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Measures and Stability in a Model, revisited

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This article is written in honor of the 8th Kazakh–French Logical Colloquium. We expand on an unpublished research note of the second author. We record some results concerning local Keisler measures with respect to a formula which is *stable in a model*. We prove that in this context, every local Keisler measure on the associated local type space is a weighted sum of (at most countably many) local types. Using this observation, we give an elementary proof of the commutativity of the Morley product in this context. We then give a functional analytic proof that the double limit property lifts to the appropriate evaluation map on pairs of local measures. We conclude with observations regarding the NOP and local Keisler measures in the (properly) stable context. Finally, we provide two proofs that the evaluation map on pairs of local Keisler measures is stable (in continuous logic). The first follows almost immediately from the work of Ben Yaacov and Keisler on the randomization; the other proof follows from the VC theorem.

Keywords: model theory, Keisler measures, stability in a model, stability, Morley product, double limit, randomization, VC theory, Krein-Smulian, functional analysis.

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Introduction

The Franco–Kazakh connections in mathematical logic date back to the late 1980s with the collaboration between Tolendi Mustafin and Bruno Poizat, which led to the first Soviet–French Colloquium in Model Theory held at Karaganda State University in Kazakhstan in 1990. Since then, the determination to maintain and strengthen these mathematical links has persisted and was rekindled in recent years, first in Lyon in 2022, and then in Astana in 2025. The scope of those connections naturally go far beyond France and Kazakhstan, as they witness the fruitful transfer of knowledge and cross-fertilization of ideas among researchers from Europe, Russia, the United States, and China. The present paper embodies this transversality, and presents work stemming (in part) from the authors' participation in the *8th Kazakh-French Logical Colloquium*. The authors are very grateful to the organizers of that meeting. They hope that their contribution to the Kazakh mathematical library will help strengthen the preexisting intellectual and social relationships between our research groups.

After Ben Yaacov's original article connecting stability theory with some of Grothendieck's functional analytic work [1], the concepts of *stability in a model* and *NIP in a model* were studied by a myriad of researchers in the field. In particular, it was the subject of several intense discussions at the Notre Dame model theory seminar in the Spring of 2017 (e.g., see [2]; in the context of a group [3];

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the NIP variant [4]; a localized history of the subject in Persian [5]). This research arose from that localized frenzy of activity.

It turns out that fundamental results in local stability theory can be generalized to the context of *stability in a model* – in particular via a reinterpretation of some of Grothendieck's work on functional analysis. From our perspective, it is natural to ask, *What does the theory of Keisler measures look like in this setting?* We show that it also closely resembles the picture in the stable context. First, we show that if $\varphi(x, y)$ is stable in M, then every φ-measure on M (finitely additive probability measure on φ-definable subsets of M^x) is the sum of (at most countably many) weighted φ-types. As a consequence, all φ-measures on M are φ^{opp}-definable. Thus evaluating the Morley product between arbitrary pairs of φ and φ^{opp}-measures on the formula $\varphi(x, y)$ is well-defined. In [1], Ben Yaacov demonstrates that the *fundamental theorem of stability theory* extends to the *stable in a model* context; it then follows from the observations above that the Morley product evaluated at $\varphi(x, y)$ commutes on appropriate pairs of measures. In other words, if $\varphi(x, y)$ is stable in M, $\mu \in \mathfrak{M}_\varphi(M)$, and $\nu \in \mathfrak{M}_{\varphi^{\text{opp}}}(M)$, then

$$(\mu_x \otimes \nu_y)(\varphi(x, y)) = (\nu_y \otimes \mu_x)(\varphi(x, y)).$$

We are also interested in the evaluation map itself. We consider the function $E_\varphi : \mathfrak{M}_\varphi(M) \times \mathfrak{M}_{\varphi^{\text{opp}}}(M) \to [0, 1]$ by

$$E_\varphi(\mu, \nu) = (\mu_x \otimes \nu_y)(\varphi(x, y)).$$

We show that if $\varphi(x, y)$ is stable in M, then E_φ also witnesses the appropriate variant of the *non-order property.* This follows more or less directly from results in functional analysis, i.e., Grothendieck's double limit theorem and the Krein–Smulian theorem. Finally, we consider the context, where $\varphi(x, y)$ is a stable formula (i.e., $\varphi(x, y)$ does not have the k-order property for some fixed k). We prove that the map E_φ is (r, ϵ)-stable for any choice of $r \in (0, 1)$ and $\epsilon > 0$. We remark that this follows implicitly by a result of Ben Yaacov and Keisler, namely the fact that the randomization of a stable formula remains stable ([6, Theorem 5.14]). We exposit why this is true and then provide a different proof which follows from the VC-theorem, mimicking techniques involving measures in NIP theories (i.e., see the proof of [7, Theorem 3.12]). This latter method implies the existence of bounds, yet we leave analysis along these lines open.

1 *Preliminaries*

Throughout the article, fix a language $\mathcal{L}$ and an $\mathcal{L}$-structure, M. We use the letters $x, y, z, \ldots$ to denote finite tuples of variables. The formula $\varphi(x, y)$ is a partitioned $\mathcal{L}$-formula with *variable* tuple x and *parameter* tuple y. We let $\varphi^{\text{opp}}(y, x)$ be the same formula as $\varphi(x, y)$, but with exchanged roles for the variables and parameters. We let $S_\varphi(M)$ be the space of φ-types with parameters from M. We let $\text{Def}_\varphi(M)$ be the Boolean algebra of definable subsets of M generated by $\{\varphi(x, b) : b \in M\}$. We will routinely identify definable sets with the formulas which define them. A φ-formula is an element of $\text{Def}_\varphi(M)$. Likewise, we have analogous definitions for $S_{\varphi^{\text{opp}}}(M)$ and $\text{Def}_{\varphi^{\text{opp}}}(M)$. A φ^{opp}-definition for a type p in $S_\varphi(M)$ is a φ^{opp}-formula, $d_p^{\varphi^{\text{opp}}}(y)$, such that for each $b \in M^y$, $\varphi(x, b) \in p$ if and only if $M \models d_p^{\varphi^{\text{opp}}}(b)$. Finally, we let $\mathfrak{M}_\varphi(M)$ and $\mathfrak{M}_{\varphi^{\text{opp}}}(M)$ denote the spaces of finitely additive probability measures on $\text{Def}_\varphi(M)$ and $\text{Def}_{\varphi^{\text{opp}}}(M)$ respectively. We recall that we can identify a measure in each of these spaces canonically with a regular Borel probability measure on their corresponding type space, e.g. $\mathfrak{M}_\varphi(M)$ is in canonical correspondence with regular Borel probability measures on $S_\varphi(M)$. For some helpful background on topics including continuous logic and stability, we refer the reader to [8].

