https://doi.org/10.31489/2026M1/194-201 *Research article*

# On a solution of the periodic boundary value problem for a hyperbolic equation with a fractional derivative

N.T. Orumbayeva, B.B. Zhantassova*

*Buketov Karaganda National Research University, Karaganda, Kazakhstan*
*(E-mail: orumbayevanurgul@gmail.com, zh.botagoz.b@gmail.com)*

The article investigates a boundary value problem for a hyperbolic equation with the Riemann–Liouville fractional derivative, which is periodic in one variable. Such equations are widely used in modeling complex physical processes with memory effects, including viscoelasticity, anomalous diffusion, and thermoviscoelasticity phenomena, where classical integer-order models fail to adequately describe the hereditary properties of materials and transport processes. To solve this problem, an iterative algorithm is proposed based on domain decomposition and the reduction of the original problem to a system of integro-differential equations. A theorem on the existence and uniqueness of the solution is proved, and an estimate of the convergence rate of the method is obtained using matrix analysis and a strengthened Gronwall–Bellman inequality. It is established that the choice of the decomposition step plays a key role in ensuring the stability of the algorithm. The conducted analysis extends the class of problems for which efficient computational algorithms can be constructed and may serve as a foundation for studying more complex nonlinear cases and problems in irregular domains.

*Keywords:* hyperbolic equation, Riemann–Liouville fractional derivative, periodic boundary value problem, integro-differential equations, algorithm, Gronwall–Bellman inequality, variable coefficients, nonlinear terms.

*2020 Mathematics Subject Classification:* 34B05, 35A05, 35A20, 35A22.

## *Introduction*

In modern mathematical physics, considerable attention is devoted to the study of differential equations with fractional derivatives, which are applied to modeling memory processes such as viscoelasticity, anomalous diffusion, and thermo-viscoelasticity [1–3]. Hyperbolic equations containing fractional derivatives are of particular interest [4–6], as they describe wave phenomena in media with hereditary properties. This work considers a periodic boundary value problem for a hyperbolic equation with the Riemann–Liouville fractional derivative, which arises in the study of processes with periodic boundary conditions. An iterative algorithm is proposed, based on decomposing the spatial domain and reducing the original problem to a system of integro-differential equations. The paper proves a theorem on the existence and uniqueness of the solution, as well as derives estimates of the convergence rate of the proposed method. The use of the parametrization method [7] enabled establishing sufficient convergence conditions for the algorithm. The results of this work extend the class of problems for which efficient computational algorithms can be constructed and can be applied to the numerical simulation of memory processes [8–10]. The conducted research broadens the range of problems for which effective computational algorithms can be developed [11, 12]. The obtained results are consistent with known approaches described in and complement them with new estimates [13, 14].

*Corresponding author. *E-mail: zh.botagoz.b@gmail.com*

This research was funded by the Science Committee of the Ministry of Science and Higher Education of the Republic of Kazakhstan (Grant No. AP23488729, 2024-2026).

*Received:* 22 April 2025; *Accepted:* 12 December 2025.

## 1 Problem statement

Consider a periodic boundary value problem for a hyperbolic equation with fractional derivative

$$\frac{\partial^2 w(t,x)}{\partial t \partial x} = A(t,x)\frac{\partial w(t,x)}{\partial t} + B(t,x)D_t^{\mu} w(t,x) + f(t,x), \quad (t,x) \in \Omega = [0,T] \times [0,X], \quad w \in R^n, \tag{1}$$

$$w(0,x) = 0, \qquad x \in [0,X], \tag{2}$$

$$w(t,0) = w(t,X), \qquad t \in [0,T], \tag{3}$$

where the $(n \times n)$ matrices $A(t,x), B(t,x)$, and the $n$-vector function $f(x,t)$ are continuous on $\Omega$,

$$\|w\| = \max_{i=\overline{1,n}} |w_i|, \quad \|A(t,x)\| = \max_{i=\overline{1,n}} \sum_{j=1}^{n} |a_{ij}(t,x)| \le \alpha, \quad \|B(t,x)\| = \max_{i=\overline{1,n}} \sum_{j=1}^{n} |b_{ij}(t,x)| \le \beta,$$

