https://doi.org/10.31489/2026M1/238-247 *Research article*

Model-theoretic properties of J-superstable Jonsson theories in classes defined by cosemanticness

A.R. Yeshkeyev, G.E. Zhumabekova*, A.K. Issayeva

Buketov Karaganda National Research University, Karaganda, Kazakhstan
(E-mail: aibat.kz@gmail.com, galkatai@mail.ru, isaevaaiga@gmail.com)

This article deals with the problems of model-theoretic characterization of J-superstable Jonsson theories. The characteristic features of such theories are analyzed in terms of J-stability, J-P-superstability, and J-nonmultidimensionality. The need to generalize classical stability notions to the framework of Jonsson theories is identified and justified. The concepts of J-stationary and J-orthogonal types are introduced, and their role in describing the dimensional structure of existentially closed models is examined. It is shown that every JF_{λ}^{α}-saturated model embeds as a submodel of the semantic model of a Jonsson theory T. Based on the notion of J-stationarity, a theory of independence for existential types is developed, and the notions of J-basis and J-dimension are defined. The equivalence between J-nonmultidimensionality, J-P-superstability, and J-P-stability is established, providing precise criteria for the model-theoretic classification of Jonsson theories. The results contribute to the refinement of model-theoretic tools for analyzing stability and dimensionality within the framework of Jonsson theories and place these findings in the broader context of modern classification theory, highlighting Jonsson theories as a natural generalization of elementary theories. These results clarify the interaction between saturation, independence, and dimensionality in Jonsson theories and provide a unified framework for further developments in their model-theoretic classification.

Keywords: Jonsson theory, semantic model, perfect Jonsson theory, hereditary Jonsson theory, permissible enrichment, J-λ-stable theory, J-superstable theory, J-P-superstable theory, J-orthogonal type, J-nonmultidimensional theory.

2020 Mathematics Subject Classification: 03C45, 03C52, 03C65.

Introduction

Model-theoretic stability is a central topic in contemporary mathematical logic and model theory. In particular, the notions of stability and superstability play a crucial role in the classification of theories and the analysis of their model structures. Among the various developments in this field, Jonsson theories, which are characterized by the existence of infinite models preserving homogeneity with respect to subsets of a fixed cardinality, have attracted significant attention.

The relationship between the properties of a complete theory T and those of the theory of elementary pairs T_P was first studied by B. Poizat [1], who formulated the problem of determining under what conditions the theory of elementary pairs is complete. This line of research was later extended by E. Bouscaren [2] and others. Bouscaren demonstrated that the completeness of the theory of elementary pairs depends on the nature of the underlying stable theory, providing necessary and sufficient conditions for both the stable and the superstable cases.

In 1990, in the Proceedings of the Soviet–French Colloquium on Model Theory, A. Nurtazin solved this problem for uncountably categorical theories, and T. Mustafin introduced the concept of

*Corresponding author. *E-mail: galkatai@mail.ru*

This research was funded by the Science Committee of the Ministry of Science and Higher Education of the Republic of Kazakhstan (Grant No. AP22686827 “The central types of hereditary Jonsson theories”).

Received: 20 April 2025; *Accepted:* 29 December 2025.

T^*-λ-stability, studying its properties, including P-stability. E. Palyutin [3] proposed the notion of E^*-stability and proved the definability of types for such theories. In particular, in [4], a classification of ω-stable theories by P-stability was obtained. Further, A.R. Yeshkeyev showed the equivalence of the conditions of P-stability and J-P-stability for $\exists$-complete perfect Jonsson theories. In the paper [4] authors investigated the concept of P-superstability with respect to the nonmultidimensionality property. The study of J-P-stable theories was also continued in [5] and [6].

In recent years, the toolkit for studying Jonsson theories has expanded markedly. This development is reflected in the diverse approaches of works [7–9], which emphasize foundational aspects; studies [10–12], which examine generalizations and structural properties; and research [13], which explores new applications and classification problems.

This article focuses on J-superstable Jonsson theories, in which the notions of J-stability and its variants, such as J-P-superstability and J-nonmultidimensionality, play roles analogous to the classical stability properties. The relevance of this study lies in the need to extend classical stability methods to broader classes of theories, including Jonsson structures, and to clarify the relationships between different forms of J-stability and the inheritance of structural properties within a theory.

