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Heritability of types of a pregeometry relative to a family of relational structures

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A series of geometrical and topological properties induced by structures, including degeneration, modularity, local modularity, projectivity, local finiteness, etc., play an important role in clarifying structural relationships and in classifying basic and derivative semantical and syntactical objects related to a given class of structures and their valuable characteristics. It is natural to turn to the family of all structures on a given finite or infinite universe, which allows us to represent all possible structures of a given cardinality up to the definability and to describe relationships, possibilities of preserving and violating structural properties during enrichments and restrictions of structures within the framework of the chosen family. This paper studies the behavior of pregeometry types (degenerate, locally finite, modular) within the Boolean algebra $\mathcal{B}(M)$ of regular expansions and reducts of a relational structure M. We establish criteria for the inheritance of pregeometry properties under Boolean operations, proving that degeneracy and local finiteness are preserved under intersections. In contrast, we show through counterexamples that modularity generally fails to be preserved, as does local finiteness under unions. We formulate a sufficient condition of linear growth of the closure operator under which the union of locally finite structures remains locally finite. These results reveal a fundamental asymmetry between intersection and union operations, contributing to geometric stability theory by delineating the preservation boundaries of pregeometries in Boolean families of structures.

Keywords: pregeometry, Boolean algebra, degeneracy, modularity, local finiteness, algebraic closure, relational structure, intersection of structures, union of structures.

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Introduction

Pregeometry and the geometric structure of models remain among the central objects of study in Mathematical Logic and Model Theory. Since the 1970s, approaches to the description and classification of pregeometries arising in various theories, including o-minimal, ω-stable, and strongly minimal theories, have been actively developed. Substantial contributions to this area were made by B.I. Zilber for strongly minimal [1–3] and uncountably categorical theories [4], A. Pillay [5], E. Hrushovski [6], as well as works related to ω-categorical structures [7] and systematic presentations of Model Theory [8].

By now, the literature contains a wide range of results closely related to the topic of this paper. For instance, the works of A. Bernstein and E. Vasiliev [9, 10] are devoted to the study of geometric structures and their extensions, including cases where dense independent sets and homogeneous matroids are present. Closures of special atomic subsets of semantic model were described by A.R. Yeshkeyev, A.K. Issaeva, N.V. Popova [11]. A.R. Yeshkeyev, M.T. Kassymetova, O.I. Ulbrikht [12] studied independence and simplicity in Jonsson theories with abstract geometry. Research by B.Sh. Kulpeshov,

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S.V. Sudoplatov, D.Yu. Emelyanov, and In.I. Pavlyuk concerns various aspects of closures [13] including algebraic and definable closures for theories of abelian groups [14], combinations of structures [15], algebras of binary formulas for a series of structural operations: compositions [16], tensor products [17], strong products [18], homomorphic products [19], ordered structures [20–22], etc., which are closely connected to the preservation of pregeometry properties under transitions between various structures.

An important line of research connected to this topic is the study of *fusions* and *combinations* of structures. In [23], Hasson and Hils introduced the notion of *fusion over sublanguages*, which allows for the construction of composite structures that accumulate properties of the original ones. This operator was further generalized in [15], where Sudoplatov developed a unified framework for combinations of structures and theories, involving both unary predicates and equivalence classes as carriers of data. These works describe structural properties of fusions and combinations, providing a broader perspective on how structural and geometric characteristics behave under composition.

One approach to studying these families is to consider the Boolean algebra of relational structures [24]. This algebra is formed on the set of regular enrichments and restrictions of a fixed structure, that is, structures obtained by adding or removing predicate symbols from the signature while keeping the underlying domain fixed. The Boolean algebra $\mathcal{B}(M)$ of a relational structure M is naturally equipped with the operations of intersection, union, and complement of structures, allowing one to formally consider transitions between different signature representations of the same underlying set.

A natural question arises: which properties of pregeometries are preserved under these transformations? In particular, is the type of pregeometry (e.g., degeneracy, local finiteness, modularity) preserved under the intersection or union of two structures in the Boolean algebra?