$\alpha, \beta = \text{const}$, $0 < \mu < 1$, $D_t^{\mu} w(t,x)$ is the Riemann–Liouville fractional derivative of order $\mu$, defined by the formula

$$D_t^{\mu} w(t,x) = \frac{1}{\Gamma(1-\mu)} \frac{d}{dt} \int\limits_0^t \frac{w(\tau,x)}{(t-\tau)^{\mu}} d\tau, \quad t \in [0,T],$$

where $\Gamma(z)$ is gamma function: $\Gamma(z) = \int\limits_0^{\infty} x^{z-1} e^{-x} dx$.

Let $C(\Omega, \mathbb{R}^n)$ be the space of continuous functions $w : \Omega \to \mathbb{R}^n$ on $\Omega$ with the norm

$$\|w(t,x)\|_1 = \max_{(t,x)\in\Omega} \|w(t,x)\|.$$

The function $w(t,x) \in C(\Omega, \mathbb{R}^n)$, having partial derivatives $\frac{\partial w(t,x)}{\partial t} \in C(\Omega, \mathbb{R}^n)$, $\frac{\partial^2 w(t,x)}{\partial t \partial x} \in C(\Omega, \mathbb{R}^n)$, provided that $D_t^{\mu} w(t,x)$ exists in the Riemann–Liouville sense, is called a solution of the problem (1)–(3), if it satisfies the system (1) for all $(t,x) \in \Omega$, has zero value on the characteristic $t = 0$, and has equal values on the characteristics $x = 0$, $x = X$ for all $t \in [0,T]$.

To solve this problem, we introduce a new function $u(t,x) = \frac{\partial w(t,x)}{\partial t}$, then $w(t,x) = \int\limits_0^t u(\tau,x) d\tau$. Substituting $u(t,x)$ into equation (1) gives

$$\frac{\partial u(t,x)}{\partial x} = A(t,x)u(t,x) + B(t,x)D_t^{\mu}\left(\int\limits_0^t u(\tau,x)d\tau\right) + f(t,x), \quad (t,x) \in \Omega = [0,T] \times [0,X],$$

with the boundary condition:

$$u(t,0) = u(t,X), \quad t \in [0,T].$$

Let us partition the interval $[0,X]$ into $N$ parts with a step size $h = X/N$. We introduce the partition points: $x_r = (r-1)h,\ r = 1, 2, \ldots, N$. We define the restrictions of the function $u(t,x)$ on each interval $u_r(t,x) = u(t,x)$, $x \in [(r-1)h, rh)$.

The original problem is equivalent to the problem

$$\frac{\partial u_r(t,x)}{\partial x} = A(t,x)u_r(t,x) + B(t,x)D_t^{\mu}\left(\int\limits_0^t u_r(\tau,x)d\tau\right) + f(t,x), \tag{4}$$

$$u_1(t,0) = \lim_{x \to Nh-0} u_N(t,x), \quad t \in [0,T], \tag{5}$$

$$\lim_{x\to rh-0} u_r(t,x) = u_{r+1}(t,rh), \quad r=1,2,...,N-1, \tag{6}$$

where $(t,x)\in\Omega_r=[0,T]\times[(r-1)h,rh)$, (6) is the matching condition of the solution on the internal partition lines,

$$D_t^\mu\left(\int_0^t u_r(\tau,x)d\tau\right) = \frac{1}{\Gamma(1-\mu)}\frac{\partial}{\partial t}\int_0^t \frac{\int_0^\tau u_r(\tau_1,x)d\tau_1}{(t-\tau)^\mu}d\tau = \frac{1}{\Gamma(1-\mu)}\int_0^t \frac{u_r(\tau,x)}{(t-\tau)^\mu}d\tau.$$

