

## Estimates of solutions for a class of nonlinear equations in finite-dimensional space

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This paper investigates a class of nonlinear operator equations in a finite-dimensional Hilbert space. Specifically, equations of the form  $u + L(u) = g$  are considered, where  $L$  is a nonlinear continuous operator acting on the space  $H$ , and  $g$  is a given element of this space. The main focus is on deriving a priori estimates for solutions and studying the conditions for their existence. For a sufficiently broad class of nonlinear operators, uniform estimates of solutions are established, expressed in terms of the parameters of the original problem and the properties of the operator under consideration. It is shown that these estimates do not depend on the index of the finite-dimensional approximation, which allows them to be used in the study of the corresponding infinite-dimensional problems. The results justify the method of finite-dimensional approximations and to carry out the limiting transition to the original problem. This makes it possible to obtain a priori estimates for solutions of infinite-dimensional nonlinear operator equations arising in problems of mathematical physics and functional analysis. The proposed approach can be applied to the study of a wide class of nonlinear problems and to the construction of approximate methods for their solution.

**Keywords:** nonlinear operator equation, finite-dimensional Hilbert space, a priori estimates of solutions, nonlinear operator, existence of solutions, approximation of solutions, operator equations, functional analysis.

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### Introduction

In many problems of mathematical physics, energy estimates and identities play an important role, as they make it possible to obtain a priori estimates of solutions to the equations under consideration. The use of such estimates allows one to prove existence theorems for weak solutions of problems in mathematical physics. However, weak solutions do not always allow the application of perturbation theory methods or a more detailed analysis of the properties of the solution [1]. In this connection, there arises a need to obtain strong solutions to the corresponding problems [2–4].

One of the widely used methods of investigation is the use of finite-dimensional approximations of the original problem [5, 6]. An important condition in this approach is the derivation of a priori estimates that do not depend on the approximation index. The existence of such estimates makes it possible to pass to the limit and prove the existence of a solution to the original problem [7, 8].

In the present paper, a nonlinear equation in a finite-dimensional space is considered. For a sufficiently wide class of equations, a priori estimates of solutions are obtained, which explicitly depend on the constants appearing in the initial conditions of the problem. It is shown that in the case when finite-dimensional systems serve as approximations of the original problem, the obtained estimates do not depend on the approximation index. This makes it possible to pass to the limit and obtain a priori estimate for the solution of the original mathematical physics problem.

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# 1 Notation and basic assumptions

Throughout this section,  $H$  denotes a finite-dimensional real Hilbert space with inner product  $\langle \cdot, \cdot \rangle$  and norm  $\|\cdot\|$ .

We consider equations of the form

$$u + L(u) = g \in H, \quad (1)$$

where  $L(\cdot)$  is a nonlinear continuous mapping such that  $L(0) = 0$ , and  $g$  is an element of the space  $H$ . The solution  $u$  of equation (1) is sought in  $H$ .

We focus on finite-dimensional equations of type (1), which are finite-dimensional approximations of infinite-dimensional problems of the same form in an infinite-dimensional Hilbert space [9, 10]. It is of *particular importance to derive estimates that do not depend on the approximation index, allowing passage to the limit and obtaining, in the limit, an a priori estimate for the solution of the infinite-dimensional problem*. The infinite-dimensional problems of type (1), which are the ultimate goal of this study, typically arise in mathematical physics and are written in a weak (variational) form [11, 12].

In this section, we denote  $f(u)$  as an operation of the form

$$f(u) := u + L(u).$$

If  $\xi \in [0; +\infty)$  is a parameter and  $u(\xi)$  is a vector-valued function continuously differentiable with respect to  $\xi$ , then we assume that  $L(u(\xi))$ , as well as the expressions involving  $L(u)$  and  $f(u)$  that appear later, are also continuously differentiable.

We introduce the notation  $L_u$ :

$$(L(u(\xi)))_\xi = L_{u(\xi)} u_\xi(\xi).$$

It is obvious that the operator  $L_u$  (for each  $u \in H$ ) is linear

$$L_u v = (L(u(\xi))) \Big|_{u_\xi=v}.$$

Here,  $L_u$  denotes the Frechet derivative of the operator  $L$  at the point  $u$ :

$$L_u v = \frac{d}{d\xi} L(u - \xi v) \Big|_{\xi=0}.$$

We have

$$(f(u(\xi)))_\xi = u_\xi + L_u u_\xi = (E + L_u) u_\xi.$$

Further, if  $u_0, v_0 \in H$ , then the vector  $L_{u_0} v_0$  is defined as follows: take a continuously differentiable vector-function  $u(\xi)$  such that

$$u \Big|_{\xi=0} = u_0, \quad u_\xi(\xi) \Big|_{\xi=0} = v_0,$$

and define as  $L_{u_0} v_0$  the vector

$$L_{u_0} v_0 = (L(u(\xi)))_\xi \Big|_{\xi=0}.$$

Here and throughout,  $E$  denotes the identity operator.

We denote

$$D_u = E + L_u, \quad D_u^* = E + L_u^*$$

$$D_u^* f(u) = (E + L_u^*) f(u).$$

The symbol  $*$  stands for the adjoint operation.

$$M_u a = \left( D_{u(\xi)}^* f(u(\xi)) \right)_{\xi} \Big|_{\substack{u(\xi) = u \\ u_{\xi}(\xi) = a}} = M_u u_{\xi} \Big|_{u_{\xi} = a} = M_u a.$$

*Condition U1.* There exist linear invertible operators  $T$  and  $Q$  such that

$$\|T\| \leq C_T, \quad \|Q\| \leq C_T, \quad \|T^{-1}\| < \infty, \quad \|Q^{-1}\| < \infty,$$

and for any  $u \in H$  the inequalities hold

$$\langle Tu, L(u) \rangle \geq 0, \quad \langle Tu, u \rangle \geq \|Qu\|^2 \geq C_{1T} \|Tu\|^2, \quad (2)$$

where  $C_T$  and  $C_{1T}$  are some fixed positive constants.

In what follows, by  $C$  or  $c$  (uppercase or lowercase, with or without indices, with or without tildes) we will denote constants (generally different in different places) that do not depend on the neighboring factors.

