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Well-posedness of elliptic-parabolic differential problem with integral condition

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In this paper, we study a class of nonlocal boundary value problems for elliptic-parabolic equations subject to integral-type conditions. Such problems naturally emerge in various physical and engineering contexts, including diffusion processes in composite materials and systems with memory or nonlocal interactions. The model considered involves a mixed-type equation in which the elliptic and parabolic components are coupled through nonlocal boundary terms, while the boundary conditions incorporate integral constraints that generalize the traditional Dirichlet and Neumann formulations. To investigate the solvability of this problem, we employ analytical methods based on the theory of parabolic and elliptic operators in weighted Hölder spaces, which are particularly suitable for handling boundary singularities and ensuring regularity of solutions. We establish the existence, uniqueness, and continuous dependence of solutions on the input data, thereby proving the well-posedness of the problem. Furthermore, we derive coercivity inequalities for solutions of the associated mixed nonlocal boundary problems, which guarantee their stability and provide essential tools for studying related inverse and control problems. The findings extend several classical results and offer a unified approach to the analysis of nonlocal elliptic-parabolic models.

Keywords: elliptic-parabolic equation, nonlocal boundary value problem, integral condition, Hölder spaces, well-posedness, coercivity inequalities, stability, mixed-type differential equations.

2020 Mathematics Subject Classification: 35M12, 39K40.

Introduction

Elliptic partial differential equations play a fundamental role across nearly all branches of mathematics — from harmonic analysis and geometry to Lie theory — and have a wide range of applications in physics and engineering. The well-posedness of local boundary value problems for elliptic equations, along with their various applications, has been extensively studied by numerous researchers [1–3].

Equations of mixed-composite type form an important class of partial differential equations (PDEs) that combine features of different types of equations — typically elliptic, parabolic, and sometimes hyperbolic — within a single formulation [4–6]. These equations often arise in mathematical models describing processes where the nature of the physical phenomenon changes across a domain or depends on certain parameters.

In general, an equation is called mixed type when its classification (elliptic, parabolic, or hyperbolic) varies in different regions of the domain. A mixed-composite type equation extends this idea by coupling different equations or operators — such as elliptic and parabolic ones — through boundary, interface, or integral-type conditions [7].

In mathematical modeling, elliptic equations are paired with local boundary conditions that dictate the solution at the domain's edge. However, traditional boundary conditions may be insufficient for accurately modeling certain processes or phenomena. As a result, nonlocal boundary conditions are often employed in mathematical models of physical, chemical, biological, or environmental processes. These conditions, known as nonlocal boundary conditions, arise when data at the domain's edge cannot be directly observed or when boundary data are dependent on internal data within the domain [8–10].

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Various nonlocal boundary value problems with Samarskii–Ionkin condition for partial differential equations have been investigated by many researchers [11, 12].

Moreover, the identification of partial differential equations (PDEs) arises in numerous applied problems and has been the subject of extensive research [13–15].

The significance of well-posedness (WP) in the analysis of boundary value problems (BVPs) for (PDEs) is widely recognized [16–18].

Considerable attention has been devoted to the study of coercivity inequalities (CIs) arising in nonlocal BVPs for elliptic and parabolic PDEs [19–21].

In this paper, we study the WP of a nonlocal BVP of the form

$$\begin{cases} -\mathcal{U}_{tt}(t) + A\mathcal{U}(t) = g(t), \ t \in (0, d], \\ \mathcal{U}_t(t) - A\mathcal{U}(t) = f(t), \ t \in [-d, 0) \end{cases} \tag{1}$$

with an integral condition $\mathcal{U}(d) = \int_{-d}^{0} \mu(s)\mathcal{U}(s)\, ds + \xi$ in a Hilbert space $\aleph$ with a self-adjoint positive definite operator (SAPDO) A. Here, $\xi \in D(A)$, while $g(t)$ and $f(t)$ are prescribed smooth functions.

The principal result demonstrates the WP of problem (1) in weighted Hölder spaces. New CIs for the solutions of elliptic-parabolic nonlocal BVPs are derived.

1 *The main theorem on the WP of (1)*

Throughout this work, $\aleph$ is a Hilbert space and A is assumed to be a SAPDO satisfying $A \geq \delta I$ for $\delta > \delta_0 > 0$, where I is the identity operator. We also set $V = A^{1/2}$.