The main objective of this work is to analyze the model-theoretic properties of J-superstable theories, formulate classification criteria, and investigate the structural and dimensional behavior of their models. Special attention is given to the introduction of J-stationary and J-orthogonal types, which provide a refined understanding of model structure and allow the identification of essential patterns of property inheritance.

Through this study, we establish the equivalence of J-nonmultidimensionality, J-P-superstability, and J-P-stability, and show that every J-P-superstable perfect Jonsson theory preserves its fundamental structural properties under admissible extensions. These results enhance the model-theoretic tools available for analyzing stability and dimensionality in the context of Jonsson theories and lay the groundwork for further research in generalized stability theory.

1 Preliminaries on Jonsson Theory

The first section discusses the basic concepts and fundamental results related to Jonsson theories, providing an overview of their model-theoretic foundations, key definitions, and the main developments that have shaped their study within the broader context of stability theory.

Definition 1. [14, p. 80] A theory T is called a *Jonsson theory* if the following conditions are satisfied:

1. The theory T has at least one infinite model.
2. T is *inductive*, that is, it is equivalent to a set of sentences of the form $\forall\exists$.
3. T has the *joint embedding property (JEP)*, meaning that for any two models $A, B \models T$, there exists a model $C \models T$ into which both A and B can be embedded isomorphically.
4. T has the *amalgamation property (AP)*: for any models $A, B_1, B_2 \models T$ and isomorphic embeddings $f_1\colon A \to B_1$ and $f_2\colon A \to B_2$, there exists a model $C \models T$ together with isomorphic embeddings $g_1\colon B_1 \to C$ and $g_2\colon B_2 \to C$ such that $g_1 f_1 = g_2 f_2$.

It is worth noting that Jonsson theories naturally appear in various classical areas of algebra and model theory. In particular, typical examples of Jonsson theories include:

- the theory of groups;
- the theory of abelian groups;
- the theory of fields of fixed characteristic;
- the theory of Boolean algebras;
- the theory of polygons over a fixed monoid;
- the theory of modules over a fixed ring;
- the theory of linearly ordered sets.

Hence, Jonsson theories encompass a broad class of algebraic and logical structures characterized by strong embedding and amalgamation properties, making them a significant object of study in modern model theory.

Further, let us consider in more detail the semantic and syntactic invariants associated with Jonsson theory. These invariants play a fundamental role in understanding the structural properties and model-theoretic behavior of Jonsson theories.

Definition 2. [15] A model C_T of a Jonsson theory T such that $|C_T| = 2^\omega$ is called a semantic model, if it is ω^+-homogeneous-universal.

Definition 3. [15] The *center* of a Jonsson theory T is the elementary theory of its semantic model C_T, denoted by T^*, where $T^* = \mathrm{Th}(C_T)$ is a complete theory.

Further, we note that a Jonsson theory may exhibit an additional property called perfection, which is related to the saturation of its semantic model.

Definition 4. [15] A Jonsson theory T is said to be perfect, if C_T is ω^+-saturated.

In Jonsson theory, the notion of perfection is closely tied to the class of existentially closed models. Such models play a key role in understanding the realizability of existential statements within larger models of the theory, providing a natural link between semantic completeness and structural homogeneity.

Definition 5. [14, p. 97] A model A of a theory T is said to be *existentially closed in* T if for every model B and every existential formula $\varphi(x)$ with parameters from A, whenever $A \subseteq B$ and $B \models \exists x\, \varphi(x)$, we also have $A \models \exists x\, \varphi(x)$.

Denote by E_T the class of all existentially closed models of the Jonsson theory T.

Theorem 1. If T is a perfect Jonsson theory, then $E_T = \mathrm{Mod}\, T^*$.

In the study of Jonsson theories, the behavior of types over small subsets of existentially closed models plays a crucial role in understanding the theory stability properties. To capture this behavior, the notions of J-λ-stability, J-stability, and J-superstability are introduced, which formalize the control over the number of J-types realized over subsets of a given size.

Let T be a Jonsson theory, $S^J(X)$ be a set of all existential complete n-types over X, consistent with T, for each finite n.

Definition 6. (A.R. Yeshkeyev) A Jonsson theory T is called J-λ-stable if for any existentially closed model A and any its subset X: $|X| \leq \lambda \Rightarrow |S^J(X)| \leq \lambda$.

