

On a family of spectral problems for fourth-order ordinary differential operators

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Academician Olga Ladyzhenskaya highlighted the significance of developing a fundamental system in the space of solenoidal functions for canonical domains such as cubes, squares, and related geometries. In principle, such a construction could be obtained through the solution of the spectral problem associated with the Stokes operator in these standard domains. In the present paper, we investigate a class of fourth-order ordinary differential operators defined on a finite interval. In the corresponding spectral problems naturally arise in the construction of a fundamental system in the space of solenoidal functions for a cubic domain. It is established that each problem from the considered family possesses a discrete countable spectrum, while the associated eigenfunctions form an orthogonal basis in an appropriate Sobolev functional space. Moreover, the basis property remains valid for the triple direct product of the constructed system. The analysis relies on functional-analytic methods, compact operator theory, and Hilbert–Schmidt spectral theory, providing a rigorous framework for the proposed construction and further developments. By additionally exploiting the properties of the curl operator, one can achieve the original objective without explicitly solving the spectral problem for the Stokes operator.

Keywords: spectral problem, fourth-order ordinary differential operator, fundamental system, solenoidal functions, orthogonal basis, Sobolev space, compact operator, curl operator.

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Introduction

Throughout this work, let $\Omega_1 \subset \mathbb{R}^d$, with $d \geq 2$, be a domain with Lipschitz boundary $\partial\Omega_1$, and let $\Omega = (0, l)^d \subset \mathbb{R}^d$.

O.A. Ladyzhenskaya notes in [1, p. 4835]: “Our term Fundamental system $\{\varphi_k\}_{k=1}^\infty$ does not require the assumption of linear independence of elements of the system. Instead, the following property is required: for any $\varepsilon > 0$ and any $\varphi \in \overset{\circ}{W}_2^1(\Omega_1)$ there exists a sum $\varphi^{(\varepsilon)} = \sum_{k=1}^{N_\varepsilon} a_k \varphi_k$, $N_\varepsilon < \infty$, such that $\|\nabla(\varphi - \varphi^{(\varepsilon)})\|_{L^2(\Omega_1)} \leq \varepsilon$ ”.

We introduce the function spaces

$$L^2(\Omega_1) \equiv (L^2(\Omega_1))^d \equiv H(\Omega_1) \oplus H^\perp(\Omega_1), \quad (1)$$

$$H(\Omega_1) = \{\vec{w} \mid \vec{w} \in L^2(\Omega_1), \operatorname{div} \vec{w} = 0, \vec{w} \cdot \vec{n}|_{\partial\Omega_1} = 0\}, \quad (2)$$

$$H^\perp(\Omega_1) = \{\vec{w} \mid \vec{w} \in L^2(\Omega_1), \vec{w} = \operatorname{grad} q, q \in W_2^1(\Omega_1)\}, \quad (3)$$

$$V(\Omega_1) = \{\vec{w} \mid \vec{w} \in \overset{\circ}{W}_2^1(\Omega_1), \operatorname{div} \vec{w} = 0\}, \quad (4)$$

$$\overset{\circ}{W}_2^1(\Omega_1) \equiv (\overset{\circ}{W}_2^1(\Omega_1))^d. \quad (5)$$

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Note that here we have a dense and continuous embeddings

$$V(\Omega_1) \hookrightarrow H(\Omega_1) \equiv H'(\Omega_1) \hookrightarrow V'(\Omega_1).$$

Problem A: Construct a system of functions in $V(\Omega)$ (4) that is fundamental (in the sense of Ladyzhenskaya) for the space $H(\Omega)$ (2) (where $H(\Omega)$ is the orthogonal complement of $H^\perp(\Omega)$ (3)).

