https://doi.org/10.31489/2025M4/163-179 *Research article*

Studying a system of non-local condition hyperbolic equations

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Local boundary value problems for hyperbolic differential equations have been studied in considerable detail. However, the mathematical modeling of a number of real-world processes leads to nonlocal boundary value problems involving nonlinear hyperbolic differential equations, which remain poorly understood. In this paper, we consider a system of hyperbolic equations defined by both point and integral boundary conditions in a rectangular domain. To the best of our knowledge, such a problem is studied here for the first time. We note that this formulation is quite general and encompasses several special cases. The classical Goursat-Darboux problem-a problem with integral boundary conditions in which some boundary conditions are specified as point conditions and others as integral conditions-is derived from this formulation as a particular case. Under natural conditions on the initial data, the necessary conditions for the solvability of a nonlocal boundary value problem are established. A corresponding Green's function for the boundary value problem is constructed and the problem is reduced to an equivalent integral equation. Using the principle of contracting Banach maps, conditions for the existence and uniqueness of a solution to the boundary value problem are established. An example is given illustrating the validity of the obtained results.

Keywords: non-local boundary value problems, integral and point boundary conditions, Goursat-Darboux problem, system of hyperbolic equations, existence and uniqueness of solutions, unique solvability, Green's function.

2020 Mathematics Subject Classification: 35G35, 35G46, 35L53, 35L57.

Introduction

Recently, intensive research has been carried out on nonlocal boundary value problems for both ordinary and partial differential equations. The significance of these problems was emphasized in [1]. If, instead of classical boundary conditions, algebraic relations are defined between the values of the unknown function on the boundary and/or inside the domain, such a boundary value problem is referred to as a nonlocal boundary value problem [2–4]. These algebraic relations can be expressed in terms of pointwise values of the unknown function and/or its integral.

Non-local condition boundary value problems arise while constructing mathematical models of processes that occur in atomic and nuclear physics, demography, heating processes and in other fields of natural science. The papers [5, 6] study one-dimensional nonlinear hyperbolic equations given with integral and multipoint boundary conditions. Sufficient conditions for the existence and uniqueness of the problem are found.

In [7–9], a system of hyperbolic equations is investigated under two-point and integral boundary conditions. The Green's function for the problem is constructed, the boundary value problem is reduced to an equivalent integral equation, and sufficient conditions for the existence and uniqueness of the solution are obtained.

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Received: 7 July 2025; *Accepted:* 11 September 2025.

In [10–12], a nonlocal problem with integral conditions for a system of hyperbolic equations in a rectangular domain is analyzed. The existence of a unique classical solution and the methods for its construction are discussed.

Kozhanov A.I. and Pulkina L.S. investigated a multidimensional hyperbolic equation with integral boundary conditions in [13].

In [14–16], a nonlocal boundary value problem with an integral condition for a system of hyperbolic equations was considered, and necessary and sufficient conditions for its well-posedness were established.

Papers [17–19] study the existence and uniqueness of strong solutions using methods of functional analysis.

Paper [20] analyzes an optimal control problem with integral boundary conditions.

In the present work, we consider a Goursat–Darboux system with pointwise and integral condition. A necessary condition for the solvability of the problem is proved. The problem considered is reduced to an equivalent equation by means of equivalent transformations. Sufficient conditions for the existence and uniqueness of the solution are found by means of the Banach compressed mapping principle.

1 Problem statement

We consider a non-local problem with integral and pointwise boundary conditions for a Goursat–Darboux system in the domain $Q = [0,T] \times [0,l]$:

$$z_{tx} = f(t, x, z(t, x)), \tag{1}$$

$$Az(0, x) + \int_0^T n(t)z(t, x)dt = \varphi(x), \ \ x \in [0, l], \tag{2}$$

$$Bz(t, 0) + \int_0^l m(x)z(t, x)dx = \psi(t), \ \ t \in [0, T]. \tag{3}$$

Here, $z(t, x) = col(z_1(t, x), z_2(t, x), \ldots, z_n(t, x))$ is an unknown n-dimensional vector-function; $f : Q \times R^n \to R^n$ is a given function; $\varphi(x), \psi(t)$ are functions that are differentiable on $[0, T], [0, l]$ respectively. $A, B \in R^{n\times n}$ are the given matrices, $n(t)$ and $m(x)$ are $n\times n$-dimensional matrix functions. $det\left(A + \int_0^T n(t)dt\right) \neq 0$, $det\left(B + \int_0^l m(x)dx\right) \neq 0$. Furthermore, the matrices $A, n(t)$ and $B, m(x)$ are pairwise commutative. So, $A\cdot B = B\cdot A$, $A\cdot m(x) = m(x)\cdot A$, $B\cdot n(t) = n(t)\cdot B$, $m(x)\cdot n(t) = n(t)\cdot m(x)$.

Note that problem (1)–(3) is quite general. For example, if the matrices A and B are both zero, then the problem reduces to one with pure integral conditions. When $A = B = E$ and $n(t) \equiv m(x) \equiv 0$, we obtain the classical Goursat–Darboux problem, and there are other variants.

2 Main results

In the paper, it is shown that for the solvability of problem (1)–(3) the compatability condition of functions $\varphi(x)$ and $\psi(t)$ is satisfied.

Theorem 1. For the solvability of problem (1)–(3), it is necessary that the compatibility condition

$$B\varphi(0) + \int_0^l m(x)\varphi(x)dx = A\psi(0) + \int_0^T n(t)\psi(t)dt$$

is fulfilled.

