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Criterion for a formula-definable quasivariety

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In this paper, we study classes of models of a first-order language L with a countable signature σ. For a model A, let $\mathrm{Th}(A)$ denote the set of all sentences of L that are true in A, called the elementary type of A. The cardinality of the set T of all elementary types of the signature σ does not exceed the continuum. The product of elementary types of models A and B is defined by $\mathrm{Th}(A) \cdot \mathrm{Th}(B) = \mathrm{Th}(A \times B)$, where $A \times B$ is the Cartesian product of A and B. Infinite products, ultraproducts, and ultrapowers of elementary types with respect to an ultrafilter D are defined analogously. This yields an algebra $\langle T, \cdot \rangle$, which is a commutative semigroup with identity and zero. A binary absorption (recognition) relation is introduced in this semigroup. An elementary type N absorbs an elementary type M if $N \cdot M = N$. This notion leads to the concept of a formula-definable class of models. Formula-definable classes are closed under ultraproducts as well as finite and infinite direct products; they are idempotently formula-definable and axiomatizable. Varieties and quasivarieties are also considered. All varieties form formula-definable classes of models. Examples of a formula-definable class of models and of a class that is not formula-definable are given. An example of a formula-definable quasivariety that is not a variety is presented. It is shown that not all quasivarieties are formula-definable. Criteria are obtained for a quasivariety to be formula-definable and for a formula-definable class of models to be a quasivariety.

Keywords: model, identities, quasi-identities, variety, quasi-variety, Cartesian product of theories, elementary type, h-quasi-identities, equivalence relation, Boolean algebra.

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Introduction

Let L be a first-order language with a countable signature σ. For any model A (that is, an algebraic structure of signature σ) of the language L, denote by $\mathrm{Th}(A)$ the set of all sentences (closed formulas) of L that are true in the model A. We call the theory $\mathrm{Th}(A)$ the elementary type of the model A.

The abstract class of all models of a countable signature σ of the language L is partitioned into classes by the relation of elementary equivalence of models (A. Tarski [1]).

Thus we obtain Th_L, the set of all elementary types of the signature σ in the language L. The cardinality of the set Th_L of all elementary types of a countable signature σ of the language L does not exceed 2^ω.

In what follows, let $T \in Th_L$ denote an elementary type of the signature σ of the language L of some model.

If K is a class of models of the signature σ of the language L, then the set of elementary types of all models from K is denoted by $Th_L(K)$. That is, if V is a quasivariety, then $Th_L(V)$ is the set of elementary types of all models of the quasivariety V.

If H is a set of elementary types of signature σ of language L, then K_H is the class of all models of all elementary types from the set H.

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1 *Formula-Definable classes of models*

Definition 1. [2] The product of the elementary type T_1 of a model A and the elementary type T_2 of a model B, both of signature σ of the language L, is defined as

$$T_1 \cdot T_2 = Th(A \times B),$$

where $A \times B$ is the Cartesian product of the models A and B. Similarly, one defines the infinite product $\prod_{i \in I} T_i$, the ultraproduct $\prod_{i \in I} T_i/D$, the ultrapower T^I/D with respect to an ultrafilter D, and the filtered product of elementary types.

The algebra $\langle Th_L, \cdot \rangle$ is a commutative semigroup with identity and zero [2]. In this semigroup we introduce a binary absorption ("recognition") relation.

Definition 2. [2] An elementary type T_2 absorbs an elementary type T_1 if

$$T_1 \cdot T_2 = T_2.$$

An elementary type T is called *idempotent* if $T \cdot T = T$. A model A absorbs a model B when

$$Th(A \times B) = Th(A).$$

A model A is called *idempotent* if

$$Th(A \times A) = Th(A).$$

Theorem 1. [2] Let T_1, T_2, T_3 be elementary types from Th_L. If

$$T_1 \cdot T_2 \cdot T_3 = T_3,$$

then

$$T_1 \cdot T_3 = T_3.$$

Definition 3. [2] A class of models K is called a *formula-definable class of models* if there exists a model A of signature σ such that for any model B of signature σ,

$$B \in K \quad \text{if and only if} \quad Th(A \times B) = Th(A).$$

In this case, the model A is called a *determiner* of the class K. If, in addition, the model A is idempotent, then the class K is called an *idempotently formula-definable class of models.*

In other words, in an idempotently formula-definable class of models, there exists an idempotent model that absorbs ("recognizes") only the models of this class.