The following result holds

*Theorem 1.* Suppose that condition U1 is satisfied. Then for any  $g \in H$  the problem

$$f(u) = g$$

has a solution  $u \in H$ , satisfying the estimate

$$C_{1T} \|Tu\|^2 \leq \|Qu\|^2 \leq C_T \|g\|^2, \quad (3)$$

where the operator  $Q$ ,  $T$  and the constants  $C_T$ ,  $C_{1T}$  are taken from condition U1.

The result of this theorem is well known and widely used. We present it in a form convenient for us.

## 2 Proof of Theorem 1

Proofs of theorems whose statements are similar to that of Theorem 1 are well known, and we could restrict ourselves to a reference [1, p. 550]. However, for completeness, we provide a proof of Theorem 1.

For the proof, we use a standard technique in a form convenient for us.

If  $g \equiv 0$ , the equation  $u + L(u) = 0$  has the solution  $u \equiv 0$ . Therefore, in this case estimate (3) holds.

Let  $0 \neq g \in H$ . Denote by  $M(g)$  the set of vectors

$$M(g) = \{u \in H : \langle Tu, u \rangle \leq 16 \langle Tg, g \rangle\},$$

where  $T$  is the operator from condition U1.

Assume that the equation  $u + L(u) = g$  has no solution  $u \in M(g)$ . Define the mapping

$$F(u) = - \frac{u + L(u) - g}{\sqrt{\langle T(u + L(u) - g), u + L(u) - g \rangle}} 4\sqrt{\langle Tg, g \rangle}. \quad (4)$$

Since the equation  $u + L(u) = g$  has no solution, it follows from inequality (2) in condition U1 that the mapping  $F(\cdot)$  is continuous. It is easy to see that this mapping sends the set  $M(g)$  into itself.

Since, according to condition U1, the inequality  $\langle Tv, v \rangle \geq \|Qv\|^2$  holds for any  $v \in H$  and the operator  $Q$  is invertible, the quantity  $\langle Tu_0, u_0 \rangle$  can be taken as the square of the norm of the vector  $u_0$ ,

and the quantity  $\frac{1}{2}(\langle Tu_0, g \rangle + \langle u_0, T^*g \rangle)$  as the inner product (consistent with this norm) of the vectors  $u_0$  and  $g$  in some real Hilbert space, which in this subsection will be denoted by  $\tilde{H}$ .\*

In the space  $\tilde{H}$  the set  $M(g)$  is a ball. Therefore, by Brouwer's fixed point theorem and the finite dimensionality of  $H$  (see [2]), the mapping  $F(\cdot)$  has a fixed point  $u_0 \in M(g)$ , that is,

$$F(u_0) = u_0. \quad (5)$$

Applying the operator  $T$  to (5) and then taking the inner product of the resulting equality with  $u_0 + L(u_0) - g$  we obtain, using (4),

$$\begin{aligned} -A &\equiv -4\sqrt{\langle Tg, g \rangle} \sqrt{\langle T(u_0 + L(u_0) - g), u_0 + L(u_0) - g \rangle} = \\ &= \langle Tu_0, u_0 + L(u_0) - g \rangle \geq \langle Tu_0, u_0 \rangle - \langle Tu_0, g \rangle = \\ &= \langle Tu_0, u_0 \rangle - \frac{1}{2}(\langle Tu_0, g \rangle + \langle u_0, T^*g \rangle). \end{aligned} \quad (6)$$

In deriving (6) the inequality (2) from condition U1 was used in the penultimate step.

Therefore, applying the well-known Cauchy inequality, we obtain

$$-A \geq \langle Tu_0, u_0 \rangle - (\varepsilon^{-1} \langle Tu_0, u_0 \rangle + \varepsilon \langle Tg, g \rangle) = \langle Tu_0, u_0 \rangle (1 - \varepsilon^{-1}) - \varepsilon \langle Tg, g \rangle. \quad (7)$$

From (5) and (4) we have

$$\langle Tu_0, u_0 \rangle = \langle TF(u_0), F(u_0) \rangle = 16 \langle Tg, g \rangle.$$

Hence, from (7) we obtain

$$-A \geq [(1 - \varepsilon^{-1})16 - \varepsilon] \langle Tg, g \rangle.$$

Choosing  $\varepsilon = 2$ , we get  $-A \geq 6 \langle Tg, g \rangle$ .

The negativity of the left-hand side contradicts inequality (2) from condition U1. This proves that the equation  $u + L(u) = g$  has a solution  $u \in M(g)$ .

For solving the equation  $u + L(u) = g$ , taking the inner product with  $Tu$ , we have

$$\langle Tu, u \rangle + \langle Tu, L(u) \rangle = \langle Tu, g \rangle = \frac{1}{2}(\langle Tu, g \rangle + \langle u, T^*g \rangle) = \langle u, g \rangle_{\tilde{H}}.$$

Here, on the right-hand side stands the inner product of the vectors  $Tu$  and  $g$  in the Hilbert space  $\tilde{H}$ . Applying the Cauchy inequality and taking into account the first inequality in (2) from condition U1, we obtain:

$$\langle Tu, u \rangle \leq \langle Tu, u \rangle + \langle Tu, L(u) \rangle = \langle u, g \rangle_{\tilde{H}} \leq \|u\|_{\tilde{H}} \|g\|_{\tilde{H}} = \sqrt{\langle Tu, u \rangle} \sqrt{\langle Tg, g \rangle}.$$

Hence,

$$\langle Tu, u \rangle \leq \langle Tg, g \rangle \leq \|Tg\| \|g\| \leq C_T \|g\|^2,$$

where  $C_T$  is the constant from condition U1.

Now from the second inequality (2) of the condition U1 we get

$$\|Qu\|^2 \leq C_T \|g\|^2.$$

Theorem 1 is proved.

