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Algebras of binary isolating formulas for theories of modular products of graphs

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This article discusses the problems of constructing and classifying algebras generated by modular products of cycles. We demonstrate that algebras of binary isolating, used to analyze relationships between binary formulas of a theory, can be naturally interpreted in terms of metric properties of graphs. A characteristic feature of the modular product is that with sufficiently large cycle parameters ($m, n > 4$), the diameter of such a graph does not exceed three. This makes it possible to define an algebra of binary formulas using only four labels. For small cycle parameters, the presence of simplices is identified and justified. Based on the analysis, we propose a generalized scheme combining modular products of cycles and their extended versions. It is proved that for $m, n > 4$, the algebra of binary isolating formulas for the theory of $C_m \nabla C_n$ is isomorphic to the algebra of simplices of corresponding diameter. Explicit Cayley tables are constructed for products involving small cycles (C_3–C_6), leading to general descriptions of algebras $\mathfrak{M}_o$ (odd) and $\mathfrak{M}_e$ (even). The proposed approach provides new opportunities for classifying theories and establishing correspondences between algebras and graphs, underlining its relevance for modern model theory and structural combinatorics.

Keywords: algebra of binary isolating formulas, modular product, model theory, Cayley tables, classification of theories, simplicial algebras, cycle graphs, graph diameter.

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Introduction

In modern model theory, a section devoted to the algebras of binary isolating and weakly isolating formulas is actively developing. These algebraic structures, constructed according to an arbitrary complete theory, serve as a powerful tool for analyzing the structure of binary types and the complex system of relations between their implementations. Such algebraic invariants turn out to be a powerful tool for classifying theories, allowing one to establish correspondences between theories and the algebraic structures associated with them. Contributions to the development of this field have been made by works devoted to the study of the general properties of such algebras [1–3], as well as their calculation for specific classes of theories, such as the theories of groups [4], ordered structures [5–7], weakly o-minimal theories [8–10], Cartesian products of graphs and simplices [11].

Modular products of graphs are of particular interest in this context. They are an important construction of structural combinatorics, which makes it possible to build complex graphs from simpler ones. This operation, based on the Cartesian product of sets of vertices with the definition of edges through synchronization of adjacency relations in the original graphs, accumulates both local and global properties of factors. This makes the modular product a promising object for model-theoretical analysis, where vertices and edges are interpreted as carriers of predicates, and the operation itself is interpreted as a composition of logical constructions. This approach makes it possible to study the

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transfer of structural properties, such as connectivity, diameter, or the presence of certain subgraphs, and to study their influence on the expressive power of the corresponding theory.

Throughout this paper, we consider graphs as first-order structures in the language $\mathcal{L} = \{R\}$ with a single binary predicate interpreted as the edge relation. For a graph G, its *theory* $\mathrm{Th}(G)$ is the set of all $\mathcal{L}$-sentences true in G. We study the *algebra of binary isolating formulas* $\mathfrak{M}_T$ associated with a complete theory $T = \mathrm{Th}(G)$. The elements (labels) of this algebra correspond to the isomorphism types of principal formulas $\theta(x, a)$ in two variables that isolate complete types over finite sets. The algebraic operation $\cdot$ on these labels is induced by the superposition of such formulas and yields a set of possible labels for the resulting formula. For a detailed construction, see [1, 3].

The relevance of this study is due to the need for a systematic study of binary formula algebras for theories generated by combinatorial constructions, in particular, modular products. Such an analysis makes it possible not only to classify theories by their derived algebraic invariants, but also to better understand the relationship between the combinatorial structure of the model and the logical properties of its theory.

The novelty of the work lies in the fact that it is the first to study the distribution algebras of binary isolating formulas for the theories of modular products of graphs, primarily regular polygons (cycles). For these products, Cayley tables of the corresponding algebras are constructed and analyzed, which makes it possible to give their explicit description.