First, we present several results that will be needed in the sequel.

Lemma 1. The following estimates hold [22]:

$$\begin{cases} \| V^{\mu} \exp(-tV) \|_{\aleph \to \aleph} \leq (\frac{\mu}{e})^{\mu} t^{-\mu}, \ t \in (0, \infty), \ \mu \in [0, e], \\ \| A^{\mu} \exp(-tA) \|_{\aleph \to \aleph} \leq (\frac{\mu}{e})^{\mu} t^{-\mu}, \ t \in (0, \infty), \ \mu \in [0, e], \\ \| (I - \exp(-2dV))^{-1} \|_{\aleph \to \aleph} \leq M(\delta) \end{cases} \tag{2}$$

for some $M(\delta) \geq 0$.

Lemma 2. Operator

$$(I - V)e^{-2dV} + I + V - 2e^{-dV} \int_{-d}^{0} \mu(s) e^{sV^2} ds$$

has an inverse

$$N = \left((I - V)e^{-2dV} + I + V - 2e^{-dV} \int_{-d}^{0} \mu(s) e^{sV^2} ds \right)^{-1}$$

and the following estimates are fulfilled

$$\| N \|_{\aleph \to \aleph} \leq M(\delta), \quad \| VN \|_{\aleph \to \aleph} \leq M(\delta). \tag{3}$$

The Proof for Lemma 2 relies on the spectral representations of unit SAPDO A [22].

Function $\mathcal{U}(t)$ is said to be a solution of problem (1) it the following conditions are met:

1. $\mathcal{U}(t)$ is twice continuously differentiable on $(0, d]$ and continuously differentiable on $[-d, d]$; the derivatives at the endpoints are understood in the sense of one-sided limits;
2. $\mathcal{U}(t) \in D(A)$ for all $t \in [-d, d]$, and the mapping $t \mapsto A\mathcal{U}(t)$ is continuous on $[-d, d]$;
3. $\mathcal{U}(t)$ satisfies the system and the nonlocal boundary condition in (1).

The function $\mathcal{U}(t)$ fulfilling the above requirements will be referred to as a solution of problem (1) in the space $\mathtt{C}(\aleph)=\mathtt{C}_{-d,d}(\aleph)$, consisting of all continuous functions $\psi(y)$ defined on $[-d,d]$ with values in $\aleph$, with the norm

$$\|\psi\|_{\mathtt{C}_{-d,d}(\aleph)}=\max_{y\in[-d,d]}\|\psi(y)\|_{\aleph}.$$

To derive the formula for solution of problem (1), we will consider the following auxiliary problems

$$\begin{cases} -\mathcal{U}''(t)+A\mathcal{U}(t)=g(t),\ t\in(0,d),\\ \mathcal{U}(0)=\mathcal{U}_0,\ \mathcal{U}(d)=\mathcal{U}_d, \end{cases} \tag{4}$$

$$\begin{cases} \mathcal{U}'(t)-A\mathcal{U}(t)=f(t),\ t\in(-d,0),\\ \mathcal{U}(0)=\mathcal{U}_0. \end{cases} \tag{5}$$

It is well established (cf. [22]) that, for sufficiently smooth data, problems (4) and (5) admit a unique solution. Moreover, the following relations are valid:

$$\mathcal{U}(t)=\left(I-e^{-2dV}\right)^{-1}\Bigg[\left(e^{-tV}-e^{-(2d-t)V}\right)\mathcal{U}_0+\left(e^{-(d-t)V}-e^{-(t+d)V}\right)\mathcal{U}_d \tag{6}$$

$$-\left(e^{-(d-t)V}-e^{-(t+d)V}\right)(2V)^{-1}\int_0^d\left(e^{-(d-\theta)V}-e^{-(\theta+d)V}\right)g(\theta)d\theta\Bigg]$$

$$+(2V)^{-1}\int_0^d\left(e^{-|t-\theta|V}-e^{-(t+\theta)V}\right)g(\theta)d\theta,\quad t\in[0,d],$$

$$\mathcal{U}(t)=e^{tA}\mathcal{U}_0+\int_0^t e^{(t-y)A}f(y)dy,\quad t\in[-d,0]. \tag{7}$$

