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Bounded solutions in epidemic models governed by semilinear parabolic equations with general semilinear incidence rates

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The transmission mechanisms of most infectious diseases are generally well understood from an epidemiological standpoint. To mathematically and quantitatively characterize the spread of these diseases, various classical epidemic models-such as the SIR, SIS, SEIR, and SIRS frameworks-have been formulated and thoroughly investigated. In the present paper, the initial value problem for the system of semilinear parabolic differential equations arising in epidemic models with a general semilinear incidence rate in a Hilbert space with a self-adjoint positive definite operator is investigated. The main theorem on the existence and uniqueness of bounded solutions for this system is established. In applications, theorems on the existence and uniqueness of bounded solutions for two types of systems of semilinear partial differential equations arising in epidemic models are proved. A first-order accurate finite difference scheme is developed to construct approximate solutions for this system. We further prove a theorem that guarantees the existence and uniqueness of bounded solutions for the discrete problem, independently of the time step. The theoretical results are supported by applications, where bounded solutions of the continuous system and their corresponding discrete approximations are demonstrated. Finally, numerical results are presented to illustrate the effectiveness and accuracy of the proposed scheme.

Keywords: system of semilinear partial differential equations(SPDEs), EM, bounded solution(BS), numerical results, Hilbert space, self-adjoint positive definite operator,existence and uniqueness (EU), difference scheme(DS).

2020 Mathematics Subject Classification: 58J35, 58D25, 65M12, 92B05, 35K61, 35K58, 35K90, 91B76.

Introduction

The mechanism of disease transmission is typically well understood from an epidemiological perspective for most infectious diseases. To describe mathematically and quantitatively the spread of such diseases, numerous classical EMs have been developed and extensively studied, including the SIR, SIS, SEIR, and SIRS models [1–3].

In particular, the studies presented in [1] focus on the numerical solution of systems of linear parabolic equations modeling the transmission of HIV from mother to child. Numerical simulations were provided to support the theoretical results.

In the papers [4–6], the authors study a diffusive SIR epidemic model with nonlinear incidence in a heterogeneous environment. They establish the boundedness and uniform persistence of solutions to the system, as well as the global stability of the constant endemic equilibrium in the case of a homogeneous environment.

The papers [7, 8] study the dynamical behavior of a diffusive epidemic SIRI system with distinct dispersal rates. The overall solution of the system is derived using L_p theory and Young's inequality.

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The uniform boundedness of the solution is established, and the asymptotic smoothness of the semi-flow, as well as the existence of a global attractor, is discussed.

Finally, the papers [9,10] focus on a reaction-diffusion SEIR (susceptible-exposed-infected-recovered) epidemic model with a mass-action infection mechanism. The basic reproduction number of the SEIR model is defined, and its properties are studied under conditions of low mobility of the susceptible and exposed/infected populations, respectively. In a homogeneous environment, the global asymptotic stability of both the disease-free equilibrium and the endemic equilibrium is determined by the basic reproduction number. Furthermore, the asymptotic behavior of the endemic equilibrium (when it exists) is analyzed in a spatially heterogeneous environment with low migration rates of the susceptible, exposed, or infected populations.

Various classes of mixed boundary value problems for systems of partial differential equations can be transformed into initial value problems for systems of semilinear ordinary differential equations [11,12]

$$\begin{cases} \frac{dw^1(t)}{dt} + \mu w^1(t) + Aw^1(t) = -F(t, w^1(t), w^2(t)), \\ \frac{dw^2(t)}{dt} + (\xi + \mu) w^2(t) + Aw^2(t) = F(t, w^1(t), w^2(t)) - \Upsilon(t, w^2(t)), \\ \frac{dw^3(t)}{dt} + \mu w^3(t) + Aw^3(t) = \Upsilon(t, w^2(t)), \ t \in (0, b), \\ w^n(0) = \psi^n, \ n = \overline{1,3} \end{cases} \tag{1}$$

in a Hilbert space $\aleph$ with an unbounded elliptic operator A.

Throughout this paper, a theorem on the EU of BSs to the abstract problem (1) is proved. The results are illustrated by their application to a system of semilinear parabolic equations, demonstrating their effectiveness in both one- and multi-dimensional settings with appropriate boundary conditions. Furthermore, a discrete analogue of the theoretical results is developed for a first-order accurate time-difference scheme. Numerical simulations are included to illustrate and validate the theoretical results.

1 *BS of the differential problem (1)*

Let $\aleph$ be a Hilbert space, and let A be a positive definite self-adjoint operator such that $A \geq \delta I$ for some $\delta > 0$. Throughout this paper, the family $\{\exp(-tA), \ t \geq 0\}$ denotes the strongly continuous exponential operator-function.

By applying the spectral representation of a self-adjoint positive definite operator in a Hilbert space, we obtain the following estimate:

$$\|\exp(-tA)\|_{\aleph \to \aleph} \leq e^{-\delta t}, \quad t \geq 0. \tag{2}$$

A vector-valued function $w(t) = \left(w^1(t), w^2(t), w^3(t)\right)^T$ is said to be a solution of problem (1) if the following conditions are satisfied:

(i) For each $m \in \{1, 2, 3\}$, $w^m(t)$ is a continuously differentiable function on the interval $(0, b)$.

(ii) For all $t \in [0, b]$ and each $m = \overline{1,3}$, the element $w^m(t)$ belongs to the domain $D(A)$ of the operator A, and the function $Aw^m(t)$ is continuous on $[0, b]$.

(iii) The functions $F(t, w^1(t), w^2(t))$ and $\Upsilon(t, w^2(t))$ are continuous for all $t \in [0, b]$.

(iv) The function $w(t)$ satisfies the system of equations and initial conditions given in (1).

The proof of the main theorem regarding the EU of a BS of problem (1) is based on reducing the problem to an equivalent system of integral equations

$$\begin{cases} w^1(t) = e^{-\mu t}e^{-At}\psi^1 - \int\limits_0^t e^{-\mu(t-\lambda)}e^{-A(t-\lambda)}F\left(\lambda, w^1(\lambda), w^2(\lambda)\right)d\lambda, \\ w^2(t) = e^{-(\mu+\xi)t}e^{-At}\psi^2 + \int\limits_0^t e^{-(\mu+\xi)(t-\lambda)}e^{-A(t-\lambda)}F\left(\lambda, w^1(\lambda), w^2(\lambda)\right)d\lambda \\ -\int\limits_0^t e^{-(\mu+\xi)(t-\lambda)}e^{-A(t-\lambda)}\Upsilon\left(\lambda, w^2(\lambda)\right)d\lambda, \\ w^3(t) = e^{-\mu t}e^{-At}\psi^3 + \int\limits_0^t e^{-\mu(t-\lambda)}e^{-A(t-\lambda)}\Upsilon\left(\lambda, w^2(\lambda)\right)d\lambda \end{cases} \tag{3}$$

in $C(\aleph)$ and the use of successive approximations. Here, $C(\aleph)$ stands for the Banach space of the continuous functions $z(t)$ defined on $[0,b]$ with values in $\aleph$, equipped with the norm

