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Application of isotropic geometry to the solution of the Monge–Ampere equation

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This paper explores the Monge–Ampere equation in the context of isotropic geometry. The study begins with an overview of the fundamental properties of isotropic space, including its scalar product, distance formula, and the nature of surfaces and curvatures within this geometric framework. A special focus is placed on dual transformations with respect to the isotropic sphere, and the self-inverse property of the dual surface is established. The article formulates the Monge–Ampere equation for isotropic space and studies its invariant solutions under isotropic motions. Several lemmas are proved to demonstrate how solutions transform under linear modifications and isotropic motions. A specific class of Monge–Ampere-type nonlinear partial differential equations is solved analytically using dual transformations and separation of variables. Additionally, translation surfaces and their curvature properties are studied in detail, particularly through the lens of dual curvature. The results demonstrate the deep relationship between curvature invariants and Monge–Ampere-type equations and show how duality simplifies the solution of nonlinear PDEs. These methods can be used for surface reconstruction and modeling in isotropic spaces.

Keywords: isotropic geometry, Monge–Ampere equation, linear transformation, dual transformation, dual surface, curvature invariants, surface reconstruction, Dirichlet problem, PDE.

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Introduction

The Monge–Ampere equation occupies a prominent position in the theory of nonlinear partial differential equations due to its rich mathematical structure and wide applicability in geometric analysis, optimization, and mathematical physics. In classical differential geometry, this equation naturally arises in the context of surface theory, particularly in problems involving the reconstruction of a surface from curvature invariants [1]. A key feature of the Monge–Ampere equation is its close relationship with convex geometry and curvature prescriptions, as first systematically studied by I.Ya. Bakelman in the framework of the generalized Dirichlet problem for convex surfaces [2].

While significant progress has been achieved in Euclidean settings, the exploration of Monge–Ampere-type equations in non-Euclidean geometries, such as isotropic or semi-Riemannian spaces, is relatively recent. Isotropic geometry, which is a limiting case of semi-Euclidean geometry, provides a degenerate metric structure where distances are defined in a directionally dependent manner. This degenerate nature introduces novel phenomena not present in Riemannian or pseudo-Riemannian frameworks, thereby making isotropic geometry a fertile ground for discovering new geometric properties and solving PDEs under non-standard metrics [3].

In his book [4], O'Neill introduced fundamental concepts of semi-Riemannian geometry, from which the notion of isotropic and degenerate metric spaces naturally arises as a special geometric model.

The geometry of isotropic space R_{n+1}^n, as introduced, is characterized by a scalar product that is degenerate not along a single axis. The differential geometry of isotropic space was first studied by K. Strubecker [5, 6]. This leads to a unique classification of surfaces and transformations, including

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duality mappings with respect to the isotropic sphere. The theory of dual surfaces in isotropic space has been actively developed in recent works, including the classification and reconstruction of surfaces via dual curvature invariants [7]. The concept of dual transformation plays a central role in understanding curvature-driven surface generation, a theme that appears throughout this study.

One of the central motivations for the present paper stems from the growing body of research demonstrating that dual transformations in isotropic spaces offer elegant and computationally tractable methods for solving highly nonlinear equations such as the Monge–Ampere equation. Generalizing Lonen's works [8], Artykbaev, Sultanov, and Ismoilov have shown in several studies [9] that the total and mean curvatures of a surface and its dual are closely related, and that this relationship can be used to construct surfaces with prescribed curvature characteristics. The present study builds upon these foundational results and extends them in several directions. In the work by A. Polyanin [10], certain solutions of the Monge–Ampere equation are presented without derivation. In contrast, in this paper, we also explore a method for finding a different type of solution.

Firstly, we investigate the invariant form of the Monge–Ampere equation under isotropic motions and provide a detailed analysis of its solutions under linear perturbations. The result that any solution of the Monge–Ampere equation remains invariant under the addition of linear functions is well-known in classical settings, but here it is adapted and rigorously proven for isotropic geometry, leading to new insights into the geometry of the solution space.

