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Automorphisms of free braided nonassociative algebras of rank 2

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We prove the elementary reducibility of any nonaffine automorphism of a free nonassociative algebra of rank two over an arbitrary field. Using this result we establish a property of automorphisms of this algebra that will be needed later. We then derive a necessary and sufficient condition for the isomorphism of two free braided nonassociative algebras of rank two over a field with diagonal braidings. We describe the automorphism groups of two generated free braided nonassociative algebras with involutive diagonal braidings over an arbitrary field of characteristic not equal to two. Depending on the form of the diagonal involutive braiding, five different automorphism groups of a two-generated free nonassociative algebra arise in this case: 1) the group of all automorphisms, 2) the group of all odd automorphisms, 3) the subgroup of the group of triangular automorphisms, 4) the toric automorphism group, 5) the semidirect product of the toric automorphism group with the subgroup generated by an automorphism that permutes two variables.

Keywords: Yang–Baxter equation, braided space, braiding, diagonal braiding, involutive braiding, braided algebra, free nonassociative algebra, automorphism, odd automorphism, toric automorphism group.

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Introduction

A *braided space* (see, for example, [1,2]) is a linear space V over an arbitrary field K with a linear map $\beta : V \otimes V \to V \otimes V$, called a *braiding*, that satisfies the Yang–Baxter equation

$$(\beta \otimes \mathrm{id})(\mathrm{id} \otimes \beta)(\beta \otimes \mathrm{id}) = (\mathrm{id} \otimes \beta)(\beta \otimes \mathrm{id})(\mathrm{id} \otimes \beta). \tag{1}$$

Let $X = \{x_1, x_2, \ldots, x_n\}$ be a basis of V. The linear map

$$\beta : x_i \otimes x_s \mapsto \beta_{is} x_s \otimes x_i, \text{ where } \beta_{is} \in K, \ 1 \le i, s \le n, \tag{2}$$

is a braiding and it is called a *diagonal braiding.* In the case $\beta^2 = \mathrm{id}$ the braiding β is called *involutive.* The diagonal braiding β is involutive if and only if

$$\beta_{ij}\beta_{ji} = 1, \quad \text{for all } \ 1 \le i, j \le n. \tag{3}$$

In particular, $\beta_{ii} = \pm 1$ for all i.

Every braiding β of V can be uniquely extended to the free associative algebra $K\langle X\rangle$ [3] and to the free nonassociative algebra $K\{X\}$ [4] freely generated by set X over a field K so that $K\langle X\rangle$ and $K\{X\}$ are braided algebras. Moreover, they also proved that the free braided algebra $K\{X\}$ has a natural structure of a braided nonassociative Hopf algebra [4]. This algebra plays an important role in quantum Lie theory (see, for example, [3,5]). R. Mutalip, A. Naurazbekova and U. Umirbaev [6] described all automorphism groups of free braided associative algebras of rank 2 with diagonal involutive braidings

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over a field of characteristic $\neq 2$. Many papers are devoted to the investigation of structures of braided algebras (see, for example, [7–9]).

Consider the following grading

$$K\{x_1, x_2\} = C_0 \oplus C_1 \tag{4}$$

of the free nonassociative algebra $K\{x_1, x_2\}$ in two variables x_1, x_2 over K, where C_0 and C_1 are the linear spans of all even length monomials and a linear span of all odd length monomials, respectively.

Denote by $\mathrm{Aut}A$ the automorphism group of an algebra A. Let us call an automorphism $\varphi \in \mathrm{Aut}K\{x_1, x_2\}$ *odd* if $\varphi(x_1), \varphi(x_2) \in C_1$. The set of all odd automorphisms of $K\{x_1, x_2\}$ denote by G_{odd}. It is clear that G_{odd} is subgroup of $\mathrm{Aut}K\{x_1, x_2\}$.

Let $(K\{x_1, x_2\}, \beta)$ be a free braided nonassociative algebra in two variables x_1, x_2 with a diagonal involutive braiding $\beta = (\beta_{11}, \beta_{12}, \beta_{21}, \beta_{22})$ over a field K of characteristic $\neq 2$. In this paper we show that

(1) if $\beta_{ij} = 1$ for all i, j then $\mathrm{Aut}(K\{x_1, x_2\}, \beta) = \mathrm{Aut}K\{x_1, x_2\}$;

(2) if $\beta_{ij} = -1$ for all i, j then $\mathrm{Aut}(K\{x_1, x_2\}, \beta) = G_{\mathrm{odd}}$;

(3) if $\beta_{11} = \beta_{22}$, $\beta_{12} = \beta_{21}$, and $\beta_{11}\beta_{12} = -1$ then $\mathrm{Aut}(K\{x_1, x_2\}, \beta) \cong (K^* \times K^*) \rtimes \mathbb{Z}_2$, where $\mathbb{Z}_2$ is the subgroup of $\mathrm{Aut}K\{x_1, x_2\}$ generated by (x_2, x_1);

(4) if $\beta_{12} = 1$ and $\beta_{11}\beta_{22} = -1$ then $\mathrm{Aut}(K\{x_1, x_2\}, \beta) \cong \{\varphi \in \mathrm{Aut}K\{x_1, x_2\} \mid \varphi(x_1) = ax_1 + g(x_2^2),\ \varphi(x_2) = bx_2,\ a, b \in K^*,\ g(x) \in K\{x\}\}$;

(5) if $\beta_{12} \neq \pm 1$ or $\beta_{12} = -1,\ \beta_{11}\beta_{22} = -1$ then $\mathrm{Aut}(K\{x_1, x_2\}, \beta) \cong K^* \times K^*$.

The paper is organized as follows. In Section 1, we give some definitions and facts on free braided nonassociative algebras. In Section 2, we prove elementary reducibility of any nonaffine automorphism of a free nonassociative algebra of rank two over an arbitrary field. Using this result we prove the property of automorphisms of this algebra that we need in the future. In Section 3, we derive a necessary and sufficient condition for the isomorphism of two free braided nonassociative algebras of rank two over a field with diagonal braidings. In Section 4, we describe all automorphism groups of free braided nonassociative algebras of rank 2 equipped with diagonal involutive braidings over a field of characteristic $\neq 2$.

1 *Free braided nonassociative algebra*

The *braid monoid* B_n [3] is an associative monoid generated by elements $b_1, b_2, \ldots, b_{n-1}$, called braids, subject to the following relations:

$$b_t b_{t+1} b_t = b_{t+1} b_t b_{t+1}, \quad b_i b_j = b_j b_i, \quad 1 \leq t < n-1, \quad |i-j| > 1. \tag{5}$$

Let V be a linear space over a field K equipped with a braiding $\beta : V \otimes V \to V \otimes V$. Using (1), it is easy to see that the linear maps

$$\beta_i = \mathrm{id}^{\otimes(i-1)} \otimes \beta \otimes \mathrm{id}^{\otimes(n-i-1)} : V^{\otimes n} \to V^{\otimes n}, \quad 1 \leq i < n,$$

satisfy all relations (5).

Introduce the following notation:

$$[t;t] = 1, \quad [m;t] = \beta_{t-1}\beta_{t-2}\cdots\beta_{m+1}\beta_m, \quad [t;m] = \beta_m\beta_{m+1}\cdots\beta_{t-2}\beta_{t-1}, \quad m < t.$$

Define the map $\nu_r^{t,n} : V^{\otimes n} \to V^{\otimes n}$, $t \leq r < n$ as a superposition of maps β_i:

$$\nu_r^{t,n} = [t; r+1][t+1; r+2]\cdots[t+n-r-1; n].$$

V. Kharchenko proved [3], that

$$\nu_r^{t,n} = [n; r][n-1; r-1]\cdots[n-r+t; t].$$

An algebra A with a multiplication $\mathrm{m} : A \otimes A \to A$ is called a *braided algebra* [1] if A is a braided space and

$$(\mathrm{m} \otimes \mathrm{id})\beta = \beta_2\beta_1(\mathrm{id} \otimes \mathrm{m}), \quad (\mathrm{id} \otimes \mathrm{m})\beta = \beta_1\beta_2(\mathrm{m} \otimes \mathrm{id})$$

(the operators in the superposition act from left to right).