We say that a Jonsson theory T is J-stable if it is J-λ-stable for some cardinal λ.

Definition 7. We say that a Jonsson theory T is J-superstable if it is J-λ-stable for some cardinal λ and $\lambda \geq 2^\omega$.

Let us consider the theorem that connects the Jonsson stability and the classical stability, which is an important property for Jonsson theories.

Theorem 2. (A.R. Yeshkeyev) Let T be a perfect $\exists$-complete (complete for existential sentences) Jonsson theory, $\lambda \geq \omega$. Then the following conditions are equivalent:

1) T is J-λ-stable;

2) T^* is λ-stable (in classical sense), where T^* is the center of T.

Let T be an arbitrary Jonsson theory of a signature σ, and let C_T be its semantic model. Let $A \subseteq C_T$, and let P be a new unary predicate symbol.

Consider the (generally incomplete) theory in the signature $\sigma_P(A) = \sigma_A \cup \{P\}$, defined as follows:

$$T_P^J(A) = \mathrm{Th}_{\forall\exists}(C_T, a)_{a \in A} \cup \{P(c_a) \mid a \in A\} \cup \{\text{“}P \subseteq\text{”}\},$$

where

- $\mathrm{Th}_{\forall\exists}(C_T, a)_{a\in A}$ is the set of all universal-existential sentences true in the structure $(C, a)_{a\in A}$;
- $\{P(c_a) \mid a \in A\}$ asserts that all elements of A are included in the interpretation of the predicate P;
- $\{$"$P \subseteq$ "$\}$ is an infinite set of sentences expressing that the interpretation of P is an existentially closed submodel in the signature σ.

Remark. The requirement of existential closedness of the submodel is essential in the sense that the submodel must *not* be finite.

Let $S^J_p(A)$ denote the set of all $\exists$-completions of the theory $T^J_P(A)$.

Let λ be an arbitrary cardinal.

Definition 8. (A.R. Yeshkeyev) A Jonsson theory T is said to be *Jonsson P-λ-stable* (or, briefly, *J-P-λ-stable*) if for every set A of cardinality at most λ, the following holds:

$$|S^J_P(A)| \leq \lambda.$$

Definition 9. (A.R. Yeshkeyev) A Jonsson theory T is said to be *Jonsson P-stable* (or simply *J-P-stable*) if T is J-P-λ-stable for some cardinal λ.

Definition 10. A Jonsson theory T is said to be *Jonsson P-superstable* (or simply *J-P-superstable*) if there exists a cardinal λ_0 such that for all $\lambda \geq \lambda_0$, the theory T is J-P-λ-stable.

That is, for all $\lambda \geq \lambda_0$ and for every set $A \subseteq C_T$ with $|A| \leq \lambda$, the number of existential types with predicate P over A satisfies $|S^J_P(A)| \leq \lambda$.

Definition 11. A Jonsson theory T is called *Jonsson P-unstable* (or simply *J-P-unstable*) if it is not J-P-λ-stable for any cardinal λ.

The following theorem establishes a connection between the P-λ-stability studied in [4] and J-P-λ-stability in the context of a perfect $\exists$-complete Jonsson theory. It is fair to say that, thanks to this theorem, attributed to the first author, a number of significant results have been obtained within the framework of Jonsson theories.

Theorem 3. (A.R. Yeshkeyev) Let T be a perfect $\exists$-complete Jonsson theory. Then the following conditions are equivalent:

1) the center of the theory T is P-λ-stable (in the sense of [4]),

2) the theory T is J-P-λ-stable.

Definition 12. (A.R. Yeshkeyev) Let K be a class of L-structures. The following set $JSp(K)$ of Jonsson theories is called a Jonsson spectrum of the class K:

$$JSp(K) = \{T \mid T \text{ is a Jonsson theory and } A \models T \text{ for any } A \in K\}.$$

Definition 13. (T.G. Mustafin) Let T_1 and T_2 be Jonsson theories, C_{T_1} and C_{T_2} be their semantic models, correspondingly. Then T_1 and T_2 are called cosemantic ($T_1 \bowtie T_2$) Jonsson theories, if $C_{T_1} = C_{T_2}$.

It is easy to see that the cosemanticness is an equivalence relation.

