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Inequalities for analytic functions associated with hyperbolic cosine function

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In this paper, we investigate the geometric properties of a specific subclass of analytic functions satisfying the condition $f'(z) \prec cosh(\sqrt{z})$ meaning that the function $f'(z)$ is subordinate to the function $cosh(\sqrt{z})$. Also, we focus on deriving sharp inequalities for Taylor coefficients, particularly for b_2 and the modulus of the second derivative $f''(z)$. Utilizing the Schwarz lemma, both on the unit disc and on its boundary, we provide essential insights into the distortion and growth behaviors of these functions. The paper demonstrates the sharpness of these inequalities through extremal functions and applies the Julia–Wolff lemma to establish boundary behavior results. These findings contribute significantly to the understanding of the analytic functions associated with the hyperbolic cosine function, with potential applications in geometric function theory. It is considered that the extremal functions obtained in this study could be potential hyperbolic activation functions in neural network architectures. This perspective builds a conceptual bridge between geometric function theory and artificial intelligence, indicating that insights from complex analysis can inspire the development of more effective and theoretically grounded activation mechanisms in deep learning. Empirical evaluation of architectures built with novel activation functions may be considered as potential future work.

Keywords: Schwarz estimate, angular derivative, the principle of subordination, activation function, extremal function, analytic function, Julia–Wolff lemma, angular limit, Schwarz lemma at the boundary, the unit disc

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Introduction

Let $\mathcal{A}$ represent the class of functions of the form $f(z) = z + b_2 z^2 + b_3 z^3 + ...$, analytic in the unit disc $D = \{z : |z| < 1\}$. Also, let $\mathcal{W}$ be the subclass of $\mathcal{A}$ satisfying the condition

$$f'(z) \prec \cosh \sqrt{z},$$

where the symbol "$\prec$" indicates the principle of subordination [1]. Also, we choose the branch of the square root function so that

$$\cosh \sqrt{z} = 1 + \frac{z}{2!} + \frac{z^2}{4!} + \frac{z^3}{6!} + ...$$

The conformal mapping $\cosh \sqrt{z} : D \to \mathbb{C}$ maps the unit disc D onto the region

$$\left\{ \alpha \in \mathbb{C} : \left| \ln \left(\alpha + \sqrt{\alpha^2 - 1} \right) \right| < 1 \right\}$$

defined on the principal branch of the logarithm and the square root function [2].

Determining the upper bound for Taylor coefficients has been a key area of focus in understanding geometric properties, offering important insights into different subclasses of $\mathcal{W}$. In this section, we

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establish an upper bound for b_2, a coefficient of the function $f(z)$. To achieve this, we will apply the Schwarz lemma. Furthermore, the following section will evaluate the modulus of the second derivative of $f(z)$ from below, requiring the use of the Schwarz lemma on the boundary.

The Schwarz lemma asserts that for any analytic function $p(z)$ mapping the unit disc D into itself and satisfying $p(0) = 0$, $|p(z)| \leq |z|$ for $z \in D$ and $|p'(0)| \leq 1$. In simpler terms, it asserts that if a function maps the unit disc into itself and maps its origin to the origin, then the function cannot magnify the distances between points inside the disc by more than 1 [1].

Extending the Schwarz lemma to the boundary of the unit disc offers deep insights into the behavior of analytic functions close to this boundary, proving especially beneficial when studying functions that approach the disc's edge. In this paper, we set out to examine the application and significance of this remarkable theorem to various classes of functions. The Schwarz lemma implies the boundary Schwarz lemma $|p'(1)| \geq 1$. Osserman and Unkelbach [3, 4] showed that in this case, we have in fact

$$\left|p'(1)\right| \geq 1 + \frac{1-|p'(0)|}{1+|p'(0)|} = \frac{2}{1+|p'(0)|},$$

where p satisfies the conditions of the Schwarz lemma, p extends continuously to the boundary point $1 \in \partial D = \{z : |z| = 1\}$, $|p(1)| = 1$ and $p'(1)$ exists. These inequalities are sharp. In mathematical literature, these inequalities and their generalizations are topics of continuous discussion and hold great importance in the geometric theory of functions [5–7]. Some properties of analytic function classes related to the Jack and the Schwarz lemmas were studied in [8]. In [9], a new bound for the Schwarz inequality was obtained for analytic functions mapping the unit disk onto itself.

