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Investigation of the solution of a boundary value problem with variable coefficients whose principal part is the Cauchy–Riemann equation

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This study is devoted to obtaining an analytical expression for the solution of a non-local boundary value problem for a linear inhomogeneous differential equation with variable coefficients, which principle part is the Cauchy–Riemann equation. Since the Cauchy–Riemann equation is a first-order elliptic equation, the problem formulated with a classical boundary condition in a finite domain is ill-posed. Defining a boundary condition for a first-order elliptic equation within a finite domain requires special investigation. For a first-order elliptic equation in the x_1ox_2 plane, a new boundary condition is proposed within a bounded region that is concave in the x_2 direction, and an expression for the solution is obtained. For this purpose, using the fundamental solution of the principal part of the equation, the main relation consisting of two parts is obtained, the first part yields an arbitrary solution to the equation, and the second part gives the boundary values of the solution representing the necessary conditions. Utilizing these necessary and specified boundary conditions, a system of Fredholm integral equations of the second kind with a singular kernel is constructed to find a solution, and a method for elimination the singularity in the solution is proposed.

Keywords: first-order elliptic equation, Cauchy–Riemann equation, embroidery condition, nonlocal boundary condition, main relation, Green's second formula, necessary conditions, regularization of singularity.

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Introduction

The Dirichlet, Neumann, Poincaré and directional derivative problems for second-order elliptic equations, particularly for the Laplace equation with local boundary conditions, have been widely studied in [1–3]. Since the Cauchy–Riemann equation is a first-order elliptic equation, the problems formulated for it using classical conditions are known to be globally ill-posed.

In [4], the Dirichlet problem for the Cauchy–Riemann equation is studied under the condition that the given function on the boundary satisfies what the authors call the necessary condition, a very rigid condition.

In general, writing out boundary conditions for first-order elliptic equations, as well as proving the correctness of the problem, require special research. Unlike previous works focusing on Dirichlet or Neumann problems, in [5] a unified analytical framework was developed to handle mixed (Robin type) boundary conditions by combining complex analysis and functional analysis methods, thus expanding the applicability of the Cauchy–Riemann boundary problem theory. In [6–8], problems related to the Cauchy–Riemann equation under classical boundary conditions are studied essentially using methods of complex analysis. In [9], the Cauchy–Riemann operator's spectral behavior with homogeneous

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Dirichlet-type boundary conditions was investigated, and the theoretical understanding of its spectral properties was advanced by demonstrating that the operator possesses the Volterra property.

In this study, a new boundary condition for the Cauchy–Riemann equation is proposed and the analytical solution of the problem is reduced to the system of Fredholm integral equations of the second type.

Let D be a region bounded in the x_1ox_2 plane and convex in the x_2 direction as shown in Figure 1. If we project the domain D onto the x_1 axis parallel to the x_2 axis, then the boundary Γ is divided into two parts Γ_1 and Γ_2 ($\Gamma = \Gamma_1 \cup \Gamma_2$, since the domain D is convex in the x_2 direction). The equations of the curves Γ_1 and Γ_2 are given by

$$x_2 = \gamma_k(x_1),\ (k = 1, 2); \quad x_1 \in [a_1, b_1] = pr|_{x_1} D = pr|_{x_1}\Gamma_1 = pr|_{x_1}\Gamma_2$$

and Γ is a Lyapunov curve.

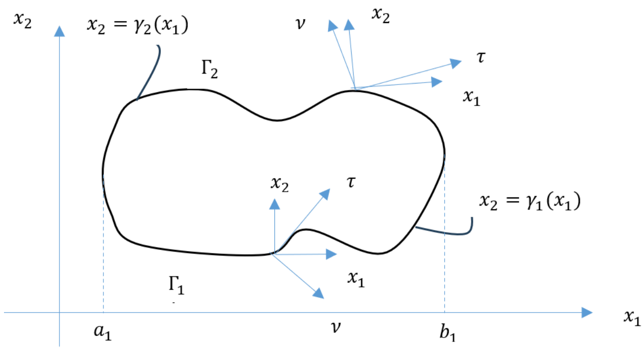

Figure 1. Region D

The following problem in the domain D is considered

$$\frac{\partial u(x)}{\partial x_2} + i\frac{\partial u(x)}{\partial x_1} = a(x)u(x) + f(x), \quad x \in D, \tag{1}$$

$$\alpha_1(x_1)u\left(x_1, \gamma_1(x_1)\right) + \alpha_2(x_1)u\left(x_1, \gamma_2(x_1)\right) = \varphi(x_1), \quad x_1 \in [a_1, b_1]. \tag{2}$$

Here, $a(x)$, $f(x)$, $\alpha_1(x_1)$, $\alpha_2(x_1)$ and $\varphi(x_1)$ are given continuously differentiable functions and $i = \sqrt{-1}$.

