is denoted that

$d = \frac{\tau^{-\mu}}{\Gamma(1-\mu)}, \quad w_1 = -\frac{3^{1-\mu}}{2^{2-\mu}} \frac{1}{1-\mu} + \frac{3^{2-\mu}}{2^{2-\mu}} \frac{1}{(1-\mu)(2-\mu)},$

$w_2 = \frac{3^{1-\mu}}{2^{-\mu}} \frac{1}{1-\mu} - \frac{3^{2-\mu}}{2^{1-\mu}} \frac{1}{(1-\mu)(2-\mu)}, \quad w_3 = -\frac{3^{2-\mu}}{2^{2-\mu}} \frac{1}{1-\mu} + \frac{3^{2-\mu}}{2^{2-\mu}} \frac{1}{(1-\mu)(2-\mu)},$

$q_1(r) = (r + \frac{1}{2})^{1-\mu} - (r - \frac{1}{2})^{1-\mu}, \quad q_2(r) = (r + \frac{1}{2})^{2-\mu} - (r - \frac{1}{2})^{2-\mu}, \quad q_3(r) = (r + \frac{1}{2})^{1-\mu},$

$a(k, m) = \frac{q_1(m-k)}{2-2\mu} - \frac{q_3(m-k)}{1-\mu} + \frac{q_2(m-k)}{(1-\mu)(2-\mu)}, \quad b(k, m) = \frac{2q_3(m-k)}{1-\mu} - \frac{2q_2(m-k)}{(1-\mu)(2-\mu)},$

$c(k, m) = -\frac{q_1(m-k)}{2-2\mu} - \frac{q_3(m-k)}{1-\mu} + \frac{q_2(m-k)}{(1-\mu)(2-\mu)}.$

We note that all computations concerning the problem (6) are conducted for $\mathsf{T} = 1$.

Now, consider the discrete problem

$$B^{\mathrm{z}}_h u^h + \lambda u^h = g^h, \tag{7}$$

in the case $\xi(\mathrm{z}) = 1$.

Lemma 1. Assume $\lambda > 0$. Then, equation (7) admits a unique solution, which is given by the formula

$$u^h = (B_h^z + \lambda I)^{-1} g^h = \left\{ \sum_{j=1}^{M-1} G(j,n;\lambda+\eta) g_j h \right\}_0^M, \tag{8}$$

where

$$G(j,n;\lambda+\eta) = \frac{h(R^n - R^{2M-n})(R^j - R^{2M-j})}{(1+R^{2M-1})(1-R^2)} + \frac{h(R^{|n-j|+1} - R^{2M-n-j+1})}{1-R^2},$$

for $1 \le j \le M-1$ and $1 \le n \le M$

$$R = \frac{1}{1+\delta h}, \quad \delta = \frac{h}{2}(\lambda+\eta) + \sqrt{\frac{h^2}{4}(\lambda+\eta)^2 + (\lambda+\eta)}.$$

Here, $G(j,n;\lambda+\eta)$ is said to be the Green's function of the equation (7), for which we derive the next formula

$$\sum_{j=1}^{M-1} G(j,n;\lambda+\eta)h = \frac{1}{\lambda+\eta} - \frac{1}{\lambda+\eta} \frac{R^{M-n} + R^{M+n-1}}{1+R^{2M-1}}. \tag{9}$$

To demonstrate positivity of B_h^z in the Banach space $\mathcal{C}_h$, we first require the next supplementary lemma.

Lemma 2. The estimates

$$|\delta| \ge \max\left\{ \frac{h}{2}|\lambda+\eta|, \sqrt{|\lambda+\eta|} \right\}, \tag{10}$$

$$|R| \le \frac{1}{1+\sqrt{|\lambda+\eta|}h\cos(\theta)} < 1, \quad |\theta| < \frac{\pi}{2} \tag{11}$$

are satisfied.

Theorem 7. $(\lambda I + B_h^z)^{-1}$ defined by (8) holds the next estimate

$$\|(\lambda I + B_h^z)^{-1}\|_{\mathcal{C}_h \to \mathcal{C}_h} \le \frac{M(\theta,\eta)}{1+|\lambda|}, \tag{12}$$

for every λ lying in the set $\left\{\lambda : |\arg\lambda| \le \theta,\, 0 \le \theta < \frac{\pi}{2}\right\}$.