If we use the principle of subordination for the class we defined above, there exists a Schwarz function $p(z)$ such that

$$f'(z) = \cosh\sqrt{p(z)}.$$

Here, the function $p(z)$ meets the criteria of the Schwarz lemma [1]. Therefore, applying the Schwarz lemma, we derive

$$f''(z) = \frac{p'(z)}{2\sqrt{p(z)}}\sinh\sqrt{p(z)}$$

and

$$f''(0) = \frac{p'(0)}{2}\lim_{z\to 0}\frac{\sinh\sqrt{p(z)}}{\sqrt{p(z)}}.$$

Therefore, we have

$$f''(0) = \frac{p'(0)}{2}$$

and

$$\left|f''(0)\right| \leq \frac{1}{2}.$$

We will now demonstrate that the final inequality is sharp. Let

$$f(z) = 2\sqrt{z}\sinh\sqrt{z} - 2\cosh\sqrt{z} + 2.$$

Then,

$$f'(z) = \cosh\sqrt{z},$$

$$f''(z) = \frac{1}{2\sqrt{z}}\frac{\sinh\sqrt{z}}{\sqrt{z}}$$

and

$$\left|f''(0)\right| = \frac{1}{2}.$$

Lemma 1. If $f \in \mathcal{W}$, then

$$\left|f''(0)\right| \leq \frac{1}{2}.$$

This result is sharp, as demonstrated by the extremal function

$$f(z) = 2\sqrt{z}\sinh\sqrt{z} - 2\cosh\sqrt{z} + 2.$$

The subsequent lemma, referred to as the Julia–Wolff lemma, is required for the following discussion [10].

Lemma 2 (Julia–Wolff lemma). If p is an analytic function in the unit disc D with $p(0) = 0$ and $p(D) \subset D$, and additionally, p has an angular limit $p(1)$ at $1 \in \partial D$ where $|p(1)| = 1$, then the angular derivative $p'(1)$ exists and $1 \leq |p'(1)| \leq \infty$.

1 *Main results*

This section focuses on examining the second derivative of the analytic function $f(z)$. During this analysis, we will derive stronger inequalities by considering the coefficients of the Taylor series expansion of $f(z)$. Additionally, we will provide an inequality that demonstrates the relationship between these coefficients.

Theorem 1. Let $f(z) \in \mathcal{W}$. Suppose that, for $1 \in \partial D$, f has an angular limit $f(1)$ at 1, $f'(1) = \cosh 1$. Then we have the inequality

$$\left|f''(1)\right| \geq \frac{\sinh 1}{2}. \tag{1}$$

This result is sharp, with equality for the function

$$f(z) = 2\sqrt{z}\sinh\sqrt{z} - 2\cosh\sqrt{z} + 2.$$

Proof. Let

$$f'(z) = \cosh\sqrt{p(z)}.$$

Then,

$$f''(z) = \frac{p'(z)}{2\sqrt{p(z)}}\sinh\sqrt{p(z)},$$

$$f''(1) = \frac{p'(1)}{2\sqrt{p(1)}}\sinh\sqrt{p(1)}$$

and

$$f''(1) = \frac{p'(1)}{2}\sinh 1.$$

Since the function $p(z)$ satisfies the conditions of the Schwarz lemma at the boundary, we obtain

$$1 \leq \left|p'(1)\right| = \frac{2\left|f''(1)\right|}{\sinh 1}$$

and

$$\left|f''(1)\right| \geq \frac{\sinh 1}{2}.$$

Now, we will prove that inequality (1) is sharp. Let

$$f(z) = 2\sqrt{z}\sinh\sqrt{z} - 2\cosh\sqrt{z} + 2.$$

Then,

$$f''(z) = \frac{1}{2\sqrt{z}} \frac{\sinh\sqrt{z}}{\sqrt{z}}$$

and

$$\left|f''(1)\right| = \frac{\sinh 1}{2}.$$

□

Theorem 2. Assuming the same conditions as in Theorem 1, we obtain

$$\left|f''(1)\right| \geq \frac{\sinh 1}{1 + 2\left|f''(0)\right|}. \tag{2}$$

Inequality (2) is sharp, achieving equality for the function

$$f(z) = 2\sqrt{z}\sinh\sqrt{z} - 2\cosh\sqrt{z} + 2.$$

Proof. Let $p(z)$ be as defined in the proof of Theorem 1. Thus, by the Schwarz lemma on the boundary,

$$\frac{2}{1 + |p'(0)|} \leq \left|p'(1)\right| = \frac{2\left|f''(1)\right|}{\sinh 1}.$$