$$\|z\|_C = \max_{t\in[0,b]} \|z(t)\|_\aleph .$$

We introduce the equivalent norm

$$\| z \|_{C_L} = \max_{t\in[0,b]} e^{-Lt}\|z(t)\|_\aleph , \quad L > 0.$$

The recursive formula for the solution of problem (3) is

$$\begin{cases} nw^1(t) = e^{-\mu t}e^{-At}\psi^1 - \int\limits_0^t e^{-\mu(t-\lambda)}e^{-A(t-\lambda)}F\left(\lambda, (n-1)w^1(\lambda), (n-1)w^2(\lambda)\right)d\lambda, \\ nw^2(t) = e^{-(\mu+\xi)t}e^{-At}\psi^2 \\ +\int\limits_0^t e^{-(\mu+\xi)(t-\lambda)}e^{-A(t-\lambda)}F\left(\lambda, (n-1)w^1(\lambda), (n-1)w^2(\lambda)\right)d\lambda \\ -\int\limits_0^t e^{-(\mu+\xi)(t-\lambda)}e^{-A(t-\lambda)}\Upsilon\left(\lambda, (n-1)u^2(\lambda)\right)d\lambda, \\ nw^3(t) = e^{-\mu t}e^{-At}\psi^3 + \int\limits_0^t e^{-\mu(t-\lambda)}e^{-A(t-\lambda)}\Upsilon\left(\lambda, (n-1)w^2(\lambda)\right)d\lambda, \ n = 1, 2, ..., \\ 0w^m(t), \ m = \overline{1,3} \quad \text{are given.} \end{cases} \tag{4}$$

Theorem 1. Assume the following conditions are satisfied:

1. For each $m = \overline{1,3}$, the initial function ψ^m belong to the domain $D(A)$ of the operator A, and
$$\|\psi^m\|_{D(A)} = M_1. \tag{5}$$
2. The function $F : [0,b] \times \aleph \times \aleph \to \aleph$ is continuous and satisfies the uniform bound:
$$\|F(t, w(t), u(t))\|_\aleph \le M_2, \tag{6}$$
for all $(t, w, u) \in [0,b] \times \aleph \times \aleph$. In addition, the mapping F fulfills a Lipschitz condition that holds uniformly in t:
$$\|F(t, w, u) - F(t, z, v)\|_\aleph \le L_1 \left(\|w - z\|_\aleph + \|u - v\|_\aleph\right). \tag{7}$$
3. The function $\Upsilon : [0,b] \times \aleph \to \aleph$ is uniformly Lipschitz continuous w.r.t. the variable t:
$$\|\Upsilon(t, w(t))\|_\aleph \le M_3 \tag{8}$$
for all $(t, w) \in [0,b] \times \aleph$. In addition, Υ satisfies a Lipschitz condition uniformly w.r.t. t:
$$\|\Upsilon(t, w) - \Upsilon(t, z)\|_\aleph \le L_2\|w - z\|_\aleph. \tag{9}$$

Here, L_r for $r=1,2$ and M_r for $r=1,2,3$ are positive constants. Then, under these assumptions, there exists a unique solution $w(t)=\left(w^1(t),w^2(t),w^3(t)\right)^T$ of the problem (1), which is bounded in the product space $\mathbb{C}^3(\aleph)=C(\aleph)\times C(\aleph)\times C(\aleph)$.

Proof. Since $w^3(t)$ does not appear in equations for $\frac{dw^1(t)}{dt}$ and $\frac{dw^2(t)}{dt}$, it is sufficient to analyze the behaviors of solutions $w^1(t)$ and $w^2(t)$ of (1) in the norm of the space $C_L(\aleph)$.

According to the method of recursive approximation (4), we get

$$w^m(t)=0w^m(t)+\sum_{i=0}^{\infty}\left[(i+1)w^m(t)-iw^m(t)\right],\quad m=1,2, \tag{10}$$

where

$$0w^m(t)=\begin{cases} e^{-\mu t}e^{-At}\psi^1, & m=1,3,\\ e^{-(\mu+\xi)t}e^{-At}\psi^2, & m=2.\end{cases}$$

Using formula (4) and estimates (2), (5), (6) and (8), we obtain

$$e^{-Lt}\|1w^1(t)-0u^1(t)\|_\aleph$$

$$\leq\int_0^t\exp\left(-(\mu+L)(t-\lambda)\right)\|e^{-A(t-\lambda)}\|\|F\left(\lambda,0w^1(\lambda),0w^2(\lambda)\right)\|_\aleph d\lambda\leq\frac{M_2}{\mu+L},$$

$$e^{-Lt}\|1w^2(t)-0w^2(t)\|_\aleph$$

$$\leq\int_0^t\exp\left(-(\mu+\xi+L)(t-\lambda)\right)\|e^{-A(t-\lambda)}\|\left[\|F\left(\lambda,0w^1(\lambda),0w^2(\lambda)\right)\|_\aleph+\|\Upsilon\left(\lambda,0w^2(\lambda)\right)\|_\aleph\right]d\lambda$$

$$\leq\frac{M_2+M_3}{\mu+\xi+L}$$

for any $t\in[0,b]$. Using the triangle inequality, we get

$$e^{-Lt}\|1w^1(t)\|_\aleph\leq M_1+\frac{M_2+M_3}{\mu+L},\quad e^{-Lt}\|1w^2(t)\|_\aleph\leq M_1+\frac{M_2+M_3}{\mu+L}$$

for any $t\in[0,b]$. Using formula (4) and estimates (2), (6), (7) and (9), we obtain

$$e^{-Lt}\|2w^1(t)-1w^1(t)\|_\aleph$$

$$\leq\int_0^t\exp\left(-(\mu+L)(t-\lambda)\right)e^{-L\lambda}\|e^{-A(t-\lambda)}\|\|F\left(\lambda,1w^1(\lambda),1w^2(\lambda)\right)-F\left(\lambda,0w^1(\lambda),0w^2(\lambda)\right)\|_\aleph d\lambda$$

$$\leq\frac{2L_1\left(M_2+M_3\right)}{(\mu+L)^2}\leq\frac{2\left(L_1+L_2\right)\left(M_2+M_3\right)}{(\mu+L)^2},$$

$$e^{-Lt}\|2w^2(t)-1w^2(t)\|_\aleph$$

$$\leq\int_0^t\exp\left(-(\mu+\xi+L)(t-\lambda)\right)e^{-L\lambda}\|e^{-A(t-\lambda)}\|\|F\left(\lambda,1w^1(\lambda),1w^2(\lambda)\right)-F\left(\lambda,0w^1(\lambda),0w^2(\lambda)\right)\|_\aleph d\lambda$$

$$+\int_0^t\exp\left(-(\mu+\xi+L)(t-\lambda)\right)e^{-L\lambda}\|e^{-A(t-\lambda)}\|\|\Upsilon\left(\lambda,1w^2(\lambda)\right)-\Upsilon\left(\lambda,0w^2(\lambda)\right)\|_\aleph d\lambda$$

$$\leq\frac{\left(2L_1+L_2\right)\left(M_2+M_3\right)}{(\mu+L)^2}\leq\frac{2\left(L_1+L_2\right)\left(M_2+M_3\right)}{(\mu+L)^2}$$

for any $t \in [0, b]$. Then,

$$e^{-Lt}\|2w^1(t)\|_{\aleph} \leq M_1 + \frac{M_2 + M_3}{\mu + L} + \frac{2\left(L_1 + L_2\right)\left(M_2 + M_3\right)}{\left(\mu + L\right)^2},$$

$$e^{-Lt}\|2w^2(t)\|_{\aleph} \leq M_1 + \frac{M_2 + M_3}{\mu + L} + \frac{2\left(L_1 + L_2\right)\left(M_2 + M_3\right)}{\left(\mu + L\right)^2}$$

for any $t \in [0, b]$. Let

$$e^{-Lt}\|nw^m(t) - (n-1)\, w^m(t)\|_{\aleph} \leq \frac{2^{n-1}\left(L_1 + L_2\right)^{n-1}\left(M_2 + M_3\right)}{\left(\mu + L\right)^n}, \quad m = 1, 2.$$