Secondly, we focus on a special class of Monge–Ampere-type equations that arise in the context of translation surfaces in isotropic space. Using the techniques of separation of variables and dual transformation, we derive exact analytical solutions for these equations. In particular, we solve the equation

$$\left(\frac{\partial^2 z}{\partial x \partial y}\right)^2 - \frac{\partial^2 z}{\partial x^2} \cdot \frac{\partial^2 z}{\partial y^2} = f(x) y^2,$$

by assuming a quadratic *ansatz* and reducing the resulting PDE to a system of ODEs. The general solution is expressed in terms of integrals of nonlinear functions and demonstrates the applicability of this approach to constructing explicit surfaces with curvature-driven features.

Thirdly, we introduce and analyze translation surfaces whose total curvature of the dual surface is separable in the form $K^*(x, y) = \varphi(x)\psi(y)$. Using the inverse problem framework, we demonstrate that such dual curvature data uniquely determines the original surface up to an isotropic motion. This result contributes to the general problem of surface reconstruction from curvature invariants and finds relevance in applications such as surface design in computer graphics and shape optimization.

The geometric significance of these results lies in the structure of the isotropic space itself. Unlike Euclidean geometry, where the normal to a surface is uniquely defined by the metric, in isotropic geometry the notion of normality is more subtle. Here, we distinguish between the special normal vector $\vec{n}_m$ and the standard unit normal $\vec{n}$, and we show that the second fundamental form and the total curvature remain invariant under this choice. This confirms earlier findings in [11] and supports the use of duality-based methods for analyzing surface properties.

Furthermore, in the final section of the paper, we consider an application of the Monge–Ampere equation arising in the theory of plasticity and elasticity. A particular nonlinear equation governing large deformations of elastic plates is shown to be a higher-order Monge–Ampere-type equation. We demonstrate how this complex nonlinear equation can be transformed into a linear PDE with constant coefficients by applying dual transformations, and we solve it using separation of variables. The solution process also illustrates how dual mappings can be used not only in geometric but also in physical models.

It is worth noting that similar approaches have been explored by researchers studying special surfaces in isotropic spaces, such as ruled, helicoidal, and Weingarten-type surfaces [12–15]. However, the novelty of the present work lies in the formulation and solution of Monge–Ampere equations specifically in terms of dual curvature data, and the construction of explicit surface representations using integrable systems techniques. The article [16] investigates the parametric and algebraic representations of minimal surfaces in four-dimensional Euclidean space. It presents a generalized form of the Weierstrass–Enneper formula and analyzes the differential-geometric properties, projections, and modeling significance of such surfaces. This approach is closely related to the methods applied in solving the Monge—Ampere equation within isotropic geometry and provides an effective geometric framework for studying related problems

1 *Geometry of isotropic space*

Let $Ox_i\,(i=1...\,n+1)$ be a coordinate system in affine space A_{n+1}. The scalar product of vectors $\overrightarrow{X}(x_1,x_2,...,x_{n+1})$ and $\overrightarrow{Y}(y_1,y_2,...,y_{n+1})$ is defined by the following formula:

$$(\vec{X},\vec{Y})=\begin{cases}\sum\limits_{i=1}^{n}x_iy_i, & if\ \sum\limits_{i=1}^{n}x_iy_i\neq 0,\\ x_{n+1}y_{n+1}, & if\ \sum\limits_{i=1}^{n}x_iy_i=0.\end{cases}\tag{1}$$

Definition 1. An affine space A_{n+1}, in which the scalar product of vectors is calculated using formula (1), is called an isotropic space R^n_{n+1}.

The scalar product (1) is called a degenerate scalar product.

Minkowski space is a pseudo-Euclidean space with index 1. It serves as a geometric framework for the theory of relativity. This space also includes isotropic space as a special case. This can be seen in the following lemma.

Lemma 1. The isotropic space R^n_{n+1} is a subspace of the $(n+2)$-dimensional Minkowski space ${}^1R_{n+2}$ [11].

We define the norm of a vector in isotropic space R^n_{n+1} as the root of the scalar product of a vector $|\vec{X}|=\sqrt{(\vec{X},\vec{X})}$, and the distances between points are defined as the norm of the vector connecting these points.