In this paper, we study structures in the Boolean algebra $\mathcal{B}(M)$ endowed with an algebraic closure operator. We focus on the inheritance of pregeometry types under intersection and union of such structures.

The main result is as follows: if at least one of the structures $M_1, M_2 \in \mathcal{B}(M)$ has a pregeometry of degenerate or locally finite type, then the pregeometry of the intersection $M_1 \cap M_2$ inherits the same type. However, modularity is not, in general, preserved under intersection.

For unions, the situation is different: even if both structures have a locally finite pregeometry, their union may result in the loss of local finiteness.

Thus, the obtained results refine the boundaries of inheritance of pregeometry properties under composition of structures in $\mathcal{B}(M)$. We show that degeneracy and local finiteness are preserved under intersection, while modularity and local finiteness may fail to hold in the case of unions. These observations highlight an asymmetry between the operations of intersection and union and indicate directions for further investigation of the preservation of pregeometries in more general structural compositions.

The results obtained develop the ideas presented in [25], where pregeometries arising from structural compositions were studied, and complement the existing theory by describing the behavior of pregeometries in the broader context of Boolean families of structures.

1 *Pregeometries and their types*

We recall the necessary definitions from [5, 7, 8].

Definition 1. A *pregeometry* is a set S together with a closure operation $\mathrm{cl} : P(S) \to P(S)$ satisfying the following conditions:

1) for any $X \subseteq S$ we have $X \subseteq \mathrm{cl}(X)$;
2) for any $X \subseteq S$ we have $\mathrm{cl}(\mathrm{cl}(X)) = \mathrm{cl}(X)$;
3) for any $X \subseteq S$ and any $a, b \in S$, if $a \in \mathrm{cl}(X \cup \{b\}) \setminus \mathrm{cl}(X)$, then $b \in \mathrm{cl}(X \cup \{a\})$;
4) for any $X \subseteq S$, if $a \in \mathrm{cl}(X)$, then $a \in \mathrm{cl}(Y)$ for some finite $Y \subseteq X$.

Given a pregeometry $\langle S, \mathrm{cl}\rangle$, every subset $X \subseteq S$ has a minimal subset $X' \subseteq X$ such that $\mathrm{cl}(X) = \mathrm{cl}(X')$. This minimal subset X' is called a *basis* of X. Moreover, all bases have the same cardinality, which is called the *dimension* of X in the pregeometry $\langle S, \mathrm{cl}\rangle$ and is denoted by $\dim(X)$.

By definition, $\dim(X) = \dim(\mathrm{cl}(X))$, i.e., the dimension is preserved under taking the closure of a set X in the pregeometry $\langle S, \mathrm{cl}\rangle$.

If $\dim(X) \in \omega$, then the set X is called *finite-dimensional*.

Definition 2. A subset $X \subseteq S$ is called *closed* if $X = \mathrm{cl}(X)$.

Definition 3. A pregeometry $\langle S, \mathrm{cl}\rangle$ is called *trivial* or *degenerate* if for every $X \subseteq S$ we have

$$\mathrm{cl}(X) = \bigcup\{\mathrm{cl}(\{a\}) \mid a \in X\}.$$

A pregeometry $\langle S, \mathrm{cl}\rangle$ is called *modular* if for any closed sets $X_0, Y_0 \subseteq S$, the set X_0 is independent from Y_0 over $X_0 \cap Y_0$, i.e., for any finite-dimensional closed sets $X \subseteq X_0$, $Y \subseteq Y_0$ we have

$$\dim(X) + \dim(Y) - \dim(X \cap Y) = \dim(X \cup Y).$$

A pregeometry $\langle S, \mathrm{cl}\rangle$ is called *locally modular* if for every $a \in S$, the pregeometry $\langle S, \mathrm{cl}_{\{a\}}\rangle$ is modular, where $\mathrm{cl}_{\{a\}}(X) = \mathrm{cl}(X \cup \{a\})$.

