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The intrinsic geometry of a convex surface in Galilean space

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This paper investigates the intrinsic geometry of a convex surface in the Galilean space R_3^1. The Galilean space, as a special case of a pseudo-Euclidean space, possesses a degenerate metric. The angle between two directions is defined using a parabolic method, which is itself degenerate. The three-dimensional Galilean space, similar to the Euclidean space, is based on a three-dimensional affine space. While the fundamental geometric objects in these spaces coincide structurally, the geometric quantities associated with them differ significantly from those in Euclidean geometry. It becomes necessary to introduce and rigorously define various geometric characteristics of objects in Galilean space. Therefore, special attention in this work is given to the total angle around the vertex of a cone, the angle between curves on a convex surface, and the variation of curve turning on a convex surface. A geodesic on a convex surface is defined as a curve with bounded variation of turning. A triangle is defined as a curve homeomorphic to a circle, bounded by three geodesics. Using the concept of the total angle around the vertex of a cone, we define the intrinsic curvature of convex surfaces in Galilean space and obtain an analogue of the Gauss–Bonnet theorem for convex surfaces in Galilean geometry. The results obtained extend classical notions of intrinsic geometry under a degenerate metric.

Keywords: Galilean space, convex surface, intrinsic geometry, intrinsic curvature, Gauss–Bonnet theorem, degenerate metric, tangent cone, geodesic, curves with bounded variation of turning.

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Introduction

Modern differential geometry successfully applies methods of both intrinsic and extrinsic geometry to the study of curves and surfaces in various spaces. One such space is the Galilean space R_3^1, where a degenerate metric coexists with the affine structure. This metric does not depend on all coordinates, leading to fundamental differences in the definitions of distances, angles, and curvature, as compared to the Euclidean case.

It is well known that the study of surface geometry is traditionally divided into “intrinsic” and “extrinsic” components. In Euclidean space, the first fundamental form plays a central role in intrinsic geometry. However, in Galilean space, the first fundamental form of a surface is degenerate, and Gauss’s theorem, stating that the Gaussian curvature of a surface can be expressed entirely in terms of the coefficients of the first fundamental form and their derivatives-does not hold. Therefore, it becomes necessary to redefine intrinsic curvature, highlighting specific geometric characteristics that arise due to the degeneracy of the metric.

The aim of this paper is to define the fundamental geometric characteristics of convex surfaces and to construct an analogue to the intrinsic geometry of a surface within the Galilean space. Due to the degenerate nature of the metric, it is not possible to directly apply classical Euclidean definitions such as geodesics, arc length, or intrinsic curvature. Consequently, this paper introduces new approaches: using angles between generators of tangent cones, curves with bounded variation of turning, and cylindrical mappings.

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This work builds upon the results of previous studies by A.D. Alexandrov and O. Roschel, and contributes to the further developing of the theory of convex surfaces in pseudo-Euclidean spaces [1,2].

Since the 2000s, there has been an increasing interest in the geometry of Galilean space. In this context, special attention should be given to the works [3–5].

1 Galilean space and fundamental concepts

The Galilean space R_3^1 is an affine space A_3 equipped with two scalar products defined for vectors $X = \{x_1, y_1, z_1\}$ and $Y = \{x_2, y_2, z_2\}$:
1. $(X, Y) = (X, Y)_1 = x_1 \cdot x_2$,
2. $(X, Y) = (X, Y)_2 = y_1 \cdot y_2 + z_1 \cdot z_2$, when $(X, Y)_1 = 0$.

The norm of a vector is defined as the square root of its scalar square, and the distance between two points equals the norm of the vector connecting them [6].

The motions of Galilean space, i.e., linear transformations preserving distances between corresponding points, are described by the system [7]:

$$\begin{aligned} x' &= x + a, \\ y' &= \alpha x + y \cos\phi + z \sin\phi + b, \\ z' &= \beta x - y \sin\phi + z \cos\phi + c. \end{aligned}$$

Here a, b, c are translation parameters, α, β correspond to a Galilean shear (related to the parabolic angle h), and φ denotes the Euclidean rotation angle in the (y, z)-plane.