A linear map $\varphi : V \to V'$ of linear spaces V and V' equipped with braidings β and β', respectively, is called a *homomorphism of braided spaces* if

$$\beta(\varphi \otimes \varphi) = (\varphi \otimes \varphi)\beta'. \tag{6}$$

A *homomorphism of braided algebras* is a linear map that is simultaneously a homomorphism of algebras and braided spaces.

Let $X = \{x_1, x_2, \ldots, x_n\}$ be a basis of V. Denote by $K\langle X\rangle$ the free associative algebra generated by the set X over a field K. The set of all associative words X' in the alphabet X forms a linear basis of $K\langle X\rangle$. Set $\mathrm{mdeg}(x_i) = e_i$, where $e_1, \ldots, e_n$ is the standard basis for $\mathbb{Z}^n$. If $v = x_{i_1}x_{i_2}\ldots x_{i_k} \in X'$, then we put

$$d(v) = k \text{ and } \mathrm{mdeg}(v) = \sum_{j=1}^{k} \mathrm{mdeg}(x_{i_j}).$$

Consider the tensor algebra $T(V) = \bigoplus_{i=0}^{\infty} V^{\otimes i}$ of a linear space V. It is clear that $T(V) = K\langle X\rangle$. We will write $v_1v_2\ldots v_k$ instead of $v_1 \otimes v_2 \otimes \ldots \otimes v_k \in V^{\otimes k}$. Denote by $\mathrm{m}' : T(V) \oplus' T(V) \to T(V)$ the product in $T(V)$, where $\otimes'$ is the tensor product $\otimes$ with the separation function of a pair of tensors. So, $(u \otimes' v)\mathrm{m}' = u \otimes v$, where $u, v \in T(V)$.

Consider the linear map

$$\theta_t : V^{\otimes n} \to V^{\otimes t} \otimes' V^{\otimes(n-t)}, \quad 0 \le t \le n,$$

defined by

$$(x_{i_1}x_{i_2}\cdots x_{i_n})\theta_t = x_{i_1}x_{i_2}\cdots x_{i_t} \otimes' x_{i_{t+1}}\cdots x_{i_n}.$$

V. Kharchenko [3] proved that every braiding β of V has a unique extension β' on $K\langle X\rangle$ so that $K\langle X\rangle$ is a braided algebra. The braiding β' is defined in [3] by

$$(u \otimes' v)\beta' = (u \otimes v)\nu_t^{1,n}\theta_{n-t}, \quad u \in V^{\otimes t}, \quad v \in V^{\otimes(n-t)}. \tag{7}$$

We set $(1 \otimes' v)\beta' = v \otimes' 1$ and $(u \otimes' 1)\beta' = 1 \otimes' u$.

Denote by $K\{X\}$ and $K\{y\}$ free nonassociative algebras over a field K freely generated by the set X and one variable y, respectively. The set of all nonassociative words X^* in the alphabet X and the set of all nonassociative words Y^* in the alphabet y form linear basis for $K\{X\}$ and $K\{y\}$, respectively.

Every nonassociative word $vf \in X^*$ of length t has a unique representation $vf = v \cdot f$, where $v \in X'$, $f \in Y^*$, $d(v) = d(f) = t$. We can consider f as a linear map

$$f : V^{\otimes t} \to K\{X\}.$$

We can linearly extend the action f on $K\langle X\rangle$ by $V^{\otimes n}f = 0$ if $n \neq t$. We also can extend this action on $K\langle X\rangle$ to an action of the algebra $K\{y\}$ by linearity. The linear map [4]

$$K\langle X\rangle \otimes K\{y\} \to K\{X\} \text{ defined by } (a \otimes g) \mapsto a \cdot g$$

is an isomorphism of linear spaces.

Let $u \in X'$, $f \in Y^*$, and $d(u) = d(f) = t$. Then $uf \in X^*$. Using this, we have

$$(uf)(vg) = (uv)(fg), \quad u, v \in X', \quad f, g \in Y^*.$$

U. Umirbaev and V. Kharchenko [4] proved that the every braiding β of V has a unique extension β^* on $K\{X\}$ so that $K\{X\}$ is a braided algebra. The braiding β^* is defined in [4] by

$$(uf \otimes vg)\beta^* = (u \otimes' v)\beta'(g \otimes f), \tag{8}$$

where $u, v \in X'$, $f, g \in Y^*$, and β' is the braiding of $K\langle X\rangle$ defined by (7). It is clear that

$$(1 \otimes vf)\beta^* = vf \otimes 1 \quad \text{and} \quad (ug \otimes 1)\beta^* = 1 \otimes ug. \tag{9}$$

For convenience, denote the extensions β' and β^* by the same symbol β.

Let $w \in X^*$ and let $w = w'f$, where $w' \in X'$ and $f \in Y^*$. We put

$$\operatorname{mdeg}(w) = \operatorname{mdeg}(w').$$

Lemma 1. Let V be a linear space over a field K with a linear basis $X = \{x_1, x_2, ..., x_n\}$ and a diagonal braiding $\beta : x_i \otimes x_j \mapsto \beta_{ij} \cdot x_j \otimes x_i$. If $u, v \in X^*$, $\operatorname{mdeg}(u) = (k_1, k_2, \ldots, k_n)$ and $\operatorname{mdeg}(v) = (l_1, l_2, \ldots, l_n)$, then

$$(u \otimes v)\beta^* = \prod_{ij} \beta_{ij}^{k_i l_j}(v \otimes u).$$

Proof. Let $u, v \in X^*$. Denote by u' and v' the associative words obtained from the nonassociative words u and v, respectively, by removing all brackets. Then $u = u'f$ and $v = v'g$ for some $f, g \in Y^*$. It is proven in [6] that

$$(u' \otimes' v')\beta' = \prod_{i,j} \beta_{ij}^{k_i l_j}(v' \otimes' u').$$

By this and (8), we have

$$(u \otimes v)\beta = (u'f \otimes v'g)\beta^* = (u' \otimes' v')\beta(g \otimes f) = \prod_{i,j} \beta_{ij}^{k_i l_j}(v' \otimes' u')(g \otimes f)$$

$$= \prod_{i,j} \beta_{ij}^{k_i l_j}(v'g \otimes u'f) = \prod_{i,j} \beta_{ij}^{k_i l_j}(v \otimes u). \qquad \square$$

2 Properties of automorphisms of $K\{x_1, x_2\}$

Denote by $\deg(u)$ the degree function on X^* such that $\deg(x_i) = 1$ for all i. Every nonassociative word u of degree ≥ 2 is uniquely represented as $u = u_1 \cdot u_2$, where $\deg(u_1),\ \deg(u_2) < \deg(u)$.

Set $x_1 < x_2 < \cdots < x_n$. Let u, v be arbitrary elements of X^*. We say that $u < v$, if $\deg(u) < \deg(v)$. If $\deg(u) = \deg(v) \geq 2$, where $u = u_1 \cdot u_2$, $v = v_1 \cdot v_2$, then we say that $u < v$ if $u_1 < v_1$ or if $u_1 = v_1$ and $u_2 < v_2$.

It is not difficult to see that the statement of next lemma is true.

Lemma 2. Let $u, v, w \in X^*$. If $u > v$ then $wu > wv$ and $uw > vw$.

Arbitrary element $f \in K\{X\}$ can be uniquely written as

$$f = \alpha_1 f_1 + \alpha_2 f_2 + \cdots + \alpha_k f_k,$$

where $f_i \in X^*$, $0 \neq \alpha_i \in K$ for all i and $f_1 > f_2 > ... > f_k$. Let us call f_1 *the leader* of f and denote it by $\widehat{f}$.

