https://doi.org/10.31489/2026M1/37-54 *Research article*

A Boundary Value Problem for a Time-Fractional Diffusion Equation in a Non-Cylindrical Shrinking Domain

A.D. Akhmetshin, M.T. Omarov*, R.Z. Toleukhanova

Buketov Karaganda National Research University, Karaganda, Kazakhstan
(E-mail: aleksandr_050401@mail.ru, madiomarovt@gmail.com, toleuch_r@mail.ru)

This article deals with the fundamental problems in the mathematical theory of fractional differential equations, specifically focusing on the analytical solvability of boundary value problems in time-dependent domains. The relevance of the study implies the necessity of developing methods for equations with non-local operators modeling anomalous diffusion. A one-dimensional diffusion equation containing a Riemann-Liouville fractional derivative with respect to time is examined. The characteristic features of the problem, posed in a non-cylindrical domain bounded by a moving linear boundary and a fixed spatial coordinate, are analyzed. The need to handle inhomogeneous boundary data is identified, and the problem is initially reduced to one with homogeneous conditions. On the basis of the study, the author constructs the fundamental solution in a quarter-plane by means of the bilateral Laplace transform and obtains the Green function for the Dirichlet problem. It is shown that the solution can be expressed through an integral representation in terms of a specific boundary density. This density satisfies a Volterra-type integral equation with a weakly singular kernel. Using the contraction mapping principle, it is proved that this equation has a solution. Consequently, the existence of a regular solution to the original boundary value problem is established.

Keywords: time-fractional diffusion equation, Riemann–Liouville derivative, infinite memory, non-cylindrical domain, shrinking domain, Dirichlet problem, Green function, fundamental solution, Wright function, bilateral Laplace transform, Volterra integral equation, weakly singular kernel.

2020 Mathematics Subject Classification: Primary 35R11; Secondary 26A33, 35K05, 35C15, 45D05, 33E12.

Introduction

Over the last decades, time-fractional diffusion equations have been intensively studied as models of anomalous transport in complex media. For single-term equations on the whole line or in $\mathbb{R}^n$, the Cauchy problem has become classical. Using Laplace and Fourier transforms, the fundamental solutions can be represented in terms of the Wright and M-Wright functions, and their probabilistic interpretation as self-similar densities is well understood [1, 2]. These kernels provide the natural building blocks for Green functions in bounded geometries and for the analysis of qualitative properties such as positivity and scaling.

A large body of work addresses initial-boundary value problems in cylindrical domains with fixed spatial cross-section. For one-dimensional problems with Caputo time-fractional derivatives, a maximum principle and related a priori estimates were established in [3] and later extended to weak solutions and general elliptic operators in [4]. For equations with Riemann–Liouville derivatives, Pskhu developed a Green-function method for one-dimensional boundary value problems: in particular, explicit representations for the first boundary value problem on a finite interval and for related boundary value problems in rectangles were obtained in [5, 6]. These contributions show that, in a fixed domain, boundary data can often be reduced to Volterra integral equations with weakly singular kernels.

*Corresponding author. *E-mail: madiomarovt@gmail.com*

This research was funded by the Science Committee of the Ministry of Science and Higher Education of the Republic of Kazakhstan (Grant no. AP23488729, AP23488740).

Received: 23 January 2025; *Accepted:* 6 January 2026.

Time-fractional diffusion in unbounded or semi-bounded domains has also been investigated. In [7] a time-fractional diffusion equation with mass absorption in a quarter-line was solved under a Dirichlet boundary condition that is harmonic in time; combining Laplace transform in time and sine transform in space yields an explicit solution as a superposition of Wright-type kernels. For numerical purposes, boundary integral equation approaches have been developed: Yao and Wang proposed a boundary-integral scheme for the time-fractional diffusion equation, where the solution is expressed as a single-layer potential and the time dependence is treated in the Laplace domain [8].

The geometry of the spatial domain plays a subtle role once the boundary is allowed to move. A step towards non-cylindrical domains for time-fractional equations was taken by Kubica, Rybka, and Ryszewska, who considered a one-dimensional heat equation with Caputo time derivative in a noncylindrical region and established existence of weak solutions via the Galerkin method [9]. On the other hand, Pskhu solved the first boundary value problem for a diffusion-wave equation with a fractional time derivative of Dzhrbashyan–Nersesyan type in a non-cylindrical domain, derived a Green representation and proved that Hölder regularity of the lateral boundary ensured existence [10].

Moving-boundary problems of Stefan type provide another important class of non-cylindrical configurations. Roscani analysed a one-dimensional time-fractional diffusion equation with a free boundary, employing a fractional weak maximum principle and constructing exact self-similar solutions in terms of Wright functions [11]. Kubica and Ryszewska studied a time-fractional Stefan problem with Riemann–Liouville flux, derived a self-similar formulation, and obtained explicit similarity solutions describing phase-change fronts with memory [12].

A further line of research relevant for the present work concerns equations with fractional derivatives whose lower limit is shifted to minus infinity. Diffusion-wave equations with such derivatives model processes with “infinite memory” and require the introduction of appropriate asymptotic boundary conditions instead of classical initial data. Pskhu and Rekhviashvili constructed Green functions for an asymptotic boundary value problem on the real line, proved solvability in weighted function spaces, and discussed applications to fractional electrodynamics [13]. Their analysis shows that the choice of the lower limit in the fractional derivative substantially affects both the functional setting and the structure of Green representations.

Angular domains create additional difficulties due to corner singularities and the interplay between spatial and temporal degeneracies. For fractional diffusion equations with time-fractional derivatives starting at $t = 0$, a boundary value problem in a curvilinear angle was solved in [14]. The authors proved existence in weighted Hölder spaces, demonstrating that the Hölder continuity of the boundary curve suffices to control the singular behaviour near the vertex. More recently, it was considered the first boundary value problem for a fractional diffusion equation in a degenerate angular domain whose opening shrinks to a point at the initial time [15]. They showed that the associated boundary integral equations are no longer of standard Volterra type; nevertheless, by a careful analysis of Wright kernels and Carleman–Vekua regularisation they obtained existence results in suitable weighted classes.

Loaded fractional diffusion equations, in which the partial differential equation contains additional integral or fractional-differential terms supported on lower-dimensional sets, form another active research direction. Boundary value problems for a diffusion equation with a fractional load supported on a straight ray were studied [16]. Depending on the position of the supporting line and on the order of the fractional operator in the load, the resulting boundary integral equation may be of genuine Volterra type, of pseudo-Volterra type, or even fail to have a unique solution, illustrating the delicate interplay between nonlocality and geometry.

Finally, a number of generalisations replace the Caputo or Riemann–Liouville derivative by more flexible operators. Of particular interest are diffusion models driven by g-fractional derivatives with respect to a monotone function g. Angelani and Garra analysed g-fractional diffusion in bounded domains with absorbing boundaries, obtained explicit series representations of the solution, and studied first-passage time distributions and their dependence on the choice of g [17]. These models demonstrate how non-standard fractional operators can be combined with boundary conditions to generate rich transient behaviours.

Against this background, the present paper addresses a boundary value problem for a time-fractional diffusion equation with Riemann–Liouville derivative of order $0 < \alpha < 1$ whose lower limit is $-\infty$, posed in a non-cylindrical domain

$$\Omega = \{(x,t) : 0 < x < -t,\ -\infty < t < 0\}.$$

The domain degenerates to a point as $t \to 0^-$, and the boundary conditions are prescribed on both the fixed boundary $x = 0$ and the moving boundary $x = -t$. To the best of our knowledge, no previous work combines simultaneously (i) an infinite-memory Riemann–Liouville derivative, (ii) a genuinely non-cylindrical domain whose lateral boundary originates at a single point, and (iii) boundary data given on two intersecting time-like curves.