## 2 Main results

We introduce the substitution $v_r(t,x) = u_r(t,x)-\lambda_r(t)$, where $\lambda_r(t) = u_r(t,(r-1)h)$. Then $u_r(t,x) = v_r(t,x)+\lambda_r(t)$. Substituting into equations (4)–(6), we obtain

$$\frac{\partial v_r(t,x)}{\partial x} = A(t,x)v_r(t,x)+A(t,x)\lambda_r(t)+$$

$$+\frac{B(t,x)}{\Gamma(1-\mu)}\int_0^t \frac{v_r(\tau,x)}{(t-\tau)^\mu}d\tau+\frac{B(t,x)}{\Gamma(1-\mu)}\int_0^t \frac{\lambda_r(\tau)}{(t-\tau)^\mu}d\tau+f(t,x), \tag{7}$$

$$v_r(t,(r-1)h)=0, \quad t\in[0,T], \quad r=\overline{1,N}, \tag{8}$$

$$\lambda_1(t) = \lambda_N(t)+\lim_{x\to X-0} v_N(t,x), \quad t\in[0,T], \tag{9}$$

$$\lambda_r(t)+\lim_{x\to rh-0} v_r(t,x) = \lambda_{r+1}(t), \quad r=1,2,...,N-1. \tag{10}$$

The problem (7), (8) with fixed $\lambda_r(t)$ is a one-parameter family of Cauchy problems for systems of integro-differential equations, where $t\in[0,T]$, and is equivalent to the integral equation

$$v_r(t,x) = \int_{(r-1)h}^x A(t,\xi)v_r(t,\xi)d\xi+\lambda_r(t)\int_{(r-1)h}^x A(t,\xi)d\xi+$$

$$+\int_{(r-1)h}^x \frac{B(t,\xi)}{\Gamma(1-\mu)}\int_0^t \frac{v_r(\tau,\xi)}{(t-\tau)^\mu}d\tau d\xi+\frac{1}{\Gamma(1-\mu)}\int_0^t \frac{\lambda_r(\tau)}{(t-\tau)^\mu}d\tau\int_{(r-1)h}^x B(t,\xi)d\xi+\int_{(r-1)h}^x f(t,\xi)d\xi. \tag{11}$$

Passing to the limit as $x\to rh-0$ in (11) and substituting in (9), (10) instead of $\lim\limits_{x\to rh-0} v_r(t,x)$, $r=\overline{1,N}$, the corresponding right-hand sides for the unknown functions $\lambda_r(t)$, $r=\overline{1,N}$ we obtain a system of equations

$$h\lambda_1(t) = h\lambda_N(t)+h\int_{(N-1)h}^{Nh} A(t,\xi)v_N(t,\xi)d\xi+h\lambda_N(t)\int_{(N-1)h}^{Nh} A(t,\xi)d\xi+$$

$$+h\int_{(N-1)h}^{Nh}\frac{B(t,\xi)}{\Gamma(1-\mu)}\int_0^t \frac{v_N(\tau,\xi)}{(t-\tau)^\mu}d\tau d\xi+\frac{h}{\Gamma(1-\mu)}\int_0^t \frac{\lambda_N(\tau)}{(t-\tau)^\mu}d\tau\int_{(N-1)h}^{Nh} B(t,\xi)d\xi+h\int_{(N-1)h}^{Nh} f(t,\xi)d\xi,$$

$$\lambda_r(t) + \int\limits_{(r-1)h}^{rh} A(t,\xi)v_r(t,\xi)d\xi + \lambda_r(t)\int\limits_{(r-1)h}^{rh} A(t,\xi)d\xi + \int\limits_{(r-1)h}^{rh} \frac{B(t,\xi)}{\Gamma(1-\mu)}\int\limits_0^t \frac{v_r(\tau,\xi)}{(t-\tau)^\mu}d\tau d\xi +$$

$$+\frac{1}{\Gamma(1-\mu)}\int\limits_0^t \frac{\lambda_r(\tau)}{(t-\tau)^\mu}d\tau \int\limits_{(r-1)h}^{rh} B(t,\xi)d\xi + \int\limits_{(r-1)h}^{rh} f(t,\xi)d\xi = \lambda_{r+1}(t), \quad r=\overline{1,N-1}.$$