Corollary 1. Let $0 \neq f, g \in K\{X\}$. Then

$$\widehat{f \cdot g} = \widehat{f} \cdot \widehat{g} \quad \text{and} \quad \deg(f \cdot g) = \deg(f) + \deg(g).$$

Lemma 3. Let u, v be nonassociative words of $K\{y\}$ and let $f \in X^*$. If $u > v$, then $u(f) > v(f)$.

Proof. It is clear that $u(f), v(f) \in X^*$. Let $\deg(v) = k$. Establish the lemma's statement by induction on k. If $\deg(v) = 1$ then $v = y$ and the inequality $u > v$ holds if and only if $\deg(u) > 1$. Therefore, $u(f) > v(f)$.

Let $\deg(v) \geq 2$. We can represent u and v as $u = u_1 \cdot u_2$ and $v = v_1 \cdot v_2$, respectively. $u > v$ implies that one of the following conditions is satisfied: 1) $\deg(u) > \deg(v)$; 2) $\deg(u) = \deg(v)$ and $u_1 > v_1$; 3) $\deg(u) = \deg(v)$, $u_1 = v_1$, and $u_2 > v_2$.

If $\deg(u) > \deg(v)$ then $\deg(u(f)) > \deg(v(f))$ and $u(f) > v(f)$.

If $\deg(u) = \deg(v)$ and $u_1 > v_1$ then, by induction proposition, $u_1(f) > v_1(f)$. Therefore, $u(f) > v(f)$.

If $\deg(u) = \deg(v)$, $u_1 = v_1$, and $u_2 > v_2$ then, by induction proposition, $u_1(f) = v_1(f)$ and $u_2(f) > v_2(f)$. Therefore, $u(f) > v(f)$. ☐

Denote by $K\{x_1, x_2\}$ the free nonassociative algebra in two variables x_1, x_2 over a field K. The next corollary follows immediately from Corollary 1 and Lemma 2.

Corollary 2. Let $u \in K\{y\}$ and $0 \neq f \in K\{x_1, x_2\}$. Then

$$\widehat{u(f)} = \widehat{u}(\widehat{f}) \text{ and } \deg(u(f)) = \deg(u) \cdot \deg(f).$$

Denote by $\varphi = (f_1, f_2)$ the automorphism of $K\{x_1, x_2\}$ such that $\varphi(x_1) = f_1, \varphi(x_2) = f_2$. If $f_i = x_i$, $f_j = \alpha x_j + g(x_i)$, $i \neq j$, $0 \neq \alpha \in K$, $g \in K\{x_i\}$, then the automorphism φ is called *elementary.* The automorphism generated by elementary automorphisms is called *tame.*

P. Cohn [10] proved that all automorphisms of free Lie algebras of a finite rank over a field are tame. J. Lewin [11] extended this result to free algebras in Nielsen–Schreier varieties. It is well known that the variety of all nonassociative algebras [12] over a field is Nielsen–Schreier (see also [13]). As a consequence, every automorphism of $K\{x_1, x_2\}$ is tame.

The degree of $\varphi = (f_1, f_2)$ is defined by

$$\deg(\varphi) = \deg(f_1) + \deg(f_2).$$

An automorphism φ is called *elementary reducible* if there exists an elementary automorphism λ such that $\deg(\varphi \circ \lambda) < \deg(\varphi)$.

Every nonzero element $g \in K\{x_1, x_2\}$ can be uniquely represented as

$$g = g_0 + g_1 + \cdots + g_{m-1} + g_m,$$

where g_i is homogenous element of degree i, $g_m \neq 0$. Let us call g_m *the highest homogeneous part* of g and denote it by $\overline{g}$.

Let $\mathrm{Af}K\{x_1, x_2\}$ be the affine automorphism group of $K\{x_1, x_2\}$, i.e., the group of automorphisms of the form

$$(a_1x_1 + b_1x_2 + c_1, a_2x_1 + b_2x_2 + c_2),$$

where $a_i, b_i, c_i \in K$, $a_1b_2 \neq a_2b_1$, let $\mathrm{Tr}K\{x_1, x_2\}$ be the triangular automorphism group of $K\{x_1, x_2\}$, i.e., the group of automorphisms of the form

$$(ax_1 + f(x_2), bx_2 + c),$$

where $0 \neq a, b \in K$, $c \in K$, $f(x_2) \in K\{x_2\}$, and let $C = \mathrm{Af}K\{x_1, x_2\} \cap \mathrm{Tr}K\{x_1, x_2\}$.

The notation $h_i(x_2)$ means that $h_i(x_2) \in K\{x_2\}$ is a homogeneous element of degree i with respect to the degree function deg in one variable x_2. It is clear that $h_0(x_2) \in K$.

A. Alimbaev, A. Naurazbekova, and D. Kozybaev [14] showed that the system of elements

$$A_0 = \{\mathrm{id} = (x_1, x_2), \gamma = (x_2, x_1 + ax_2) | a \in K\}$$

is a left coset representative system for $\mathrm{Af}K\{x_1, x_2\}$ modulo C, and the system of elements

$$B_0 = \{\delta = (x_1 + q(x_2), x_2) | q(x_2) = h_2(x_2) + \ldots + h_n(x_2)\}$$

is a left coset representative system for $\mathrm{Tr}K\{x_1, x_2\}$ modulo C. They also proved the following two lemmas.

Lemma 4. [14] Any automorphism φ of $K\{x_1, x_2\}$ can be representation as

$$\varphi = \gamma_1 \circ \delta_1 \circ \gamma_2 \circ \delta_2 \circ \ldots \circ \gamma_k \circ \delta_k \circ \lambda,$$

where $\gamma_i \in A_0$, $\gamma_2, \ldots, \gamma_k \neq \mathrm{id}$, $\delta_i \in B_0$, $\delta_1, \ldots, \delta_k \neq \mathrm{id}$, and $\lambda \in \mathrm{Af}K\{x_1, x_2\}$. Moreover, this representation of φ is unique.

Lemma 5. [14] Let

$$\varphi_k = \delta_1 \circ \gamma_2 \circ \delta_2 \circ \ldots \circ \gamma_k \circ \delta_k,$$

where $\mathrm{id} \neq \gamma_i \in A_0$, $\mathrm{id} \neq \delta_i \in B_0$ for all i. If $\delta_i = (x_1 + q_i(x_2), x_2)$ and $\deg(q_i(x_2)) = n_i$ for all $1 \leq i \leq k$ then

$$\deg(\varphi_k(x_1)) = n_1 n_2 \ldots n_{k-1} n_k,$$

$$deg(\varphi_k(x_2)) = \begin{cases} n_1 n_2 \ldots n_{k-1}, & \text{if } k > 1, \\ 1, & \text{if } k = 1. \end{cases}$$

Proposition 1. Nonaffine automorphisms of a free nonassociative algebra $K\{x_1, x_2\}$ of rank 2 over an arbitrary field K are elementary reducible.

Proof. Let φ be a nonaffine automorphism of $K\{x_1, x_2\}$. It is not difficult to see that if $\gamma^{-1} \circ \varphi$, where $\gamma \in A_0$, is elementary reducible then φ is also elementary reducible. By Lemma 4, φ can be uniquely represented as the product

$$\varphi = \gamma_1 \circ \delta_1 \circ \gamma_2 \circ \delta_2 \circ \cdots \circ \gamma_k \circ \delta_k \circ \lambda,$$

where $\gamma_i \in A_0$, $\gamma_2, \ldots, \gamma_k \neq \mathrm{id}$, $\tau_i \in B_0$, $\delta_1, \ldots, \delta_k \neq \mathrm{id}$, and $\lambda \in \mathrm{Af}K\{x_1, x_2\}$.