Methodologically, our approach is close in spirit to the Green-function constructions in [5, 10, 13] but differs in several essential aspects. We first construct a fundamental solution in the quarter-plane $x > 0$, $t < 0$ using the bilateral Laplace transform with respect to time and derive from it a Green function for the Dirichlet problem on the quarter-line, expressed explicitly through Wright kernels [1,2]. This Green function is then employed to represent the solution in the non-cylindrical domain in terms of boundary layer potentials, leading to a Volterra-type integral equation for an unknown boundary density on the moving boundary. The kernel of this equation exhibits only a weak singularity, so the problem is well posed in a weighted Banach space of functions on $(-\infty, 0]$; existence follows from the contraction mapping principle. In this way we obtain an explicit Green representation for the regular solution and identify natural conditions on the right-hand side. The results thus extend the potential-theoretic approach to time-fractional diffusion into a new regime where both nonlocality in time and degeneracy of the spatial domain at the initial moment must be treated simultaneously.

The paper is organized as follows. In the introduction we formulate the boundary value problem for the time-fractional diffusion equation with the Riemann–Liouville derivative on the non-cylindrical domain and introduce the notion of a regular solution. In the next section we reduce the problem with inhomogeneous boundary data to an equivalent problem with homogeneous boundary conditions by an explicit transformation. We then construct in the quarter-plane a fundamental solution of the fractional diffusion equation by means of the bilateral Laplace transform in time and derive the Green function for the Dirichlet problem on the quarter-line. Using this Green function, we obtain in the non-cylindrical domain an integral representation of the solution in terms of volume and boundary layer potentials. The boundary conditions on the moving boundary lead to a Volterra-type integral equation with a weakly singular kernel for an unknown boundary density. Finally, we introduce a suitable weighted Banach space, study the corresponding integral operator, and prove existence of the boundary density, and hence of the regular solution to the original boundary value problem, by the contraction mapping principle.

1 *Problem statement*

Let $0 < \alpha < 1$ and set $\beta = \frac{\alpha}{2} \in \left(0, \frac{1}{2}\right)$. We consider the fractional diffusion equation

$$D^{\alpha}_{-\infty t} u(x,t) - \frac{\partial^2 u}{\partial x^2}(x,t) = f(x,t), \tag{1}$$

in the non-cylindrical domain

$$\Omega = \{(x,t) : 0 < x < -t,\ -\infty < t < 0\}. \tag{2}$$

Here $D^{\alpha}_{-\infty t}$ denotes the Riemann–Liouville fractional derivative of order α with respect to t with lower limit $-\infty$.

We impose boundary conditions on both sides of the space interval:

$$u(0,t) = \varphi(t), \qquad u(-t,t) = \psi(t), \qquad -\infty < t < 0. \tag{3}$$

The right-hand side f and the boundary data φ, ψ will be specified later. Our aim is to construct a representation for the solution of (1), (3) and to obtain conditions on f for the existence of a regular solution.

1.1 Preliminaries

For $0 < \alpha < 1$ the Riemann–Liouville derivative $D^{\alpha}_{-\infty t}$ is defined by

$$D^{\alpha}_{-\infty t} u(x,t) = \frac{\partial}{\partial t} D^{\alpha-1}_{-\infty t} u(x,t),$$

where

$$D^{\alpha-1}_{-\infty t} u(x,t) = \frac{1}{\Gamma(1-\alpha)} \int_{-\infty}^{t} (t-\tau)^{-\alpha} u(x,\tau)\, d\tau.$$

We also use the notation

$$h_{\gamma}(s) = \frac{s^{\gamma-1}}{\Gamma(\gamma)}, \quad s > 0, \quad \gamma > 0,$$

so that $D^{\alpha-1}_{-\infty t} u(x,t) = \int_{-\infty}^{t} u(x,\tau) h_{1-\alpha}(t-\tau)\, d\tau$.

Definition 1 (Regular solution). Let Ω be given by (2). A function $u = u(x,t)$ is called a *regular solution* of (1) in Ω if

- $u \in C(\overline{\Omega})$;
- $\partial^2 u / \partial x^2$ exists and is continuous in Ω;
- for every $(x,t) \in \Omega$ the quantity $D^{\alpha-1}_{-\infty t} u(x,t)$ is well-defined and continuously differentiable in t;
- for every fixed x and every $R < 0$ the function $(R-t)^{-\alpha} u(x,t)$ is integrable on $(-\infty, R)$;
- u satisfies (1) pointwise in Ω and the boundary conditions (3) hold.

Definition 2. The two-sided Laplace transform of a function f is

$$F(s) = \int_{-\infty}^{+\infty} f(t) e^{-st}\, dt$$

and it exists at s if the integral converges absolutely:

$$\int_{-\infty}^{+\infty} |f(t)| e^{-\operatorname{Re} s \cdot t}\, dt < \infty.$$

It is sufficient that:

- f be piecewise continuous on every finite interval;
- $\exists M_1, M_2 > 0, a < b \in \mathbb{R}$ such that

$$|f(t)| \le M_1 e^{at} \quad (t \ge 0), \qquad |f(t)| \le M_2 e^{bt} \quad (t \le 0).$$

Then $F(s)$ exists and is analytic in the entire strip $a < \operatorname{Re} s < b$.

Definition 3. For $\eta \in \mathbb{R}$ and $\beta = \frac{\alpha}{2}$, we define the function $w_\mu(x, y)$ by

$$w_\eta(x, y) = y^{\eta-1}\phi\left(-\beta, \eta; -\frac{x}{y^\beta}\right),$$

where $\phi(\rho, \eta; z)$ denotes the Wright function, given by the series representation

$$\phi(\rho, \eta; z) = \sum_{k=0}^{\infty} \frac{z^k}{k!\Gamma(\eta + \rho k)}, \qquad \rho > -1.$$

In the special case where $\eta = 0$, we denote the function simply as $w(x, y) = w_0(x, y)$.

2 *Reduction to homogeneous boundary conditions*

We first reduce problem (1), (3) to an equivalent problem with homogeneous boundary conditions. Define

$$u(x, t) = v(x, t) + \varphi(t) - \frac{x}{t}\big(\psi(t) - \varphi(t)\big), \qquad (x, t) \in \Omega. \tag{4}$$

Since $t < 0$ in Ω and $0 < x < -t$, the coefficient x/t is well-defined and satisfies $|x/t| \leq 1$ for all $(x, t) \in \Omega$, in particular it is bounded on compact subsets of Ω.

By direct substitution into the boundary conditions (3) we obtain

$$u(0, t) = v(0, t) + \varphi(t) = \varphi(t) \quad \Rightarrow \quad v(0, t) = 0,$$

and

$$u(-t, t) = v(-t, t) + \varphi(t) - \frac{-t}{t}(\psi(t) - \varphi(t)) = v(-t, t) + \psi(t) \quad \Rightarrow \quad v(-t, t) = 0.$$

Hence v satisfies the homogeneous boundary conditions

$$v(0, t) = v(-t, t) = 0, \qquad -\infty < t < 0. \tag{5}$$

Substituting (4) into the equation (1) we obtain

$$D^\alpha_{-\infty t} v(x, t) - \frac{\partial^2 v}{\partial x^2}(x, t) = f(x, t) - D^\alpha_{-\infty t}\varphi(t) + D^\alpha_{-\infty t}\left[\frac{x}{t}\big(\psi(t) - \varphi(t)\big)\right],$$

because the second derivative with respect to x of the affine function $x \mapsto \varphi(t) - \frac{x}{t}(\psi(t) - \varphi(t))$ vanishes. Thus the function v solves the inhomogeneous problem

$$D^\alpha_{-\infty t} v(x, t) - \frac{\partial^2 v}{\partial x^2}(x, t) = \tilde{f}(x, t), \tag{6}$$

in Ω, with homogeneous boundary conditions (5), where

$$\tilde{f}(x, t) = f(x, t) - D^\alpha_{-\infty t}\varphi(t) + D^\alpha_{-\infty t}\left[\frac{x}{t}\big(\psi(t) - \varphi(t)\big)\right].$$

Therefore it suffices to solve problem (6), (5) and then recover u by (4).