This system can be written in matrix form as

$$Q(h)\lambda(t) = -\Lambda(t,h,\lambda) - V_1(t,h,v) - V_2(t,h,v) - F(t,h), \tag{12}$$

where $\lambda(t) = \big(\lambda_1(t), \lambda_2(t), \ldots, \lambda_N(t)\big)'$,

$$Q(h) = \begin{pmatrix} h & 0 & \ldots & 0 & -h\Big(1+\int\limits_{(N-1)h}^{Nh} A(t,\xi)d\xi\Big) \\ 1+\int\limits_0^h A(t,\xi)d\xi & -1 & \ldots & 0 & 0 \\ 0 & 1+\int\limits_h^{2h} A(t,\xi)d\xi & \ldots & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & 1+\int\limits_{(N-2)h}^{(N-1)h} A(t,\xi)d\xi & -1 \end{pmatrix},$$

$$\Lambda(t,h,\lambda) = \Bigg(h\int\limits_{(N-1)h}^{Nh} B(t,\xi)d\xi\cdot\frac{1}{\Gamma(1-\mu)}\int\limits_0^t \frac{\lambda_N(\tau)}{(t-\tau)^\mu}d\tau,$$

$$\int\limits_0^h B(t,\xi)d\xi\cdot\frac{1}{\Gamma(1-\mu)}\int\limits_0^t \frac{\lambda_1(\tau)}{(t-\tau)^\mu}d\tau, \ldots, \int\limits_{(N-2)h}^{(N-1)h} B(t,\xi)d\xi\cdot\frac{1}{\Gamma(1-\mu)}\int\limits_0^t \frac{\lambda_{N-1}(\tau)}{(t-\tau)^\mu}d\tau\Bigg)',$$

$$V_1(t,h,v) = \Bigg(h\int\limits_{(N-1)h}^{Nh} A(t,\xi)v_N(t,\xi)d\xi, \int\limits_0^h A(t,\xi)v_1(t,\xi)d\xi, \ldots, \int\limits_{(N-2)h}^{(N-1)h} A(t,\xi)v_{N-1}(t,\xi)d\xi\Bigg)',$$

$$V_2(t,h,v) = \Bigg(h\int\limits_{(N-1)h}^{Nh} \frac{B(t,\xi)}{\Gamma(1-\mu)}\int\limits_0^t \frac{v_N(\tau,\xi)}{(t-\tau)^\mu}d\tau d\xi,$$

$$\int\limits_0^h \frac{B(t,\xi)}{\Gamma(1-\mu)}\int\limits_0^t \frac{v_1(\tau,\xi)}{(t-\tau)^\mu}d\tau d\xi, \ldots, \int\limits_{(N-2)h}^{(N-1)h} \frac{B(t,\xi)}{\Gamma(1-\mu)}\int\limits_0^t \frac{v_{N-1}(\tau,\xi)}{(t-\tau)^\mu}d\tau d\xi\Bigg)',$$

$$F(t,h) = \Bigg(h\int\limits_{(N-1)h}^{Nh} f(t,\xi)d\xi, \int\limits_0^h f(t,\xi)d\xi, \ldots, \int\limits_{(N-2)h}^{(N-1)h} f(t,\xi)d\xi\Bigg)'.$$

To find the system consisting of the functions $\{\lambda_r(t), v_r(t,x)\}$, $r=\overline{1,N}$, we have a closed system consisting of equations (11) and (12).

Suppose the matrix $Q(h)$ is invertible for all $t \in [0,T]$ [15]. Taking as the initial approximation $v_r(t,x)=0$, from system (12), we find $\lambda_r^{(0)}(t)$, $r=\overline{1,N}$. From equation (11) with $\lambda_r(t)=\lambda_r^{(0)}(t)$, we find $v_r^{(0)}(t,x)$, $r=\overline{1,N}$.

*Step 1.* Using equation (12) with $v_r(t,x)=v_r^{(0)}(t,x)$, we find $\lambda_r^{(1)}(t)$, $r=\overline{1,N}$. From equation (11) with $\lambda_r(t)=\lambda_r^{(1)}(t)$, we find $v_r^{(1)}(t,x)$, $r=\overline{1,N}$.

Continuing the process, at the $k$-th step, we obtain a system of pairs $\{\lambda_r^{(k)}(t), v_r^{(k)}(t,x)\}$.