The following main results were achieved in the course of the study. First of all, Cayley tables were constructed for the algebras of binary isolating formulas of the theories of modular products of an edge graph on cycles of small length, namely C_3, C_4, C_5 and C_6. Based on the analysis of these specific cases, a general description of the $\mathfrak{M}_\mathfrak{o}$ and $\mathfrak{M}_\mathfrak{e}$ algebras for modular products of an edge graph on cycles of arbitrary odd and even length, respectively, was given. An important result of the work was the formulation and proof of a theorem that gives a classification of binary formula algebras for theories of modular products of graph edges on polygons (Theorem 1). Further, the case of the modular product of two cycles $C_m \nabla C_n$ for $m, n > 4$ was investigated; for such graphs, it was proved that their diameter does not exceed 3, and they always contain triangles (Theorems 3). Finally, it is established that in the case of $m, n > 4$, the algebra of binary isolating formulas of the theory of $C_m \nabla C_n$ turns out to be isomorphic to the algebra of simplices of the corresponding diameter (Theorem 4).

The results obtained demonstrate that the class of algebras of binary isolating formulas for modular products of cycles has a rich but classifiable structure, and reveal clear connections between the combinatorial properties of the products and the algebraic properties of the corresponding theories.

1 *Algebras of binary isolating formulas for theories of modular products of graphs*

Definition 1. [12] Let G and H be graphs. The *modular product* $M = G \nabla H$ is a graph defined as follows. The vertex set of M is the Cartesian product:

$$V(M) = V(G) \times V(H).$$

Two distinct vertices $(u, v), (x, y) \in V(M)$ are joined by an edge if and only if

$$u \neq x, \quad v \neq y, \quad \text{and} \quad \big(u \sim_G x\big) \iff \big(v \sim_H y\big).$$

Let $T = \mathrm{Th}(G \nabla H)$. The algebra $\mathfrak{M}_T$ of binary isolating formulas for T has a finite set of *labels* (denoted by integers) corresponding to the possible distances or isomorphism types of pairs of vertices in the graph. The multiplication $i \cdot j = K$, where K is a set of labels, encodes the possible labels for a pair (a, c) given that (a, b) has label i and (b, c) has label j for some vertex b. The tables below are computed by analyzing the graph structure of $G \nabla H$.

We further consider algebras generated by the operation of modular multiplication of edges in graphs of regular polygons. Graphs of regular polygons can be regarded as cycles, denoted C_n, where n is the length of the cycle.

In the case of a modular product of an edge graph on itself, $H\nabla H$, two identical algebraic structures arise. Their signatures, defined by the set $\rho_{\nu(p)} = \{0,1\}$ of labels, are set by the following multiplication rules:

$$\begin{aligned} &0 \cdot 0 = \{0\}, \\ &0 \cdot 1 = \{1\}, \\ &1 \cdot 1 = \{0,1\}, \\ &1 \cdot 0 = \{1\}. \end{aligned}$$

The algebra of the modular product of an edge with a triangle $H\nabla C_3$ with the set $\rho_{\nu(p)} = \{0,1,2,3\}$ of labels is defined by the following label products:

$$\begin{array}{ll} 0 \cdot 0 = \{0\}, & 0 \cdot 1 = \{1\}, \\ 1 \cdot 0 = \{1\}, & 1 \cdot 1 = \{0,2\}, \\ 1 \cdot 2 = \{0,1\}, & 1 \cdot 3 = \{0,2\}, \\ 2 \cdot 0 = \{2\}, & 2 \cdot 1 = \{1,2\}, \\ 2 \cdot 2 = \{0,1\}, & 2 \cdot 3 = \{1,3\}, \\ 3 \cdot 0 = \{3\}, & 3 \cdot 1 = \{0,2\}, \\ 3 \cdot 2 = \{1,3\}, & 3 \cdot 3 = \{0,2\}. \end{array}$$

The modular product $H\nabla C_4$ (an edge on a quarter-cycle) generates two isomorphic algebras. The set of their labels is $\rho_{\nu(p)} = \{0,1,2\}$, and the multiplicative structure is completely determined by the following rules:

$$\begin{array}{ll} 0 \cdot 0 = \{0\}, & 0 \cdot 1 = \{1\}, \\ 1 \cdot 0 = \{1\}, & 1 \cdot 1 = \{0,2\}, \\ 1 \cdot 2 = \{1,2\}, & 2 \cdot 0 = \{2\}, \\ 2 \cdot 1 = \{1,2\}, & 2 \cdot 2 = \{0,2\}. \end{array}$$