Using formula (6), conditions $\mathcal{U}(d)=\int_{-d}^{0}\mu(\theta)\mathcal{U}(\theta)d\theta+\xi$, and $\mathcal{U}'(0+)=\mathcal{U}'(0-)$, we can write

$$\mathcal{U}(d)=\int_{-d}^{0}\mu(\theta)e^{\theta A}d\theta\mathcal{U}_0+\int_{-d}^{0}\mu(\theta)\int_0^{\theta}e^{-(\theta-y)A}f(y)dyd\theta+\xi, \tag{8}$$

$$A\mathcal{U}(0)+f(0)=\left(I-e^{-2dV}\right)^{-1}\Bigg[-V(I+e^{-2dV})\mathcal{U}_0+2Ve^{-dV}\mathcal{U}_d$$

$$-e^{-dV}\int_0^d\left(e^{-(d-\theta)V}-e^{-(d+\theta)V}\right)g(\theta)\,d\theta\Bigg]+\int_0^d e^{-\theta V}g(\theta)\,d\theta. \tag{9}$$

Using formulas (8) and (9), we obtain that

$$A\mathcal{U}(0)=\left(I-e^{-2dV}\right)^{-1}\Bigg[-V(I+e^{-2dV})\mathcal{U}_0$$

$$+2Ve^{-dV}\left\{\int\limits_{-d}^{0}\mu(\theta)e^{\theta A}d\theta\,\mathcal{U}_0+\int\limits_{-d}^{0}\mu(\theta)\int\limits_{0}^{\theta}e^{-(\theta-y)A}f(y)dyd\theta+\xi\right\}$$

$$-e^{-dV}\int\limits_{0}^{d}\left(e^{-(d-\theta)V}-e^{-(d+\theta)V}\right)g(\theta)d\theta\Bigg]+\int\limits_{0}^{d}e^{-\theta V}g(\theta)d\theta-f(0).$$

Since the operator

$$(I-V)e^{-2dV}+I+V-2e^{-dV}\int\limits_{-d}^{0}\mu(\theta)e^{\theta V^2}d\theta$$

has an inverse

$$N=\left((I-V)e^{-2dV}+I+V-2e^{-dV}\int\limits_{-d}^{0}\mu(\theta)e^{\theta V^2}d\theta\right)^{-1},$$

we derive that

$$\mathcal{U}_0=N\Bigg[2e^{-dV}\left(\int\limits_{-d}^{0}\mu(\theta)\int\limits_{0}^{\theta}e^{-(\theta-y)A}f(y)dyd\theta+\xi\right) \tag{10}$$

$$-e^{-dV}\int\limits_{0}^{d}\left(e^{-(d-\theta)V}-e^{-(d+\theta)V}\right)g(\theta)d\theta$$

$$+\left(I-e^{-2dV}\right)V^{-1}\int\limits_{0}^{d}e^{-\theta V}g(\theta)d\theta-\left(I-e^{-2dV}\right)V^{-1}f(0)\Bigg].$$

Hence, the solution of nonlocal BVP (1) is represented by formulas (7), (10), and (9). Now, let us denote by $\mathtt{C}^{\mu}_{-d,d}(\aleph),\ \mu\in(0,1)$, the Banach space obtained by completing the space of smooth $\aleph$-valued function $\psi(y)$ on $[-d,d]$ in the norm

$$\parallel\psi\parallel_{\mathtt{C}^{\mu}_{-d,d}(\aleph)}=\|\psi\|_{\mathtt{C}_{-d,d}(\aleph)}+\sup_{-d<y<y+\Delta y<0}\|\psi(y+\Delta y)-\psi(y)\|_{\aleph}(\Delta y^{-\mu}(-y)^{\mu}$$

$$+\sup_{0<y<y+\Delta y<d}\|\psi(y+\Delta y)-\xi(y)\|_{\aleph}(\Delta y)^{-\mu}(d-y)^{\mu}(y+\Delta y)^{\mu},$$

and denote by $\mathtt{C}^{\mu}_{0,d}(\aleph),\ \mu\in(0,1)$, the Banach space obtained by completing the space of smooth $\aleph$-valued function $\psi(y)$ on $[0,d]$ in the norm