Let us denote by ν and τ the outward and tangential normals drawn to boundary of Γ, respectively.

It is known that the fundamental solution of the Cauchy–Riemann equation is

$$U(x - \xi) = \frac{1}{2\pi}\frac{1}{x_2 - \xi_2 + i(x_1 - \xi_2)}, \tag{3}$$

where $x - \xi = (x_1 - \xi_1, x_2 - \xi_2)$.

1 The main relation

We multiply equation (1) by the fundamental solution (3) and integrate over the domain D:

$$\int_D \frac{\partial u(x)}{\partial x_2}U(x-\xi)dx + i\int_D \frac{\partial u(x)}{\partial x_1}U(x-\xi)dx = \int_D a(x)u(x)U(x-\xi)dx + \int_D f(x)U(x-\xi)dx. \tag{4}$$

Applying the Ostrogradsky–Gauss formula to the integral on the left-hand side of equation (4), we get

$$\int_D \frac{\partial u(x)}{\partial x_2}U(x-\xi)dx = \int_\Gamma u(x)U(x-\xi)\cos(\nu, x_2)dx - \int_D u(x)\frac{\partial U(x-\xi)}{\partial x_2}dx, \tag{5}$$

$$\int_D \frac{\partial u(x)}{\partial x_1} U(x-\xi)dx = \int_\Gamma u(x)U(x-\xi)\cos(\nu, x_1)dx - \int_D u(x)\frac{\partial U(x-\xi)}{\partial x_1}dx. \quad (6)$$

Let us substitute expressions (5) and (6) into equation (4) and write it down as follows

$$\int_\Gamma u(x)U(x-\xi)\cos(\nu, x_2)dx - \int_D u(x)\frac{\partial U(x-\xi)}{\partial x_2}dx + i\left[\int_\Gamma u(x)U(x-\xi)\cos(\nu, x_1)dx\right.$$

$$\left. - \int_D u(x)\frac{\partial U(x-\xi)}{\partial x_1}dx\right] = \int_D a(x)u(x)U(x-\xi)dx + \int_D f(x)U(x-\xi)dx$$

or

$$\int_\Gamma u(x)U(x-\xi)\Big[\cos(\nu, x_2) + i\cos(\nu, x_1)\Big]dx - \int_D a(x)u(x)U(x-\xi)dx - \int_D f(x)U(x-\xi)dx$$

$$= \int_D u(x)\left[\frac{\partial U(x-\xi)}{\partial x_2} + i\frac{\partial U(x-\xi)}{\partial x_1}\right]dx.$$

Since the function $U(x-\xi)$ is a fundamental solution of the principal part of equation (1), we can write the last equation as

$$\int_\Gamma u(x)U(x-\xi)\Big[\cos(\nu, x_2) + i\cos(\nu, x_1)\Big]dx - \int_D a(x)u(x)U(x-\xi)dx - \int_D f(x)U(x-\xi)dx$$

$$= \begin{cases} u(\xi), & \xi \in D, \\ \frac{1}{2}u(\xi), & \xi \in \Gamma. \end{cases} \quad (7)$$

Main relation (7) we obtained consists of two parts. The first part gives an arbitrary solution of the given equation (1) in the domain D for $\xi \in D$, and the second part gives the necessary conditions for $\xi \in \Gamma$ that gives relation between values of boundary conditions and values of the obtained solution. It should be noted that in the literature expressions of the form (7) are known as necessary conditions derived from the first fundamental relations, similar to Green's second formula, in the study of higher-order equations [2]. Similar methods have also been applied in [10–12] in the process of finding analytical solutions to problems for the Cauchy–Riemann equation with non-local boundary conditions in regions with various geometries.