If $\vec{X}-\vec{Y}=\vec{AB}$, then the distance between points A and B is calculated using the following formula:

$$d=\begin{cases}\sqrt{\sum\limits_{i=1}^{n}(y_i-x_i)^2}, & if\ \sqrt{\sum\limits_{i=1}^{n}(y_i-x_i)^2}\neq 0,\\ |y_{n+1}-x_{n+1}|, & if\ x_i=y_i\ \ (i=\overline{1,n}).\end{cases}\tag{2}$$

The hyperplanes in $R^{n-i}_{n+1}(i=1..n-2)$ can be of two types — isotropic R^{n-i}_{n+1} or Euclidean R_n. Hyperplanes $x_{n+1}=constant$ are Euclidean spaces. If a two-dimensional plane is considered and it is parallel to the Ox_{n+1} axis, then the intrinsic geometry of this plane becomes Galilean. The intrinsic geometry of the Galilean plane is presented in [17].

Since the isotropic space R^n_{n+1} is an affine space, there is an affine coordinate transformation that maintains the distance defined by formula (2). This transformation is called the motion of isotropic space R^n_{n+1} and is given by the following formula [3]:

$$X'=A\cdot X+B,\quad A=\begin{pmatrix} & & A_E & & & 0\\ & & & & & ...\\ - - - & - - - & - - - & - - - & - - - & 0\\ h_1 & h_2 & ... & h_{n-1} & h_n & 1\end{pmatrix},\tag{3}$$

where $A_E = (a_{ij})_{i,\,j\,=\,1..n}$ is the motion matrix in the Euclidean space R_n, $B^T = (b_1, b_2, ..., b_{n+1})$ is the parallel translation vector, and $(h_1, h_2, ..., h_n, 1)$ is the vector with sliding coordinate components.

If we define a sphere in isotropic space as a set of geometric points equidistant from a given point $(x_1^0, x_2^0, ..., x_n^0, x_{n+1})$, then its equation has the following form:

$$\sum_{i=1}^{n} (x_i - x_i^0)^2 = r^2.$$

We will call this sphere a metric sphere.

Let us consider in the R_{n+1}^n a surface defined by the following vector equation [1]:

$$r(u_1, u_2, ..., u_n) = \Big(x_i(u_1, u_2, ..., u_n) | (u_1, u_2, ..., u_n) \in D \subset R_n,\ i = \overline{1..(n+1)} \Big). \tag{4}$$

The first quadratic form of (4) a surface is defined by analogy with Euclidean space

$$I = ds^2 = \sum_{i=1}^{n}\sum_{j=1}^{n} g_{ij}\, du_i du_j,$$

where g_{ij} are the coefficients of the first quadratic form of the surface and

$$g_{ij} = (\overrightarrow{r'}_{u_i}, \overrightarrow{r'}_{u_j}) = \sum_{k=1}^{n} \left(\frac{\partial x_k}{\partial u_i}, \frac{\partial x_k}{\partial u_j} \right).$$

In the case where $ds^2 = 0$, an additional first quadratic form $ds^2 = dx_{n+1}$ is considered.

Since we mainly consider surfaces with a single-valued projection onto the plane $x_{n+1} = 0$, $ds^2 \neq 0$ for all points of the surface. Therefore, an additional first quadratic form of the surface is not considered.

The normal to the surface is taken to be the only orthogonal vector to all tangent vectors of the surface $\overrightarrow{n_m}(0, 0, ..., 0, 1)$ [9].

By analogy with the Euclidean space, the second quadratic form of the surface is defined as the scalar product of the vector of the second-order differential $d^2\overrightarrow{r}$ by the surface normal.

The surface normal needs a clear definition in order to handle the issues in question. To this end, the following formula can be offered:

The standard, and orthogonal, form of the normals is given by

$$\overrightarrow{n} = \frac{[\overrightarrow{r_{u_1}}, ..., \overrightarrow{r_{u_n}}]}{|[\overrightarrow{r_{u_1}}, ..., \overrightarrow{r_{u_n}}]|},$$

in which $[\overrightarrow{r_{u_1}}, ..., \overrightarrow{r_{u_n}}]$ signifies the vector product.