3 Fundamental solution in the quarter-plane

To construct a regular solution, we first partly study the equation

$$D^{\alpha}_{-\infty t}v(x,t) - \frac{\partial^2 v}{\partial x^2}(x,t) = \tilde{f}(x,t), \tag{7}$$

in the quarter-plane

$$\Omega_1 = \{(x,t) : x > 0,\ t < 0\},$$

with the boundary condition

$$v(0,t) = 0, \qquad t < 0. \tag{8}$$

In order to construct the fundamental solution we consider the problem

$$\begin{cases} D^{\alpha}_{-\infty t}v(x,t) - \dfrac{\partial^2 v}{\partial x^2}(x,t) = \delta(x-x_1)\delta(t-t_1), & (x,t) \in \Omega_1, \\ v(0,t) = 0, \quad \lim_{x\to\infty} v(x,t) = 0, & t < 0, \end{cases} \tag{9}$$

where $x_1 > 0$, $t_1 < 0$ are fixed.

3.1 Bilateral Laplace transform in t

We select the solution of (9), i.e.,

$$v(x,t) = 0, \qquad t < t_1. \tag{10}$$

Thus $v(x,t)$ is understood as being defined for all $t \in \mathbb{R}$ by the zero continuation (10). In particular, for every $\operatorname{Re} p > 0$ and every fixed $x > 0$ the bilateral Laplace integral

$$\bar{v}(x,p) = \int_{-\infty}^{+\infty} v(x,t)e^{-pt}\,dt$$

converges absolutely.

Let $v(x,t)$ be a solution of (9) such that, for every $\operatorname{Re} p > 0$

$$\int_{-\infty}^{\infty} |v(x,t)|e^{-pt}\,dt < \infty$$

for all $x > 0$, and assume that $v(x,t)$ is the pre image under the bilateral Laplace transform (this property will be satisfied a posteriori for the explicit fundamental solution constructed below). Consider the bilateral Laplace transform with respect to t,

$$\bar{v}(x,p) = \int_{-\infty}^{\infty} v(x,t)e^{-pt}\,dt, \qquad \operatorname{Re} p > 0.$$

Using integration by parts and the representation of the Riemann–Liouville derivative we obtain

$$\int_{-\infty}^{\infty} D^{\alpha}_{-\infty t}v(x,t)e^{-pt}\,dt = e^{-pt}D^{\alpha-1}_{-\infty t}v(x,t)\Big|_{t=-\infty}^{t=+\infty} + p^{\alpha}\bar{v}(x,p) = p^{\alpha}\bar{v}(x,p),$$

under the aforementioned decay and integrability assumptions the boundary term vanishes. Furthermore,

$$\int_{-\infty}^{\infty} \frac{\partial^2 v}{\partial x^2}(x,t)e^{-pt}\,dt = \frac{\partial^2}{\partial x^2}\bar{v}(x,p).$$

Transforming (9) leads to the ordinary differential equation

$$p^{\alpha}\bar{v}(x,p) - \frac{\partial^2 \bar{v}}{\partial x^2}(x,p) = e^{-pt_1}\delta(x-x_1), \qquad x>0,\ \operatorname{Re} p>0.$$

Equivalently,

$$\frac{\partial^2 \bar{v}}{\partial x^2}(x,p) - p^{\alpha}\bar{v}(x,p) = -e^{-pt_1}\delta(x-x_1) =: g(x,p).$$

For $x \neq x_1$ the homogeneous equation $\bar{v}_{xx} - p^{\alpha}\bar{v} = 0$ has the general solution

$$\bar{v}(x,p) = Q_1(p)e^{-p^{\beta}x} + Q_2(p)e^{p^{\beta}x}, \qquad \beta = \frac{\alpha}{2}.$$

To take into account the delta source at $x = x_1$, we write the solution using the method of variation of parameters. We set

$$\bar{v}(x,p) = Q_1(x,p)e^{-p^{\beta}x} + Q_2(x,p)e^{p^{\beta}x}$$

and impose

$$Q_1'(x,p)e^{-p^{\beta}x} + Q_2'(x,p)e^{p^{\beta}x} = 0, \qquad -p^{\beta}Q_1'(x,p)e^{-p^{\beta}x} + p^{\beta}Q_2'(x,p)e^{p^{\beta}x} = g(x,p).$$

Solving this system gives

$$Q_1'(x,p) = -\frac{e^{p^{\beta}x}}{2p^{\beta}}g(x,p), \qquad Q_2'(x,p) = \frac{e^{-p^{\beta}x}}{2p^{\beta}}g(x,p).$$

Consequently,

$$Q_1(x,p) = -\int_0^x \frac{e^{p^{\beta}\xi}}{2p^{\beta}}g(\xi,p)\,d\xi + C_1(p),$$
$$Q_2(x,p) = -\int_x^{\infty} \frac{e^{-p^{\beta}\xi}}{2p^{\beta}}g(\xi,p)\,d\xi + C_2(p).$$

Substituting back yields

$$\begin{aligned}\bar{v}(x,p) &= Q_1(x,p)e^{-p^{\beta}x} + Q_2(x,p)e^{p^{\beta}x}\\ &= -\int_0^x \frac{g(\xi,p)e^{-p^{\beta}(x-\xi)}}{2p^{\beta}}\,d\xi - \int_x^{\infty} \frac{g(\xi,p)e^{-p^{\beta}(\xi-x)}}{2p^{\beta}}\,d\xi +\\ &\quad + C_1(p)e^{-p^{\beta}x} + C_2(p)e^{p^{\beta}x} =\\ &= -\int_0^{\infty} \frac{g(\xi,p)e^{-p^{\beta}|x-\xi|}}{2p^{\beta}}\,d\xi + C_1(p)e^{-p^{\beta}x} + C_2(p)e^{p^{\beta}x}.\end{aligned}$$

Using $g(x,p) = -e^{-pt_1}\delta(x-x_1)$, we obtain

$$\bar{v}(x,p) = e^{-pt_1}\frac{e^{-p^{\beta}|x-x_1|}}{2p^{\beta}} + C_1(p)e^{-p^{\beta}x} + C_2(p)e^{p^{\beta}x}.$$

The boundary condition at $x = 0$ reads

$$\bar{v}(0,p) = e^{-pt_1}\frac{e^{-p^{\beta}x_1}}{2p^{\beta}} + C_1(p) + C_2(p) = 0.$$

The boundedness as $x \to \infty$ again implies $C_2(p) = 0$, hence

$$C_1(p) = -e^{-pt_1}\frac{e^{-p^{\beta}x_1}}{2p^{\beta}},$$

and therefore

$$\bar{v}(x,p) = \frac{e^{-pt_1}}{2p^{\beta}}\Big(e^{-p^{\beta}|x-x_1|} - e^{-p^{\beta}(x+x_1)}\Big).$$

3.2 Inverse Laplace transform and the Wright kernels

We now invert the Laplace transform. For this we introduce a family of kernels $w_\eta(x,t)$, $\eta \in \mathbb{R}$, $x \geq 0$, $t > 0$, such that

$$\int_0^\infty e^{-pt} w_\eta(x,t)\,dt = p^{-\eta} e^{-p^\beta x}, \qquad \operatorname{Re} p > 0. \tag{11}$$

Here and below $D^\rho_{\tau t}$ denotes the Riemann–Liouville fractional derivative in t of order ρ with lower limit τ. The kernels w_η are Wright-type functions and satisfy, in particular, the relations

$$D^\rho_{\tau t} w_\eta(x, t-\tau) = w_{\eta-\rho}(x, t-\tau), \tag{12}$$

for all ρ and all admissible η, and

$$D^\alpha_{\tau t} w_\beta(x, t-\tau) = \frac{\partial^2}{\partial x^2} w_\beta(x, t-\tau), \qquad \alpha = 2\beta. \tag{13}$$

From (11) with $\eta = \beta$ we see that the inverse transform of $e^{-p^\beta x}/p^\beta$ is precisely $w_\beta(x,t)$. Hence

$$v(x,t) = \frac{1}{2}\Big(w_\beta(|x-x_1|, t-t_1) - w_\beta(x+x_1, t-t_1)\Big), \qquad t > t_1.$$

We summarize the above in the following lemma.