Expressions of the type (7) are derived from various basic relations, by which all necessary linearly independent conditions can be obtained. As emphasized in [2] while the D'Alembert formula gives the solution of the Cauchy problem for the second-order wave equation, it cannot directly give a solution to the boundary value problem posed for the Laplace equation. That is, since the D'Alembert formula includes it's own initial conditions, then, by writing them down, we obtain a solution to the Cauchy problem from the D'Alembert formula. However, it is not possible to specify the two functions that participate in the Green's II formula obtained for the Laplace equation (they are linearly dependent functions). By specifying one of them, we obtain the Dirichlet problem, and by specifying the other, we obtain the Neumann problem.

In [2], a boundary value problem for the Laplace equation was considered and the expression derived from Green's II formula was called a necessary and sufficient condition. In [10] and [11], for a first-order elliptical equation a new approach to non-local boundary value problem for the Cauchy–Riemann equation was proposed. Paper [11] was devoted to investigation of a new method for investigating of solutions to boundary value problems for first order elliptic equations. Computational aspects of first-order partial differential equations with nonlocal boundary condition were considered in [13].

2 Necessary and sufficient conditions

Now let us single out the necessary and sufficient conditions from the main relation (7):

$$\frac{1}{2}u\left(\xi_1,\gamma_1(\xi_1)\right)=\frac{1}{2\pi}\int_{\Gamma_1}\frac{u(x)}{x_2-\gamma_1(\xi_1)+i(x_1-\xi_1)}\Big[\cos(\nu,x_2)+i\cos(\nu,x_1)\Big]dx$$
$$+\frac{1}{2\pi}\int_{\Gamma_2}\frac{u(x)}{x_2-\gamma_1(\xi_1)+i(x_1-\xi_1)}\Big[\cos(\nu,x_2)+i\cos(\nu,x_1)\Big]dx$$
$$-\frac{1}{2\pi}\int_{D}\frac{a(x)u(x)}{x_2-\gamma_1(\xi_1)+i(x_1-\xi_1)}dx-\frac{1}{2\pi}\int_{D}\frac{f(x)}{x_2-\gamma_1(\xi_1)+i(x_1-\xi_1)}dx$$
$$=\frac{1}{2\pi}\int_{a_1}^{b_1}\frac{u\left(x_1,\gamma_1(x_1)\right)}{\gamma_1(x_1)-\gamma_1(\xi_1)+i(x_1-\xi_1)}\Big[-\cos(x_1,\tau)+i\sin(x_1,\tau)\Big]\frac{dx_1}{\cos(x_1,\tau)}$$