Since we consider two surface normals (the special normal $\overrightarrow{n_m}$ and the normal $\overrightarrow{n}$), the formula for the second quadratic form will be as follows:

$$II = (d^2 r, \overrightarrow{N}) = \sum_{i,j=1}^{n} D_{ij} du_i du_j.$$

Here, D_{ij} is the coefficient of the second quadratic form, calculated as:

1) $D_{ij} = \frac{\partial^2 x_{n+1}}{\partial u_i \partial u_j}$, if $\overrightarrow{N} = \overrightarrow{n_m}$,

2) $D_{ij} = (r_{u_i u_j}, \overrightarrow{n})$, if $\overrightarrow{N} = \overrightarrow{n}$.

In particular, if the surface is defined by the following equation

$$x_{n+1} = f(x_1, x_2,\ ...,\ x_n), \tag{5}$$

where $(x_1, x_2, ..., x_n) \in D \subset R_n$, then

$$II = \sum_{i,j=1}^{n} \frac{\partial^2 x_{n+1}}{\partial x_i \partial x_j} du_i du_j.$$

Hyperplanes parallel to the normal vector are isotropic hyperplanes of the corresponding dimension. In particular, a two-dimensional plane parallel to the normal vector is a two-dimensional isotropic plane, called the Galilean plane [7]. Therefore, the geometry is Galilean in a two-dimensional normal section of the surface. A two-dimensional normal section of the surface is a curve on the Galilean plane.

The curvature of the curve of the normal section is called the normal curvature of the curve on the surface. The normal curvature of the curve on the surface is calculated by the following formula:

$$k_n = \frac{II}{I}.$$

In isotropic space R^n_{n+1}, the second sphere is a surface with constant normal curvature in all directions, given by the following equation:

$$2x_{n+1} = \sum_{i=1}^{n} {x_i}^2. \tag{6}$$

Definition 2. The surface defined by equation (6) is called an isotropic sphere in R^n_{n+1}.

The mean and total curvatures are the main geometric characteristics of a surface. The total curvature of the surface (4) is calculated as:

$$K = \frac{\det \left| (D_{ij})_{i,j=\overline{1.n}} \right|}{\det \left| (g_{ij})_{i,j=\overline{1.n}} \right|}.$$

Lemma 2. The total curvatures of the surface (5), determined by the normal and the special normal, are mutually equal $K = K_m$ [11].

2 *Dual transformation with respect to the isotropic sphere*

Let the surface F be given by the equation (5) and suppose it lies within the isotropic sphere of the space R^n_{n+1}. Consider the set of points obtained via dual mapping of the tangent hyperplanes to the surface F at each of its points, with respect to the isotropic sphere. This set forms a new surface defined as follows.

Definition 3. The surface F^* is called the *dual surface* to the surface F with respect to the isotropic sphere.

If the surface F is regular, then the dual surface F^* is also a surface and is given by the system:

$$\begin{cases} x_i^* = \dfrac{\partial f}{\partial x_i}, \qquad i = 1, \; ... \;, n, \\ x_{n+1}^* = \displaystyle\sum_{i=1}^{n} x_i \cdot \frac{\partial f}{\partial x_i} - f. \end{cases}$$

Theorem 1. The dual image of the surface F^* coincides with the surface F; that is,

$$F^{**} = F.$$

The total curvature of the surface (5) has the form:

$$K = \begin{vmatrix} f_{x_1x_1} & f_{x_1x_2} & \cdots & f_{x_1x_n} \\ f_{x_2x_1} & f_{x_2x_2} & \cdots & f_{x_2x_n} \\ \vdots & \vdots & \ddots & \vdots \\ f_{x_nx_1} & f_{x_nx_2} & \cdots & f_{x_nx_n} \end{vmatrix}.$$

The right-hand side is the Monge–Ampere operator. In isotropic space, the problem of recovering a surface from its total curvature is equivalent to solving the Monge–Ampere equation.