1 The spectral problem for the Stokes operator. The curl operator

Let $x = (x_1, \dots, x_d) \in \Omega_1 \subset \mathbb{R}^d$, where $d \geq 2$, and let Ω_1 be a bounded, simply connected open domain with boundary $\partial\Omega_1$. We seek nontrivial solutions $\{\vec{w}_k(x), p_k(x)$, with $x \in \Omega_1$, $k \in \mathbb{N}\}$, along with the corresponding eigenvalues $\{\lambda_k$, $k \in \mathbb{N}\}$ for the following boundary value problem:

$$-\Delta \vec{w}(x) + \nabla p(x) = \lambda \vec{w}(x), \quad x \in \Omega_1, \quad (6)$$

$$\operatorname{div}\{\vec{w}(x)\} = 0, \quad x \in \Omega_1, \quad (7)$$

$$\vec{w}(x) = 0, \quad x \in \partial\Omega_1. \quad (8)$$

First, let us consider the boundary value problem for the Stokes operator

$$-\Delta \vec{w}(x) + \nabla p(x) = \vec{f}(x), \quad x \in \Omega_1, \quad (9)$$

$$\operatorname{div}\{\vec{w}(x)\} = 0, \quad x \in \Omega_1, \quad (10)$$

$$\vec{w}(x) = 0, \quad x \in \partial\Omega_1. \quad (11)$$

As is well known, this problem admits the following variational formulation

$$\sum_{j=1}^d (\nabla w_j, \nabla v_j) \stackrel{\text{def}}{=} ((\vec{w}, \vec{v})) = (\vec{f}, \vec{v}) \quad \forall \vec{v} \in V(\Omega_1). \quad (12)$$

The following theorem holds (see [2, Chapter I, Section 2.2]).

Theorem 1. For an arbitrary open domain $\Omega_1 \subset \mathbb{R}^d$, bounded in at least one direction, and for every $\vec{f} \in L^2(\Omega_1)$, problem (9)–(11) admits a unique solution $w \in V(\Omega_1)$.

Let us return to the spectral problem (6)–(8). According to Theorem 1, the mapping $G : \vec{f} \mapsto w$ is linear and continuous from $L^2(\Omega_1)$ (1) into $V(\Omega_1)$ (4), and hence into $\overset{\circ}{W}_2^1(\Omega_1)$ (5). Since Ω_1 is bounded, the compactness of the embedding $\overset{\circ}{W}_2^1(\Omega_1) \hookrightarrow L^2(\Omega_1)$ implies that the linear operator $G : L^2(\Omega_1) \rightarrow L^2(\Omega_1)$ is compact. This operator is also self-adjoint because, in view of (12) and the symmetry of the operator $\Lambda = G^{-1}$, i.e.,

$$(\Lambda \vec{w}, \vec{v}) = (\vec{w}, \Lambda \vec{v}) \quad \forall w, v \in \mathcal{D}(\Lambda) \equiv V(\Omega_1) \cap \overset{\circ}{W}_2^2(\Omega_1), \quad \text{where} \quad \overset{\circ}{W}_2^2(\Omega_1) = (\overset{\circ}{W}_2^2(\Omega_1))^d,$$

the following equalities hold:

$$(G \vec{f}_1, \vec{f}_2) = (\vec{w}_1, \vec{f}_2) = (\vec{w}_1, \Lambda \vec{w}_2) = ((\vec{w}_1, \vec{w}_2)) = (\Lambda \vec{w}_1, \vec{w}_2) = (\vec{f}_1, \vec{w}_2) = (\vec{f}_1, G \vec{f}_2),$$

where $G f_j = w_j$, $j = 1, 2$.

As a result, we obtain the following theorem.