$$+\frac{1}{2\pi}\int_{a_1}^{b_1}\frac{u\left(x_1,\gamma_2(x_1)\right)}{\gamma_2(x_1)-\gamma_1(\xi_1)+i(x_1-\xi_1)}\Big[\cos(x_1,\tau)-i\sin(x_1,\tau)\Big]\frac{dx_1}{\cos(x_1,\tau)}$$
$$-\frac{1}{2\pi}\int_{D}\frac{a(x)u(x)}{x_2-\gamma_1(\xi_1)+i(x_1-\xi_1)}dx-\frac{1}{2\pi}\int_{D}\frac{f(x)}{x_2-\gamma_1(\xi_1)+i(x_1-\xi_1)}dx$$
$$=\frac{1}{2\pi}\int_{a_1}^{b_1}\frac{u\left(x_1,\gamma_1(x_1)\right)[1-i\gamma_1{}'(x_1)]dx_1}{\gamma_1'\left(\sigma_1(x_1,\xi_1)\right)(x_1-\xi_1)+i(x_1-\xi_1)}+\frac{1}{2\pi}\int_{a_1}^{b_1}\frac{u\left(x_1,\gamma_2(x_1)\right)[1-i\gamma_2{}'(x_1)]dx_1}{\gamma_2(x_1)-\gamma_1(\xi_1)+i(x_1-\xi_1)}$$
$$-\frac{1}{2\pi}\int_{D}\frac{a(x)u(x)}{x_2-\gamma_1(\xi_1)+i(x_1-\xi_1)}dx-\frac{1}{2\pi}\int_{D}\frac{f(x)}{x_2-\gamma_1(\xi_1)+i(x_1-\xi_1)}dx, \tag{8}$$
$$\frac{1}{2}u\left(\xi_1,\gamma_2(\xi_1)\right)=\frac{1}{2\pi}\int_{\Gamma_1}\frac{u(x)}{x_2-\gamma_2(\xi_1)+i(x_1-\xi_1)}\Big[\cos(\nu,x_2)+i\cos(\nu,x_1)\Big]dx$$
$$+\frac{1}{2\pi}\int_{\Gamma_2}\frac{u(x)}{x_2-\gamma_2(\xi_1)+i(x_1-\xi_1)}\Big[\cos(\nu,x_2)+i\cos(\nu,x_1)\Big]dx$$
$$-\frac{1}{2\pi}\int_{D}\frac{a(x)u(x)}{x_2-\gamma_2(\xi_1)+i(x_1-\xi_1)}dx-\frac{1}{2\pi}\int_{D}\frac{f(x)}{x_2-\gamma_2(\xi_1)+i(x_1-\xi_1)}dx$$
$$=\frac{1}{2\pi}\int_{a_1}^{b_1}\frac{u\left(x_1,\gamma_1(x_1)\right)}{\gamma_1(x_1)-\gamma_2(\xi_1)+i(x_1-\xi_1)}\Big[-\cos(x_1,\tau)+i\sin(x_1,\tau)\Big]\frac{dx_1}{\cos(x_1,\tau)}$$
$$+\frac{1}{2\pi}\int_{a_1}^{b_1}\frac{u\left(x_1,\gamma_2(x_1)\right)}{\gamma_2(x_1)-\gamma_2(\xi_1)+i(x_1-\xi_1)}\Big[\cos(x_1,\tau)-i\sin(x_1,\tau)\Big]\frac{dx_1}{\cos(x_1,\tau)}$$
$$-\frac{1}{2\pi}\int_{D}\frac{a(x)u(x)}{x_2-\gamma_2(\xi_1)+i(x_1-\xi_1)}dx-\frac{1}{2\pi}\int_{D}\frac{f(x)}{x_2-\gamma_2(\xi_1)+i(x_1-\xi_1)}dx$$
$$=-\frac{1}{2\pi}\int_{a_1}^{b_1}\frac{u\left(x_1,\gamma_1(x_1)\right)[1-i\gamma_1{}'(x_1)]dx_1}{\gamma_1(x_1)-\gamma_2(\xi_1)+i(x_1-\xi_1)}+\frac{1}{2\pi}\int_{a_1}^{b_1}\frac{u\left(x_1,\gamma_2(x_1)\right)[1-i\gamma_2{}'(x_1)]dx_1}{\gamma_2'\left(\sigma_2(x_1,\xi_1)\right)(x_1-\xi_1)+i(x_1-\xi_1)}$$
$$-\frac{1}{2\pi}\int_{D}\frac{a(x)u(x)}{x_2-\gamma_2(\xi_1)+i(x_1-\xi_1)}dx-\frac{1}{2\pi}\int_{D}\frac{f(x)}{x_2-\gamma_2(\xi_1)+i(x_1-\xi_1)}dx. \tag{9}$$