3 Monge–Ampere equation

I.Ya. Bakelman studied the connection between the extrinsic curvature of convex surfaces and the second-order nonlinear Monge–Ampere equation [2]. In this case, I.Ya. Bakelman showed that the solution of the generalized Dirichlet problem for the Monge–Ampere equation exists and is unique by estimating the area of the normal image of the surface. The listed problems were solved only if the domain $D \subset R_2$ is convex where the function is defined. By applying the geometry of the Galilean space, A. Artykbaev solved the problem for the existence and uniqueness of the convex surface for the given extrinsic curvature if the domain $D \subset R_2$ is non-convex [18]. Also, in the article [7], the concept of generalized extrinsic curvature is given, and the existence and uniqueness of the solution to the Monge–Ampere equation in the multi-connected domain is proved. The Monge–Ampere equation in a discrete setting with a special invariant can be observed in the Sharipov's works [19]. In [20, 21], Lions and Urbas established the existence and regularity results for a wide class of fully nonlinear elliptic PDEs. The paper [22], provides a clear and accessible overview of the modern theory of the Monge–Ampere equation. It discusses the notion of Alexandrov (weak) solutions, interior and boundary regularity results, and classical methods developed by Calabi, Cheng–Yau, and Lions. The article also emphasizes the analytical and geometric aspects of the equation, offering valuable insights into the existence and smoothness of convex solutions to Dirichlet-type problems. In this paper, we address the problem of reconstructing a surface in three-dimensional isotropic space by solving the Monge–Ampere equation, using the relationship between the surface equation and the Monge–Ampere equation in isotropic space. To this end, we first introduce the Monge–Ampere equation in three-dimensional space.

It is known that the Monge–Ampere equation is generally as follows:

$$z_{xx}z_{yy} - z_{xy}^2 = \phi\left(x, y, z, z_x, z_y\right).$$

In this case, if $\phi\left(x, y, z, z_x, z_y\right) > 0$, the equation is elliptic and its solution is a convex surface equation. Now, if we consider this equation in the semi-Euclidean space, that is, in the isotropic space, it will be as follows:

$$K\left(x, y\right) = z_{xx}z_{yy} - z_{xy}^2.$$

3.1 General invariant solution

We present some statements related to the solution of the Monge–Ampere equation and motions in isotropic space.

Lemma 3. If the function $z = f(x, y)$ is a solution of the Monge–Ampere equation

$$\det D^2 f = f_{xx}f_{yy} - (f_{xy})^2 = F(x, y),$$

then the function

$$z = f(x, y) + C_1x + C_2y + C$$

is also a solution of the same equation.

Proof. This follows from the fact that the Monge–Ampere operator involves only second-order partial derivatives. Since the linear part $C_1x + C_2y + C$ vanishes under second-order differentiation of second order, it does not affect the operator:

$$\frac{\partial^2}{\partial x^2}(f + C_1x + C_2y + C) = f_{xx}, \quad \frac{\partial^2}{\partial y^2}(f + C_1x + C_2y + C) = f_{yy},$$

$$\frac{\partial^2}{\partial x \partial y}(f + C_1x + C_2y + C) = f_{xy}.$$

Therefore, the Monge–Ampere determinant remains unchanged. □

Lemma 4. The surface defined by the function

$$z = f(x, y) + C_1x + C_2y + C$$

can be obtained from the surface $z = f(x, y)$ by an isotropic motion.

Proof. Consider applying (3) an isotropic motion to the surface, specifically an isotropic shear transformation (translation along the z-axis depending linearly on x and y). This motion is given by:

$$\begin{cases} x' = x, \\ y' = y, \\ z' = Ax + By + z + C, \end{cases} \tag{7}$$

where $A, B, C \in \mathbb{R}$ are constants.

Applying this transformation to the surface $z = f(x, y)$, we obtain:

$$z' = f(x, y) + Ax + By + C,$$

which coincides with the general form $f(x, y) + C_1x + C_2y + C$. Hence, the transformation corresponds to (7) a motion in isotropic space. □

Taking Lemmas 3 and 4 into account, we will not consider the linear case in the subsequent solutions. The reason is that, in isotropic space, adding a linear term results in two different solutions representing the same surface, differing only by their position.

4 Analytical solution of a Monge–Ampere-type equation

We study a nonlinear Monge–Ampere-type partial differential equation of the form:

$$\left(\frac{\partial^2 z}{\partial x \partial y}\right)^2 - \frac{\partial^2 z}{\partial x^2} \cdot \frac{\partial^2 z}{\partial y^2} = f(x)y^2. \tag{8}$$

Our aim is to construct general and particular solutions, including transformation invariance and exact construction for a specific case.