Theorem 2. [2, chapter I, Point 2.6] For a bounded open domain $\Omega_1 \subset \mathbb{R}^d$ with Lipschitz boundary $\partial\Omega_1$, the operator G possesses an orthonormal sequence of eigenfunctions $\{\vec{w}_k(x), k \in \mathbb{N}\}$ such that the following relations hold:

$$\Lambda \vec{w}_k = \lambda_k \vec{w}_k, \quad k \geq 1, \quad \lambda_k > 0, \quad \lambda_k \rightarrow +\infty, \quad k \rightarrow +\infty, \quad (13)$$

$$\vec{w}_k(x) \in V(\Omega_1) : ((\vec{w}_k, \vec{v})) = \lambda_k (\vec{w}_k, \vec{v}) \quad \forall \vec{v} \in V(\Omega_1), \quad (14)$$

$$(\vec{w}_k, \vec{w}_m) = \delta_{km}, \quad ((\vec{w}_k, \vec{w}_m)) = \lambda_k \delta_{km} \quad \forall k, m \in \mathbb{N}. \quad (15)$$

Thus, the spectral problem (6)–(8) admits a solution satisfying conditions (13)–(15). However, the explicit construction of the system of eigenfunctions $\{\vec{w}_k(x), k \in \mathbb{N}\}$ is an intractable problem, except for the case when the boundary conditions (8) are replaced by periodicity conditions. Similar spectral constructions and basis properties for higher-order differential operators have been extensively studied in recent years [3–5].

The main objective of the present work is to construct a system of functions that is fundamental (in the sense of O.A. Ladyzhenskaya) in the space $H(\Omega_1)$ without solving the spectral problem (6)–(8).

In this work, for simplicity, we will primarily consider the 3-D case. Assume that

$$d = 3, \quad x = (x_1, x_2, x_3) \in \Omega_1 \subset \mathbb{R}^3,$$

and let

$$\vec{U}(x) = (U_1(x), U_2(x), U_3(x)).$$

The components of the velocity vector $\vec{w}(x) = (w_1(x), w_2(x), w_3(x))$ are related to the vector-valued function $\vec{U}(x)$ via the curl operator as follows:

$$\vec{w}(x) = \text{curl}\{\vec{U}(x)\} \stackrel{\text{def}}{=} (\partial_{x_2} U_3 - \partial_{x_3} U_2, \partial_{x_3} U_1 - \partial_{x_1} U_3, \partial_{x_1} U_2 - \partial_{x_2} U_1). \quad (16)$$

In the 3-D case the stream (current) function was not defined.

The following propositions are true [2, Theorem I.1.2–I.1.4, Lemma I.1.3; Appendix I: Propositions 1.2, 2.1].

Proposition 1. Let $\Omega_1 = \Omega = (0, l)^3$. Then the mapping (16)

$$\text{curl} : (\overset{\circ}{W}_2^2(\Omega))^3 \cap (\ker\{\text{curl}\})^\perp \rightarrow \text{curl}\{(\overset{\circ}{W}_2^2(\Omega))^3\} \equiv V(\Omega)$$

is an isomorphism, where $\ker\{\text{curl}\} = (H(\Omega))^\perp$ in space $L^2(\Omega)$.

Proposition 2. Let $\Omega_1 = \Omega = (0, l)^3$. Then the mapping (16)

$$\text{curl} : (\overset{\circ}{W}_2^1(\Omega))^3 \cap (\ker\{\text{curl}\})^\perp \rightarrow \text{curl}\{(\overset{\circ}{W}_2^1(\Omega))^3\} \equiv H(\Omega)$$

is an isomorphism, where $\ker\{\text{curl}\} = (H(\Omega))^\perp$ in space $L^2(\Omega)$.

According to Propositions 1 and 2, it is sufficient to construct a fundamental system of functions in the space $(\overset{\circ}{W}_2^2(\Omega))^3$ that is orthogonal in the space $(\overset{\circ}{W}_2^1(\Omega))^3$. Then, by using the operator curl (16), we will be able to construct a fundamental system of functions (in the sense of O.A. Ladyzhenskaya) in the space $H(\Omega)$ (2).

At this point, there arises the necessity of solving a certain family of spectral problems for fourth-order ordinary differential operators. Related spectral problems for fourth-order differential operators and their self-adjoint extensions have also been investigated in various settings [6–8]. Compactness arguments, spectral asymptotics and basis properties for fourth-order operators continue to play an

important role in the modern theory of operator differential equations [9–11]. The remainder of the paper is devoted to this issue: the formulation of the main result (Theorem 3) is presented in Section 2, while its proof is given in Section 3.