Sufficient conditions for the feasibility and convergence of the proposed algorithm, as well as an estimate of the difference between the exact and approximate solutions, are established by

*Theorem 1.* Let for some $h>0: Nh=X, N=1,2,...$, $(nN\times nN)$ matrix $Q(h)$ is invertible and the inequalities hold

1) $\|[Q(h)]^{-1}\| < \gamma(h)$,

2) $q(h)=h^2\max\{1,h\}\left(\alpha+\frac{\beta T^{1-\mu}}{\Gamma(2-\mu)}\right)^2 E_{1-\mu}\big(h\gamma(h)\max\{1,h\}\cdot\beta\cdot T^{1-\mu}\big)e^{h\left(\alpha+\frac{\beta T^{1-\mu}}{\Gamma(2-\mu)}\right)}<1$,

where $E_{1-\mu}\left(h\gamma(h)\max\{1,h\}\cdot\beta\cdot T^{1-\mu}\right)=\sum_{k=0}^{\infty}\frac{\left(h\gamma(h)\max\{1,h\}\cdot\beta\cdot T^{1-\mu}\right)^k}{\Gamma((1-\mu)k+1)}$, $0<\mu<1$, $\|A(t,x)\|\le\alpha$, $\|B(t,x)\|\le\beta$. Then the boundary value problem (1)–(3) has a unique solution $w^*(t,x)$ and the estimate holds

$$\|w^*(t,x)-w^{(k)}(t,x)\|\le$$
$$\le T\left(1+h\max\{1,h\}\left(\alpha+\frac{\beta T^{1-\mu}}{\Gamma(2-\mu)}\right)E_{1-\mu}\big(h\gamma(h)\max\{1,h\}\cdot\beta\cdot T^{1-\mu}\big)\right)\frac{[q(h)]^k}{1-q(h)}M(h)\|f(t,x)\|_1.$$

*Proof.* The following inequalities hold

$$\|\Lambda(t,h,\lambda)\|\le\frac{\max\{1,h\}h\beta}{\Gamma(1-\mu)}\left\|\int_0^t\frac{\lambda_N(\tau)}{(t-\tau)^\mu}d\tau\right\|\le\frac{\max\{1,h\}h\beta}{\Gamma(1-\mu)}\int_0^t\frac{1}{(t-\tau)^\mu}\max_{r=\overline{1,N}}\|\lambda_r(\tau)\|d\tau,$$

$$\|V_1(t,h,v)\|\le\max\{1,h\}\alpha h\max_{r=\overline{1,N}}\sup_{x\in[(r-1)h,rh)}\|v_r(t,x)\|,$$

$$\|V_2(t,h,v)\|\le\frac{\max\{1,h\}h\beta}{\Gamma(1-\mu)}\int_0^t\frac{1}{(t-\tau)^\mu}\max_{r=\overline{1,N}}\sup_{x\in[(r-1)h,rh)}\|v_r(\tau,x)\|d\tau,$$

$$\|F(t,h)\|\le h\max\{1,h\}\|f(t,x)\|_1.$$

Assume that the matrix $Q(h)$ is invertible. Taking as the initial approximation $v_r(t,x)=0$, from system (12), we find $\lambda_r^{(0)}(t)$.

Let us use the generalized Gronwall–Bellman inequality for equations with a fractional integral [16]

$$\|\lambda^{(0)}(t)\|=\max_{r=\overline{1,N}}\|\lambda_r^{(0)}(t)\|\le h\gamma(h)\max\{1,h\}\cdot\|f(t,x)\|_1\cdot E_{1-\mu}\big(h\gamma(h)\max\{1,h\}\cdot\beta\cdot t^{1-\mu}\big),$$

where $E_{1-\mu}\big(h\gamma(h)\max\{1,h\}\cdot\beta\cdot t^{1-\mu}\big)$ is the two-parameter Mittag-Leffler function.