Proof. Firstly, we consider the operator B_h^z defined by formula (3) for the case $\xi(\mathrm{z}) = 1$. If we set $n = 0$, we get

$$u_0 = \frac{h^2 R\left(1-R^{2M-2}\right)}{\left(1+R^{2M-1}\right)(1-R)} g_1 + \frac{h^2}{\left(1+R^{2M-1}\right)(1-R)} \sum_{j=2}^{M-1} \left(R^j - R^{2M-j}\right) g_j.$$

Then it follows that

$$|u_0| \le 2h^2 \left|\tfrac{R}{1-R}\right| |g_1| + \tfrac{h^2}{1-|R|} \textstyle\sum_{j=2}^{M-1} \left(|R|^j + |R|^{2M-j}\right) |g_j| \le 2h^2 \|g^h\|_{C_h} \left\{ \left|\tfrac{R}{1-R}\right| + \left(\tfrac{|R|}{1-|R|}\right)^2 \right\}. \tag{13}$$

Now, we estimate $|u_n|$ for $1 \le n \le M-1$. Applying the triangle inequality in the formula (8), we achieve

$$|u_n| \le \frac{2h^2}{|1-R^2|} \sum_{j=1}^{M-1} 2|R|^{j+1} |g_j| + \frac{h^2}{|1-R^2|} \sum_{j=1}^{M-1} 2|R|^{|n-j|+1} |g_j|.$$

If we compute the geometric series, we obtain

$$|u_n| \leq 2h^2 \|g^h\|_{C_h} \left\{ \left(\frac{|R|}{1-|R|} \right)^2 \frac{4}{|1+R|} + \left| \frac{R}{1-R} \right| \frac{1}{|1+R|} \right\}. \tag{14}$$

Then using the estimate (11), we arrive at

$$\left(\frac{|R|}{1-|R|} \right)^2 \leq \frac{1}{|\lambda+\eta|\, h^2 \cos^2\theta}. \tag{15}$$

In addition, we have that

$$|\lambda+\eta| = |(|\lambda|\cos\theta+\eta) + i\,|\lambda|\sin\theta| \geq (|\lambda|+\eta)\cos\theta.$$

Therefore, we achieve

$$\frac{1}{|\lambda+\eta|} \leq \frac{\frac{1}{\eta\cos\theta}}{1+\frac{1}{\eta}|\lambda|} \leq \frac{M(\theta,\eta)}{1+|\lambda|}. \tag{16}$$

Using the estimates (15) and (16), we deduce

$$\left(\frac{|R|}{1-|R|} \right)^2 h^2 \leq \frac{\frac{1}{\cos^2\theta}}{|\lambda+\eta|} \leq \frac{M(\theta,\eta)}{1+|\lambda|}. \tag{17}$$

By the estimates (10), (16) and the definition of R, we have

$$\left| \frac{R}{1-R} \right| h^2 = \frac{h}{|\delta|} \leq \frac{2}{|\lambda+\eta|} \leq \frac{M(\theta,\eta)}{1+|\lambda|}. \tag{18}$$

As a result of (13), (14), (17) and (18), we prove that

$$\|u^h\|_{\mathcal{C}_h} \leq \frac{M(\theta,\eta)}{1+|\lambda|} \|g^h\|_{\mathcal{C}_h}.$$

Hence, we obtained the estimate (12) when $\xi(\mathrm{z})=1$. Additionally, assuming that $\lambda>0$ is sufficiently large, we employ the fixed-point theorem to derive analogous results for the Green's function (8), thereby completing the proof. □

Theorem 8. Suppose that $\lambda>0$ and $0<\gamma<\frac{1}{2}$. The norms in the spaces $\mathsf{K}_\gamma(\mathcal{C}_h,\, B_h^{\mathrm{z}})$ and $\mathcal{C}_h^{2\gamma}$ are equivalent uniformly in h for $0<h<h_0$.

Proof. It follows from (8) and (9) that

$$\begin{aligned}&\left(\lambda^\gamma B_h^{\mathrm{z}}(B_h^{\mathrm{z}}+\lambda I)^{-1} g^h\right)_n \\ &= \frac{\eta\lambda^\gamma}{\lambda+\eta} g_n + \frac{\lambda^{\gamma+1}}{\lambda+\eta} \frac{R^{M-n}+R^{M+n-1}}{1+R^{2M-1}} g_n + \lambda^{\gamma+1} \sum_{j=1}^{M-1} G(j,n;\lambda+\eta)(g_n-g_j)h.\end{aligned}$$