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\*In a real Hilbert space, the equality  $\langle a, b \rangle = \langle b, a \rangle$  holds for the inner product of vectors  $a$  and  $b$ . Then, for a vector  $u \in H$  and a linear operator  $T$ , we have  $\langle Tu, u \rangle = \langle u, Tu \rangle$ . On the other hand,  $\langle Tu, u \rangle = \langle u, T^*u \rangle$ , where  $T^*$  denotes the operator adjoint to the operator  $T$ . Therefore,  $\langle Tu, u \rangle = \langle \frac{1}{2}(T + T^*)u, u \rangle$ . If the quantity  $\langle Tu, u \rangle$  is strictly positive,

then it can be regarded as the square of a norm (which indeed it is):  $\langle Tu, u \rangle = \left\| \sqrt{\frac{1}{2}(T + T^*)} u \right\|^2$ . Thus, the inner product of vectors  $a, b \in H$ , consistent with this norm, is defined by the formula  $\langle a, b \rangle_{\tilde{H}} = \frac{1}{2}(\langle Ta, b \rangle + \langle a, T^*b \rangle)$ .

### 3 Formulation of the main finite-dimensional theorem

From Theorem 1, since the operators  $T$  and  $Q$  are invertible (under the given assumptions), we obtain the estimate

$$\|u\| \leq \tilde{C} \|f(u)\|. \quad (8)$$

In this inequality, the constant  $\tilde{C}$  depends on the smallest eigenvalue of the operator  $T^*T$  (or  $Q^*Q$ ).

In the case where  $f(\cdot)$  is a finite-dimensional approximation of a nonlinear problem (written in weak form), the estimates (8) will depend on the approximation index and therefore cannot ensure passage to the limit.

To ensure the possibility of passing to the limit, it is sufficient, for finite-dimensional approximations, to obtain estimates that do not depend on the approximation index.

To obtain such estimates, we will use the conditions given below.

*Condition U2.* For the mapping  $L(\cdot)$  and the operators  $L_u, M_u$  for any elements  $u, v \in H$  the following inequalities hold:

$$\begin{aligned} \|M_u - M_v\|_{H \rightarrow H} + \|L_u - L_v\|_{H \rightarrow H} + \|L(u) - L(v)\| &\leq \\ &\leq (\psi(\|u\|) + \psi(\|v\|)) \|u - v\|; \end{aligned} \quad (9)$$

$$\|u\|^2 + \|L(u)\|^2 + \|L_u u\|^2 + \|M_u u\| \leq \psi(\|u\|). \quad (10)$$

In (9) and (10)  $\psi(\cdot)$  is a function defined by

$$\psi(t) = C(t^2 + 1)^m, \quad (11)$$

where  $C$  and  $m > 0$  are constants.

*Condition U3.* There exists an invertible linear operator  $T$  such that

$$\|T\| \leq C_0, \quad \|T^{-1}\| < \infty,$$

and for any  $u \in H$  the following inequalities hold:

$$\langle f(u), Tu \rangle \geq 0, \quad \|Tu\| \leq d_0 \|f(u)\|, \quad (12)$$

where  $d_0 > 0$  is a constant.

We introduce the notation

$$\tilde{L}(u) = L(u) + \|Tu\|^{2k+\varepsilon} Tu, \quad (13)$$

$$\tilde{f}(u) = u + \tilde{L}(u) = f(u) + \|Tu\|^{2k+\varepsilon} Tu, \quad (14)$$

where  $T$  is the operator from condition U3. The numbers  $\varepsilon, k$  from (13) and  $m$  from (11) are fixed constants such that  $k > m + 3$  and  $\varepsilon > 0$ .

We denote by  $\tilde{L}_u, \tilde{M}_u, \tilde{D}_u$  the operators analogous to  $L_u, M_u, D_u$ , in the definitions of which  $f(u)$  and  $L(u)$  are replaced by  $\tilde{f}(u)$  and  $\tilde{L}(u)$ , respectively.

We note the following lemma that follows from condition U2.

*Lemma 1.* For the mapping  $\tilde{L}(\cdot)$  and operators  $\tilde{L}_u, \tilde{L}_u^*, \tilde{D}_u, \tilde{D}_u^*, \tilde{M}_u, \tilde{M}_u^*$  the following inequalities hold:

$$\begin{aligned} \|\tilde{M}_u - \tilde{M}_v\|_{H \rightarrow H} + \|\tilde{M}_u^* - \tilde{M}_v^*\|_{H \rightarrow H} + \|\tilde{L}_u - \tilde{L}_v\|_{H \rightarrow H} + \\ + \|\tilde{L}_u^* - \tilde{L}_v^*\|_{H \rightarrow H} + \|\tilde{D}_u - \tilde{D}_v\|_{H \rightarrow H} + \|\tilde{D}_u^* - \tilde{D}_v^*\|_{H \rightarrow H} \leq \\ \leq \tilde{\psi}(\|u\|) \tilde{\psi}(\|v\|) \|u - v\|; \end{aligned} \quad (15)$$

$$\begin{aligned} & \|\widetilde{L}(u)\| + \|\widetilde{L}_u\|_{H \rightarrow H} + \|\widetilde{M}_u\|_{H \rightarrow H} + \|\widetilde{D}_u\|_{H \rightarrow H} + \\ & + \|\widetilde{L}_u^*\|_{H \rightarrow H} + \|\widetilde{M}_u^*\|_{H \rightarrow H} + \|\widetilde{D}_u^*\|_{H \rightarrow H} \leq \widetilde{\psi}(\|u\|). \end{aligned} \quad (16)$$

In (15) and (16)  $\widetilde{\psi}(\cdot)$  is strictly increasing function on  $[0; \infty)$ ,  $\widetilde{\psi}(0) \geq 1$ .

Let us introduce one more condition used in what follows.

*Condition U4.* If the vector  $u \in H$  satisfies condition

$$\|u\|^{4k+2} \geq \|Tu\|^{4k+2+2\varepsilon}, \quad \|u\|^2 \geq \widehat{C}, \quad \widehat{C} \geq 10,$$

then

$$\inf \frac{\langle \widetilde{M}_u P_u a, P_u a \rangle}{\|P_u a\|^2} < (k+1) \|u\|^{4k}, \quad (17)$$

where  $k$  and  $\varepsilon$  from (13)–(14),  $P_u$  is the projector defined by

$$P_u a = a - \frac{\langle a, u \rangle}{\|u\|^2} u.$$

In (17) *infimum* is taken over all  $a \in H$  such that  $P_u a \neq 0$ .

The following result holds.