Definition 1 (Double Limit Property). Let X and Y be sets and let $f : X \times Y \to [0, 1]$. We say that f has the *double limit property* if for any two sequence $(a_i)_{i \in \mathbb{N}}$, $(b_j)_{j \in \mathbb{N}}$ with $a_i \in X$ and $b_j \in Y$,

$$\lim_i \lim_j f(a_i, b_j) = \lim_j \lim_i f(a_i, b_j)$$

provided limits on both sides exist.

The definition of *stability in a model* given in [1] restricted to discrete structures is as follows:

Definition 2. A formula $\varphi(x, y)$ is *stable in* M if $\varphi : M^x \times M^y \to \{0, 1\}$ has the double limit property, where $\varphi(a, b) = 1$ if $M \models \varphi(a, b)$ and $\varphi(a, b) = 0$ otherwise.

We first remark that clearly $\varphi(x, y)$ is stable in M if and only if $\varphi^{\text{opp}}(y, x)$ is stable in M. We also remark that if $\varphi(x, y)$ is stable, then $\varphi(x, y)$ is stable in any model of the underlying theory. On the other hand, if $\varphi(x, y)$ is stable in a model M, it does not imply that it is stable in an elementary extension of M. An example of a formula which is stable in a model but not stable is the edge relation in the graph constructed by taking the disjoint union of all finite graphs on subsets of $\mathbb{N}$.

In [1], Ben Yaacov established a surprising connection between functional analysis and local stability. In particular, he gave a proof of the *fundamental theorem of stability* using Grothendieck's *double limit theorem* [9]. Before stating the theorem, let us briefly recall some basic functional analysis.

Definition 3 (Weak Topology). Let Y be a Banach space over a field F ($F = \mathbb{R}$ or $\mathbb{C}$). Let Y^* be the space of continuous linear functionals from Y to F. Then, the *weak topology on* Y is the coarsest topology such that each element of Y^* remains a continuous function from Y to F.

Definition 4 (Relatively Weakly Compact). Let Y be a Banach space and let $A \subset Y$. We say that A is *weakly compact* if A is a compact subset of Y under the weak topology. Furthermore, we say that A is *relatively weakly compact* if the closure of A under the weak topology is weakly compact.

Let X be a topological space. Then $C_b(X)$ denotes the Banach space of bounded, continuous, complex-valued functions on X, equipped with the uniform norm, $||\cdot||_\infty$. We say that a set A is $||\cdot||_\infty$-bounded if there exists c in $\mathbb{R}$ such that for all f in A, $||f||_\infty < c$. Grothendieck's theorem (as formulated in [1]) is as follows:

Theorem 1 (Grothendieck [9]). Let X be an arbitrary topological space, $X_0 \subseteq X$ a dense subset. Then the following are equivalent for $A \subset C_b(X)$:
(i) The set A is relatively weakly compact in $C_b(X)$.
(ii) The set A is $||\cdot||_\infty$-bounded, and whenever $f_n \in A$ and $x_n \in X_0$ form two sequences we have that

$$\lim_n \lim_m f_n(x_m) = \lim_m \lim_n f_n(x_m),$$

provided both limits exist.

Via the double limit theorem above, Ben Yaacov derived the following (among other results):

Theorem 2. Assume that $\varphi(x, y)$ is stable in M, $p \in S_\varphi(M)$, and $q \in S_{\varphi^{\text{opp}}}(M)$. Then p has a φ^{opp}-definition $d_p^\varphi(y)$, q has a φ-definition $d_q^{\varphi^{\text{opp}}}(x)$, and $d_p^\varphi(y) \in q$ if and only if $d_q^{\varphi^{\text{opp}}}(x) \in p$.

Finally, we recall the following fact.

Fact 1. The following are equivalent:
(i) $\varphi(x, y)$ is stable in M.
(ii) There does not exist $(a_i, b_j)_{(i,j)\in\omega\times\omega}$ from $M^x \times M^y$such that
 (a) either for every $i \neq j$, $M \models \varphi(a_i, b_j)$ if and only if $i < j$,
 (b) or for all $i \neq j$, $M \models \varphi(a_i, b_j)$ if and only if $i > j$.
(iii) The map $\chi_\varphi : S_\varphi(M) \times S_{\varphi^{\text{opp}}}(M) \to \{0, 1\}$ has the double limit property where

$$\chi_\varphi(p, q) = 1 \iff d_p^\varphi(y) \in q \iff d_q^{\varphi^{\text{opp}}}(x) \in p.$$

In the above fact, the equivalence of *(i)* and *(ii)* can be found in [2, Proposition 2.3]. Clearly, *(iii)* implies *(i)* and *(i)* implies *(iii)* follows from Theorem 1 together with Theorem 2. We remark that a proof of these equivalences can also be found in Starchenko's unpublished research note on the topic.

2 Local Keisler measures

In this section, we prove that if $\varphi(x,y)$ is stable in M, then all φ-measures are (at most) countable sums of weighted φ-types. The proof of this theorem uses the Sobczyk–Hammer decomposition theorem for positive, bounded charges. We recall this theorem in the case of finitely additive probability measures. But first, we need to recall two different kinds of finitely additive measures.

Definition 5. Let $\mathbb{B}$ be a Boolean algebra of subsets of X (containing both X and $\emptyset$) and μ be a finitely additive probability measure on $\mathbb{B}$.