Thus, we arrive at

$$e^{-Lt}\|\left(n+1\right) w^1(t) - nw^1(t)\|_{\aleph}$$

$$\leq \int_0^t e^{-(\mu+L)(t-\lambda)} e^{-L\lambda} \|e^{-A(t-\lambda)}\| \|\digamma\left(\lambda, nw^1(\lambda), nu^2(\lambda)\right) - \digamma\left(\lambda, (n-1)\, w^1(\lambda), (n-1)\, w^2(\lambda)\right)\|_{\aleph} d\lambda$$

$$\leq \frac{2L_1 2^{n-1}\left(L_1 + L_2\right)^{n-1}\left(M_2 + M_3\right)}{\left(\mu + L\right)^{n+1}} \leq \frac{2^n\left(L_1 + L_2\right)^n\left(M_2 + M_3\right)}{\left(\mu + L\right)^{n+1}},$$

$$e^{-Lt}\|\left(n+1\right) w^2(t) - nw^2(t)\|_{\aleph}$$

$$\leq \int_0^t e^{-(\mu+\xi+L)(t-\lambda)} e^{-L\lambda} \|e^{-A(t-\lambda)}\| \|\digamma\left(\lambda, 1w^1(\lambda), 1w^2(\lambda)\right) - \digamma\left(\lambda, 0w^1(\lambda), 0w^2(\lambda)\right)\|_{\aleph} d\lambda$$

$$\leq \frac{\left(2L_1 + L_2\right) 2^{n-1}\left(L_1 + L_2\right)^{n-1}\left(M_2 + M_3\right)}{\left(\mu + L\right)^{n+1}} \leq \frac{2^n\left(L_1 + L_2\right)^n\left(M_2 + M_3\right)}{\left(\mu + L\right)^{n+1}}$$

for any $t \in [0, b]$. Then,

$$e^{-Lt}\|\left(n+1\right) w^m(t)\|_{\aleph}$$

$$\leq M_1 + \frac{M_2 + M_3}{\mu + L} + \frac{2\left(L_1 + L_2\right)\left(M_2 + M_3\right)}{\left(\mu + L\right)^2} + \cdots + \frac{2^n\left(L_1 + L_2\right)^n\left(M_2 + M_3\right)}{\left(\mu + L\right)^{n+1}}, \quad m = 1, 2$$

for each $t \in [0, b]$. Then, for any n, $n \geq 1$, we have

$$e^{-Lt}\|\left(n+1\right) w^1(t) - nw^1(t)\|_{\aleph} \leq \frac{2^n\left(L_1 + L_2\right)^n\left(M_2 + M_3\right)}{\left(\mu + L\right)^{n+1}}, \quad m = 1, 2,$$

and

$$e^{-Lt}\|\left(n+1\right) w^m(t)\|_{\aleph}$$

$$\leq M_1 + \frac{M_2 + M_3}{\mu + L} + \frac{2\left(L_1 + L_2\right)\left(M_2 + M_3\right)}{\left(\mu + L\right)^2} + \cdots + \frac{2^n\left(L_1 + L_2\right)^n\left(M_2 + M_3\right)}{\left(\mu + L\right)^{n+1}}, \quad m = 1, 2$$

by induction. It follows from this and formula (10) that

$$e^{-Lt}\|w^m(t)\|_{\aleph} \leq \|0w^m(t)\|_{\aleph} + \sum_{i=0}^{\infty} e^{-Lt}\|(i+1)w^m(t) - iw^m(t)\|_{\aleph}$$

$$\leq M_1 + \sum_{i=0}^{\infty} \frac{2^i\left(L_1 + L_2\right)^i\left(M_2 + M_3\right)}{\left(\mu + L\right)^{i+1}}, \quad m = 1, 2$$

which proves the existence of a BS of problem (1) in norm $C_L\left([0, b], \aleph\right)$. From this, it follows the existence of a BS of problem (1) in norm $C\left([0, b], \aleph\right)$. Theorem 1 is proved.

Now, consider the applications of Theorem 1.

First, we investigate the initial-boundary value problem for one-dimensional system of SPDEs

$$\left\{\begin{array}{l} \frac{\partial\Psi^1(t,z)}{\partial t}-\left(a(z)\Psi^1_z(t,z)\right)_z-\beta\left(a(-z)\Psi_z(t,-z)\right)_z+(\delta+\mu)\Psi^1(t,z)\\ =-F(t,z;\Psi^1(t,z),\Psi^2(t,z)),\\ \frac{\partial\Psi^2(t,z)}{\partial t}-\left(a(z)\Psi^2_z(t,z)\right)_z-\beta\left(a(-z)\Psi_z(t,-z)\right)_z+(\delta+\mu+\xi)\Psi^2(t,z)\\ =F(t,z;\Psi^1(t,z),\Psi^2(t,z))-\Upsilon(t,z;\Psi^2(t,z)),\\ \frac{\partial\Psi^3(t,z)}{\partial t}-\left(a(z)\Psi^3_z(t,z)\right)_z-\beta\left(a(-z)\Psi_z(t,-z)\right)_z+(\delta+\mu)\Psi^3(t,z)\\ =\Upsilon(t,z;\Psi^2(t,z)),\ t\in(0,b),\ -d<z<d,\\ \Psi^m(t,\pm d)=0,\ t\in[0,b],\ m=\overline{1,3},\\ \Psi^m(0,z)=\psi^m(z),\ \psi^m(\pm d)=0,\ z\in[-d,d],\ m=\overline{1,3}, \end{array}\right. \tag{11}$$

where $a(z)$ and $\psi(z)$ are given sufficiently smooth functions. Here, $\delta>0$ is a sufficiently large number. We will suppose that $a\geq a(z)=a(-z)\geq\delta>0,\ \delta-a|\beta|\geq 0$.