We consider a quadratic ansatz in y:

$$z(x, y) = \varphi(x)y^2 + U(x)y + V(x)$$

and compute the necessary derivatives:

$$\frac{\partial^2 z}{\partial x \partial y} = 2\varphi'(x)y + U'(x),$$
$$\frac{\partial^2 z}{\partial x^2} = \varphi''(x)y^2 + U''(x)y + V''(x),$$
$$\frac{\partial^2 z}{\partial y^2} = 2\varphi(x).$$

Substituting into equation (8), we obtain a polynomial in y. Matching coefficients gives the following system:

$$4(\varphi')^2 - 2\varphi\varphi'' = f(x),$$
$$4\varphi' U' - 2\varphi U'' = 0,$$
$$(U')^2 - 2\varphi V'' = 0.$$

Solving this system, we obtain the general solution:

$$w(x,y) = \varphi(x)y^2 + C_1 y \int \varphi^2(x)\, dx + \frac{1}{2}C_1^2 \int_a^x (x-t)\varphi^3(t)\, dt,$$

where $\varphi(x)$ satisfies the nonlinear ODE

$$\varphi\varphi'' = 2(\varphi')^2 - \frac{1}{2}f(x).$$

Particular case: $f(x) = 0$.

We assume $\varphi(x) = A/(x+C)$ and verify that it satisfies:

$$\varphi\varphi'' = 2(\varphi')^2.$$

This leads to a family of solutions of the form:

$$w(x,y) = \frac{A}{x+C}y^2 - \frac{C_1 A^2 y}{x+C} + \frac{C_1^2}{2}\int_a^x (x-t)\left(\frac{A}{t+C}\right)^3 dt.$$

Particular case: $f(x) = x^2$.

We seek $\varphi(x) = ax^n$. The equation

$$a^2 n(n-1)x^{2n-2} = 2a^2 n^2 x^{2n-2} - \frac{1}{2}x^2$$

is satisfied when $n = 2$, yielding

$$a = \pm\frac{1}{2\sqrt{3}}.$$

Hence,

$$\varphi(x) = \frac{1}{2\sqrt{3}}x^2.$$

Using this, we compute the full solution:

$$w(x,y) = \frac{1}{2\sqrt{3}}x^2y^2 + \frac{C_1 y}{60}x^5 \frac{C_1^2}{48\sqrt{3}}\int_a^x (x-t)t^6 dt.$$

5 *Translation surface*

When the surface is uniquely projected onto the Oxy plane in isotropic space, it is given by the parametrization:

$$\overrightarrow{r}(x,y) = x \cdot \overrightarrow{i} + y \cdot \overrightarrow{j} + (f(x) + g(y)) \cdot \overrightarrow{k}. \tag{9}$$

In this case, the coefficients of the first fundamental form are: $E = 1,\ F = 0,\ G = 1$, and the coefficients of the second fundamental form are: $L = f''(x),\ M = 0,\ N = g''(y)$.

Taking this into account, the formula for the total curvature of the surface can be obtained as:

$$K = f''(x) \cdot g''(y).$$

The total curvature of the dual surface is given by:

$$K^* = \frac{1}{f''(x) \cdot g''(y)}.$$

Let

$$K^* = \varphi(x) \cdot \psi(y) \neq 0$$

be a function defined on the domain $D \subset \mathbb{R}^2$, where $\varphi(x)$ and $\psi(y)$ are continuous, non-vanishing functions.

Lemma 5. If the total curvature of the dual surface is given by $K^* = \varphi(x) \cdot \psi(y)$, then there exists a surface of the form

$$\vec{r}_\lambda(x,y) = x\,\vec{i} + y\,\vec{j} + \left(\int \left[\int \frac{1}{\lambda \varphi(x)}\, dx \right] dx + \int \left[\int \frac{\lambda}{\psi(y)}\, dy \right] dy \right) \vec{k}, \tag{10}$$

for which K^* is the total curvature of its dual surface and $\varphi(x), \psi(y) \in C^2(D)$.

Proof. From the general formula (10) for the total curvature of a dual surface in a translation surface, we have:

$$\frac{1}{f''_{xx}(x) \cdot g''_{yy}(y)} = \varphi(x) \cdot \psi(y).$$

Rewriting, we obtain:

$$\frac{1}{f''_{xx}(x) \cdot \varphi(x)} = g''_{yy}(y) \cdot \psi(y).$$

This leads to the separation of variables as:

$$\frac{1}{f''_{xx}(x) \cdot \varphi(x)} = \lambda = g''_{yy}(y) \cdot \psi(y), \tag{11}$$

where λ is a constant of separation.