We apply the triangle inequality to obtain the next estimate

$$\begin{aligned}&\left| \left(\lambda^\gamma B_h^{\mathrm{z}}(B_h^{\mathrm{z}}+\lambda I)^{-1} g^h\right)_n \right| \\ &\leq \frac{\eta\lambda^\gamma}{\lambda+\eta} |g_n| + \frac{\lambda^{\gamma+1}}{\lambda+\eta} \frac{|R|^{M-n}+|R|^{M+n-1}}{|1+R^{2M-1}|} |g_n| + \lambda^{\gamma+1} \sum_{j=1}^{M-1} |G(j,n;\lambda+\eta)|\, |g_n-g_j|\, h \\ &\leq \left\{ \frac{\eta\lambda^\gamma}{\lambda+\eta} + \frac{\lambda^{\gamma+1}}{\lambda+\eta} + M_1(\eta) \frac{\lambda^{\gamma+1}}{\sqrt{\lambda+\eta}} \sum_{j=1}^{M-1} R^{|n-j|} |(n-j)h|^{2\gamma} h \right\} \|g^h\|_{\mathcal{C}_h}^{2\gamma} \leq M(\eta) \|g^h\|_{\mathcal{C}_h}^{2\gamma}\end{aligned}$$

for $\lambda > 0$ and $\mathrm{z} \in [0, L]$. Thus, we conclude that $g^h \in \mathsf{K}_\gamma(\mathcal{C}_h,\, B_h^{\mathrm{z}})$ and

$$\|g^h\|_{\mathsf{K}_\gamma(\mathcal{C}_h,\, B_h^{\mathrm{z}})} \leq M(\eta)\|g^h\|_{\mathcal{C}_h}^{2\gamma}.$$

Now, we prove the reverse inequality. We have the identity for a PO B_{z}^h as follows:

$$\begin{aligned} g_n &= \int_0^\infty (B_h^{\mathrm{z}} + \lambda I)^{-1} B_h^{\mathrm{z}} (B_h^{\mathrm{z}} + \lambda I)^{-1} g_n d\lambda \\ &= \int_0^\infty \sum_{j=1}^{M-1} G(j,n;\lambda+\eta)\, B_h^{\mathrm{z}}\, (B_h^{\mathrm{z}} + \lambda I)^{-1}\, g_j\, h\, d\lambda. \end{aligned}$$

Then we derive that

$$g_n - g_{n+r} = \int_0^\infty \lambda^{-\gamma} \sum_{j=1}^{M-1} \left(G(j,n;\lambda+\eta) - G(j,n+r;\lambda+\eta)\right) \lambda^\gamma B_h^{\mathrm{z}}\, (B_h^{\mathrm{z}} + \lambda I)^{-1}\, g_j\, h\, d\lambda,$$

from which it follows

$$\|g_n - g_{n+r}\| \leq \int_0^\infty \lambda^{-\gamma} \sum_{j=1}^{M-1} |G(j,n;\lambda+\eta) - G(j,n+r;\lambda+\eta)|\; h\, d\lambda \|g^h\|_{\mathsf{K}_\gamma(\mathcal{C}_h, B_h^{\mathrm{z}})}.$$

If we denote that

$$\mathcal{J}_h = \frac{1}{|rh|^\gamma} \int_0^\infty \lambda^{-\gamma} \sum_{j=1}^{M-1} |G(j,n;\lambda+\eta) - G(j,n+r;\lambda+\eta)|\, h\, d\lambda,$$

then we arrive at

$$\frac{\|g_n - g_{n+r}\|}{|rh|^\gamma} \leq \mathcal{J}_h\, \|g^h\|_{\mathsf{K}_\gamma(\mathcal{C}_h, B_h^{\mathrm{z}})}.$$

Based on the Lemma 2, we get the next estimate

$$\frac{\|g_n - g_{n+r}\|}{|rh|^\gamma} \leq \frac{M}{\gamma(1-2\gamma)}\, \|g^h\|_{\mathsf{K}_\gamma(\mathcal{C}_h, B_h^{\mathrm{z}})}, \quad 1 \leq n < n+r \leq M-1.$$

That is, we deduce

$$\|g^h\|_{\mathcal{C}_h^\gamma} \leq \frac{M}{\gamma(1-2\gamma)}\, \|g^h\|_{\mathsf{K}_\gamma(\mathcal{C}_h, B_h^{\mathrm{z}})}.$$