Let $e_1 = (1, 0, 0)$, $e_2 = (0, 1, 0)$, $e_3 = (0, 0, 1)$ be an orthonormal basis of the space. Then it is easy to establish that a motion maps a plane parallel to the vectors e_2, e_3 into another parallel plane. These planes are Euclidean and are called special planes. Planes not parallel to e_2 and e_3 are called planes in general position. Vectors parallel to special planes are also called special vectors.

A sphere in Galilean space R_3^1 is a set of all points equidistant from a point X_0 and is defined by the equation:

$$(x - x_0, x - x_0) = r^2.$$

If the center of the sphere is at the origin and the radius is 1, then

$$(x, x)_1 = x^2 = 1.$$

The set of all points whose coordinates satisfy the sphere equation forms a set of parallel special planes located at unit distance from the origin.

Unit vectors in the directions of X and Y have the coordinates:

$$\widetilde{X} = \left\{1, \frac{y_1}{x_1}, \frac{z_1}{x_1}\right\}, \quad \widetilde{Y} = \left\{1, \frac{y_2}{x_2}, \frac{z_2}{x_2}\right\}.$$

These vectors are the radius vectors of points on the unit sphere.

The angle between spatial vectors is defined as the distance between the endpoints of their corresponding unit vectors on the sphere, and is given by:

$$h = \sqrt{\left(\frac{y_1}{x_1} - \frac{y_2}{x_2}\right)^2 + \left(\frac{z_1}{x_1} - \frac{z_2}{x_2}\right)^2}.$$

It is evident that $0 \leq h < \infty$, and $h \to \infty$ if one of the vectors approaches a special direction. When $h = 0$, the vectors are parallel.

The angle between a spatial vector $\widetilde{X} = (x_1, y_1, z_1)$ and a special vector $\widetilde{Y} = (x_2, y_2, z_2)$ is defined as:

$$f = \frac{(\vec{X}, \vec{Y})}{|Y|_2} = \frac{\frac{y_1}{x_1} y_2 + \frac{z_1}{x_1} z_2}{\sqrt{y_2^2 + z_2^2}}.$$

The geometric interpretation of the angle f is the projection length of the unit vector $\widetilde{X}$ onto the direction of $\widetilde{Y}$ in the special plane. The projection is taken along the vector e_1. If $\vec{X}$ is parallel to e_1, then $f = 0$.

The angle between special vectors is given by the standard Euclidean formula:

$$\cos\varphi = \frac{y_1 y_2 + z_1 z_2}{\sqrt{y_1^2 + z_1^2} \cdot \sqrt{y_2^2 + z_2^2}} = \frac{(\vec{X}, \vec{Y})_2}{|\vec{X}|_2 \cdot |\vec{Y}|_2}.$$

Thus, the angle between lines in Galilean space is defined via the angle between their direction vectors.

Let F be a surface in R_3^1 that does not possess special tangent planes. We introduce a special curvilinear coordinate system by considering all intersections of F with special planes $x = \text{const}$. We choose the family of curves formed by these intersections as $u = u_0$ coordinate lines, and arbitrary transverse curves on F as $v = v_0$ lines. Then the surface can be parameterized as:

$$\vec{r}(u, v) = ue_1 + y(u, v)e_2 + z(u, v)e_3.$$

Here, the vectors $\vec{r}_u$ and $\vec{r}_v$ form a basis of the Galilean tangent plane at each point. The direction of $\vec{r}_v$ corresponds to the distinguished direction in the Galilean plane.

Let a curve on F be given by the equation $v = v(u)$. The arc length of the curve between points $A(u_0)$ and $A(u_1)$, where $u_0 \neq u_1$, is computed as follows:

$$ds = |\vec{r}_u du + \vec{r}_v dv| = |du|.$$

Hence, the square of the arc length differential on the surface equals the square of the increment of the coordinate u:

$$ds^2 = du^2.$$

This expression is referred to as the first fundamental form of the surface.

