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On integro-differential equations with the highest-order derivative in the integral term

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In this article, a two-point boundary value problem for an integro-differential equation in which the highest-order derivatives appear in the integral term is considered. The Dzhumabaev parametrization method is applied to solve the problem. The original problem is reduced to an equivalent problem for an integro-differential equation with parameters. The resulting problem includes an integro-differential equation with parameters, an initial condition, and an additional relation. Conditions for the existence and uniqueness of a solution to the integro-differential equation with parameters are established in terms of the coefficients and kernels of the equation, as well as the boundary functions. An explicit representation of the solution in terms of the parameters is constructed. The unique solvability of the original two-point boundary value problem is established in terms of the initial data. A special case of the integro-differential equation with the highest-order derivative appearing in the integral term, subject to two-point boundary conditions, is also investigated. The Dzhumabaev parametrization method is used to solve the problem. An explicit form of the solution is obtained.

Keywords: integro-differential equations, highest-order derivative in the integral term, two-point condition, continuous coefficients, Dzhumabaev parametrization method, parametrized problem, functional term, existence and uniqueness, explicit solution.

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Introduction

We consider a two-point boundary value problem for the integro-differential equation with higher-order derivatives in the integral term

$$A_1(x)z'(x) + A_0(x)z(x) = F(x) + \\ + \int_0^1 \Big\{ K_0(x)L_0(s)z''(s) + K_1(x)L_1(s)z'(s) + K_2(x)L_2(s)z(s) \Big\} ds, \qquad x \in [0,1], \quad (1)$$

$$Bz(0) + Cz(1) = d, \quad (2)$$

where $z(x)$ is the unknown function; the functions $A_0(x)$, $A_1(x)$, and $F(x)$ are continuous on $[0,1]$; the functions $K_j(x)$, $j = 0,1,2$ are continuous on $[0,1]$; the functions $L_1(s)$ and $L_2(s)$ are continuous on $[0,1]$; $L_0(s)$ is continuously differentiable on $[0,1]$; and B, C, and d are constants.

Let $C([0,1],\mathbb{R})$ denote the space of real-valued continuous functions on $[0,1]$, equipped with the norm $||z||_0 = \max_{x\in[0,1]} ||z(x)||$.

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A function $z(x) \in C([0,1],\mathbb{R})$ is said to be a solution of the two-point problem (1)–(2) if:

1) $z(x)$ possesses derivatives $z'(x)$ and $z''(x)$ at each point $x \in [0,1]$;

2) $z(x)$ satisfies the integro-differential equation (1) for all $x \in [0,1]$;

3) $z(x)$ satisfies the boundary condition (2) at the points $x = 0$ and $x = 1$.

The necessity of mathematical modeling of processes involving singularities and small parameters has stimulated the development of the theory of integro-differential equations with different derivative orders in their differential and integral parts [1,2]. A substantial subclass of such equations consists of first-order integro-differential equations in which higher-order derivatives appear in the integral term.

The existence and uniqueness of solutions to initial value problems for integro-differential equations of various orders, as well as their asymptotic behavior and stability, have been extensively investigated (see the references therein). Integro-differential equations with highest-order derivatives in the integral part were studied in [3–5]. The works [6–8] were devoted to investigate an important class of such equations.

It should be emphasized that integro-differential equations with highest-order derivatives in the integral part are particularly effective for modeling processes with aftereffect phenomena, as well as biological and medical systems involving memory.

If the coefficients $A_0(x)$, $A_1(x)$, the right-hand side $F(x)$, and the kernels $K_j(x)$, $j = 0,1,2$, are continuously differentiable on $[0,1]$, then equation (1), after differentiation of both sides with respect to x, can be reduced to an integro-differential equation of neutral or Fredholm type. In that case, the order of the derivative in the integral term becomes equal to or lower than that in the differential part. Numerous works have been devoted to solvability issues in this setting.

More interesting and technically challenging are the cases where the coefficients $A_0(x)$, $A_1(x)$, the function $F(x)$, and the kernels $K_j(x)$, $j = 0,1,2$, are only continuous on $[0,1]$.

The present paper is devoted precisely to this situation. We investigate the solvability of a two-point boundary value problem for an integro-differential equation in which the integral term contains derivatives of higher order than those appearing in the differential part.

New approaches to solving boundary value problems for systems of integro-differential equations, as well as for loaded differential equations, were proposed in [9–11]. In the works of Dzhumabaev [12–14], coefficient criteria for the unique solvability of boundary value problems for Fredholm systems of integro-differential equations were established. Significant results were also obtained for systems of first-order Fredholm nonlinear integro-differential equations in [15]. Problems with parameters for Fredholm systems of integro-differential equations were investigated in [16,17]. In the works [16,17], also numerical algorithms were proposed, and analysis were carried out for integro-differential systems with integral conditions [18].

In the works of Dzhumabaev [9–11], systems of first-order Fredholm integro-differential equations with two-point conditions for various types of integral kernels were investigated.

The integro-differential equation considered in this paper cannot be reduced to a system of first-order integro-differential equations for two reasons. First, the order of the derivative in the differential part cannot be increased to second order, since the coefficients of the differential part, $A_1(x)$, $A_0(x)$, and the right-hand side $F(x)$ are only continuous functions on the interval $[0,1]$. Second, integro-differential equation (1) cannot be differentiated twice with respect to x.

Thus, we cannot increase the order of the derivative in the differential part to second order. This class of integro-differential equations requires special study and the development of methods for solving boundary value problems for such equations.

In this paper, we study the solvability of the first-order integro-differential equation with highest-order derivatives appearing in the integral part (1)–(2).

1 *Scheme of the Parametrization Method and Equivalent Problem*

To solve the two-point boundary value problem (1)–(2), we apply the Dzhumabaev parametrization method [19].

Let $\lambda = z(0)$. Introducing a new function $\widetilde{z}(x)$, we perform the change of variables in problem (1)–(2) given by

$$z(x) = \widetilde{z}(x) + \lambda, \qquad x \in [0,1].$$

As a result, we obtain an equivalent problem for an integro-differential equation with a parameter

$$A_1(x)\widetilde{z}'(x) = -A_0(x)\widetilde{z}(x) - A_0(x)\lambda + \int_0^1 K_2(x)L_2(s)ds\lambda + F(x)+$$
$$+\int_0^1 \Big\{K_0(x)L_0(s)\widetilde{z}''(s) + K_1(x)L_1(s)\widetilde{z}'(s) + K_2(x)L_2(s)\widetilde{z}(s)\Big\}ds, \quad x \in [0,1]. \quad (3)$$

$$\widetilde{z}(0) = 0, \quad (4)$$

$$[B + C]\lambda + C\widetilde{z}(1) = d. \quad (5)$$

A pair $(\widetilde{z}(x), \lambda)$ is said to be a solution of the problem for the integro-differential equation with parameter (3)–(5) if:

1) $\widetilde{z}(x)$ possesses derivatives $\widetilde{z}'(x)$ and $\widetilde{z}''(x)$ at each point $x \in [0,1]$;

2) $\widetilde{z}(x)$ and λ satisfy the integro-differential equation (3) for all $x \in [0,1]$;

3) the initial condition (4) is satisfied by $\widetilde{z}(x)$ at the point $x = 0$;

4) condition (5) is satisfied by $\widetilde{z}(x)$ and λ at the point $x = 1$.