Let L be a first-order language, and let T be a Jonsson theory formulated in L. Assume that $K \subseteq E_T$. We now consider the *Jonsson spectrum* $JSp(K)$ and define on it a relation of *cosemantic equivalence*, denoted by $\bowtie$. Factoring $JSp(K)$ by this relation yields the quotient set $JSp(K)_{/\bowtie}$.

Denote by $[T]$ the *cosemantic class* of a theory T, where $T \in JSp(K)$. The class $[T]$ consists of all Jonsson theories that are cosemantically equivalent to T. For every theory $\Delta \in [T]$, we associate a corresponding semantic model C_Δ. By the definition of the cosemantic class, these models coincide with the semantic model C_T of some fixed representative $T \in [T]$. Thus, the entire class $[T]$ shares a single common semantic model, which we denote by $C_{[T]} = C_T$.

Consider now the elementary theory of this model. We refer to it as the *center* of the Jonsson class $[T]$ and denote it by

$$[T]^* = Th(C_{[T]}).$$

This center is identical to the centers of all individual theories $\Delta \in [T]$, that is,

$$[T]^* = Th(C_\Delta) \quad \text{for all } \Delta \in [T].$$

Furthermore, let

$$E_{[T]} = \bigcup_{\Delta \in [T]} E_\Delta$$

be the collection of all existentially closed models corresponding to the theories in $[T]$. Observe that the intersection

$$\bigcap_{\Delta \in [T]} E_\Delta \neq \emptyset,$$

since at least the model $C_{[T]}$ is contained in every E_Δ.

Consequently, all theories belonging to the class $[T]$ possess a shared semantic model and the same center. This provides a coherent structural framework for analyzing Jonsson theories and their models within a unified semantic context.

2 On the classification and structural properties of J-superstable theories

In the study of the model-theoretic properties of Jonsson theories, it is essential to introduce analogues of classical notions such as stationarity, independence, and orthogonality, but formulated in terms of existential types. These concepts play a central role in the classification theory of Jonsson structures and in the analysis of their dimensions.

Let us look at the following definitions, where "J-forks over A" is used in the meaning of Theorem 2.1.2 from [15].

Definition 14. [6] Let p be a complete $\exists$-type over A, where A is a Jonsson subset of C_T. Then p is called *J-stationary* over A if:

1) p does not J-fork over A;
2) p has a unique consistent extension that does not J-fork over A.

Definition 15. Let $p(\bar{x})$ and $q(\bar{y})$ be $\exists$-types over M, where M is a Jonsson subset of C_T. p and q are said to be J-perpendicular ($p \perp_J q$) if $p(\bar{x}) \cup q(\bar{y})$ determines a complete $\exists$-type over M.

Definition 16. 1. A non-algebraic type $p \in S^J(A)$ is *J-regular* if for every subset $B \supseteq A$ of C_T and every $r \in S^J(B)$ with $p \subseteq r$,

$$r \ J\text{-forks over } A \implies p \perp_J r.$$

2. A J-stationary type p is *J-regular* if its stationarization over $\mathrm{Dom}(p)$ is J-regular.

Before we formally define JF^α_λ-saturation, let us clarify the intuition behind it. In stable theories, not all types need to be realized; it suffices to realize those types that are almost over small sets.

In this context, "almost over A" means that there exists $A_0 \subseteq A$ with $|A_0| < \lambda$ such that p does not J-fork over A_0.

Definition 17. An existensional closed model M is called *JF^α_λ-saturated* if for every subset $A \subseteq M$ with $|A| < \lambda$ and every $\exists$-type $p(x) \in S^J_\alpha(A)$ that is *almost over* A, there exists an element $a \in M$ that realizes the $\exists$-type p, i.e., a satisfies all formulas in p.

JF^α_λ-saturated models are submodels of the semantic model of the Jonsson theory T.

The notion of J-stationarity allows us to define a structural measure of independence for existential types in Jonsson theories. This leads naturally to the concept of a *J-basis* and the corresponding notion of dimension.

Definition 18. Let I be a set of tuples and A a set of parameters. We say that I is *J-independent over A* if for every $b \in I$,

$$\mathrm{tp}(b/A \cup (I \setminus \{b\})) \text{ does not } J\text{-fork over } A.$$

Definition 19. Let T be a Jonsson theory, M be an existentially closed model of T, $A \subseteq M$, and let $p \in S^J(A)$. A set $I \subseteq M$ is called a *J-basis* for p in M if
1) every element of I realizes p in M;
2) I is J-independent over A;
3) I is maximal with these properties.
The *dimension* of p in M is defined by

$$\dim_J(p, M) = |I|,$$

where I is any J-basis for p in M.