So, we have

$$\|\lambda^{(0)}(t)\|_2=\max_{t\in[0,T]}\|\lambda^{(0)}(t)\|\le h\gamma(h)\max\{1,h\}E_{1-\mu}\big(h\gamma(h)\max\{1,h\}\cdot\beta\cdot T^{1-\mu}\big)\|f(t,x)\|_1.$$

Under our assumptions, the Cauchy problem (7), (8) with $\lambda_r(t) = \lambda_r^{(0)}(t)$, has a unique solution $v_r^{(0)}(t,x)$. By the Gronwall–Bellman inequality,

$$\|v^{(0)}(t,x)\|_3 = \max_{r=\overline{1,N}} \max_{(t,x)\in\Omega_r} \|v_r^{(0)}(t,x)\| \leq$$

$$\leq h\left[\left(\alpha + \frac{\beta T^{1-\mu}}{\Gamma(2-\mu)}\right)\|\lambda^{(0)}(t)\|_2 + \|f(t,x)\|_1\right] e^{h\left(\alpha+\frac{\beta T^{1-\mu}}{\Gamma(2-\mu)}\right)} \leq M(h)\|f(t,x)\|_1,$$

$$M(h) = h\left[\left(\alpha + \tfrac{\beta T^{1-\mu}}{\Gamma(2-\mu)}\right) h\gamma(h)\max\{1,h\} E_{1-\mu}\big(h\gamma(h)\max\{1,h\}\cdot\beta\cdot T^{1-\mu}\big) + 1\right] e^{h\left(\alpha+\frac{\beta T^{1-\mu}}{\Gamma(2-\mu)}\right)}.$$

According to the algorithm, we determine $\lambda^{(1)}(t)$ and estimate $\lambda_r^{(1)}(t) - \lambda_r^{(0)}(t)$. Let us use the generalized Gronwall–Bellman inequality for equations with a fractional integral [16]

$$\|\lambda^{(1)}(t) - \lambda^{(0)}(t)\|_2 \leq$$

$$\leq h\max\{1,h\}\left(\alpha + \frac{\beta T^{1-\mu}}{\Gamma(2-\mu)}\right) E_{1-\mu}\big(h\gamma(h)\max\{1,h\}\cdot\beta\cdot T^{1-\mu}\big)\|v^{(0)}(t,x)\|_3.$$

Then

$$\left\|v^{(1)}(t,x) - v^{(0)}(t,x)\right\|_3 \leq q(h)\|v^{(0)}(t,x)\|_3.$$

Continuing the iterative process, we obtain a sequence of systems of pairs $\{\lambda_r^{(k)}(t), v_r^{(k)}(t,x)\}$, $r = 1,2,...,N$, $k = 1,2,...$ :

$$\|\lambda^{(k+1)}(t) - \lambda^{(k)}(t)\|_2 \leq$$

$$\leq h\max\{1,h\}\left(\alpha + \frac{\beta T^{1-\mu}}{\Gamma(2-\mu)}\right) E_{1-\mu}\big(h\gamma(h)\max\{1,h\}\cdot\beta\cdot T^{1-\mu}\big)\|v^{(k)}(t,x) - v^{(k-1)}(t,x)\|_3, \quad (13)$$

$$\|v^{(k+1)}(t,x) - v^{(k)}(t,x)\|_3 \leq q(h)\|v^{(k)}(t,x) - v^{(k-1)}(t,x)\|_3, \quad k = 1,2,\dots \quad (14)$$

By virtue of condition 2) of Theorem 1 and inequalities (13), (14), the sequence $\{\lambda_r^{(k)}(t), v_r^{(k)}(t,x)\}$ converges to $\{\lambda_r^*(t), v_r^*(t,x)\}$ as $k \to \infty$ and the following estimates hold:

$$\|\lambda^*(t) - \lambda^{(k)}(t)\|_2 \leq$$

$$\leq h\max\{1,h\}\left(\alpha + \frac{\beta T^{1-\mu}}{\Gamma(2-\mu)}\right) E_{1-\mu}\big(h\gamma(h)\max\{1,h\}\cdot\beta\cdot T^{1-\mu}\big)\frac{[q(h)]^k}{1-q(h)} M(h)\|f(t,x)\|_1,$$

$$\|v^*(t,x) - v^{(k)}(t,x)\|_3 \leq \frac{[q(h)]^k}{1-q(h)} M(h)\|f(t,x)\|_1.$$

Since $\{\lambda_r^*(t), v_r^*(t,x)\}$ is a solution of the problem (7)–(10), the function $w^*(t,x)$, obtained by joining together the systems of functions $\int_0^t (\lambda_r^*(\tau) + v_r^*(\tau,x))d\tau$, will be a solution of the original problem (1)–(3), and the estimate of the Theorem 1 holds.