Proof. Let us find the solution of equation (1) as follows:

$$z(t,x) = a(t) + b(x) + \int_0^t \int_0^x f(\tau, s, z(\tau, s)) d\tau\, ds, \tag{4}$$

where the functions $a(t)$ and $b(x)$ are unknown differentiable functions and are determined in the intervals $[0,T], [0,l]$, respectively. We require that the function determined by equality (4) satisfies boundary conditions (2) and (3). Then, we obtain the relations

$$A\left[a(0) + b(x)\right] + \int_0^T n(t) \left[a(t) + b(x) + \int_0^t \int_0^x f(\tau, s, z(\tau, s)) d\tau\, ds\right] dt$$

$$= Aa(0) + \int_0^T n(t)a(t)dt + \left(A + \int_0^T n(t)dt\right) b(x)$$

$$+ \int_0^T n(t) \int_0^t \int_0^x f(\tau, s, z(\tau, s)) d\tau ds\, dt = \varphi(x), \quad x \in [0, l]. \tag{5}$$

$$B\left[a(t) + b(0)\right] + \int_0^l m(x) \left[a(t) + b(x) + \int_0^t \int_0^x f(\tau, s, z(\tau, s)) d\tau\, ds\right] dx$$

$$= \left(B + \int_0^l m(x)dx\right) a(t) + Bb(0) + \int_0^l m(x)b(x)dx$$

$$+ \int_0^l m(x) \int_0^t \int_0^x f(\tau, s, z(\tau, s)) d\tau ds\, dx = \psi(t), \quad t \in [0, T]. \tag{6}$$

Applying conditions (3) to relation (5) and conditions (2) to relation (6), we obtain

$$B\left[Aa(0) + \int_0^T n(t)a(t)dt + \left(A + \int_0^T n(t)dt\right) b(0)\right]$$

$$+ \int_0^l m(x) \left[Aa(0) + \int_0^T n(t)a(t)dt + \left(A + \int_0^T n(t)dt\right) b(x)\right] dx$$

$$+ \int_0^T \int_0^l n(t)m(x) \int_0^t \int_0^x f(\tau, s, z(\tau, s)) d\tau\, ds\, dt\, dx = B\varphi(0) + \int_0^l m(x)\varphi(x)dx,$$

$$A\left[\left(B + \int_0^l m(x)dx\right) a(0) + Bb(0) + \int_0^l m(x)b(x)dx\right]$$

$$+\int_0^T n(t)\left[\left(B+\int_0^l m(x)dx\right)a(t)+\left(Bb(0)+\int_0^l m(x)b(x)dx\right)\right]dt$$

$$+\int_0^T\int_0^l n(t)m(x)\int_0^t\int_0^x f(\tau,s,z(\tau,s))d\tau\,ds\,dt\,dx = A\psi(0)+\int_0^T n(t)\psi(t)dt.$$

From this we obtain

$$\left(Aa(0)+\int_0^T n(t)a(t)dt\right)\left(B+\int_0^l m(x)dx\right)+\left(A+\int_0^T n(t)dt\right)\left(Bb(0)+\int_0^l m(x)b(x)dx\right)$$

$$+\int_0^T\int_0^l n(t)m(x)\int_0^t\int_0^x f(\tau,s,z(\tau,s))d\tau\,ds\,dt\,dx = B\varphi(0)+\int_0^l m(x)\varphi(x)dx,$$

$$\left(Aa(0)+\int_0^T n(t)a(t)dt\right)\left(B+\int_0^l m(x)dx\right)+\left(Bb(0)+\int_0^l m(x)b(x)dx\right)\left(A+\int_0^T n(t)dt\right)$$

$$+\int_0^T\int_0^l n(t)m(x)\int_0^t\int_0^x f(\tau,s,z(\tau,s))d\tau\,ds\,dt\,dx = A\psi(0)+\int_0^T n(t)\psi(t)dt.$$

The right hand side equality is obtained from the left-hand side equality. □

In this paper, we construct the Green function for problem (1)–(3). We note that problem (1)–(3) is reduced to an equivalent integral equation.

Theorem 2. The equivalent integral equation for the problem (1)–(3) is as follows

$$z(t,x)=\left(B+\int_0^l m(x)dx\right)^{-1}\psi(t)+\left(A+\int_0^T n(t)dt\right)^{-1}\varphi(x)$$

$$-\left(B+\int_0^l m(x)dx\right)^{-1}\left(A+\int_0^T n(t)dt\right)^{-1}\left(B\varphi(0)+\int_0^l m(x)\varphi(x)dx\right)$$

$$+\int_0^T\int_0^l G(t,x,\tau,s)f(\tau,s,z)d\tau\,ds, \tag{7}$$

where

$$G(t,x,\tau,s)=\left(B+\int_0^l m(x)dx\right)^{-1}\left(A+\int_0^T n(t)dt\right)^{-1}$$

$$\times \begin{cases} \left(A+\int\limits_0^{\tau} n(\alpha)\,d\alpha\right)\left(B+\int\limits_0^{s} m(\beta)\,d\beta\right), & 0\le\tau\le t,\ 0\le s\le x, \\ -\left(A+\int\limits_0^{\tau} n(\alpha)\,d\alpha\right)\int\limits_s^{l} m(\beta)\,d\beta, & 0\le\tau\le t,\ x<s\le l, \\ -\left(B+\int\limits_0^{s} m(\beta)\,d\beta\right)\int\limits_\tau^{T} n(\alpha)\,d\alpha, & t<\tau\le T, 0\le s\le x, \\ \int\limits_\tau^{T} n(\alpha)\,d\alpha\int\limits_s^{l} m(\beta)\,d\beta, & t<\tau\le T, x<s\le l. \end{cases}$$

Proof. The unknown functions $a(t)$ and $b(x)$ can be considered as solutions to a system of linear algebraic equations defined by equalities (5) or (6). This system is of the n-th order. The sought functions $a(t)$ and $b(x)$ have dimension $2n$. It is clear that this system has an infinite set of solutions. We fix an arbitrary solution. Let

$$Aa(0)+\int\limits_0^T n(t)a(t)dt=0$$

be an arbitrary solution.