The algebra for the modular product of an edge with a pentagon $H\nabla C_5$ with label set $\rho_{\nu(p)} = \{0,1,2,3,4,5\}$ is defined by the following label products:

$$\begin{aligned} &0 \cdot 0 = \{0\}, \\ &0 \cdot 1 = 1 \cdot 0 = \{1\}, \\ &0 \cdot 2 = 2 \cdot 0 = \{2\}, \\ &0 \cdot 3 = 3 \cdot 0 = \{3\}, \\ &0 \cdot 4 = 4 \cdot 0 = \{4\}, \\ &0 \cdot 5 = 5 \cdot 0 = \{5\}, \\ &1 \cdot 1 = \{0,2\}, \\ &1 \cdot 2 = 2 \cdot 1 = \{1,3\}, \\ &1 \cdot 3 = 3 \cdot 1 = 1 \cdot 5 = 5 \cdot 1 = 3 \cdot 3 = 3 \cdot 5 = 5 \cdot 3 = 5 \cdot 5 = \{0,2,4\}, \\ &2 \cdot 2 = 2 \cdot 4 = 4 \cdot 2 = 4 \cdot 4 = \{0,2,4\}, \\ &1 \cdot 4 = 4 \cdot 1 = 2 \cdot 3 = 3 \cdot 2 = 2 \cdot 5 = 5 \cdot 2 = \{1,3,5\}, \\ &3 \cdot 4 = 4 \cdot 3 = 4 \cdot 5 = 5 \cdot 4 = \{1,3,5\}. \end{aligned}$$

Consider the modular product of an edge graph on a hexagon, $H\nabla C_6$. This construction generates two isomorphic algebras supported by the set $\rho_{\nu(p)} = \{0,1,2,3\}$ of labels. The multiplication operation in these algebras is given by the following table:

$$
\begin{array}{ll}
0 \cdot 0 = \{0\}, & 0 \cdot 1 = \{1\}, \\
1 \cdot 0 = \{1\}, & 1 \cdot 1 = \{0, 2\}, \\
1 \cdot 2 = \{0, 1\}, & 1 \cdot 3 = \{0, 2\}, \\
2 \cdot 0 = \{2\}, & 2 \cdot 1 = \{1, 2\}, \\
2 \cdot 2 = \{0, 1\}, & 2 \cdot 3 = \{1, 3\}, \\
3 \cdot 0 = \{3\}, & 3 \cdot 1 = \{0, 2\}, \\
3 \cdot 2 = \{1, 3\}, & 3 \cdot 3 = \{0, 2\}.
\end{array}
$$

For the modular product $H \nabla C_k$, depending on the parity of the diameter $\mathrm{diam}(H \nabla C_k)$, we obtain two possible isomorphism types of algebras of binary isolating formulas. The odd-diameter case is denoted by $\mathfrak{M}_o$, and the even-diameter case is denoted by $\mathfrak{M}_e$. Below we give explicit descriptions of these algebras via Cayley tables.

Definition of $\mathfrak{M}_\mathfrak{o}$. Its signature includes labels $\{0, 1, 2, 3, \ldots, n\}$, where n is an odd number equals to the diameter of the graph resulting from the product. The structure of this algebra is determined by the following multiplication table:

$$
\begin{array}{l}
0 \cdot 0 = \{0\}, \\
0 \cdot 1 = 1 \cdot 0 = \{1\}, \\
0 \cdot 2 = 2 \cdot 0 = \{2\}, \\
0 \cdot 3 = 3 \cdot 0 = \{3\}, \\
0 \cdot 4 = 4 \cdot 0 = \{4\}, \\
\vdots \\
0 \cdot n = n \cdot 0 = \{n\}, \\
1 \cdot 1 = \{0, 2\}, \\
1 \cdot 2 = 2 \cdot 1 = \{1, 3\}, \\
1 \cdot 3 = 3 \cdot 1 = \{0, 2, 4\}, \\
2 \cdot 2 = \{0, 2, 4\}, \\
2 \cdot 3 = 3 \cdot 2 = \{1, 3, 5\}, \\
3 \cdot 3 = \{0, 2, 4, 6\}, \\
4 \cdot 1 = 1 \cdot 4 = \{1, 3, 5\}, \\
4 \cdot 2 = 2 \cdot 4 = \{0, 2, 4, 6\}, \\
4 \cdot 3 = 3 \cdot 4 = \{1, 3, 5, \ldots, n\}, \\
4 \cdot 4 = \{0, 2, 4, \ldots, n-1\}, \\
\vdots \\
n \cdot 1 = 1 \cdot n = \{1, 3, 5, \ldots, n\}, \\
n \cdot 2 = 2 \cdot n = \{0, 2, 4, \ldots, n-1\}, \\
n \cdot 3 = 3 \cdot n = \{1, 3, 5, \ldots, n\}, \\
n \cdot 4 = 4 \cdot n = \{0, 2, 4, \ldots, n-1\}, \\
\vdots \\
n \cdot n = \{0, 2, 4, \ldots, n-1\}.
\end{array}
$$

Definition of $\mathfrak{M}_\mathfrak{e}$. Its carrier is a set of labels $\{0, 1, 2, 3, \ldots, n\}$, where n is an even number equal to the diameter of the graph resulting from the product. The structure of the algebra is determined by the following multiplication table:

$0 \cdot 0 = \{0\}$	$0 \cdot 1 = 1 \cdot 0 = \{1\}$
$0 \cdot 2 = 2 \cdot 0 = \{2\}$	$0 \cdot 3 = 3 \cdot 0 = \{3\}$
$0 \cdot 4 = 4 \cdot 0 = \{4\}$	$\vdots$
$0 \cdot n = n \cdot 0 = \{n\}$	$1 \cdot 1 = \{0, 2\}$
$1 \cdot 2 = 2 \cdot 1 = \{1, 3\}$	$1 \cdot 3 = 3 \cdot 1 = \{0, 2\}$
$2 \cdot 2 = \{0, 2, 4\}$	$2 \cdot 3 = 3 \cdot 2 = \{1, 3, 5\}$
$3 \cdot 3 = \{0, 2, 4, 6\}$	$4 \cdot 1 = 1 \cdot 4 = \{1, 3, 5\}$
$4 \cdot 2 = 2 \cdot 4 = \{0, 2, 4, 6\}$	$4 \cdot 3 = 3 \cdot 4 = \{1, 3, 5, \ldots, n-1\}$
$4 \cdot 4 = \{0, 2, 4, \ldots, n\}$	$\vdots$
$n \cdot 1 = 1 \cdot n = \{1, 3, 5, \ldots, n-1\}$	$n \cdot 2 = 2 \cdot n = \{0, 2, 4, \ldots, n\}$
$n \cdot 3 = 3 \cdot n = \{1, 3, 5, \ldots, n-1\}$	$n \cdot 4 = 4 \cdot n = \{0, 2, 4, \ldots, n\}$
$\vdots$	$n \cdot n = \{0, 2, 4, \ldots, n\}$

Remark 1. Let G and H be non-empty graphs. Then for the modular product $G\nabla H$ the following holds:

1. If both graphs G and H are bipartite, then $G\nabla H$ consists of exactly two connected components, isomorphic to each other.
2. If at least one of the graphs G or H is not bipartite, then $G\nabla H$ is connected.

If G and H are bipartite with bipartitions (U_1, U_2) and (V_1, V_2) respectively, then the vertex set of $M = G\nabla H$ splits into two disjoint subsets: $(U_1 \times V_1) \cup (U_2 \times V_2)$ and $(U_1 \times V_2) \cup (U_2 \times V_1)$. By the definition of adjacency in the modular product, no edge connects these two subsets, and each induces a connected component isomorphic to the *modular product with a fixed parity constraint*. This structural dichotomy is reflected in the algebra $\mathfrak{M}_T$: the labels and multiplication become symmetric with respect to these two components, effectively yielding two isomorphic copies of the same algebraic structure.

Reflecting Remark 1 on algebras, we obtain that in the first case we get two identical algebras, in the second one, depending on the diameter of the resulting graph, they will be either $\mathfrak{M}_{\mathfrak{e}}$ or $\mathfrak{M}_{\mathfrak{o}}$.

