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Rational analogues of Bernstein–Szabados operators on several intervals

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Bernstein polynomials play a very important role in approximation theory, probability theory, computer aided geometric design and many other areas. In 2017 J. Szabados constructed polynomial operators that can be considered as the most natural generalization to several intervals of the classical Bernstein operators. Their main advantages include fixed difference between degrees of the used polynomials and the number of used nodes. Unfortunately, they exist only under strong restrictions on the geometry of intervals (intervals have to form a polynomial inverse image of an interval). The main goal of the paper is to present a rational operator that generalizes J. Szabados' construction, and exists for an arbitrary system of several intervals. Moreover, this construction (unlike J. Szabados') is a linear positive operator. One of the main ingredients in the construction is the fact (which was proved by M.G. Krein, B.Ya. Levin, and A.A. Nudel'man) that an arbitrary finite system of real intervals is the inverse image of an interval by a rational function with precisely one pole at each gap. The approximation properties of such operators are studied as well. Further possible generalizations (of V.S. Videnskii's operators to one interval) are considered.

Keywords: Bernstein polynomials, rational operators, several intervals, inverse images, rate of approximation, linear positive operators, Videnskii rational functions, Ditzian–Totik modulus of continuity.

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Introduction

Approximation theory and harmonic analysis on several intervals of the real line is an area that attracts attention of many researchers. For example, the asymptotics of Chebyshev polynomials and their norms were studied in papers [1–3]; several related aspects of the theory of orthogonal polynomials can be found in [4–6]; the capacity of several intervals was considered in [7,8]; different approximation problems on several segments were solved in [9–11], among many others.

The polynomial inverse image method plays an important role in solving a number of problems in this field (see, for example, the survey [12], as well as later works with references to it). The method consists of several steps. Firstly it is necessary to prove the result for a system of intervals, that is a preimage of an interval under polynomial mapping (inverse image of an interval). The next step is to prove the result for arbitrary polynomials on an inverse image of an interval. Finally it is necessary to approximate an arbitrary system of intervals by inverse images, varying some endpoints of the intervals.

Sometimes, for example in polynomial interpolation, slight change of the system of intervals gives dramatically worse the asymptotic behaviour of the Lebesgue constants (see, for example, [13]). In [14] it was proved that even in the case of interpolation by polynomials on several intervals, it is useful to replace the preimage of an interval under polynomial mappings with the preimage of an interval under rational functions with fixed denominator. Then instead of varying the systems of intervals it is possible to vary the poles.

J. Szabados in [15] constructed analogues of Bernstein polynomials on several intervals with similar reproducing and interpolation properties only for the case of polynomial preimages of an interval. More

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precisely, they preserve polynomials up to a fixed degree (like classical Bernstein polynomials preserve polynomials up to degree one) and interpolate at the endpoints of the intervals (like classical Bernstein polynomials interpolate at ± 1.) We refer to [16–18] for the theory of classical Bernstein polynomials.

However, Szabados' operators are not positive, unlike the classical Bernstein polynomials, and polynomial preimages correspond to very special systems of intervals.

The main goal of the paper is to show that the use of rational functions makes it possible to overcome these disadvantages of BS.

Main results

Let $J_s = \cup_{j=1}^{s} I_j$, $I_j = [a_j, b_j]$, $s \geq 1$, be a system of real intervals. More precisely, $0 = a_1 < b_1 < \ldots < a_s < b_s = 1$, and let Π_n be the set of polynomials of degree at most n. Let $C(J_s)$ be the space of continuous functions on J_s with the sup-norm.

J. Szabados' construction works for the case where $J_s = p^{-1}([0,1])$, where $p \in \Pi_m$, $m \geq s$. For $n \in \mathbb{N}$, let $x_{k1} < \ldots < x_{km_k}$ be defined by

$$p(x_{ki}) = \frac{k}{n}, \quad i = 1, \ldots, m_k, \quad k = 0, \ldots, n,$$

where m_k are given explicitly (in most cases they are equal to m, with normalization $p(0) = 0$).

For an arbitrary $f(x) \in C(J_s)$, let

$$\tilde{L}_k(f, x) = \sum_{i=1}^{m_k} f(x_{ki}) \tilde{\ell}_{ki}(x) \in \Pi_{m_k - 1}, \quad k = 0, \ldots, n,$$

be the Lagrange interpolation polynomial with respect to the nodes x_{ki}. J. Szabados' operator is given by

$$BS_n(f, x) = \sum_{k=0}^{n} L_k(f, x) b_{nk}(p(x)), \quad x \in J_s,$$

where

$$b_{nk}(x) = \left(\begin{array}{c} n \\ k \end{array} \right) x^k (1-x)^{n-k}, \quad k = 0, \ldots, n,$$

are the fundamental functions of the Bernstein polynomials.