*Theorem 2.* If conditions U2, U3, and U4 are satisfied, then for any  $u \in H$  the following estimate holds

$$\|u\|^2 \leq (\|f(u)\|^2 + 2)^\Theta,$$

where  $\Theta$  is a constant independent of  $u \in H$ , depending only on the constant  $m$ ,  $C_0$ ,  $d_0$ ,  $k$ ,  $\varepsilon$ , appearing in conditions U2, U3, and U4.

Recall that by  $C$  (lowercase or uppercase, with or without indices) we denote constants (generally different in different places).

#### 4 Main lemmas

In what follows, throughout this section we assume that conditions U2, U3, and U4 are satisfied.

We introduce the functionals  $J(u)$ ,  $\mathfrak{D}(u)$ ,  $\gamma(u)$  and the vector  $N(u) \in H$ , depending on  $u \in H$ :

$$J(u) = \exp \left\{ \|u\|^{4k+2} - \|\widetilde{f}(u)\|^2 \right\}, \quad (18)$$

where  $k > m + 3$ ,  $m$  is taken from (11) and  $\widetilde{f}(\cdot)$  is defined in (14);

$$\mathfrak{D}(u) = \exp \left\{ \exp \{ \|u\|^2 \} - \|\widetilde{f}(u)\|^2 \right\}; \quad (19)$$

$$\gamma(u) = \frac{\langle \widetilde{D}_u^* \widetilde{f}(u), u \rangle}{\|u\|^2}; \quad (20)$$

$$N(u) = \widetilde{D}_u^* \widetilde{f}(u) - \gamma(u)u = P_u \widetilde{D}_u^* \widetilde{f}(u), \quad (21)$$

where  $P_u$  is the projector defined in (17).

The following holds.

*Lemma 2.* For any  $c > 0$  the set

$$M^{(c)} = \{u \in H : J(u) \geq c\}$$

is bounded and compact.

*Proof.* Since  $H$  is finite-dimensional, it suffices to prove that the set  $M^{(c)}$  is bounded.

Assume the contrary. Suppose that the set  $M^{(c)}$  is unbounded. Then there exists a sequence  $u_1, u_2, \dots, u_n, \dots$  such that

$$\lim_{n \rightarrow \infty} \|u_n\| = +\infty, \quad J(u_n) \geq c. \quad (22)$$

For  $J(u_n)$  using (18) we have

$$\begin{aligned} 0 < c \leq J(u_n) &= \exp \left\{ \|u_n\|^{4k+2} - \|f(u_n) + \|Tu_n\|^{2k+\varepsilon} Tu_n\|^2 \right\} = \\ &= \exp \left\{ \|u_n\|^{4k+2} - \|f(u_n)\|^2 - 2\langle u_n, Tu_n \rangle \|Tu_n\|^{2k+\varepsilon} - \right. \\ &\quad \left. - 2\langle L(u_n), Tu_n \rangle \|Tu_n\|^{2k+\varepsilon} - \|Tu_n\|^{4k+2+2\varepsilon} \right\} \leq \\ &\leq \exp \left\{ \|u_n\|^{4k+2} + C \|u_n\|^{2k+2+\varepsilon} + C \|L(u_n)\| \|u_n\|^{2(k+1)+\varepsilon} - \|Tu_n\|^{4k+2+2\varepsilon} \right\} \leq \\ &\leq \exp \left\{ C_1 \|u_n\|^{4k+2} + C \|u_n\|^{2k+2+\varepsilon} + C_1 \|u_n\|^{3k-1+\varepsilon} - \|Tu_n\|^{4k+2+2\varepsilon} \right\}. \end{aligned} \quad (23)$$

In deriving (23) we used the boundedness of the operator  $T$ , condition U2 (see (11)) and the choice  $k > m + 3$ .

Now we use the invertibility of the operator  $T$  ( $\|T^{-1}\|_{H \rightarrow H} < C_2 < \infty$ ). Then from (23) we obtain

$$0 < c \leq J(u_n) \leq \exp \left\{ C_3 \left[ \|u_n\|^{4k+2+\varepsilon} - C_5 \|u_n\|^{4k+2+2\varepsilon} \right] \right\}.$$

Hence, as  $n \rightarrow \infty$  using (22) and  $\varepsilon > 0$  we arrive at a contradiction  $0 < c \leq 0$ .

From this contradiction it follows that the set  $M^{(c)}$  is bounded.

The lemma is proved.

Let us note the following obvious result.

*Lemma 3.* Let  $u \in H$ . Then the equality holds

$$\langle N(u), u \rangle_H = 0.$$

*Proof.* The equality follows directly from the definitions of the functional  $\gamma(\cdot)$  and the vector  $N(\cdot)$  (see (20), (21)). The lemma is proved.

For  $u_0 \in H$ , suppose that

$$\tilde{\mathfrak{D}}(u_0) = \sup \mathfrak{D}(u), \quad (24)$$

where *supremum* is taken over all  $u \in H$  such that

$$J(u) \geq J(u_0). \quad (25)$$

In formulas (24) and (25) the functionals  $\mathfrak{D}(\cdot)$ ,  $J(\cdot)$  are defined by (18) and (19).

The following holds.

*Lemma 4.* Let  $0 \neq u_0 \in H$ . Then there exists a vector  $\tilde{u}_0$  that attains the *supremum* (24)–(25). For this vector  $\tilde{u}_0$  the inequalities hold

$$\mathfrak{D}(\tilde{u}_0) \geq \mathfrak{D}(u_0), \quad (26)$$

$$J(\tilde{u}_0) \geq J(u_0). \quad (27)$$

*Proof.* The functionals  $\mathfrak{D}(\cdot)$ ,  $J(\cdot)$  are continuous. By Lemma 2 and inequality (25), *supremum* is taken over a compact set.

It is well known that *supremum* of a continuous functional over a compact set is attained at some point of that set [13, 14]. Thus, the existence of a vector  $\tilde{u}_0$ , realizing the *supremum* (24)–(25), is established [15, 16].

The inequalities (26)–(27) follow directly from the definitions.

The lemma is proved.