1. We say that μ is *strongly continuous* on $\mathbb{B}$ if for all $\epsilon > 0$ there exist $F_1, ..., F_n \in \mathbb{B}$ such that $\{F_i\}_{i=1}^n$ forms a partition of X and $\mu(F_i) < \epsilon$, for each $i \leq n$.
2. We say that μ is $\{0,1\}$*-valued* on $\mathbb{B}$ if for every F in $\mathbb{B}$, $\mu(F) \in \{0,1\}$.

We refer the reader to [10, Theorem 5.2.7, p. 146] for a proof of the following theorem.

Theorem 3 (Sobczyk–Hammer Decomposition Theorem). Let $\mathbb{B}$ be a Boolean algebra on X (containing $\emptyset$ and X) and μ be a finitely additive probability measure on $\mathbb{B}$. Then, there exists an initial segment I of $\mathbb{N}$, a sequence of distinct finitely additive probability measures $(\mu_i)_{i\in I}$, and a sequence of non-negative real numbers $(r_i)_{i\in I}$, with the following properties:

(i) μ_0 is strongly continuous on $\mathbb{B}$,
(ii) μ_i is $\{0,1\}$-valued on $\mathbb{B}$ for every $i \geq 1$,
(iii) $\sum_{i\in I} r_i = 1$, and
(iv) $\mu = \sum_{i\in I} r_i\mu_i$.

Further, the decomposition in *(iv)* is unique (obviously, up to permutation of the sequence and non-trivially weighted measures (i.e., $r_i > 0$)).

The Sobczyk–Hammer decomposition theorem allows us to decompose any finitely additive probability measure into a single strongly continuous measure and a sum of (at most countably many) $\{0,1\}$-valued measures. We will show that if $\varphi(x,y)$ is stable in M, then there do not exist any strongly continuous measures on $\mathrm{Def}_\varphi(M)$. Thus every finitely additive probability measure will be the "weighted sum" of at most countably many types.

2.1 Measures are sums of types

Definition 6. Let $\mathbb{B}$ be a Boolean algebra on a set X. We say that $\mathbb{B}$ has a 2*-tree* if there exists $T \in \mathcal{P}(\mathbb{B})$ such that $(T, \supsetneq)$ is an infinite, complete, binary tree, and if $A, C \in T$, $A \not\supset C$, and $C \not\supset A$, then $A \cap C = \emptyset$.

Fact 2. Let $\mathbb{B}$ be a Boolean algebra on a set X and assume that $\mathbb{B}$ has a 2-tree. Then $|\mathrm{Ult}(\mathbb{B})| \geq 2^{\aleph_0}$ where $\mathrm{Ult}(\mathbb{B})$ is the set of ultrafilters on $\mathbb{B}$.

Proof. Let $A_\gamma = \{B \in T : B \in \gamma\}$ for a given path γ in T. Clearly, A_γ has the finite intersection property (since if $B, C \in A_\gamma$, then either $B \subset C$ or $C \subset B$) and so A_γ can be extended to an ultrafilter over $\mathbb{B}$. This construction gives an injective map from paths in T into ultrafilters on $\mathbb{B}$, proving the claim. □

Lemma 1. Let $\mathbb{B}$ be a Boolean algebra on a set X. Assume that there exists a strongly continuous measure μ over $\mathbb{B}$. Then $\mathbb{B}$ has a 2-tree.

Proof. Using μ, we will build a 2-tree in steps.
Stage 0*:* Let $T_0 = \{X\}$.
Stage $n+1$*:* We construct a tree of height $n+1$. Assume that T_n is a (complete) binary tree of height n such that $\mu(A) > 0$, for each $A \in T_n$. Assume furthermore that if $A, B \in T$ and $A \not\supset B$ and $B \not\supset A$, then $A \cap B = \emptyset$. We will construct T_{n+1} by adding two children to each leaf. Let $\mathbb{L}_n$

be the collection of leaves on T_n. Let $\epsilon = \frac{\min\{\mu(L):L\in\mathbb{L}_n\}}{2}$. Since μ is strongly continuous, there exist $H_1, ..., H_m \in \mathbb{B}$ such that $\mathbb{H} = \{H_1, ..., H_m\}$ partitions X and for each $j \leq m$, $\mu(H_j) < \epsilon$. Now fix a leaf L_i. Consider $L_i \cap \mathbb{H} = \{L_i \cap H : H \in \mathbb{H}\}$. We notice that $L_i \cap \mathbb{H}$ forms a partition of L_i. Therefore,

$$0 < \mu(L_i) = \mu\Big(\bigcup_{K\in L_i\cap\mathbb{H}} K\Big) = \sum_{K\in L_i\cap\mathbb{H}} \mu(K).$$

Hence, there exists $K_r = L_i \cap H_r \in L_i \cap \mathbb{H}$ such that $\mu(K_r) > 0$. Furthermore,

$$\mu(K_r) = \mu(L_i \cap H_r) \leq \mu(H_r) < \epsilon \leq \frac{L_i}{2}.$$

Therefore there must exist some $K_l \in L_i \cap \mathbb{H}$ such that $K_l \neq K_r$ and $\mu(K_l) > 0$. We now add K_r, K_l as children to the leaf L_i. Let T_{n+1} be the tree constructed after repeating this process for each $L \in \mathbb{L}_n$. Clearly, T_{n+1} is a binary tree of height $n + 1$ such that $\mu(A) > 0$ for each $A \in T_{n+1}$.

Now let $T = \bigcup_{n\geq 0} T_n$. T is clearly a 2-tree by construction. □

Definition 7. Let M_φ be the reduct of M to the language $\mathcal{L}_\varphi = \{\varphi(x, y)\}$. A subset N of M is a φ-substructure of M, written $N \prec_\varphi M$, if the induced structure on N (in the language $\mathcal{L}_\varphi$) is an elementary substructure of M_φ.

Lemma 2. Assume that $\varphi(x, y)$ is stable in M. Then there are no strongly continuous measures on $\mathrm{Def}_\varphi(M)$.