Since

$$\left|p'(0)\right| = 2\left|f''(0)\right|,$$

we take

$$\frac{2}{1 + 2\left|f''(0)\right|} \leq \frac{2\left|f''(1)\right|}{\sinh 1}$$

and

$$\left|f''(1)\right| \geq \frac{\sinh 1}{1 + 2\left|f''(0)\right|}.$$

Next, we will demonstrate that inequality (2) is sharp. Consider

$$f(z) = 2\sqrt{z}\sinh\sqrt{z} - 2\cosh\sqrt{z} + 2.$$

Then, we have

$$\left|f''(1)\right| = \frac{\sinh 1}{2}.$$

However, we also have

$$z + b_2 z^2 + b_3 z^3 + ... = 2\sqrt{z}\sinh\sqrt{z} - 2\cosh\sqrt{z} + 2,$$

$$1 + 2b_2 z + 3b_3 z^2 + ... = \cosh\sqrt{z}$$

and

$$2b_2 + 6b_3 z + ... = \frac{1}{2}\frac{\sinh\sqrt{z}}{\sqrt{z}}.$$

Upon taking the limit as z approaches 0 in the final equation, we find that $b_2 = \frac{1}{4}$. Consequently, this yields

$$\frac{\sinh 1}{1 + 2\left|f''(0)\right|} = \frac{\sinh 1}{2}.$$

□

Theorem 3. Under the conditions of Theorem 1, we obtain

$$\left|f'(1)\right| \geq \frac{\sinh 1}{2}\left(1+\frac{2\left(1-4\left|b_2\right|\right)^2}{1-16\left|b_2\right|^2+\left|6b_3-\frac{2}{3}b_2^2\right|}\right). \tag{3}$$

The bound is sharp with the extremal function given by

$$f(z)=\sinh z.$$

Proof. Let $p(z)$ denote the same function as in the proof of Theorem 1 and let $u(z)=z$. According to the maximum principle, for every $z\in D$, it follows that $|p(z)|\leq|u(z)|$. Therefore, $h(z)=\frac{p(z)}{u(z)}$ is an analytic function and $|h(z)|<1$ for $|z|<1$. By Taylor expansion of the function $p(z)$, we have $p(z)=c_1z+c_2z^2+c_3z^3+\dots$. Thus, we take

$$h(z)=\frac{p(z)}{u(z)}=\frac{c_1z+c_2z^2+c_3z^3+\ \dots}{z}=c_1+c_2z+c_3z^2+\ \dots\ ,$$

$$|h(0)|=|c_1|$$

and

$$\left|h'(0)\right|=|c_2|\,.$$

Through straightforward computations, we obtain

$$1+2b_2z+3b_3z^2+\dots=\cosh\sqrt{p(z)}=1+\frac{p(z)}{2!}+\frac{p(z)^2}{4!}+\frac{p(z)^3}{6!}+\ \dots\ ,$$

$$\begin{aligned}2b_2z+3b_3z^2+\dots &= \frac{c_1z+c_2z^2+c_3z^3+\dots}{2!}+\frac{\left(c_1z+c_2z^2+c_3z^3+\ \dots\ \right)^2}{4!}+\ \dots\\ &= \frac{1}{2!}z\left(c_1+c_2z+c_3z^2+\dots\right)+\frac{1}{4!}z^2\left(c_1+c_2z+c_3z^2+\dots\right)^2+\dots,\end{aligned}$$

$$2b_2+3b_3z+\dots=\frac{1}{2!}\left(c_1+c_2z+c_3z^2+\dots\right)+\frac{1}{4!}z\left(c_1+c_2z+c_3z^2+\dots\right)^2+\dots,$$

$$b_2=\frac{1}{4}c_1$$

and

$$3b_3=\frac{1}{2}c_2+\frac{1}{24}c_1^2.$$

Thus, based on the expression for $h(z)$, we have

$$|h(0)|=4\left|b_2\right| \tag{4}$$

and

$$\left|h'(0)\right|=\left|6b_3-\frac{2}{3}b_2^2\right|.$$

The combined function

$$w(z)=\frac{h(z)-h(0)}{1-\overline{h(0)}h(z)}$$

is analytic in the unit disc D, $w(0) = 0$, $|w(z)| < 1$ for $z \in D$ and $|w(1)| = 1$ for $1 \in \partial D$. By the Schwarz lemma on the boundary, we obtain

$$\frac{2}{1+|w'(0)|} \leq \left|w'(1)\right| = \frac{1-|h(0)|^2}{\left|1-\overline{h(0)}h(1)\right|^2}\left|h'(1)\right| \leq \frac{1+|h(0)|}{1-|h(0)|}\left(\left|p'(1)\right| - \left|u'(1)\right|\right).$$