Then, from equality (5), we find

$$b(x)=\left(A+\int\limits_0^T n(t)dt\right)^{-1}\varphi(x)-\left(A+\int\limits_0^T n(t)dt\right)^{-1}\int\limits_0^T n(t)\int\limits_0^t\int\limits_0^x f(\tau,s,z(\tau,s))d\tau\,ds\,dt, \tag{8}$$

$$b(0)=\left(A+\int\limits_0^T n(t)dt\right)^{-1}\varphi(0).$$

Taking the equalities $b(x)$ and $b(0)$ into account in equality (6), we get

$$\left(B+\int\limits_0^l m(x)dx\right)a(t)+\left(A+\int\limits_0^T n(t)dt\right)^{-1}B\varphi(0)$$

$$+\int\limits_0^l m(x)\left(A+\int\limits_0^T n(t)dt\right)^{-1}\varphi(x)dx-\int\limits_0^l m(x)\left(A+\int\limits_0^T n(t)dt\right)^{-1}$$

$$\times\int\limits_0^T n(t)\int\limits_0^t\int\limits_0^x f(\tau,s,z(\tau,s))d\tau\,ds\,dt\,dx+\int\limits_0^l m(x)\int\limits_0^t\int\limits_0^x f(\tau,s,z(\tau,s))d\tau\,ds\,dx=\psi(t).$$

Hence,

$$a(t)=\left(B+\int\limits_0^l m(x)dx\right)^{-1}\psi(t)-\left(B+\int\limits_0^l m(x)dx\right)^{-1}\left(A+\int\limits_0^T n(t)dt\right)^{-1}B\varphi(0)$$

$$-\left(B+\int\limits_0^l m(x)dx\right)^{-1}\left(A+\int\limits_0^T n(t)dt\right)^{-1}\int\limits_0^l m(x)\varphi(x)dx$$

$$+\left(B+\int\limits_0^l m(x)dx\right)^{-1}\left(A+\int\limits_0^T n(t)dt\right)^{-1}\times\int\limits_0^l\int\limits_0^T m(x)n(t)\int\limits_0^t\int\limits_0^x f(\tau,s,z(\tau,s))d\tau\,ds\,dt\,dx$$

$$-\left(B+\int\limits_0^l m(x)dx\right)^{-1}\int\limits_0^l m(x)\int\limits_0^t\int\limits_0^x f(\tau,s,z(\tau,s))d\tau\,ds\,dx. \tag{9}$$

Taking into account equalities (8) and (9) obtained for functions $b(x)$ and $a(t)$ in equality (4), we have

$$z(t,x)=\left(B+\int\limits_0^l m(x)dx\right)^{-1}\psi(t)+\left(A+\int\limits_0^T n(t)dt\right)^{-1}\varphi(x)$$

$$-\left(B+\int\limits_0^l m(x)dx\right)^{-1}\left(A+\int\limits_0^T n(t)dt\right)^{-1}\left[B\varphi(0)+\int\limits_0^l m(x)\varphi(x)dx\right]$$

$$+\left(B+\int\limits_0^l m(x)dx\right)^{-1}\left(A+\int\limits_0^T n(t)dt\right)^{-1}\int\limits_0^l\int\limits_0^T m(x)n(t)\int\limits_0^t\int\limits_0^x f(\tau,s,z(\tau,s))d\tau\,ds\,dt\,dx$$

$$-\left(B+\int\limits_0^l m(x)dx\right)^{-1}\int\limits_0^l m(x)\int\limits_0^t\int\limits_0^x f(\tau,s,z(\tau,s))d\tau\,ds\,dx$$

$$-\left(A+\int\limits_0^T n(t)dt\right)^{-1}\int\limits_0^T n(t)\int\limits_0^t\int\limits_0^x f(\tau,s,z(\tau,s))d\tau\,ds\,dt+\int\limits_0^t\int\limits_0^x f(\tau,s,z(\tau,s))d\tau\,ds,\quad (t,x)\in Q. \tag{10}$$

We make the same transformations in equality (10) as follows

$$\int\limits_0^T n(t)\int\limits_0^t\int\limits_0^x f(\tau,s,z(\tau,s))d\tau\,ds\,dt=\int\limits_0^T\int\limits_0^x\left(\int\limits_t^T n(\tau)d\tau\right)f(t,s,z(t,s))dt\,ds,$$

$$\int\limits_0^l m(x)\int\limits_0^t\int\limits_0^x f(\tau,s,z(\tau,s))d\tau\,ds\,dx=\int\limits_0^l\int\limits_0^t\left(\int\limits_x^l m(s)ds\right)f(\tau,x,z(\tau,x))d\tau\,dx,$$

$$\int\limits_0^l\int\limits_0^T m(x)n(t)\int\limits_0^t\int\limits_0^x f(\tau,s,z(\tau,s))d\tau\,ds\,dt\,dx=\int\limits_0^T\int\limits_0^l\left(\int\limits_t^T n(\tau)d\tau\int\limits_x^l m(s)ds\right)f(t,x,z(t,x))dt\,dx.$$

Taking into account these expressions in equality (10), we can write

$$z(t,x)=\left(B+\int\limits_0^l m(x)dx\right)^{-1}\psi(t)+\left(A+\int\limits_0^T n(t)dt\right)^{-1}\varphi(x)$$

$$-\left(B+\int\limits_0^l m(x)dx\right)^{-1}\left(A+\int\limits_0^T n(t)dt\right)^{-1}\left[B\varphi(0)+\int\limits_0^l m(x)\varphi(x)dx\right]$$

$$+\left(B+\int_0^l m(x)dx\right)^{-1}\left(A+\int_0^T n(t)dt\right)^{-1}\int_0^T\int_0^l\left[\int_t^T n(\tau)d\tau\int_x^l m(s)ds\right]f(t,x,z(t,x))dt\,dx$$

$$-\left(B+\int_0^l m(x)dx\right)^{-1}\int_0^l\int_0^t\left(\int_x^l m(s)ds\right)f(\tau,x,z(\tau,x))d\tau\,dx$$