For a fixed value of λ, equation (3) together with condition (4) constitutes a Cauchy problem for the integro-differential equation with parameter. The parameter λ is then determined from relation (5).

Next, we describe the construction of a solution to the Cauchy problem for the integro-differential equation with parameter (3), (4).

First, we introduce the following notation

$$\theta_0 = \int_0^1 L_0(s)\widetilde{z}''(s)ds, \qquad \theta_1 = \int_0^1 L_1(s)\widetilde{z}'(s)ds, \qquad \theta_2 = \int_0^1 L_2(s)\widetilde{z}(s)ds.$$

Let $A_1(x) \neq 0$ for all $x \in [0,1]$ and

$$a(x) = -\int_0^x [A_1(s)]^{-1}A_0(s)ds, \qquad x \in [0,1].$$

The solution to the Cauchy problem for the integro-differential equation with parameter (3), (4) can be represented in the following form:

$$\widetilde{z}(x) = e^{a(x)}\int_0^x e^{-a(s)}[A_1(s)]^{-1}F(s)ds-$$

$$-e^{a(x)}\int_0^x e^{-a(s)}[A_1(s)]^{-1}A_0(s)ds\lambda + e^{a(x)}\int_0^x e^{-a(s)}[A_1(s)]^{-1}K_2(s)\int_0^1 L_2(s_1)ds_1ds\lambda+$$

$$+e^{a(x)}\int_0^x e^{-a(s)}[A_1(s)]^{-1}K_0(s)ds\theta_0 + e^{a(x)}\int_0^x e^{-a(s)}[A_1(s)]^{-1}K_1(s)ds\theta_1+$$

$$+e^{a(x)}\int_0^x e^{-a(s)}[A_1(s)]^{-1}K_2(s)ds\theta_2, \qquad x\in[0,1]. \quad (6)$$

Let

$$\tilde{L} = \int_0^1 L_2(s_1)ds_1, \qquad U(x,Z) = e^{a(x)}\int_0^x e^{-a(s)}[A_1(s)]^{-1}Z(s)ds, \qquad x\in[0,1],$$

where $Z(x)$ is an arbitrary function, continuous on $[0,1]$.

Next, representation (6) can be rewritten in the following form:

$$\widetilde{z}(x) = U(x,F) - U(x,A_0)\lambda + U(x,K_2)\tilde{L}\lambda + U(x,K_0)\theta_0 + U(x,K_1)\theta_1 + U(x,K_2)\theta_2, \qquad x\in[0,1]. \quad (7)$$

Introduce the notations

$$D_i(F) = \int_0^1 L_i(\xi)U(\xi,F)d\xi, \quad D_i(A_0) = \int_0^1 L_i(\xi)U(\xi,A_0)d\xi, \quad D_i(K_j) = \int_0^1 L_i(\xi)U(\xi,K_j)d\xi,$$

$$i=1,2; \qquad j=0,1,2.$$

Replacing x by ξ, multiplying both sides of (7) by $L_1(\xi)$ and $L_2(\xi)$, and then integrating over the interval $[0,1]$, we obtain two equations for θ_1 and θ_2:

$$\theta_1 = D_1(F) - D_1(A_0)\lambda + D_1(K_2)\tilde{L}\lambda + D_1(K_0)\theta_0 + D_1(K_1)\theta_1 + D_1(K_2)\theta_2, \quad (8)$$

$$\theta_2 = D_2(F) - D_2(A_0)\lambda + D_2(K_2)\tilde{L}\lambda + D_2(K_0)\theta_0 + D_2(K_1)\theta_1 + D_2(K_2)\theta_2. \quad (9)$$

Suppose that $\Phi_2 = 1 - D_2(K_2) \neq 0$. Then equation (9) uniquely determines θ_2:

$$\theta_2 = -\Phi_2^{-1}D_2(A_0)\lambda + \Phi_2^{-1}D_2(K_2)\tilde{L}\lambda + \Phi_2^{-1}D_2(K_0)\theta_0 + \Phi_2^{-1}D_2(K_1)\theta_1 + \Phi_2^{-1}D_2(F). \quad (10)$$

Substituting the expression for θ_2 into (8), we derive

$$\Big[1 - D_1(K_1) - D_1(K_2)\Phi_2^{-1}D_2(K_1)\Big]\theta_1 = \Big[D_1(K_2)\tilde{L} - D_1(A_0) + D_2(K_2)\Phi_2^{-1}[D_2(K_2)\tilde{L} - D_2(A_0)]\Big]\lambda+$$

$$+\Big[D_1(K_0) + D_2(K_2)\Phi_2^{-1}D_2(K_0)\Big]\theta_0 + D_1(F) + D_1(K_2)\Phi_2^{-1}D_2(F). \quad (11)$$

Suppose that $\Phi_1 = 1 - D_1(K_1) - D_1(K_2)\Phi_2^{-1}D_2(K_1) \neq 0$. Then equation (11) uniquely determines θ_1:

$$\theta_1 = U_1\lambda + V_1\theta_0 + G_1, \quad (12)$$

where $U_1 = \Phi_1^{-1}\Big[D_1(K_2)\tilde{L} - D_1(A_0) + D_2(K_2)\Phi_2^{-1}[D_2(K_2)\tilde{L} - D_2(A_0)]\Big],$

$V_1 = \Phi_1^{-1}\Big[D_1(K_0) + D_2(K_2)\Phi_2^{-1}D_2(K_0)\Big], \quad G_1 = \Phi_1^{-1}D_1(F) + \Phi_1^{-1}D_1(K_2)\Phi_2^{-1}D_2(F).$

Thus, we have expressed θ_1 in terms of λ and θ_0.

Substituting the obtained expression for θ_1 into (10), we obtain

$$\theta_2 = U_2\lambda + V_2\theta_0 + G_2, \tag{13}$$

where $U_2 = \Phi_2^{-1}\Big\{D_2(K_2)\tilde{L} - D_2(A_0) + D_2(K_1)\Phi_1^{-1}\Big[D_1(K_2)\tilde{L} - D_1(A_0) +$

$+ D_2(K_2)\Phi_2^{-1}[D_2(K_2)\tilde{L} - D_2(A_0)]\Big]\Big\},$

$V_2 = \Big\{\Phi_2^{-1}D_2(K_0) + \Phi_2^{-1}D_2(K_1)\Phi_1^{-1}\Big[D_1(K_0) + D_2(K_2)\Phi_2^{-1}D_2(K_0)\Big]\Big\},$

$G_2 = \Phi_2^{-1}D_2(F) + \Phi_2^{-1}D_2(K_1)\Big[\Phi_1^{-1}D_1(F) + \Phi_1^{-1}D_1(K_2)\Phi_2^{-1}D_2(F)\Big].$

Consequently, θ_1 and θ_2 are represented in terms of λ and θ_0.