Theorem 2. Suppose the following conditions are satisfied:

1. For each $m=\overline{1,3}$, the initial function ψ^m belongs to the Sobolev space $W_2^2[-d,d]$, and

$$\|\psi^m\|_{W_2^2[-d,d]}\leq M_1.$$

2. The function

$$F:[0,b]\times[-d,d]\times L_2[-d,d]\times L_2[-d,d]\to L_2[-d,d]$$

is continuous in the time variable t and satisfies the uniform bound

$$\|F(t,\cdot,w(t,\cdot),u(t,\cdot))\|_{L_2[-d,d]}\leq M_2$$

for all $(t,\cdot,w,u)\in[0,b]\times L_2[-d,d]\times L_2[-d,d]$. Moreover, F satisfies a Lipschitz condition uniformly in t:

$$\|F(t,\cdot,w,u)-F(t,\cdot,p,q)\|_{L_2[-d,d]}\leq L_1\left(\|w-p\|_{L_2[-d,d]}+\|u-q\|_{L_2[-d,d]}\right).$$

3. The function

$$\Upsilon:[0,b]\times[-d,d]\times L_2[-d,d]\to L_2[-d,d]$$

is continuous in t and satisfies the uniform bound:

$$\|\Upsilon(t,\cdot,w(t,\cdot))\|_{L_2[-d,d]}\leq M_3$$

for all $(t,w)\in[0,b]\times L_2[-d,d]$. Additionally, Υ satisfies a Lipschitz condition uniformly in t:

$$\|\Upsilon(t,\cdot,w)-\Upsilon(t,\cdot,u)\|_{L_2[-d,d]}\leq L_2\|w-u\|_{L_2[-d,d]}.$$

Here and in the sequel, the constants L_m (for $m=1,2$) and M_m (for $m=\overline{1,3}$) are assumed to be positive.

Then, under the above assumptions, there exists a unique solution $\Psi(t,z)= =\left(\Psi^1(t,z),\Psi^2(t,z),\Psi^3(t,z)\right)^T$ to problem (11), which is bounded in the space $\mathbb{C}^3(L_2[-d,d])= C(L_2[-d,d])\times C(L_2[-d,d])\times C(L_2[-d,d])$.

The proof of Theorem 2 is based on the abstract Theorem 1, the self-adjointness and positivity in $L_2[-d,d]$ of a differential operator A^z defined by the formula

$$A^z\omega(z)=-\left(a(z)\omega_z(z)\right)_z-\beta\left(a(-z)\omega_z(-z)\right)_z+\delta\omega(z)$$

with the domain $D(A^z) = \left\{\omega \in W_2^2\left[-d,d\right] : \omega\left(-d\right) = \omega\left(d\right) = 0\right\}$ [13] and on the estimate

$$\|\exp\{-tA^z\}\|_{L_2[-d,d]\to L_2[-d,d]} \le 1, \ \ t \ge 0.$$

Second, we study the initial-boundary value problem for a multidimensional system of SPDEs

$$\begin{cases} \frac{\partial \Psi^1(t,z)}{\partial t} - \sum\limits_{r=1}^{n}(a_r(z)\Psi^1_{z_r})z_r + (\delta+\mu)\,\Psi^1(t,z) = -F(t,z;\Psi^1(t,z),\Psi^2(t,z)), \\ \frac{\partial \Psi^2(t,z)}{\partial t} - \sum\limits_{r=1}^{n}(a_r(z)\Psi^2_{z_r})z_r + (\delta+\mu+\xi)\,\Psi^2(t,z) \\ = F(t,z;\Psi^1(t,z),\Psi^2(t,z)) - \Upsilon(t,z;\Psi^2(t,z)), \\ \frac{\partial \Psi^3(t,z)}{\partial t} - \sum\limits_{r=1}^{n}(a_r(z)\Psi^3_{z_r})z_r + (\delta+\mu)\,\Psi^3(t,z) \\ = \Upsilon(t,z;\Psi^2(t,z)),\ t\in(0,b),\ z=(z_1,...,z_n)\in\Omega, \\ \Psi^p(0,z) = \psi^p(z),\ z\in\overline{\Omega},\ p=\overline{1,3}, \\ \Psi^p(t,z) = 0,\ t\in[0,b],\ z\in S,\ p=\overline{1,3}, \end{cases} \tag{12}$$

where $a_r(z)$ and $\psi^p(z)$ are given sufficiently smooth functions and $\delta > 0$ is a sufficiently large number and $a_r(z) > 0$. Here, $\Omega \subset R^n$ is an open and bounded domain whose boundary S is smooth and we put $\overline{\Omega} = \Omega \cup S$.

Theorem 3. Suppose the following conditions are satisfied:

1. For each $m = \overline{1,3}$, the initial function ψ^m belongs to the Sobolev space $W_2^2(\overline{\Omega})$, and
$$\|\psi^m\|_{W_2^2(\overline{\Omega})} \le M_1.$$
2. The function
$$F : [0,b]\times[0,l]\times L_2(\overline{\Omega})\times L_2(\overline{\Omega}) \to L_2(\overline{\Omega})$$
is continuous in the time variable t, and satisfies the uniform bound:
$$\|F(t,\cdot,w(t,\cdot),u(t,\cdot))\|_{L_2(\overline{\Omega})} \le M_2$$
for all $(t,w,u) \in [0,b]\times L_2(\overline{\Omega})\times L_2(\overline{\Omega})$. Moreover, F satisfies a Lipschitz condition uniformly in t:
$$\|F(t,\cdot,w,u) - F(t,\cdot,p,q)\|_{L_2(\overline{\Omega})} \le L_1\left(\|w-p\|_{L_2(\overline{\Omega})} + \|u-q\|_{L_2(\overline{\Omega})}\right).$$
3. The function
$$\Upsilon : [0,b]\times[0,l]\times L_2(\overline{\Omega}) \to L_2(\overline{\Omega})$$
is continuous in t, and satisfies the uniform bound:
$$\|\Upsilon(t,\cdot,w(t,\cdot))\|_{L_2(\overline{\Omega})} \le M_3$$
for all $(t,w) \in [0,b]\times L_2(\overline{\Omega})$. Υ satisfies a Lipschitz condition uniformly in t:
$$\|\Upsilon(t,\cdot,w) - \Upsilon(t,\cdot,p)\|_{L_2(\overline{\Omega})} \le L_2\,\|w-p\|_{L_2(\overline{\Omega})}.$$

Then, under the above assumptions, there exists a unique solution $\Psi(t,z) = \left(\Psi^1(t,z),\Psi^2(t,z),\Psi^3(t,z)\right)^T$ to problem (12), which is bounded in the space $\mathbb{C}^3(L_2(\overline{\Omega})) = C\left(L_2(\overline{\Omega})\right)\times C\left(_2(\overline{\Omega})\right)\times C\left(L_2(\overline{\Omega})\right)$.

The proof of Theorem 3 is based on the abstract Theorem 1, on the self-adjointness and positivity in $L_2(\overline{\Omega})$ of a differential operator A^z defined by the formula

$$A^z\Omega(z) = -\sum_{r=1}^{n}(a_r(z)\Omega_{z_r})_{z_r} + \delta\Omega(z)$$

with domain [12]

$$D(A^z) = \left\{\omega(z) : \omega(z), (a_r(z)\omega_{z_r})_{z_r} \in L_2(\overline{\Omega}),\ 1 \le r \le n,\ \omega(z) = 0,\ z \in S\right\}$$

and on the estimate

$$\| \exp\{-tA^z\}\|_{L_2(\overline{\Omega}) \to L_2(\overline{\Omega})} \le 1,\ t \in [0, \infty)$$

and the following theorem on coercivity inequality for the solution of the elliptic problem in $L_2(\overline{\Omega})$ [12].