Define the vector-valued function  $u(\xi)$  as the solution of the Cauchy problem

$$\begin{cases} u_\xi = x(\xi) u + P_u a \equiv x(\xi) u + a - \frac{\langle a, u \rangle}{\|u\|^2} u, \\ u|_{\xi=0} = \tilde{u}_0, \end{cases} \quad (28)$$

where  $a \in H$  and  $\tilde{u}_0$  is the vector realizing the *supremum* (24)–(25).

For  $J(u(\xi))$  and  $\mathfrak{D}(u(\xi))$  we have:

$$\begin{aligned} J_\xi(u(\xi)) &= 2J(u) \left[ (2k+1) \|u\|^{4k} \langle u, u_\xi \rangle - \langle \tilde{f}(u), \tilde{D}_u u_\xi \rangle \right] = \\ &= 2J(u) \left\langle (2k+1) \|u\|^{4k} \cdot u - \gamma(u)u - N(u), u_\xi \right\rangle = \\ &= 2J(u) \left[ ((2k+1) \|u\|^{4k} - \gamma(u)) \|u\|^2 \cdot x - \langle N(u), P_u a \rangle \right]; \\ \mathfrak{D}_\xi(u(\xi)) &= 2\mathfrak{D}(u) \left[ (\exp \{ \|u\|^2 \}) \langle u, u_\xi \rangle - \langle \tilde{f}(u), \tilde{D}_u u_\xi \rangle \right] = \\ &= 2\mathfrak{D}(u) \left[ (\exp \{ \|u\|^2 \}) \langle u, u_\xi \rangle - \langle \tilde{D}_u^* \tilde{f}(u), u_\xi \rangle \right] = \\ &= 2\mathfrak{D}(u) \left[ (\exp \{ \|u\|^2 \}) \langle u, u_\xi \rangle - \langle N(u) + \gamma(u)u, u_\xi \rangle \right] = \\ &= 2\mathfrak{D}(u) \left[ (\exp \{ \|u\|^2 \} - \gamma(u)) \|u\|^2 \cdot x - \langle N(u), P_u a \rangle \right]. \end{aligned} \quad (29)$$

*Lemma 5.* Let  $\tilde{u}_0$  be the vector realizing *supremum* (24)–(25). Then the following relations hold:

$$N(\tilde{u}_0) = 0; \quad (30)$$

$$(\exp \{ \|\tilde{u}_0\|^2 \} - \gamma(\tilde{u}_0)) \left( (2k+1) \|\tilde{u}_0\|^{4k} - \gamma(\tilde{u}_0) \right) \leq 0. \quad (31)$$

*Proof.* We first prove (30).

Suppose that  $N(\tilde{u}_0) \neq 0$ . In the problem (28) we choose

$$\begin{cases} x = 0, \\ a = -N(u(\xi)). \end{cases}$$

Then for  $J(u(\xi))$  and  $\mathfrak{D}(u(\xi))$ , using calculations (29) and equality

$$P_{\tilde{u}_0} a = N(\tilde{u}_0),$$

we have

$$\begin{aligned} J_\xi(u) &= \|N(u(\xi))\|^2 \cdot 2 J(u), \\ \mathfrak{D}_\xi(u) &= \|N(u(\xi))\|^2 \cdot 2 \mathfrak{D}(u). \end{aligned}$$

Next, if  $N(\tilde{u}_0) \neq 0$ , then we have the inequalities

$$J_\xi(u) > 0, \quad \mathfrak{D}_\xi(u) > 0,$$

which hold for  $\xi \in [0; \xi_0]$ , where  $\xi_0 > 0$  is some number.

From this, we obtain the strict inequalities

$$J(u(\xi_0)) > J(\tilde{u}_0), \quad \mathfrak{D}(u(\xi_0)) > \mathfrak{D}(\tilde{u}_0), \quad (32)$$

which contradict the definition of the vector  $\tilde{u}_0$ .

Therefore, (30) holds.

Now let us prove (31). Suppose that inequality (31) does not hold.

In problem (28) we choose  $a = 0$  and  $x(\xi) = \exp \{\|\tilde{u}_0\|^2\} - \gamma(\tilde{u}_0)$ . Then the vector-function  $u = u(\xi)$  is defined as the solution of the Cauchy problem

$$\begin{cases} u_\xi = (\exp \{\|\tilde{u}_0\|^2\} - \gamma(\tilde{u}_0)) u(\xi), \\ u|_{\xi=0} = \tilde{u}_0. \end{cases} \quad (33)$$

From computations (29) using (33), we obtain:

$$\begin{aligned} J_\xi(u(\xi))|_{\xi=0} &= 2 J(\tilde{u}_0) \left[ (2k+1) \|\tilde{u}_0\|^{4k} - \gamma(\tilde{u}_0) \right] \cdot \left[ \exp \{\|\tilde{u}_0\|^2\} - \gamma(\tilde{u}_0) \right] \|\tilde{u}_0\|^2; \\ \mathfrak{D}_\xi(u(\xi))|_{\xi=0} &= 2 \mathfrak{D}(\tilde{u}_0) \left[ \exp \{\|\tilde{u}_0\|^2\} - \gamma(\tilde{u}_0) \right]^2 \|\tilde{u}_0\|^2. \end{aligned}$$

Hence, from the assumption to the contrary, we obtain the strict inequalities

$$J_\xi(u(\xi))|_{\xi=0} > 0, \quad \mathfrak{D}_\xi(u(\xi))|_{\xi=0} > 0.$$

In turn, these inequalities imply that for some  $\xi_0 > 0$  the strict inequalities (32) hold.

Since inequalities (32) contradict the definition of the vector  $\tilde{u}_0$ , we obtain a contradiction with the assumption. Therefore, (31) is established.

The lemma is proved.

*Lemma 6.* Suppose that  $u \in H$  and the following condition holds

$$\exp \{\|u\|^2\} \leq |\gamma(u)|.$$

Then the following inequality holds

$$\|u\| \leq \tilde{C}_1, \quad (34)$$

where  $\tilde{C}_1$  is a constant depending only on the constants  $C, m, d_0, k, \varepsilon$ , appearing in conditions U2, U3, and U4.