As for the construction of a fundamental system of functions in the space $H(\Omega)$ (2), it can be carried out in a manner similar to that used in our previous works [12–14].

2 A family of spectral problems for fourth-order ordinary differential operators

Let $\Omega = (0, l)$. We introduce the following definitions.

Definition 1. Denote by $V_1(0, l)$ and $V_2(0, l)$ the Hilbert spaces endowed with the inner products and the norms

$$V_1(0, l) : \begin{cases} (u'(x), v'(x))_{L^2(0, l)} \quad \forall u, v \in \overset{\circ}{W}_2^1(0, l), \\ \|u\|_{V_1(0, l)} \equiv [(u'(x), u'(x))_{L^2(0, l)}]^{1/2}, \end{cases} \quad (17)$$

$$V_2(0, l) : \begin{cases} ((u, v)) = (u''(x), v''(x))_{L^2(0, l)} \quad \forall u, v \in \overset{\circ}{W}_2^2(0, l), \\ \|u\|_{V_2(0, l)} \equiv [((u(x), u(x)))]^{1/2}. \end{cases} \quad (18)$$

We consider the following family of spectral problems:

$$\begin{cases} X^{IV}(x) + \mu X(x) = -\lambda^2 X''(x), & x \in (0, l), \\ X(0) = X(l) = X'(0) = X'(l) = 0, & \mu \in \mathbb{R}^1, \end{cases} \quad (19)$$

for which the following theorem holds true.

Theorem 3 (Main result). For each fixed value of the parameter $\mu \in \mathbb{R}^1$, the spectral problem (19) has a countable spectrum $\{\lambda_n^2, n \in \mathbb{N}\}$ with a single limit point $\lambda^2 = +\infty$. The corresponding system of eigenfunctions $\{X_n(x), n \in \mathbb{N}\} \subset V_2(0, l)$ (18) forms an orthogonal basis in the space $V_1(0, l)$ (17).

3 Proof of Theorem 3

To solve $X(x) \in V_2(0, l)$ of the boundary value problem (19) we obtain the integral identity

$$((X, Y)) + \mu g(X, Y) = \lambda^2 (X', Y') \quad \text{for all } Y(x) \in V_2(0, l), \quad (20)$$

where

$$(X, Y)_{V_2(0, l)} \stackrel{\text{def}}{=} ((X, Y)) = \int_0^l X''(x) Y''(x) dx,$$

$$g(X, Y) = \int_0^l X(x) Y(x) dx \leq \tilde{c}_1 \|X\|_{V_1(0, l)} \|Y\|_{V_1(0, l)} \leq c_1 \|X\|_{V_2(0, l)} \|Y\|_{V_2(0, l)}, \quad (21)$$

$$(X', Y') = \int_0^l X'(x) Y'(x) dx \leq \tilde{c}_2 \|X\|_{V_1(0, l)} \|Y\|_{V_1(0, l)} \leq c_2 \|X\|_{V_2(0, l)} \|Y\|_{V_2(0, l)}. \quad (22)$$

By F. Riesz's theorem, the functional $g(X, Y)$ is uniquely representable as a scalar product

$$g(X, Y) = ((AX, Y)), \quad \forall Y \in V_2(0, l), \quad (23)$$

where A is a bounded linear operator in $V_2(0, l)$ with norm not exceeding c_1 from (21). Next, we have

$$((BX, Y)) = \int_0^l X'(x) Y'(x) dx, \quad \forall Y \in V_2(0, l), \quad (24)$$

where B is a bounded linear operator in $V_2(0, l)$ with norm not exceeding c_2 from (22). The operators A and B are linear and completely continuous in the space $V_2(0, l)$ (as will be shown below). Moreover, the operator A is symmetric and the operator B is symmetric and positive, i.e.