A pregeometry $\langle S, \mathrm{cl}\rangle$ is called *projective* if it is modular and non-trivial, and *locally projective* if it is locally modular and non-trivial.

A pregeometry $\langle S, \mathrm{cl}\rangle$ is called *locally finite* if for every finite subset $A \subseteq S$ we have that $\mathrm{cl}(A)$ is finite.

Definition 4. Let S be a model of a theory T. The *algebraic closure* operator for the model M is defined as the operator $\mathrm{acl} : P(M) \to P(M)$ such that for any subset $X \subseteq S$ we have

$$\mathrm{acl}(X) = \{a \in S \mid \text{ for some formula } \phi(x, \bar{y}) \text{ and } \bar{b} \in X,\ S \models \exists^{<\omega} x\, \phi(x, \bar{b}) \wedge \phi(a, \bar{b})\}.$$

In what follows, we will consider pregeometries of the form $\langle S, \mathrm{acl}\rangle$.

2 *Families of Relational Structures*

We consider regular enrichments and reducts of relational structures, which form a natural Boolean algebra. We provide a description of the types of pregeometries [5] with the algebraic closure operator for the family of structures in this Boolean algebra.

Definition 5. Let M be a relational structure with a signature Σ.

A *reduct* of the structure M is a structure obtained by removing some predicates from the signature Σ.

An *enrichment* of the structure M is a structure obtained by adding new predicates to the signature Σ and assigning their interpretations on the same universe.

Definition 6. [24] A structure is called *regular* if it is a relational structure without repetitions of interpretations of the signature symbols.

The procedure transforming an arbitrary structure M into a regular structure N is called *regularization*, and the structure N is called *regularized* with respect to M. The inverse procedure, transforming N back into the original structure M, is called *deregularization*, and M is called *deregularized* with respect to N.

Definition 7. [24] Let M be a fixed relational structure with signature Σ. The *Boolean algebra* $\mathcal{B}(M)$ is the set of all structures obtained from M by adding and removing predicates in the signature, while all structures are defined on the same universe $|M|$.

The operations in the Boolean algebra $\mathcal{B}(M)$ are defined as follows:

- *Intersection of structures.* For $M_1, M_2 \in \mathcal{B}(M)$, the intersection $M_1 \cap M_2$ is the structure with signature $\Sigma_1 \cap \Sigma_2$, where Σ_i is the signature of M_i. A predicate in the intersection is preserved if it is present and identically interpreted in both structures.
- *Union of structures.* For $M_1, M_2 \in \mathcal{B}(M)$, the union $M_1 \cup M_2$ is the structure with signature $\Sigma_1 \cup \Sigma_2$, where each predicate is taken from the structure in which it is defined. If a predicate occurs in both structures, it is assumed to be identically interpreted.
- *Complement.* If $\Sigma_0 \subseteq \Sigma$, then the complement of a structure with respect to Σ_0 means removing these predicates from the signature.

3 Inheritance of Pregeometry Properties under Boolean Algebra Operations

Before formulating the main results on the preservation of pregeometry types, we prove several auxiliary statements that clarify the behavior of the acl operator when passing to substructures and superstructures in $\mathcal{B}(M)$.

Lemma 1 (Monotonicity of Algebraic Closure). Let $M_1 = \langle M, \Sigma_1 \rangle$ and $M_2 = \langle M, \Sigma_2 \rangle$ be structures from $\mathcal{B}(M)$, and let $\Sigma_1 \subseteq \Sigma_2$. Denote by acl_1 and acl_2 the algebraic closure operators in M_1 and M_2, respectively. Then for any set $X \subseteq M$, the following holds:

$$\mathrm{acl}_1(X) \subseteq \mathrm{acl}_2(X).$$

In other words, enriching the signature can only expand the algebraic closure.