Proof. Assume that there exists a strongly continuous measure over $\mathrm{Def}_\varphi(M)$. By Lemma 1, there exists a 2-tree. Let $\mathbb{B}_0$ be the Boolean algebra generated by this 2-tree. By Fact 2, $\mathbb{B}_0$ is a countable subalgebra of $\mathrm{Def}_\varphi(M)$ such that $|\mathrm{Ult}(\mathbb{B}_0)| \geq 2^{\aleph_0}$. Choose $C \subset M$ such that for each $B \in \mathbb{B}_0$, there exists $b_1, ..., b_n$ in C such that B is an element of the boolean algebra generated by $\{\varphi(x, b_i) : i \leq n\}$. Notice that since $\mathbb{B}$ is countable, we may choose C to be countable. By the Downward Löwenheim–Skolem theorem, there exists an $\mathcal{L}_\varphi$-structure N such that $C \subseteq N$, $N \prec_\varphi M$, and $|N| = \aleph_0$. Then,

$$2^{\aleph_0} \leq |\mathrm{Ult}(\mathbb{B}_0)| \leq |\mathrm{Ult}(\mathrm{Def}_\varphi(C))| \leq |\mathrm{Ult}(\mathrm{Def}_\varphi(N))| = |S_\varphi(N)|.$$

However, since $\varphi(x, y)$ is stable in M, it is also stable in N. By Theorem 2, every φ-type over N is definable by a φ^{opp}-formula with parameters from N. Since $|N| = \aleph_0$, there are only countably many φ^{opp}-formulas. Therefore, not every φ-type over N is definable — a contradiction. □

Theorem 4. Let $\varphi(x, y)$ be stable in M and let $\mu \in \mathfrak{M}_\varphi(M)$. Then there exists an initial segment I of $\mathbb{N}$ such that $\mu = \sum_{i\in I} r_i \delta_{p_i}$ where $p_i \in S_\varphi(M)$, $\sum_{i\in I} r_i = 1$, and each $r_i > 0$. Obviously, the statement also holds when $\varphi(x, y)$ is *stable* (i.e., does not admit the k-order property for some k).

Proof. Direct from the Sobczyk–Hammer Decomposition Theorem and Lemma 2. □

2.2 The Morley product is commutative

We now aim to show that the Morley product commutes on appropriate pairs of measures. First, we need to appropriately define what we mean by the *Morley product* in this context. To define it, we make some quick observations.

Fact 3. Suppose that X is a topological space and Y is a dense subset of X. Let $f : Y \to Z$ be a map. If there exists some $\tilde{f} : X \to Z$ such that $\tilde{f}$ is continuous and $\tilde{f}|_Y = f$, then $\tilde{f}$ is the unique function with such property.

Proof. Clear via the net definition of continuity. □

Proposition 1. Suppose that $\varphi(x,y)$ is stable in M and $\mu \in \mathfrak{M}_{\varphi}(M)$. Consider the map $f^{\varphi}_{\mu} : \{\mathrm{tp}_{\varphi^{\mathrm{opp}}}(b/M) : b \in M^{y}\} \to [0,1]$ via $f^{\varphi}_{\mu}(\mathrm{tp}_{\varphi^{\mathrm{opp}}}(b/M)) = \mu(\varphi(x,b))$. This map is well-defined and there exists a unique continuous function $F^{\varphi}_{\mu} : S_{\varphi^{\mathrm{opp}}}(M) \to [0,1]$ such that $F^{\varphi}_{\mu}|_{\{\mathrm{tp}_{\varphi^{\mathrm{opp}}}(b/M): b \in M^{y}\}} = f^{\varphi}_{\mu}$.

Proof. By Theorem 4, $\mu = \sum_{i\in I} r_i \delta_{p_i}$ where each $p_i \in S_{\varphi}(M)$. We argue that the map f^{φ}_{μ} is well-defined. Notice that if $b \in M^{y}$, then

$$\mu(\varphi(x,b)) = \sum_{i\in I} r_i[\delta_{p_i}(\varphi(x,b))] = \sum_{\substack{i\in I\\ M\models d^{\varphi}_{p_i}(b)}} r_i.$$

By Theorem 2, each formula $d^{\varphi}_{p_i}(y)$ is a φ^{opp}-formula which implies that the value above only depends on the φ^{opp}-type of b, hence f^{φ}_{μ} is indeed well-defined.

Since $\{\mathrm{tp}_{\varphi^{\mathrm{opp}}}(b/M) : b \in M^{y}\}$ is a dense subset of $S_{\varphi^{\mathrm{opp}}}(M)$, by Fact 3 it suffices to prove that there exists a continuous map from $S_{\varphi^{\mathrm{opp}}}(M)$ to $[0,1]$ which restricts to f^{φ}_{μ}. We claim that $\sum_{i\in I} r_i \mathbf{1}_{[d^{\varphi}_{p_i}(y)]}$ is the appropriate map. We remark that we may view $\sum_{i\in I} r_i \mathbf{1}_{[d^{\varphi}_{p_i}(y)]}$ as a map from $S_{\varphi^{\mathrm{opp}}}(M)$ to $[0,1]$ since stability in M implies that every formula of the form $d^{\varphi}_{p_i}(y)$ is a φ^{opp}-formula (Theorem 2). □

We may now define the *Morley product* in this setting.

Definition 8. Suppose that $\varphi(x,y)$ is stable in M. Let $\mu \in \mathfrak{M}_{\varphi}(M)$ and $\nu \in \mathfrak{M}_{\varphi^{\mathrm{opp}}}(M)$. We define the Morley product of μ with ν, denoted $\mu_x \otimes \nu_y$, as follows:

$$(\mu \otimes \nu)(\varphi(x,y)) = \int_{S_{\varphi^{\mathrm{opp}}}(M)} F^{\varphi}_{\mu} d\tilde{\nu},$$

where F^{φ}_{μ} is the function from Proposition 1 and $\tilde{\nu}$ is the regular Borel probability measures corresponding to ν. Likewise, since $\varphi(x,y)$ is stable in M if and only if $\varphi^{\mathrm{opp}}(x,y)$ is stable in M, we may also define

$$(\nu \otimes \mu)(\varphi(x,y)) = \int_{S_{\varphi}(M)} F^{\varphi^{\mathrm{opp}}}_{\nu} d\tilde{\mu},$$

with the obvious analogous definitions.