$$((BX, Y)) = ((X, BY)), \quad \forall X, Y \in V_2(0, l); \quad ((BX, X)) > 0 \quad \forall X \neq 0. \quad (25)$$

As a result, we arrive at the functional equation

$$X + \mu AX = \lambda^2 BX \quad (26)$$

in the space $V_2(0, l)$. We represent the equation (26) as

$$(I + \mu A + \lambda_0^2 B)X = (\lambda^2 + \lambda_0^2)BX, \quad (27)$$

where I is the identity operator. Obviously, for a sufficiently large real λ_0^2 , the operator $D = I + \mu A + \lambda_0^2 B$ has a bounded inverse, i.e.,

$$X = (\lambda^2 + \lambda_0^2)D^{-1}BX.$$

The operator $D^{-1}B$, as a product of a bounded operator and a completely continuous operator, is a completely continuous operator.

Let us show that for a sufficiently large real λ_0^2 , the operator $D = I + \mu A + \lambda_0^2 B$ has a bounded inverse. We denote $DX = Z$. According to (23)–(24), the equality (27) is equivalent to the integral identity (20) if we accept the assumption: $\lambda^2 = \lambda_0^2$. Then from the identity (20) and the inequality

$$((X, X)) + \mu g(X, X) \geq \|X\|_{V_2(0, l)}^2 - |\mu|c_3\|X\|_{V_1(0, l)}^2,$$

we obtain $((Z, X)) = ((X, X)) + \mu g(X, X) + \lambda_0^2\|X\|_{V_1(0, l)}^2$

$$\geq \|X\|_{V_2(0, l)}^2 + (\lambda_0^2 - |\mu|c_3)\|X\|_{V_1(0, l)}^2.$$

Hence, for $\lambda_0^2 \geq |\mu|c_3$ we obtain

$$\|X\|_{V_2(0, l)} \leq \|Z\|_{V_2(0, l)} = \|DX\|_{V_2(0, l)}, \quad \|D^{-1}Z\|_{V_2(0, l)} \leq \|Z\|_{V_2(0, l)},$$

i.e., the operator D for such λ_0^2 has a bounded inverse.

We use the following assertion: Corollary from the monograph [15, Chapter XII, §2, pp. 460–461]:

Corollary. A continuous inverse operator D^{-1} exists if and only if a continuous inverse operator $(D^*)^{-1}$ exists. Moreover,

$$(D^*)^{-1} = (D^{-1})^*. \quad (28)$$

Using the self-adjointness of the operator D , from (28) we obtain the self-adjointness of the inverse operator D^{-1} , i.e.,

$$D^{-1} = (D^*)^{-1} = (D^{-1})^*.$$

Along with the previous Corollary, one could also use Theorem 5.30 from [16, Chapter III, §5, p. 169]:

Theorem 5.30. Let $D \in \mathcal{C}(X, Y)$ be densely defined. If D^{-1} exists and belongs to $\mathcal{B}(Y, X)$, then $(D^*)^{-1}$ exists and belongs to $\mathcal{B}(X^*, Y^*)$, with

$$(D^*)^{-1} = (D^{-1})^*. \quad (29)$$

Conversely, if $(D^*)^{-1}$ exists and belongs to $\mathcal{B}(X^*, Y^*)$, then D^{-1} exists and belongs to $\mathcal{B}(Y, X)$ and (29) holds.

The set of all closed operators from X to Y is denoted by $\mathcal{C}(X, Y)$. The set of all bounded operators from X to Y is denoted by $\mathcal{B}(X, Y)$.

Furthermore, the following assertions are needed for the proof of our main result:

Theorem 2 from [17, chapter IV, § 6, p. 243]:

Theorem 2. If C is a compact operator and D is a bounded operator, then the operators CD and DC are compact.

Theorem 2 from [18, chapter IV, §18.2, p. 189]:

Theorem 2. Let operators C and D be self-adjoint. The operator CD is self-adjoint if and only if C and D are permutable.