$$-\left(A+\int_0^T n(t)dt\right)^{-1}\int_0^T\int_0^x\left(\int_t^T n(\tau)d\tau\right)f(t,s,z(t,s))dt\,ds$$

$$+\int_0^t\int_0^x f(\tau,s,z(\tau,s))d\tau\,ds,\quad (t,x)\in Q. \tag{11}$$

From equality (11) we obtain

$$z(t,x)=\left(B+\int_0^l m(x)dx\right)^{-1}\psi(t)+\left(A+\int_0^T n(t)dt\right)^{-1}\varphi(x)$$

$$-\left(B+\int_0^l m(x)dx\right)^{-1}\left(A+\int_0^T n(t)dt\right)^{-1}\left(B\varphi(0)+\int_0^l m(x)\varphi(x)dx\right)$$

$$+\int_0^t\int_0^x\left[E-\left(A+\int_0^T n(t)dt\right)^{-1}\int_\tau^T n(\alpha)d\alpha-\left(B+\int_0^l m(x)dx\right)^{-1}\int_s^l m(\beta)d\beta\right.$$

$$\left.+\left(B+\int_0^l m(x)dx\right)^{-1}\left(A+\int_0^T n(t)dt\right)^{-1}\int_\tau^T n(\alpha)d\alpha\int_s^l m(\beta)d\beta\right]f(\tau,s,z(\tau,s))d\tau\,ds$$

$$+\int_0^t\int_x^l\left[\left(B+\int_0^l m(x)dx\right)^{-1}\left(A+\int_0^T n(t)dt\right)^{-1}\int_\tau^T n(\alpha)d\alpha\int_s^l m(\beta)d\beta\right.$$

$$\left.-\left(B+\int_0^l m(x)dx\right)^{-1}\int_s^l m(\beta)d\beta\right]f(\tau,s,z(\tau,s))d\tau\,ds$$

$$+\int_t^T\int_0^x\left[\left(B+\int_0^l m(x)dx\right)^{-1}\left(A+\int_0^T n(t)dt\right)^{-1}\int_\tau^T n(\alpha)d\alpha\int_s^l m(\beta)d\beta\right.$$

$$\left.-\left(A+\int_0^T n(t)dt\right)^{-1}\int_\tau^T n(\alpha)d\alpha\right]f(\tau,s,z(\tau,s))d\tau ds$$

$$+\int_t^T\int_x^l\left[\left(B+\int_0^l m(x)dx\right)^{-1}\left(A+\int_0^T n(t)dt\right)^{-1}\int_\tau^T n(\alpha)d\alpha\int_s^l m(\beta)d\beta\right]$$

$$\times f(\tau,s,z(\tau,s))d\tau ds,\quad (t,x)\in Q. \tag{12}$$

Given equality (12), we can write:

$$E-\left(A+\int_0^T n(t)dt\right)^{-1}\int_\tau^T n(\alpha)d\alpha-\left(B+\int_0^l m(x)dx\right)^{-1}\int_s^l m(\beta)d\beta$$

$$+\left(B+\int_0^l m(x)dx\right)^{-1}\left(A+\int_0^T n(t)dt\right)^{-1}\int_\tau^T n(\alpha)d\alpha\int_s^l m(\beta)d\beta$$

$$=\left(B+\int_0^l m(x)dx\right)^{-1}\left(A+\int_0^T n(t)dt\right)^{-1}\times\left(A+\int_0^\tau n(\alpha)d\alpha\right)\left(B+\int_0^s m(\beta)d\beta\right),$$

$$\left(B+\int_0^l m(x)dx\right)^{-1}\left(A+\int_0^T n(t)dt\right)^{-1}\int_\tau^T n(\alpha)d\alpha\int_s^l m(\beta)d\beta$$

$$-\left(B+\int_0^l m(x)dx\right)^{-1}\int_s^l m(\beta)d\beta$$

$$=-\left(B+\int_0^l m(x)dx\right)^{-1}\left(A+\int_0^T n(t)dt\right)^{-1}\left[\left(A+\int_0^\tau n(\alpha)d\alpha\right)\int_s^l m(\beta)d\beta\right],$$

$$\left(B+\int_0^l m(x)dx\right)^{-1}\left(A+\int_0^T n(t)dt\right)^{-1}\int_\tau^T n(\alpha)d\alpha\int_s^l m(\beta)d\beta$$

$$-\left(A+\int_0^T n(t)dt\right)^{-1}\int_\tau^T n(\alpha)d\alpha$$

$$=-\left(B+\int_0^l m(x)dx\right)^{-1}\left(A+\int_0^T n(t)dt\right)^{-1}\left[\left(B+\int_0^s m(\beta)d\beta\right)\int_\tau^T n(\alpha)d\alpha\right].$$

As a result, we obtain equation (7)

$$z(t,x)=\left(B+\int_0^l m(x)dx\right)^{-1}\psi(t)+\left(A+\int_0^T n(t)dt\right)^{-1}\varphi(x)$$

$$-\left(B+\int_0^l m(x)dx\right)^{-1}\left(A+\int_0^T n(t)dt\right)^{-1}\left[B\varphi(0)+\int_0^l m(x)\varphi(x)dx\right]$$

$$+\int_0^t\int_0^x\left(B+\int_0^l m(x)dx\right)^{-1}\left(A+\int_0^T n(t)dt\right)^{-1}$$

$$\times\left(A+\int_0^\tau n(\alpha)d\alpha\right)\left(B+\int_0^s m(\beta)d\beta\right)f(\tau,s,z(\tau,s))d\tau\,ds$$

$$-\int_0^t\int_x^l\left(B+\int_0^l m(x)dx\right)^{-1}\left(A+\int_0^T n(t)dt\right)^{-1}$$

$$\times\left[\left(A+\int_0^\tau n(\alpha)d\alpha\right)\int_s^l m(\beta)d\beta\right]f(\tau,s,z(\tau,s))d\tau\,ds$$

$$-\int_t^T\int_0^x\left(B+\int_0^l m(x)dx\right)^{-1}\left(A+\int_0^T n(t)dt\right)^{-1}$$

