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On Triggers of Order

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It is shown that the concepts of heir and coheir, introduced by D. Lascar and B. Poizat, play a fundamental role in model theory, particularly in classification theory. The related notions of proper heir and proper coheir are introduced, containing important constructs within themselves. Poizat's lemma on the existence of a proper heir of any non-definable type over a model is presented as an important fact of existence in unstable theories. The concept of an order trigger in a model is then introduced as the skeleton of an algorithmic device that produces ω-evidence of the order property in it. This evidence is constructed using a method very similar to the "back and forth" method of classical model theory, where at each step two possibilities for choosing elements are alternated. As an example of use, a simplified proof of the characterization theorem of the class of unstable theories using these concepts is explained. It is pointed out that applications of more advanced constructions, such as order triggers, can help in solving problems related to the classification of small, countable and minimal models of unstable theories.

Keywords: proper heir, proper coheir, strong heir, unstable theory, non-definable type, order property, trigger of order, ω-evidence.

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Introduction

A new philosophy of model theory based on the notions of heir and coheir of a type was presented in [1]; it was shown that for types over models of stable theories these notions and one of non-forking extension coincide. The study of the properties of heirs and coheirs in a general context was continued in [2], which outlined their possible applications.

Definition 1. [1] Let $\mathcal{M}$ be a model, let $\mathcal{N}$ be an elementary extension of it, and let $q(x)$ be a complete type over N and $p(x) = q \restriction M$. Then

(1) q is an *heir* of p if for any formula $\varphi(x, y)$ over $\mathcal{M}$ with $\varphi(x, a) \in q$ there exists $b \in M$ with $\varphi(x, b) \in p$;

(2) q is a *coheir* of p if any formula in q is realized in $\mathcal{M}$.

As can be seen, a coheir does not speak about the relationship between a type and its restriction, but rather about the relationship between a type (as a set of formulas) and a model. Therefore, the concept is often used, especially when specifying the relationship between a type and a model, under the name "finitely satisfiable": instead of "a type is a coheir of its restriction over $\mathcal{M}$", one can say "a type is finitely satisfiable in $\mathcal{M}$" [2].

Coheirs have been well accepted by the model theory community because types that are finitely satisfiable in $\mathcal{M}$ are M-invariant, and they also allow one to construct Morley sequences (see e.g. [3,4]). Heirs are also used (see [5]), but, being more abstract, much less frequently than coheirs.

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In the hierarchy of concepts in model theory (and classification theory), what role do heirs and coheirs play?

This note aims to address answer this question.

In the note, we consider infinite models of the first-order language L with equality. As usual, we live in a universe (i.e., a fairly saturated and homogeneous model of L), which we do not designate in any way; all objects (sets, relations, tuples, etc.) are taken from it, models are its elementary submodels. Unless otherwise stated, the satisfiability of formulas and sentences, which is denoted by $\models$, also applies to the universe.

Models are denoted by uppercase calligraphic Latin letters, and their basic sets are denoted by the corresponding ordinary letters. Tuples of elements and variables are denoted by lowercase Latin letters (without dashes); Greek letters φ, ψ, θ are used to denote formulas in L. The space of complete types with variable x over a set A is denoted by $S^x(A)$. Similarly, we write M^x instead of $M^{l(x)}$, where $l(x)$ is the length of the tuple x. The type of a tuple a over a set A is denoted by $tp(a/A)$.

1 *Poizat's Lemma*

In [1], the following properties of a non-definable type p over a model $\mathcal{M}$ were shown:

(1) p can have arbitrarily many distinct heirs over some elementary extension of $\mathcal{M}$;

(2) the number of coheirs of type p over any elementary extension of $\mathcal{M}$ is bounded by some fixed cardinal depending only on the cardinality of $\mathcal{M}$.

Definition 2. Let p, q be complete types over models and let q be an extension of p. Then q is a *proper heir* of p if it is heir of p, but not coheir of p. Likewise, q is a *proper coheir* of p if it is coheir of p, but not heir of p.

From the above properties (1), (2) it almost directly follows that any non-definable type over a model has a proper heir.