Remark 1. Since our definition of the Morley product is slightly non-standard, we are careful to make sure it resembles the normal Morley product on types. Suppose that $\varphi(x,y)$ is stable in M, let $p \in S_{\varphi}(M)$, $q \in S_{\varphi^{\mathrm{opp}}}(M)$, and fix $\mathcal{U}$ such that $M \prec \mathcal{U}$. Let $\hat{p} \in S_{\varphi}(\mathcal{U})$ be the unique M-definable extension of p to $\mathcal{U}$. Then $(\delta_p \otimes \delta_q)(\varphi(x,y)) = 1$ if and only if $\mathcal{U} \models \varphi(a,b)$, where $b \models q$ and $a \models \hat{p}|_{Mb}$. Indeed, consider the following sequence of bi-implications:

$$\begin{aligned}(\delta_p \otimes \delta_q)(\varphi(x,y)) = 1 \iff \int_{S_{\varphi^{\mathrm{opp}}}(M)} \chi_{[d^{\varphi}_{p}(y)]} d\delta_q = 1 \iff \delta_q(d^{\varphi}_{p}(y)) = 1\\ \iff d^{\varphi}_{p}(y) \in q \iff \mathcal{U} \models d^{\varphi}_{p}(b) \iff \varphi(x,b) \in \hat{p} \iff \mathcal{U} \models \varphi(a,b).\end{aligned}$$

Theorem 5. Suppose that $\varphi(x,y)$ is stable in M. Then

$$(\mu \otimes \nu)(\varphi(x,y)) = (\nu \otimes \mu)(\varphi(x,y)).$$

Proof. Consider the following sequence of equations:

$$\begin{aligned}(\mu \otimes \nu)(\varphi(x,y)) &= \int_{S_{\varphi^{\mathrm{opp}}}(M)} F^{\varphi}_{\mu} d\tilde{\nu} = \int_{S_{\varphi^{\mathrm{opp}}}(M)} \sum_{i\in I} r_i \mathbf{1}_{[d^{\varphi}_{p_i}(y)]} d\tilde{\nu}\\ &= \sum_{i\in I} r_i \nu(d^{\varphi}_{p_i}(y)) = \sum_{i\in I} r_i \sum_{j\in J} s_j \delta_{q_j}(d^{\varphi}_{p_i}(y)) \overset{(*)}{=} \sum_{i\in I}\sum_{j\in J} r_i s_j \delta_{p_i}(d^{\varphi^{\mathrm{opp}}}_{q_j}(x)).\end{aligned}$$

Equation $(*)$ is justified by Theorem 2. A symmetric computation shows

$$(\nu \otimes \mu)(\varphi(x,y)) = \sum_{i \in I} \sum_{j \in J} r_i s_j \delta_{p_i}(d_{q_j}^{\varphi^{\mathrm{opp}}}(x)),$$

completing the proof. □

2.3 Some functional analysis and double limits

By Theorem 5, we may define the following evaluation map, E_φ, on appropriate pairs of Keisler measures. We prove that if $\varphi(x,y)$ is stable in M, then E_φ also has the double limit property. Our proof follows directly from classical results in functional analysis, namely Grothendieck's double limit theorem and the Krein-Smulian theorem. For other applications of functional analysis in this area, we refer the reader to [11] and [12]. We first recall the definition of the evaluation map from the introduction.

Definition 9. Suppose that $\varphi(x,y)$ is stable in M. Then we define the map $E_\varphi : \mathfrak{M}_\varphi(M) \times \mathfrak{M}_{\varphi^{\mathrm{opp}}}(M) \to [0,1]$ via

$$E_\varphi(\mu,\nu) = (\mu \otimes \nu)(\varphi(x,y)).$$

By Theorem 5, $E_\varphi(\mu,\nu)$ is also equal to $(\nu \otimes \mu)(\varphi(x,y))$.

A proof of the following theorem can be found in most graduate textbooks on functional analysis.

Theorem 6 (Krein-Smulian Theorem). If Y is a Banach space and K is weakly compact subset of Y, then the closed convex hull of K, denoted $\overline{\mathrm{co}}(K)$, is weakly compact. The closed convex hull of K is the intersection of all norm closed, convex subsets of Y containing K.

Corollary 1. If Y is a Banach space and Z is a relatively weakly compact subset of Y, then $\overline{\mathrm{co}}(Z)$ is a weakly compact subset of Y.

Proof. Let Z^w denote the weak closure of Z. Then Z^w is weakly compact and so by the Krein–Smulian theorem, $\overline{\mathrm{co}}(Z^w)$ is weakly compact. Note that $\overline{\mathrm{co}}(Z) \subseteq \overline{\mathrm{co}}(Z^w)$ and since $\overline{\mathrm{co}}(Z)$ is a closed subset of a compact set, it is also compact. □

Definition 10. Suppose that X is a set. If we endow X with the discrete topology and let $\mathcal{M}(X)$ be the collection of regular Borel probability measures on X, then we can consider the set of finitely supported probability measures (the convex hull of the Dirac measures):

$$\mathrm{conv}_\delta(X) \colon = \left\{ \sum_{i \in I} r_i \delta_{x_i} : I \subseteq \mathbb{N}, x_i \in X, r_i \in \mathbb{R}_{\geq 0}, \sum_{i \in I} r_i = 1 \right\}.$$

For simplicity of notation, we write $\sum_{i \in I} r_i \delta_{x_i}$ simply as $\sum_{i \in I} r_i x_i$.

Definition 11. Suppose that X and Y are sets and $f : X \times Y \to [0,1]$. Then we define $f_c : \mathrm{conv}_\delta(X) \times \mathrm{conv}_\delta(Y) \to [0,1]$ via

$$f_c\left(\sum_{i \in I} r_i x_i, \sum_{j \in J} s_j y_j \right) = \sum_{i \in I} \sum_{j \in J} r_i s_j f(x_i, y_j).$$

We now prove the key combinatorial proposition via functional analysis.