Theorem 1. If T is the theory of the modular product of an edge with polygons, and $\mathfrak{M}$ is the algebra of binary isolating formulas of the theory T, then the algebra $\mathfrak{M}$ is isomorphic to the algebra $\mathfrak{M}_{\mathfrak{o}}$ or $\mathfrak{M}_{\mathfrak{e}}$.

Proof. Let H be the edge graph with two vertices joined by an edge, and let $k \geq 3$. Put $G = H\nabla C_k$ and $T = \mathrm{Th}(G)$.

We first describe the structure of the graph G. The vertex set of G is $\{0, 1\} \times V(C_k)$. Two distinct vertices (i, a) and (j, b) are adjacent in G if and only if $i \neq j$, $a \neq b$, and $(i \sim_H j) \iff (a \sim_{C_k} b)$. Since H is an edge graph, the condition $i \neq j$ already implies $i \sim_H j$. Hence adjacency in G is equivalent to the condition that $i \neq j$ and $a \sim_{C_k} b$.

Therefore, each vertex $(0, a)$ is adjacent exactly to the vertices $(1, a-1)$ and $(1, a+1)$ (indices taken modulo k), and similarly for vertices of the form $(1, a)$. This shows that each connected component of G is a cycle.

If k is odd, then C_k is not bipartite, and by Remark 1 the graph G is connected. Hence G is a single cycle of length $2k$, and $\mathrm{diam}(G) = k$, which is an odd number.

If k is even, then both H and C_k are bipartite, and again by Remark 1 the graph G consists of two isomorphic connected components. Each component is a cycle of length k, and therefore has diameter $k/2$, which is an even number.

We now describe the algebra $\mathfrak{M}_T$. Since G is finite and vertex-transitive on each connected component, the complete 2-types over the empty set are uniquely determined by the graph distance between two vertices lying in the same component. Hence the set of labels of $\mathfrak{M}_T$ can be identified with

$\{0, 1, 2, \ldots, n\}$, where $n = \operatorname{diam}(G)$, and the label t corresponds to the formula expressing that the distance between two vertices equals t.

Let $i, j \in \{0, 1, \ldots, n\}$ and let u, v, w be vertices of G such that $\operatorname{dist}(u, v) = i$ and $\operatorname{dist}(v, w) = j$. Since each connected component of G is a cycle, all possible values of $\operatorname{dist}(u, w)$ are exactly the integers t satisfying

$$|i - j| \leq t \leq \min(n, i + j)$$

and

$$t \equiv i + j \pmod 2.$$

Conversely, for every such t there exist vertices u, v, w with $\operatorname{dist}(u, v) = i$, $\operatorname{dist}(v, w) = j$, and $\operatorname{dist}(u, w) = t$. Therefore, the product of labels $i \cdot j$ in $\mathfrak{M}_T$ is precisely the set of all labels t satisfying the above conditions.

If n is odd, then the maximal admissible element of this set is n when $i + j$ is odd and $n - 1$ when $i + j$ is even, which yields exactly the multiplication rules defining the algebra $\mathfrak{M}_o$. If n is even, then n belongs to the even parity class, and the multiplication rules coincide with those defining the algebra $\mathfrak{M}_e$.

Hence the algebra $\mathfrak{M}_T$ is isomorphic to $\mathfrak{M}_o$ when the diameter of G is odd, and to $\mathfrak{M}_e$ when the diameter of G is even. □

Remark 2. If the graph contains at least one simplex, then the algebra for this graph will be isomorphic to the algebra of simplices [11].

For $M = C_3 \nabla C_3$ and $diam(M) = 2$ the algebra will have the set $\{0, 1, 2\}$ of labels and the following multiplication rules:

$$\begin{array}{ll} 0 \cdot 0 = \{0\}, & 0 \cdot 1 = \{1\}, \\ 1 \cdot 0 = \{1\}, & 1 \cdot 1 = \{0, 2\}, \\ 1 \cdot 2 = \{0, 1, 2\}, & 2 \cdot 0 = \{2\}, \\ 2 \cdot 1 = \{0, 1, 2\}, & 2 \cdot 2 = \{0, 1, 2\}. \end{array}$$

For $M = C_m \nabla C_n$, where $2 < m < 5$, the graph M may contain simplices, therefore Theorem 1 extends to include also the algebra for simplices [11].