We now proceed to consider

$$\theta_0 = \int_0^1 L_0(s)\widetilde{z}''(s)ds.$$

By integrating the integral in this expression by parts, we obtain

$$\int_0^1 L_0(s)\widetilde{z}''(s)ds = L_0(1)\widetilde{z}'(1) - L_0(0)\widetilde{z}'(0) - \int_0^1 L_0'(s)\widetilde{z}'(s)ds.$$

We have

$$\theta_0 = L_0(1)\widetilde{z}'(1) - L_0(0)\widetilde{z}'(0) - \int_0^1 L_0'(s)\widetilde{z}'(s)ds.$$

Using (3), (4) and (7), we determine $\widetilde{z}'(1)$, $\widetilde{z}'(0)$ and $\int_0^1 L_0'(s)\widetilde{z}'(s)ds$:

$$\widetilde{z}'(1) = -[A_1(1)]^{-1}A_0(1)\widetilde{z}(1) - [A_1(1)]^{-1}A_0(1)\lambda + [A_1(1)]^{-1}K_2(1)\tilde{L}\lambda + [A_1(1)]^{-1}F(1) + \\ + [A_1(1)]^{-1}K_0(1)\theta_0 + [A_1(1)]^{-1}K_1(1)\theta_1 + [A_1(1)]^{-1}K_2(1)\theta_2, \tag{14}$$

$$\widetilde{z}'(0) = -[A_1(0)]^{-1}A_0(0)\lambda + [A_1(0)]^{-1}K_2(0)\tilde{L}\lambda + [A_1(0)]^{-1}F(0) + \\ + [A_1(0)]^{-1}K_0(0)\theta_0 + [A_1(0)]^{-1}K_1(0)\theta_1 + [A_1(0)]^{-1}K_2(0)\theta_2, \tag{15}$$

$$\int_0^1 L_0'(s)\widetilde{z}'(s)ds = -\int_0^1 L_0'(s)[A_1(s)]^{-1}A_0(s)\widetilde{z}(s)ds + [E_1(K_2)\tilde{L} - E_1(A_0)]\lambda + \\ + E_1(F) + E_1(K_0)\theta_0 + E_1(K_1)\theta_1 + E_1(K_2)\theta_2, \tag{16}$$

where

$$E_1(A_0) = \int_0^1 L_0'(s)[A_1(s)]^{-1}A_0(s)ds, \quad E_1(F) = \int_0^1 L_0'(s)[A_1(s)]^{-1}F(s)ds,$$

$$E_1(K_j) = \int_0^1 L_0'(s)[A_1(s)]^{-1}K_j(s)ds, \qquad j = 0, 1, 2.$$

We find

$$\int_0^1 L_0'(s)[A_1(s)]^{-1}A_0(s)\widetilde{z}(s)ds = E_2(F) + E_2(K_2)\tilde{L}\lambda - E_2(A_0)\lambda + \\ + E_2(K_0)\theta_0 + E_2(K_1)\theta_1 + E_2(K_2)\theta_2, \quad (17)$$

where

$$E_2(F) = \int_0^1 L_0'(s)[A_1(s)]^{-1}A_0(s)U(s,F)ds, \quad E_2(A_0) = \int_0^1 L_0'(s)[A_1(s)]^{-1}A_0(s)U(s,A_0)ds,$$

$$E_2(K_j) = \int_0^1 L_0'(s)[A_1(s)]^{-1}A_0(s)U(s,K_j)ds, \quad j = 0, 1, 2.$$

We also find

$$\widetilde{z}(1) = U(1,F) - U(1,A_0)\lambda + U(1,K_2)\tilde{L}\lambda + U(1,K_0)\theta_0 + U(1,K_1)\theta_1 + U(1,K_2)\theta_2. \quad (18)$$

Next, we replace $\widetilde{z}(1)$ in (14) with the expression given in (18):

$$\widetilde{z}'(1) = [A_1(1)]^{-1}F(1) - [A_1(1)]^{-1}A_0(1)U(1,F) + \\ + \Big\{[A_1(1)]^{-1}A_0(1)U(1,A_0) - [A_1(1)]^{-1}A_0(1)U(1,K_2)\tilde{L} - [A_1(1)]^{-1}A_0(1) + [A_1(1)]^{-1}K_2(1)\tilde{L}\Big\}\lambda + \\ + \Big\{[A_1(1)]^{-1}K_0(1) - [A_1(1)]^{-1}A_0(1)U(1,K_0)\Big\}\theta_0 + \\ + \Big\{[A_1(1)]^{-1}K_1(1) - [A_1(1)]^{-1}A_0(1)U(1,K_1)\Big\}\theta_1 + \\ + \Big\{[A_1(1)]^{-1}K_2(1) - [A_1(1)]^{-1}A_0(1)U(1,K_2)\Big\}\theta_2. \quad (19)$$

Finally, using (19), (15) and (16), (17), we find an expression for θ_0:

$$\theta_0 = W(F) + [W(K_2)\tilde{L} - W(A_0)]\lambda + W(K_0)\theta_0 + W(K_1)\theta_1 + W(K_2)\theta_2, \quad (20)$$

$$W(F) = L_0(1)[A_1(1)]^{-1}F(1) - [A_1(1)]^{-1}A_0(1)U(1,F) - L_0(0)[A_1(0)]^{-1}F(0) + E_2(F) - E_1(F),$$

$$W(A_0) = L_0(1)\Big\{[A_1(1)]^{-1}A_0(1) - [A_1(1)]^{-1}A_0(1)U(1,A_0)\Big\} - L_0(0)[A_1(0)]^{-1}A_0(0) + E_2(A_0) - E_1(A_0),$$

$$W(K_0) = L_0(1)\Big\{[A_1(1)]^{-1}K_0(1) - [A_1(1)]^{-1}A_0(1)U(1,K_0)\Big\} - L_0(0)[A_1(0)]^{-1}K_0(0) + E_2(K_0) - E_1(K_0),$$

$$W(K_1) = L_0(1)\Big\{[A_1(1)]^{-1}K_1(1) - [A_1(1)]^{-1}A_0(1)U(1,K_1)\Big\} - L_0(0)[A_1(0)]^{-1}K_1(0) + E_2(K_1) - E_1(K_1),$$

$$W(K_2) = L_0(1)\Big\{[A_1(1)]^{-1}K_2(1) - [A_1(1)]^{-1}A_0(1)U(1,K_2)\Big\} - L_0(0)[A_1(0)]^{-1}K_2(0) + E_2(K_2) - E_1(K_2).$$

Substituting the expression (18) for $\widetilde{z}(1)$ into condition (5), we obtain

$$[B+C]\lambda + C[U(1,K_2)\tilde{L} - U(1,A_0)]\lambda + CU(1,K_0)\theta_0 + CU(1,K_1)\theta_1 + CU(1,K_2)\theta_2 = d - CU(1,F). \quad (21)$$

Using the expressions (12) and (13), and substituting them into (20), we have

$$\Big\{1 - W(K_0) - W(K_1)V_1 - W(K_2)V_2\Big\}\theta_0 =$$
$$= \Big\{W(K_2)\tilde{L} - W(A_0) + W(K_1)U_1 + W(K_2)U_2\Big\}\lambda + W(F) + W(K_1)G_1 + W(K_2)G_2. \quad (22)$$

Let $Q_1 = 1 - W(K_0) - W(K_1)V_1 - W(K_2)V_2$ and assume that $Q_1 \neq 0$.