Proposition 2. Suppose that X and Y are sets and $f : X \times Y \to [0,1]$. Then f has the double limit property if and only if $f_c : \mathrm{conv}_\delta(X) \times \mathrm{conv}_\delta(Y) \to [0,1]$ has the double limit property.

Proof. If f_c has the double limit property, then clearly f has the double limit property. Therefore, we only need to prove the other direction. Endow Y with the discrete topology. Let $\mathbf{X} = \{f(a, y) : a \in X\}$. It is obvious that $\mathbf{X} \subset C_b(Y)$ and $\mathbf{X}$ is $||\cdot||_\infty$-bounded. By assumption, f has the double limit property and by Theorem 1, $\mathbf{X}$ is relatively weakly compact in $C_b(Y)$. By Corollary 6, we have that $\overline{\text{co}}(\mathbf{X})$ is a weakly compact subset of $C_b(Y)$. Since $\overline{\text{co}}(\mathbf{X})$ is weakly compact in $C_b(Y)$, it is also relatively weakly compact, and so we can apply Theorem 1. So, for any infinite sequences $g_i \in \overline{\text{co}}(\mathbf{X})$ and $b_j \in Y$

$$\lim_i \lim_j g_i(b_j) = \lim_j \lim_i g_i(b_j),$$

provided both limits exist. In particular, this implies that for $g_i := \sum_{\ell_i \in L_i} r_{\ell_i} f(a_{\ell_i}, y)$,

$$\lim_i \lim_j f_c \left(\sum_{\ell_i \in L_i} r_{\ell_i} a_{\ell_i}, b_j \right) = \lim_j \lim_i f_c \left(\sum_{\ell_i \in L_i} r_{\ell_i} a_{\ell_i}, b_j \right),$$

provided both limits exist.

Notice that the computation above demonstrates that the map $f_c|_{\text{conv}_\delta(X)\times Y}$ has the double limit property. Now consider $\text{conv}_\delta(X)$ endowed with the discrete topology and let $\mathbf{Y} = \{f_c(x, b) : b \in Y\}$. It is clear that $\mathbf{Y} \subset C_b(\text{conv}_\delta(X))$ and that $\mathbf{Y}$ is $||\cdot||_\infty$-bounded since each function is bounded by 1. Since $f_c|_{\text{conv}_\delta(X)\times Y}$ has the double limit property, we can again apply Theorem 1 and so $\mathbf{Y}$ is relatively weakly compact in $C_b(\text{conv}_\delta(X))$. By Corollary 6, $\overline{\text{co}}(\mathbf{Y})$ is weakly compact in $C_b(\text{conv}_\delta(X))$. By Theorem 1, f_c has the double limit property. ☐

Corollary 2. If $\varphi(x, y)$ is stable in M, then the map $E_\varphi : \mathfrak{M}_\varphi(M) \times \mathfrak{M}_{\varphi^{\text{opp}}}(M) \to [0, 1]$ has the double limit property.

Proof. By Fact 1, the map $\chi_\varphi : S_\varphi(M) \times S_{\varphi^{\text{opp}}}(M) \to \{0, 1\}$ has the double limit property. By Theorem 4, we have that $\mathfrak{M}_\varphi(M) = \text{conv}_\delta(S_\varphi(M))$ and since φ^{opp} is also stable in M, $\mathfrak{M}_{\varphi^{\text{opp}}}(M) = \text{conv}_\delta(S_{\varphi^{\text{opp}}})$. The computation in Theorem 5 demonstrates that $E_\varphi = (\chi_\varphi)_c$. By Proposition 2, E_φ has the double limit property. ☐

3 *Proper stability and the order property*

In this section, we work with *honest-to-goodness* stable formulas and give a proof of an implicit theorem of Ben Yaacov and Keisler. Another proof of this theorem is given by Khanaki and Pourmahdian using indiscernible arrays (see [13, Theorem 3.11]). We show that if $\varphi(x, y)$ is stable, then the evaluation map E_φ does not witness the continuous logic analogue of the order property. Throughout this section, we fix $\mathcal{L}$-structures M and $\mathcal{U}$ such that $M \prec \mathcal{U}$ and $\mathcal{U}$ is a monster model. We let T be the theory of M in the language $\mathcal{L}$. We first show how to use the randomization to derive a proof. We then give another proof using the VC theorem. Given a theory T, we denote by T^R its randomization. We refer the reader to Section 3.2 of [14] for background and notation regarding the randomization.

The following fact is due to Ben Yaacov and Keisler [6, Theorem 5.14].

Fact 4. Suppose that $\varphi(x, y)$ is a stable formula with respect to T. Then the randomized formula $\mathbb{E}[\varphi(x, y)]$ is stable (in the sense of continuous logic) with respect to T^R. In other words, if N is a model of T^R then for every $r \in (0, 1)$ and $\epsilon > 0$, there exists some integer $n = n(\epsilon, r)$ such that there does not exist an array of elements $(\mathbf{a}_i, \mathbf{b}_j)_{(i,j)\in[n]\times[n]}$ from $N^x \times N^y$ such that

$$\mathbb{E}[\varphi(\mathbf{a}_i, \mathbf{b}_j)] \geq r + \epsilon \quad \text{whenever} \quad i \geq j$$

and

$$\mathbb{E}[\varphi(\mathbf{a}_i, \mathbf{b}_j)] \leq r \quad \text{whenever} \quad i < j.$$

Note that the integer n does not depend on the choice of the model N.

Using the above, it is easy to see that E_φ also does not witness the continuous version of the order property. This follows from the observation that the randomization encodes the computations of the Morley product.