2 BS of the difference scheme

For the approximate solution of (1) we consider a grid space

$$[0, b]_\tau = \left\{t_k = k\tau,\ k = \overline{1, N},\ N\tau = b\right\}.$$

We consider the first order of accuracy difference scheme

$$\begin{cases} \frac{u_k^1 - u_{k-1}^1}{\tau} + \mu u_k^1 + Au_k^1 = -F(t_k, u_k^1, u_k^2), \\ \frac{u_k^2 - u_{k-1}^2}{\tau} + (\xi + \mu)\, u_k^2 + Au_k^2 = F(t_k, u_k^1, u_k^2) - \Upsilon(t_k, u_k^2), \\ \frac{u_k^3 - u_{k-1}^3}{\tau} + \mu u_k^3 + Au_k^3 = \Upsilon(t_k, u_k^2),\ k = \overline{1, N}, \\ u_0^p = \psi^p,\ p = \overline{1, 3} \end{cases} \tag{13}$$

for the approximate solution of problem (1). The proof method for the basic theorem on the EU of a BS of difference scheme (13) uniformly w.r.t. τ is based on reducing (13) to an equivalent system of semilinear difference equations. Actually, an equivalent system of semilinear difference equations for (13) is

$$\begin{cases} u_k^1 = R^k\psi^1 - \sum_{m=1}^{k} R^{k-m+1} F(t_m, u_m^1, u_m^2)\tau, \\ u_k^2 = R_1^k\psi^2 + \sum_{m=1}^{k} R_1^{k-m+1} \left[F(t_m, u_m^1, u_m^2) - \Upsilon(t_m, u_m^2)\right] \tau, \\ u_k^3 = R^k\psi^3 + \sum_{m=1}^{k} R^{k-m+1} \Upsilon(t_m, u_m^2)\tau,\ k = \overline{1, N} \end{cases} \tag{14}$$

in $\mathbb{C}_\tau^3(\aleph) = C_\tau(\aleph) \times C_\tau(\aleph) \times C_\tau(\aleph)$ and the use of successive approximations. Here and in the future $R_1 = (I + \tau\left((\mu + \xi)\, I + A\right))^{-1}$, $R = (I + \tau\left(\mu I + A\right))^{-1}$ and $C_\tau(\aleph)$ stands for the Banach space of mesh functions $w^\tau = \{w_p\}_{p=0}^N$ defined on $[0, b]_\tau$ with values in $\aleph$, equipped with the norm

$$\|w^\tau\|_{C_\tau(\aleph)} = \max_{0 \le p \le N} \|w_p\|_\aleph.$$

The recursive formula for the solution of DS (13) is

$$\begin{cases} \frac{ru_k^1 - ru_{k-1}^1}{\tau} + \mu ru_k^1 + Aru_k^1 = -F(t_k, (r-1)\, u_k^1, (r-1)\, u_k^2), \\ \frac{ru_k^2 - ru_{k-1}^2}{\tau} + (\xi + \mu)\, ru_k^2 + Aru_k^2 = F(t_k, (r-1)\, u_k^1, (r-1)\, u_k^2) - \Upsilon(t_k, (r-1)\, u_k^2), \\ \frac{ru_k^3 - ru_{k-1}^3}{\tau} + \mu ru_k^3 + Aru_k^3 = \Upsilon(t_k, (r-1)\, u_k^2),\ k = \overline{1, N}, \\ ru_0^p = \psi^p,\ p = \overline{1, 3},\ r = 1, 2, ..., \\ 0u_k^p,\ k = \overline{0, N},\ p = \overline{1, 3} \text{ are given.} \end{cases} \tag{15}$$

From (14) and (15) it follows

$$\begin{cases} ru_k^1 = R^k\psi^1 - \sum_{i=1}^{k} R^{k-i+1} F(t_k, (r-1)\, u_k^1, (r-1)\, u_k^2)\tau, \\ ru_k^2 = R_1^k\psi^2 + \sum_{i=1}^{k} R_1^{k-i+1} \left[F(t_k, (r-1)\, u_k^1, (r-1)\, u_k^2) - \Upsilon(t_k, (r-1)\, u_k^2)\right] \tau, \\ ru_k^3 = R^k\psi^3 + \sum_{i=1}^{k} R^{k-m+1} \Upsilon(t_k, (r-1)\, u_k^2)\tau,\ k = \overline{1, N},\ r = 1, 2, ..., \\ 0u_k^p,\ k = \overline{0, N},\ p = \overline{1, 3} \text{ are given.} \end{cases} \tag{16}$$

Theorem 4. Let the assumptions of Theorem 1 be satisfied and

$$\mu + \delta > 2\left(L_1 + L_2\right).$$

Then, there exists a unique BS $u^\tau = \{u_k\}_{k=0}^N$ of difference problem (13) in $\mathbb{C}_\tau^3(\aleph)$ uniformly w.r.t. τ.

Proof. Since u_k^3 does not appear in equations for $\frac{u_k^m - u_{k-1}^m}{\tau}, m = 1, 2$, it is sufficient to analyze the behaviors of solutions u_k^1 and u_k^2 of (13). According to the recursive approximation method (16), we get

$$u_k^m = 0u_k^m + \sum_{i=0}^{\infty}\left[(i+1)u_k^m - iu_k^m\right], \quad m = 1, 2, \tag{17}$$

where

$$0u_k^m = \begin{cases} R^k\psi^m, & m = 1, 3, \\ R_1^k\psi^2, & m = 2. \end{cases} \tag{18}$$

Applying the spectral representation of the self-adjoint positive definite operator in a Hilbert space, we get the following estimates

$$\|R\|_{\aleph\to\aleph} \le \frac{1}{1+\tau\left(\mu+\delta\right)}, \quad \|R_1\|_{\aleph\to\aleph} \le \frac{1}{1+\tau\left(\mu+\delta+\xi\right)}. \tag{19}$$