From (22), we obtain

$$\theta_0 = [Q_1]^{-1}\Big\{W(K_2)\tilde{L} - W(A_0) + W(K_1)U_1 + W(K_2)U_2\Big\}\lambda +$$
$$+ [Q_1]^{-1}\Big\{W(F) + W(K_1)G_1 + W(K_2)G_2\Big\}. \quad (23)$$

Substituting the expression (23) for θ_0 into (12) and (13), we get

$$\theta_1 = U_1\lambda + V_1[Q_1]^{-1}\Big\{W(K_2)\tilde{L} - W(A_0) + W(K_1)U_1 + W(K_2)U_2\Big\}\lambda +$$
$$+ G_1 + [Q_1]^{-1}\Big\{W(F) + W(K_1)G_1 + W(K_2)G_2\Big\}, \quad (24)$$

$$\theta_2 = U_2\lambda + V_2[Q_1]^{-1}\Big\{W(K_2)\tilde{L} - W(A_0) + W(K_1)U_1 + W(K_2)U_2\Big\}\lambda +$$
$$+ G_2 + [Q_1]^{-1}\Big\{W(F) + W(K_1)G_1 + W(K_2)G_2\Big\}. \quad (25)$$

Using the values of θ_0, θ_1, and θ_2 found from (23)—(25), and substituting them into (21), we have

$$\Big\{B + C + C[U(1,K_2)\tilde{L} - U(1,A_0) + U(1,K_1)U_1 + U(1,K_2)U_2] +$$
$$+ C[U(1,K_0) + U(1,K_1)V_1 + U(1,K_2)V_2][Q_1]^{-1}\Big\{W(K_2)\tilde{L} - W(A_0) + W(K_1)U_1 + W(K_2)U_2\Big\}\Big\}\lambda =$$
$$= d - CU(1,F) - CU(1,K_1)G_1 - CU(1,K_2)G_2 -$$
$$- C[U(1,K_0) + U(1,K_1) + U(1,K_2)][Q_1]^{-1}\Big\{W(F) + W(K_1)G_1 + W(K_2)G_2\Big\}. \quad (26)$$

Let

$$Q_2 = B + C + C[U(1,K_2)\tilde{L} - U(1,A_0) + U(1,K_1)U_1 + U(1,K_2)U_2] +$$
$$+ C[U(1,K_0) + U(1,K_1)V_1 + U(1,K_2)V_2][Q_1]^{-1}\Big\{W(K_2)\tilde{L} - W(A_0) + W(K_1)U_1 + W(K_2)U_2\Big\}$$

and assume that $Q_2 \neq 0$.

It then follows from (26) that λ is uniquely determined:

$$\lambda = [Q_2]^{-1}\Big\{d - CU(1,F) - CU(1,K_1)G_1 - CU(1,K_2)G_2\Big\} -$$
$$- [Q_2]^{-1}C[U(1,K_0) + U(1,K_1) + U(1,K_2)][Q_1]^{-1}\Big\{W(F) + W(K_1)G_1 + W(K_2)G_2\Big\}. \quad (27)$$

We have

$$\theta_0 = [Q_1]^{-1}\Big\{W(K_2)\tilde{L} - W(A_0) + W(K_1)U_1 + W(K_2)U_2\Big\}[Q_2]^{-1}\Big\{d - CU(1,F) -$$
$$- CU(1,K_1)G_1 - CU(1,K_2)G_2 - C[U(1,K_0) + U(1,K_1) + U(1,K_2)][Q_1]^{-1} \times$$
$$\times \Big\{W(F) + W(K_1)G_1 + W(K_2)G_2\Big\}\Big\} + [Q_1]^{-1}\Big\{W(F) + W(K_1)G_1 + W(K_2)G_2\Big\}. \quad (28)$$

Finally, using the expression for λ in (27) and for θ_0 in (28), we find an explicit forms for θ_1 and θ_2.

2 Main Results

Based on the above, the following statement can be formulated.

Theorem 1. Assume that

a) the functions $A_0(x)$, $A_1(x)$, and $F(x)$ are continuous on $[0,1]$; $A_1(x)\neq 0$ for all $x\in[0,1]$;

b) the functions $K_j(x)$, $j=0,1,2$ are continuous on $[0,1]$, $L_1(s)$ and $L_2(s)$ are continuous on $[0,1]$; $L_0(s)$ is continuously differentiable on $[0,1]$;

c) B, C and d are constants;

d) $\Phi_2=1-D_2(K_2)\neq 0$ and $\Phi_1=1-D_1(K_1)-D_1(K_2)\Phi_2^{-1}D_2(K_1)\neq 0$;

e) $Q_1=1-W(K_0)-W(K_1)V_1-W(K_2)V_2\neq 0$;

f) $Q_2=B+C+C[U(1,K_2)\tilde{L}-U(1,A_0)+U(1,K_1)U_1+U(1,K_2)U_2]+$
$+C[U(1,K_0)+U(1,K_1)V_1+U(1,K_2)V_2][Q_1]^{-1}\Big\{W(K_2)\tilde{L}-W(A_0)+W(K_1)U_1+W(K_2)U_2\Big\}\neq 0$,

where

$$a(x)=-\int_0^x[A_1(s)]^{-1}A_0(s)ds,\quad x\in[0,1];\ \tilde{L}=\int_0^1 L_2(s)ds,$$

$$D_i(Z)=\int_0^1 L_i(\xi)U(\xi,Z)d\xi,\ \ U(x,Z)=e^{a(x)}\int_0^x e^{-a(s)}[A_1(s)]^{-1}Z(s)ds;\ \ i=1,2,$$

$$W(Z)=L_0(1)\Big\{[A_1(1)]^{-1}Z(1)-[A_1(1)]^{-1}A_0(1)U(1,Z)\Big\}-L_0(0)[A_1(0)]^{-1}Z(0)+E_2(Z)-E_1(Z),$$

$$V_1=\Phi_1^{-1}\Big[D_1(K_0)+D_2(K_2)\Phi_2^{-1}D_2(K_0)\Big],$$

$$V_2=\Big\{\Phi_2^{-1}D_2(K_0)+\Phi_2^{-1}D_2(K_1)\Phi_1^{-1}\Big[D_1(K_0)+D_2(K_2)\Phi_2^{-1}D_2(K_0)\Big]\Big\},$$

$$U_1=\Phi_1^{-1}\Big[D_1(K_2)\tilde{L}-D_1(A_0)+D_2(K_2)\Phi_2^{-1}[D_2(K_2)\tilde{L}-D_2(A_0)]\Big],$$

$$U_2=\Phi_2^{-1}\Big\{D_2(K_2)\tilde{L}-D_2(A_0)+D_2(K_1)\Phi_1^{-1}\Big[D_1(K_2)\tilde{L}-D_1(A_0)+$$
$$+D_2(K_2)\Phi_2^{-1}[D_2(K_2)\tilde{L}-D_2(A_0)]\Big]\Big\},$$

$E_1(Z)=\int_0^1 L_0'(s)[A_1(s)]^{-1}Z(s)ds$, $\quad E_2(Z)=\int_0^1 L_0'(s)[A_1(s)]^{-1}A_0(s)U(s,Z)ds$, Z is A_0 or K_j, $j=0,1,2$.