Let $\lambda = \mathrm{id}$. Then as in Lemma 5

$$\gamma_1^{-1} \circ \varphi = \varphi_k = \delta_1 \circ \gamma_2 \circ \delta_2 \circ \cdots \circ \gamma_k \circ \delta_k = \phi_{k-1} \circ \gamma_k \circ \delta_k.$$

Put $\varphi_{k-1} = (u_1, u_2)$. By Lemma 5, $\deg(u_1) > \deg(u_2)$. Since $\gamma_k = (x_2, x_1 + a_k x_2)$, $a_k \in K$, and $\delta_k = (x_1 + q_k(x_2), x_2)$ it follows that

$$\varphi_{k-1} \circ \gamma_k = (u_2, u_1 + a_k u_2)$$

and

$$\varphi_k = \varphi_{k-1} \circ \gamma_k \circ \delta_k = (u_2 + q_k(u_1 + a_k u_2), u_1 + a_k u_2) = (w_1, w_2).$$

By Lemma 5,

$$\deg w_1 > \deg w_2 = \deg(u_1 + a_k u_2) > \deg(u_2). \tag{10}$$

Consequently,

$$\deg(\varphi_{k-1} \circ \gamma_k) < \deg \varphi_k.$$

Since δ_k is the elementary automorphism, it follows that the automorphism $\gamma_1^{-1} \circ \varphi = \varphi_k$ is elementary reducible. This means that φ is also elementary reducible.

Let now

$$\lambda = (a_1x_1 + b_1x_2 + c_1, a_2x_1 + b_2x_2 + c_2) \neq \mathrm{id},$$

where $a_1b_2 - a_2b_1 \in K^*$. We have

$$\gamma_1^{-1} \circ \varphi = \varphi_k \circ \lambda = (a_1w_1 + b_1w_2 + c_1, a_2w_1 + b_2w_2 + c_2) = (f_1, f_2).$$

Let a_1, a_2 be non-zero. Consider the automorphism

$$\gamma_1^{-1} \circ \varphi \circ (x_1 - \frac{a_1}{a_2}x_2, x_2) = (f_1, f_2) \circ (x_1 - \frac{a_1}{a_2}x_2, x_2) = (f_1 - \frac{a_1}{a_2}f_2, f_2).$$

By Lemma 5, $\deg(f_1 - \frac{a_1}{a_2}f_2) < \deg f_1$. Consequently, the automorphism φ is elementary reducible.

Let now one of two coefficients a_1, a_2 be zero. Without loss of generality, we may assume that $a_1 = 0$ and $a_2 \neq 0$. Then $b_1 \in K^*$.

Since $K\{w_2\} = K\{b_1w_2 + c_1\}$, it follows that there exists $v_k(y) \in K\{y\}$ such that

$$v_k(b_1w_2 + c_1) = q_k(w_2).$$

Consider the automorphism

$$\psi = \gamma_1^{-1} \circ \varphi \circ (x_1, a_2^{-1}x_2 - v_k(x_1)) = (f_1, a_2^{-1}f_2 - v_k(f_1))$$
$$= (b_1w_2 + c_1, a_2^{-1}(a_2w_1 + b_2w_2 + c_2) - v_k(b_1w_2 + c_1)),$$

where

$$a_2^{-1}(a_2w_1 + b_2w_2 + c_2) - v_k(b_1w_2 + c_1) = w_1 + a_2^{-1}(b_2w_2 + c_2) - q_k(w_2)$$
$$= u_2 + q_k(w_2) + a_2^{-1}(b_2w_2 + c_2) - q_k(w_2) = u_2 + a_2^{-1}(b_2w_2 + c_2).$$

By(10), $\deg \psi < \deg \gamma_1^{-1} \circ \phi$. Since $(x_1, a_2^{-1}x_2 - v_k(x_1))$ is the elementary automorphism, it follows that the automorphism φ is elementary reducible. □

Proposition 2. If $\varphi = (f_1, f_2)$ is an automorphism of a free nonassociative algebra $K\{x_1, x_2\}$ in two variables x_1, x_2 over a field K and $\deg(\varphi) \geq 3$, then $\overline{f_1}, \overline{f_2}$ are homogeneous elements of a free nonassociative algebra $K\{ax_1 + bx_2\}$ in one variable $ax_1 + bx_2$, $a, b \in K$, $(a, b) \neq (0, 0)$, and $\deg(f_1) \mid \deg(f_2)$ or $\deg(f_2) \mid \deg(f_1)$.

Proof. Let $\varphi = (f_1, f_2)$ be an automorphism of $K\{x_1, x_2\}$ with $\deg(\varphi) \geq 3$. Without loss of generality, we can assume that $\deg(f_1) \leq \deg(f_2)$. Establish the proposition's statement by induction on $\deg(\varphi) = \deg(f_1) + \deg(f_2)$.

By Proposition 1, φ is elementary reducible. Therefore, there exists an elementary automorphism $\epsilon = (x_1, cx_2 - g(x_1))$, where $c \neq 0$, $g(x_1) \in K\{x_1\}$, such that $\deg(\varphi \circ \epsilon) < \deg(\varphi)$. Since $\varphi \circ \epsilon = (f_1, cf_2 - g(f_1))$, it follows that $\deg(f_2) = \deg(g(f_1))$. By Corollary 2,

$$\deg(g(f_1)) = \deg(g) \cdot \deg(f_1).$$

It means that $\deg(f_1) \mid \deg(f_2)$. If $\deg(\varphi \circ \epsilon) \geq 3$ then, by the induction proposition, $\overline{f_1}$ is homogeneous element of $K\{ax_1 + bx_2\}$. Notice that this is true even if $\deg(\varphi \circ \epsilon) = 2$. Hence, by Corollary 2, $c\overline{f_2} = \overline{g(f_1)} = \overline{g}(\overline{f_1})$. Consequently, $\overline{f_2}$ is the homogeneous element of $K\{ax_1 + bx_2\}$. □

3 Diagonal braidings on $K\{x_1, x_2\}$

Let V be a linear space over a field K with a linear basis x_1, x_2 and with a diagonal braiding

$$\beta = (\beta_{11}, \beta_{12}, \beta_{21}, \beta_{22})$$

defined by (2). Denoted by

$$\overline{\beta} = (\beta_{22}, \beta_{21}, \beta_{12}, \beta_{11})$$

the diagonal braiding obtained from β by exchanging the variables x_1 and x_2.

Proposition 3. Let $(K\{x_1, x_2\}, \beta)$ and $(K\{x_1, x_2\}, \gamma)$ be free braided nonassociative algebras in two variables x_1, x_2 over a field K equipped with diagonal braidings β and γ, respectively. Then $(K\{x_1, x_2\}, \beta)$ is isomorphic to $(K\{x_1, x_2\}, \gamma)$ if and only if $\gamma = \beta$ or $\gamma = \overline{\beta}$.

Proof. Let $\varphi : (K\{x_1,\, x_2\}, \beta) \to (K\{x_1,\, x_2\}, \gamma)$ is an isomorphism and $\varphi(x_1) = g_1$, $\varphi(x_2) = g_2$. Denote by $Lin(g)$ the linear part of g. Let $Lin(g_1) = a_1x_1 + b_1x_2$ and $Lin(g_2) = a_2x_1 + b_2x_2$. Since φ is an isomorphism, it follows that

$$a_1b_2 - a_2b_1 \neq 0 \tag{11}$$

and

$$((x_i \otimes x_j)\beta)(\varphi \otimes \varphi) = ((x_i \otimes x_j)(\varphi \otimes \varphi))\gamma, \text{ where } 1 \leq i, j \leq 2.$$

Denote by $Qu(g)$ the quadratic part of g. Let $\beta = (\beta_{11}, \beta_{12}, \beta_{21}, \beta_{22})$ and $\gamma = (\gamma_{11}, \gamma_{12}, \gamma_{21}, \gamma_{22})$. Then

$$Qu(\beta_{ij}(\varphi(x_j) \otimes \varphi(x_i))) = Qu\left((\varphi(x_i) \otimes \varphi(x_j))\gamma\right),\ 1 \leq i, j \leq 2.$$