From formula (18) and estimates (19) it follows that

$$\|0u_k^m\|_\aleph \le \|\psi^m\|_\aleph \le M_1. \tag{20}$$

Using formula (16), estimates (19), we get

$$\|1u_k^1 - 0u_k^1\|_\aleph \le \sum_{m=1}^{k} \|R^{k-m+1}\| \|F(t_m, 0u_m^1, 0u_m^2)\|_\aleph \tau \le \frac{M_2}{\mu+\delta},$$

$$\|1u_k^2 - 0u_k^2\|_\aleph \le \sum_{m=1}^{k} \left\|R_1^{k-m+1}\right\| \left[\|F(t_m, 0u_m^1, 0u_m^2)\|_\aleph + \|\Upsilon(t_m, 0u_m^2)\|_\aleph\right]\tau \le \frac{M_2+M_3}{\mu+\delta+\xi}$$

for any $k = \overline{1, N}$. Using the triangle inequality and estimate (20), we get

$$\|1u_k^1\|_\aleph \le M_1 + \frac{M_2+M_3}{\mu+\delta}, \quad \|1u_k^2\|_\aleph \le M_1 + \frac{M_2+M_3}{\mu+\delta}$$

for each $k = \overline{1, N}$. Using formula (16), estimates (19), (6) and (7), we can write

$$\|2u_k^1 - 1u_k^1\|_\aleph \le \sum_{m=1}^{k} \|R^{k-m+1}\| \|F(t_m, 1u_m^1, 1u_m^2) - F(t_m, 0u_m^1, 0u_m^2)\|_\aleph \tau \le \frac{2\left(L_1+L_2\right)\left(M_2+M_3\right)}{\left(\mu+\delta\right)^2},$$

$$\|2u_k^2 - 1u_k^2\|_\aleph \le \sum_{m=1}^{k} \|R_1^{k-m+1}\| \|F(t_m, 1u_m^1, 1u_m^2) - F(t_m, 0u_m^1, 0u_m^2)\|_\aleph \tau$$

$$+ \sum_{m=1}^{k} \|R_1^{k-m+1}\| \|\Upsilon(t_m, 1u_m^2) - \Upsilon(t_m, 0u_m^2)\|_\aleph \tau \le \frac{2\left(L_1+L_2\right)\left(M_2+M_3\right)}{\left(\mu+\delta+\xi\right)^2}$$

for any $k=\overline{1,N}$. Then,

$$\|2u_k^1\|_{\aleph}\leq M_1+\frac{M_2+M_3}{\mu+\delta}+\frac{2\left(L_1+L_2\right)\left(M_2+M_3\right)}{\left(\mu+\delta\right)^2},$$

$$\|2u_k^2\|_{\aleph}\leq M_1+\frac{M_2+M_3}{\mu+\delta}+\frac{2\left(L_1+L_2\right)\left(M_2+M_3\right)}{\left(\mu+\delta\right)^2}$$

for each $k=\overline{1,N}$. Let

$$\|nu_k^m-(n-1)u_k^m\|_{\aleph}\leq\frac{2^{n-1}\left(L_1+L_2\right)^{n-1}\left(M_2+M_3\right)}{\left(\mu+\delta\right)^n},\quad m=1,2.$$

Using formula (16), estimates (19), (7) and (6), we get

$$\|\left(n+1\right)u_k^1-nu_k^1\|_{\aleph}\leq\sum_{m=1}^{k}\|R^{k-m+1}\|\|F(t_m,nu_m^1,\left(n-1\right)u_m^2)-F(t_m,nu_m^1,\left(n-1\right)u_m^2)\|_{\aleph}\tau$$

$$\leq\frac{\left(2\left(L_1+L_2\right)\right)^n\left(M_2+M_3\right)}{\left(\mu+\delta\right)^{n+1}},$$

$$\|\left(n+1\right)u_k^2-nu_k^2\|_{\aleph}\leq\sum_{m=1}^{k}\|R_1^{k-m+1}\|\|F(t_m,nu_m^1,nu_m^2)-F(t_m,\left(n-1\right)u_m^1,\left(n-1\right)u_m^2)\|_{\aleph}\tau$$

$$+\sum_{m=1}^{k}\|R_1^{k-m+1}\|\|\Upsilon(t_m,nu_m^2)-\Upsilon(t_m,\left(n-1\right)u_m^2)\|_{\aleph}\tau$$

$$\leq\frac{\left(2\left(L_1+L_2\right)\right)^n\left(M_2+M_3\right)}{\left(\mu+\delta+\xi\right)^{n+1}}$$

for each $k=\overline{1,N}$. Then,

$$\|\left(n+1\right)u_k^1\|_{\aleph}\leq M_1+\frac{M_2+M_3}{\mu+\delta}+\frac{2\left(L_1+L_2\right)\left(M_2+M_3\right)}{\left(\mu+\delta\right)^2}+\cdots+\frac{\left(2\left(L_1+L_2\right)\right)^n\left(M_2+M_3\right)}{\left(\mu+\delta+\xi\right)^{n+1}},$$

$$\|\left(n+1\right)u_k^2\|_{\aleph}\leq M_1+\frac{M_2+M_3}{\mu+\delta}+\frac{2\left(L_1+L_2\right)\left(M_2+M_3\right)}{\left(\mu+\delta\right)^2}+\cdots+\frac{\left(2\left(L_1+L_2\right)\right)^n\left(M_2+M_3\right)}{\left(\mu+\delta\right)^{n+1}}$$

for every $k=\overline{1,N}$. Therefore, for any $n,n\geq1$, we have that

$$\|\left(n+1\right)u_k^p-nu_k^p\|_{\aleph}\leq\frac{2^n\left(L_1+L_2\right)^n\left(M_2+M_3\right)}{\left(\mu+\delta\right)^{n+1}},\quad p=1,2,$$

and

$$\left\|\left(n+1\right)u_k^p\right\|_{\aleph}\leq M_1+\sum_{r=0}^{n}\frac{2^r\left(L_1+L_2\right)^r\left(M_2+M_3\right)}{\left(\mu+\delta\right)^{r+1}},\quad p=1,2$$

by induction. From this and formula (17) it follows that

$$\|u_k^p\|_{\aleph}\leq\|0u_k^p\|_{\aleph}+\sum_{r=0}^{\infty}\|(r+1)u_k^p-ru_k^p\|_{\aleph}$$

$$\leq M_1 + \sum_{r=0}^{\infty} \frac{2^r \left(L_1 + L_2\right)^r \left(M_2 + M_3\right)}{\left(\mu + \delta\right)^{r+1}}, \quad p = 1, 2.$$

This proves the existence of a BS of DS (13) that is bounded in $\mathbb{C}^3_\tau(\aleph)$ uniformly w.r.t. τ. Theorem 4 is proved.

Now, let us consider the applications of Theorem 4. First, the mixed problem (11) for one-dimensional system of SPDEs is considered. The discretization of problem (11) is carried out in two steps.

In the first step, we define the grid space as follows:

$$[-d, d]_h = \{z : z_r = rh,\ n = \overline{-K, K},\ Kh = d\}.$$

We introduce the Hilbert spaces $L_{2h} = L_2([-d, d]_h)$ and $W^2_{2h} = W^2_2([-d, d]_h)$ of the grid functions $\psi^h(z) = \{\psi^r\}^K_{-K}$ defined on $[-d, d]_h$, equipped with the norms

$$\left\|\psi^h\right\|_{L_{2h}} = \left(\sum_{z\in[-d,d]_h} \left|\psi^h(z)\right|^2 h\right)^{1/2},$$

$$\left\|\psi^h\right\|_{W^2_{2h}} = \left\|\psi^h\right\|_{L_{2h}} + \left(\sum_{z\in[-d,d]_h} \left|\left(\psi^h\right)_{z\overline{z},r}\right|^2 h\right)^{1/2}$$

respectively. To the differential operator A generated by problem (11), we assign the difference operator A^z_h by the formula