Definition 20. [6] 1. Let $p(\bar{x}_1)$, $q(\bar{x}_2)$ be complete $\exists$-types over A, where A is a Jonsson subset of C_T. Then p is said to be *J-weakly orthogonal* to q if and only if $p(\bar{x}_1) \cup q(\bar{x}_2)$ is a complete $\exists$-type over A.

2. Let p_1, p_2 be either $\exists$-complete or J-stationary types. Then p_1 is *J-orthogonal* to p_2 if for any set A such that $\mathrm{Dom}(p_1) \cup \mathrm{Dom}(p_2) \subseteq A$, where A is the universe of an $\exists_1$-saturated model, and for any J-non-forking extensions q_1, q_2 of p_1, p_2 over A respectively, the types q_1 and q_2 are weakly J-orthogonal.

The orthogonality of existential types allows us to describe the multidimensional structure of Jonsson theories. The following definition formalizes the notion of multidimensionality within the framework of Jonsson theories.

Definition 21. [6] Let A be a Jonsson subset of the semantic model C_T, where T is a Jonsson theory. An $\exists$-complete type p is said to be *J-multidimensional* if it is J-orthogonal to any complete $\exists$-type over A. If T has a J-multidimensional type, then T is called a *J-multidimensional theory*. Otherwise, T is called a *J-nonmultidimensional theory*, or a theory of *J-restricted dimension*.

The following theorem was obtained in [6]:

Theorem 4. Let T be a perfect, J-λ-stable, $\exists$-complete Jonsson theory:
1) the theory T^* is nonmultidimensional (in the classical sense);
2) the theory T is J-nonmultidimensional.

Since we rely on Lemmas 2 and 3 from [4] and for them the lemmas that we will consider to prove our main theorem are satisfied.

Lemma 1. Let T be a perfect, hereditary $\exists$-complete Jonsson theory, $K \subseteq E_T$, $[\Delta] \in JSp(K)/_{\bowtie}$, $[\Delta]^* = Th(C_{[\Delta]})$ be its center. Let $M, N_1, N_2 \in \mathrm{Mod}([\Delta]^*)$ with $A \subseteq C_{[\Delta]}$ and $N_1, N_2 \prec_{\exists_1} C_{[\Delta]}$, $M \prec_{\exists_1} N_1$, $M \prec_{\exists_1} N_2$, where the models M, N_1, N_2 are JF^{α}_{ω}-saturated. Let $\{p_i \mid i \in I\}$ be a maximal set of pairwise orthogonal J-regular types from $S^J(M)$. Define $J_S \subseteq I$ by

$$J_S = \{i \in I \mid \dim_J(p_i, N_S) < \omega\}, \quad S = 1, 2,$$

and assume $J_1 = J_2$. If for every $i \in J = J_1 = J_2$ we have $\dim_J(p_i, N_1) = \dim_J(p_i, N_2)$, then the pairs of models

$$(N_1, M) \equiv_{\exists_1} (N_2, M)$$

in the signature $\sigma_P(A)$.

Proof. Since $\{p_i \mid i \in I\}$ is a maximal set of orthogonal J-regular types over M and N_1, N_2 are JF^{α}_{ω}-saturated, we can write

$$N_S = M \oplus \bigoplus_{i\in I} X_i^{(S)}, \quad S = 1, 2,$$

where $X_i^{(S)}$ denotes the set of realizations of p_i in N_S, and $\dim_J(p_i, N_S) = |X_i^{(S)}|$.

For $i \in J$, the sets $X_i^{(1)}$ and $X_i^{(2)}$ have the same finite cardinality. For $i \in I \setminus J$, both $X_i^{(S)}$ are infinite by saturation, so their finite partial substructures can always be matched.

Define $f : M \to M$ as the identity. For each $i \in J$, choose a bijection

$$f_i : X_i^{(1)} \to X_i^{(2)}.$$

For $i \in I \setminus J$, by JF^{α}_{ω}-saturation, any finite subset of $X_i^{(1)}$ can be matched with a corresponding finite subset of $X_i^{(2)}$.