Then problem for integro-differential equation with parameter (3)–(5) has a unique solution.

Proof. Consider problem (3)–(5).

Using assumptions a)–f) and notations, we construct λ^*, θ_0^*, θ_1^* and θ_2^*:

$$\lambda^*=[Q_2]^{-1}\Big\{d-CU(1,F)-CU(1,K_1)G_1-CU(1,K_2)G_2\Big\}-$$
$$-[Q_2]^{-1}C[U(1,K_0)+U(1,K_1)+U(1,K_2)][Q_1]^{-1}\Big\{W(F)+W(K_1)G_1+W(K_2)G_2\Big\},\quad (29)$$

$$\theta_0^*=[Q_1]^{-1}\Big\{W(K_2)\tilde{L}-W(A_0)+W(K_1)U_1+W(K_2)U_2\Big\}[Q_2]^{-1}\Big\{d-CU(1,F)-$$
$$-CU(1,K_1)G_1-CU(1,K_2)G_2-C[U(1,K_0)+U(1,K_1)+U(1,K_2)][Q_1]^{-1}\times$$
$$\times\Big\{W(F)+W(K_1)G_1+W(K_2)G_2\Big\}\Big\}+[Q_1]^{-1}\Big\{W(F)+W(K_1)G_1+W(K_2)G_2\Big\},\quad (30)$$

$$\theta_1^*=U_1\lambda^*+V_1\theta_0^*+G_1,\quad (31)$$

$$\theta_2^* = U_2\lambda^* + V_2\theta_0^* + G_2, \tag{32}$$

where $G_1 = \Phi_1^{-1}D_1(F) + \Phi_1^{-1}D_1(K_2)\Phi_2^{-1}D_2(F)$,
$G_2 = \Phi_2^{-1}D_2(F) + \Phi_2^{-1}D_2(K_1)\big[\Phi_1^{-1}D_1(F) + \Phi_1^{-1}D_1(K_2)\Phi_2^{-1}D_2(F)\big]$,

$$D_i(F) = \int_0^1 L_i(\xi)U(\xi,F)d\xi, \quad i=1,2, \quad U(x,F) = e^{a(x)}\int_0^x e^{-a(s)}[A_1(s)]^{-1}F(s)ds, \quad x \in [0,1].$$

Then, using the expressions (29)–(32), the unique solution to the Cauchy problem (3)–(4) has the following form

$$\widetilde{z}^*(x) = U(x,F) + [U(x,K_2)\tilde{L} - U(x,A_0)]\lambda^* + U(x,K_0)\theta_0^* + U(x,K_1)\theta_1^* + U(x,K_2)\theta_2^*, \quad x \in [0,1]. \tag{33}$$

The pair $(\widetilde{z}^*(x), \lambda^*)$, determined by (29) and (33), is the unique solution to the problem for the integro-differential equation with parameter (3)–(5).

Theorem 1 is proved. □

Theorem 2. Assume that the conditions a)–f) of Theorem 1 are fulfilled. Then two-point problem for integro-differential equation with higher-order derivatives in integral term (1)–(2) has a unique solution.

Proof. Under the assumptions of Theorem 1, the problem for integro-differential equation with parameter (3)–(5) admits a unique solution, given by the pair $(\widetilde{z}^*(x), \lambda^*)$.

We define

$$z^*(x) = \widetilde{z}^*(x) + \lambda^*, \qquad x \in [0,1].$$

According to the scheme of the method, this function is the unique solution of the two-point boundary value problem for integro-differential equation with higher-order derivatives in integral term (1)–(2).

Theorem 2 is proved. □

3 *Special Case*

We consider special case of the integro-differential equation with second-order derivative in integral term

$$A_0(x)z'(x) + A_1(x)z(x) = F(x) + K(x)\int_0^1 L(s)z''(s)ds, \qquad x \in [0,1], \tag{34}$$

$$Bz(0) + Cz(1) = d, \tag{35}$$

where the function $z(x)$ is the unknown function; $A_i(x)$, $i = 0,1$, and $F(x)$ are continuous on $[0,1]$; $K(x)$ is continuous on $[0,1]$; $L(s)$ is continuously differentiable on $[0,1]$; B, C and d are constants.

A function $z(x) \in C([0,1],\mathbb{R})$ is a solution of two-point problem (34)–(35) if:

1) $z(x)$ has derivatives $z'(x), z''(x)$ at each point $x \in [0,1]$;

2) $z(x)$ satisfies to the integro-differential equations (34) on $[0,1]$;

3) the two-point condition (35) is satisfied by $z(x)$ and $z'(x)$ at the points $x = 0$, $x = 1$.

Integration by parts is applied to the integral in the integro-differential equation (34).

We have

$$K(x)\int_0^1 L(s)z''(s)ds = K(x)L(1)z'(1) - K(x)L(0)z'(0) - K(x)\int_0^1 L'(s)z'(s)ds.$$

Then, the integro-differential equation (34) can be written in the form

$$A_0(x)z'(x) = -A_1(x)z(x) + F(x) + \\ + K(x)L(1)z'(1) - K(x)L(0)z'(0) - K(x)\int_0^1 L'(s)z'(s)ds, \qquad x \in [0,1]. \quad (36)$$

Therefore, we obtain the two-point problem for the neutral integro-differential equation with functional terms (36) and (35). The functional terms include piecewise-constant generalized arguments with respect to the derivative of unknown function at the points $x = 0$ and $x = 1$.

We apply Dzhumabaev parametrization method for solving problem (36)–(35).

Let $\lambda = z(0)$. Introducing the new function $\widetilde{z}(x)$, we perform in problems (36), (35) the change of variables

$$z(x) = \widetilde{z}(x) + \lambda, \qquad x \in [0,1].$$

We obtain the following equivalent problem for the neutral integro-differential equation with parameter and functional terms

$$A_1(x)\widetilde{z}'(x) = -A_0(x)\widetilde{z}(x) - A_0(x)\lambda + F(x) + \\ + K(x)L(1)\widetilde{z}'(1) - K(x)L(0)\widetilde{z}'(0) - K(x)\int_0^1 L'(s)\widetilde{z}'(s)ds, \quad x \in [0,1]. \quad (37)$$

$$\widetilde{z}(0) = 0, \quad (38)$$

$$[B + C]\lambda + C\widetilde{z}(1) = d. \quad (39)$$

A pair $(\widetilde{z}(x), \lambda)$ is a solution to problem for the integro-differential equation with parameter (37)–(39) if:

1) $\widetilde{z}(x)$ has derivatives $\widetilde{z}'(x), \widetilde{z}''(x)$ at each point $x \in [0,1]$;

2) $\widetilde{z}(x)$ and λ satisfy to the integro-differential equation (37) on $[0,1]$;

3) the initial condition (38) is satisfied by $\widetilde{z}(x)$ at the point $x = 0$;

4) the condition (39) is satisfied by $\widetilde{z}(x)$ and λ at the point $x = 1$.