$$\parallel\psi\parallel_{\mathtt{C}^{\mu}_{0,d}(\aleph)}=\|\psi\|_{\mathtt{C}_{0,d}(\aleph)}+\sup_{0<y<y+\Delta y<d}\|\psi(y+\Delta y)-\psi(y)\|_{\aleph}(\Delta y)^{-\mu}(d-y)^{\mu}(y+\Delta y)^{\mu},$$

finally denote by $\mathtt{C}^{\mu}_{-d,0}(\aleph),\ \mu\in(0,1)$, the Banach space obtained by completion of the set of all smooth $\aleph$-valued functions $\psi(y)$ on $[-d,0]$ in the norm

$$\parallel\psi\parallel_{\mathtt{C}^{\mu}_{-d,0}(\aleph)}=\|\psi\|_{\mathtt{C}_{-d,0}(\aleph)}+\sup_{-d<y<y+\Delta y<0}\|\psi(y+\Delta y)-\psi(y)\|_{\aleph}(\Delta y)^{-\mu}(-y)^{\mu}.$$

Here, $\mathtt{C}_{a,b}(\aleph)$ is defined as the Banach space of all continuous functions $\psi(y)$ defined on $[p,q]$ with values in $\aleph$, endowed with the norm

$$\|\psi\|_{\mathtt{C}_{p,q}(\aleph)} = \max_{y\in[p,q]} \|\psi(y)\|_{\aleph}.$$

Problem (1) is considered *well-posed* in $\mathtt{C}(\aleph)$ if, for every $g(t) \in \mathtt{C}_{0,d}(\aleph)$, $f(t) \in \mathtt{C}_{-d,0}(\aleph)$, and $\xi \in D(A)$, it has a unique solution $\mathcal{U}(t) \in \mathtt{C}(\aleph)$ satisfying the CI

$$\|\mathcal{U}''\|_{\mathtt{C}_{0,d}(\aleph)} + \|\mathcal{U}'\|_{\mathtt{C}_{-d,0}(\aleph)} + \|A\mathcal{U}\|_{\mathtt{C}(\aleph)} \leq M\Big(\|g\|_{\mathtt{C}_{0,d}(\aleph)} + \|f\|_{\mathtt{C}_{-d,0}(\aleph)} + \|A\xi\|_{\aleph}\Big),$$

where M represents a positive constant whose value does not depend on $g(t)$, $f(t)$, and ξ.

The given problem (1) is not well-posed in $\mathtt{C}(\aleph)$ [23]. The WP of BVP (1) can be established by formulating the problem in appropriate function spaces $F(\aleph)$ consisting of smooth $\aleph$-valued functions defined on $[-d,d]$.

A function $\mathcal{U}(t)$ is said to be a solution of problem (1) in $F(\aleph)$ if it satisfies the problem in $\mathtt{C}(\aleph)$ and, moreover, the functions $\mathcal{U}''(t)$ $(t \in [0,d])$, $\mathcal{U}'(t)$ $(t \in [-d,d])$ and $A\mathcal{U}(t)$ $(t \in [-d,d])$ are elements of $F(\aleph)$.

Similarly to the space $\mathtt{C}(\aleph)$, problem (1) is considered *well-posed* in $F(\aleph)$ if the subsequent CI holds:

$$\|\mathcal{U}''\|_{F_{0,d}(\aleph)} + \|\mathcal{U}'\|_{F_{-d,0}(\aleph)} + \|A\mathcal{U}\|_{F(\aleph)} \leq M\Big(\|g\|_{F_{0,d}(\aleph)} + \|f\|_{F_{-d,0}(\aleph)} + \|A\xi\|_{\aleph}\Big),$$

where $M > 0$ denotes a constant that does not depend on $g(t)$, $f(t)$, and ξ.

Setting $F(\aleph) = \mathtt{C}^{\mu}_{0,d}(\aleph) = \mathtt{C}^{\mu}_{0,d}([-d,d],\aleph)$ for $\mu \in (0,1)$, we can formulate our main theorem as follows.