$$\times\left[\left(B+\int_0^s m(\beta)d\beta\right)\int_\tau^T n(\alpha)d\alpha\right]f(\tau,s,z(\tau,s))d\tau\,ds$$

$$+\int_t^T\int_x^l\left(B+\int_0^l m(x)dx\right)^{-1}\left(A+\int_0^T n(t)dt\right)^{-1}\left[\int_\tau^T n(\alpha)d\alpha\int_s^l m(\beta)d\beta\right]$$

$$\times f(\tau,s,z(\tau,s))d\tau\,ds,\quad (t,x)\in Q. \tag{13}$$

In this equality, having determined the matrix-function $G(t,x,\tau,s)$, we proved the first part of the theorem. We now calculate the derivative of the function $z(t,x)$ determined by equality (13) with respect to t and x

$$z_{tx}(t,x)=\frac{\partial^2}{\partial t\partial x}\left[\left(B+\int_0^l m(x)dx\right)^{-1}\psi(t)+\left(A+\int_0^T n(t)dt\right)^{-1}\varphi(x)\right.$$

$$\left.-\left(B+\int_0^l m(x)dx\right)^{-1}\left(A+\int_0^T n(t)dt\right)^{-1}\left[B\varphi(0)+\int_0^l m(x)\varphi(x)dx\right]\right]$$

$$+\frac{\partial^2}{\partial t\partial x}\left[\int_0^t\int_0^x\left(B+\int_0^l m(x)dx\right)^{-1}\left(A+\int_0^T n(t)dt\right)^{-1}\right.$$

$$\left.\times\left(A+\int_0^\tau n(\alpha)d\alpha\right)\left(B+\int_0^s m(\beta)d\beta\right)f(\tau,s,z(\tau,s))d\tau\,ds\right]$$

$$-\frac{\partial^2}{\partial t\partial x}\left[\int_0^t\int_x^l\left(B+\int_0^l m(x)dx\right)^{-1}\left(A+\int_0^T n(t)dt\right)^{-1}\right.$$

$$\times \left[\left(A + \int_0^\tau n(\alpha)d\alpha \right) \int_s^l m(\beta)d\beta \right] f(\tau, s, z(\tau, s))d\tau\, ds \Bigg]$$

$$-\frac{\partial^2}{\partial t \partial x} \left[\int_t^T \int_0^x \left(B + \int_0^l m(x)dx \right)^{-1} \left(A + \int_0^T n(t)dt \right)^{-1} \right.$$

$$\times \left[\left(B + \int_0^s m(\beta)d\beta \right) \int_\tau^T n(\alpha)d\alpha \right] f(\tau, s, z(\tau, s))d\tau ds \Bigg]$$

$$+\frac{\partial^2}{\partial t \partial x} \int_t^T \int_x^l \left(B + \int_0^l m(x)dx \right)^{-1} \left(A + \int_0^T n(t)dt \right)^{-1}$$

$$\times \left[\int_\tau^T n(\alpha)d\alpha \int_s^l m(\beta)d\beta \right] f(\tau, s, z(\tau, s))d\tau\, ds$$

$$= \left(B + \int_0^l m(x)dx \right)^{-1} \left(A + \int_0^T n(t)dt \right)^{-1} \left[AB + B \int_0^t n(\alpha)d\alpha + A \int_0^x m(\beta)d\beta \right.$$

$$+ \int_0^t n(\alpha)d\alpha \int_0^x m(\beta)d\beta + A \int_x^l m(\beta)d\beta + \int_0^t n(\alpha)d\alpha \int_x^l m(\beta)d\beta$$

$$\left. +B \int_t^T n(\alpha)d\alpha + \int_0^x m(\beta)d\beta \int_t^T n(\alpha)d\alpha + \int_t^T n(\alpha)d\alpha \int_x^l m(\beta)d\beta \right]$$

$$\times f(t, x, z(t, x)) = f(t, x, z(t, x)).$$

We now show that the function defined by equation (11) satisfies the non-local boundary conditions (2) and (3), with

$$A \left[\left(B + \int_0^l m(x)dx \right)^{-1} \psi(0) + \left(A + \int_0^T n(t)dt \right)^{-1} \varphi(x) \right.$$

$$- \left(B + \int_0^l m(x)dx \right)^{-1} \left(A + \int_0^T n(t)dt \right)^{-1} \left[B\varphi(0) + \int_0^l m(x)\varphi(x)dx \right]$$

$$+ \left(B + \int_0^l m(x)dx \right)^{-1} \left(A + \int_0^T n(t)dt \right)^{-1}$$

$$\times \int_0^T \int_0^l \left[\int_t^T n(\tau)d\tau \int_x^l m(s)ds \right] f(t, x, z(t, x))dt\, dx$$

$$-\left(A+\int\limits_0^T n(t)dt\right)^{-1}\int\limits_0^T\int\limits_0^x\left(\int\limits_t^T n(\tau)d\tau\right)f(t,s,z(t,s))dtds\Bigg]$$

$$+\int\limits_0^T n(t)\Bigg[\left(B+\int\limits_0^l m(x)dx\right)^{-1}\psi(t)+\left(A+\int\limits_0^T n(t)dt\right)^{-1}\varphi(x)$$

$$-\left(B+\int\limits_0^l m(x)dx\right)^{-1}\left(A+\int\limits_0^T n(t)dt\right)^{-1}\left[B\varphi(0)+\int\limits_0^l m(x)\varphi(x)dx\right]$$

$$+\left(B+\int\limits_0^l m(x)dx\right)^{-1}\left(A+\int\limits_0^T n(t)dt\right)^{-1}$$

$$\times\int\limits_0^T\int\limits_0^l\left[\int\limits_\tau^T n(\tau)d\tau\int\limits_x^l m(s)ds\right]f(t,x,z(t,x))dt\,dx$$