Let us show that the self-adjoint operators D^{-1} and B from (27) are permutable. For this, we use Proposition g) from [19, chapter VIII, point 116, pp. 300–301]:

Proposition. Let A be a bounded linear operator and B is a (general) linear operator. Then the permutability relation satisfies the following rule:

(g) If the operator B^{-1} exists, then the permutability of A and B implies that A and B^{-1} are permutable.

Proof of point (g) of the above Proposition. For all $f \in \mathcal{D}(B^{-1})$, we have

$$Af = ABB^{-1}f = BAB^{-1}f,$$

therefore, $Af \in \mathcal{D}(B^{-1})$ and

$$B^{-1}Af = AB^{-1}f \quad \forall f \in \mathcal{D}(B^{-1}).$$

It remains to establish that the operators A and B are permutable. This will be shown below.

Thus, it remains to use the spectral properties of a completely continuous self-adjoint operator [17, chapter IV, §6, pp. 246–247, Hilbert–Schmidt Theorem], which completes the proof of Theorem 3.

With respect to relations (21) and (23) we have

$$\begin{aligned} -g(X, Y) &= -\int_0^l X(x) Y(x) dx = \int_0^l \int_0^x X'(\xi) d\xi \int_0^x Y'(\xi) d\xi dx \\ &= \int_0^l d \left(\int_0^x (x - \xi) X'(\xi) d\xi \right) \int_0^x Y'(\xi) d\xi = -\int_0^l \int_0^x (x - \xi) X'(\xi) d\xi Y'(x) dx \\ &= -\int_0^l d \left(\int_0^x \frac{(x - \xi)^2}{2} X'(\xi) d\xi \right) Y'(x) = \int_0^l \int_0^x \frac{(x - \xi)^3}{6} X''(\xi) d\xi Y''(x) dx \\ &= -((AX(x), Y(x))), \quad \text{where } A \in \mathcal{L}(V_2(0, l), V_2(0, l)). \end{aligned} \quad (30)$$

$$AX(x) = \int_0^x \frac{(x - \xi)^5}{120} X''(\xi) d\xi. \quad (31)$$

With respect to relations (22), (24)–(25) we have

$$\begin{aligned} \int_0^l X'(x) Y'(x) dx &= - \int_0^l X(x) Y''(x) dx \\ &= - \int_0^l \int_0^x (x - \xi) X''(\xi) d\xi Y''(x) dx = \int_0^l [BX(x)]'' Y''(x) dx \\ &= ((BX(x), Y(x))), \quad \text{where } B \in \mathcal{L}(V_2(0, l), V_2(0, l)), \end{aligned} \quad (32)$$

i.e.,

$$BX(x) = - \int_0^x \frac{(x - \xi)^3}{6} X''(\xi) d\xi. \quad (33)$$

Next, from (31) and (33) we obtain the equalities (where $X(x) \in V_2(0, l)$):

$$\begin{aligned} -720ABX &= \int_0^x (x - \xi_1)^5 \int_0^{\xi_1} (\xi_1 - \xi)^3 X''(\xi) d\xi d\xi_1 \\ &= \frac{5}{4} \int_0^x \int_{\xi}^x (x - \xi_1)^4 (\xi_1 - \xi)^4 d\xi_1 X''(\xi) d\xi, \\ -720BAX &= \int_0^x (x - \xi_1)^3 \int_0^{\xi_1} (\xi_1 - \xi)^5 X''(\xi) d\xi d\xi_1 \\ &= \frac{5}{4} \int_0^x \int_{\xi}^x (x - \xi_1)^4 (\xi_1 - \xi)^4 d\xi_1 X''(\xi) d\xi, \\ &= \int_{\xi}^x (x - \xi_1)^4 (\xi_1 - \xi)^4 d\xi_1 = \frac{(x - \xi)^9}{A_9^3} \end{aligned}$$

which implies that $AB = BA$. Therefore, A and B are permutable.