If $diam(C_n \nabla C_m) = n$, the algebra of simplices $\mathfrak{M}_{\mathfrak{s}}$ has the set $\rho_{\nu(p)} = \{0, 1, 2, \ldots, n\}$ of labels, and is defined by the following label products:

$$\begin{array}{c} 0 \cdot k = \{k\}, \quad k \cdot 0 = \{k\}, \quad \text{for all } k \in \rho, \\ a \cdot b = \{0, 1, 2, \ldots, \min(n, a + b)\}, \qquad a, b \in \rho. \end{array}$$

Theorem 2. If T is the theory of the modular product of the graph C_m with $2 < m < 5$ and polygons, and $\mathfrak{M}$ is the algebra of binary isolating formulas of the theory T, then the algebra $\mathfrak{M}$ is isomorphic to the algebra $\mathfrak{M}_{\mathfrak{o}}$, or $\mathfrak{M}_{\mathfrak{e}}$, or $\mathfrak{M}_{\mathfrak{s}}$.

Proof. Let $2 < m < 5$, so $m \in \{3, 4\}$, and let $n \geq 3$. Put $G = C_m \nabla C_n$ and $T = \operatorname{Th}(G)$.

If the graph G contains a simplex, then by Remark 2 the algebra $\mathfrak{M}_T$ is isomorphic to the algebra of simplices $\mathfrak{M}_s$. Thus, in this case the statement of the theorem holds.

Assume now that G contains no simplex. Then G has no triangles, and the complete 2-types over the empty set are determined by the distances between vertices in connected components of G. Consequently, the labels of $\mathfrak{M}_T$ can again be identified with the possible distances between vertices, and the product of labels is determined by the possible distances obtained by composition through an intermediate vertex.

If at least one of the cycles C_m or C_n is not bipartite, then by Remark 1 the graph G is connected. If both cycles are bipartite, then G consists of two isomorphic connected components. In either case, each connected component of G is a graph of bounded diameter, and the label multiplication is completely determined by the metric structure inside a component.

As in the proof of Theorem 1, the parity of paths in a bipartite component implies that only distances of the same parity as the sum of the factors may occur in a product. Therefore, depending on whether the diameter of G is odd or even, the multiplication table of $\mathfrak{M}_T$ coincides with that of $\mathfrak{M}_o$ or $\mathfrak{M}_e$, respectively.

Thus, if G contains no simplex, the algebra $\mathfrak{M}_T$ is isomorphic to $\mathfrak{M}_o$ or to $\mathfrak{M}_e$. Together with the simplex case considered above, this completes the proof. □

Theorem 3. Let $n, m > 4$. Then the diameter of the modular product $M = C_n \nabla C_m$ satisfies the inequality

$$\operatorname{diam}(M) \leq 3.$$

Proof. Let C_n and C_m be cycles with $n, m > 4$, and let $M = C_n \nabla C_m$ be their modular product. Let $A = (u, v)$ and $B = (x, y)$ be two distinct vertices of M. Denote by

$$a = d_{C_n}(u, x), \qquad b = d_{C_m}(v, y)$$

the distances in the corresponding cycles. Since $A \neq B$, we cannot have $a = b = 0$.

We consider several cases.

Case 1: $u \neq x$ and $v \neq y$ (i.e., $a \geq 1$ and $b \geq 1$).

Subcase 1.1: $a = 1$ and $b = 1$. Then $u \sim x$ and $v \sim y$, hence

$$(u \sim x) \iff (v \sim y),$$

and therefore $A \sim B$. Thus $\operatorname{dist}_M(A, B) = 1$.

Subcase 1.2: $a \geq 2$ and $b \geq 2$. Then $u \nsim x$ and $v \nsim y$, so again

$$(u \sim x) \iff (v \sim y)$$

holds. Hence $A \sim B$ and $\operatorname{dist}_M(A, B) = 1$.