When $du = 0$, i.e., $u = \text{const}$, the corresponding curve lies entirely in a special plane. In this case, the differential of arc length is given by

$$ds^2 = (y_v^2 + z_v^2)dv^2 = G(u, v)dv^2.$$

We refer to this as the first supplementary fundamental form of the surface. Thus, with the chosen curvilinear coordinates, the coefficients of the first fundamental forms are $E_1 = 1$, and $G = y_v^2 + z_v^2$.

Suppose two points emanate from a point $M(u_0, v_0)$ on a surface in general position (i.e., whose tangents are not parallel to a special plane). Let $d\vec{r}$ and $\delta\vec{r}$ be the differentials of the radius vector along these curves. The angle θ between them is defined as the angle between the vectors $d\vec{r}$ and $\delta\vec{r}$.

Hence,

$$\theta = \sqrt{G(u, v)} \left(\frac{dv}{du} - \frac{\delta v}{\delta u} \right).$$

Similar to the Euclidean case, the concept of surface area can be introduced. The area of a domain D on the surface is given by

$$S = \iint\limits_D \sqrt{G(u, v)}\, du\, dv.$$

2 Convergence of convex surfaces in R_3^1

The degeneracy of the scalar product induces a degenerate metric in the Galilean space R_3^1. If two points lie on different planes, then the distance between the special planes to which they belong is defined as the distance between the points. When the points lie on the same special plane, the distance between them is defined as the length of the segment connecting them. Special planes in R_3^1 are Euclidean planes.

Suppose that a sequence of convex polyhedra F_n converges to a convex surface F, and a sequence of points $x_n \in F_n$ converges to a point $x \in F$.

We consider only such approximations for which the points x_n and y_n-converging respectively to x and y-remain at distances of the same order.

Theorem 1. Let a sequence of closed convex polyhedra F_n converge to a closed convex surface F, and let sequences of points $x_n, y_n \in F_n$ converge to points $x, y \in F$, respectively. Then the distances between x_n and y_n, measured on F_n, converge to the distance between x and y, measured on F, i.e.,

$$\rho_F(x,y) = \lim_{n\to\infty} \rho_{F_n}(x_n, y_n).$$

Proof. Suppose the points x_n and y_n lie on different special planes and converge to points x and y lying on corresponding special planes. Then we have:

$$\rho_{E_n}(x_n, y_n) \leq \rho_E(x,y),$$

where ρ_E denotes Euclidean distance. Moreover, in Galilean space, for points lying on different special planes, the distances are equal:

$$\rho_{F_n}(x_n, y_n) = \rho_F(x,y),$$

since in this case the measured distance is formally defined: it does not depend on the surface itself.

If the points x_n and y_n lie on the same special plane, then the metric is considered as a second-order metric, and we have $\rho_2(x_n, y_n) = \rho_E(x_n, y_n)$. Instead of computing the direct distance between the points, we consider the length L_n of a polygonal line on the special plane connecting x_n and y_n. This broken line arises from the intersection of F_n with the special plane. Since the special plane is Euclidean, distances on F_n within it are measured via the polygonal path joining x_n and y_n, and thus

$$\rho_2(x_n, y_n) = L_n(x_n, y_n).$$

When $F_n \to F$, the Euclidean length of the polygonal line $L_n(x_n, y_n)$ converges to the length of the curve $L(x,y)$ on the special plane. Therefore,

$$\lim_{n\to\infty} \rho(x_n, y_n) = \lim_{n\to\infty} L_n(x_n, y_n) = L(x,y) = \rho_2(x,y).$$

The theorem is thus proved.

□

3 The total angle around the vertex of a cone

The definition of the total angle around the vertex of a cone in Galilean space was introduced in the work [8] of A. Artykbayev. The main challenge in this definition lies in the concept of the angle between a vector and a special plane. Therefore, cones are divided into two classes: cones that do not have special supporting planes and cones that do.

In both cases, the total angle around the cone's vertex is defined using a circle of unit radius centered at the vertex of the cone.