In [2, p. 300] B. Poizat gave an elegant proof of the latter fact using the following lemma: any non-definable type over a model $\mathcal{M}$ has a (strong) heir that is not M-invariant; the proof relied on primary model-theoretic methods of definability dating back to E.W. Beth, A. Robinson, L. Svenonius and others.

For completeness, we present here Lemma 1, which is a weakened version of Poizat's lemma and is sufficient for our purposes.

Let $\mathcal{M}$ be a model of the language L and $p \in S^x(M)$. For each formula of the form $\varphi(x, y)$ in the language L, we define the relation R_φ as a subset of M^y:

$$a \in R_\varphi \Leftrightarrow \varphi(x, a) \in p. \tag{1}$$

The enrichment of the model $\mathcal{M}$ to the language $L^* = L \cup \{R_\theta\}_{\theta(x,\cdot)\in L}$ is denoted by $\mathcal{M}^*$.

Definition 3. [2] If $\mathcal{N}^*$ is an elementary extension of $\mathcal{M}^*$, then the relations $\{R_\theta\}_{\theta(x,\cdot)\in L}$ define a type over N that is an heir of p, it is called a *strong heir* of p .

There may also exist non-strong heirs of p over N: if $\mathcal{N}$ is an elementary extension of a model $\mathcal{M}$ that does not admit enrichment to an elementary extension of $\mathcal{M}^*$, then it is clear that there is no strong heir of p over N [2].

Lemma 1. If a formula $\varphi(x, y)$ of a language L is such that R_φ from (1) is not definable in $\mathcal{M}$. Then there exist an elementary extension $\mathcal{N}$ of $\mathcal{M}$, type $q \in S^x(N)$ and tuples $a, b \in N^y$ of the same type over M such that q is a strong heir of p and $\varphi(x, a) \wedge \neg\varphi(x, b) \in q$.

Proof. Let $\varphi(x,y)$ be a formula in L and R_φ is not definable in $\mathcal{M}$. We will first show that the following theory is then consistent:

$$T^* = Th(\mathcal{M}^*) \cup \{\psi(c_1) \leftrightarrow \psi(c_2)\}_{\psi(y)\in L(M)} \cup \{R_\varphi(c_1), \neg R_\varphi(c_2)\},$$

where $\mathcal{M}^*$ is the enrichment of $\mathcal{M}$ described before Definition 3, and c_1, c_2 are new constants.

On the contrary, let us assume that T^* is inconsistent. Then, by compactness, there exists a finite set $\{\psi_1(y),\ldots,\psi_n(y)\}$ of $L(M)$-formulas such that

$$\forall yz \left(\bigwedge_{i=1}^{n}(\psi_i(y) \leftrightarrow \psi_i(z)) \to (R_\varphi(y) \leftrightarrow R_\varphi(z))\right) \in Th(\mathcal{M}^*).$$

This means that there is a Boolean function $f : \{0,1\}^n \to \{0,1\}$ such that if $\varepsilon_1,\ldots\varepsilon_n,\varepsilon \in \{0,1\}$ and $f(\varepsilon_1,\ldots\varepsilon_n) = \varepsilon$ then

$$\forall y \left(\bigwedge_{i=1}^{n} \psi_i^{\varepsilon_i}(y) \to R_\varphi^{\varepsilon}(y)\right) \in Th(\mathcal{M}^*).$$

Here, as usual, for any formula θ we put $\theta^1 = \theta$ and $\theta^0 = \neg\theta$. Now it is obvious that the $L(M)$-formula

$$\bigvee_{f(\varepsilon_1,\ldots\varepsilon_n)=1} \left(\bigwedge_{i=1}^{n} \psi_i^{\varepsilon_i}(y)\right)$$

defines R_φ, which is a contradiction.

So, T^* is consistent. Let $\mathcal{N}^*$ be its model, where the constants c_1 and c_2 are interpreted by tuples a and b, respectively. The set of formulas $\{R_\theta\}_{\theta(x,\cdot)\in L}$ defines a type q over N, which is a strong heir of p. It is now easy to see that $\mathcal{N}$ and q, together with a and b, satisfy the statement of the lemma. □

Since the formula $\varphi(x,a) \wedge \neg\varphi(x,b)$ from Lemma 1 is not realized in $\mathcal{M}$, we obtain the following:

Corollary 1. [2] Any non-definable type over a model has a strong proper heir.