Let us clarify the features of expressions (8) and (9), which we obtained for the necessary and sufficient conditions: To do this, we write equations (8) and (9) as follows:

$$\begin{cases} u\left(\xi_1,\gamma_1(\xi_1)\right)=-\frac{1}{\pi}\int_{a_1}^{b_1}\frac{u(x_1,\gamma_1(x_1))[1-i\gamma_1{}'(x_1)]dx_1}{(x_1-\xi_1)\gamma_1'(\sigma_1(x_1,\xi_1))+i}+\Re_1, \\ u\left(\xi_1,\gamma_2(\xi_1)\right)=\frac{1}{\pi}\int_{a_1}^{b_1}\frac{u(x_1,\gamma_2(x_1))[1-i\gamma_2{}'(x_1)]dx_1}{(x_1-\xi_1)\gamma_2'(\sigma_2(x_1,\xi_1))+i}+\Re_2. \end{cases} \tag{10}$$

In expression (10), $\Re_1$, and $\Re_2$ denote the sum of the regular terms of expressions (8) and (9). It is easy to see that

$$\frac{[1-i\gamma_k{}'(x_1)]}{\gamma_k'(\sigma_k(x_1,\xi_1))+i}=\frac{[1-i\gamma_k{}'(x_1)]}{\gamma_k'(\sigma_k(x_1,\xi_1))+i}+i-i$$

$$=-i+\frac{[1-i\gamma_k{}'(x_1)]+i\left(\gamma_k'(\sigma_k(x_1,\xi_1))+i\right)}{\gamma_k'(\sigma_k(x_1,\xi_1))+i}=-i+\frac{1-i\gamma_k{}'(x_1)+i\gamma_k'(\sigma_k(x_1,\xi_1))+i^2}{\gamma_k'(\sigma_k(x_1,\xi_1))+i}$$

$$=-i+i\frac{\gamma_k'(\sigma_k(x_1,\xi_1))-\gamma_k{}'(x_1)}{\gamma_k'(\sigma_k(x_1,\xi_1))+i},\quad k=1,2;\; x_1\in[a_1,b_1]. \tag{11}$$

Since the points $\sigma_k(x_1,\xi_1)$ lie between x_1 and ξ_1, when x_1 and ξ_1 coincide, the point $\sigma_k(x_1,\xi_1)$ also coincides with them. Therefore, when $x_1-\xi_1\longrightarrow 0$, $\gamma_k'(\sigma_k(x_1,\xi_1))-\gamma_k{}'(x_1)=0$, $k=1,2$.

If we substitute expression (11) into (10), we get

$$\begin{cases} u(\xi_1,\gamma_1(\xi_1))=\frac{i}{\pi}\int_{a_1}^{b_1}\frac{u(x_1,\gamma_1(x_1))}{(x_1-\xi_1)}dx_1+\Re_3, \\ u(\xi_1,\gamma_2(\xi_1))=-\frac{i}{\pi}\int_{a_1}^{b_1}\frac{u(x_1,\gamma_2(x_1))}{(x_1-\xi_1)}dx_1+\Re_4, \end{cases} \tag{12}$$

where $\Re_3$ and $\Re_4$ denote the sum of the regular integrals corresponding to expressions (10) and (11), respectively.

3 Regularization of singularities

Taking into account boundary condition (2), we write the following linear combination from (12)

$$\alpha_1(\xi_1)u(\xi_1,\gamma_1(\xi_1))-\alpha_2(\xi_1)u(\xi_1,\gamma_2(\xi_1))$$

$$=\frac{i}{\pi}\int_{a_1}^{b_1}\frac{\alpha_1(\xi_1)u(x_1,\gamma_1(x_1))+\alpha_2(\xi_1)u(x_1,\gamma_2(x_1))}{x_1-\xi}dx_1+\alpha_1(\xi_1)\Re_3+\alpha_2(\xi_1)\Re_4$$

$$=\frac{i}{\pi}\int_{a_1}^{b_1}\Big\{\Big[(\alpha_1(\xi_1)-\alpha_1(x_1))+\alpha_1(x_1)\Big]u(x_1,\gamma_1(x_1))$$

$$+\Big[(\alpha_2(\xi_1)-\alpha_2(x_1))+\alpha_2(x_1)\Big]u(x_1,\gamma_2(x_1))\Big\}\frac{dx_1}{x_1-\xi_1}$$

$$=\frac{i}{\pi}\int_{a_1}^{b_1}\frac{\alpha_1(x_1)u(x_1,\gamma_1(x_1))+\alpha_2(x_1)u(x_1,\gamma_2(x_1))}{x_1-\xi_1}dx_1+\alpha_1(\xi_1)\Re_3+\alpha_2(\xi_1)\Re_4$$

$$=\frac{i}{\pi}\int_{a_1}^{b_1}\frac{\varphi(x_1)}{x_1-\xi_1}dx_1+\alpha_1(\xi_1)\Re_3+\alpha_2(\xi_1)\Re_4. \tag{13}$$

If the functions $\alpha_1(x_1)$ and $\alpha_2(x_1)$ belong to the Hölder class, then we can say that the limit in expression (13) exists in the Cauchy sense. Since $x_1-\xi_1=0$, $\alpha_k(\xi_1)-\alpha_k(x_1)=0$, $k=1,2$. In this case the resulting singular integral no longer contains the unknown function.