Let $A_1(x) \neq 0$ for all $x \in [0,1]$ and $a(x) = -\int_0^x [A_1(s)]^{-1}A_0(s)ds$. Then we can rewrite the integro-differential equation with parameter and functional terms (37) in the form

$$\widetilde{z}'(x) = -[A_1(x)]^{-1}A_0(x)\widetilde{z}(x) - [A_1(x)]^{-1}A_0(x)\lambda + [A_1(x)]^{-1}F(x) + \\ + [A_1(x)]^{-1}K(x)L(1)\widetilde{z}'(1) - [A_1(x)]^{-1}K(x)L(0)\widetilde{z}'(0) - [A_1(x)]^{-1}K(x)\int_0^1 L'(s)\widetilde{z}'(s)ds, \quad x \in [0,1]. \quad (40)$$

Introduce notations

$$b_p(x) = [A_1(x)]^{-1}K(x)L(p), \qquad p = 0,1; \qquad \theta = \int_0^1 L'(s)\widetilde{z}'(s)ds.$$

Solution to the Cauchy problem for integro-differential equation with parameter and functional terms (37)–(38) can be written in the form

$$\widetilde{z}(x) = -e^{a(x)}\int\limits_0^x e^{-a(s)}[A_1(s)]^{-1}A_0(s)ds\lambda + e^{a(x)}\int\limits_0^x e^{-a(s)}[A_1(s)]^{-1}F(s)ds+$$
$$+e^{a(x)}\int\limits_0^x e^{-a(s)}[A_1(s)]^{-1}b_1(s)ds\widetilde{z}'(1) - e^{a(x)}\int\limits_0^x e^{-a(s)}[A_1(s)]^{-1}b_0(s)ds\widetilde{z}'(0)-$$
$$-e^{a(x)}\int\limits_0^x e^{-a(s)}[A_1(s)]^{-1}K(s)ds\theta, \qquad x\in[0,1]. \quad (41)$$

Let $U(x,Z) = e^{a(x)}\int\limits_0^x e^{-a(s)}[A_1(s)]^{-1}Z(s)ds$; $i=1,2$; Z is an arbitrary function.

Then we can rewrite the expression (41) in the next form

$$\widetilde{z}(x) = -U(x,A_0)\lambda + U(x,F) + U(x,b_1)\widetilde{z}'(1) - U(x,b_0)\widetilde{z}'(0) - U(x,K)\theta, \qquad x\in[0,1]. \quad (42)$$

From (42) we determine the value $\widetilde{z}(t)$ at $t=1$:

$$\widetilde{z}(1) = -U(1,A_0)\lambda + U(1,F) + U(1,b_1)\widetilde{z}'(1) - U(1,b_0)\widetilde{z}'(0) - U(1,K)\theta. \quad (43)$$

From (40), and taking into account the notation, we also obtain

$$\widetilde{z}'(x) = -[A_1(x)]^{-1}A_0(x)\widetilde{z}(x) - [A_1(x)]^{-1}A_0(x)\lambda + [A_1(x)]^{-1}F(x)+$$
$$+b_1(x)\widetilde{z}'(1) - b_0(x)\widetilde{z}'(0) - [A_1(x)]^{-1}K(x)\theta, \qquad x\in[0,1]. \quad (44)$$

We substitute for $\widetilde{z}(x)$ in (44) the corresponding expression given in (42)

$$\widetilde{z}'(x) = [A_1(x)]^{-1}A_0(x)[U(x,A_0)-1]\lambda + [A_1(x)]^{-1}[F(x) - A_0(x)U(x,F)]+$$
$$+[b_1(x) - [A_1(x)]^{-1}A_0(x)U(x,b_1)]\widetilde{z}'(1) - [b_0(x) - [A_1(x)]^{-1}A_0(x)U(x,b_0)]\widetilde{z}'(0)-$$
$$-[A_1(x)]^{-1}[K(x) - A_0(x)U(x,K)]\theta, \qquad x\in[0,1]. \quad (45)$$

Changing x by ξ, multiplying both parts of (45) by $L'(\xi)$ and integrating from 0 to 1, we get equation for θ:

$$\theta = \int\limits_0^1 L'(\xi)[A_1(\xi)]^{-1}A_0(\xi)[U(\xi,A_0)-1]d\xi\lambda + \int\limits_0^1 L'(\xi)[A_1(\xi)]^{-1}[F(\xi) - A_0(\xi)U(\xi,F)]d\xi+$$
$$+\int\limits_0^1 L'(\xi)[b_1(\xi) - [A_1(\xi)]^{-1}A_0(\xi)U(\xi,b_1)]d\xi\widetilde{z}'(1) - \int\limits_0^1 L'(\xi)[b_0(\xi) - [A_1(\xi)]^{-1}A_0(\xi)U(\xi,b_0)]d\xi\widetilde{z}'(0)-$$
$$-\int\limits_0^1 L'(\xi)[A_1(\xi)]^{-1}[K(\xi) - A_0(\xi)U(\xi,K)]d\xi\theta. \quad (46)$$

From (45) we determine equations for finding $\widetilde{z}'(1)$ and $\widetilde{z}'(0)$:

$$[1-b_1(1)+[A_1(1)]^{-1}A_0(1)U(1,b_1)]\widetilde{z}'(1)=[A_1(1)]^{-1}A_0(1)[U(1,A_0)-1]\lambda+ \\ +[A_1(1)]^{-1}[F(1)-A_0(1)U(1,F)]-[b_0(1)-[A_1(1)]^{-1}A_0(1)U(1,b_0)]\widetilde{z}'(0)- \\ -[A_1(1)]^{-1}[K(1)-A_0(1)U(1,K)]\theta, \quad (47)$$

$$\widetilde{z}'(0)=-[A_1(0)]^{-1}A_0(0)\lambda+[A_1(0)]^{-1}F(0)+b_1(0)\widetilde{z}'(1)-b_0(0)\widetilde{z}'(0)-[A_1(0)]^{-1}K(0)\theta. \quad (48)$$

Assume that $B_1=1-b_1(1)+[A_1(1)]^{-1}A_0(1)U(1,b_1)\neq 0$. Then from (47) we uniquely determine $\widetilde{z}'(1)$ through λ, $\widetilde{z}'(0)$ and θ:

$$\widetilde{z}'(1)=B_1^{-1}[A_1(1)]^{-1}A_0(1)[U(1,A_0)-1]\lambda+B_1^{-1}[A_1(1)]^{-1}[F(1)-A_0(1)U(1,F)]- \\ -B_1^{-1}[b_0(1)-[A_1(1)]^{-1}A_0(1)U(1,b_0)]\widetilde{z}'(0)-B_1^{-1}[A_1(1)]^{-1}[K(1)-A_0(1)U(1,K)]\theta. \quad (49)$$