When the cone does not possess special supporting planes, the total angle around its vertex is defined via the intersection of the cone with special planes. Since special planes determine a sphere in Galilean space [8], intersecting the cone with these planes yields hyperbolas with asymptotes parallel to lines passing through the cone's vertex.

The sphere in Galilean space consists of two parallel special planes. If one of these sections is reflected symmetrically onto the other, we obtain both branches of the hyperbola on the same special plane.

Let V be a convex cone in R_3^1 that does not have any special supporting plane. Intersect V with a special plane π_0 passing through the cone's vertex. Let μ_1 and μ_2 be the generatrices of the cone lying on this intersection. Let γ_1 and γ_2 be the curves formed by intersecting the cone V with the unit sphere of Galilean space, i.e., with the pair of special planes located at unit distance from the cone's vertex. Clearly, the curves γ_1 and γ_2 have asymptotes parallel to the lines μ_1 and μ_2, respectively.

This configuration, when visualized on a special plane, appears as shown in Figure 1.

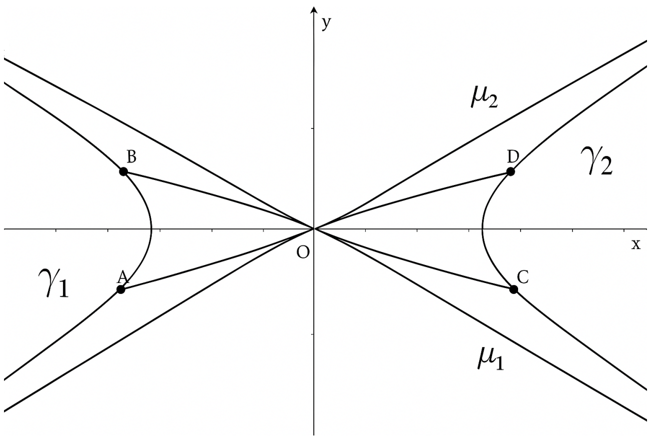

Figure 1. Intersection of a convex cone with special planes in Galilean space R_3^1

Let us denote the angular quantity by

$$\omega = AD + BC - \breve{AB} - \breve{DC} > 0.$$

The total angle around the cone's vertex is defined as the limit of ω as the points A, B, C, D on the respective branches of the curves γ_1 and γ_2 tend to infinity.

In [8], the limit was proven to be bounded. Furthermore, in [9], an analytical formula for this total angle was obtained when the equations of the curves γ_1 and γ_2 are known explicitly.

To define the curvature of fundamental sets on a convex surface, we use of the total angle around the cone's vertex in Galilean space.

When the cone has special supporting planes, its intersection with the unit sphere centered at the vertex is a closed curve. The length of this closed curve is then taken as the total angle around the vertex of the cone possessing a special supporting plane [10, 11].

4 Angle between curves on a convex surface in Galilean space R_3^1

To define the angle between two curves on a convex surface, we use the angle between the generatrices of the tangent cone. At every point on a convex surface in Galilean space, a tangent cone exists. This follows from the fact that Galilean space is affine, and affine structures are preserved in Galilean geometry.

When the convex surface is regular, the tangent cone degenerates into a plane. The geometry on this plane is Galilean.

Let l_1 and l_2 be two generatrices of a convex cone V, directed into the same half-space with respect to a special plane π_0. The generatrices l_1 and l_2 intersect the curve γ_i (for $i = 1$ or 2, depending on the direction of l_1 and l_2). The length of the arc of the curve γ_i enclosed between l_1 and l_2 is taken as the angle φ^+ between them.

We intersect the cone V with a plane passing through the bisector of the angle formed by the generatrices μ_1 and μ_2, and parallel to the Ox-axis. This intersection is referred to as the principal section of the cone V.

Generatrices of the cone directed into opposite half-spaces with respect to both the special plane and the principal section, and forming equal angles with the generatrices lying in the principal section, are called conjugate generatrices. The angle between conjugate generatrices is defined to be half of the total angle around the vertex of the cone.