Since

$$\left|w'(0)\right| = \frac{|h'(0)|}{1-|h(0)|^2} = \frac{\left|6b_3 - \frac{2}{3}b_2^2\right|}{1-16\left|b_2\right|^2},$$

we take

$$\frac{2}{1+\frac{\left|6b_3-\frac{2}{3}b_2^2\right|}{1-16|b_2|^2}} \leq \frac{1+4\left|b_2\right|}{1-4\left|b_2\right|}\left(\frac{2\left|f''(1)\right|}{\sinh 1} - 1\right)$$

and

$$\left|f'(1)\right| \geq \frac{\sinh 1}{2}\left(1+\frac{2\left(1-4\left|b_2\right|\right)^2}{1-16\left|b_2\right|^2+\left|6b_3-\frac{2}{3}b_2^2\right|}\right).$$

We will now demonstrate that the inequality (3) achieves equality. Consider

$$f(z) = \sinh z.$$

Then,

$$f'(z) = \cosh z$$

and

$$f''(1) = \sinh 1.$$

On the other hand, we have

$$z + b_2 z^2 + b_3 z^3 + ... = \sinh z = z + \frac{z^3}{3!} + \frac{z^5}{5!} + ...,$$

$$b_2 z^2 + b_3 z^3 + ... = \frac{z^3}{3!} + \frac{z^5}{5!} + ...$$

and

$$b_2 + b_3 z + ... = \frac{z}{3!} + \frac{z^3}{5!} + \ ... \ .$$

Passing to the limit ($z \to 0$) in the last equality yields $b_2 = 0$. Similarly, using straightforward calculations, we obtain $b_2 = \frac{1}{3!}$. Therefore, we obtain

$$\frac{\sinh 1}{2}\left(1+\frac{2\left(1-4\left|b_2\right|\right)^2}{1-16\left|b_2\right|^2+\left|6b_3-\frac{2}{3}b_2^2\right|}\right) = \sinh 1.$$

□

Theorem 4. Let $f \in \mathcal{W}$ and $f(z) - z$ has no critical point in D except $z = 0$ and $b_2 > 0$. Then

$$\left|3b_3 - \frac{1}{3}b_2^2\right| \leq 4\left|b_2 \ln\left(4b_2\right)\right|. \tag{5}$$

This result is sharp.

Proof. Given that $b_2 > 0$ in the expression of the function $f(z)$, and considering inequality (4), assuming that $f(z) - z$ has no critical point in D except $z = 0$, we denote the analytic branch of the logarithm by $\ln h(z)$, normalized under the condition

$$\ln h(0) = \ln(4b_2) < 0.$$

The fractional function

$$\Theta(z) = \frac{\ln h(z) - \ln h(0)}{\ln h(z) + \ln h(0)}$$

is analytic in the unit disc D, $|\Theta(z)| < 1$ for $z \in D$ and $\Theta(0) = 0$. By the Schwarz lemma, we obtain

$$1 \geq \left|\Theta'(0)\right| = \frac{|2\ln h(0)|}{|\ln h(0) + \ln h(0)|^2} \left|\frac{h'(0)}{h(0)}\right| = \frac{-1}{2\ln h(0)} \left|\frac{h'(0)}{h(0)}\right| = -\frac{\left|6b_3 - \frac{2}{3}b_2^2\right|}{8b_2 \ln(4b_2)}$$

and

$$\left|3b_3 - \frac{1}{3}b_2^2\right| \leq 4b_2 \ln(4b_2).$$

Now, we will show that inequality (5) is sharp. Let

$$p(z) = ze^{\frac{1+z}{1-z}\ln 4b_2} = zg(z),$$

where $g(z) = e^{\frac{1+z}{1-z}\ln 2b_1}$. Thus, we have

$$g(z) \quad = \quad \frac{p(z)}{z} = \frac{c_1 z + c_2 z^2 + c_3 z^3 + ...}{z} = c_1 + c_2 z + c_3 z^2 + \ \dots \ .$$