This completes the proof for the case $\xi(\mathrm{z}) = 1$. Now, suppose that $\xi(\mathrm{z})$ is a continuous function, with $\mathrm{z}, \mathrm{z}_0 \in [0, 1]$ being fixed points. Since

$$\left\| \left(B_h^{\mathrm{z}} - B_h^{\mathrm{z}_0}\right) \left(B_h^{\mathrm{z}_0}\right)^{-1} \right\|_{\mathcal{C}_h \to \mathcal{C}_h} \leq M,$$

and, moreover, the following formula holds:

$$\begin{aligned} B_h^{\mathrm{z}}\, (B_h^{\mathrm{z}} + \lambda I)^{-1}\, g^h = {} & B_h^{\mathrm{z}_0} \left(B_h^{\mathrm{z}_0} + \lambda I\right)^{-1} g^h \\ & + \lambda\, (B_h^{\mathrm{z}} + \lambda I)^{-1} \left(B_h^{\mathrm{z}} - B_h^{\mathrm{z}_0}\right) \left(B_h^{\mathrm{z}_0}\right)^{-1} B_h^{\mathrm{z}_0} \left(B_h^{\mathrm{z}_0} + \lambda I\right)^{-1} g^h, \end{aligned}$$

we conclude that

$$\left|\lambda^\gamma B_h^{\mathrm{z}}\, (B_h^{\mathrm{z}} + \lambda I)^{-1}\, g^h\right| \leq \|g^h\|_{\mathsf{K}_\gamma(\mathcal{C}_h, B_h^{\mathrm{z}_0})} + M_1 \lambda \left\|(B_h^{\mathrm{z}} + \lambda I)^{-1}\right\|_{\mathcal{C}_h \to \mathcal{C}_h} \|g^h\|_{\mathsf{K}_\gamma(\mathcal{C}_h, B_h^{\mathrm{z}_0})}.$$

Therefore,

$$\left|\lambda^{\gamma} B_h^{\mathrm{z}} \left(B_h^{\mathrm{z}} + \lambda I\right)^{-1} g^h\right| \le M \left\|g^h\right\|_{\mathsf{K}_{\gamma}(\mathcal{C}_h, B_h^{\mathrm{z}_0})}.$$

Thus, we obtain

$$\|g^h\|_{\mathsf{K}_{\gamma}(\mathcal{C}_h, B_h^{\mathrm{z}})} \le M \|g^h\|_{\mathsf{K}_{\gamma}(\mathcal{C}_h, B_h^{\mathrm{z}_0})},$$

which completes the proof of this theorem. □

Theorem 9. [18] Suppose A_τ is an operator in $\mathsf{K}_\tau = \mathcal{C}[a,b]_\tau$ that is given by $A_\tau \omega^\tau = \left\{-\frac{\omega_k - \omega_{k-1}}{\tau}\right\}_1^N$ with $\omega_N = 0$. Then, the operator A_τ is positive in K_τ, and the following relation holds:

$$A_\tau^{\gamma} g^\tau = \left\{-\frac{1}{\Gamma(1-\gamma)} \sum_{m=k}^{N} \frac{\Gamma(m-k-\gamma+1)}{\Gamma(m-k+1)} \frac{g_m - g_{m-1}}{h^{\gamma}}\right\}_1^N.$$

In addition, the fractional difference derivative is defined as follows:

$$D_\tau^{\gamma} g^\tau := \left\{-\frac{1}{\Gamma(1-\gamma)} \sum_{m=k}^{N} \frac{\Gamma(m-k-\gamma+1)}{\Gamma(m-k+1)} \frac{g_m - g_{m-1}}{h^{\gamma}}\right\}_1^N.$$

Thus, we arrive at the next theorem.

Theorem 10. Suppose A_τ is an operator in $\mathsf{K}_\tau = \mathcal{C}[a,b]_\tau$ that is given by $A_\tau \omega^\tau = \left\{-\frac{\omega_k - \omega_{k-1}}{\tau}\right\}_1^N$, whose domain is

$$\mathcal{D}(A_\tau) = \left\{\omega^\tau : \frac{\omega_k - \omega_{k-1}}{\tau} \in \mathsf{K}_\tau,\ \omega_N = 0\right\}.$$

Then the operator A_τ is positive in K_τ, and we have

$$A_\tau^{\gamma} g^\tau = D_\tau^{\gamma} g^\tau$$

for all $g^\tau(t) \in \mathcal{D}(A_\tau)$.

Hence, we deduce the next result regarding the coercive stability of the DS (6).

Theorem 11. Let τ and h be sufficiently small positive numbers, and let $0 < \gamma < 1$. Then, the solution of the DS (6) satisfies the coercive stability estimate

$$\max_{1 \le k \le N} \left\| \left\{ \frac{u_{n+1}^k - 2u_n^k + u_{n-1}^k}{h^2} \right\}_{n=1}^{M-1} \right\|_{\mathcal{C}^{\gamma}[0,L]_h} \le M(\gamma) \max_{1 \le k \le N} \|g_k^h\|_{\mathcal{C}^{\gamma}[0,L]_h},$$

where $M(\gamma)$ is independent of τ, h, and g_k^h for $1 \le k \le N$.