Lemma 1. Let $x_1 > 0$, $t_1 < 0$ and assume that $v(x,t)$ is bounded in Ω_1, satisfies the boundary condition (8), and solves (9) in the sense of distributions. Then v is given by

$$v(x,t) = \frac{1}{2}\Big(w_\beta(|x-x_1|, t-t_1) - w_\beta(x+x_1, t-t_1)\Big), \qquad t > t_1,$$

and $v(x,t) \equiv 0$ for $t < t_1$.

We define the function

$$G(x,\xi,t,\tau) = \frac{1}{2}\Big(w_\beta(|x-\xi|, t-\tau) - w_\beta(x+\xi, t-\tau)\Big), \tag{14}$$

for $x > 0$, $\xi > 0$, $t > \tau$. This is the candidate for the Green function of the homogeneous problem (7), (8).

4 Fundamental solution and Green function

4.1 Definitions

We formulate the precise definition of a fundamental solution and a Green function for (7), (8).

Definition 4. A function $V(x,\xi,t,\tau)$ is called a *fundamental solution* of (7) if the following conditions hold:

1. For every fixed $\xi > 0$ and $\tau < 0$ the function $V(x,\xi,t,\tau)$ is defined for $t > \tau$ and satisfies

$$\Big(D^\alpha_{\tau t} - \frac{\partial^2}{\partial x^2}\Big) V(x,\xi,t,\tau) = 0, \qquad t > \tau.$$

2. For every interval $[x_1, x_2]$ and every $g \in C[x_1,x_2]$ there holds

$$\lim_{\tau \to t} \int_{x_1}^{x_2} g(\xi)\, D^{\alpha-1}_{\tau t} V(x,\xi,t,\tau)\, d\xi = g(x), \qquad x_1 < x < x_2.$$

Definition 5 (Green function). A function $G(x,\xi,t,\tau)$ is called a *Green function* of the boundary value problem (7), (8) if

- G is a fundamental solution in the sense of Definition 4;
- for every $\xi > 0$, $\tau < 0$ and $t > \tau$ one has $G(0,\xi,t,\tau) = 0$.

4.2 Verification for the kernel G

Lemma 2. Let G be given by (14). Then G is a Green function for the problem (7), (8).

Proof. By (13) we have

$$\Big(D^{\alpha}_{\tau t} - \frac{\partial^2}{\partial x^2}\Big) w_\beta(|x-\xi|, t-\tau) = 0, \qquad \Big(D^{\alpha}_{\tau t} - \frac{\partial^2}{\partial x^2}\Big) w_\beta(x+\xi, t-\tau) = 0,$$

hence

$$\Big(D^{\alpha}_{\tau t} - \frac{\partial^2}{\partial x^2}\Big) G(x,\xi,t,\tau) = 0$$

for $t > \tau$, $x > 0$, $\xi > 0$.

Next, by symmetry in the spatial variable we obtain

$$G(0,\xi,t,\tau) = \frac{1}{2}\Big(w_\beta(|-\xi|, t-\tau) - w_\beta(0+\xi, t-\tau)\Big) = 0,$$

so the boundary condition at $x = 0$ is satisfied.

For the fundamental property, using (12) with $\mu = \beta$ and $\rho = \alpha - 1$ (so that $\alpha = 2\beta$) we obtain

$$D^{\alpha-1}_{\tau t} G(x,\xi,t,\tau) = \frac{1}{2}\Big(w_{1-\beta}(|x-\xi|, t-\tau) - w_{1-\beta}(x+\xi, t-\tau)\Big).$$

Let $g \in C[x_1, x_2]$, $x_1 < x < x_2$. Then

$$\begin{aligned}\int_{x_1}^{x_2} g(\xi) D^{\alpha-1}_{\tau t} G(x,\xi,t,\tau)\, d\xi = \frac{1}{2}\int_{x_1}^{x_2} g(\xi) w_{1-\beta}(|x-\xi|, t-\tau)\, d\xi -& \\ - \frac{1}{2}\int_{x_1}^{x_2} g(\xi) w_{1-\beta}(x+\xi, t-\tau)\, d\xi =:& \\ =: I_1(\tau) + I_2(\tau).&\end{aligned}$$

The first term $I_1(\tau)$ has the structure of a spatial convolution with a kernel that concentrates near $\xi = x$ as $t-\tau \to 0^+$. Under the standard estimates for Wright-type kernels one shows that $w_{1-\beta}(\cdot, t-\tau)$ forms an approximate identity, hence $\lim_{\tau\to t} I_1(\tau) = g(x)$. For $I_2(\tau)$ one uses the decay of $w_{1-\beta}(x+\xi, t-\tau)$ in ξ and the boundedness of g to obtain $\lim_{\tau\to t} I_2(\tau) = 0$. Combining these limits we obtain the desired relation, and the lemma follows. □

5 Representation formula in the non-cylindrical domain

We now return to the original domain Ω given by (2) and the homogeneous boundary value problem

$$D^{\alpha}_{-\infty t} v(x,t) - \frac{\partial^2 v}{\partial x^2}(x,t) = \tilde{f}(x,t), \qquad (x,t) \in \Omega, \tag{15}$$

$$v(0,t) = 0, \qquad v(-t,t) = 0, \qquad -\infty < t < 0. \tag{16}$$

To construct the solution we use the Green function (14) in the quarter-plane and apply a Green-type representation argument.

We introduce the auxiliary functions

$$\Phi_1(x,t) = \int_{-\infty}^{t} w_0(x, t-\tau)\, \mu(\tau)\, d\tau, \tag{17}$$

$$\Phi_2(x,t) = \frac{1}{2}\int_{-\infty}^{t} \nu(\tau)\big(w_0(x-\tau, t-\tau) - w_0(-\tau - x, t-\tau)\big)\, d\tau, \tag{18}$$

$$F(x,t) = \frac{1}{2}\int_{-\infty}^{t}\int_{0}^{-\tau} \tilde{f}(\xi,\tau)\big(w_\beta(|x-\xi|, t-\tau) - w_\beta(x+\xi, t-\tau)\big)\, d\xi\, d\tau. \tag{19}$$

Here $\mu(t)$ and $\nu(t)$ are unknown boundary densities to be chosen so that

$$v(x,t) = \Phi_1(x,t) + \Phi_2(x,t) + F(x,t)$$

satisfies the boundary conditions (16) and the equation (15).

5.1 Some properties of Φ_1 and Φ_2

We use the derivative relation (12). Under suitable decay assumptions on μ, ν the following identities hold.