Then

$$g(0) = c_1 = 4b_2, \quad g'(0) = c_2 = 6b_3 - \frac{2}{3}b_2^2.$$

Following straightforward computations, we obtain

$$g'(z) = \frac{2}{(1-z)^2} \ln(4b_2)\, e^{\frac{1+z}{1-z}\ln(4b_2)}$$

and

$$g'(0) = 8b_2 \ln(4b_2).$$

Thus, we obtain

$$\left|3b_3 - \frac{1}{3}b_2^2\right| = 4\left|b_2 \ln(4b_2)\right|.$$

□

Based on the findings from [5], this theorem derives the modulus of the function's derivative at point 1 by considering its Taylor expansions around two points.

Theorem 5. Let $f \in \mathcal{W}$ and $f'(a) = 1$ for $0 < |a| < 1$. Suppose that for $1 \in \partial D$, f has an angular limit $f(1)$ at 1, $f'(1) = \cosh 1$. Then, we have the inequality

$$|f''(1)| \geq \frac{\sinh 1}{2}\left(1 + \frac{1-|a|^2}{|1-a|^2} + \frac{|a|-2|f''(0)|}{|a|+2|f''(0)|} \times \left[1 + \frac{|a|^2+4|f''(0)||f''(a)|\left(1-|a|^2\right)-2|f''(a)|\left(1-|a|^2\right)-2|f''(0)|}{|a|^2+4|f''(0)||f''(a)|\left(1-|a|^2\right)+2|f''(a)|\left(1-|a|^2\right)+2|f''(0)|}\frac{1-|a|^2}{|1-a|^2}\right]\right). \tag{6}$$

Inequality (6) is sharp, with equality for each possible value of $|f''(0)|$ and $|f''(a)|$.

Proof. According to the Schwarz–Pick lemma [1], we have

$$\left|\frac{s(z)-s(a)}{1-\overline{s(a)}s(z)}\right| \leq \left|\frac{z-a}{1-\overline{a}z}\right| = |\rho(z)|$$

and

$$|s(z)| \leq \frac{|s(a)|+|\rho(z)|}{1+|s(a)|\,|\rho(z)|}, \tag{7}$$

where $s : D \to D$ is an analytic function and $a \in D$. If $t : D \to D$ is analytic and $0 < |a| < 1$, letting $s(z) = \frac{t(z)-t(0)}{z\left(1-\overline{t(0)}t(z)\right)}$ in (7), we obtain

$$\left|\frac{t(z)-t(0)}{z\left(1-\overline{t(0)}t(z)\right)}\right| \leq \frac{\left|\frac{t(a)-t(0)}{a\left(1-\overline{t(0)}t(a)\right)}\right| + |\rho(z)|}{1+\left|\frac{t(a)-t(0)}{a\left(1-\overline{t(0)}t(a)\right)}\right| |\rho(z)|}$$

and

$$|t(z)| \leq \frac{|t(0)|+|z|\,\frac{|A|+|\rho(z)|}{1+|A||\rho(z)|}}{1+|t(0)|\,|z|\,\frac{|A|+|\rho(z)|}{1+|A||\rho(z)|}}, \tag{8}$$

where

$$A = \frac{t(a)-t(0)}{a\left(1-\overline{t(0)}t(a)\right)}.$$

If we take

$$t(z) = \frac{p(z)}{z\frac{z-a}{1-\overline{a}z}},$$

then, we have

$$t(0) = \frac{p'(0)}{-a}, t(a) = \frac{p'(a)\left(1-|a|^2\right)}{a}$$

and

$$A = \frac{\frac{p'(a)\left(1-|a|^2\right)}{a} + \frac{p'(0)}{a}}{a\left(1+\overline{\frac{p'(0)}{a}}\frac{p'(a)\left(1-|a|^2\right)}{a}\right)},$$

where $|A| \leq 1$. Let $|t(0)| = b$ and let

$$B = \frac{\left|\frac{p'(a)\left(1-|a|^2\right)}{a}\right| + \left|\frac{p'(0)}{a}\right|}{|a|\left(1+\left|\frac{p'(0)}{a}\right|\left|\frac{p'(a)\left(1-|a|^2\right)}{a}\right|\right)}.$$