The main advantage of the operators BS compared to ordinary Bernstein polynomials of an extended function onto $[0,1]$ is that the difference between the number of function values and the degree of the operator is $m - s - 1$, i.e., independent of n, just as in the case of the classic Bernstein polynomials. But BS are not positive operators and the assumption $J_s = p^{-1}([0,1])$ with $s > 1$ is valid for very special systems of intervals only (for J_s in general position it is not satisfied for any $m \geq s$).

Now we will give the construction of rational analogues of operators BS that preserve their main advantages, exist for all J_s and n, and are positive.

From [19] it follows that for any system J_s there exists a polynomial $S \in \Pi_{s-1}$ with exactly one zero at each gap (b_i, a_{i+1}), $i = 1, \ldots, l-1$ such that $J_s = R^{-1}\left([0,1]\right)$, where

$$R(x) = \frac{\prod_{i=1}^{s}(x - a_i)}{S(x)}.$$

Now for $n \in \mathbb{N}$ let $x_{k1} < \ldots < x_{ks}$ be such that

$$R(x_{ki}) = \frac{k}{n}, \quad i = 1, \ldots, s, \quad k = 0, \ldots, n.$$

Then for an arbitrary $f(x) \in C(J_s)$, let

$$L_k(f,x) = \left(\sum_{i=1}^{s} f(x_{ki})\ell_{ki}^2(x)\right) \Big/ \sum_{i=1}^{s} \ell_{ki}^2(x)\ , \quad k = 0,\dots,n,$$

where

$$\ell_{ki}(x) = \frac{S(x_{ki})}{S(x)} \prod_{\substack{j=1 \\ j \neq i}}^{m_k} \frac{x - x_{kj}}{x_{ki} - x_{kj}}$$

are the fundamental Lagrange rational functions with the denominator $S(x)$. Rational analogues of Bernstein–Szabados operators are then defined by the formula

$$B_n(f,x) = \sum_{k=0}^{n} L_k(f,x) b_{nk}(R(x)), \quad x \in J_s. \tag{1}$$

Those operators are linear and positive, each term in (1) is a rational function of degree $sn + s - 1$, they preserve constants and interpolate f at the endpoints of J_s.

Now we state an analogue of J. Szabados' convergence estimate for (1). Let

$$\varphi(x) = \sqrt{(x - a_j)(b_j - x)} \quad \text{if} \quad x \in I_j,\ j = 1,\dots,s,$$

and define the Ditzian–Totik modulus of continuity as

$$\omega_\varphi(f,t) = \sup_{0<h\le t} \|\Delta_{h\varphi(x)} f(x)\|_{J_s},$$

where the difference is meant to be zero if any of the arguments is outside J_s, and we assume that t is so small that both $x \pm \varphi(x)$ fall into the same interval I_j. Further let

$$V(f) = \sup_{x,y\in J_s} |f(x) - f(y)|.$$

Theorem 1. For an arbitrary $f \in C(J_s)$ we have

$$\|f(x) - B_n(f,x)\|_{J_s} \le c\omega_\varphi\left(f, \frac{1}{\sqrt{n}}\right) + c\frac{V(f)}{\sqrt{n}}.$$

(Here and in what follows, c always denotes a positive constant depending on J_s, but independent of n, not necessarily the same at each occurrence.)

Proof. The proof goes essentially the same way as in [15, Proof of Theorem 1]. Let $x \in I_j$. Since both operators L_k and the classic Bernstein polynomials reproduce constants, we get

$$|f(x) - B_n(f,x)| \le c \sum_{k=0}^{n}\sum_{i=1}^{s} |f(x) - f(x_{ki}|\ell_{ki}^2(x) b_{nk}(R(x))$$

$$\le c \sum_{k=0}^{n} \left\{ \omega_\varphi\left(f, \frac{1}{\sqrt{n}}\right) \left[\frac{\sqrt{n}}{\varphi(x)}(x - x_{kj}) + 1\right] \ell_{kj}^2(x) + V(f) \sum_{i \neq j} \ell_{ki}^2(x) \right\} b_{nk}(R(x)).$$

We estimate the right-hand side sum for $0 \le k \le n/2$; the other part can be handled similarly. Then it is sufficient to consider the case $0 \le R(x) \le 4/5$, since for $4/5 \le y \le 1$ the estimate

$$\sum_{k=0}^{[n/2]} \left(\frac{n}{k}\right)^{\alpha} b_{nk}(y) \le n^{\alpha}(4/5)^{n/2}, \quad \alpha \ge 0,$$

was proved in [15].