Substituting the found $\widetilde{z}'(1)$ into (48), we obtain

$$\Big[1+b_0(0)+b_1(0)B_1^{-1}[b_0(1)-[A_1(1)]^{-1}A_0(1)U(1,b_0)]\Big]\widetilde{z}'(0)= \\ =\Big[b_1(0)B_1^{-1}[A_1(1)]^{-1}A_0(1)[U(1,A_0)-1]-[A_1(0)]^{-1}A_0(0)\Big]\lambda+ \\ +[A_1(0)]^{-1}F(0)+b_1(0)B_1^{-1}[A_1(1)]^{-1}[F(1)-A_0(1)U(1,F)]- \\ -b_1(0)B_1^{-1}[A_1(1)]^{-1}[K(1)-A_0(1)U(1,K)]\theta-[A_1(0)]^{-1}K(0)\theta. \quad (50)$$

Assume that $B_0=1+b_0(0)+b_1(0)B_1^{-1}[b_0(1)-[A_1(1)]^{-1}A_0(1)U(1,b_0)]\neq 0$. Then from (50) we uniquely determine $\widetilde{z}'(0)$ through λ and θ:

$$\widetilde{z}'(0)=\alpha_0\lambda-\beta_0\theta+\gamma_0, \quad (51)$$

where $\alpha_0=B_0^{-1}\Big[b_1(0)B_1^{-1}[A_1(1)]^{-1}A_0(1)[U(1,A_0)-1]-[A_1(0)]^{-1}A_0(0)\Big]$,
$\beta_0=B_0^{-1}\Big[b_1(0)B_1^{-1}[A_1(1)]^{-1}[K(1)-A_0(1)U(1,K)]+[A_1(0)]^{-1}K(0)\Big]$,
$\gamma_0=B_0^{-1}\Big[[A_1(0)]^{-1}F(0)+b_1(0)B_1^{-1}[A_1(1)]^{-1}[F(1)-A_0(1)U(1,F)]\Big]$.

Substituting the found $\widetilde{z}'(0)$ in (49), we have

$$\widetilde{z}'(1)=\alpha_1\lambda-\beta_1\theta+\gamma_1, \quad (52)$$

where $\alpha_1=B_1^{-1}[A_1(1)]^{-1}A_0(1)[U(1,A_0)-1]-B_1^{-1}[b_0(1)-[A_1(1)]^{-1}A_0(1)U(1,b_0)]\alpha_0$,
$\beta_1=B_1^{-1}[A_1(1)]^{-1}[K(1)-A_0(1)U(1,K)]+B_1^{-1}[b_0(1)-[A_1(1)]^{-1}A_0(1)U(1,b_0)]\beta_0$,
$\gamma_1=B_1^{-1}[A_1(1)]^{-1}[F(1)-A_0(1)U(1,F)]-B_1^{-1}[b_0(1)-[A_1(1)]^{-1}A_0(1)U(1,b_0)]\gamma_0$.

Now, we return to expression (46). We replace $\widetilde{z}'(0)$ and $\widetilde{z}'(1)$ by corresponding representations (51) and (52):

$$[1+\phi_1-\varphi_1\beta_1+\varphi_0\beta_0]\theta=[\phi_2+\varphi_1\alpha_1-\varphi_0\alpha_0]\lambda+\phi_3+\varphi_1\gamma_1-\varphi_0\gamma_0, \quad (53)$$

where $\phi_1=\int\limits_0^1 L'(\xi)[A_1(\xi)]^{-1}[K(\xi)-A_0(\xi)U(\xi,K)]d\xi$,
$\phi_2=\int\limits_0^1 L'(\xi)[A_1(\xi)]^{-1}A_0(\xi)[U(\xi,A_0)-1]d\xi$, $\quad \phi_3=\int\limits_0^1 L'(\xi)[A_1(\xi)]^{-1}[F(\xi)-A_0(\xi)U(\xi,F)]d\xi$,
$\varphi_1=\int\limits_0^1 L'(\xi)[b_1(\xi)-[A_1(\xi)]^{-1}A_0(\xi)U(\xi,b_1)]d\xi$, $\quad \varphi_0=\int\limits_0^1 L'(\xi)[b_0(\xi)-[A_1(\xi)]^{-1}A_0(\xi)U(\xi,b_0)]d\xi$.

Assume that $C_0 = 1 + \phi_1 - \varphi_1\beta_1 + \varphi_0\beta_0 \neq 0$. Then, from (53) it follows

$$\theta = C_0^{-1}[\phi_2 + \varphi_1\alpha_1 - \varphi_0\alpha_0]\lambda + C_0^{-1}[\phi_3 + \varphi_1\gamma_1 - \varphi_0\gamma_0]. \tag{54}$$

Using the expression (43) for $\widetilde{z}(1)$ and taking into account formulas (51), (52), and (54), we get

$$\widetilde{z}(1) = \Big\{-U(1,A_0) + U(1,b_1)\alpha_1 - U(1,b_0)\alpha_0 - \\ - [U(1,b_1)\beta_1 - U(1,b_0)\beta_0 + U(1,K)]C_0^{-1}[\phi_2 + \varphi_1\alpha_1 - \varphi_0\alpha_0]\Big\}\lambda + U(1,F) + U(1,b_1)\gamma_1 - \\ - U(1,b_0)\gamma_0 - [U(1,b_1)\beta_1 - U(1,b_0)\beta_0 + U(1,K)]C_0^{-1}[\phi_3 + \varphi_1\gamma_1 - \varphi_0\gamma_0]. \tag{55}$$

Further, substituting the expression (55) into (39), we obtain

$$\Big[B + C\Big\{1 - U(1,A_0) + U(1,b_1)\alpha_1 - U(1,b_0)\alpha_0 - \\ - [U(1,b_1)\beta_1 - U(1,b_0)\beta_0 + U(1,K)]C_0^{-1}[\phi_2 + \varphi_1\alpha_1 - \varphi_0\alpha_0]\Big\}\Big]\lambda = d - C\Big\{U(1,F) + \\ + U(1,b_1)\gamma_1 - U(1,b_0)\gamma_0 - [U(1,b_1)\beta_1 - U(1,b_0)\beta_0 + U(1,K)]C_0^{-1}[\phi_3 + \varphi_1\gamma_1 - \varphi_0\gamma_0]\Big\}. \tag{56}$$

Assume that

$$Q_0 = \Big[B + C\Big\{1 - U(1,A_0) + U(1,b_1)\alpha_1 - U(1,b_0)\alpha_0 - \\ - [U(1,b_1)\beta_1 - U(1,b_0)\beta_0 + U(1,K)]C_0^{-1}[\phi_2 + \varphi_1\alpha_1 - \varphi_0\alpha_0]\Big\}\Big] \neq 0.$$

Then from (56) we uniquely determine λ:

$$\lambda = [Q_0]^{-1}\Big[d - C\Big\{U(1,F) + U(1,b_1)\gamma_1 - U(1,b_0)\gamma_0 - \\ - [U(1,b_1)\beta_1 - U(1,b_0)\beta_0 + U(1,K)]C_0^{-1}[\phi_3 + \varphi_1\gamma_1 - \varphi_0\gamma_0]\Big\}\Big].$$

Therefore, we found λ, θ, $\widetilde{z}(1)$ and $\widetilde{z}(0)$. Then from (42) we can define the explicit form of solution to the Cauchy problem for integro-differential equation with parameter and functional terms (37), (38).