*Proof.* We may assume that  $\|u\|^2 \geq 1$ . From the assumption of the lemma ( $\exp \{\|u\|^2\} \leq |\gamma(u)|$ ), using conditions U2 and U3 and the fact that  $\|u\|^2 \geq 1$ , we obtain

$$\exp \{\|u\|^2\} \leq |\gamma(u)| \leq |\gamma(u)| \|u\|^2 = \left| \langle \tilde{D}_u^* \tilde{f}(u), u \rangle \right| \leq \|\tilde{f}(u)\| \|\tilde{D}_u u\| \leq$$

$$\leq \left( \|u\| + \|\tilde{L}(u)\| \right) \left( \|u\| + \|\tilde{L}_u u\| \right) \leq C_0 (1 + \|u\|^2)^{2(2k+1+\varepsilon)},$$

where  $C_0$  is a constant.

From this inequality, estimate (34) follows.

The lemma is proved.

The following result holds.

*Lemma 7.* Let  $\tilde{u}_0 \in H$  be a vector realizing the *supremum* in (24)–(25). Then either a) or b) holds:

$$a) \|\tilde{u}_0\| \leq \tilde{C}_1, \quad \text{where } \tilde{C}_1 \text{ is from Lemma 6;}$$

$$b) N(\tilde{u}_0) = 0 \quad \text{and} \quad \gamma(\tilde{u}_0) \geq (2k+1) \|\tilde{u}_0\|^{4k}, \quad k \text{ is from (14).}$$

*Proof.* We may assume that  $\|\tilde{u}_0\|^2 \geq 1$ . From Lemma 5 we know that (30) and (31) hold. Therefore, if b) does not hold, then we obtain  $\gamma(\tilde{u}_0) \leq (2k+1) \|\tilde{u}_0\|^{4k}$ . Then from (31) it follows that

$$\exp \{ \|\tilde{u}_0\|^2 \} \leq \gamma(\tilde{u}_0) = |\gamma(\tilde{u}_0)|.$$

Hence, by Lemma 6, condition a) holds.

The lemma is proved.

## 5 Proof of Theorem 2

Let  $u_0 \in H$  be a vector whose norm is to be estimated in terms of the norm of the vector  $f(u_0) = u_0 + L(u_0)$ . From the vector  $f(u_0)$  construct the vector  $f(u_0) + \|Tu_0\|^{2k+\varepsilon} Tu_0 = \tilde{f}(u_0)$ . Next, for the vector  $u_0$  construct the vector  $\tilde{u}_0$ , realizing the *supremum* in (24)–(25). According to Lemma 7, either a) or b) holds:

$$a) \|\tilde{u}_0\| \leq \tilde{C}_1; \tag{35}$$

$$b) N(\tilde{u}_0) = 0 \quad \text{and} \quad \gamma(\tilde{u}_0) \geq (2k+1) \|\tilde{u}_0\|^{4k},$$

where  $\tilde{C}_1$  is a constant defined in Lemma 7.

If condition a), then from the definition of  $J(u_0)$  and Lemma 4 we have

$$\begin{aligned} J(u_0) &\leq J(\tilde{u}_0) = \exp \left\{ \|\tilde{u}_0\|^{4k+2} - \|\tilde{f}(\tilde{u}_0)\|^2 \right\} \leq \\ &\leq \exp \left\{ \|\tilde{u}_0\|^{4k+2} \right\} \leq \exp \left\{ \tilde{C}_1^{4k+2} \right\}. \end{aligned} \tag{36}$$

From this and from the definition of  $J(\cdot)$  we obtain

$$\begin{aligned} \|u_0\|^{2(2k+1)} &\leq \|\tilde{f}(u_0)\|^2 + \tilde{C}_1^{4k+2} = \|f(u_0) + \|Tu_0\|^{2k+\varepsilon} Tu_0\|^2 + \tilde{C}_1^{4k+2} \leq \\ &\leq [\|f(u_0)\| + \|Tu_0\|^{2k+1+\varepsilon}]^2 + \tilde{C}_1^{4k+2} \leq \\ &\leq [\|f(u_0)\| + (d_0 \|f(u_0)\|)^{2k+1+\varepsilon}]^2 + \tilde{C}_1^{4k+2}. \end{aligned} \tag{37}$$

In deriving (37) the last step uses condition U3.

From (37) we obtain the estimate

$$\|u_0\|^2 \leq (\|f(u_0)\|^2 + 2)^\Theta, \tag{38}$$

where  $\Theta$  is a constant independent of  $u_0 \in H$ , depending only on the constants  $d_0$  from condition U3,  $m$  from condition U2, and the choice  $k > m$  and  $\varepsilon \in \left(0; \frac{1}{10}\right)$  (see (14)).

Thus, in the case where condition  $a)$  of (35) holds, Theorem 2 is proved.

In the case where condition  $b)$  of (35) holds, we define the vector-function  $u = u(\xi)$  as the solution of the Cauchy problem

$$\begin{cases} u_\xi = P_u a, \\ u|_{\xi=0} = \tilde{u}_0, \end{cases}$$

where  $P_u$  is the projector from condition U4, and the parameter  $a \in H$  will be chosen later.

For  $J(u(\xi))$  and  $\mathfrak{D}(u(\xi))$ , using (29), we obtain:

$$\begin{aligned} J_\xi(u(\xi)) &= 2 J(u) \left[ \langle (2k+1) \|u\|^{4k} u - \tilde{D}_u^* \tilde{f}(u), u_\xi \rangle \right] = \\ &= 2 J(u) \langle -N(u), P_u a \rangle; \\ \mathfrak{D}_\xi(u(\xi)) &= 2 \mathfrak{D}(u) \left[ \langle \exp \{ \|u\|^2 \} u - \tilde{D}_u^* \tilde{f}(u), u_\xi \rangle \right] = \\ &= 2 \mathfrak{D}(u) \langle -N(u), P_u a \rangle. \end{aligned} \tag{39}$$