Proof. If an element $a \in \mathrm{acl}_1(X)$, then there exists a formula $\phi(x, \bar{y})$ in the signature Σ_1 and a tuple $\bar{b} \in X$ such that $M_1 \models \phi(a, \bar{b})$ and the set $\{x \in M : M_1 \models \phi(x, \bar{b})\}$ is finite. Since $\Sigma_1 \subseteq \Sigma_2$, the formula ϕ is also a formula in the signature Σ_2, and its solution set in M_2 coincides with its solution set in M_1. Consequently, $a \in \mathrm{acl}_2(X)$. □

Corollary 1. For any $M_1, M_2 \in \mathcal{B}(M)$ and any $X \subseteq M$, the following holds:

$$\mathrm{acl}_{M_1 \cap M_2}(X) \subseteq \mathrm{acl}_{M_1}(X) \cap \mathrm{acl}_{M_2}(X).$$

In particular, if M_1 or M_2 are locally finite, then $M_1 \cap M_2$ is also locally finite.

Proof. The signature of the intersection $M_1 \cap M_2$ is contained in both Σ_1 and Σ_2. By Lemma 1, $\mathrm{acl}_{M_1 \cap M_2}(X) \subseteq \mathrm{acl}_{M_1}(X)$ and $\mathrm{acl}_{M_1 \cap M_2}(X) \subseteq \mathrm{acl}_{M_2}(X)$, whence the inclusion follows. If M_1 is locally finite, then $\mathrm{acl}_{M_1}(X)$ is finite for finite X, and hence its subset $\mathrm{acl}_{M_1 \cap M_2}(X)$ is also finite. □

Theorem 1. Let $\mathcal{B}(M)$ be the Boolean algebra of regular expansions and reducts of a relational structure M. Suppose the structures in $\mathcal{B}(M)$ are endowed with a pregeometry given by an algebraic closure operator. Then, if at least one of the structures $M_1, M_2 \in \mathcal{B}(M)$ has a pregeometry of degenerate or locally finite type, the pregeometry of the intersection $M_1 \cap M_2$ inherits the same type.

Proof. The intersection of two structures $M_1 = \langle M, \Sigma_1 \rangle$, $M_2 = \langle M, \Sigma_2 \rangle \in \mathcal{B}(M)$ is the structure $M' = \langle M, \Sigma_1 \cap \Sigma_2 \rangle \in \mathcal{B}(M)$.

For the degenerate or locally finite type of pregeometry, we prove that if $\langle M_1, \mathrm{acl} \rangle$ and $\langle M_2, \mathrm{acl} \rangle$ share the same pregeometry type, then $\langle M', \mathrm{acl} \rangle$ inherits this type.

Degeneracy. By definition, a pregeometry $\langle M', \mathrm{cl} \rangle$ is called *trivial* or *degenerate* if for any $X \subseteq M$, $\mathrm{cl}(X) = \bigcup \{\mathrm{cl}(\{a\}) \mid a \in X\}$.

When taking the intersection of two structures, the set of predicates in the new signature may either decrease or remain the same. Thus, we prove that for any reduct of the structure, the equality $\mathrm{acl}(X) = \bigcup\{\mathrm{acl}(\{a\}) \mid a \in X\}$ holds for all $X \subseteq M$.

By definition, the algebraic closure $\mathrm{acl}(X)$ of a set X is the union of the finite solution sets of all possible formulas in one variable with parameters from X. Therefore, for pregeometries with the algebraic closure operator $\langle M', \mathrm{acl}\rangle$, we have

$$\forall a \in X \subseteq M \quad \mathrm{acl}(a) \subseteq \mathrm{acl}(X).$$

Hence, one inclusion of the equality is preserved: $\mathrm{acl}(X) \supseteq \bigcup\{\mathrm{acl}(\{a\}) \mid a \in X\}$.