$$-\left(B+\int\limits_0^l m(x)dx\right)^{-1}\int\limits_0^l\int\limits_0^t\left(\int\limits_x^l m(s)ds\right)f(\tau,x,z(\tau,x))d\tau\,dx$$

$$-\left(A+\int\limits_0^T n(t)dt\right)^{-1}\int\limits_0^T\int\limits_0^x\left[\int\limits_t^T n(\tau)d\tau\right]f(t,s,z(t,s))dtds+\int\limits_0^t\int\limits_0^x f(\tau,s,z(\tau,s))d\tau\,ds\Bigg]dt$$

$$=\left(B+\int\limits_0^l m(x)dx\right)^{-1}\left[A\psi(0)+\int\limits_0^T n(t)\psi(t)dt\right]$$

$$-\left(B+\int\limits_0^l m(x)dx\right)^{-1}\left(A+\int\limits_0^T n(t)dt\right)^{-1}B\varphi(0)\left(A+\int\limits_0^T n(t)dt\right)$$

$$+\left(A+\int\limits_0^T n(t)dt\right)^{-1}\varphi(x)\left(A+\int\limits_0^T n(t)dt\right)-\left(A+\int\limits_0^T n(t)dt\right)^{-1}$$

$$\times\left(A+\int\limits_0^T n(t)dt\right)\int\limits_0^T\int\limits_0^x\left(\int\limits_t^T n(\tau)d\tau\right)f(t,s,z(t,s))dt\,ds$$

$$-\left(B+\int\limits_0^l m(x)dx\right)^{-1}\left(A+\int\limits_0^T n(t)dt\right)^{-1}\left(A+\int\limits_0^T n(t)dt\right)$$

$$\times\int\limits_0^l m(x)\varphi(x)dx+\left(B+\int\limits_0^l m(x)dx\right)^{-1}\left(A+\int\limits_0^T n(t)dt\right)^{-1}$$

$$\times\left(A+\int_0^T n(t)dt\right)\int_0^T\int_0^l\left[\int_t^T n(\tau)d\tau\int_x^l m(s)ds\right]f(t,x,z(t,x))dt\,dx$$

$$-\left(B+\int_0^l m(x)dx\right)^{-1}\int_0^T n(t)\int_0^l\int_0^t\left(\int_x^l m(s)ds\right)f(\tau,x,z(\tau,x))d\tau\,dx\,dt$$

$$+\int_0^T\int_0^x\left(\int_t^T n(\tau)d\tau\right)f(t,s,z(t,s))dtds$$

$$=\left(B+\int_0^l m(x)dx\right)^{-1}\left[A\psi(0)+\int_0^T n(t)\psi(t)dt\right]-\left(B+\int_0^l m(x)dx\right)^{-1}B\varphi(0)+\varphi(x)$$

$$-\int_0^T\int_0^x\left(\int_t^T n(\tau)d\tau\right)f(t,s,z(t,s))dt\,ds$$

$$-\left(B+\int_0^l m(x)dx\right)^{-1}\int_0^l m(x)\varphi(x)dx+\left(B+\int_0^l m(x)dx\right)^{-1}$$

$$\times\int_0^T\int_0^l\left[\int_t^T n(\tau)d\tau\int_x^l m(s)ds\right]f(t,x,z(t,x))dtdx$$

$$-\left(B+\int_0^l m(x)dx\right)^{-1}\int_0^T n(t)\int_0^l\int_0^t\left(\int_x^l m(s)ds\right)f(\tau,x,z(\tau,x))d\tau\,dx\,dt$$

$$+\int_0^T\int_0^x\left(\int_t^T n(\tau)d\tau\right)f(t,s,z(t,s))dtds$$

$$=\left(B+\int_0^l m(x)dx\right)^{-1}\left[\left(A\psi(0)+\int_0^T n(t)\psi(t)dt\right)-\left(B\varphi(0)+\int_0^l m(x)\varphi(x)dx\right)\right]$$

$$+\varphi(x)=\varphi(x).$$

In a similar way, we can show that the point-wise and integral boundary condition

$$Bz(t,0)+\int_0^l m(x)z(t,x)dx=\psi(t),\quad t\in[0,T]$$

is satisfied. □

3 Existence and uniqueness

It is seen from the proved theorem that problem (1)–(3) is equivalent to the integral equation

$$z(t,x) = \left(B + \int_0^l m(x)dx\right)^{-1} \psi(t) + \left(A + \int_0^T n(t)dt\right)^{-1} \varphi(x)$$

$$- \left(B + \int_0^l m(x)dx\right)^{-1} \left(A + \int_0^T n(t)dt\right)^{-1} \left[B\varphi(0) + \int_0^l m(x)\varphi(x)dx\right]$$

$$+ \int_0^T \int_0^l G(t,x,\tau,s) f(\tau,s,z) d\tau\, ds. \tag{14}$$

In order to prove the existence and uniqueness of the solution to problem (1)–(3) we determine the operator $P : C(Q; R^n) \to C(Q; R^n)$ as follows:

$$(Pz)(t,x) = \left(B + \int_0^l m(x)dx\right)^{-1} \psi(t) + \left(A + \int_0^T n(t)dt\right)^{-1} \varphi(x)$$

$$- \left(B + \int_0^l m(x)dx\right)^{-1} \left(A + \int_0^T n(t)dt\right)^{-1} \left(B\varphi(0) + \int_0^l m(x)\varphi(x)dx\right)$$

$$+ \int_0^T \int_0^l G(t,x,\tau,s) f(\tau,s,z) d\tau\, ds.$$

It is known that solving problem (1)–(3) or integral equation (14) is equivalent to finding the fixed point of the operator P. In other words, problem (1)–(3) has a solution if and only if the operator P has a fixed point.