Subcase 1.3: $a = 1$ and $b \geq 2$ (the case $a \geq 2$ and $b = 1$ is symmetric). Then $u \sim x$ and $v \nsim y$, so A and B are not adjacent.

Let u' be the neighbor of x distinct from u. Since $n > 4$, the two neighbors of any vertex in C_n are not adjacent, hence $u' \nsim u$. Similarly, since $b \geq 2$, the vertex v is not adjacent to y. As y has exactly two neighbors in C_m, and v can be adjacent to at most one of them, there exists a neighbor v' of y such that $v' \nsim v$.

Set $C = (u', v')$. Then $A \sim C$, because $u \neq u'$, $v \neq v'$, and $u \nsim u'$, $v \nsim v'$. Moreover, $C \sim B$, since $u' \neq x$, $v' \neq y$, and $u' \sim x$, $v' \sim y$. Thus $A \sim C \sim B$ is a path of length 2, and $\operatorname{dist}_M(A, B) \leq 2$.

Case 2: $u = x$ and $v \neq y$ (i.e., $a = 0$ and $b \geq 1$). (The case $v = y$ and $u \neq x$ is symmetric.)

Since $n > 4$, there exists a vertex $u' \in V(C_n) \setminus \{u\}$ such that $u' \nsim u$. Similarly, since $m > 4$, there exists a vertex $v' \in V(C_m) \setminus \{v, y\}$ such that $v' \nsim v$. Set $C = (u', v')$. Then $A \sim C$ because $u \neq u'$, $v \neq v'$, $u \nsim u'$, and $v \nsim v'$.

Now we consider two subcases.

Subcase 2.1: $v' \nsim y$. Then $C \sim B$ because $u' \neq u$, $v' \neq y$, $u' \nsim u$, and $v' \nsim y$. Hence $A \sim C \sim B$ and $\operatorname{dist}_M(A, B) = 2$.

Subcase 2.2: $v' \sim y$. (Note that $u' \nsim u$ by our choice.) Choose a neighbor u'' of u such that $u'' \nsim u'$. This is possible because in a cycle of length greater than 4, for any vertex u and any vertex u'

with $u' \nsim u$, at least one of the two neighbors of u is not adjacent to u'. Indeed, if both neighbors of u were adjacent to u', then u' would be adjacent to two neighbors of u, which in a cycle of length greater than 4 forces $u' = u$, a contradiction.

Choose a neighbor v'' of y such that $v'' \nsim v'$. This is possible because v' is adjacent to y and, in C_m with $m > 4$, the two neighbors of y are not adjacent; hence the neighbor of y different from v' is not adjacent to v'.

Set $D = (u'', v'')$. Then:

- $C \sim D$, because $u' \neq u''$, $v' \neq v''$, and $u' \nsim u''$, $v' \nsim v''$.
- $D \sim B$, because $u'' \neq u$, $v'' \neq y$, and $u'' \sim u$, $v'' \sim y$.

Thus $A \sim C \sim D \sim B$ is a path of length 3, and $\mathrm{dist}_M(A, B) \leq 3$.

Case 3: $u = x$ and $v = y$ is impossible, since $A \neq B$.

Having considered all cases, we conclude that $\mathrm{dist}_M(A, B) \leq 3$ for any vertices $A, B \in V(M)$. Therefore, $\mathrm{diam}(M) \leq 3$.

The bound is sharp. For example, in $C_5 \nabla C_6$ one has $\mathrm{dist}((0, 0), (0, 3)) = 3$. □

Theorem 4. If $n, m > 4$, then $C_n \nabla C_m$ contains a triangle.