Let us prove uniqueness. Suppose that $w^{**}(t,x), w^*(t,x)$ are two solutions of the problem (1)–(3). Then the corresponding systems of pairs $\{\lambda_r^{**}(t), v_r^{**}(t,x)\}$, $\{\lambda_r^*(t), v_r^*(t,x)\}$, $r = \overline{1,N}$, will be solutions of the boundary value problem (7)–(10), and similarly to (13), (14):

$$\|\lambda^{**}(t) - \lambda^*(t)\|_2 \leq h\max\{1,h\}\left(\alpha + \frac{\beta T^{1-\mu}}{\Gamma(2-\mu)}\right) E_{1-\mu}\big(h\gamma(h)\max\{1,h\}\cdot\beta\cdot T^{1-\mu}\big)\|v^*(t,x) - v^{**}(t,x)\|_3,$$

$$\|v^{**}(t,x) - v^*(t,x)\|_3 \leq q(h)\|v^{**}(t,x) - v^*(t,x)\|_3, \quad q(h) < 1.$$

It follows from this that $\lambda_r^*(t) = \lambda_r^{**}(t), v_r^*(t,x) = v_r^{**}(t,x)$, i.e. $w^{**}(t,x) = w^*(t,x)$, for $(t,x) \in \Omega$. The theorem is proved. □

*Conclusion*

An iterative algorithm has been proposed in this work for solving the periodic boundary value problem of a hyperbolic equation with the Riemann–Liouville fractional derivative. A theorem on the existence and uniqueness of the solution has been proved, and estimates of its approximate values have been established. The convergence conditions of the algorithm have been derived using matrix analysis and the Gronwall–Bellman inequality. It has been shown that the choice of the partition step of the spatial interval plays a key role in ensuring the stability of the method. The results of this work can be applied to the modeling of a wide class of physical processes with memory and periodic boundary conditions. Further research may be aimed at solving nonlocal boundary value problems for nonlinear hyperbolic equations with a fractional derivative.

*Acknowledgments*

The authors thank the reviewers for their valuable comments and suggestions.

*Author Contributions*

All authors contributed equally to this work.

*Conflict of Interest*

The authors declare no conflict of interest.

References

1. Samko, S.G., Kilbas, A.A., & Marichev, O.I. (1993). *Fractional integrals and derivatives: Theory and applications.* Yverdon: Gordon and Breach.
2. Nakhushev, A.M. (2003). *Drobnye ischisleniia i ikh primeneniia [Fractional calculus and its applications]*. Moscow: FIZMATLIT [in Russian].
3. Leonenko, N., & Podlubny, I. (2022). Monte Carlo method for fractional-order differentiation extended to higher orders. *Fractional Calculus and Applied Analysis, 25*(3), 841–857. https://doi.org/10.1007/s13540-022-00048-w
4. Abdimanapova, P.B., & Temesheva, S.M. (2023). On a Solution of a Nonlinear Nonlocal Boundary Value Problem for one Class of Hyperbolic Equation. *Lobachevskii Journal of Mathematics Open source preview, 44*(7), 2529–2541. https://doi.org/10.1134/S1995080223070028
5. Pskhu, A.V. (2022). Green Function of the First Boundary-Value Problem for the Fractional Diffusion-Wave Equation in a Multidimensional Rectangular Domain. *Journal of Mathematical Sciences, 260*(3), 325–334. https://doi.org/10.1007/s10958-022-05697-z
6. Khubiev, K.U. (2018). Zadachi so smeshcheniem dlia nagruzhennogo uravneniia giperbolo-parabolicheskogo tipa s operatorom drobnoi diffuzii [Boundary value problem with shift for loaded hyperbolic-parabolic type equation involving fractional diffusion operator]. *Vestnik Udmurtskogo universiteta. Matematika. Mekhanika. Kompiuternye nauki – Bulletin of Udmurt University. Mathematics. Mechanics. Computer Science, 28*(1), 82–90 [in Russian]. https://doi.org/10.20537/vm180108
7. Dzhumabayev, D.S. (1989). Criteria for the unique solvability of a linear boundary-value problem for an ordinary differential equation. *USSR Computational Mathematics and Mathematical Physics, 29*(1), 34–46. https://doi.org/10.1016/0041-5553(89)90038-4