Examples:

1. The class of models of a single equivalence relation is formula-definable. A determiner of this class of models is a model with an infinite number of equivalence classes, each of which is infinite [3,4].

2. The class of all ω-stable models, the class of all superstable models, the class of all stable models, and the class of all unstable elementary types are not formula-definable sets of elementary types [2,5,6].

The main point of the subsequent discussion is that one moves from studying the properties of classes of models to studying the properties of the sets of elementary types of these classes. This approach allows us to consider the semigroup $\langle Th_L, \cdot \rangle$ and the properties of its subsemigroups [7,8], as well as to discover some new properties of classes of models using the operation of the direct product of models [9,10].

Theorem 2. [2] A formula-definable class of models is closed under ultraproducts, finite and infinite direct products. Moreover, it is an idempotently formula-definable class of models and an axiomatizable class of models.

2 Formula-Definable Quasivarieties

Formulas of the form

$$P\big(f_1(x_1, \ldots, x_m), \ldots, f_n(x_1, \ldots, x_m)\big), \qquad f(x_1, \ldots, x_m) = g(x_1, \ldots, x_m),$$

where $f, g, f_1, \ldots, f_n$ are terms of signature Ω and $P \in \Omega$, are called *quasi-atomic formulas* of signature Ω in the variables $x_1, \ldots, x_m$.

If

$$A_1(x_1, \ldots, x_m), \ldots, A_{s+1}(x_1, \ldots, x_m)$$

are some quasi-atomic formulas of the signature Ω (including equality) in the variables $x_1, \ldots, x_m$, then formulas of the form

$$(\forall x_1 \ldots x_m)\, A_1(x_1, \ldots, x_m)$$

are called *identities*, and formulas of the form

$$(\forall x_1 \ldots x_m)\big(A_1(x_1, \ldots, x_m) \wedge \cdots \wedge A_s(x_1, \ldots, x_m) \to A_{s+1}(x_1, \ldots, x_m)\big)$$

are called *quasi-identities.*

Every identity

$$(\forall x_1 \ldots x_m)\, A_1(x_1, \ldots, x_m)$$

is equivalent to the quasi-identity

$$(\forall x_1 \ldots x_m)\,\big(x_1 = x_1 \to A_1(x_1, \ldots, x_m)\big).$$

A class K of systems of signature Ω is called a *variety* (respectively, a *quasivariety*) if there exists a set Σ of identities (respectively, quasi-identities) of signature Ω such that K consists precisely of those systems of signature Ω in which all formulas from Σ are true [1]. In this case, the set Σ is called the defining set of identities (respectively, quasi-identities) of the variety (quasivariety) K.

Theorem 3. [1] A class of models V is a quasivariety if and only if it is:

1) closed under ultraproducts;
2) hereditary;
3) multiplicatively closed;
4) contains the trivial model.

Theorem 4. A formula-definable class of models is a quasivariety if and only if it is hereditary and contains the trivial model.

Proof. This follows from Theorem 2 and Theorem 3. □

Thus, the notion of a formula-definable quasivariety is effectively introduced here.

Theorem 5. [2] Every variety is an idempotently formula-definable quasivariety.

The example given above (Example 1) is an idempotently formula-definable quasivariety, but it is not a variety.

Theorem 6. A quasivariety defined by the quasi-identity

$$\forall x\,(\, P_1(x) \to P_2(x)\,)$$

is not a formula-definable quasivariety. However, a quasivariety defined by the quasi-identity

$$\forall x\,(\, x = a \to P(x)\,),$$

where a is a constant in the signature, *and is* a formula-definable quasivariety.