From (39), taking into account that  $N(\tilde{u}_0) = 0$ , we obtain:

$$\begin{aligned} J_\xi(u(\xi)) &= 2 J(u) \left\langle - \int_0^\xi \left( \tilde{D}_u^* \tilde{f}(u) - \gamma(u) u \right)_\eta d\eta, P_{u(\xi)} a \right\rangle = \\ &= 2 J(u) \left\langle - \int_0^\xi \left( \tilde{D}_u^* \tilde{f}(u) - \gamma(u) u \right)_\eta d\eta, P_{\tilde{u}_0} a + \int_0^\xi (P_{u(\xi)} a)_\eta d\eta \right\rangle = \\ &= 2 J(u) \left[ -\xi \left\langle \left( \tilde{D}_{u(\eta)}^* \tilde{f}(u(\eta)) - \gamma(u(\eta)) u_\eta \right) \Big|_{\eta=0}, P_{\tilde{u}_0} a \right\rangle + \xi^2 O(1) \right] = \\ &= 2 J(u) \left[ -\xi \left\langle \tilde{M}_{\tilde{u}_0} P_u a, P_u a \right\rangle + \xi \gamma(\tilde{u}_0) \|P_{\tilde{u}_0} a\|^2 + \xi^2 O(1) \right] = \\ &= 2 J(u(\xi)) \left[ \left( \gamma(\tilde{u}_0) - \frac{\langle \tilde{M}_{\tilde{u}_0} P_{\tilde{u}_0} a, P_{\tilde{u}_0} a \rangle}{\|P_{\tilde{u}_0} a\|^2} \right) \cdot \|P_{\tilde{u}_0} a\|^2 \xi + \xi^2 O(1) \right]; \\ \mathfrak{D}_\xi(u(\xi)) &= 2 \mathfrak{D}(u) \left[ \left( \gamma(\tilde{u}_0) - \frac{\langle \tilde{M}_{\tilde{u}_0} P_{\tilde{u}_0} a, P_{\tilde{u}_0} a \rangle}{\|P_{\tilde{u}_0} a\|^2} \right) \cdot \|P_{\tilde{u}_0} a\|^2 \xi + \xi^2 O(1) \right]. \end{aligned} \tag{40}$$

Under condition  $b)$  we have

$$\gamma(\tilde{u}_0) - \frac{\langle \tilde{M}_{\tilde{u}_0} P_{\tilde{u}_0} a, P_{\tilde{u}_0} a \rangle}{\|P_{\tilde{u}_0} a\|^2} \geq (2k+1) \|\tilde{u}_0\|^{4k} - \frac{\langle \tilde{M}_{\tilde{u}_0} P_{\tilde{u}_0} a, P_{\tilde{u}_0} a \rangle}{\|P_{\tilde{u}_0} a\|^2}.$$

If at least one of the inequalities  $a')$ , or  $b')$  holds:

$$\begin{aligned} a') \quad & \|\tilde{u}_0\| \leq \tilde{C}, \text{ where } \tilde{C} \geq 10; \\ b') \quad & \|\tilde{u}_0\|^{4k+2} < \|T\tilde{u}_0\|^{4k+2+2\varepsilon}, \end{aligned} \tag{41}$$

then:

in the case where inequality  $a')$  in (41) holds, it follows that Theorem 2 is proved;

in the case where inequality  $b')$  in (41) holds, we have:

$$\begin{aligned} \|\tilde{u}_0\|^{4k+2} - \|\tilde{f}(\tilde{u}_0)\|^2 &= \|\tilde{u}_0\|^{4k+2} - \|f(\tilde{u}_0) + \|T\tilde{u}_0\|^{2k+\varepsilon} T\tilde{u}_0\|^2 = \\ &= \|\tilde{u}_0\|^{4k+2} - \|f(\tilde{u}_0)\|^2 - 2\|T\tilde{u}_0\|^{2k+\varepsilon} \langle f(\tilde{u}_0), T\tilde{u}_0 \rangle - \|T\tilde{u}_0\|^{4k+2+2\varepsilon} < 0. \end{aligned}$$

In deriving this inequality, condition U3 was used (see (12)).

From this, by the definition of  $J(\cdot)$  we obtain  $J(\tilde{u}_0) < 1$ .

Therefore, by Lemma 4 (see (25)) we have

$$J(u_0) \leq J(\tilde{u}_0) < 1, \quad (42)$$

which holds if  $b')$  in (41) is satisfied.

From (42), repeating the argument used to derive estimate (38) from (36), we again obtain inequality (38).

Thus, in case  $b)$  of (35), if at least one of the inequalities  $a')$ , or  $b')$  in (41) holds, then Theorem 2 is proved.

If neither  $a')$  nor  $b')$  in (41) holds, then the following inequalities are satisfied

$$\|\tilde{u}_0\|^{4k+2} \geq \|T\tilde{u}_0\|^{4k+2+2\varepsilon}, \quad \|\tilde{u}_0\|^2 > \tilde{C} \geq 10. \quad (43)$$

From (43) we can apply condition U4, and taking into account computations (40) we obtain

$$\begin{cases} J_\xi(u(\xi)) = 2J(\tilde{u}_0) \left( \gamma(\tilde{u}_0) - \theta(\tilde{u}_0) \right) \cdot \|P_{\tilde{u}_0} a\|^2 \xi + \xi^2 O(1), \\ \mathfrak{D}_\xi(u(\xi)) = 2\mathfrak{D}(\tilde{u}_0) \left( \gamma(\tilde{u}_0) - \theta(\tilde{u}_0) \right) \cdot \|P_{\tilde{u}_0} a\|^2 \xi + \xi^2 O(1), \end{cases} \quad (44)$$

where

$$\theta(\tilde{u}_0) = \inf_{\|P_{\tilde{u}_0} a\|=1} \left\langle \tilde{M}_{\tilde{u}_0} P_{\tilde{u}_0} a, P_{\tilde{u}_0} a \right\rangle.$$

Since, by condition  $b)$  of (35)

$$\gamma(\tilde{u}_0) \geq (2k+1) \|\tilde{u}_0\|^{4k},$$

it follows from (44) taking into account condition U4, that the strict inequalities

$$J_\xi(u(\xi)) > 0, \quad \mathfrak{D}_\xi(u(\xi)) > 0 \quad (45)$$

hold for  $\xi \in (0; \xi_0]$ ,  $\xi_0$  is a sufficiently small number.