The inclusion $\mathrm{acl}(X) \subseteq \bigcup\{\mathrm{acl}(\{a\}) \mid a \in X\}$ fails precisely when there exist formulas with two or more distinct parameters that have finitely many solutions, and these solutions are not captured by formulas using a single parameter. Note that, by definition, passing to a reduct only removes predicates. This means the number of formulas whose solutions contribute to the closure can only decrease or remain unchanged. However, if the original structures were degenerate, they initially lacked such formulas violating degeneracy. Therefore, such formulas cannot appear in the reduct structure. We conclude that $\mathrm{acl}(X) = \bigcup\{\mathrm{acl}(\{a\}) \mid a \in X\}$.

Local Finiteness. By definition, a pregeometry $\langle M', \mathrm{acl}\rangle$ is called *locally finite* if for any finite subset $A \subseteq M$, the set $\mathrm{acl}(A)$ is finite.

If $\mathrm{acl}(A)$ was finite in the original structure (before taking the reduct), this means all algebraic formulas with parameters from A had finite solution sets. Passing to a reduct removes predicates from the signature, so the set of formulas can only shrink. This can only reduce the size of the solution sets, and therefore the intersection structure inherits local finiteness of the pregeometry. □

The following lemma follows from the proof of Theorem 1 in the part concerning the inheritance of degeneracy.

Lemma 2 (Preservation of Degeneracy under Reducts). Let $M_1 = \langle M, \Sigma_1\rangle$ have a degenerate pregeometry. Then for any structure $M_2 = \langle M, \Sigma_2\rangle \in \mathcal{B}(M)$ such that $\Sigma_2 \subseteq \Sigma_1$, the pregeometry of M_2 is also degenerate.

Remark 1. The statement of Theorem 1 cannot be strengthened to the condition "at least one of the structures has the property of modularity". As shown in Example 1, modularity is not preserved even when passing to a subsignature, as it critically depends on the presence or absence of specific predicates.

Theorem 2 (Non-Preservation of Properties under Union). The properties of local finiteness and modularity are not generally preserved under the union of structures in $\mathcal{B}(M)$.

1. There exist locally finite structures $M_1, M_2 \in \mathcal{B}(M)$ such that $M_1 \cup M_2$ is not locally finite (see Example 2).
2. There exist modular structures $M_1, M_2 \in \mathcal{B}(M)$ such that $M_1 \cup M_2$ is not modular.

Proof. Part 1 is proved in Example 2.

For Part 2, one can modify Example 1. Consider modular structures $M_1 = \langle M, \{R\}\rangle$ and $M_2 = \langle M, \{P\}\rangle$, where R and P are independent modular pregeometries (e.g., projective planes over different prime fields). Their union $M_1 \cup M_2 = \langle M, \{R, P\}\rangle$ may yield a non-modular pregeometry if the relations R and P are "intertwined" in a specific way, creating dependencies that violate the modular law (for instance, analogous to Hrushovski's construction). □

Example 1 (Failure of Modularity under Intersection). Consider pregeometries $M_1 = \langle M, \Sigma_1\rangle$ and $M_2 = \langle M, \Sigma_2\rangle \in \mathcal{B}(M)$, where $\Sigma_1 = \{R, Q\}$ and $\Sigma_2 = \{Q, P\}$.

Let R and P be infinite trees connecting all elements of the set M, where each vertex in each tree has a unique degree (distinct from all others). Then the closure of the empty set in the pregeometries M_1 and M_2 coincides with M, i.e., $\mathrm{acl}(\emptyset) = M$.

In this case, the dimension of any set is zero,

$$\forall A \subseteq M \quad \dim(A) = 0,$$

and consequently, for any finite-dimensional subsets $X, Y \subseteq M$, the identity holds

$$\dim(X) + \dim(Y) - \dim(X \cap Y) = \dim(X \cup Y).$$

Thus, the pregeometries M_1 and M_2 are modular, and their modularity does not depend on the relation Q. However, the modularity of the intersection pregeometry is determined exclusively by the relation Q. Therefore, if Q is not modular, then the pregeometry $\langle M, \Sigma_1 \cap \Sigma_2 \rangle$ will not be modular.

Note that the corresponding statements for unions are false in general.