Theorem 1. Suppose $\xi \in D\,(A)$. Then BVP (1) is well-posed in a Hölder space $\mathtt{C}^{\mu}_{0,d}(\aleph)$ and the following CI holds:

$$\left\|\mathcal{U}''\right\|_{\mathtt{C}^{\mu}_{0,d}(\aleph)} + \left\|\mathcal{U}'\right\|_{\mathtt{C}^{\mu}_{-d,0}(\aleph)} + ||A\mathcal{U}||_{\mathtt{C}^{\mu}_{0,d}(\aleph)} \tag{11}$$

$$\leq M(\delta)\left[\mu^{-1}(1-\mu)^{-1}\left[\|f\|_{\mathtt{C}^{\mu}_{-d,0}(\aleph)} + \|g\|_{\mathtt{C}^{\mu}_{0,d}(\aleph)}\right] \ + \|A\xi\|_{\aleph}\right],$$

where $M(\delta)$ is a constant that is independent of $g(t)$, $f(t)$, and ξ.

Proof. The CI (11) is derived from the estimate

$$\left\|\mathcal{U}'\right\|_{\mathtt{C}^{\mu}_{-d,0}(\aleph)} + ||A\mathcal{U}||_{\mathtt{C}^{\mu}_{-d,0}(\aleph)} \leq M(\delta)\mu^{-1}(1-\mu)^{-1}\left\|f\right\|_{\mathtt{C}^{\mu}_{-d,0}(\aleph)} + M\left\|A\mathcal{U}_0\right\|_{\aleph} \tag{12}$$

$$\leq M(\delta)\mu^{-1}(1-\mu)^{-1}\left\|f\right\|_{\mathtt{C}^{\mu}_{-d,0}(\aleph)} + M\left\|A\mathcal{U}_0\right\|_{\aleph}$$

for the solution of problem (5) and the estimate

$$\left\|\mathcal{U}''\right\|_{\mathtt{C}^{\mu}_{0,d}(\aleph)} + ||A\mathcal{U}||_{\mathtt{C}^{\mu}_{0,d}(\aleph)} \leq M(\delta)\mu^{-1}(1-\mu)^{-1}\left\|g\right\|_{\mathtt{C}^{\mu}_{0,d}(\aleph)} \tag{13}$$

$$+M(\delta)\left[\|A\mathcal{U}_0\|_{\aleph} \ + \|A\mathcal{U}_1\|_{\aleph}\right]$$

associated with the solution of BVP (4) and the estimates

$$\|A\mathcal{U}_0\|_{\aleph} \leq M(\delta)\Big[\mu^{-1}(1-\mu)^{-1}\Big[\|f\|_{\mathtt{C}^{\mu}_{-d,0}(\aleph)} + \|g\|_{\mathtt{C}^{\mu}_{0,d}(\aleph)}\Big] + \|A\xi\|_{\aleph}\Big], \tag{14}$$

$$\|A\mathcal{U}_d\|_{\aleph} \leq M(\delta)\Big[\mu^{-1}(1-\mu)^{-1}\Big[\|f\|_{\mathtt{C}^{\mu}_{-d,0}(\aleph)} + \|g\|_{\mathtt{C}^{\mu}_{0,d}(\aleph)}\Big] + \|A\xi\|_{\aleph}\Big] \tag{15}$$

for the solution of BVP (1). Estimates (12) and (13) were obtained in [24]. Applying formula (10), we get

$$A\mathcal{U}_0 = N\left[2V^2e^{-dV}\left(\int_{-d}^{0}\mu(\theta)\int_{0}^{\theta}e^{-(\theta-y)A}f(y)dyd\theta + A\xi\right)\right.$$

$$\left.-V^2e^{-dV}\int_{0}^{d}\left(e^{-(d-\theta)V}-e^{-(d+\theta)V}\right)g(\theta)d\theta\right]$$

$$+NV\left[\left(I-e^{-2dV}\right)\int_{0}^{d}e^{-\theta V}g(\theta)d\theta-\left(I-e^{-2dV}\right)f(0)\right].$$

Therefore, the proof of estimate (14) is based on the triangle inequality and estimates (2), (3). Applying formula (8), we get

$$A\mathcal{U}(d) = \int_{-d}^{0}\mu(\theta)e^{\theta A}d\theta A\mathcal{U}_0 + A\xi$$

$$+\int_{-d}^{0}\mu(\theta)\int_{0}^{\theta}Ae^{-(\theta-y)A}\left(f(y)-f(\theta)\right)dyd\theta+\int_{-d}^{0}\mu(\theta)\left(I-e^{\theta A}\right)f(\theta)d\theta.$$

Therefore, the estimate (15) is proved based on the triangle inequality together with (2) and (3), which completes the proof of Theorem 1.