It follows that

$$\begin{aligned} &\beta_{ij}(a_ja_ix_1 \otimes x_1 + b_ja_ix_2 \otimes x_1 + a_jb_ix_1 \otimes x_2 + b_jb_ix_2 \otimes x_2) \\ &= \gamma_{11}a_ia_jx_1 \otimes x_1 + \gamma_{21}b_ia_jx_1 \otimes x_2 + \gamma_{12}a_ib_jx_2 \otimes x_1 + \gamma_{22}b_ib_jx_2 \otimes x_2 \end{aligned} \tag{12}$$

since, by (9), for any $w \in X^*$ and any $c \in K$

$$((w \otimes c)\beta)(\varphi \otimes \varphi) = (c \otimes w)(\varphi \otimes \varphi) = (c \otimes \varphi(w)) = (\varphi(w) \otimes c)\gamma = ((w \otimes c)(\varphi \otimes \varphi))\gamma.$$

Comparing the coefficients of the terms $x_i \otimes x_j$ in (12), we obtain

$$(\gamma_{11} - \beta_{ij})a_ia_j = (\gamma_{21} - \beta_{ij})a_jb_i = (\gamma_{12} - \beta_{ij})a_ib_j = (\gamma_{22} - \beta_{ij})b_ib_j = 0,\ 1 \leq i, j \leq 2. \tag{13}$$

By (11), $a_1b_2 \neq 0$ or $a_2b_1 \neq 0$. If $a_1b_2 \neq 0$ then (13) implies that

$$(\gamma_{11}, \gamma_{12}, \gamma_{21}, \gamma_{22}) = (\beta_{11}, \beta_{12}, \beta_{21}, \beta_{22}).$$

If $a_2b_1 \neq 0$ then (13) implies that

$$(\gamma_{11}, \gamma_{12}, \gamma_{21}, \gamma_{22}) = (\beta_{22}, \beta_{21}, \beta_{12}, \beta_{11}).$$

Hence, if $(K\{x_1, x_2\}, \beta)$ is isomorphic to $(K\{x_1, x_2\}, \gamma)$ then $\gamma = \beta$ or $\gamma = \overline{\beta}$.

Let φ be an automorphism of $K\{x_1,\, x_2\}$ such that $\varphi(x_1) = x_2$ and $\varphi(x_2) = x_1$. Let $u, v \in X^*$, $\mathrm{mdeg}(u) = (k_1, k_2)$ and $\mathrm{mdeg}(v) = (l_1, l_2)$. Then $\mathrm{mdeg}(\varphi(u)) = (k_2, k_1)$ and $\mathrm{mdeg}(\varphi(v)) = (l_2, l_1)$. By Lemma 1, we have

$$((u \otimes v)\beta)(\varphi \otimes \varphi) = \beta_{11}^{k_1l_1}\beta_{12}^{k_1l_2}\beta_{21}^{k_2l_1}\beta_{22}^{k_2l_2}(\varphi(v) \otimes \varphi(u))$$

and

$$((u \otimes v)(\varphi \otimes \varphi))\overline{\beta} = \beta_{11}^{k_1l_1}\beta_{12}^{k_1l_2}\beta_{21}^{k_2l_1}\beta_{22}^{k_2l_2}(\varphi(v) \otimes \varphi(u)).$$

Hence,

$$((u \otimes v)\beta)(\varphi \otimes \varphi) = ((u \otimes v)(\varphi \otimes \varphi))\overline{\beta},$$

and $(K\{x_1, x_2\}, \beta)$ is isomorphic to $(K\{x_1, x_2\}, \overline{\beta})$. ☐

If $n=2$ then, by (3), the braiding β is involutive if and only if

$$\beta_{11}=\pm 1,\ \beta_{22}=\pm 1,\ \beta_{12}\beta_{21}=1.$$

Note that if β is involutive then β' is also involutive.

4 Automorphisms of $(K\{x_1,\, x_2\},\beta)$

Introduce the following notations:

(1) $G_1=\{\varphi\in \mathrm{Aut}K\{x_1,x_2\}\mid \varphi=(a_1x_1,b_2x_2)\,\mathrm{or}\,\varphi=(b_1x_2,\ a_2x_1),\ a_1,b_2,a_2,b_1\in K^*\}$;

(2) $G_2=\{\varphi\in \mathrm{Aut}K\{x_1,x_2\}\mid \varphi=(a_1x_1+g(x_2^2),b_2x_2),\ a_1,\ b_2\in K^*,g(x)\in K\,\{x\}\}$;

(3) $G_{tor}=\{\varphi\in \mathrm{Aut}K\{x_1,x_2\}\mid \varphi=(a_1x_1,b_2x_2),\ a_1,\ b_2\in K^*\}$ is the group of all *toric* automorphisms of $K\{x_1,x_2\}$;

(4) $\mathbb{Z}_2$ is the subgroup of $\mathrm{Aut}K\{x_1,x_2\}$ generated by (x_2,x_1).

Note that, by (6), if $\varphi\in \mathrm{Aut}K\{x_1,x_2\}$ then $\varphi\in \mathrm{Aut}(K\{x_1,x_2\},\beta)$ if and only if

$$\beta(\varphi\otimes\varphi)=(\varphi\otimes\varphi)\beta. \tag{14}$$

The main result of this section is the following theorem.

Theorem 1. Let $(K\{x_1,x_2\},\beta)$ be a free braided nonassociative algebra in two generators x_1, x_2 over a field K of arbitrary characteristic $\neq 2$ equipped with an involutive diagonal braiding $\beta=(\beta_{11},\beta_{12},\beta_{21},\beta_{22})$. Then

(1) $\mathrm{Aut}(K\{x_1,x_2\},\beta)=\mathrm{Aut}K\{x_1,x_2\}$ if $\beta_{ij}=1$ for all i,j;

(2) $\mathrm{Aut}(K\{x_1,x_2\},\beta)\cong(K^*\times K^*)\rtimes\mathbb{Z}_2$ if $\beta_{11}=\beta_{22}$, $\beta_{12}=\beta_{21}$, and $\beta_{11}\beta_{12}=-1$;

(3) $\mathrm{Aut}(K\{x_1,x_2\},\beta)=G_{\mathrm{odd}}$ if $\beta_{ij}=-1$ for all i,j;

(4) $\mathrm{Aut}(K\{x_1,x_2\},\beta)\cong G_2$ if $\beta_{12}=1$ and $\beta_{11}\beta_{22}=-1$;

(5) $\mathrm{Aut}(K\{x_1,x_2\},\beta)\cong K^*\times K^*$ if $\beta_{12}\neq\pm1$ or $\beta_{12}=-1,\beta_{11}\beta_{22}=-1$.

Following Lemmas 6, 7, 8, 9, 10, and 11 immediately imply the statement of this theorem.

Lemma 6. If $\beta_{ij}=1$ for all i,j then $\mathrm{Aut}(K\{x_1,x_2\},\beta)=\mathrm{Aut}K\{x_1,x_2\}$.

Proof. Let $\varphi\in\mathrm{Aut}K\{x_1,x_2\}$ and let $u,\ v\in X^*$. By Lemma 1, we have

$$((u\otimes v)\beta)(\varphi\otimes\varphi)=(v\otimes u)(\varphi\otimes\varphi)=\varphi(v)\otimes\varphi(u)=(\varphi(u)\otimes\varphi(v))\beta=(u\otimes v)(\varphi\otimes\varphi)\beta.$$

By (14), $\varphi\in\mathrm{Aut}(K\{x_1,x_2\},\beta)$. Consequently, $\mathrm{Aut}(K\{x_1,x_2\},\beta)=\mathrm{Aut}K\{x_1,x_2\}$. □

Lemma 7. If $\beta_{11}=\beta_{22}$, $\beta_{12}=\beta_{21}$, and $\beta_{11}\beta_{12}=-1$ then $\mathrm{Aut}(K\{x_1,x_2\},\beta)=G_1\cong(K^*\times K^*)\rtimes\mathbb{Z}_2$.