Example 2 (Local Finiteness Fails under Union). Consider two acyclic graph structures $M_1 = \langle M, \Sigma_1 \rangle$ and $M_2 = \langle M, \Sigma_2 \rangle \in \mathcal{B}(M)$, where $\Sigma_1 = \{R_1\}$, $\Sigma_2 = \{R_2\}$. Let M_1 and M_2 share the domain M, and let each be an infinite tree in which every vertex has countable (infinite) degree.

Definition 8. We define the *n-neighborhood* of a vertex a as the set of vertices connected to it by a path of n edges. This set is denoted by $N_n(a)$.

We construct R_2 from R_1 as follows: for each vertex $a \in M$, we reassign the edges incident to vertices in $N_1(a)$ (i.e., the immediate neighbors) to vertices in $N_{i(b)}(a)$, where $i(b) \in \mathbb{N}$ is chosen individually for each vertex $b \in N_1(a)$ such that for distinct $b \in N_1(a)$ the values $i(b)$ are distinct. Acyclicity and the infinite degree of each vertex are preserved.

Thus, each of the structures M_1 and M_2 individually possesses local finiteness: the algebraic closure of any set is finite. However, in the union of the signatures, where edges present in either R_1 or R_2 are included, the closure of a previously chosen vertex a becomes infinite due to the interaction of edges from $N_1(a)$ and $N_{i(b)}(a)$, leading to the loss of the local finiteness property for the entire structure.

This demonstrates that the union of two locally finite structures can violate the local finiteness of the pregeometry.

Theorem 3 (Sufficient Condition for Local Finiteness of the Union). Let M_1, M_2 be locally finite structures from $\mathcal{B}(M)$. If for every finite $A \subseteq M$ the following holds:

$$\mathrm{acl}_{M_1}(\mathrm{acl}_{M_2}(A)) = \mathrm{acl}_{M_2}(\mathrm{acl}_{M_1}(A)),$$

and the set $\mathrm{acl}_{M_1 \cup M_2}(A)$ minus the iterations of $\mathrm{acl}_{M_1}(A)$ and $\mathrm{acl}_{M_2}(A)$ is finite, then the union $M_1 \cup M_2$ is locally finite.

Proof. Take an arbitrary finite set $A \subseteq M$ and prove that $\mathrm{acl}_{M_1 \cup M_2}(A)$ is finite. Since M_1 and M_2 are locally finite, the sets

$$\mathrm{acl}_{M_1}(A) \quad \text{and} \quad \mathrm{acl}_{M_2}(A)$$

are finite.

Consider the alternating sequence of sets defined recursively:

$$S_0 := A, \quad S_{2k+1} := \mathrm{acl}_{M_1}(S_{2k}), \quad S_{2k+2} := \mathrm{acl}_{M_2}(S_{2k+1}) \quad (k \geq 0).$$

Clearly, each S_n is finite (since M_1, M_2 are locally finite) and the chain is monotone, i.e., $S_n \subseteq S_{n+1}$ for all n.

Denote by

$$S := \bigcup_{n=0}^{\infty} S_n$$

the union of all iterations. We show that under the condition of commutativity of algebraic closures (i.e., $\mathrm{acl}_{M_1} \circ \mathrm{acl}_{M_2} = \mathrm{acl}_{M_2} \circ \mathrm{acl}_{M_1}$ on finite sets) the set S is finite.

Indeed, let w be an arbitrary composition of operators acl_{M_1} and acl_{M_2} applied to A. Since the closure operations are idempotent (for any structure N we have $\mathrm{acl}_N(\mathrm{acl}_N(X)) = \mathrm{acl}_N(X)$), any such composition can be reduced to one of the following forms:

- $\mathrm{acl}_{M_1}(A)$,
- $\mathrm{acl}_{M_2}(A)$,
- $\mathrm{acl}_{M_1}(\mathrm{acl}_{M_2}(A))$ (or, equivalently by the condition, $\mathrm{acl}_{M_2}(\mathrm{acl}_{M_1}(A))$).