From (8), we obtain

$$|p(z)| \leq |z|\,|\rho(z)|\,\frac{b+|z|\,\frac{B+|\rho(z)|}{1+B|\rho(z)|}}{1+b\,|z|\,\frac{B+|\rho(z)|}{1+B|\rho(z)|}}$$

and

$$\frac{1-|p(z)|}{1-|z|} \geq \frac{1+b\,|z|\,\frac{B+|\rho(z)|}{1+B|\rho(z)|} - b\,|z|\,|\rho(z)| - |z|^2\,|\rho(z)|\,\frac{B+|\rho(z)|}{1+B|\rho(z)|}}{(1-|z|)\left(1+b\,|z|\,\frac{B+|\rho(z)|}{1+B|\rho(z)|}\right)} = I.$$

Let $V(z)=1+b\left|z\right|\frac{B+\left|\rho(z)\right|}{1+B\left|\rho(z)\right|}$ and $R(z)=1+B\left|\rho(z)\right|$. Considering the functions $V(z)$ and $R(z)$ in the earlier inequality, we obtain

$$I=\frac{1}{V(z)R(z)}\left\{\frac{1-\left|z\right|^2\left|\rho(z)\right|}{1-\left|z\right|}+B\left|\rho(z)\right|\frac{1-\left|z\right|^2}{1-\left|z\right|}+bB\left|z\right|\frac{1-\left|\rho(z)\right|^2}{1-\left|z\right|}\right\}. \tag{9}$$

Since

$$\lim_{z\to 1}V(z)=\lim_{z\to 1}\left(1+b\left|z\right|\frac{B+\left|\rho(z)\right|}{1+B\left|\rho(z)\right|}\right)=1+b,$$

$$\lim_{z\to 1}R(z)=\lim_{z\to 1}\left(1+B\left|\rho(z)\right|\right)=1+B$$

and

$$1-\left|\rho(z)\right|^2=1-\left|\frac{z-a}{1-\overline{a}z}\right|^2=\frac{\left(1-\left|a\right|^2\right)\left(1-\left|z\right|^2\right)}{\left|1-\overline{a}z\right|^2},$$

passing to the angular limit in (9) yields

$$\left|p'(1)\right|\geq 1+\frac{1-\left|a\right|^2}{\left|1-a\right|^2}+\frac{1-b}{1+b}\left[1+\frac{1-B}{1+B}\frac{1-\left|a\right|^2}{\left|1-a\right|^2}\right].$$

Moreover, since

$$\frac{1-b}{1+b}=\frac{1-\left|t(0)\right|}{1+\left|t(0)\right|}=\frac{1-\left|\frac{p'(0)}{a}\right|}{1+\left|\frac{p'(0)}{a}\right|}=\frac{\left|a\right|-\left|p'(0)\right|}{\left|a\right|+\left|p'(0)\right|}=\frac{\left|a\right|-2\left|f''(0)\right|}{\left|a\right|+2\left|f''(0)\right|},$$

$$\frac{1-B}{1+B}=\frac{1-\frac{\left|\frac{p'(a)\left(1-\left|a\right|^2\right)}{a}\right|+\left|\frac{p'(0)}{a}\right|}{\left|a\right|\left(1+\left|\frac{p'(0)}{a}\right|\left|\frac{p'(a)\left(1-\left|a\right|^2\right)}{a}\right|\right)}}{1+\frac{\left|\frac{p'(a)\left(1-\left|a\right|^2\right)}{a}\right|+\left|\frac{p'(0)}{a}\right|}{\left|a\right|\left(1+\left|\frac{p'(0)}{a}\right|\left|\frac{p'(a)\left(1-\left|a\right|^2\right)}{a}\right|\right)}},$$

$$\frac{1-B}{1+B}=\frac{\left|a\right|^2+2\left|f''(0)\right|2\left|f''(a)\right|\left(1-\left|a\right|^2\right)-2\left|f''(a)\right|\left(1-\left|a\right|^2\right)-2\left|f''(0)\right|}{\left|a\right|^2+2\left|f''(0)\right|2\left|f''(a)\right|\left(1-\left|a\right|^2\right)+2\left|f''(a)\right|\left(1-\left|a\right|^2\right)+2\left|f''(0)\right|}$$