8. Assanova, A. (2025). Novel Approach to Solving Boundary Value Problems for Systems of Hyperbolic Second Order Differential Equations. *Journal of Mathematical Sciences United States Open source preview, 291*(2), 190–201. https://doi.org/10.1007/s10958-025-07800-6
9. Iskakova, N.B., Bakirova, E.A., Khanzharova, B.S., & Kadirbayeva, Z.M. (2024). An Algorithm for Solving Boundary Value Problems for Delay Differential Equations with Loadings. *Lobachevskii Journal of Mathematics Open source preview, 45*(10), 5032–5042. https://doi.org/10.1134/S1995080224606064
10. Pskhu, A.V., Kosmakova, M.T., & Izhanova, K.A. (2024). Cauchy Problem for a Loaded Fractional Diffusion Equation. *Lobachevskii Journal of Mathematics Open source preview, 45*(9), 4574–4581. https://doi.org/10.1134/S1995080224605228
11. Ramazanov, M.I., Gulmanov, N.K., Kopbalina, S.S., & Omarov, M.T. (2024). Solution of a Singular Integral Equation of Volterra Type of the Second Kind. *Lobachevskii Journal of Mathematics, 45*(11), 5898–5906. https://doi.org/10.1134/S1995080224606830
12. Pskhu, A.V., Ramazanov, M.I., & Kosmakova, M.T. (2023). Boundary value problem for a loaded fractional diffusion equation. *Turkish Journal of Mathematics, 47*(5), 1585–1594. https://doi.org/10.55730/1300-0098.3450
13. Pskhu, A.V. (2025). On overdetermination conditions in order determination inverse problems for fractional diffusion equations. *Uzbek Mathematical Journal Open source preview, 69*(2), 222–231. https://doi.org/10.29229/uzmj.2025-2-21
14. Orumbayeva, N.T., Kosmakova, M.T., Tokmagambetova, T.D., & Manat, A.M. (2025). Solutions of boundary value problems for loaded hyperbolic type equations. *Bulletin of the Karaganda University. Mathematics Series,* 2(118), 177–188. https://doi.org/10.31489/2025m2/177-188
15. Orumbayeva, N.T., Assanova, A.T., & Keldibekova, A.B. (2022). On an algorithm of finding an approximate solution of a periodic problem for a third-order differential equation. *Eurasian Mathematical Journal, 13*(1), 69–85. https://doi.org/10.32523/2077-9879-2022-13-1-69-85
16. Ye, H., Gao, J., & Ding, Y. (2007). A generalized Gronwall inequality and its application to a fractional differential equation. *Journal of Mathematical Analysis and Applications, 328*, 1075–1081. https://doi.org/10.1016/j.jmaa.2006.05.061

*Author Information**

**Nurgul Tumarbekovna Orumbayeva** — Candidate in Physical and Mathematical Sciences, Associate Professor, Buketov Karaganda National Research University, Institute of Applied Mathematics, 28 Universitetskaya St., Karaganda, 100028, Kazakhstan; e-mail: *orumbayevanurgul@gmail.com*; https://orcid.org/0000-0003-1714-6850

**Botagoz Beketovna Zhantassova** (corresponding author) — PhD doctoral student, Senior lecturer, Buketov Karaganda National Research University, Institute of Applied Mathematics, 28 Universitetskaya St., Karaganda, 100028, Kazakhstan; e-mail: *zh.botagoz.b@gmail.com*; https://orcid.org/0009-0008-6712-4928

*Authors' names are presented in the following order: first name, middle name (if any), last name.