2 Illustrative examples

We now illustrate several applications of Theorem 1.

Firstly, the nonlocal BVP for an elliptic-parabolic equation

$$\begin{cases} -\mathcal{U}_{tt}-(a(z)\mathcal{U}_z)_z+\delta\mathcal{U}=g(t,z),\ t\in[0,d],\ z\in[0,b] \\ \mathcal{U}_t+(a(z)\mathcal{U}_z)_z-\delta\mathcal{U}=f(t,z),\ t\in[-d,0],\ z\in[0,b], \\ \mathcal{U}(0+,z)=\mathcal{U}(0-,z),\ \mathcal{U}_t(0+,z)=\mathcal{U}_t(0-,z),\ z\in[0,b], \\ \mathcal{U}(t,0)=\mathcal{U}(t,b),\ \mathcal{U}_z(t,0)=\mathcal{U}_z(t,b),\ t\in[-d,d] \end{cases} \tag{16}$$

with the integral condition $\mathcal{U}(d,z)=\int_{-d}^{0}\mu(\Delta t)\,\mathcal{U}(\tau,z)\,d\tau+\xi(z),\ z\in[0,b]$ is considered. Problem (16) admits a unique smooth solution $\mathcal{U}(t,z)$ for smooth functions $a(z)$, with $a(z)=a(0)$ and $a(z)\geq a>0$ for $z\in(0,b)$, and for $g(t,z)$ $(t\in[0,d],\ z\in[0,b])$ and $f(t,z)$ $(t\in[-d,0],\ x\in[0,b])$, where $\delta>0$.

We define the space $L_2[0,b]$ of all square integrable functions $\xi(z)$ defined on $[0,b]$ and the spaces $W_2^1[0,b]$ and $W_2^2[0,b]$ with the norms

$$\|\xi\|_{W_2^1[0,b]}=\|\xi\|_{L_2[0,b]}+\left(\int_0^b|\xi_z|^2\,dz\right)^{1/2},\ \left\|\xi^h\right\|_{W_2^2[0,b]}=\|\xi\|_{L_2[0,b]}+\left(\int_0^b|\xi_{zz}|^2\,dz\right)^{1/2}.$$

This reduces mixed problem (16) to the nonlocal BVP (1) in a Hilbert space $\aleph=L_2[0,b]$ with a SAPDO A given by (16).

Theorem 2. The solution of nonlocal BVP (16) satisfies the CI

$$\| \mathcal{U}_{tt} \|_{\mathtt{C}^{\mu}_{0,d}(L_2(0,b))} + \| \mathcal{U}_t \|_{\mathtt{C}^{\mu}_{-d,0}(L_2(0,b))} + \| \mathcal{U} \|_{\mathtt{C}^{\mu}_{-d,d}(W_2^2(0,b))}$$

$$\leq M_c(\delta)\Big[\mu^{-1}(1-\mu)^{-1}\Big[\, \| g \|_{\mathtt{C}^{\mu}_{0,d}(L_2(0,b))} + \| f \|_{\mathtt{C}^{\mu}_{-d,0}(L_2(0,b))} \Big] + \|\xi\|_{W_2^2(0,b)}\Big].$$

Here, the constant $M(\delta)$ is independent of the functions $g(t,z)$, $f(t,z)$, and $\xi(z)$.

Proof of Theorem 2 builds upon the theoretical framework developed in Theorem 1, utilizing the symmetry properties of the operator associated with problem (16).