Proof. Number the vertices of the cycles modulo n and modulo m. Take

$$u_1 = 0,\ u_2 = 1,\ u_3 = 3 \in V(C_n), \qquad v_1 = 0,\ v_2 = 1,\ v_3 = 3 \in V(C_m).$$

For $n, m \geq 5$ these vertices are pairwise distinct in each cycle. In each of the cycles,

$$u_1 \sim u_2, \quad u_1 \not\sim u_3, \quad u_2 \not\sim u_3$$

and similarly for v_i. Two vertices a and b are adjacent in C_k if and only if $|a - b| = 1$ or $|a - b| = k - 1$. Since $|0 - 1| = 1$, while $|0 - 3| = 3$, $|1 - 3| = 2$, and for $k \geq 5$ the numbers $2, 3$ are not equal to 1 or $k - 1$. Therefore, for pairs $i \neq j$ it holds that $u_i \sim u_j \iff v_i \sim v_j$, and also $u_i \neq u_j$, $v_i \neq v_j$. By the definition of the modular product, this means that the vertices $P_i = (u_i, v_i)$ are pairwise adjacent in $C_n \nabla C_m$, i.e. they form K_3. □

Theorem 5. Let T be the theory of the modular product of graphs $C_m \nabla C_n$, where $m, n > 5$, and let $\mathfrak{M}$ be the algebra of binary isolating formulas of the theory T. Then the algebra will have, depending on the diameter, two sets of labels: $\{0, 1, 2\}$ or $\{0, 1, 2, 3\}$.

From Theorems 3 and 4 it follows that if $\mathfrak{M}$ has $diam(C_m \nabla C_n) = 2$, the algebra will have the set $\rho_{\nu(p)} = \{0, 1, 2\}$ of labels, and is defined by the following label products:

$$\begin{array}{ll}
0 \cdot 0 = \{0\}, & 0 \cdot 1 = \{1\}, \\
1 \cdot 0 = \{1\}, & 1 \cdot 1 = \{0, 2\}, \\
1 \cdot 2 = \{0, 1, 2\}, & 2 \cdot 0 = \{2\}, \\
2 \cdot 1 = \{0, 1, 2\}, & 2 \cdot 2 = \{0, 1, 2\}.
\end{array}$$

If $diam(C_m \nabla C_n) = 3$, the algebra $\mathfrak{M}$ will have the set $\rho_{\nu(p)} = \{0, 1, 2, 3\}$ of labels, and is defined by the following label products:

$$\begin{array}{ll}
0 \cdot 0 = \{0\}, & 0 \cdot 1 = \{1\}, \\
1 \cdot 0 = \{1\}, & 1 \cdot 1 = \{0, 1, 2\}, \\
1 \cdot 2 = \{0, 1, 3\}, & 1 \cdot 3 = \{0, 1, 2, 3\}, \\
2 \cdot 0 = \{2\}, & 2 \cdot 1 = \{0, 1, 2, 3\}, \\
2 \cdot 2 = \{0, 1, 2, 3\}, & 2 \cdot 3 = \{0, 1, 2, 3\}, \\
3 \cdot 0 = \{3\}, & 3 \cdot 1 = \{0, 1, 2, 3\}, \\
3 \cdot 2 = \{0, 1, 2, 3\}, & 3 \cdot 3 = \{0, 1, 2, 3\}.
\end{array}$$

As we can see, these algebras are isomorphic to the algebras of simplices for similar diameters.

Conclusion

In this work, we investigated the algebraic properties of the modular product of cycles $C_m \nabla C_n$. It was established that for small parameters $2 < m < 5$ the resulting structures admit simplices, and therefore the algebra of binary isolating formulas extends naturally to include the algebra $\mathfrak{M}_{\mathfrak{s}}$. For larger parameters $n, m > 4$, we proved that the diameter of the modular product is bounded above by 3, and moreover, such graphs necessarily contain triangles.

As a consequence, the algebra $\mathfrak{M}$ associated with the theory of $C_m \nabla C_n$ can be fully characterized by its set of labels, which is $\{0, 1, 2\}$ when the diameter equals 2, and $\{0, 1, 2, 3\}$ when the diameter equals 3. In both cases, the multiplication tables of $\mathfrak{M}$ are shown to be isomorphic to the corresponding simplicial algebras.

Thus, the modular product of cycles not only demonstrates bounded diameter and the presence of complete subgraphs, but also provides a natural algebraic structure that generalizes and unifies previously studied cases. This highlights the deep interplay between graph-theoretic properties of modular products and the algebraic theory of binary isolating formulas.

Examples of derived graphs in modular and other products with their Cayley tables can be viewed on the website https://graph-product.ru

Conflict of Interest

The author declares, no conflict of interest.

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