Theorem 3. Assume that

a) $A_i(x)$, $i = 0,1$, and $F(x)$ are continuous on $[0,1]$; let $A_1(x) \neq 0$ for all $x \in [0,1]$;
b) $K(x)$ is continuous on $[0,1]$; $L(s)$ is continuously differentiable on $[0,1]$; B, C and d are constants;
c) $B_1 = 1 - b_1(1) + [A_1(1)]^{-1}A_0(1)U(1,b_1) \neq 0$ and
$B_0 = 1 + b_0(0) + b_1(0)B_1^{-1}[b_0(1) - [A_1(1)]^{-1}A_0(1)U(1,b_0)] \neq 0$;
d) $C_0 = 1 + \phi_1 - \varphi_1\beta_1 + \varphi_0\beta_0 \neq 0$;
e) $Q_0 = \Big[B + C\Big\{1 - U(1,A_0) + U(1,b_1)\alpha_1 - U(1,b_0)\alpha_0 - $
$-[U(1,b_1)\beta_1 - U(1,b_0)\beta_0 + U(1,K)]C_0^{-1}[\phi_2 + \varphi_1\alpha_1 - \varphi_0\alpha_0]\Big\}\Big] \neq 0,$

where $a(x) = -\int\limits_0^x [A_1(s)]^{-1}A_0(s)ds, \quad b_p(x) = [A_1(x)]^{-1}K(x)L(p), \quad p = 0,1, \quad x \in [0,1],$

$U(x,Z) = e^{a(x)}\int\limits_0^x e^{-a(s)}[A_1(s)]^{-1}Z(s)ds, \quad x \in [0,1], \; i = 1,2, \; Z$ is A_0, b_1, b_0 and K,

$\alpha_0 = B_0^{-1}\Big[b_1(0)B_1^{-1}[A_1(1)]^{-1}A_0(1)[U(1,A_0) - 1] - [A_1(0)]^{-1}A_0(0)\Big],$

$\alpha_1 = B_1^{-1}[A_1(1)]^{-1}A_0(1)[U(1,A_0)-1] - B_1^{-1}[b_0(1)-[A_1(1)]^{-1}A_0(1)U(1,b_0)]\alpha_0,$
$\beta_0 = B_0^{-1}\Big[b_1(0)B_1^{-1}[A_1(1)]^{-1}[K(1)-A_0(1)U(1,K)] + [A_1(0)]^{-1}K(0)\Big],$
$\beta_1 = B_1^{-1}[A_1(1)]^{-1}[K(1)-A_0(1)U(1,K)] + B_1^{-1}[b_0(1)-[A_1(1)]^{-1}A_0(1)U(1,b_0)]\beta_0,$
$\phi_1 = \int\limits_0^1 L'(\xi)[A_1(\xi)]^{-1}[K(\xi)-A_0(\xi)U(\xi,K)]d\xi, \quad \phi_2 = \int\limits_0^1 L'(\xi)[A_1(\xi)]^{-1}A_0(\xi)[U(\xi,A_0)-1]d\xi,$
$\varphi_1 = \int\limits_0^1 L'(\xi)[b_1(\xi)-[A_1(\xi)]^{-1}A_0(\xi)U(\xi,b_1)]d\xi, \quad \varphi_0 = \int\limits_0^1 L'(\xi)[b_0(\xi)-[A_1(\xi)]^{-1}A_0(\xi)U(\xi,b_0)]d\xi.$

Then two-point boundary value problem for integro-differential equation (34)–(35) has a unique solution.

Proof. Let's consider problem (34)–(35). We apply the parametrization method and move on to the equivalent problem (37)–(39). Let the conditions of the theorem be satisfied. Then we will uniquely determine the unknowns λ, θ, $\widetilde{z}(0)$, and $\widetilde{z}(1)$.

We have

$$\lambda^* = [Q_0]^{-1}\Big[d - C\Big\{U(1,F) + U(1,b_1)\gamma_1 - U(1,b_0)\gamma_0 - \\ - [U(1,b_1)\beta_1 - U(1,b_0)\beta_0 + U(1,K)]C_0^{-1}[\phi_3 + \varphi_1\gamma_1 - \varphi_0\gamma_0]\Big\}\Big],$$

$$\theta^* = C_0^{-1}[\phi_2 + \varphi_1\alpha_1 - \varphi_0\alpha_0]\lambda^* + C_0^{-1}[\phi_3 + \varphi_1\gamma_1 - \varphi_0\gamma_0],$$

$$\widetilde{z}^{*\prime}(0) = \alpha_0\lambda^* - \beta_0\theta^* + \gamma_0,$$

$$\widetilde{z}^{*\prime}(1) = \alpha_1\lambda^* - \beta_1\theta^* + \gamma_1,$$

where

$\gamma_0 = B_0^{-1}\Big[[A_1(0)]^{-1}F(0) + b_1(0)B_1^{-1}[A_1(1)]^{-1}[F(1)-A_0(1)U(1,F)]\Big],$
$\gamma_1 = B_1^{-1}[A_1(1)]^{-1}[F(1)-A_0(1)U(1,F)] - B_1^{-1}[b_0(1)-[A_1(1)]^{-1}A_0(1)U(1,b_0)]\gamma_0,$
$\phi_3 = \int\limits_0^1 L'(\xi)[A_1(\xi)]^{-1}[F(\xi)-A_0(\xi)U(\xi,F)]d\xi.$

Then, we obtain

$$\widetilde{z}^*(x) = -U(x,A_0)\lambda^* + U(x,F) + U(x,b_1)\widetilde{z}'(1) - U(x,b_0)\widetilde{z}^{*\prime}(0) - U(x,K)\theta^*, \qquad x \in [0,1],$$

where $U(x,F) = e^{a(x)}\displaystyle\int_0^x e^{-a(s)}[A_1(s)]^{-1}F(s)ds.$

Therefore, problem for integro-differential equation with parameter and functional terms (37)–(39) admits a unique solution, given by the pair $(\widetilde{z}^*(x), \lambda^*)$.

We define

$$z^*(x) = \widetilde{z}^*(x) + \lambda^*, \qquad x \in [0,1].$$

According to the scheme of the method, this function is the unique solution of the two-point boundary value problem for integro-differential equation with higher-order derivative in integral term (34)–(35).

Theorem 3 is proved. □

Conclusion

We propose a method for solving boundary value problems for integro-differential equations with higher-order derivatives appearing in the integral term. Unlike classical works and the works of Dzhumabaev, this paper proposes a new approach to solving boundary value problems for integro-differential equations with higher-order derivatives in the integral term, where the coefficients of the equation are assumed to be continuous functions only.

This approach is planned to be extended to problems where the order difference between the differential and integral parts is greater than two, with the goal of refining solvability results for a broader range of boundary value problems.

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Author Contributions

All authors contributed equally to this work.

Conflict of Interest

The authors declare no conflict of interest.

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