Proof. By Proposition 3, $\beta=(-1,1,1,-1)$ or $\beta=(1,-1,-1,1)$. Let $\varphi=(f_1,f_2)$ is an automorphism of $(K\{x_1,x_2\},\beta)$ and $\deg(\varphi)\geq 3$. Since $\mathrm{Aut}(K\{x_1,x_2\},\beta)\subseteq\mathrm{Aut}K\{x_1,x_2\}$, it follows that, by Proposition 2,

$$\overline{f_1}=h_1(ax_1+bx_2),\overline{f_2}=h_2(ax_1+bx_2), \tag{15}$$

where h_1,h_2 are homogeneous elements of $K\{y\}$, $\deg(h_1)=m_1$, $\deg(h_2)=m_2$, $m_1\mid m_2$ or $m_2\mid m_1$, $a,b\in K$, and $(a,b)\neq(0,0)$. By (14), we get

$$\overline{((x_i\otimes x_j)\beta)(\varphi\otimes\varphi)}=\overline{((x_i\otimes x_j)(\varphi\otimes\varphi))\beta},\quad 1\leq i,j\leq 2.$$

Hence,

$$\overline{\beta_{ij}\varphi(x_j)\otimes\varphi(x_i)}=\overline{(\varphi(x_i)\otimes\varphi(x_j))\beta},\quad 1\leq i,j\leq 2.$$

Using (15), we get

$$\beta_{ij}h_j(ax_1+bx_2)\otimes h_i(ax_1+bx_2)=(h_i(ax_1+bx_2)\otimes h_j(ax_1+bx_2))\,\beta,\quad 1\leq i,j\leq 2. \tag{16}$$

We can write $h_k(ax_1+bx_2)$ as

$$h_k(ax_1+bx_2)=a^{m_k}h_k(x_1)+b^{m_k}h_k(x_2)+w_k,\ 1\le k\le 2,$$

where each term of w_k contains both x_1 and x_2.

Using Lemma 1 and comparing coefficients of the terms $h_j(x_1)\otimes h_i(x_1)$, $h_j(x_2)\otimes h_i(x_1)$, $h_j(x_1)\otimes h_i(x_2)$, $h_j(x_2)\otimes h_i(x_2)$ in (16), we obtain

$$\begin{aligned}(\beta_{ij}-\beta_{11}^{m_im_j})a^{m_i+m_j}&=(\beta_{ij}-\beta_{12}^{m_im_j})a^{m_i}b^{m_j}=(\beta_{ij}-\beta_{21}^{m_im_j})a^{m_j}b^{m_i}\\&=(\beta_{ij}-\beta_{22}^{m_im_j})b^{m_i+m_j}=0,\quad 1\le i,j\le 2.\end{aligned}\tag{17}$$

If $\beta=(-1,1,1,-1)$ then, by (17),

$$(\beta_{ij}-(-1)^{m_im_j})a^{m_i+m_j}=(\beta_{ij}-(-1)^{m_im_j})b^{m_i+m_j}=0,\quad 1\le i,j\le 2.$$

This implies that

$$(-1-(-1)^{m_1^2})a^{2m_1}=(-1-(-1)^{m_1^2})b^{2m_1}=(1-(-1)^{m_1m_2})a^{m_1+m_2}$$

$$=(1-(-1)^{m_1m_2})b^{m_1+m_2}=(-1-(-1)^{m_2^2})a^{2m_2}=(-1-(-1)^{m_2^2})b^{2m_2}=0.$$

It easily follows from this that $a=b=0$ over a field of characteristic $\neq 2$.

If $\beta=(1,-1,-1,1)$ then, by (17),

$$(\beta_{ij}-1)a^{m_i+m_j}=(\beta_{ij}-1)b^{m_i+m_j}=0,\quad 1\le i,j\le 2.$$

This implies that

$$-2a^{m_1+m_2}=-2b^{m_1+m_2}=0.$$

Hence, $a=b=0$ over a field of characteristic $\neq 2$.

Consequently, if $\beta=(-1,1,1,-1)$ or $\beta=(1,-1,-1,1)$ then $(K\{x_1,x_2\},\beta)$ has only automorphisms of degree 2. Therefore,

$$\varphi=(a_1x_1+b_1x_2+c_1,\ a_2x_1+b_2x_2+c_2),\ a_i,\ b_i,\ c_i\in K.$$

Using (14), we get

$$((x_i\otimes x_j)\beta)\varphi\otimes\varphi=((x_i\otimes x_j)\varphi\otimes\varphi)\beta,\quad 1\le i,j\le 2.$$

Hence,

$$\beta_{ij}(a_jx_1+b_jx_2+c_j)\otimes(a_ix_1+b_ix_2+c_i)=((a_ix_1+b_ix_2+c_i)\otimes(a_jx_1+b_jx_2+c_j))\beta,\quad 1\le i,j\le 2.$$

By comparing the coefficients of the terms $x_i\otimes x_j$, x_i, $1\le i,j\le 2$, and the term 1 on both sides of the equality, we obtain the following relations:

$$(\beta_{ij}-\beta_{11})a_ia_j=(\beta_{ij}-\beta_{12})a_ib_j=(\beta_{ij}-\beta_{21})a_jb_i=(\beta_{ij}-\beta_{22})b_ib_j$$

$$=(\beta_{ij}-1)(a_ic_j+a_jc_i)=(\beta_{ij}-1)(b_ic_j+b_jc_i)=(\beta_{ij}-1)c_ic_j=0,\quad 1\le i,j\le 2.$$

Varying the values of $1\le i,j\le 2$, we get

$$\begin{aligned}(\beta_{11}-\beta_{12})a_1b_1&=(\beta_{11}-\beta_{21})a_1b_1=(\beta_{11}-\beta_{22})b_1^2=(\beta_{11}-1)a_1c_1\\&=(\beta_{11}-1)b_1c_1=(\beta_{11}-1)c_1^2=0,\end{aligned}\tag{18}$$

$$\begin{aligned}(\beta_{12}-\beta_{11})a_1a_2 &= (\beta_{12}-\beta_{21})a_2b_1 = (\beta_{12}-\beta_{22})b_1b_2\\ = (\beta_{12}-1)(a_1c_2+a_2c_1) &= (\beta_{12}-1)(b_1c_2+b_2c_1) = (\beta_{12}-1)c_1c_2 = 0,\end{aligned} \tag{19}$$

$$\begin{aligned}(\beta_{21}-\beta_{11})a_1a_2 &= (\beta_{21}-\beta_{12})a_2b_1 = (\beta_{21}-\beta_{22})b_1b_2\\ = (\beta_{21}-1)(a_2c_1+a_1c_2) &= (\beta_{21}-1)(b_2c_1+b_1c_2) = (\beta_{21}-1)c_2c_1 = 0,\end{aligned} \tag{20}$$

$$\begin{aligned}(\beta_{22}-\beta_{11})a_2^2 = (\beta_{22}-\beta_{12})a_2b_2 = (\beta_{22}-\beta_{21})a_2b_2 &= (\beta_{22}-1)a_2c_2\\ = (\beta_{22}-1)b_2c_2 &= (\beta_{22}-1)c_2^2 = 0.\end{aligned} \tag{21}$$

If $\beta=(-1,1,1,-1)$ or $\beta=(1,-1,-1,1)$, then it follows from (18), (19), (20), and (21) that

$$a_1b_1 = a_1a_2 = b_1b_2 = a_2b_2 = c_1^2 = c_2^2 = 0$$

or

$$a_1b_1 = a_1a_2 = b_1b_2 = a_2b_2 = a_1c_2+a_2c_1 = b_1c_2+b_2c_1 = c_1c_2 = 0,$$

respectively. Using (11), this implies

$$a_1 \neq 0,\ b_2 \neq 0,\ b_1 = a_2 = c_1 = c_2 = 0$$

or

$$b_1 \neq 0,\ a_2 \neq 0,\ a_1 = b_2 = c_1 = c_2 = 0.$$

So we have

$$\varphi = (a_1x_1,\ b_2x_2) \ \text{ or } \ \varphi = (b_1x_2,\ a_2x_1).$$

Using Lemma 1, it is easy to see that $\varphi \in \mathrm{Aut}(K\{x_1,x_2\},\beta)$. Consequently, $\mathrm{Aut}(K\{x_1,x_2\},\beta) = G_1$. □

Lemma 8. If $\beta_{ij} = -1$ for all i, j then $\mathrm{Aut}(K\{x_1,x_2\},\beta) = G_{\mathrm{odd}}$.