Proposition 3. Suppose that $\varphi(x, y)$ is stable. For every $r \in (0, 1)$ and $\epsilon > 0$, there exists some integer $n = n(\epsilon, r)$ such that there does not exist an array of Keisler measures $(\mu_i, \nu_j)_{(i,j)\in[n]\times[n]}$, where $\mu_i \in \mathfrak{M}_\varphi(M)$ and $\nu_j \in \mathfrak{M}_{\varphi^{\text{opp}}}(M)$ such that

$$(\mu_i \otimes \nu_j)(\varphi(x, y)) \geq r + \epsilon, \quad \text{whenever} \quad i \geq j$$

and

$$(\mu_i \otimes \nu_j)(\varphi(x, y)) \leq r, \quad \text{whenever} \quad i < j.$$

Proof. Consider $[0, 1]^2$ with the corresponding Lebesgue measure L^2 and the simple models of the randomization of T relative to $[0, 1]^2$, namely $M^{[0,1]^2}$ and $\mathcal{U}^{[0,1]^2}$. More explicitly, if N is a model of T then $N^{[0,1]}$ is the collection of measurable maps from $[0, 1]^2$ to N with finite image. It follows from quantifier elimination of T^R that $M^{[0,1]^2} \prec \mathcal{U}^{[0,1]^2}$. If $\mu \in \mathfrak{M}_\varphi(M)$ and $\nu \in \mathfrak{M}_{\varphi^{\text{opp}}}(M)$, then Theorem 4 implies that $\mu = \sum_{k\in K} r_k\delta_{p_k}$ and $\nu = \sum_{w\in W} d_w\delta_{q_w}$ where K and W are initial segments of $\mathbb{N}$ and

1) for each $k \in K$, p_k is in $S_\varphi(M)$,
2) for each $w \in W$, q_w is in $S_{\varphi^{\text{opp}}}(M)$,
3) for each $k \in K$ and $w \in W$, r_k and d_w are positive real numbers,
4) $\sum_{k\in K} r_k = \sum_{w\in W} d_w = 1$.

For each q_w, choose some b_w in $\mathcal{U}^y$ such that $b_w \models q_w$. Let $\mathbf{b}_\nu : [0, 1]^2 \to \mathcal{U}$ via $\mathbf{b}_\nu((s, t)) = b_w$ whenever $s \in [\sum_{\ell=0}^{w-1} d_\ell, \sum_{\ell=0}^{w} d_\ell)$ with the convention that $\sum_{\ell=0}^{-1} d_\ell = 0$. For each p_k, choose some a_k in $\mathcal{U}^x$ such that $a_k \models \hat{p}_k|_{M(b_w)_{w\in W}}$, where $\hat{p}_k$ is the unique M-definable extension of p in $S_\varphi(\mathcal{U})$. Let $\mathbf{a}_\mu : [0, 1]^2 \to \mathcal{U}$ via $\mathbf{a}_\mu((s, t)) = a_k$, when $t \in [\sum_{\ell=0}^{k-1} r_\ell, \sum_{\ell=0}^{k} r_\ell)$ again with the convention that $\sum_{\ell=0}^{-1} r_\ell = 0$. In the following computations, if $(a, b) \in \mathcal{U}^x \times \mathcal{U}^y$, we let $\varphi(a, b) = 1$ if $\mathcal{U} \models \varphi(a, b)$ and 0 otherwise. Then

$$(\mu \otimes \nu)(\varphi(x, y)) \overset{(a)}{=} \sum_{k\in K}\sum_{w\in W} r_k d_w (\delta_{q_w}(d^\varphi_{p_k}(y))) \overset{(b)}{=} \sum_{k\in K}\sum_{w\in W} r_k d_w \varphi(a_k, b_w)$$
$$= \int_{(s,t)\in[0,1]^2} \varphi(\mathbf{a}_\mu(s, t)), \mathbf{b}_\nu(s, t))dL^2 = \mathbb{E}[\varphi(\mathbf{a}_\mu, \mathbf{b}_\nu)].$$

Equation (a) is derived in the proof of Theorem 4. Equation (b) follows from Remark 1. Thus, if the statement is false, then $\mathbb{E}[\varphi(x, y)]$ witnesses the continuous logic version of the order property. This contradicts Fact 4. □

We now work to give a second proof of Proposition 3 via the VC theorem. The statement of the VC theorem given below is much weaker than the general statement, but it is all that we need.

Theorem 7 (VC–theorem). Suppose that X is a set and $\mathcal{F}$ is a collection of subsets of X. Suppose that the VC-dimension of the class $\mathcal{F}$ is bounded by d. Then for every $\epsilon > 0$, there exists an integer $n = n(\epsilon, d)$ such that for every atomic measure μ on X (i.e., $\mu = \sum_{i\in I} r_i\delta_{x_i}$, where $I \subseteq \mathbb{N}$), there exists $a_1, ..., a_n \in X$ such that for any $F \in \mathcal{F}$,

$$\sup_{F\in\mathcal{F}} |\mu(F) - \text{Av}(a_1, ..., a_n)(F)| < \epsilon.$$

We remark that n does not depend on the choice of measure.

Lemma 3. Suppose that $\varphi(x,y)$ is stable. For every $\epsilon > 0$ there exists some natural number $N = N(\epsilon)$ such that for every $\mu \in \mathfrak{M}_{\varphi}(M)$ there exists $a_1, ..., a_N \in M$ such that for any $b \in M^y$,

$$|\mu(\varphi(x,b)) - \mathrm{Av}(a_1, ..., a_N)(\varphi(x,b))| < \epsilon.$$

Proof. There are several ways to see this. By Corollary 4, we may write $\mu = \sum_{i=0}^{\omega} r_i \delta_{p_i}$, where each p_i is in $S_{\varphi}(M)$. Let $\mathcal{U}$ be a monster model such that $M \prec \mathcal{U}$ and consider the measure $\hat{\mu} \in \mathfrak{M}_{\varphi}(\mathcal{U})$ given by $\sum_{i=0}^{\omega} r_i \delta_{\hat{p}_i}$, where $\hat{p}_i$ is the unique global M-definable extension of p to $\mathcal{U}$. For each $i \in \omega$, we have that $\hat{p}_i$ is both definable over M and finitely satisfiable in M [2, Proposition 2.3]. As a consequence, the measure $\hat{\mu}$ is finitely satisfiable and φ-definable over M (for appropriate definitions in this context, see [15, Section 6]). If $\varphi(x,y)$ is stable, it is NIP and so an application of [15, Theorem 6.4] implies that for every $\epsilon > 0$, there exists $a_1, ..., a_n \in M^x$ such that

$$\sup_{b \in \mathcal{U}^y} |\mu(\varphi(x,b)) - \mathrm{Av}(\bar{a})(\varphi(x,b))| < \epsilon.$$