From (45) it follows that there exists  $\xi_0 > 0$  such that for all  $\xi \in (0; \xi_0]$  the strict inequalities hold

$$J(u(\xi)) > J(\tilde{u}_0), \quad \mathfrak{D}(u(\xi)) > \mathfrak{D}(\tilde{u}_0). \quad (46)$$

The inequalities (46) contradict the definition of the vector  $\tilde{u}_0$ , realizing the *supremum* in (24)–(25).

Therefore, in the case when condition  $b)$  of (35) holds, either  $a')$  or  $b')$  in (41) must hold, and estimate (38) is valid, while case (43) does not occur.

Consequently, Theorem 2 is proved also in case  $b)$ .

Theorem 2 is proved.

### Conclusion

The paper investigates a class of nonlinear operator equations in a finite-dimensional Hilbert space. The main focus is on obtaining a priori estimates for solutions of the considered equations and analyzing their properties. It is shown that for a sufficiently broad class of nonlinear operators, one can establish uniform estimates of solutions that depend only on the parameters of the original problem and do not depend on the approximation index.

The obtained estimates make it possible to apply the method of finite-dimensional approximations to the study of corresponding problems in infinite-dimensional spaces. The availability of a priori estimates ensures the possibility of passing to the limit and obtaining a priori estimates for solutions of the original problems of mathematical physics [17, 18]. This, in turn, is an important tool in proving the existence of solutions to nonlinear operator equations.

The proposed approach can be used to analyze a wider class of nonlinear problems arising in functional analysis and mathematical physics [3, 4]. The obtained results may serve as a basis for further research related to the study of the stability of solutions, their qualitative properties, and the construction of effective approximate methods for solving nonlinear operator equations.

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### Author contributions

All authors contributed equally to this work. All authors participated in the conduct of the study and in the preparation of the final version of the manuscript.

### Conflict of interest

The authors declare no conflict of interest.

### References

- 1 Lions, Zh.-L. (1972). *Nekotorye metody resheniia nelineinykh kraevykh zadach* [Some methods for solving nonlinear boundary value problems]. Moscow: Mir [in Russian].
- 2 Granas, A., & Dugundji, J. (2003). *Fixed Point Theory*. New York, NY: Springer. <https://doi.org/10.1007/978-0-387-21593-8>
- 3 Otelbaev, M. (2011). Examples of equations of Navier-Stokes type not strongly solvable in the large. *Mathematical Notes*, 89(5), 726–733. <https://doi.org/10.1134/S0001434611050129>
- 4 Otelbaev, M. (2013). Sushchestvovanie silnogo resheniia uravneniia Nave–Stoksa [Existence of a strong solution to the Navier–Stokes equation]. *Matematicheskii zhurnal – Mathematical journal*, 13(4(50)), 5–104 [in Russian].
- 5 Bauschke, H.H., & Combettes, P.L. (2017). *Convex Analysis and Monotone Operator Theory in Hilbert Spaces*. 2nd ed. Cham: Springer. <https://doi.org/10.1007/978-3-319-48311-5>
- 6 Dinca, G., & Mawhin, J. (2021). *Brouwer Degree: The Core of Nonlinear Analysis*. Cham: Birkhäuser. <https://doi.org/10.1007/978-3-030-63230-4>
- 7 Ern, A., & Guermond, J.-L. (2021). *Finite Elements I: Approximation and Interpolation*. Cham: Springer. <https://doi.org/10.1007/978-3-030-56341-7>

- 8 Ern, A., & Guermond, J.-L. (2021). *Finite Elements II: Galerkin Approximation, Elliptic and Mixed PDEs*. Cham: Springer. <https://doi.org/10.1007/978-3-030-56923-5>
- 9 Ern, A., & Guermond, J.-L. (2021). *Finite Elements III: First-Order and Time-Dependent PDEs*. Cham: Springer. <https://doi.org/10.1007/978-3-030-57348-5>
- 10 Arendt, W., Chalendar, I., & Eymard, R. (2022). Galerkin approximation of linear problems in Banach and Hilbert spaces. *IMA Journal of Numerical Analysis*, 42(1), 165–198. <https://doi.org/10.1093/imanum/draa067>
- 11 Meidner, D., & Vexler, B. (2018). Optimal error estimates for fully discrete Galerkin approximations of semilinear parabolic equations. *ESAIM: Mathematical Modelling and Numerical Analysis*, 52(6), 2307–2325. <https://doi.org/10.1051/m2an/2018040>
- 12 Ye, X., Zhang, S., & Zhu, Y. (2020). Stabilizer-free weak Galerkin methods for monotone quasilinear elliptic PDEs. *Results in Applied Mathematics*, 8, Article 100097. <https://doi.org/10.1016/j.rinam.2020.100097>
- 13 Keith, B., & Surowiec, T.M. (2026). Proximal Galerkin: A structure-preserving finite element method for pointwise bound constraints. *Foundations of Computational Mathematics*, 26, 385–481. <https://doi.org/10.1007/s10208-024-09681-8>
- 14 Becker, R., & Braack, M. (2025). The concept of mapped coercivity for nonlinear operators in Banach spaces. *Journal of Functional Analysis*, 289(3), Article 110893. <https://doi.org/10.1016/j.jfa.2025.110893>
- 15 Azzam-Laouir, D., Belhoula, W., Castaing, C., & Monteiro Marques, M.D.P. (2020). Multi-valued perturbation to evolution problems involving time dependent maximal monotone operators. *Evolution Equations and Control Theory*, 9(1), 219–254. <https://doi.org/10.3934/eect.2020004>
- 16 Vilches, E., & Nguyen, B.T. (2020). Evolution inclusions governed by time-dependent maximal monotone operators with a full domain. *Set-Valued and Variational Analysis*, 28, 569–581. <https://doi.org/10.1007/s11228-020-00533-5>
- 17 Jenaliyev, M., & Ramazanov, M. (2018). On a homogeneous parabolic problem in an infinite corner domain. *Filomat*, 32(3), 965–974. <https://doi.org/10.2298/FIL1803965J>
- 18 Akhmetshin, A.D., Omarov, M.T., & Toleukhanova, R.Z. (2026). A Boundary Value Problem for a Time-Fractional Diffusion Equation in a Non-Cylindrical Shrinking Domain. *Bulletin of the Karaganda University. Mathematics Series*, 1(121), 37–54. <https://doi.org/10.31489/2026M1/37-54>

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