Since the types p_i are pairwise orthogonal, the realizations of different p_i are independent. Therefore, the partial isomorphisms on each $X_i^{(S)}$ can be combined with the identity on M to form a partial $\exists$-isomorphism from N_1 to N_2 over M.

Hence, for any $\exists$-formula $\exists\bar{x}\,\varphi(\bar{x}, \bar{m})$ with parameters $\bar{m} \subseteq M$, we have

$$N_1 \models \exists\bar{x}\,\varphi(\bar{x}, \bar{m}) \iff N_2 \models \exists\bar{x}\,\varphi(\bar{x}, \bar{m}),$$

which proves $(N_1, M) \equiv_{\exists_1} (N_2, M)$ in $\sigma_P(A)$. □

Lemma 2. Let T be a perfect, hereditary $\exists$-complete Jonsson theory, $K \subseteq E_T$, $[\Delta] \in JSp(K)/_{\bowtie}$, $[\Delta]^* = Th(C_{[\Delta]})$ be its center. Let $M_1, M_2, N_1, N_2 \in \mathrm{Mod}([\Delta]^*)$ with $A \subseteq M_1 \cap M_2$ and $N_1, N_2 \prec_{\exists_1} C_{[\Delta]}$, $M_1 \prec_{\exists_1} N_1$, $M_2 \prec_{\exists_1} N_2$. Then there exist JF^{α}_{ω}-saturated models M, N_1', N_2' satisfying:
1) $A \subseteq M$;
2) $M \prec_{\exists_1} N_1'$ and $M \prec_{\exists_1} N_2'$;
3) $(N_1, M_1) \equiv_{\exists} (N_1', M)$ and $(N_2, M_2) \equiv_{\exists_1} (N_2', M)$ in the signature $\sigma_P(A)$.

Proof. Let M be any JF^{α}_{ω}-saturated model containing A and existentially embedding both M_1 and M_2 in the sense that

$$M_1 \prec_{\exists_1} M, \quad M_2 \prec_{\exists_1} M.$$

Such a model exists by standard chain arguments for Jonsson theories and saturation: take an increasing chain of countable extensions starting from A that embeds M_1 and M_2, and then take a JF^{α}_{ω}-saturated elementary extension.

Consider N_1' to be a JF^{α}_{ω}-saturated model such that

$$N_1 \prec_{\exists_1} N_1' \quad \text{and} \quad M \subseteq N_1'.$$

Similarly, let N_2' be a JF^{α}_{ω}-saturated model extending N_2 with $M \subseteq N_2'$. By the properties of Jonsson theories and JF^{α}_{ω}-saturation, these extensions can be chosen so that

$$(N_1, M_1) \equiv_{\exists_1} (N_1', M), \quad (N_2, M_2) \equiv_{\exists_1} (N_2', M)$$

in the signature $\sigma_P(A)$.

1. By construction, $A \subseteq M$.
2. $M \prec_{\exists_1} N_1'$ and $M \prec_{\exists_1} N_2'$ hold since N_1' and N_2' are built as saturated extensions containing M.
3. The existential equivalences follow from the JF^{α}_{ω}-saturation and the fact that $M_1 \prec_{\exists_1} N_1$, $M_2 \prec_{\exists_1} N_2$. Any existential statement over M true in N_1' or N_2' can be mirrored in N_1 or N_2 via embeddings extending M_1 and M_2.

Hence, the models M, N_1', N_2' satisfy all required properties. □

We now turn to our main result. This result generalizes Theorem 1 of [4] in the context of the Jonsson spectrum of class K.

Since K is a subclass of E_T, the Kaiser hull of any model in K is a Jonsson theory. Let Δ be one of them.

Theorem 5. Let T be a perfect, hereditary $\exists$-complete, J-superstable Jonsson theory, $K \subseteq E_T$, $[\Delta] \in JSp(K)/_{\bowtie}$, $[\Delta]^*$ be its center. Then the following conditions are equivalent:
(a) $[\Delta]$ is J-nonmultidimensional;
(b) $[\Delta]$ is J-P-superstable;
(c) $[\Delta]$ is J-P-stable.

Proof. It is obvious that (b) implies (c). We shall show that (a) implies (b), and that (c) implies (a).

In the proof of $(a) \Rightarrow (b)$:

A class $[\Delta]$ of Jonsson theories is J-nonmultidimensional if every theory in the class is J-nonmultidimensional. Let Δ a be J-nonmultidimensional theory. By Theorem 4, it follows that Δ^* is a nonmultidimensional theory.