Let first $k = 0$. Then we have $|\ell_{0i}(x)| \le c,\ x_{0i} = a_i,\ i = 1, \dots, s$, and

$$\frac{(x-a_i)\ell_{0i}^2(x)}{\varphi(x)} \le c\frac{x-a_i}{\varphi(x)} \le c\sqrt{R(x)}.$$

Now let $1 \le k \le n/2$.

Because of $|\ell_{kj}(x)| \le c$ we get

$$\sum_{k=1}^{[n/2]} \ell_{kj}^2(x) b_{nk}(R(x)) \le c \sum_{k=1}^{[n/2]}) b_{nk}(R(x)) \le c. \tag{2}$$

On the other hand, $|x - x_{kj}| \le |R(x) - \frac{k}{n}|$, therefore by [15, Lemma 1] applied with $\alpha = 0,\ \beta = 1$ yields

$$\begin{aligned}\sum_{k=1}^{[n/2]} |x-x_{kj}|\ell_{kj}^2(x) b_{nk}(R(x)) &\le c \sum_{k=1}^{[n/2]} |x-x_{kj}| b_{nk}(R(x)) \\ &\le \sum_{k=1}^{[n/2]} |R(x) - \frac{k}{n}| \le c\sqrt{\frac{R(x)}{n}} \le \frac{\varphi(x)}{\sqrt{n}}. \end{aligned} \tag{3}$$

Finally, using $\ell_{ki}^2(x) \le c\sqrt{\frac{n}{k}}|x - x_{kj}|,\ i \ne j$, and [15, Lemma 1] with $\alpha = 1/2,\ \beta = 1$, we obtain that

$$\begin{aligned}\sum_{k=1}^{[n/2]} \sum_{i \ne j} \ell_{ki}^2(x) b_{nk}(R(x) &\le c \sum_{k=1}^{[n/2]} \sqrt{\frac{n}{k}} |x-x_{kj}| b_{nk}(R(x)) \\ &\le c \sum_{k=1}^{[n/2]} \sqrt{\frac{n}{k}} \left| R(x) - \frac{k}{n} \right| b_{nk}(R(x)) \le \frac{c}{\sqrt{n}}. \end{aligned} \tag{4}$$

Combining (2)–(4) completes the proof. □

Remark 1. Substituting

$$L_k^{(1)}(f,x) = \sum_{i=1}^{s} f(x_{ki})\ell_{ki}(x)$$

instead of $L_k(f,x)$ in (1) other operators (denoted by $B_n^{(1)}(f,x)$) can be constructed. They are rational functions of degree $ns + s - 1$, use ns function values, satisfy $B_n^{(1)}(R,x) = R(x)$ for all $x \in J_s$, but don't form positive operators.

Remark 2. V.S. Videnskii in a series of papers (compare also his book [18], and paper [20]) considered a generalization of the classical Bernstein polynomials for rational approximation on $[0,1]$.

More precisely, Videnskii's operators have the form

$$V_n(f,x) = \sum_{k=0}^{n} f(\tau_{nk}) u_{nk}(x),$$

where the nodes τ_{nk} are determined by the formulas

$$\phi_n(\tau_{nk}) = \frac{k}{n}, \quad k = 0, 1, .., n,$$

$$\phi_n(x) = \frac{1}{n}\sum_{i=1}^{n} h_{ni}(x),$$

and the rational functions $u_{nk}(x)$, which are analogues of $b_{nk}(x)$ from the classical Bernstein operators, are defined with the help of the generating function as follows:

$$h_{ni}(x) = \frac{\rho_{ni} x}{1 + \rho_{ni} - x}, \quad \rho_{ni} > 0, \quad i = 0, 1, ...,$$

$$g_n(x, y) = \sum_{k=0}^{n} y^k u_{nk}(x),$$

$$g_n(x, y) = \prod_{i=0}^{n-1} \left(h_{ni}(x)\, y + (1 - h_{ni}(x))\right).$$

It is possible to generalize this construction to the case of several intervals by the same method as above.

Conclusion

Bernstein polynomials have many applications in modern science and technology, but up to now there is no complete analogue of them for the case of several (greater than one) intervals of the real axis. In this paper a generalization of Bernstein polynomials to rational functions on several intervals is constructed. Those operators exist for an arbitrary (unlike previously constructed generalizations) system of intervals. Approximation properties of the presented operators are studied as well.

Conflict of interests

There is no conflict of interest.

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