An application of the VC theorem gives uniform bounds. ☐

Alternative Proof of Proposition 3. Suppose not. Then there exist $r \in (0,1)$, $\epsilon > 0$ and sequences $(\mu_i, \nu_j)_{(i,j) \in [k] \times [k]}$ for arbitrarily larger k which witnesses the (r,ϵ)-order property. Fix k arbitrarily large. We now construct a discrete formula (which is a Boolean combination of $\varphi(x,y)$) which witnesses the order property — and since stable formulas are closed under Boolean combinations, we obtain a contradiction. By Proposition 3, there exists a natural number N such that

1. For every $\mu \in \mathfrak{M}_{\varphi}(M)$, there exists $a_1, \ldots, a_N \in M^x$ such that

$$\sup_{b \in M^y} |\mu(\varphi(x,b)) - \mathrm{Av}(\bar{a})(\varphi(x,b))| < \frac{\epsilon}{16}.$$

For each $i \leq k$, let $\bar{a}_i = a_1^i, \ldots, a_N^i$ witness the above equation for μ_i.

2. For every $\nu \in \mathfrak{M}_{\varphi^{\mathrm{opp}}}(M)$, there exists $b_1, \ldots, b_N \in M^y$ such that

$$\sup_{a \in M^x} |\nu(\varphi(a,y)) - \mathrm{Av}(\bar{b})(\varphi(a,y))| < \frac{\epsilon}{16}.$$

For each $j \leq k$, let $\bar{b}_j = b_1^j, \ldots, b_N^j$ witness the above equation for ν_j.

Consider the formula given by,

$$\theta(x_1, ..., x_N, y_1, ..., y_N) := \bigvee_{\substack{A \times B \subseteq [N] \times [N] \\ \frac{|A \times B|}{N^2} > r + \frac{\epsilon}{2}}} \left(\bigwedge_{(i,j) \in A \times B} \varphi(x_i, y_j) \right).$$

We claim that $\theta(\bar{x}, \bar{y})$ is unstable. Notice that

$$M \models \theta(\bar{a}_i, \bar{b}_j) \Longrightarrow M \models \bigvee_{\substack{A \times B \subseteq [N] \times [N] \\ \frac{|A \times B|}{N^2} > r + \frac{\epsilon}{2}}} \left(\bigwedge_{(\ell,k) \in A \times B} \varphi(a_\ell^i, b_k^j) \right) \Longrightarrow (\mathrm{Av}(\bar{a}_i) \otimes \mathrm{Av}(\bar{b}_j))(\varphi(x,y)) > r + \frac{\epsilon}{2},$$

and likewise,

$$M \models \neg\theta(\bar{a}_i, \bar{b}_j) \Longrightarrow (\mathrm{Av}(\bar{a}_i) \otimes \mathrm{Av}(\bar{b}_j))(\varphi(x,y)) \leq r + \frac{\epsilon}{2}.$$

Moreover, if $i < j$, then

$$r \geq (\mu_i \otimes \nu_j)(\varphi(x,y)) \approx_{\epsilon/16} (\mathrm{Av}(\bar{a}_i) \otimes \nu_j)(\varphi(x,y)) = (\nu_j \otimes \mathrm{Av}(\bar{a}_i))(\varphi(x,y))$$
$$\approx_{\epsilon/16} (\mathrm{Av}(\bar{b}_j) \otimes \mathrm{Av}(\bar{a}_i))(\varphi(x,y)) = (\mathrm{Av}(\bar{a}_i) \otimes \mathrm{Av}(\bar{b}_j))(\varphi(x,y)),$$

and likewise, if $i \geq j$,

$$r + \epsilon \leq (\mu_i \otimes \nu_j)(\varphi(x,y)) \approx_{\epsilon/16} (\mathrm{Av}(\bar{a}_i) \otimes \nu_j)(\varphi(x,y)) = (\nu_j \otimes \mathrm{Av}(\bar{a}_i))(\varphi(x,y))$$
$$\approx_{\epsilon/16} (\mathrm{Av}(\bar{b}_j) \otimes \mathrm{Av}(\bar{a}_i))(\varphi(x,y)) = (\mathrm{Av}(\bar{a}_i) \otimes \mathrm{Av}(\bar{b}_j))(\varphi(x,y)).$$

Hence, if $i < j$ and $\models \theta(\bar{a}_i, \bar{b}_j)$, then $(\mathrm{Av}(\bar{a}_i) \otimes \mathrm{Av}(\bar{b}_j))(\varphi(\bar{x},\bar{y}))$ is greater than $r + \frac{\epsilon}{2}$ (by witnessing θ) and less than $r + \frac{\epsilon}{8}$ (by the above implication) — a contradiction. Hence, if $i < j$, then $\models \neg\theta(\bar{a}_i, \bar{b}_j)$. A similar argument shows that if $i \geq j$ then $\theta(\bar{a}_i, \bar{b}_j)$ must hold. Thus $\theta(\bar{x}, \bar{y})$, a Boolean combination of $\varphi(x,y)$, is unstable — a contradiction. □

Conclusion

We prove that if a formula $\varphi(x,y)$ is stable in a model, then all local Keisler measures with respect to this formula decompose into countable sums of types. We also give a functional-analytic proof of the fact that if a formula $\varphi(x,y)$ is stable, then local Keisler measures with respect to this formula do not satisfy the order property. This research demonstrates how tools and techniques from functional analysis can be used to understand the combinatorial properties of measures in model-theoretic settings.

Much is already known about Keisler measures — and in particular local Keisler measures — in the NIP and stable settings. Future research directions include understanding Keisler measures in general contexts with the independence property, such as simple, NTP2, and NSOP1 settings.

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Author Contributions

All authors contributed equally to this work.

Conflict of Interest

The authors declare no conflict of interest.

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