On the other hand, a proper heir of a type over a model may exist even if all types over that model are definable.

Example 1. Over a model $\mathcal{M} = \langle\omega; =, \in\rangle$, where $L = \{=, \in\}$, all types are definable, and there exists a unique non-algebraic 1-type over it, which we denote by ∞. Over any proper elementary extension of a model $\mathcal{M}$, the heir of ∞ will be proper.

2 *Trigger of Order*

We will start this section with a definition.

Definition 4. A triple $\langle a, b, \varphi(x,y)\rangle$ is called a *trigger of order* in $\mathcal{M}$ if it satisfies the following conditions:

(1) $\models \varphi(a,b)$,
(2) $\varphi(x,b)$ is not realized in $\mathcal{M}$,
(3) $tp(b/Ma)$ is finitely satisfiable in $\mathcal{M}$ (i.e. $tp(b/Ma)$ is a coheir of $tp(b/M)$).

In this definition, the word "order" comes from the order property (OP) defined in [6] (see also the theorem below).

Obviously, a proper heir of a type over a model $\mathcal{M}$ provides us with a trigger of order in $\mathcal{M}$: take as a the realization of this proper heir and its formula $\varphi(x,b)$, which is not realized in $\mathcal{M}$. Likewise, any proper coheir produces a trigger of order.

Theorem 1. If there is a trigger of order in a model, then the model contains ω-evidence of the order property.

Proof. Let $\langle a, b, \varphi(x,y)\rangle$ be an order trigger in $\mathcal{M}$ as in Definition 4. By condition (2) of Definition 4 it follows that for each $m \in M^x$ we have $\neg\varphi(m,y) \in r := tp(b/Ma)$.

Let $B := \varphi(a, \mathcal{M})$. By induction on $i \in \omega$ we determine $a_i \in M^x$ and $b_i \in B$. We choose an arbitrary $a_0 \in M^x$. Since $\neg\varphi(a_0, y) \in r$, by the property (3) of Definition 4 one can find $b_0 \in B$ such that $\models \neg\varphi(a_0, b_0)$.

Let us assume that the elements a_i, b_i are defined for all $i < k$. We choose $a_k \in M^x$ realizing $\bigwedge_{i<k} \varphi(x, b_i)$ (note that the last formula is realized by a). Finally, we choose $b_k \in B$ realizing $\bigwedge_{i\leq k} \neg\varphi(a_i, y)$, which is again possible by property (3) of Definition 4.

From the construction it is clear that for any $m, n \in \omega$ the following holds:

$$\models \varphi(a_m, b_n) \Leftrightarrow n < m,$$

which means that the formula $\neg\varphi(x,y)$ has the order property [6]. □

To our knowledge, [7] was the first paper to give an example of using proper heirs to discover structure in a (minimal) model; it inspired us to write [8] and the present note. We also noticed that [9] considered very closely related issues. There are many unsolved problems about small (i.e., countable, minimal, etc.) models of countable unstable theories [7,10]. Theorem 1 provides a method for analyzing the structures of such models.

Let us note one more consequence of Theorem 1. After it, the equivalence of the following properties of unstable theories can be easily proved along the line (a)⇒(b)⇒(c)⇒(d)⇒(e)⇒(a), using only the arithmetic of cardinals and the compactness theorem:
(a) for some formula φ, there are too many φ-types over some small set,
(b) there exists a non-definable φ-type over a model,
(c) there exists a proper heir,
(d) there exists an order trigger,
(e) the theory has the order property.

Conclusion

The fact that heirs and coheirs incorporate combinatorics has been known for a long time (e.g. Morley sequences). In this note we show another confirmation of this fact, as a result of a simplest use of proper heirs. It is clear that more advanced applications of such concepts will yield deeper, more meaningful results. It is also certain that constructions like order triggers can be used to classify small, countable and minimal models of unstable theories.

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Author Contributions

All authors contributed equally to this work.

Conflict of Interest

The authors declare no conflict of interest.

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