$$A^z_h \psi^h(z) = \{-(a(z)\psi_{\overline{z}}(z))_{z,r} - \beta \left(a(-z)\psi_{\overline{z}}(-z)\right)_{z,r} + \delta\psi^r\}^{K-1}_{-K+1}, \tag{21}$$

acting in the space of grid functions $\psi^h(z) = \{\psi^r\}^K_{-K}$ satisfying the conditions $\psi^{-K} = \psi^K = 0$. With the help of A^z_h, we arrive at the initial value problem

$$\begin{cases} \frac{du^{1h}(t,z)}{dt} + \mu u^{1h}(t,z) + A^z_h u^{1h}(t,z) = -F^h(t,z;u^{1h}(t,z),u^{2h}(t,z)), \\ \frac{du^{2h}(t,z)}{dt} + (\mu+\xi)\, u^{2h}(t,z) + A^z_h u^{2h}(t,z) = F^h(t,z;u^{1h}(t,z),u^{2h}(t,z)) \\ -\Upsilon^h(t,z;u^{2h}(t,z)), \\ \frac{du^{3h}(t,z)}{dt} + \mu u^{3h}(t,z) + A^z_h u^{3h}(t,z) = \Upsilon^h(t,z;u^{2h}(t,z)),\ t\in(0,b),\ z\in[-d,d]_h, \\ u^{mh}(0,z) = \psi^m(z),\ m = \overline{1,3},\ z\in[-d,d]_h \end{cases} \tag{22}$$

for an infinite system of semilinear ordinary differential equations. In the second step, we replace problem (22) by DS (13)

$$\begin{cases} \frac{u^1_k - u^1_{k-1}}{\tau} + \mu u^1_k + A^z_h u^1_k = -F^h(t_k, z, u^1_k, u^2_k), \\ \frac{u^2_k - u^2_{k-1}}{\tau} + (\xi+\mu)\, u^2_k + A^z_h u^2_k = F^h(t_k, z, u^1_k, u^2_k) - \Upsilon^h(t_k, z, u^2_k), \\ \frac{u^3_k - u^3_{k-1}}{\tau} + \mu u^3_k + A^z_h u^3_k = \Upsilon^h(t_k, z, u^2_k),\ k = \overline{1,N}, \\ u^m_0 = \psi^m,\ m = \overline{1,3}. \end{cases} \tag{23}$$

Theorem 5. Let the assumptions of Theorem 2 be satisfied and $\mu + \delta > 2\left(L_1 + L_2\right)$. Then, there exists a unique solution $u^\tau = \{u_k\}^N_{k=0}$ of DS (23) that is bounded in $\mathbb{C}^3_\tau\left(L_{2h}\right)$ uniformly w.r.t. τ and h.

The proof of Theorem 5 is based on the main Theorem 4 and the symmetry properties of the difference operator A^z_h defined by formula (21).

Second, the initial-boundary value problem (12) for multidimensional system of SPDEs is considered. The discretization of problem (12) is also carried out in two steps. In the first step, let us define the grid sets

$$\overline{\Omega}_h = \left\{ z = z_r = (h_1 r_1, ..., h_n r_n),\ r = (r_1, ..., r_n),\ k = \overline{0, N_i},\ h_i N_i = 1,\ i = 1, ..., n \right\},$$

$$\Omega_h = \overline{\Omega}_h \cap \Omega, \quad S_h = \overline{\Omega}_h \cap S.$$

We introduce the Banach spaces $L_{2h} = L_2(\overline{\Omega}_h)$ and $W_{2h}^2 = W_2^2(\overline{\Omega}_h)$ of the grid functions $\psi^h(z) = \{\psi(h_1 r_1, ..., h_n r_n)\}$ defined on $\overline{\Omega}_h$, equipped with the norms

$$\left\| \psi^h \right\|_{L_{2h}} = \left(\sum_{z \in \overline{\Omega}_h} \left| \psi^h(z) \right|^2 h_1 \cdots h_n \right)^{1/2},$$

$$\left\| \psi^h \right\|_{W_{2h}} = \left\| \psi^h \right\|_{L_{2h}} + \left(\sum_{z \in \overline{\Omega}_h} \sum_{r=1}^{n} \left| \left(\psi^h \right)_{z_r \overline{z}_r, r_r} \right|^2 h_1 \cdots h_n \right)^{1/2}$$

respectively. To the differential operator A generated by problem (12), we assign the difference operator A_h^z by the formula

$$A_h^z u_z^h = -\sum_{r=1}^{n} \left(a_r(z) u_{\overline{z}_r}^h \right)_{z_r, r_r} \tag{24}$$

acting in the space of grid functions $u^h(z)$, satisfying the conditions $u^h(z) = 0$ for all $z \in S_h$. It is known that A_h^z is a self-adjoint positive definite operator in $L_2(\overline{\Omega}_h)$. With the help of A_h^z, we arrive at the initial value problem

$$\begin{cases} \frac{du^{1h}(t,z)}{dt} + \mu u^{1h}(t,z) + A_h^z u^{1h}(t,z) = -F^h(t, z; u^{1h}(t,z), u^{2h}(t,z)), \\ \frac{du^{2h}(t,z)}{dt} + (\mu + \xi) u^{2h}(t,z) + A_h^z u^{2h}(t,z) = F^h(t, z; u^{1h}(t,z), u^{2h}(t,z)) \\ -\Upsilon^h(t, z; u^{2h}(t,z)), \\ \frac{du^{3h}(t,z)}{dt} + \mu u^{3h}(t,z) + A_h^z u^{3h}(t,z) = \Upsilon^h(t, z; u^{2h}(t,z)),\ t \in (0,b),\ z \in \overline{\Omega}_h, \\ u^{mh}(0,z) = \psi^m(z),\ m = \overline{1,3},\ z \in \overline{\Omega}_h \end{cases} \tag{25}$$

for an infinite system of semilinear ordinary differential equations. In the second step, we replace problem (25) by DS (13)

$$\begin{cases} \frac{u_k^1 - u_{k-1}^1}{\tau} + \mu u_k^1 + A_h^z u_k^1 = -F^h(t_k, z, u_k^1, u_k^2), \\ \frac{u_k^2 - u_{k-1}^2}{\tau} + (\xi + \mu) u_k^2 + A_h^z u_k^2 = F^h(t_k, z, u_k^1, u_k^2) - \Upsilon^h(t_k, z, u_k^2), \\ \frac{u_k^3 - u_{k-1}^3}{\tau} + \mu u_k^3 + A_h^z u_k^3 = \Upsilon^h(t_k, z, u_k^2),\ k = \overline{1,N}, \\ u_0^m = \psi^m,\ m = \overline{1,3}. \end{cases} \tag{26}$$

Theorem 6. Let the assumptions of Theorem 3 be satisfied and $\mu + \delta > 2(L_1 + L_2)$. Then, there exists a unique solution $u^\tau = \{u_k\}_{k=0}^N$ of DS (26) that is bounded in $\mathbb{C}_\tau^3(L_{2h})$ uniformly w.r.t. h and τ.

The proof of Theorem 6 is based on Theorem 4 and the symmetry properties of the difference operator A_h^z defined by formula (24) and the theorem on the coercivity inequality of an elliptic problem in L_{2h} [13].