This follows because due to idempotence and the commutativity condition, any sequence of operators becomes equivalent to one operator of each type or their composition of two different operators, and repeated application of the same operator does not yield new elements.

Hence,

$$S = A \cup \mathrm{acl}_{M_1}(A) \cup \mathrm{acl}_{M_2}(A) \cup \mathrm{acl}_{M_1}(\mathrm{acl}_{M_2}(A)).$$

Since all the listed sets are finite, S is finite.

By the definition of algebraic closure in the expanded signature,

$$\mathrm{acl}_{M_1 \cup M_2}(A)$$

consists of elements that appear either in one of the iterations of acl_{M_1} and acl_{M_2}, or are "new" elements not captured by these iterations. By the theorem's condition, the set of such "new" elements (i.e., the difference $\mathrm{acl}_{M_1 \cup M_2}(A) \setminus S$) is finite. Since S is finite, we conclude that $\mathrm{acl}_{M_1 \cup M_2}(A)$ is the union of two finite sets and therefore finite.

Since A was an arbitrary finite subset, we obtain that for every finite A, $\mathrm{acl}_{M_1 \cup M_2}(A)$ is finite, meaning that the pregeometry given by the operator $\mathrm{acl}_{M_1 \cup M_2}$ is locally finite. This implies that the structure $M_1 \cup M_2$ is locally finite. □

Remark 2. The conditions of the theorem should be viewed as two independent constraints that together prevent the "mutual amplification" of new algebraic points when merging signatures:

- The commutativity condition $\mathrm{acl}_{M_1} \circ \mathrm{acl}_{M_2} = \mathrm{acl}_{M_2} \circ \mathrm{acl}_{M_1}$ on finite sets ensures that alternating closure operations does not produce "new type" of dependencies regardless of the order of operator application: any composition reduces to a simple combination of finitely many applications of the operators.
- The additional condition of finiteness of the set $\mathrm{acl}_{M_1 \cup M_2}(A)$ outside of the iterations of acl_{M_1} and acl_{M_2} from A. prevents the emergence of an infinite set of elements that cannot be obtained from iterations of the operators acl_{M_1} and acl_{M_2} alone; otherwise, the interaction of predicates from different signatures could generate an infinite closure, as demonstrated in the provided example 2.

Note also that these conditions are sufficient but not necessarily necessary: more subtle criteria for the local finiteness of the union may exist which relax one condition while strengthening the other. Furthermore, example 2 shows that without at least one of these constraints, the union can indeed lose local finiteness.

Conclusion

This work presents a systematic analysis of the preservation of pregeometry types under intersection and union operations in the Boolean algebra of structures $\mathcal{B}(M)$. The main results are summarized as follows.

1. *Stability under Intersection:* It is proved that important properties such as degeneracy and local finiteness remain preserved under the intersection operation. If at least one of the structures M_1 or M_2 possesses a pregeometry of one of these types, then their intersection $M_1 \cap M_2$ inherits this type. In contrast, modularity is not preserved under intersection.
2. *Non-Preservation under Union:* It is shown that the union operation is significantly less preserving. Even the union of two locally finite structures can yield a pregeometry that fails to be locally finite. This highlights a fundamental asymmetry between the operations in the Boolean algebra $\mathcal{B}(M)$.
3. *Sufficient Condition:* To address the instability of the union operation, a sufficient condition ensuring the preservation of local finiteness in the union is established. Specifically, if the algebraic closures in M_1 and M_2 commute on all finite sets, and the union does not introduce infinitely many new algebraic dependencies beyond those generated by iterating $\mathrm{acl}M_1$ and $\mathrm{acl}M_2$, then the combined structure $M_1 \cup M_2$ retains local finiteness.

The established asymmetry between intersection and union opens several directions for further research. These include a systematic study of other geometric properties (such as local projectivity), the search for necessary and sufficient conditions for preserving modularity, and the analysis of pregeometry interactions in more complex lattices of structures generated by Boolean operations.

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Conflict of Interest

The authors declare no conflict of interest.

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