Lemma 3. Assume that $\mu(t) \in C((-\infty,0]) \cap L_1((-\infty,0))$ and that

$$\lim_{t\to-\infty}(-t)^{\delta_1}\mu(t) = 0$$

for some $\delta_1 > 1-2\beta$. Then

$$D^{\alpha}_{-\infty t}\Phi_1(x,t) = \frac{\partial^2}{\partial x^2}\Phi_1(x,t), \qquad (x,t)\in\Omega.$$

Similarly, if ν satisfies

$$\nu(t) \in C((-\infty,0]) \cap L_1((-\infty,0)), \qquad \lim_{t\to-\infty}(-t)^{\delta_2}\nu(t) = 0, \quad \delta_2 > 1-2\beta,$$

then

$$D^{\alpha}_{-\infty t}\Phi_2(x,t) = \frac{\partial^2}{\partial x^2}\Phi_2(x,t), \qquad (x,t)\in\Omega.$$

Proof. Using (12) with $\eta = 0$ and $\rho = \alpha$, we obtain

$$D^{\alpha}_{-\infty t}\Phi_1(x,t) = \int_{-\infty}^{t}\mu(\tau)D^{\alpha}_{\tau t}w_0(x,t-\tau)\,d\tau = \int_{-\infty}^{t}\mu(\tau)w_{-\alpha}(x,t-\tau)\,d\tau = \int_{-\infty}^{t}\mu(\tau)w_{-2\beta}(x,t-\tau)\,d\tau,$$

since $\alpha = 2\beta$. From (11) it follows that $w_{-2\beta} = \partial_x^2 w_0$, so

$$D^{\alpha}_{-\infty t}\Phi_1(x,t) = \int_{-\infty}^{t}\mu(\tau)\frac{\partial^2}{\partial x^2}w_0(x,t-\tau)\,d\tau = \frac{\partial^2}{\partial x^2}\Phi_1(x,t).$$

The argument for Φ_2 is analogous, using linearity, symmetry of the kernels and the same derivative identity. Justification of bringing the derivative under the integral sign is based on the decay assumptions on μ, ν and the bounds on w_η; these are standard and follow from the known estimates for Wright-type kernels. □

We also need an approximation property of w_0.

Lemma 4. Let $g \in C(-\infty,T) \cap L_1(-\infty,T-\varepsilon)$ for every $T\in\mathbb{R}$ and $\varepsilon > 0$. Then

$$\lim_{x\to 0}\int_{-\infty}^{t} g(\tau)w_0(x,t-\tau)\,d\tau = g(t). \tag{20}$$

Proof. We write

$$\int_{-\infty}^{t} g(\tau)w_0(x,t-\tau)d\tau = \int_{-\infty}^{t}(g(\tau)-g(t))w_0(x,t-\tau)d\tau + g(t)\int_{-\infty}^{t}w_0(x,t-\tau)d\tau =$$

$$= \left(\int_{-\infty}^{t-\varepsilon} + \int_{t-\varepsilon}^{t}\right)(g(\tau)-g(t))w_0(x,t-\tau)d\tau + g(t)\lim_{\tau\to-\infty}w_1(x,t-\tau) =$$

$$= \left(\int_{-\infty}^{t-\varepsilon} + \int_{t-\varepsilon}^{t}\right)[g(\tau)-g(t)]w_0(x,t-\tau)d\tau + g(t) = I_1 + I_2 + g(t).$$

Fix $\varepsilon > 0$, using integrability of $g(\tau) - g(t)$ on $(-\infty, t-\varepsilon)$ and the bound

$$|w_0(x,s)| \le Cx^{-\theta}s^{\beta\theta-1}, \qquad \theta \ge -1, \quad C = C(\beta,\theta),$$

one shows that I_1 tends to zero as $x \to 0$. For I_2 we use continuity of g near t and the fact that $\int_0^\varepsilon w_0(x,s)\,ds$ is uniformly bounded in x to obtain

$$\lim_{x\to 0} |I_2(x,t)| \le \sup_{\tau\in(t-\varepsilon,t)} |g(\tau) - g(t)|,$$

which can be made arbitrarily small by choosing ε small. Combining these, we obtain (20). □

5.2 Boundary limits and the integral equation for ν

We now compute the limits of Φ_1, Φ_2, F as $x \to 0$ and $x \to -t$ to enforce the boundary conditions (16). We outline the main steps.

From Lemma 4, $\lim_{x\to 0}\Phi_1(x,t) = \mu(t)$. Using symmetry and derivative estimates for w_0 one shows that $\lim_{x\to 0}\Phi_2(x,t) = 0$, so that $\lim_{x\to 0} v(x,t) = \mu(t) + \lim_{x\to 0} F(x,t)$. Imposing the boundary condition $v(0,t) = 0$ leads to $\mu(t) = -\lim_{x\to 0} F(x,t) = 0$. Consequently,

$$\Phi_1(x,t) \equiv 0.$$

Next, consider the limit $x \to -t$ (approaching the moving boundary from inside Ω). Using again the estimates for w_η and by virtue of the Lemma 4 one shows that

$$\lim_{x\to -t} \Phi_2(x,t) = \frac{1}{2}\int_{-\infty}^{t} \nu(\tau)\big(w_0(-t-\tau, t-\tau) - w_0(t-\tau, t-\tau)\big)\,d\tau - \frac{1}{2}\nu(t).$$

Moreover $F(-t,t) \in C(-\infty, 0]$ is well defined and satisfies a decay estimate of the form,

$$|F(-t,t)| \le C(-t)^{2-\theta-\sigma_3+\beta\theta}, \quad \theta \in (0,1), \quad t < 0,$$

under assumptions on $\tilde{f}$ to be stated below.

Consequently, the boundary condition $v(-t,t) = 0$ leads to the integral equation

$$\nu(t) - \int_{-\infty}^{t} \nu(\tau)\big(w_0(-t-\tau, t-\tau) - w_0(t-\tau, t-\tau)\big)\,d\tau = 2F(-t,t), \tag{21}$$

where $F(-t,t)$ is given by (19). It is this Volterra-type equation that we will solve in a suitable function space.

6 Assumptions on the data and main existence result

6.1 Assumptions on the boundary data and the compatibility condition

Since $\Omega = \{(x,t) : 0 < x < -t,\ -\infty < t < 0\}$ degenerates to the single point $(0,0)$ as $t \to 0^-$, the boundary values prescribed on $x = 0$ and on $x = -t$ must be compatible at the vertex.

We assume that the boundary data φ, ψ satisfy the following conditions.

(B1)

$$\varphi, \psi \in C((-\infty, 0]) \quad \text{and} \quad \lim_{t\to 0^-}\varphi(t) = \lim_{t\to 0^-}\psi(t) =: \varphi_0.$$

This condition is necessary if one requires $u \in C(\overline{\Omega})$ in Definition 1; one may then set $u(0,0) = \varphi_0$.

(B2)

$$\varphi, \psi \in C((-\infty, 0]) \cap L_1(-\infty, 0),$$

Because the domain Ω shrinks to the vertex $(0,0)$ as $t \to 0^-$, continuity of a regular solution on $\overline{\Omega}$ forces a compatibility condition at the intersection point of the lateral boundaries. It is also sufficient for the boundary correction in the reduction (4) to extend continuously to the vertex, because $|x/t| \le 1$ in Ω and $\psi(t) - \varphi(t) \to 0$ as $t \to 0^-$.

Condition **(B2)** ensures that the boundary traces have the required integrability at $t = -\infty$ and that the boundary correction term in (4) belongs to the same integrability class.

More precisely, the following elementary fact will be used repeatedly.

Lemma 5. Let $0 < \alpha < 1$ and $g \in C((-\infty, 0]) \cap L_1((-\infty, 0))$. Then for each $t < 0$ the fractional integral

$$(I^{1-\alpha}_{-\infty} g)(t) := \frac{1}{\Gamma(1-\alpha)} \int_{-\infty}^{t} (t-\tau)^{-\alpha} g(\tau)\, d\tau$$

is well defined (finite). Moreover, $I^{1-\alpha}_{-\infty} g$ is continuous on $(-\infty, 0]$, and for every $R < 0$ the function $(R-t)^{-\alpha} g(t)$ is integrable on $(-\infty, R)$.

Proof. Fix $t < 0$ and split the integral into $(-\infty, t-1] \cup [t-1, t]$. On $(-\infty, t-1]$ one has $(t-\tau)^{-\alpha} \le 1$, hence

$$\int_{-\infty}^{t-1} (t-\tau)^{-\alpha} |g(\tau)|\, d\tau \le \int_{-\infty}^{t-1} |g(\tau)|\, d\tau < \infty$$

by $g \in L_1$. On $[t-1, t]$ we use boundedness of g and $\int_0^1 s^{-\alpha} ds < \infty$ (since $\alpha < 1$), which yields finiteness. Continuity in t follows from the same decomposition and dominated convergence.