Solving (11) these differential equations gives:

$$f_\lambda(x) = \int \left[\int \frac{1}{\lambda \varphi(x)}\, dx + C_1 \right] du + C_1',$$

$$g_\lambda(y) = \int \left[\int \frac{\lambda}{\psi(y)}\, dv + C_2 \right] dy + C_2'.$$

By substituting the functions $f(x)$ and $g(y)$ into the translation surface equation (9) and omitting their linear parts, we obtain the formula presented in Lemma 5. □

Theorem 2. *(i)* If the surface belongs to a translation surface and the total curvature of the dual surface is $K^* = C_0 = \text{constant} \neq 0$, then the surface has the following equation: $\vec{r}(x,y) = x\,\vec{i} + y\,\vec{j} + \left(\frac{C_0}{2}x^2 + \frac{1}{2C_0}y^2\right)\vec{k}$.

(ii) If the total curvature is given in the form $K^* = \varphi(x)\cdot\psi(y)$, then the surface is given by formula (9).

(iii) However, if the total curvature is a non-separable function, i.e. $K^* = K^*(x,y) \neq \varphi(x)\cdot\psi(y)$, then the problem has no solution in the class of translation surfaces.

Proof. Each case in the theorem is considered separately.

(i) When $K^* = C_0 = \text{constant}$, the result is already established in [8].

(ii) When K^* is separable as $\varphi(x)\cdot\psi(y)$, the theorem follows directly from Lemma 5.

(iii) Finally, when $K^* = K^*(x,y)$ is non-separable, a surface of the form

$$\vec{r}(u,v) = x\,\vec{i} + y\,\vec{j} + (f(x) + g(y))\,\vec{k}$$

has curvature

$$K^*(x,y) = \frac{1}{f''_{xx}(x)}\cdot\frac{1}{g''_{yy}(y)}$$

which is necessarily separable in variables. This contradiction implies that no such transfer surface can exist in the non-separable case. □

6 *Applications of the Monge–Ampere equation*

The Monge–Ampere equation has been widely applied across various scientific fields. Many well-known equations include the Monge–Ampere equation as a structural component. Let us consider one such equation. By doing so, we also address the applicability of the results obtained.

Consider the nonlinear partial differential equation

$$\frac{\partial^2 z}{\partial x^2}\left[\left(\frac{\partial^2 z}{\partial x \partial y}\right)^2 - \frac{\partial^2 z}{\partial x^2}\cdot\frac{\partial^2 z}{\partial y^2}\right] = \frac{\partial^2 z}{\partial y^2}.$$

This equation is relevant in two-dimensional plasticity theory, where $z = f(x,y)$ acts as the generating function. This equation represents a particular case of the nonlinear elastic plate model, describing the bending deformations of a thin elastic plate. It models the variation of elastic energy based on the total curvature of the surface. Due to its nonlinear nature, the equation is suitable for analyzing large deformations. In the absence of external forces, it describes situations where only internal elastic forces are at play.

Let $z = f(x,y)$ be a solution. Then, the following transformed functions also satisfy the same equation:

$$z_1 = \pm C_1^{-2} f(C_1 x + C_2, C_3 y + C_4),$$

where $C_1, \ldots, C_4$ are arbitrary constants.

Let us try to solve the equation. We now define a new function

$$\omega(x,y) = \frac{\partial z}{\partial x}.$$

We consider $\omega(x,y)$ as a surface and move to the surface that is dual to $\omega^*(x,y)$ and use the following 3-dimensional dual transformation:

$$\begin{cases} x^* = \frac{\partial \omega}{\partial x}, \\ y^* = \frac{\partial \omega}{\partial y}, \\ z^* = x\frac{\partial \omega}{\partial x} + y\frac{\partial \omega}{\partial y} - \omega. \end{cases}$$