Note 1. If the function $\varphi(x_1)$ on the right-hand side of boundary condition (2) satisfies the following conditions:

$$\varphi(a_1)=\varphi(b_1)=0,\quad \varphi(x_1)\in C^{(1)}[a_1,b_1], \tag{14}$$

then the singular limit in (13) will also exist in the usual sense.

Theorem 1. Assume that the following conditions are satisfied:
(i) A bounded in plane D is convex in the direction x_2, and the boundary Γ is a Lyapunov curve;
(ii) $a(x)$, $f(x)$ are continuous functions;
(iii) $\alpha_1(x_1)$, $\alpha_2(x_1)$ belong to the Hölder class, and the function $\varphi(x)$ satisfies condition (14).
Then expression (13) is regular.

4 Fredholm property of the problem

Now, taking into account boundary condition (2), together with the regular expression (13), we obtain the following system of algebraic equations

$$\begin{cases} \alpha_1(\xi_1)u\left(\xi_1,\gamma_1(\xi_1)\right)+\alpha_2(\xi_1)u\left(\xi_1,\gamma_2(\xi_1)\right)=\varphi(\xi_1), \\ \alpha_1(\xi_1)u\left(\xi_1,\gamma_1(\xi_1)\right)-\alpha_2(\xi_1)u\left(\xi_1,\gamma_2(\xi_1)\right)=\frac{i}{\pi}\int_{a_1}^{b_1}\frac{\varphi(x_1)}{x_1-\xi_1}dx_1+\alpha_1(\xi_1)\Re_3+\alpha_2(\xi_1)\Re_4. \end{cases} \tag{15}$$

From this it follows

$$\begin{cases} u\left(\xi_1,\gamma_1(\xi_1)\right)=\frac{\varphi(\xi_1)}{2\alpha_1(\xi_1)}+\frac{i}{2\alpha_1(\xi_1)\pi}\int_{a_1}^{b_1}\frac{\varphi(x_1)}{(x_1-\xi_1)}dx_1+\Re_3, \\ u\left(\xi_1,\gamma_2(\xi_1)\right)=\frac{\varphi(\xi_1)}{2\alpha_2(\xi_1)}-\frac{i}{2\alpha_2(\xi_1)\pi}\int_{a_1}^{b_1}\frac{\varphi(x_1)}{(x_1-\xi_1)}dx_1+\Re_4. \end{cases} \tag{16}$$

If the conditions

$$\alpha_k(x_1)\neq 0, \quad k=1,2 \tag{17}$$

are satisfied, then expressions (16) give a system of integral equations with a regular Fredholm kernel of the second kind for the boundary values of the unknown function in problem (1), (2). This kernel does not include the integral of the sought function over the domain D. Thus, we show that problem (1), (2) has the Fredholm property.

Theorem 2. If the conditions of Theorem 1 and (17) are satisfied, then problem (1), (2) has the Fredholm property.

5 Solution of the boundary value problem

If we solve the system of integral equations (15), then for the functions $u\left(\xi_1,\gamma_k(\xi_1)\right)$, $(k=1,2)$ we obtain certain expressions depending on the expression

$$\int_D \frac{a(x)u(x)}{x_2-\gamma_2(\xi_1)+i(x_1-\xi_1)}dx.$$

By writing these expressions on the left side of the main relations (7), from the first component of the main relation for the function $u(x)$ we obtain a Fredholm-type integral equation of the second kind with a regular kernel. Thus, we obtain a solution to problems (1), (2).

Conclusion

Firstly, for a first-order elliptic equation with variable coefficients whose main part is the Cauchy–Riemann equation were written out the non-local boundary conditions (constructive) obtained by means of stitching from the boundaries of a plane region bounded and convex in the direction x_2 and divided into two parts, provided that the Carleman's condition on the boundary is satisfied.

The main relation consisting of two parts is obtained, the first of which gives arbitrary solutions of the equation and the second part gives the necessary conditions for $\xi \in \Gamma$ that gives relation between values of boundary conditions and the obtained solution.

For them, in the case of a partial differential equation, a a system of Fredholm integral equations of the second type with a regular kernel is obtained.

For the first time, it proved for a partial differential equation that the solution of the considered boundary problem can be obtained from the Green's formula, and for the problem of an ordinary differential equation from the Lagrange formula.

Author Contributions

All authors contributed equally to this work.

Conflict of Interest

The authors declare no conflict of interest.

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