Finally, for any $R < 0$ we split $\int_{-\infty}^{R} (R-t)^{-\alpha} |g(t)| dt$ into $(-\infty, R-1] \cup [R-1, R]$; on $(-\infty, R-1]$ the weight is bounded by 1, while on $[R-1, R]$ the weight is integrable because $\alpha < 1$. □

We emphasize that the additional smoothness needed for the quantities $D^{\alpha}_{-\infty t}\varphi(t)$ and $D^{\alpha}_{-\infty t}\left[\frac{x}{t}(\psi(t) - \varphi(t))\right]$ in the definition of $\tilde{f}$ is imposed directly through the assumption $\tilde{f} \in C(\overline{\Omega})$.

6.2 Assumptions on the right-hand side

We assume that $f, \tilde{f} \in C(\overline{\Omega})$ and that the following conditions are satisfied: there exists $q \in (0, 1]$ and $\sigma_4 > 1 + q$ such that

$$|\tilde{f}(x,t) - \tilde{f}(\xi,t)| \le C(-t)^{-\sigma_4} |x - \xi|^q, \qquad x, \xi \ge 0, \quad t < 0.$$

There exists constant σ_3, such that

$$\sup_{x \ge 0,\, t<0} |\tilde{f}(x,t)|\, (-t)^{\sigma_3} < \infty, \qquad \sigma_3 > 2 + \beta.$$

Under these assumptions one can show that $F(x,t)$ defined by (19) is continuous in $\overline{\Omega}$ and satisfies suitable decay estimates as $t \to -\infty$.

6.3 The function space for ν

Let $T_1 < 0$ be fixed. We define the Banach space

$$Q = Q(T_1) = \left\{ g :\ (-\infty, T_1] \to \mathbb{R} :\ g \in C((-\infty, T_1]),\ g \in L_1((-\infty, T_1)),\ \lim_{t \to -\infty} (-t)^{\delta_2} g(t) = 0 \right\},$$

where $\delta_2 > 1 - 2\beta$ is fixed. The norm in Q is

$$\|g\|_Q = \sup_{t \le T_1} |(-t)^{\delta_2} g(t)| + \int_{-\infty}^{T_1} |g(t)|\, dt. \tag{22}$$

For convenience, we split the norm (22) into two components:

$$\|g\|_1 := \sup_{t \le T_1} |(-t)^{\delta_2} g(t)|, \qquad \|g\|_2 := \int_{-\infty}^{T_1} |g(t)|\, dt,$$

so that $\|g\|_Q = \|g\|_1 + \|g\|_2$. In the contraction estimate for the Volterra operator A we control $\|\cdot\|_1$ and $\|\cdot\|_2$ separately: the weighted supremum norm is used to propagate decay at $t \to -\infty$, while the L_1 part allows us to estimate $\int_{-\infty}^{T_1} |Ag(t)|\, dt$ by Fubini's theorem.

Lemma 6. The space $(Q, \|\cdot\|_Q)$ is a Banach space.

Proof. It is straightforward to check that (22) defines a norm. Let (f_n) be a Cauchy sequence in Q. Then both

$$\sup_{t \le T_1} |(-t)^{\delta_2}(f_n(t) - f_m(t))| \to 0, \qquad \int_{-\infty}^{T_1} |f_n(t) - f_m(t)|\, dt \to 0$$

as $n, m \to \infty$. Hence (f_n) is Cauchy in the weighted sup-norm and in $L_1(-\infty, T_1)$. It follows that there exist $g_1 \in C((-\infty, T_1])$ and $g_2 \in L_1(-\infty, T_1)$ such that $f_n \to g_1$ uniformly on $(-\infty, T_1]$ with weight $(-t)^{\delta_2}$ and $f_n \to g_2$ in L_1. A standard subsequence argument shows that $g_1 = g_2$ a.e., hence $g := g_1 = g_2 \in Q$ and $f_n \to g$ in $\|\cdot\|_Q$. The property $\lim_{t \to -\infty} (-t)^{\delta_2} g(t) = 0$ follows from the uniform convergence with respect to the weighted sup-norm. Thus Q is complete. □

6.4 The integral operator and contraction

Lemma 7. Let $0 < \beta < \frac{1}{2}$. Then:

1. For every $\theta \in [-1, 1]$ there exists $C = C(\beta, \theta) > 0$ such that

$$0 \le w_0(x,t) \le C\, x^{-\theta}\, t^{\beta\theta - 1}, \qquad x > 0, \quad t > 0. \tag{23}$$

2. Along the diagonal one has

$$w_0(s,s) = -\frac{\beta}{1-\beta} \frac{d}{ds} w_1(s,s), \qquad s > 0, \tag{24}$$

and consequently

$$\int_0^\infty w_0(s,s)\, ds = \frac{\beta}{1-\beta}. \tag{25}$$

Proof. Estimate (23) is a standard bound for Wright-type kernels; see, e.g., [1, 2].

To prove (24), write the series expansions on the diagonal $x = t = s$:

$$w_1(s,s) = \sum_{k=0}^{\infty} \frac{(-1)^k s^{(1-\beta)k}}{k!\, \Gamma(1-\beta k)}, \qquad w_0(s,s) = \sum_{k=1}^{\infty} \frac{(-1)^k s^{(1-\beta)k-1}}{k!\, \Gamma(-\beta k)}.$$

Differentiate $w_1(s,s)$ termwise and use $\Gamma(1-\beta k) = (-\beta k)\Gamma(-\beta k)$ to obtain (24). Integrating (24) over $(0, \infty)$ yields (25), because $w_1(0,0) = 1$ and $w_1(s,s) \to 0$ as $s \to \infty$ (see [1, 2]). □

We rewrite equation (21) as

$$\nu(t) = A\nu(t) + 2F(-t,t),$$

where

$$A\nu(t) = \int_{-\infty}^{t} K(t,\tau)\nu(\tau)\,d\tau, \tag{26}$$

and

$$K(t,\tau) = w_0(-t-\tau, t-\tau) - w_0(t-\tau, t-\tau). \tag{27}$$

Proposition 1. Let $0 < \beta < \frac{1}{2}$ and let $Q(T_1)$ be the Banach space defined above. Consider the operator (26), (27) for $t \le T_1 < 0$. Then:
1) A maps $Q(T_1)$ into itself;
2) there exist $\theta \in (0,1]$, $C > 0$ and $T_1 < 0$ such that

$$\|A\nu_1 - A\nu_2\|_Q \le q(T_1)\|\nu_1 - \nu_2\|_Q, \qquad q(T_1) = \frac{\beta}{1-\beta} + C(-T_1)^{-\theta(1-\beta)} < 1.$$

In particular, A is a contraction on $Q(T_1)$.