When $\widetilde{l_1}$ and $\widetilde{l_2}$ are generatrices directed into different half-spaces divided by the special plane π_0, the angle between them is given by

$$\varphi^-\{\widetilde{l_1}, \widetilde{l_2}\} = \frac{\omega}{1} - \beta^\star,$$

where $\beta^\star$ is the angle $\varphi^-\{\widetilde{l_1}, \widetilde{l_2}^\star\}$, and $\widetilde{l_2}^\star$ is the generatrix conjugate to $\widetilde{l_2}$. It is easy to verify that

$$\varphi^+\{\widetilde{l_1}, \widetilde{l_2}^\star\} = \varphi^-\{\widetilde{l_1}^\star, \widetilde{l_2}\}.$$

It can be shown that for three generatrices of the cone not directed into the same half-space and distinct from μ_1 and μ_2, the sum of the angles between them equals the total angle around the vertex of the cone.

If the cone V degenerates into a plane or a dihedral angle, the defined angles φ^+ and φ^- coincide with the angle between rays in the Galilean plane R_2. In such cases, the total angle is zero.

Now consider an arbitrary point M on the surface F, and let V be the tangent cone at this point. Let $\{\gamma\}$ denote the family of curves on the surface F issuing from the point M and having a direction not lying in the special plane π_0. The direction of any curve in $\{\gamma\}$ coincides with a generatrix of the tangent cone V.

The angle between two such curves issuing from the point M on the convex surface F is defined as the angle between their directions — that is, the angle between the corresponding generatrices of the tangent cone.

This notion of angle does not satisfy all the properties of angles between curves on convex surfaces in Euclidean geometry. For instance, in Euclidean space, if L_1, L_2, and L_3 are three curves forming angles α_1, α_2, and α_3, then the sum of any two of these angles is at least as great as the third.

This property holds in Galilean space only for curves directed into the same half-space.

The angle defined in this manner is naturally called the "parabolic angle". It can take any positive value. When the direction of one of the curves tends to lie infinitely close to the special plane, the angle increases without bound.

5 *Curves of bounded variation of turning in R_3^1*

To introduce the analogue of a shortest path in R_3^1, we first define curves of bounded variation of turning. Let γ be a curve in the space R_3^1 connecting points A and B that lie on different special planes. Inscribe a polygonal line L_n into γ, and denote by $\mu(L_n)$ the sum of (parabolic) angles of this polygonal line. The upper limit of the values $\mu(L_n)$ over all such inscribed polygonal lines L_n is called the variation of turning of the curve γ. If $\mu(\gamma)$ is finite, then γ is called a curve of bounded variation of turning.

Lemma 1. If γ is a curve of bounded variation of turning connecting points A and B on different special planes, then it intersects each special plane of R_3^1 in at most one point.

Proof. Suppose γ has two points of intersection with some special plane, or contains a component lying entirely within a special plane. Then one can inscribe a polygonal line L_n such that at least one of its segments lies entirely within the special plane. The angle of the polygon at the ends of such a segment becomes unbounded. This contradicts the boundedness of the variation of turning. □

Lemma 2. If γ is a curve of bounded variation of turning in Galilean space R_3^1, then it also has bounded variation of turning in Euclidean space.

Proof. Let $A_{i-1}A_i$ and A_iA_{i+1} be segments of a polygonal line inscribed in γ. Let h_i be the angle between these segments in R_3^1, and $\varphi_i^\star$ be the Euclidean measure of that angle. Then the following inequality holds:

$$0 \leq \varphi_i^\star \leq \tan \varphi_i^\star \leq h_i.$$

Since γ is of bounded variation in R_3^1, we have $\sum\limits_{i=1}^{n} h_i < \infty$, and thus $\sum\limits_{i=1}^{n} \varphi_i^\star$ is also finite. Therefore, the variation of turning in Euclidean space is bounded. □

Lemma 3. Curves of bounded variation of turning have right and left semi-tangents at every point. These are not parallel to the special plane.

This follows from the properties of Euclidean curves of bounded variation of turning. Since such curves also have bounded variation in Euclidean space, the tangents cannot be parallel to the special plane; otherwise, it contradicts boundedness.