Nonmultidimensionality implies that all regular types are related, meaning that the behavior of models can be described in terms of only one dimension.

From nonmultidimensionality it follows that all regular types are connected, that is, the behavior of models can be described via a single dimension.

Lemma 1 states: if the dimensions coincide, then the models are logically indistinguishable in the signature σ_P.

Therefore, the set of possible pairs (N, M) is bounded, and hence the Δ^* is P superstable.

Before doing so, let us prove the following auxiliary statement.

Lemma 2 is needed to estimate the number of extensions or models, that is, to turn the qualitative result of Lemma 1 (extensions with the same types are equivalent) into a quantitative estimate of the number of such extensions. It is used to control cardinalities and for the subsequent derivation of superstability or bounds on the number of types.

Here we have shown that if Δ^* is nonmultidimensional, then Δ^* is P-superstable. We now need to show that Δ is J-P-superstable.

By Theorem 3 stability properties of Δ^* and Δ are preserved under this correspondence.

Assume that Δ^* is P-superstable. Then there exists a cardinal λ_0 such that Δ^* is P-λ-stable for all $\lambda \geq \lambda_0$. By the equivalence above, Δ is J-P-λ-stable for all such λ. Hence, every Δ is J-P-superstable. Then $[\Delta]$ is J-P-superstable.

Combining Lemmas 1 and 2, we complete the proof of Theorem 5 in the direction $(a) \Rightarrow (b)$.

(c) $\Rightarrow$ (a) Assume $[\Delta]$ is J-P-stable.

Suppose, towards a contradiction, that $[\Delta]$ is multidimensional. Then there exist two J-orthogonal regular types p and q.

Orthogonal types generate uncontrolled growth in the number of possible types over substructures (any combination of realizations of p and q is possible).

This contradicts J-P-stability, which requires the number of J-types to be bounded.

Hence, $[\Delta]$ must be J-nonmultidimensional.

We have established the chain of implications:

$$(a) \Rightarrow (b) \Rightarrow (c) \Rightarrow (a),$$

and therefore all three properties are equivalent. □

Conclusion

In this article, we studied the properties of Jonsson theories with additional predicates, focusing on J-stability and J-superstability. We established conditions under which models exhibit these properties and provided a systematic construction of $T_P^J(A)$, which preserves existential completeness while extending models with new predicates. The results generalize earlier work on superstable theories and offer a broader perspective on stability in models with added structure. The introduction of J-P-stability and the analysis of types in Jonsson structures with predicates represent a novel contribution to model theory and provide a foundation for further research in combinatorial and algebraic model theory. Future work may explore multidimensional types, their interactions with additional predicates, and extensions to more general classes of theories. Overall, this study deepens the understanding of the structure and classification of models in Jonsson theories, highlighting both theoretical significance and potential applications.

Author Contributions

All authors contributed equally to the development of the study, including the formulation of main results, analysis, and writing of the manuscript. All authors have read and approved the final version.

Conflict of Interest

The authors declare no conflict of interest.

References

1 Poizat, B. (1983). Paires de structure stables. *The Journal of Symbolic Logic, 48*(2), 239–249. https://doi.org/10.2307/2273543

2 Bouscaren, E. (1989). Elementary pairs of models. *Annals of Pure and Appl. Logic, 45*(2), 129–137. https://doi.org/10.1016/0168-0072(89)90057-2

3 Palyutin, E.A. (2003). E^*-stabilnye teorii [E^*-stable theories]. *Algebra i logika — Algebra and logic, 42*(2), 194–210 [in Russian].

4 Mustafin, T.G., & Nurmagambetov, T.A. (1990). O P-stabilnosti polnykh teorii [On P-stability of complete theories]. *Strukturnye svoistva algebraicheskikh sistem — Structural properties of algebraic systems*, 88–100 [in Russian].

5 Yeshkeyev, A.R., Zhumabekova, G.E., & Tungushbayeva, I.O. (2023). The central type of a semantic pair. *Logic Colloquium European Summer Meeting of the Association for Symbolic Logic*: Collection of Abstracts, 184. Italy: University of Milan.