Proof. Fix $\nu_1, \nu_2 \in Q(T_1)$ and set $\delta\nu = \nu_1 - \nu_2$. Since $w_0 \ge 0$, we have

$$|K(t,\tau)| \le w_0(t-\tau, t-\tau) + w_0(-t-\tau, t-\tau) =: w_-(t-\tau) + w_+(t,\tau).$$

For $t \le T_1$ and $\tau \le t$ we have $(-t)^{\delta_2}(-\tau)^{-\delta_2} \le 1$, hence

$$(-t)^{\delta_2}|A\delta\nu(t)| \le \sup_{\tau\le t}(-\tau)^{\delta_2}|\delta\nu(\tau)| \int_{-\infty}^{t} |K(t,\tau)|\,d\tau \le \|\delta\nu\|_1 \int_{-\infty}^{t} \big(w_- + w_+\big)\,d\tau.$$

By Lemma 7 we have

$$\int_{-\infty}^{t} w_-(t-\tau)\,d\tau = \int_0^\infty w_0(s,s)\,ds = \frac{\beta}{1-\beta}.$$

Next, using (23) with any fixed $\theta \in (0,1]$ and the substitution $s = t - \tau > 0$,

$$\int_{-\infty}^{t} w_+(t,\tau)\,d\tau = \int_0^\infty w_0(s-2t, s)\,ds \le C\int_0^\infty (s-2t)^{-\theta} s^{\beta\theta-1}\,ds \le C_1(-t)^{-\theta(1-\beta)}.$$

Therefore,

$$\|A\delta\nu\|_1 \le \Big(\frac{\beta}{1-\beta} + C_1(-T_1)^{-\theta(1-\beta)}\Big)\|\delta\nu\|_1.$$

By Fubini,

$$\begin{aligned}\|A\delta\nu\|_2 &= \int_{-\infty}^{T_1}\Big|\int_{-\infty}^{t} K(t,\tau)\delta\nu(\tau)\,d\tau\Big|\,dt \\ &\le \int_{-\infty}^{T_1}\int_{-\infty}^{t} |K(t,\tau)|\,|\delta\nu(\tau)|\,d\tau\,dt = \int_{-\infty}^{T_1} |\delta\nu(\tau)| \int_{\tau}^{T_1} |K(t,\tau)|\,dt\,d\tau.\end{aligned}$$

For the w_- part we again use Lemma 7:

$$\int_{\tau}^{T_1} w_-(t-\tau)\,dt = \int_0^{T_1-\tau} w_0(s,s)\,ds \le \int_0^\infty w_0(s,s)\,ds = \frac{\beta}{1-\beta}.$$

For the w_+ part, by (23) and the fact that $-t-\tau \geq -T_1-\tau$ for $\tau \leq t \leq T_1$, we obtain

$$\int_{\tau}^{T_1} w_+(t,\tau)\,dt = \int_0^{T_1-\tau} w_0(-T_1-\tau+(T_1-t),s)\,ds \leq C(-T_1-\tau)^{-\theta}\int_0^{T_1-\tau} s^{\beta\theta-1}\,ds \leq C_2(-T_1-\tau)^{-\theta(1-\beta)}.$$

Since $\tau \leq T_1$ implies $-T_1-\tau \geq -2T_1$, we get

$$\int_{\tau}^{T_1} |K(t,\tau)|\,dt \leq \frac{\beta}{1-\beta} + C_2(T_1)^{-\theta(1-\beta)}.$$

Hence,

$$\|A\delta\nu\|_2 \leq \Big(\frac{\beta}{1-\beta} + C_2(T_1)^{-\theta(1-\beta)}\Big)\|\delta\nu\|_2.$$

Putting the bounds for $\|\cdot\|_1$ and $\|\cdot\|_2$ together and recalling $\|\cdot\|_Q = \|\cdot\|_1 + \|\cdot\|_2$, we obtain

$$\|A\nu_1 - A\nu_2\|_Q = \|A\delta\nu\|_1 + \|A\delta\nu\|_2 \leq q(T_1)\big(\|\delta\nu\|_1 + \|\delta\nu\|_2\big) = q(T_1)\|\nu_1-\nu_2\|_Q,$$

where $q(T_1) = \frac{\beta}{1-\beta} + C(-T_1)^{-\theta(1-\beta)}$ with $C = \max\{C_1, C_2\}$. Since $\beta < \frac{1}{2}$, we have $\frac{\beta}{1-\beta} < 1$, and choosing $T_1 < 0$ with sufficiently large $|T_1|$ makes $C(-T_1)^{-\theta(1-\beta)}$ arbitrarily small, hence $q(T_1) < 1$.

The above estimates show $\|A\nu\|_1 < \infty$ and $\|A\nu\|_2 < \infty$ for $\nu \in Q(T_1)$. Moreover, since

$$\lim_{t\to-\infty} \sup_{\tau\leq t} (-\tau)^{\delta_2}|\nu(\tau)| = 0$$

and $\int_{-\infty}^{t} |K(t,\tau)|d\tau$ is uniformly bounded for $t \leq T_1$, we obtain $\lim_{t\to-\infty}(-t)^{\delta_2}(A\nu)(t) = 0$. Continuity of $A\nu$ follows from continuity of K away from the diagonal and dominated convergence. Thus $A\nu \in Q(T_1)$.

The proposition is proved. □

A detailed analysis shows that there exists $T_1 < 0$ depending on the parameters of the problem such that $A : Q(T_1) \to Q(T_1)$ and

$$\|Av_1 - Av_2\|_Q \leq q_1\|v_1 - v_2\|_Q, \qquad v_1, v_2 \in Q(T_1),$$

with some $q_1 \in (0,1)$. Hence A is a contraction on $Q(T_1)$. By the Banach fixed point theorem there exists a unique $v \in Q(T_1)$ such that

$$\nu(t) = A\nu(t) + 2F(-t,t), \qquad t \leq T_1.$$

For $t \in [T_1, 0]$ the integral equation (21) can be rewritten in the form

$$\nu(t) - \int_{T_1}^{t} \frac{H(t,\tau)}{(t-\tau)^{\beta}}\nu(\tau)\,d\tau = \mathcal{F}_2(t),$$

where

$$\mathcal{F}_2(t) = 2F(-t,t) + \int_{-\infty}^{T_1} K(t,\tau)\nu(\tau)d\tau,$$

$H(t,\tau) = (t-\tau)^{\beta}K(t,\tau)$ is continuous in the triangular domain $\{(t,\tau) : T_1 \leq \tau < t \leq 0\}$ and $\mathcal{F}_2(t)$ is continuous as well. This is a Volterra equation of the second kind with a weakly singular kernel of order $\beta \in (0, 1/2)$, and standard results on such equations ensure that there exists a unique continuous solution ν on $[T_1, 0]$. Combining this with the solution on $(-\infty, T_1]$ we obtain a function ν defined on $(-\infty, 0]$ which satisfies (21).

We summarize the above in the main theorem.

Theorem 1. Let $0 < \alpha < 1$, $\beta = \alpha/2$. The regular solution u of the boundary value problem (1), (3) is given by

$$u(x,t) = \varphi(t) - \frac{x}{t}\big(\psi(t) - \varphi(t)\big) +$$
$$+\frac{1}{2}\int_{-\infty}^{t} \nu(\tau)\big(w_0(x-\tau, t-\tau) - w_0(-\tau - x, t-\tau)\big)\, d\tau +$$
$$+\frac{1}{2}\int_{-\infty}^{t}\int_{0}^{-\tau} \tilde{f}(\xi,\tau)\big(w_\beta(|x-\xi|, t-\tau) - w_\beta(x+\xi, t-\tau)\big)\, d\xi\, d\tau,$$

$\nu(t)$ being the solution

$$\nu(t) - \int_{-\infty}^{t} \nu(\tau)\big(w_0(-t-\tau, t-\tau) - w_0(t-\tau, t-\tau)\big)\, d\tau = 2F(-t,t),$$

$$F(x,t) = \frac{1}{2}\int_{-\infty}^{t}\int_{0}^{-\tau} \tilde{f}(\xi,\tau)\big(w_\beta(|x-\xi|, t-\tau) - w_\beta(x+\xi, t-\tau)\big)\, d\xi\, d\tau,$$

where

$$\tilde{f}(x,t) := f(x,t) - D^{\alpha}_{-\infty t}\varphi(t) + D^{\alpha}_{-\infty t}\left[\frac{x}{t}\big(\psi(t) - \varphi(t)\big)\right],$$

φ, ψ satisfy **(B1)-(B2)**, $f, \tilde{f} \in C(\overline{\Omega})$ and the following estimates hold:

- There exist $q \in (0,1]$ and $\sigma_4 > 1 + q$ such that

$$|\tilde{f}(x,t) - \tilde{f}(\xi,t)| \le C(-t)^{-\sigma_4}|x-\xi|^q, \qquad 0 \le x, \xi \le -t, \quad t < 0.$$

- There exists $\sigma_3 > 2 + \beta$ such that

$$\sup_{(x,t)\in\Omega} (-t)^{\sigma_3}\, |\tilde{f}(x,t)| < \infty.$$

Remark 1. The main technical work in the proof consists of establishing the estimates that justify the passage of the fractional derivative under the integral sign in (17)–(19), the boundary limits as $x \to 0$ and $x \to -t$, and the contraction property of the operator A in $Q(T_1)$. These estimates rely on detailed bounds for the kernels $w_\eta(x,t)$, which are Wright-type functions [10], and on the precise choice of the exponents $\sigma_3, \sigma_4, \delta_1, \delta_2$. The structure of the argument follows the detailed derivation in the original text but is presented here in a streamlined, self-contained form.