Proof. Let $u, v \in X^*$ with $\mathrm{mdeg}(u) = (m_1, m_2)$ and $\mathrm{mdeg}(v) = (t_1, t_2)$. If $\beta_{ij} = -1$ then, by Lemma 1,

$$(u \otimes v)\beta = (-1)^{(m_1+m_2)(t_1+t_2)}(v \otimes u).$$

It is clear that $(-1)^{(m_1+m_2)(t_1+t_2)} = -1$ is equivalent to both m_1+m_2 and t_1+t_2 being odd, and hence we conclude

$$(f \otimes g)\beta = (-1)^{ij}(g \otimes f), \tag{22}$$

where $f \in C_i$ and $g \in C_j$ with respect to the grading (4).

Let $\varphi \in G_{\mathrm{odd}}$. Therefore $\varphi(C_i) \subseteq C_i$. Using (22), we obtain

$$(f \otimes g)\beta(\varphi \otimes \varphi) = (-1)^{ij}(g \otimes f)(\varphi \otimes \varphi)$$

$$= (-1)^{ij}(\varphi(g) \otimes \varphi(f)) = (\varphi(f) \otimes \varphi(g))\beta = (f \otimes g)(\varphi \otimes \varphi)\beta.$$

By (14), $\varphi \in \mathrm{Aut}(K\{x_1,x_2\},\beta)$ and $G_{\mathrm{odd}} \subseteq \mathrm{Aut}(K\{x_1,x_2\},\beta)$.

Let $\varphi = (f_1,\ f_2) \in \mathrm{Aut}(K\{x_1,x_2\},\beta)$, $\deg(f_1) = m_1$ and $\deg(f_2) = m_2$. We prove that $\varphi \in G_{\mathrm{odd}}$ by induction on $\deg(\varphi) = m_1 + m_2$. Let $\deg(\varphi) = 2$, i.e.,

$$\varphi = (a_1x_1 + b_1x_2 + c_1, a_2x_1 + b_2x_2 + c_2), \quad a_i, b_i, c_i \in K.$$

It follows from (18), (19), (20), and (21) that $c_1 = c_2 = 0$. So we have

$$\varphi = (a_1x_1 + b_1x_2, a_2x_1 + b_2x_2) \in G_{\text{odd}}.$$

Assume that $\deg(\varphi) = m_1 + m_2 \geq 3$. Then (17) implies that

$$\begin{aligned}(-1-(-1)^{m_im_j})a^{m_i+m_j} = (-1-(-1)^{m_im_j})a^{m_i}b^{m_j} = (-1-(-1)^{m_im_j})a^{m_j}b^{m_i} \\ = (-1-(-1)^{m_im_j})b^{m_i+m_j} = 0, \quad i,\, j \in \{1,\, 2\}.\end{aligned} \tag{23}$$

Assume that m_1m_2 is even. It follows from (23) that $a = b = 0$ over a field of characteristic $\neq 2$. Consequently, m_1m_2 is odd. This means that m_1 and m_2 are odd. Without loss of generality, we can assume that $m_1 \leq m_2$. By Proposition 2, $m_1|m_2$. Then there exists the odd automorphism $\lambda = (x_1, x_2 + dx_1^{m_2/m_1})$, where $d \in K^*$, such that $\deg(\varphi \circ \lambda) < \deg(\varphi)$. Since $\lambda \in G_{\text{odd}} \subseteq \operatorname{Aut}(K\{x_1, x_2\}, \beta)$ it follows that $\varphi \circ \lambda \in \operatorname{Aut}(K\{x_1, x_2\}, \beta)$. By the induction proposition, $\varphi \circ \lambda \in G_{\text{odd}}$. Consequently, $\varphi \in G_{\text{odd}}$. □

Lemma 9. If $\beta_{12} = 1$ and $\beta_{11}\beta_{22} = -1$ then $\operatorname{Aut}(K\{x_1, x_2\}, \beta) \cong G_2$.

Proof. By Proposition 3, $\beta = (1, 1, 1, -1)$. Consider the grading

$$K\{x_1, x_2\} = D_0 \oplus D_1$$

of $K\{x_1, x_2\}$, where D_0 and D_1 are linear spans of all monomials of even degree and all monomials of odd degree in variable x_2, respectively.

Let $u, v \in X^*$, $\operatorname{mdeg}(u) = (s, s')$ and $\operatorname{mdeg}(v) = (t, t')$. By Lemma 1,

$$(u \otimes v)\beta = (-1)^{s't'}(v \otimes u).$$

$(-1)^{s't'} = -1$ is equivalent to both s' and t' being odd. Therefore, for any homogeneous elements $f \in D_i$ and $g \in D_j$,

$$(f \otimes g)\beta = (-1)^{ij}(g \otimes f). \tag{24}$$

Let $\varphi \in G_2$. Then $\varphi(D_i) \subseteq D_i$. Using (24), we obtain

$$(f \otimes g)\beta(\varphi \otimes \varphi) = (-1)^{ij}(g \otimes f)(\varphi \otimes \varphi)$$

$$= (-1)^{ij}(\varphi(g) \otimes \varphi(f)) = (\varphi(f) \otimes \varphi(g))\beta = (f \otimes g)(\varphi \otimes \varphi)\beta.$$

Consequently, $\varphi \in \operatorname{Aut}(K\{x_1, x_2\}, \beta)$ and $G_2 \subseteq \operatorname{Aut}(K\{x_1, x_2\}, \beta)$.

Let $\varphi = (f_1, f_2) \in \operatorname{Aut}(K\{x_1, x_2\}, \beta)$ with $\deg(f_1) = m_1$ and $\deg(f_2) = m_2$. We prove that $\varphi \in G_2$ by induction on $\deg(\varphi) = m_1 + m_2$. Let

$$\varphi = (a_1x_1 + b_1x_2 + c_1, a_2x_1 + b_2x_2 + c_2), \quad a_i, b_i, c_i \in K.$$

It follows from (18), (19), (20), and (21) that

$$b_1^2 = b_1b_2 = a_2^2 = a_2b_2 = a_2c_2 = b_2c_2 = c_2^2 = 0.$$

Using this and (11), we get

$$a_1 \neq 0,\ b_2 \neq 0,\ b_1 = a_2 = c_2 = 0.$$

So

$$\varphi = (a_1x_1 + c_1,\ b_2x_2) \in G_2.$$

Assume that $\deg(\varphi) = m_1 + m_2 \geq 3$. Then (17) implies that

$$(-1-1^{m_2^2})a^{2m_2} = (1-(-1)^{m_1^2})b^{2m_1} = (1-(-1)^{m_1m_2})b^{m_1+m_2} = (-1-(-1)^{m_2^2})b^{2m_2} = 0. \qquad (25)$$

It follows from this $a = 0$ over a field of characteristic $\neq 2$. By (25), $b \neq 0$ in the case m_1 is even and m_2 is odd. Therefore, by Proposition 2, $m_2|m_1$ and m_1/m_2 is even. Then there exists the automorphism $\lambda = (x_1 + dx_2^{m_1/m_2}, x_2) \in G_2$ such that $\deg(\varphi \circ \lambda) < \deg(\varphi)$. Since $\lambda \in G_2 \subseteq (K\{x_1, x_2\}, \beta)$ it follows that $\varphi \circ \lambda \in (K\{x_1, x_2\}, \beta)$. By the induction proposition, $\varphi \circ \lambda \in G_2$. Consequently, $\varphi \in G_2$. □

Lemma 10. If $\beta_{12} = -1,\ \beta_{11}\beta_{22} = -1$, then $\mathrm{Aut}(K\{x_1, x_2\}, \beta) \cong G_{\mathrm{tor}} \cong K^* \times K^*$.