Secondly, let Ω denote the open unit cube in the n-dimensional Euclidean space $\mathbb{R}^n$, defined by $z_k \in (0,1)$ for $k = \overline{1,n}$ with S, so that $\overline{\Omega} = \Omega \cup S$. Within the domain $[-d,d] \times \Omega$, we formulate the BVP for a multi-dimensional mixed problem as follows:

$$\begin{cases} -\mathcal{U}_{tt} - \sum\limits_{r=1}^{n} (a_r(z)\mathcal{U}_{z_r})_{z_r} = g(t,z),\ t \in [0,d],\ z \in \Omega, \\ \mathcal{U}_t + \sum\limits_{r=1}^{n} (a_r(z)\mathcal{U}_{z_r})_{z_r} = f(t,z),\ t \in [-d,0],\ z \in \Omega, \\ \mathcal{U}(0+,z) = \mathcal{U}(0-,z),\ \mathcal{U}_t(0+,z) = \mathcal{U}_t(0-,z),\ z \in \overline{\Omega}, \\ \mathcal{U}(t,z) = 0,\ z \in S,\ [-d,d] \end{cases} \tag{17}$$

with the integral condition $\mathcal{U}(d,z) = \int\limits_{-d}^{0} \mu(\tau)\mathcal{U}(\tau,z)d\tau + \xi(z),\ z \in \overline{\Omega}$. Here, $a_r(z)$ $(z \in \Omega)$, $g(t,z)$ $(t \in (0,d),\ z \in \overline{\Omega})$, and $f(t,z)$ $(t \in (-d,0),\ z \in \overline{\Omega})$ are given smooth functions, with $a_r(z) \geq a > 0$.

We introduce the Hilbert space $L_2(\overline{\Omega})$ consisting of all square-integrable functions $\xi(z)$ defined on $\overline{\Omega}$, endowed with the norm

$$\| \xi \|_{L_2(\overline{\Omega})} = \sqrt{\int \cdots \int\limits_{z\in\overline{\Omega}} |\xi(z)|^2 dz_1 \cdots dz_n}$$

and the Hilbert spaces $W_2^1(\Omega), W_2^2(\Omega)$ defined on Ω, endowed with the norms

$$\|\xi\|_{W_2^1(\Omega)} = \| \xi \|_{L_2(\overline{\Omega})} + \sqrt{\int \cdots \int\limits_{z\in\Omega} \sum_{r=1}^{n} |\xi_{z_r}|^2 \, dz_1 \cdots dz_n}$$

and

$$\left\|\xi^h\right\|_{W_2^2(\Omega)} = \left\|\xi^h\right\|_{L_{2h}} + \sqrt{\int \cdots \int\limits_{z\in\Omega} \sum_{r=1}^{n} \left|\xi_{z_r \overline{z_r}}\right|^2 dz_1 \cdots dz_n}.$$

Problem (17) admits a unique smooth solution $u(t,x)$ for smooth functions $a_r(x)$, $g(t,x)$, and $f(t,x)$. Using this approach, the mixed problem (17) can be reduced to the nonlocal BVP (1) in the Hilbert space $H = L_2(\overline{\Omega})$ with a SAPDO A presented as in (17).

Theorem 3. The solution of nonlocal BVP (17) satisfies the CI

$$\|\mathcal{U}_{tt}\|_{\mathtt{C}^{\mu}_{0,d}(L_2(\overline{\Omega}))} + \|\mathcal{U}_t\|_{\mathtt{C}^{\mu}_{-d,0}(L_2(\overline{\Omega}))} + \|\mathcal{U}\|_{\mathtt{C}^{\mu}_{-d,d}(W_2^2(\Omega))}$$

$$\leq M_c(\delta)\Big[\mu^{-1}(1-\mu)^{-1}\Big[\|g\|_{\mathtt{C}^{\mu}_{0,d}(L_2(\overline{\Omega}))} + \|f\|_{\mathtt{C}^{\mu}_{-d,0}(L_2(\overline{\Omega}))}\Big] + \|\xi\|_{W_2^2(\Omega)}\Big].$$

The proof of Theorem 3 relies on the result given in Theorem 1, together with the symmetry properties of the operator associated with problem (17), and the CI for solutions of elliptic differential problems in $L_2(\overline{\Omega})$ as established in [24].

Conclusion

In the present paper, a nonlocal boundary value problem for an elliptic-parabolic equation subject to an integral condition is investigated. The well-posedness of the problem in weighted Hölder spaces is established. As an application, we derive coercivity inequalities for the solutions of mixed nonlocal boundary value problems associated with elliptic-parabolic equations. By applying the methods developed in this paper and in [25], we can establish the boundedness of solutions to a semilinear elliptic-parabolic equation.

Conflict of Interest

The authors declare no conflict of interest.

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