and

$$\frac{1-m}{1+m}=\frac{\left|a\right|^2+4\left|f''(0)\right|\left|f''(a)\right|\left(1-\left|a\right|^2\right)-2\left|f''(a)\right|\left(1-\left|a\right|^2\right)-2\left|f''(0)\right|}{\left|a\right|^2+4\left|f''(0)\right|\left|f''(a)\right|\left(1-\left|a\right|^2\right)+2\left|f''(a)\right|\left(1-\left|a\right|^2\right)+2\left|f''(0)\right|},$$

we obtain

$$\left|p'(1)\right|\geq 1+\frac{1-\left|a\right|^2}{\left|1-a\right|^2}+\frac{\left|a\right|-2\left|f''(0)\right|}{\left|a\right|+2\left|f''(0)\right|}$$

$$\times\left[1+\frac{\left|a\right|^2+4\left|f''(0)\right|\left|f''(a)\right|\left(1-\left|a\right|^2\right)-2\left|f''(a)\right|\left(1-\left|a\right|^2\right)-2\left|f''(0)\right|}{\left|a\right|^2+4\left|f''(0)\right|\left|f''(a)\right|\left(1-\left|a\right|^2\right)+2\left|f''(a)\right|\left(1-\left|a\right|^2\right)+2\left|f''(0)\right|}\frac{1-\left|a\right|^2}{\left|1-a\right|^2}\right].$$

From the definition of $p(z)$, we have

$$\left|p'(1)\right| = \frac{2\left|f''(1)\right|}{\sinh 1}.$$

We thus obtain inequality (6).

Let a be any real number in the interval $(-1, 0)$, and let c and d be arbitrary real numbers such that $0 < c = |p'(0)| < |a|$, $0 < d = |p'(a)| < \frac{|a|}{\left(1-|a|^2\right)}$ to show that inequality (6) is sharp. Let

$$\mathbb{T} = \frac{\frac{c}{a} + \frac{d\left(1-|a|^2\right)}{a}}{a\left(1 + cd\frac{1-|a|^2}{a^2}\right)} = \frac{1}{a^2}\frac{d\left(1-|a|^2\right)+c}{1+cd\frac{1-|a|^2}{a^2}}.$$

Consider the function

$$p(z) = z\frac{z-a}{1-\overline{a}z}\frac{\frac{-c}{a} + z\frac{\mathbb{T}+\rho(z)}{1+\rho(z)\mathbb{T}}}{1-\frac{c}{a}z\frac{\mathbb{T}+\rho(z)}{1+\rho(z)\mathbb{T}}}. \tag{10}$$

From equation (10), after performing straightforward calculations, we derive $p'(0) = c$ and $p'(a) = d$. Therefore, we obtain

$$p'(1) = 1 + \frac{1-a^2}{(1-a)^2} + \frac{a+c}{a-c}\left(1 + \frac{a^2+cd\left(1-|a|^2\right)-d\left(1-|a|^2\right)-c}{a^2+cd\left(1-|a|^2\right)+d\left(1-|a|^2\right)+c}\frac{1-a^2}{(1-a)^2}\right).$$

By selecting appropriate signs for the numbers a, c and d, we can infer from the final equation that inequality (6) is sharp. □

2 Conclusions and discussion

In this paper, geometric properties of a specific subclass of analytic functions satisfying the condition $f'(z) \prec cosh\sqrt{z}$ are investigated. Considering the Schwarz lemma and the boundary Schwarz lemma, significant results on distortion and growth behaviours of these functions have been obtained. Accordingly, two extremal functions have been based on the results of theorems presented in this paper.

The extremal functions obtained in this paper have been considered as activation functions for artificial neural networks. There are already studies in the literature that examine the use of extremal functions as activation functions [11, 12]. In [11], the authors propose a complex-valued activation function obtained using the Schwarz lemma. The authors stated that effective results have been obtained in both classification and function approximation problems according to simulation results. In [12], similar functions obtained in this study are presented as activation functions.

There are also various studies that propose hyperbolic functions to be used as activation functions, which is also valid for our study [13–15]. In one of the recent studies, hyperbolic sine has been used for deep learning in Tensorflow and Keras [13]. In [14], hyper-sinh-convolutional network has been proposed for early detection of Parkinson's disease from spiral drawings. Husein et al. used a hyperbolic activation function to achieve effective instance image retrieval [15]. In our study, we present two hyperbolic activation functions, defined as $g(z) = sinhz$ and $q(z) = 2\sqrt{z}sin\sqrt{z} - 2\cosh\sqrt{z} + 2$. At this point, it is worth noting that the activation functions defined in our study are not arbitrarily selected but they emerge as intuitive outcomes of the problem addressed in this study.