The proof of Theorem 11 is based on the positivity of the difference space operator B_h^{z} defined in the formula (3), on Theorem 8 concerning the structure of the fractional space $E_\gamma(C_h, B_h^{\mathrm{z}})$, on Theorem 3 regarding the relationship between FDs and FPs of positive operators, on Theorem 2 regarding the spectral angle of FPs of POs, and on Theorem 4 addressing the FPs of coercively positive sums of two operators.

3 Numerical illustrations

For the numerical illustrations, we present the following problem:

$$\begin{gathered} D_t^{\mu} u(t,\mathrm{z}) - \frac{\partial^2 u(t,\mathrm{z})}{\partial \mathrm{z}^2} + u(t,\mathrm{z}) = g(t,\mathrm{z}), \\ g(t,\mathrm{z}) = \frac{2(1-t)^{2-\mu}}{\Gamma(3-\mu)} \cos^2\left(\frac{\pi \mathrm{z}}{2}\right) + \frac{\pi^2 (1-t)^2}{2} \cos(\pi \mathrm{z}) \\ + (1-t)^2 \cos^2\left(\frac{\pi \mathrm{z}}{2}\right), \\ t \in (0,1), \quad \mathrm{z} \in (0,1), \\ u(1,\mathrm{z}) = 0, \quad \mathrm{z} \in [0,1], \\ u_{\mathrm{z}}(t,0) = u(t,1) = 0, \quad t \in [0,1] \end{gathered} \tag{19}$$

which represents a one-dimensional FPEs with $0 < \mu < 1$.

Problem (19) has the exact solution $u(t,\mathrm{z}) = (1-t)^2 \cos^2\left(\frac{\pi \mathrm{z}}{2}\right)$. Observe that the solution is independent of μ, whereas $g(t,\mathrm{z})$ is dependent explicitly on μ.

Utilizing the DS (4) for the estimate solution of (19), we arrive at

$$-\frac{1}{\Gamma(1-\mu)} \sum_{m=k+1}^{N} \frac{\Gamma(m-k-\mu)}{\Gamma(m-k)} \frac{u_n^m - u_n^{m-1}}{\tau^{\mu}} - \frac{u_{n+1}^k - 2u_n^k + u_{n-1}^k}{h^2} + u_n^k = \phi_n^k,$$

$$\begin{gathered} \phi_n^k = g(t_k, \mathrm{z}_n), \quad t_k = k\tau, \quad 0 \le k \le N-1, \quad N\tau = 1, \\ \mathrm{z}_n = nh, \quad 1 \le n \le M-1, \quad Mh = 1, \\ u_n^N = 0, \quad 0 \le n \le M, \\ u_0^k = u_1^k, \quad u_M^k = 0, \quad 0 \le k \le N. \end{gathered}$$

The resulting system of equations can be expressed in matrix form

$$\begin{gathered} AU_{n+1} + BU_n + CU_{n-1} = D\phi_n, \quad 1 \le n \le M-1, \\ U_0 - U_1 = \tilde{0}, \quad U_M = \tilde{0}, \end{gathered} \tag{20}$$

where

$$A = C = \begin{pmatrix} a_n & 0 & 0 & \cdot & 0 & 0 & 0 \\ 0 & a_n & 0 & \cdot & 0 & 0 & 0 \\ 0 & 0 & a_n & \cdot & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & a_n & 0 & 0 \\ 0 & 0 & 0 & \cdot & 0 & a_n & 0 \\ 0 & 0 & 0 & \cdot & 0 & 0 & 0 \end{pmatrix}_{(N+1)\times(N+1)}, \quad D = \begin{pmatrix} 1 & 0 & 0 & \cdot & 0 & 0 & 0 \\ 0 & 1 & 0 & \cdot & 0 & 0 & 0 \\ 0 & 0 & 1 & \cdot & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & 1 & 0 & 0 \\ 0 & 0 & 0 & \cdot & 0 & 1 & 0 \\ 0 & 0 & 0 & \cdot & 0 & 0 & 0 \end{pmatrix}_{(N+1)\times(N+1)},$$

$$B = \begin{pmatrix} b_{11} & b_{12} & b_{13} & \cdot & b_{1,N-1} & b_{1,N} & b_{1,N+1} \\ 0 & b_{22} & b_{23} & \cdot & b_{2,N-1} & b_{2,N} & b_{2,N+1} \\ 0 & 0 & b_{33} & \cdot & b_{3,N-1} & b_{3,N} & b_{3,N+1} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & b_{N-1,N-1} & b_{N-1,N} & b_{N-1,N+1} \\ 0 & 0 & 0 & \cdot & 0 & b_{N,N} & b_{N,N+1} \\ 0 & 0 & 0 & \cdot & 0 & 0 & b_{N+1,N+1} \end{pmatrix}_{(N+1)\times(N+1)},$$