Proof. Let the quasivariety V be defined by the quasi-identity

$$\forall x\,(P_1(x) \to P_2(x)).$$

Then the quasivariety V also contains a model M in which $P_1(x)$ is false for all $x \in M$.

Consider the direct product of all countable models of the quasivariety V until an idempotent model is obtained; such an idempotent model belonging to V exists because V is closed under Cartesian products and because the set of elementary types has bounded cardinality (the language has a countable signature). Denote this model by $N \in V$.

By Theorem 1, the model N absorbs all models of the quasivariety V. However, in the model N, the formula $P_1(x)$ is false for all $x \in N$. Hence N also absorbs a model S in which the quasi-identity

$$\forall x\,(P_1(x) \to P_2(x))$$

is false.

Therefore, V is not a formula-definable quasivariety. □

Now let V be a quasivariety defined by the quasi-identity

$$\forall x\,(x = a \to P(x)),$$

where a is a constant in the signature. By the same construction as in the previous case, we obtain an idempotent model $N \in V$ that absorbs all models of the quasivariety V. It is clear that the model N is a determiner of the quasivariety V.

Definition 4. Let

$$A_1(x_1, \ldots, x_m), \ldots, A_{s+1}(x_1, \ldots, x_m)$$

be quasi-atomic formulas. Formulas of the form

$$\exists x_1 \ldots x_m \left(A_1(x_1, \ldots, x_m) \wedge \cdots \wedge A_s(x_1, \ldots, x_m)\right) \wedge$$

$$(\forall x_1 \ldots x_m) \left(A_1(x_1, \ldots, x_m) \wedge \cdots \wedge A_s(x_1, \ldots, x_m) \to A_{s+1}(x_1, \ldots, x_m)\right)$$

are called *h-quasi-identities.*

Thus, an h-quasi-identity can be viewed as a certain restriction of the corresponding quasi-identity.

For example, the quasi-identity

$$\forall x\,(x = a \to P(x))$$

can be written as

$$\exists x\,(x = a) \ \wedge\ \forall x\,(x = a \to P(x)).$$

Similarly, as in Example 1, the quasi-identity

$$\forall x\,\forall y\,\forall z\,(xEy \wedge yEz \to xEz)$$

can be written as the h-quasi-identity

$$\exists x\,(xEx) \ \wedge\ \forall x\,\forall y\,\forall z\,(xEy \wedge yEz \to xEz).$$

Every identity can be represented as an h-quasi-identity.

We now give a criterion for when a quasivariety is a formula-definable quasivariety.

Theorem 7. A quasivariety V is a formula-definable quasivariety if and only if all quasi-identities defining this quasivariety are bounded by the corresponding h-quasi-identities.

Proof. Let the quasivariety V be formula-definable, and suppose that some quasi-identity

$$(\forall x_1 \dots x_m)\big(A_1(x_1,\dots,x_m) \wedge \dots \wedge A_s(x_1,\dots,x_m) \to A_{s+1}(x_1,\dots,x_m)\big)$$

defining this quasivariety is not bounded by the corresponding h-quasi-identity. Then there exists a model in V in which the premise of this quasi-identity is false for all $x_1,\dots,x_m$.

Consider the Cartesian product of all models of this quasivariety until we obtain an idempotent model that absorbs all the models of V. In this resulting model, the premise is as follows.

$$A_1(x_1,\dots,x_m) \wedge \dots \wedge A_s(x_1,\dots,x_m)$$

is also false for all $x_1,\dots,x_m$.

It is clear that this model will absorb a model in which the quasi-identity

$$(\forall x_1 \dots x_m)\big(A_1(x_1,\dots,x_m) \wedge \dots \wedge A_s(x_1,\dots,x_m) \to A_{s+1}(x_1,\dots,x_m)\big)$$

is false. Therefore, the quasivariety V is not formula-definable.

Now, suppose that all quasi-identities defining the quasivariety V are bounded by the corresponding h-quasi-identities. Then, by taking the product of all models of this quasivariety, we obtain an idempotent model that absorbs all models of V. It is clear that this model does not absorb any model in which the corresponding quasi-identities are false. Therefore, the quasivariety V is formula-definable. □

By combining different sets of quasi-identities and h-quasi-identities, one can construct various examples of formula-definable and non-formula-definable quasivarieties.