Theorem 3. Assume that the following conditions hold:

$$|f(t,x,z_2) - f(t,x,z_1)| \le M |z_2 - z_1|, \quad \forall (t,x) \in Q, \;\; z_1, z_2 \in R^n, \;\; M \ge 0 \tag{15}$$

and

$$L = lTSM < 1, \tag{16}$$

where

$$S = \max_{Q \times Q} \|G(t,x,\tau,s)\|.$$

Then, problem (1)–(3) has a unique solution in Q.

Proof. Denote

$$N = \max_Q \left| \left(B + \int_0^l m(x)dx\right)^{-1} \psi(t) + \left(A + \int_0^T n(t)dt\right)^{-1} \varphi(x) \right.$$

$$-\left(B+\int_0^l m(x)dx\right)^{-1}\left(A+\int_0^T n(t)dt\right)^{-1}\left(B\varphi(0)+\int_0^l m(x)\varphi(x)dx\right)\Bigg|,$$

$$\max_{(t,x)\in Q}|f(t,x,0)|=M_f$$

and choose $r\geq\frac{N+M_fTS}{1-L}$. We show that the relation $PB_r\subset B_r$ holds, where

$$B_r=\left\{x\in C(Q,R^n)\,:\,\|z\|\leq r\right\}.$$

For arbitrary $z\in B_r$, we have

$$\|Pz(t,x)\|\leq N+\int_0^T\int_0^l|G(t,x,\tau,s)|\left(|f(\tau,s,z(\tau,s)-f(\tau,s,0)|+|f(\tau,s,0)|\right)d\tau\,ds$$

$$\leq N+S\int_0^T\int_0^l\left(M|z|+M_f\right)dt\,dx\leq N+SMrTl+M_fTlS\leq\frac{N+M_fTS}{1-L}\leq r.$$

On the other hand, from condition (15) we obtain that for arbitrary $z_1,z_2\in B_r$ the relation

$$|Pz_2-Pz_1|\leq\int_0^T\int_0^l|G(t,x,\tau,s)|\left(|f(\tau,s,z_2(\tau,s)-f(\tau,s,z_1(\tau,s)|\right)$$

$$\leq S\int_o^T\int_0^l M|z_2(t,x)-z_1(t,x)|dtdx\leq MSTl\max_Q|z_2(t,x)-z_1(t,x)|\leq MSTl\|z_2-z_1\|$$

holds. Hence, we obtain

$$\|Pz_2-Pz_1\|\leq L\|z_2-z_1\|.$$

Taking condition (16) into account we obtain that the operator P is compressive. So, problem (1)–(3) has a unique solution. □

4 Application of the obtained results

To illustrate the obtained results, let us consider the system of hyperbolic equations

$$\begin{cases} z_{1tx}(t,x)=0.1\cos z_2(t,x), \\ z_{2tx}(t,x)=\frac{|z_1(t,x)|}{10(1+|z_1(t,x)|)}, \end{cases}\qquad (t,x)\in[0,1]\times[0,1]. \tag{17}$$

Assume that the following boundary conditions are satisfied

$$\begin{cases} 2z_1(0,x)+\int_0^1 tz_1(t,x)dt=x^2, \\ z_2(0,x)=1, \end{cases}\qquad x\in[0,1]. \tag{18}$$

$$\begin{cases} 2z_1(0,x)+\int_0^1 xz_1(t,x)dt=t^2, \\ z_2(t,0)=1, \end{cases}\qquad t\in[0,1]. \tag{19}$$

Make the following notation:

$$A = B = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}, \ n(t) = \begin{pmatrix} t & 0 \\ 0 & 0 \end{pmatrix}, \ m(x) = \begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix},$$

$$\varphi(x) = \begin{pmatrix} x^2 \\ 1 \end{pmatrix}, \ \psi(t) = \begin{pmatrix} t^2 \\ 1 \end{pmatrix}.$$

Then, conditions (18), (19) can be written as follows:

$$\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} z_1(0,x) \\ z_2(0,x) \end{pmatrix} + \int_0^1 \begin{pmatrix} t & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} z_1(t,x) \\ z_2(t,x) \end{pmatrix} dt = \begin{pmatrix} x^2 \\ 1 \end{pmatrix}, \quad x \in [0,1].$$

$$\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} z_1(t,0) \\ z_2(t,0) \end{pmatrix} + \int_0^1 \begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} z_1(t,x) \\ z_2(t,x) \end{pmatrix} dx = \begin{pmatrix} t^2 \\ 1 \end{pmatrix}, \quad t \in [0,1].$$

$$A + \int_0^1 n(t)dt = \begin{pmatrix} 2.5 & 0 \\ 0 & 1 \end{pmatrix}, \quad B + \int_0^1 m(x)dx = \begin{pmatrix} 2.5 & 0 \\ 0 & 1 \end{pmatrix},$$

$$\left(B + \int_0^l m(x)dx \right)^{-1} = \left(A + \int_0^T n(t)dt \right)^{-1} = \begin{pmatrix} 0.4 & 0 \\ 0 & 1 \end{pmatrix}.$$

Taking them into account:

$$G(t,x,\tau,s) = \begin{cases} \begin{pmatrix} 0.16 \left(2 + \frac{\tau^2}{2}\right)\left(2 + \frac{s^2}{2}\right) & 0 \\ 0 & 1 \end{pmatrix}, & 0 \le \tau \le t, \, 0 \le s \le x, \\ -\begin{pmatrix} 0.16 \left(2 + \frac{\tau^2}{2}\right) \frac{s^2}{2} & 0 \\ 0 & 0 \end{pmatrix}, & 0 \le \tau \le t, x < s \le 1, \\ -\begin{pmatrix} 0.16 \frac{\tau^2}{2} \left(2 + \frac{s^2}{2}\right), & 0 \\ 0 & 0 \end{pmatrix}, & t \le \tau \le 1, 0 < s \le x, \\ \begin{pmatrix} 0.16 \frac{\tau^2}{2} \cdot \frac{s^2}{2} & 0 \\ 0 & 0 \end{pmatrix}, & t \le \tau \le 1, x < s \le 1. \end{cases}$$

Let us estimate the main parameters of the boundary value problem (17)–(19). We have that the following estimate holds for the norm of the Green function $\max \|G(t,x,\tau,s)\| \le 1$; the Lipschitz constant $M = 0.1$, and the compression parameter $L = 1 \cdot 1 \cdot 0.1 \cdot 1 = 0.1 < 1$. So, all the conditions of Theorem 3 are fulfilled and the boundary value problem (17)–(19) has a unique solution.