$$\phi_n = \begin{pmatrix} \phi_n^0 \\ \phi_n^1 \\ \phi_n^2 \\ \cdot \\ \phi_n^{N-2} \\ \phi_n^{N-1} \\ \phi_n^N \end{pmatrix}_{(N+1)\times(1)}, \quad U_p = \begin{pmatrix} U_p^0 \\ U_p^1 \\ U_p^2 \\ \cdot \\ U_p^{N-2} \\ U_p^{N-1} \\ U_p^N \end{pmatrix}_{(N+1)\times(1)}, \quad p \in \{n-1, n, n+1\},$$

$$b_{ij} = \begin{cases} 0, & 1 \le j \le i-1, \\ \frac{1}{\tau^\mu} + 1 + \frac{2}{h^2}, & j = i, \\ \frac{\Gamma(2-\mu)-\Gamma(1-\mu)}{\Gamma(1-\mu)\tau^\mu}, & j = i+1, \\ \frac{1}{\Gamma(1-\mu)\tau^\mu}\left(\frac{\Gamma(j-i+1-\mu)}{\Gamma(j-i+1)} - \frac{\Gamma(j-i-\mu)}{\Gamma(j-i)}\right), & i+2 \le j \le N, \\ -\frac{\Gamma(N-i+1-\mu)}{\Gamma(1-\mu)\Gamma(N-i+1)\tau^\mu}, & j = N+1 \end{cases}$$

for $i = 1, 2, \ldots, N-2$, and

$$a_n = -\frac{1}{h^2}, \quad b_{N-1,N-1} = \frac{1}{\tau^\mu} + 1 + \frac{2}{h^2}, \quad b_{N-1,N} = \frac{\Gamma(2-\mu) - \Gamma(1-\mu)}{\Gamma(1-\mu)\tau^\mu},$$

$$b_{N-1,N+1} = -\frac{\Gamma(2-\mu)}{\Gamma(1-\mu)\tau^\mu}, \quad b_{N,N} = \frac{1}{\tau^\mu} + 1 + \frac{2}{h^2}, \quad b_{N,N+1} = -\frac{1}{\tau^\mu}, \quad b_{N+1,N+1} = 1$$

and

$$\phi_n^k = \frac{2(1-k\tau)^{2-\mu}\cos^2\left(\frac{\pi nh}{2}\right)}{\Gamma(3-\mu)} + \frac{\pi^2}{2}(1-k\tau)^2\cos(\pi nh) + (1-k\tau)^2\cos^2\left(\frac{\pi nh}{2}\right).$$

To solve the problem (20), the procedure of modified Gaussian elimination method is utilized. We seek the solution of the matrix equation in the following form:

$$U_j = \alpha_{j-1}U_{j-1} + \beta_{j-1}, \quad j = 1, 2, \ldots, M-1, \quad U_0 = (I - \alpha_0)^{-1}\beta_0,$$

where α_j's are $(N+1)\times(N+1)$ square matrices and β_j's are $(N+1)\times 1$ column matrices defined for $j = M-1, M-2, \cdots, 1$ by

$$\alpha_{j-1} = -\left(B + A\alpha_j\right)^{-1} C,$$
$$\beta_{j-1} = \left(B + A\alpha_j\right)^{-1}\left(D\,\phi_j - A\beta_j\right).$$

Here, α_{M-1} denotes the zero matrix $(N+1)\times(N+1)$, and β_{M-1} denotes the zero matrix $(N+1)\times 1$.

Furthermore, by utilizing the DS (6), we obtain a second-order accurate DS in both t and z. Specifically, the Crank-Nicolson scheme for parabolic equations can be employed to represent a second-order accurate DS with respect to t and z

$$D_\tau^\mu u_n^k - \frac{1}{2}\left[\frac{u_{n+1}^k - 2u_n^k + u_{n-1}^k}{h^2} + \frac{u_{n+1}^{k-1} - 2u_n^{k-1} + u_{n-1}^{k-1}}{h^2}\right] + \frac{1}{2}\left[u_n^k + u_n^{k-1}\right] = \phi_n^k,$$