The goal is to convert the original nonlinear equation into a second-order linear PDE. After performing the transformation, we obtain:

$$(1+x^{*2})^2\frac{\partial^2 z^*}{\partial x^{*2}}+2x^*y^*(1+x^{*2})\frac{\partial^2 z^*}{\partial x^*\partial y^*}+y^{*2}(x^{*2}-1)\frac{\partial^2 z^*}{\partial y^{*2}}=0. \tag{12}$$

This is a hyperbolic partial differential equation. To further simplify it, we use the coordinate transformation:

$$t=\arctan x^*,\quad \xi=\frac{1}{2}\ln(1+x^{*2})-\ln y^*,\quad W=\frac{z^*}{\sqrt{1+x^{*2}}}.$$

Under this change of variables, equation (12) is transformed into a linear PDE with constant coefficients:

$$\frac{\partial^2 W}{\partial t^2}=\frac{\partial^2 W}{\partial \xi^2}-W.$$

We now solve the PDE, using the method of separation of variables. Let

$$W(t,\xi)=T(t)\cdot X(\xi).$$

Substitute into the equation:

$$T''(t)X(\xi)-T(t)X''(\xi)+T(t)X(\xi)=0.$$

Divide both sides by $T(t)X(\xi)$:

$$\frac{T''(t)}{T(t)}-\frac{X''(\xi)}{X(\xi)}+1=0.$$

This implies:

$$\frac{T''(t)}{T(t)}+1=\frac{X''(\xi)}{X(\xi)}=-\lambda.$$

So we obtain two ODEs:

$$\begin{aligned} T''(t)+(\lambda+1)T(t)&=0,\\ X''(\xi)+\lambda X(\xi)&=0. \end{aligned}$$

The general solutions are

$$\begin{aligned} T(t)&=C_1\cos(\sqrt{\lambda+1}\,t)+C_2\sin(\sqrt{\lambda+1}\,t),\\ X(\xi)&=A_1\cos(\sqrt{\lambda}\,\xi)+A_2\sin(\sqrt{\lambda}\,\xi). \end{aligned}$$

Therefore, the general solution to the PDE is

$$W(t,\xi)=\left[A_1\cos(\sqrt{\lambda}\,\xi)+A_2\sin(\sqrt{\lambda}\,\xi)\right]\cdot\left[C_1\cos(\sqrt{\lambda+1}\,t)+C_2\sin(\sqrt{\lambda+1}\,t)\right].$$

We now reverse the transformation steps to reconstruct $\omega(x,y)$.

Recover $Z(X,Y)$, recall that

$$W=\frac{z^*}{\sqrt{1+x^{*2}}}.$$

From this it follows

$$z^*=W\cdot\sqrt{1+x^{*2}}.$$

Using Theorem 1, which states that the dual transformation is self-inverse, we find $\omega(x,y)$:

$$\omega(x,y) = xx^* + yy^* - z^*(x^*, y^*).$$

Integrating $\omega(x,y)$, we get $z = f(x,y)$. Finally, since $\omega = \dfrac{\partial z}{\partial x}$, we integrate:

$$z(x,y) = \int \omega(x,y)\,dx + \phi(y),$$

where $\phi(y)$ is an arbitrary function of y arising from the integration.

Conclusion

In this paper, we investigated the Monge–Ampere equation in three-dimensional isotropic space and demonstrated its strong connection with the geometry of surfaces, dual transformations, and curvature invariants. By leveraging the properties of isotropic geometry, particularly the degenerate metric and dual mappings, we formulated and solved a class of nonlinear Monge–Ampere-type equations.

Using of dual transformation techniques, we linearized a complex nonlinear PDE, solved it analytically using separation of variables, and reconstructed the original surface using the inverse dual transform. The method proved effective in simplifying the solution process and understanding the geometric structure behind the equation.

We also studied translation surfaces and provided conditions under which such surfaces can be constructed from given curvature functions. In particular, we showed that the total curvature of the dual surface imposes strict conditions on the form of the original surface.

The results obtained in this work can serve as a foundation for further research in isotropic differential geometry, geometric PDEs, and applications in computer graphics, elasticity, and geometric modeling. The approach of using duality and curvature invariants offers a powerful framework for the analysis and reconstruction of surfaces governed by Monge–Ampere-type equations.

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Conflict of Interest

The author declare no conflict of interest.

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