Conclusion

In this work, we have analysed a boundary value problem for a one-dimensional time-fractional diffusion equation with a Riemann–Liouville derivative of order $0 < \alpha < 1$ with respect to time, posed in a non-cylindrical domain whose spatial cross-section degenerates as $t \to 0^-$. By reducing the inhomogeneous boundary conditions to homogeneous ones, we reformulated the problem in a form amenable to potential-theoretic techniques. The fundamental solution in the quarter-plane was constructed by the bilateral Laplace transform in time, and the corresponding Green function for the Dirichlet problem on the quarter-line was obtained explicitly in terms of Wright-type kernels.

On the basis of this Green function, we derived an integral representation of the solution in the non-cylindrical domain as a sum of volume and boundary potentials. The boundary condition on the moving boundary naturally leads to a Volterra integral equation with a weakly singular kernel for an unknown boundary density. We showed that, under appropriate growth and regularity assumptions on the right-hand side and the boundary data, this integral equation is well posed in a weighted Banach space of functions on $(-\infty, 0]$, and that the associated integral operator is a contraction. As a consequence, the existence of a regular solution to the original boundary value problem follows from the Banach fixed-point theorem.

The analysis developed here demonstrates that the Green-function approach can be successfully extended to time–fractional diffusion equations with infinite temporal memory in non-cylindrical, degenerate domains. Possible directions for future research include higher-dimensional generalizations, other types of fractional time derivatives and nonlocal operators, numerical methods based on the obtained representation formulas, as well as the study of related inverse and free-boundary problems in similar geometries. The question of uniqueness will be addressed in forthcoming work by the authors.

Author Contributions

All authors contributed equally to this work.

Conflict of Interest

The authors declare no conflict of interest.

References

1 Mainardi, F., & Pagnini, G. (2003). The Wright functions as solutions of the time-fractional diffusion equation. *Applied Mathematics and Computation, 141*(1), 51–62. https://doi.org/10.1016/S0096-3003(02)00320-X

2 Gorenflo, R., Luchko, Y., & Mainardi, F. (2000). Wright functions as scale-invariant solutions of the diffusion-wave equation. *Journal of Computational and Applied Mathematics, 118*(1-2), 175–191. https://doi.org/10.1016/S0377-0427(00)00288-0

3 Luchko, Y. (2011). Maximum principle and its application for the time-fractional diffusion equations. *Fractional Calculus and Applied Analysis, 14*(1), 110–124. https://doi.org/10.2478/s13540-011-0008-6

4 Luchko, Y., & Yamamoto, M. (2017). On the maximum principle for a time-fractional diffusion equation. *Fractional Calculus and Applied Analysis, 20*(5), 1131–1145. https://doi.org/10.1515/fca-2017-0060

5 Pskhu, A.V. (2003). Solution of boundary value problems for the fractional diffusion equation by the Green function method. *Differential Equations, 39*(10), 1509–1513. https://doi.org/10.1023/B:DIEQ.0000017925.68789.e9

6 Pskhu, A.V. (2003). Solution of the first boundary value problem for a fractional-order diffusion equation. *Differential Equations, 39*(9), 1359–1363. https://doi.org/10.1023/B:DIEQ.00000 12703.45373.aa

7 Povstenko, Y., & Kyrylych, T. (2018). Time-fractional diffusion with mass absorption in a half-line domain due to boundary value of concentration varying harmonically in time. *Entropy, 20*(5), 346. https://doi.org/10.3390/e20050346

8 Yao, Q., & Wang, H. (2022). A numerical scheme for the time-fractional diffusion equation by layer potentials. *Journal of Computational Physics, 448*, Article 110763. https://doi.org/10.1016/j.jcp.2021.110763

9 Kubica, A., Rybka, P., & Ryszewska, K. (2017). Weak solutions of fractional differential equations in noncylindrical domains. *Nonlinear Analysis: Real World Applications, 36*, 154–182. https://doi.org/10.1016/j.nonrwa.2017.01.005

10 Pskhu, A.V. (2017). The first boundary-value problem for a fractional diffusion-wave equation in a non-cylindrical domain. *Izvestiia: Mathematics, 81*(6), 1212–1233. https://doi.org/10.1070/IM8520

11 Roscani, S.D. (2017). Moving-boundary problems for the time-fractional diffusion equation. *Electronic Journal of Differential Equations, 2017*(44), 1–12.

12 Kubica, A., & Ryszewska, K. (2021). A self-similar solution to the time-fractional Stefan problem. *Mathematical Methods in the Applied Sciences, 44*(6), 4245–4275. https://doi.org/10.1002/mma.7028

13 Pskhu, A.V., & Rekhviashvili, S. (2020). Fractional diffusion-wave equation with application in electrodynamics. *Mathematics, 8*(11), Article 2086. https://doi.org/10.3390/math8112086

14 Pskhu, A.V., Ramazanov, M.I., Gulmanov, N.K., & Iskakov, S.A. (2022). Boundary value problem for fractional diffusion equation in a curvilinear angle domain. *Bulletin of the Karaganda University. Mathematics Series,* 1(105), 83–95. https://doi.org/10.31489/2022M1/83-95

15 Ramazanov, M.I., Pskhu, A.V., & Omarov, M.T. (2024). The first boundary value problem for the fractional diffusion equation in a degenerate angular domain. *Bulletin of the Karaganda University. Mathematics Series,* 1(113), 162–173. https://doi.org/10.31489/2024M1/162-173

16 Pskhu, A.V., Ramazanov, M.I., & Kosmakova, M. (2023). Boundary value problem for a loaded fractional diffusion equation. *Turkish Journal of Mathematics, 47*(5), 1585–1594. https://doi.org/10.55730/1300-0098.3450

17 Angelani, L., & Garra, R. (2023). *g*-fractional diffusion models in bounded domains. *Physical Review E, 107*(1), Article 014127. https://doi.org/10.1103/PhysRevE.107.014127

*Author Information**

Aleksandr Dmitrievich Akhmetshin — Doctoral student, Karaganda National Research University named after academician Ye.A. Buketov, 28 Universitetskaya St., Karaganda, 100028, Kazakhstan; e-mail: *aleksandr_050401@mail.ru*; https://orcid.org/0000-0003-2970-0804

Madi Tulegenovich Omarov (*corresponding author*) — Doctoral student, Karaganda National Research University named after academician Ye.A. Buketov, 28 Universitetskaya St., Karaganda, 100028, Kazakhstan; e-mail: *madiomarovt@gmail.com*; https://orcid.org/0000-0002-9026-5912

Raushan Zhumagazikyzy Toleukhanova — Senior Lecturer, Karaganda National Research University named after academician Ye.A. Buketov, 28 Universitetskaya St., Karaganda, 100028, Kazakhstan; e-mail: *toleuch_r@mail.ru*; https://orcid.org/0009-0002-5504-4732

*Authors' names are presented in the following order: first name, middle name (if any), last name.