Conclusion

In a number of papers published between 2020 and 2025 by Kazakhstan [11–13] and foreign [14,15] authors, various relations between elementary types have been actively studied. In the present paper, an algebraic structure is defined as a semigroup of elementary types with respect to the direct product. By introducing a binary absorption (recognition) relation, this structure is studied as an algebraic system. As a result, it becomes possible to investigate formula-definable classes of algebraic systems, which turn out to be axiomatizable classes. As noted above, all varieties form formula-definable classes. However, not all quasivarieties are formula-definable. The main result of the paper is a criterion for the formula-definability of a quasivariety.

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Conflict of Interest

The authors declare no conflict of interest.

References

1 Maltsev, A.I. (1970). *Algebraicheskie sistemy [Algebraic systems].* Moscow: Nauka [in Russian].

2 Bekenov, M., Kassatova, A., & Nurakunov, A. (2024). On absorption's formula definable semigroups of complete theories. *Archive for Mathematical Logic, 64*, 107–116. https://doi.org/10.1007/s00153-024-00937-2

3 Weinstein, J.M. (1965). First-order properties preserved by direct product. *Doctor's Thesis.* University of Wisconsin–Madison.

4 Bekenov, M.I. (2018). Properties of Elementary Embeddability in Model Theory. *Journal of Mathematical Sciences, 230,* 10–13. https://doi.org/10.1007/s10958-018-3721-4

5 Wierzejewski, J. (1976). On stability and products. *Fundamenta Mathematicae, 93*, 81–95. https://doi.org/10.4064/fm-93-2-81-95

6 Stepanova, A.A., & Krasitskaya, A.I. (2021). P-stability of some classes S-acts. *Siberian Mathematical Journal, 62*(2), 357–363. https://doi.org/10.1134/S0037446621020142

7 Kulpeshov, B.Sh., Pavlyuk, I.I., & Sudoplatov, S.V. (2025). Pseudo-countably-categorical formulas and theories. *Mathematical Notes, 117*(3), 442–457. https://doi.org/10.1134/S0001434625030095

8 Shvidefski, M.V. (2020). On a class of subsemigroup lattices. *Siberian Mathematical Journal, 61*(5), 1177–1193. https://doi.org/10.33048/smzh.2020.61.517

9 D'Aquino, P., & Macintyre, A. (2024). Products of pseudofinite structures. *arXiv preprint* arXiv:2411.08808 [math.LO]. https://doi.org/10.48550/arXiv.2411.08808

10 Garcia, D., Macpherson, D., & Steinhorn, C. (2015). Pseudofinite structures and simplicity. *Journal of Mathematical Logic, 15*(1), Article 1550002. https://doi.org/10.1142/S0219061315500026

11 Baizhanov, B., Orynbassarov, D., & Verbovskiy, V. (2020). Some remarks on definability of types and conservative extensions. *Kazakh Mathematical Journal, 20*(3), 92–100.

12 Yeshkeyev, A.R., & Ulbrikht, O.I. (2016). JSp-kosemantichnost i JSB svoistvo abelevykh grupp [JSp-cosemanticness and JSB property of Abelian groups]. *Sibirskie elektronnye matematicheskie izvestiia — Siberian Electronic Mathematical Reports, 13*, 861–874 [in Russian]. https://doi.org/10.17377/semi.2016.13.068

13 Markhabatov, N.D. (2022). Approximations of acyclic graphs. *The Bulletin of Irkutsk State University. Series Mathematics, 40*, 104–111. https://doi.org/10.26516/1997-7670.2022.40.104

14 Macpherson, D., & Tent, K. ω-categorical pseudofinite groups. *arXiv preprint*, [math.LO]. https://doi.org/10.48550/arXiv.2403.17684

15 Palchunov, D.E. (2022). Model Theory of Subject Domains. I. *Algebra and Logic, 61*(2), 166–173. https://doi.org/10.1007/s10469-022-09685-9

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