$$\phi_n^k = g\Big(t_k - \frac{\tau}{2}, \mathrm{z}_n\Big), \quad t_k = k\tau, \quad \mathrm{z}_n = nh, \quad N\tau = 1, \quad Mh = 1,$$
$$1 \le k \le N, \quad 1 \le n \le M-1,$$
$$u_n^N = 0, \quad 0 \le n \le M,$$

$$3u_0^k - 4u_1^k + u_2^k = 0, \quad u_M^k = 0, \quad 0 \le k \le N,$$

where $D_\tau^\mu u_n^k$ is defined by (5). This yields the system of equations in matrix form

$$\begin{gathered} AU_{n+1} + BU_n + CU_{n-1} = D\phi_n, \quad 1 \le n \le M-1, \\ 3U_0 - 4U_1 + U_2 = \tilde{0}, \quad U_M = \tilde{0}, \end{gathered} \tag{21}$$

where

$$A = C = \begin{pmatrix} a_n & a_n & 0 & \cdot & 0 & 0 & 0 \\ 0 & a_n & a_n & \cdot & 0 & 0 & 0 \\ 0 & 0 & a_n & \cdot & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & a_n & a_n & 0 \\ 0 & 0 & 0 & \cdot & 0 & a_n & a_n \\ 0 & 0 & 0 & \cdot & 0 & 0 & 0 \end{pmatrix}_{(N+1)\times(N+1)}, \quad D = \begin{pmatrix} 1 & 0 & 0 & \cdot & 0 & 0 & 0 \\ 0 & 1 & 0 & \cdot & 0 & 0 & 0 \\ 0 & 0 & 1 & \cdot & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & 1 & 0 & 0 \\ 0 & 0 & 0 & \cdot & 0 & 1 & 0 \\ 0 & 0 & 0 & \cdot & 0 & 0 & 0 \end{pmatrix}_{(N+1)\times(N+1)},$$

$$B = \begin{pmatrix} b_{11} & b_{12} & b_{13} & \cdot & b_{1,N-1} & b_{1,N} & b_{1,N+1} \\ 0 & b_{22} & b_{23} & \cdot & b_{2,N-1} & b_{2,N} & b_{2,N+1} \\ 0 & 0 & b_{33} & \cdot & b_{3,N-1} & b_{3,N} & b_{3,N+1} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & b_{N-1,N-1} & b_{N-1,N} & b_{N-1,N+1} \\ 0 & 0 & 0 & \cdot & 0 & b_{N,N} & b_{N,N+1} \\ 0 & 0 & 0 & \cdot & 0 & 0 & b_{N+1,N+1} \end{pmatrix}_{(N+1)\times(N+1)},$$

$$\phi_n = \begin{pmatrix} \phi_n^0 \\ \phi_n^1 \\ \phi_n^2 \\ \cdot \\ \phi_n^{N-2} \\ \phi_n^{N-1} \\ \phi_n^N \end{pmatrix}_{(N+1)\times(1)}, \quad U_p = \begin{pmatrix} U_p^0 \\ U_p^1 \\ U_p^2 \\ \cdot \\ U_p^{N-2} \\ U_p^{N-1} \\ U_p^N \end{pmatrix}_{(N+1)\times(1)}, \quad p \in \{n-1, n, n+1\},$$

$$b_{ij} = \begin{cases} 0, & 1 \le j \le i-1, \\ d.\, w_1 + \frac{1}{h^2} + \frac{1}{2}, & j = i, \\ d\,(a(i,i+2) + w_2) + \frac{1}{h^2} + \frac{1}{2}, & j = i+1, \\ d\,(a(i,i+3) + b(i,i+2) + w_3), & j = i+2, \\ d\,(a(i,j+1) + b(i,j) + c(i,j-1)), & i+3 \le j \le N-1, \\ d\,(b(i,N) + c(i,N-1)), & j = N, \\ d.\, c(i,N), & j = N+1 \end{cases}$$

for $i = 1, 2, \ldots, N-4$, and

$$a_n = -\frac{1}{2h^2}, \quad b_{N-3,N-3} = d.\, w_1 + \frac{1}{h^2} + \frac{1}{2},$$

$$b_{N-3,N-2} = d\,(a(N-3, N-1) + w_2) + \frac{1}{h^2} + \frac{1}{2},$$

$$b_{N-3,N-1} = d\,(a(N-3, N) + b(N-3, N-1) + w_3),$$

$$b_{N-3,N} = d\,(b(N-3, N) + c(N-3, N-1)), \quad b_{N-3,N+1} = d.\, c(N-3, N),$$