In conclusion, this paper aims to strengthen the connection between complex analysis and artificial intelligence in this paper by introducing the use of extremal functions within neural network architectures. We consider that the obtained results show that mathematical findings from geometric function theory can inspire new directions in neural network research. Empirical evaluation of the new activation functions across various learning tasks and architectures to fully assess their practical impact and limitations can be considered as potential future work.

Author Contributions

T. Azeroğlu and B. N. Örnek were responsible for developing the theoretical framework that underpins the study. T. Düzenli contributed significantly to preparation of the manuscript and provided valuable input to the Conclusions and Discussion section. All authors jointly revised the manuscript and approved the final submission.

Conflict of Interest

The authors declare no conflict of interest.

References

1 Goluzin, G.M. (1969). *Geometricheskaia teoriia funktsii kompleksnoi peremennoi [Geometric theory of functions of a complex variable]*. 2nd ed. Moscow: Nauka [in Russian].

2 Mundalia, M., & Kumar, S.S. (2023). On a subfamily of starlike functions related to hyperbolic cosine function. *The Journal of Analysis, 31*(3), 2043–2062. https://doi.org/10.1007/s41478-023-00550-1

3 Osserman, R. (2000). A sharp Schwarz inequality on the boundary. *Proceedings of the American Mathematical Society, 128*, 3513–3517. https://doi.org/10.1090/S0002-9939-00-05463-0

4 Unkelbach, H. (1938). Über die Randverzerrung bei konformer Abbildung. *Mathematische Zeitschrift, 43*, 739–742. https://doi.org/10.1007/BF01181115

5 Azeroğlu, T.A., & Örnek, B.N. (2013). A refined Schwarz inequality on the boundary. *Complex Variables and Elliptic Equations, 58*(4), 571–577. https://doi.org/10.1080/17476933.2012.718338

6 Boas, H.P. (2010). Julius and Julia: Mastering the Art of the Schwarz lemma. *The American Mathematical Monthly, 117*(9), 770–785. https://doi.org/10.4169/000298910X521643

7 Dubinin, V.N. (2004). The Schwarz inequality on the boundary for functions regular in the disk. *Journal of Mathematical Sciences, 122*(6), 3623–3629. https://doi.org/10.1023/B:JOTH.0000035237.43977.39

8 Mateljević, M., Mutavdžić, N., & Örnek, B.N. (2022). Note on some classes of holomorphic functions related to Jack's and Schwarz's lemma. *Applicable Analysis and Discrete Mathematics, 16*(1), 111–131. https://doi.org/10.2298/AADM200319006M

9 Mercer, P.R. (2018). An improved Schwarz lemma at the boundary. *Open Mathematics, 16*(1), 1140–1144. https://doi.org/10.1515/math-2018-0096

10 Pommerenke, C. (1992) *Boundary Behaviour of Conformal Maps.* Berlin: Springer–Verlag. https://doi.org/10.1007/978-3-662-02770-7

11 Örnek, B.N., Aydemir, S.B., Düzenli, T., & Özak, B. (2022). Some remarks on activation function design in complex extreme learning using Schwarz lemma. *Neurocomputing, 492*, 23-33. https://doi.org/10.1016/j.neucom.2022.04.010

12 Kunc, V. (2024). *Gene expression inference using artificial neural networks* (PhD Thesis). Czech Technical University, Czech Republic.

13 Parisi, L., Ma, R., RaviChandran, N., & Lanzillotta, M. (2021). Hyper-sinh: An accurate and reliable function from shallow to deep learning in TensorFlow and Keras. *Machine Learning with Applications, 6*, Article 100112. https://doi.org/10.1016/j.mlwa.2021.100112

14 Parisi, L., Zaernia, A., Ma, R., & Youseffi, M. (2021). Hyper-sinh–Convolutional Neural Network for early detection of Parkinson's disease from spiral drawings. *WSEAS Transactions on Computer Research, 9*(1), 90–97. https://doi.org/10.37394/232018.2021.9.1

15 Husain, S.S., Ong, E.-J., & Bober, M. (2021). ACTNET: End-to-end learning of feature activations and multi-stream aggregation for effective instance image retrieval. *International Journal of Computer Vision, 129*, 1432–1450. https://doi.org/10.1007/s11263-021-01444-0

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