Conclusion

The present work studied a system of hyperbolic equations with non-local condition. Boundary conditions are rather general. In the special case, it contains the classical Goursat–Darboux problem, "pure" integral conditions, a boundary value problem whose part of the conditions is pointwise, the other part is in integral form, and other cases.

Acknowledgments

All authors contributed equally to this work.

Conflict of Interest

The authors declare no conflict of interest.

References

1 Samarskii, A.A. (1980). O nekotorykh problemakh teorii differentsialnykh uravnenii [Some problems of the theory of differential equations]. *Differentsialnye uravneniia — Differential equations, 16*(11), 1925–1935 [in Russian].

2 Nakhushev, A.M. (2006). *Zadachi so smeshcheniem dlia uravnenii v chastnykh proizvodnykh [Boundary Value Problems with Shift for Partial Differential Equations].* Moscow: Nauka [in Russian].

3 Ptashnik, B.I. (1984). *Nekorrektnye granichnye zadachi dlia differentsialnykh uravnenii s chastnymi proizvodnymi [Ill-Posed Boundary Value Problems for Partial Differential Equations].* Kiev: Naukova Dumka [in Russian].

4 Pulkina, L.S. (2012). *Zadachi s neklassicheskimi usloviiami dlia giperbolicheskikh uravnenii [Problems with Nonclassical Conditions for Hyperbolic Equations].* Samara: Samarskii universitet [in Russian].

5 Zhestkov, S.V. (1990). The Goursat problem with integral boundary conditions. *Ukrainian Mathematical Journal, 42*(1), 119–122. https://doi.org/10.1007/BF01066375

6 Paneah, B., & Paneah, P. (2009). Non-local problems in the theory of hyperbolic differential equations. *Transactions of the Moscow Mathematical Society, 70*, 135–170. https://doi.org/10.1090/S0077-1554-09-00179-4

7 Sharifov, Y.A., Mammadli, A.R., & Zeynally, F.M. (2025). The Goursat-Darboux system with two-point boundary condition. *Transactions of the National Academy of Sciences of Azerbaijan. Series of Physical-Technical and Mathematical Sciences, 45*(1), 142–152. https://doi.org/10.30546/2617-7900.45.1.2025.0142

8 Mardanov, M.J, & Sharifov, Y.A. (2025). Existence and uniqueness of solutions to the Goursat–Darboux system with integral boundary conditions. *Journal of Samara State Technical University, Ser. Physical and Mathematical Sciences, 29*(2), 241–255. https://doi.org/10.14498/vsgtu2147

9 Byszewski, L. (1990). Existence and uniqueness of solution of nonlocal problems for hyperbolic equation $u_{xt} = F(x; t; u; u_x)$. *Journal of Applied Mathematics and Stochastic Analysis, 3*(3), 163–168.

10 Assanova, A.T. (2017). Non-local problem with integral conditions for a system of hyperbolic equations in characteristic rectangle. *Russian Mathematics, 61*(5), 7–20. https://doi.org/10.3103/S1066369X17050024

11 Assanova, A.T. (2023). A generalized integral problem for a system of hyperbolic equations and its applications. *Hacettepe Journal of Mathematics and Statistics, 52*(6), 1513–1532. https://doi.org/10.15672/hujms.1094454

12 Bouziani, A. (1997). Solution forte d'un probleme mixte avec conditions non locales pour une classe d'equations hyperboliques [Strong solution of a mixed problem with non-local conditions for a class of hyperbolic eguations]. *Bulletin de la Classe des sciences — Bulletin of the Class of Sciences, 8*(1), 53–70 [in French].

13 Kozhanov, A.I., & Pulkina, L.S. (2006). On the solvability of boundary value problems with a non-local boundary condition of integral form for multidimensional hyperbolic equations. *Differential Equations, 42*(9), 1233–1246. https://doi.org/10.1134/S0012266106090023

14 Assanova, A.T., & Dzhumabaev, D.S. (2013). Well-posedness of non-local boundary value problems with integral condition for the system of hyperbolic equations. *Journal of Mathematical Analysis and Applications, 402*(1), 167–178. https://doi.org/10.1016/j.jmaa.2013.01.012

15 Assanova, A.T. (2021). On the solvability of a non-local problem for the system of Sobolev-type differential equations with integral condition. *Georgian Mathematical Journal, 28*(1), 49–57. https://doi.org/10.1515/gmj-2019-2011

16 Assanova, A.T. (2018). On a non-local problem with integral conditions for the system of hyperbolic equations. *Differential Equations, 54*(2), 201–214. https://doi.org/10.1134/S0012266118020076

17 Golubeva, N.D., & Pulkina, L.S. (1996). A non-local problem with integral conditions. *Mathematical Notes, 59*(3), 326–328. https://doi.org/10.1007/BF02308548

18 Pulkina, L.S. (2000). The L_2 solvability of a non-local problem with integral conditions for a hyperbolic equation. *Differential Equations, 36*(2), 316–318. https://doi.org/10.1007/BF02754219

19 Oussaeif, T.E., & Bouziani, A. (2018). Solvability of nonlinear Goursat type problem for hyperbolic equation with integral condition. *Khayyam Journal of Mathematics, 4*(2), 198–213. https://doi.org/10.22034/kjm.2018.65161

20 Mardanov, M.J., & Sharifov, Y.A. (2023). An optimal control problem for the systems with integral boundary conditions. *Bulletin of the Karaganda University. Mathematics Series,* 1(109), 110–123. https://doi.org/10.31489/2023M1/110-123

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