$$b_{N-2,N-2} = d.\, w_1 + \frac{1}{h^2} + \frac{1}{2}, \quad b_{N-2,N-1} = d\,(a(N-2, N) + w_2) + \frac{1}{h^2} + \frac{1}{2},$$

$$b_{N-2,N} = d\,(b(N-2, N) + w_3), \quad b_{N-2,N+1} = d.\, c(N-2, N),$$

$$b_{N-1,N-1} = d\left(-\frac{3^{1-\mu}}{2^{2-\mu}}\frac{1}{1-\mu} + \frac{3^{2-\mu}}{2^{2-\mu}}\frac{1}{(1-\mu)(2-\mu)}\right) + \frac{1}{h^2} + \frac{1}{2},$$

$$b_{N-1,N} = d\left(\frac{3^{1-\mu}}{2^{-\mu}}\frac{1}{1-\mu} - \frac{3^{2-\mu}}{2^{1-\mu}}\frac{1}{(1-\mu)(2-\mu)}\right) + \frac{1}{h^2} + \frac{1}{2},$$

$$b_{N-1,N+1} = d\left(-\frac{3^{2-\mu}}{2^{2-\mu}}\frac{1}{1-\mu} + \frac{3^{2-\mu}}{2^{2-\mu}}\frac{1}{(1-\mu)(2-\mu)}\right),$$

$$b_{N,N-1} = d\left(-\frac{1}{2^{2-\mu}}\frac{1}{1-\mu} + \frac{1}{2^{2-\mu}}\frac{1}{(1-\mu)(2-\mu)}\right),$$

$$b_{N,N} = d\left(\frac{1}{2^{-\mu}}\frac{1}{1-\mu} - \frac{1}{2^{1-\mu}}\frac{1}{(1-\mu)(2-\mu)}\right) + \frac{1}{h^2} + \frac{1}{2},$$

$$b_{N,N+1} = d\left(-\frac{3}{2^{2-\mu}}\frac{1}{1-\mu} + \frac{1}{2^{2-\mu}}\frac{1}{(1-\mu)(2-\mu)}\right) + \frac{1}{h^2} + \frac{1}{2}, \quad b_{N+1,N+1} = 1$$

and

$$\phi_n^k = \frac{2(1-k\tau+\frac{\tau}{2})^{2-\mu}\cos^2\left(\frac{\pi n h}{2}\right)}{\Gamma(3-\mu)} + \frac{\pi^2}{2}(1-k\tau+\tfrac{\tau}{2})^2\cos(\pi n h) + (1-k\tau+\tfrac{\tau}{2})^2\cos^2\left(\tfrac{\pi n h}{2}\right).$$

To solve the difference problem (21), we use the previous algorithm with

$$U_0 = (3I - 4\alpha_0 + \alpha_1\alpha_0)^{-1}\,((4I - \alpha_1)\,\beta_0 - \beta_1).$$

By utilizing the DSs (4) and (6) for the approximate solution of (19), we established the first and the second order accurate DSs. Computational results indicate that the Crank-Nicolson DS exhibits higher accuracy than the first-order scheme. Moreover, all numerical outcomes are independent of the choice of $\mu \in (0, 1)$. To illustrate, Tables 1 and 2 present the results for $\mu = \frac{1}{2}$ and $\mu = \frac{2}{3}$, respectively, for $N = M$ with values $10, 20, 40, 80, 160$.

Table 1

Analysis of errors of first-order and Crank-Nicolson DSs for $\mu = 1/2$

Method	N=M=10	N=M=20	N=M=40	N=M=80	N=M=160
First-order DS	0.1394	0.0676	0.0333	0.0165	0.0082
Crank-Nicolson DS	0.004502	0.000496	0.000111	0.000027	0.000008

Table 2

Analysis of errors of first-order and Crank-Nicolson DSs for $\mu = 2/3$

Method	N=M=10	N=M=20	N=M=40	N=M=80	N=M=160
First-order DS	0.1320	0.0635	0.0311	0.0154	0.0077
Crank-Nicolson DS	0.004611	0.000468	0.000107	0.000026	0.000007

Conclusion

This study established coercive stability estimates for a FPE with Zaremba-type boundary conditions. First- and second-order time-accurate, as well as first-order space-accurate, DSs were analyzed, and their numerical implementation was carried out using a modified Gaussian elimination method. This approach further enables the construction of higher-order schemes in z.

In the future, it would be of interest to investigate alternative definitions of FDs beyond the Riemann–Liouville type, in order to determine if comparable effects arise. Another promising direction of research is the study of various boundary value problems involving nonlocal conditions [19].

Author Contributions

All authors contributed equally to this work.

Conflict of Interest

The authors declare no conflict of interest.

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