We show that A is a completely continuous operator in $V_2(0, l)$. For this purpose, we verify that any weakly convergent sequence $\{v_k\}_{k=1}^{\infty}$ in $V_2(0, l)$ is transformed by A into a strongly convergent sequence $\{Av_k\}_{k=1}^{\infty}$. Since A is bounded, $\{Av_k\}_{k=1}^{\infty}$ weakly converges to Au , where $u(x)$ is the weak limit of $\{v_k\}_{k=1}^{\infty}$.

Further, in view of the compactness of the embedding operator $V_2(0, l) \hookrightarrow V_1(0, l)$, the sequences $\{v_k\}_{k=1}^{\infty}$ and $\{Av_k\}_{k=1}^{\infty}$ converge strongly in $V_1(0, l)$ to u and Au , respectively. Using (23) and (30) of the operator A , we obtain the estimate

$$\begin{aligned} ((A(v_k - v_m), A(v_k - v_m))) &= g(v_k - v_m, A(v_k - v_m)) \\ &\leq \tilde{c}_1 \|v_k - v_m\|_{V_1(0, l)} \|A(v_k - v_m)\|_{V_1(0, l)}. \end{aligned}$$

From this it is clear that as $k, m \rightarrow \infty$ the right-hand side tends to zero, and therefore $\{Av_k\}_{k=1}^{\infty}$ is indeed a sequence strongly convergent in $V_2(0, l)$. Note that the constant $\tilde{c}_1$ is taken from inequality (21).

Thus, the complete continuity of the operator A is proven.

We show that B is a completely continuous operator in $V_2(0, l)$. For this purpose, we verify that any weakly convergent sequence $\{v_k\}_{k=1}^\infty$ in $V_2(0, l)$ is transformed by B into a strongly convergent sequence $\{Bv_k\}_{k=1}^\infty$.

Since B is bounded, $\{Bv_k\}_{k=1}^\infty$ weakly converges to Bu , where $u(x)$ is the weak limit of $\{v_k\}_{k=1}^\infty$.

Further, in view of the compactness of the embedding operator $V_2(0, l) \hookrightarrow V_1(0, l)$, the sequences $\{v_k\}_{k=1}^\infty$ and $\{Bv_k\}_{k=1}^\infty$ converge strongly in $V_1(0, l)$ to u and Bu , respectively. Using (24) and (32) of the operator B , we obtain the estimate

$$\begin{aligned} ((B(v_k - v_m), B(v_k - v_m))) &= \int_0^l (v'_k - v'_m)[B(v_k - v_m)]' dx \\ &\leq \tilde{c}_2 \|v_k - v_m\|_{V_1(0, l)} \|B(v_k - v_m)\|_{V_1(0, l)}. \end{aligned}$$

From this it is clear that as $k, m \rightarrow \infty$ the right-hand side tends to zero, and therefore $\{Bv_k\}_{k=1}^\infty$ is indeed a sequence strongly convergent in $V_2(0, l)$. Note that the constant $\tilde{c}_2$ is taken from inequality (22).

Thus, the complete continuity of the operator B is proven.

Conclusion

In this work, we studied a family of spectral problems for fourth-order ordinary differential operators on a finite interval. These spectral problems arise in the construction of a fundamental system in the space of solenoidal functions for a cubic domain. We showed that each spectral problem in the considered family has a countable spectrum, while the corresponding system of eigenfunctions forms an orthogonal basis in a Sobolev functional space. These results can be used for the construction of a fundamental system in the space of solenoidal functions for a cubic domain. The proposed approach is also consistent with recent developments in spectral analysis of higher-order operators, biharmonic problems and generalized Helmholtz/curl decompositions [20–22].

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Author contributions

M.T. Jenaliyev served as the principal investigator of the research grant and supervised the research process. M.G. Yergaliyev and K.S. Sharipov collected and analyzed data, and led manuscript preparation. All authors participated in the revision of the manuscript and approved the final submission.

Conflict of interest

The authors declare no conflict of interest.

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