6 Kassymetova, M.T., & Zhumabekova, G.E. (2024). Svoistva nemultirazmernosti dlia klassa semanticheskikh par [Nonmultidimensional properties for the class of semantic pairs]. *Traditsionnaia mezhdunarodnaia aprelskaia matematicheskaia konferentsiia v chest Dnia uchenogo Respubliki Kazakhstan — Traditional International April Mathematical Conference in honor of the Day of Scientists of the Republic of Kazakhstan*: Collection of Abstracts, 222. Almaty: Institute of Mathematics and Mathematical Modeling [in Russian].

7 Amanbekov, S.M., Onerkhaan A., & Tungushbayeva, I.O. (2025). The properties of amalgamation and joint embedding in the meaning of positive Jonsson theories. *Kazakh Mathematical Journal, 25*(2), 6–18. https://doi.org/10.70474/18bp7004

8 Yarullina, A.R., & Kaliolla, D.Ye. (2022). The perfect Jonsson S-acts. *Proceedings of the International Scientific Conference "Actual Problems of Mathematics, Mechanics and Informatics" dedicated to the 80th anniversary of Professor T.G. Mustafin,* 40–42. Karaganda: Karaganda Buketov University.

9 Omarova, M.T., & Amanbekov, S.M. (2024). 1-perfectness of fragments of normal Jonsson theory. *International Conference Mal'tsev Meeting*: Collection of Abstracts, 243. Novosibirsk: Sobolev Institute of Mathematics, Novosibirsk State University.

10 Popova, N.V., & Amanbekov, S.M. (2024). The properties of 1-model companions of Jonsson sets in normal Jonsson theory. *International Conference Mal'tsev Meeting*: Collection of Abstracts, 244. Novosibirsk: Sobolev Institute of Mathematics, Novosibirsk State University.

11 Kassymetova, M.T., Tungushbayeva, I.O., & Yarullina, A.R. (2025). Syntactic similarity of Hybrids of classes of normal Jonsson theories. *International Conference Actual problems of analysis, differential equations and algebra (EMJ–2025) dedicated to the 15th anniversary of the Eurasian Mathematical Journal*: Collection of Abstracts, 162. Astana: L.N. Gumilyov Eurasian National University.

12 Kassymetova, M.T., Mussina, N.M., & Amanbekov, S.M. (2021). Companions of hybrids of fragments of theoretical subsets. *International Conference Mal'tsev Meeting*: Collection of Abstracts, 167. Novosibirsk: Sobolev Institute of Mathematics, Novosibirsk State University.

13 Tungushbayeva, I.O., & Ulbrikht, O.I. (2024). On theories with amalgamation and joint embedding properties. *International Conference Mal'tsev Meeting*: Collection of Abstracts, 171. Novosibirsk: Sobolev Institute of Mathematics, Novosibirsk State University.

14 Barwise, J. (1977). *Handbook of Mathematical Logic.* Studies in Logic and the Foundations of Mathematics (Vol. 90). Amsterdam: North-Holland.

15 Yeshkeyev, A.R. (2024). *Teorii i ikh modeli [Theories and their models]: monograph* (Vols. 1–2). Karaganda: Izdatelstvo KarU [in Russian].

*Author Information**

Aibat Rafhatuly Yeshkeyev — Doctor of Physical and Mathematical Sciences, Professor, Professor researcher of the Department of Algebra, Mathematical Logic and Geometry named after Professor T.G. Mustafin, Buketov Karaganda National Research University, 28 Universitetskaya St., Karaganda, 100028, Kazakhstan; e-mail: *aibat.kz@gmail.com*; https://orcid.org/0000-0003-0149-6143

Galiya Erkinovna Zhumabekova (*corresponding author*) — PhD, Associate Professor of the Department of Algebra, Mathematical Logic and Geometry named after Professor T.G. Mustafin, Buketov Karaganda National Research University, 28 Universitetskaya St., Karaganda, 100028, Kazakhstan; e-mail: *galkatai@mail.ru*; https://orcid.org/0009-0002-2785-1756

Aigul Koishibaevna Issayeva — PhD, Assistant Professor of the Department Methods of Teaching Mathematics and Computer Science, Buketov Karaganda National Research University, 28 Universitetskaya St., Karaganda, 100028, Kazakhstan; e-mail: *isaevaaiga@gmail.com*; https://orcid.org/0009-0006-5697-0710

*Authors' names are presented in the following order: first name, middle name (if any), last name.