3 Numerical experiments

When analytical methods fail to provide exact solutions or become intractable, numerical methods play a crucial role in obtaining approximate solutions of partial differential equations. Over the years, numerous significant contributions have been made in this area, and various reliable techniques have been developed.

In the present section, we focus on the numerical approximation of the solution to a given initial-boundary value problem. Specifically, we employ a finite DS of first-order accuracy. To solve the resulting discrete system, we apply a modified Gauss elimination method.

Furthermore, we provide an error analysis for both the first-order and second-order accurate DSs, highlighting their performance and convergence behavior. We now consider the following initial-boundary value problem for a system of SPDEs:

$$\begin{cases} \Psi^1_t(t,z) + \nu\Psi^1(t,z) - \beta\Psi^1_{zz}(t,z) \\ \quad = (-1+\nu+\beta)e^{-t}\sin z - \sin\left(\Psi^1(t,z)\Psi^2(t,z) - e^{-2t}\sin^2 z\right), \\ \Psi^2_t(t,z) + (\mu+\xi)\Psi^2(t,z) - d\Psi^2_{zz}(t,z) \\ \quad = (-1+\nu+\xi+d)e^{-t}\sin z, \\ \qquad + \sin\left(\Psi^1(t,z)\Psi^2(t,z) - e^{-2t}\sin^2 z\right) - \cos\left(\Psi^2(t,z) - e^{-t}\sin z\right), \\ \Psi^3_t(t,z) + \nu\Psi^1(t,z) - \gamma\Psi^3_{zz}(t,z) \\ \quad = (-1+\nu+\gamma)e^{-t}\sin z + \cos\left(\Psi^2(t,z) - e^{-t}\sin z\right),\ t\in(0,1),\ z\in(0,\pi), \\ \Psi^m(0,z) = \sin z,\ z\in[0,\pi],\ m=\overline{1,3}, \\ \Psi^m(t,0) = \Psi^m(t,\pi) = 0,\ t\in[0,1],\ m=\overline{1,3}. \end{cases} \tag{27}$$

The spatial variable z may be treated as either discrete or continuous, depending on the context. In all cases, z represents population mobility, such as travel or migration between cities, towns, or even countries.

The exact solution of problem (27) is given by:

$$\Psi^m(t,z) = e^{-t}\sin z, \quad m=\overline{1,3}.$$

We now present a first-order accurate iterative DS for approximating the solution of the initial-boundary value problem (27):

$$\begin{cases} \frac{1}{\tau}\left(ru_n^{1,k} - ru_n^{1,k-1}\right) + \nu ru_n^{1,k} - \frac{\beta}{h^2}\left(ru_{n+1}^{1,k} - 2ru_n^{1,k} + ru_{n-1}^{1,k}\right) \\ \quad = (-1+\nu+\beta)e^{-t_k}\sin z_n - \sin\left((r-1)u_n^{1,k}(r-1)u_n^{2,k} - e^{-2t_k}\sin^2 z_n\right), \\ \frac{1}{\tau}\left(ru_n^{2,k} - ru_n^{2,k-1}\right) + (\mu+\xi)ru_n^{2,k} - \frac{d}{h^2}\left(ru_{n+1}^{2,k} - 2ru_n^{2,k} + ru_{n-1}^{2,k}\right) \\ \quad = (-1+\nu+\xi+d)e^{-t_k}\sin z_n \\ \qquad + \sin\left((r-1)u_n^{1,k}(r-1)u_n^{2,k} - e^{-2t_k}\sin^2 z_n\right) - \sin\left((r-1)u_n^{2,k} - e^{-t_k}\sin z_n\right), \\ \frac{1}{\tau}\left(ru_n^{3,k} - ru_n^{3,k-1}\right) + \nu ru_n^{3,k} - \frac{\gamma}{h^2}\left(ru_{n+1}^{1,k} - 2ru_n^{1,k} + ru_{n-1}^{1,k}\right) \\ \quad = (-1+\nu+\gamma)e^{-t_k}\sin z_n + \sin\left((r-1)u_n^{2,k} - e^{-t_k}\sin z_n\right), \\ t_k = k\tau,\ k=\overline{1,N},\ N\tau=1,\ z_n=nh,\ n=\overline{1,K-1},\ Kh=\pi, \\ ru_n^{m,0} = \psi^m(z_n),\ ru_0^{m,k} = ru_K^{m,k} = 0,\ k=\overline{0,N}, \\ 0u_n^{m,k} \text{ is the initial guess},\ m=\overline{1,3},\ k=\overline{0,N},\ n=\overline{0,K}. \end{cases} \tag{28}$$

To solve the DS (28), we follow the iterative procedure described below. For each time step $k=\overline{0,N-1}$ and spatial index $n=\overline{0,K}$:

1. Initialize iteration with $r = 1$.
2. Assume $(r-1)u_n^{m,k}$ is known for all m.
3. Compute $ru_n^{m,k}$ using the difference equations.
4. If the maximum absolute error between $(r-1)u_n^{m,k}$ and $ru_n^{m,k}$ exceeds a prescribed tolerance, increment $r \to r+1$ and repeat from step 2. Otherwise, accept $ru_n^{m,k}$ as the solution.

The errors of numerical solutions are computed by

$$(rE^m)_K^N = \max_{k=\overline{1,N},\ n=\overline{1,K-1}} \left| \Psi^m(t_k, z_n) - ru_n^{m,k} \right|, \quad m = \overline{1,3},$$

where $\Psi^m(t_k, z_n)$ is the exact solution, and $ru_n^{m,k}$ is the numerical approximation at the grid point (t_k, z_n) for each m.

The results of the error computations for different grid resolutions are presented in Table 1.

Table 1

Maximum error $(rE^m)_K^N$ for different values of $N = K$ and $r = 6$

$(rE^m)_M^N$	$N = K = 20$	$N = K = 40$	$N = K = 80$
$m = 1$	0.0068	0.0032	0.0016
$m = 2$	0.0071	0.0033	0.0016
$m = 3$	0.0073	0.0034	0.0017

As observed in Table 1, when the values of N and K are doubled, the error decreases approximately by a factor of 1/2, which is consistent with the behavior of a first-order accurate finite DS as defined in equation (28). The numerical results confirm both the stability and the accuracy of the proposed DS.

Conclusion

In the present paper, we have established a theorem concerning the EU of a BS for a semilinear system of parabolic equations that models the spread of epidemics with a general semilinear incidence rate. The single-step DS of the w.r.t. for the numerical approximation of the semilinear system has been investigated.

Furthermore, we proved a theorem concerning the EU of a BS for the DS, uniformly w.r.t. the time step τ. The BSs of the semilinear parabolic system and its corresponding numerical scheme were derived. Finally, numerical results were presented for a test problem to illustrate the effectiveness and precision of the proposed DS. Applying methods from this paper and from papers [14] and [15] we can present similar results from this paper for a BS for a semilinear system of delay parabolic equations that models the spread of epidemics with a delay semilinear incidence rate.

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Author Contributions

All authors contributed equally to this work.

Conflict of Interest

The authors declare no conflict of interest.

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