Proof. By Proposition 3, $\beta = (1, -1, -1, -1)$. Let $\varphi = (f_1,\ f_2) \in \mathrm{Aut}(K\{x_1, x_2\}, \beta)$, $\deg(f_1) = m_1$ and $\deg(f_2) = m_2$. If $\deg(\varphi) = m_1 + m_2 \geq 3$, then (17) implies that

$$(-1-1^{m_1m_2})a^{m_1+m_2} = (1-(-1)^{m_1^2})b^{2m_1}$$

$$= (-1-(-1)^{m_1m_2})b^{m_1+m_2} = (-1-(-1)^{m_2^2})b^{2m_2} = 0.$$

It follows from this that $a = b = 0$ over a field of characteristic $\neq 2$. Thus, the algebra $(K\{x_1, x_2\}, \beta)$ has only automorphisms of degree 2.

Let

$$\varphi = (a_1x_1 + b_1x_2 + c_1,\ a_2x_1 + b_2x_2 + c_2),\ a_i,\ b_i,\ c_i \in K.$$

It follows from (18), (19), (20), and (21) that

$$a_1b_1 = b_1^2 = a_1a_2 = a_1c_2 + a_2c_1 = b_1c_2 + b_2c_1 = c_1c_2 = a_2^2 = a_2c_2 = b_2c_2 = c_2^2 = 0.$$

By this and (11), we get

$$a_1 \neq 0, b_2 \neq 0,\ b_1 = a_2 = c_1 = c_2 = 0.$$

So

$$\varphi = (a_1x_1,\ b_2x_2) \in G_{\mathrm{tor}}.$$

Using Lemma 1, it is not difficult to show that $G_{\mathrm{tor}} \in \mathrm{Aut}(K\{x_1, x_2\}, \beta)$. Consequently,

$$\mathrm{Aut}(K\{x_1, x_2\}, \beta) = G_{\mathrm{tor}}$$

□

Lemma 11. If $\beta_{12} \neq \pm 1$ then $\mathrm{Aut}(K\{x_1, x_2\}, \beta) = G_{\mathrm{tor}} \cong K^* \times K^*$.

Proof. By (3), $\beta_{11} = \pm 1$, $\beta_{22} = \pm 1$, $\beta_{21} \neq \beta_{12}$, $\beta_{12} \neq \pm 1$. Let $\varphi = (f_1,\ f_2) \in \mathrm{Aut}(K\{x_1, x_2\}, \beta)$, $\deg(f_1) = m_1$ and $\deg(f_2) = m_2$. If $\deg(\varphi) = m_1 + m_2 \geq 3$, then it follows from (17) that $a = b = 0$ over a field of characteristic $\neq 2$. Thus, the algebra $(K\{x_1, x_2\}, \beta)$ has only automorphisms of degree 2.

Let

$$\varphi = (a_1x_1 + b_1x_2 + c_1,\ a_2x_1 + b_2x_2 + c_2),\ a_i,\ b_i,\ c_i \in K.$$

It follows from (18), (19), (20), and (21) that

$$a_1b_1 = a_1a_2 = a_2b_1 = b_1b_2 = a_2b_2 = a_2c_1 + a_1c_2 = b_2c_1 + b_1c_2 = c_1c_2 = 0.$$

Using this and (11), we get

$$a_1 \neq 0,\ b_2 \neq 0,\ b_1 = a_2 = c_1 = c_2 = 0.$$

So

$$\varphi = (a_1x_1,\ b_2x_2) \in G_{\mathrm{tor}}.$$

Using Lemma 1, it is easy to see that $G_{\mathrm{tor}} \in \mathrm{Aut}(K\{x_1, x_2\}, \beta)$. Consequently, $\mathrm{Aut}(K\{x_1, x_2\}, \beta) = G_{\mathrm{tor}}$. □

Conclusion

Using properties of automorphisms of a two generated free nonassociative algebra we describe the automorphism groups of two generated free braided nonassociative algebras with involutive diagonal braidings over a field of characteristic not equal to two. The obtained results can be used to study automorphisms of other free nonassociative braided algebras.

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Author Contributions

All authors contributed equally to this work.

Conflict of Interest

The authors declare no conflict of interest.

References

1 Guo, L., & Li, Y. (2021). Braided dendriform and tridendriform algebras, and braided Hopf algebras of rooted trees. *Journal of Algebraic Combinatorics, 53*, 1147–1185. https://doi.org/10.1007/s10801-020-00957-0

2 Ion, B. (2023). Braided shuffle algebras and Lyndon words. *Bulletin of the Belgian Mathematical Society - Simon Stevin, 30*(5), 601–617. https://doi.org/10.36045/j.bbms.230206

3 Kharchenko, V. (2016). *Quantum Lie Theory: A Multilinear Approach.* Springer International Publishing. https://doi.org/10.1007/978-3-319-22704-7

4 Umirbaev, U., & Kharchenko, V. (2017). Free braided nonassociative Hopf algebras and Sabinin τ-algebras. *Journal of Algebra, 492*, 130–156. https://doi.org/10.1016/j.jalgebra.2017.08.019

5 Takeuchi, M. (2000). Survey of braided Hopf algebras. In *New Trends in Hopf Algebra Theory, Contemporary Mathematics, 267*, 301–324. Providence, RI: American Mathematical Society. https://doi.org/10.1090/conm/267/04277

6 Mutalip, R., Naurazbekova, A., & Umirbaev, U. (2022). Automorphisms of free braided associative algebras in two variables. *Communications in Algebra, 50*(2), 848–860. https://doi.org/10.1080/00927872.2021.1974027

7 Grewcoe, C.J., Jonke, L., Kodzoman, T., & Manolakos, G. (2022). From Hopf Algebra to Braided L_∞-Algebra. *Universe, 8*(4), 222. https://doi.org/10.3390/universe8040222

8 Bochniak, A., & Sitarz, A. (2019). Braided Hopf algebras from twisting. *Journal of Algebra and Its Applications, 18*(09), 1950178. https://doi.org/10.1142/S0219498819501780

9 Ciric, M.D., Giotopoulos, G., Radovanovic, V., & Szabo, R.J. (2021). Braided L_∞-algebras, braided field theory and noncommutative gravity. *Letters in Mathematical Physics, 111*, Article 148. https://doi.org/10.1007/s11005-021-01487-x

10 Cohn, P.M. (1964). Subalgebras of free associative algebras. *Proceedings of the London Mathematical Society, Third Series, 14*(4), 618–632. https://doi.org/10.1112/plms/s3-14.4.618

11 Lewin, J. (1968). On Schreier varieties of linear algebras. *Transactions of the American Mathematical Society, 132*(2), 553–562. https://doi.org/10.1090/S0002-9947-1968-0224663-5

12 Kurosh, A.G. (1947). Neassotsiativnye svobodnye algebry i svobodnye proizvedeniia algebr [Non-associative free algebras and free products of algebras]. *Matematicheskii Sbornik — Mathematical collection, 20*(62), 239–262 [in Russian].

13 Dotsenko, V., & Umirbaev, U. (2023). An Effective Criterion for Nielsen–Schreier Varieties. *International Mathematics Research Notices, 2023*(23), 20385–20432. https://doi.org/10.1093/imrn/rnad092

14 Alimbaev, A., Naurazbekova, A., & Kozybaev, D. (2019). Linearizatsiia avtomorfizmov i trianguliatsiia differentsirovanii svobodnykh algebr ranga 2 [Linearization of automorphisms and triangulation of derivations of free algebras of rank 2]. *Sibirskie elektronnye matematicheskie izvestiia — Siberian electronic mathematical reports, 16*, 1133–1146 [in Russian]. https://doi.org/10.33048/semi.2019.16.077

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