Variation of turning can also be defined equivalently. Let γ be a curve with a right (or left) semi-tangent at each point. Take a finite number of points A_k on γ, and at each point place the right semi-tangent t_k. The supremum of the sum of angles between successive semi-tangents over all such finite systems of points A_k is called the variation of turning of γ. This definition is equivalent to the one given above, as proved analogously in Euclidean geometry [12].

Let A and B be points on different special planes in R_3^1. Consider circular cones S_A and S_B with vertices at A and B, respectively, and with their directrices centered along the segment AB (lying in a special plane). These cones intersect. The class of closed convex surfaces formed by all possible intersections of such cones is denoted by S_{AB}.

Lemma 4. If m_{AB} is a family of curves connecting A and B and having variation of turning not greater than N, then there exists a surface F in the class S_{AB} such that all curves in the family lie within F.

Proof. From the set of surfaces, choose one. For this surface the total angle around the vertices satisfies:

$$\gamma_A = \gamma_B = 2\pi N.$$

Consider a broken line consisting of the generatrices of intersecting cones S_A and S_B, with a vertex at their intersection point. The turn at this vertex is not less than N. This follows from the triangle formed by the broken line and the segment AB, where the base angles are N, and the vertex angle is at least the sum of the base angles. The same argument applies if any vertex of the broken line does not correspond to a generatrix of surface F. In such case, that part cannot lie on the cone, implying the curve cannot lie outside F. □

Theorem 2. Let $\gamma_1, \gamma_2, \ldots, \gamma_n$ be an infinite sequence of curves with bounded turning variation, each with variation no greater than N. If γ_n converges to a curve γ, then γ is also with bounded turning variation and its variation does not exceed N.

Proof. The proof is analogous to the corresponding result in Euclidean space [12].

Let X and Y be arbitrary points on a convex polyhedron Q without special supporting planes. If X and Y lie on the same special plane, they are connected by a planar convex polygon with bounded turning. If they lie on different special planes, they can still be connected by a polygonal line on Q with bounded turning. □

Lemma 5. Any two points X and Y on a convex polyhedron Q without special planes can be connected by a polygonal line L_{XY} on Q with bounded turning.

Proof. Let points X and Y lie on the singular planes π_X and π_Y, respectively. Consider the intersection of the polyhedron Q with the singular planes π_X and π_Y, and denote by Q_{AB} the portion of Q between these two planes. At the points X and Y, the boundary of the polyhedron Q_{AB} has two directions: $l_{1X},\ l_{2X}$ at X and $l_{1Y},\ l_{2Y}$ at Y.

Now consider the spatial segment connecting the points X and Y. We construct through X and Y a general-position plane π_{XY} such that the boundary edge directions of the broken line L_{XY}, formed by the intersection of π_{XY} with the polyhedron Q_{XY}, do not coincide with the directions of the boundary at X and Y. The broken line L_{XY} contains no segments lying on singular planes. Indeed, the extreme segments are chosen in such a way that they do not lie on any singular plane. The intermediate segments cannot lie on singular planes either, since the polyhedron Q contains no such planes.

Since the plane π_{XY} is in general position, it is a Galilean plane. Consider the convex polygon formed by the segment XY and the broken line L_{XY} on this plane. In this polygon, the sum of the angles not supported by singular directions equals the sum of the angles with singular supporting directions. Hence,

$$\mu(L_{XY}) = \alpha_X + \alpha_Y,$$

where α_X and α_Y are the angles at the vertices X and Y supported by singular directions. These values α_X and α_Y are finite because the segment XY and the extreme edges of the broken line L_{XY}, which form these angles, are spatial (i.e., not singular). Therefore, the value $\mu(L_{XY})$ is bounded. This completes the proof of the lemma. □

Consider a sequence of convex polyhedra F_n with common boundary L_n, all lacking special supporting planes. Suppose each L_n is a polygonal line with bounded turning on F_n, and the sequence F_n converges to a convex surface F with boundary L.

Lemma 6. If a sequence of polygonal lines γ_n with bounded turning on F_n converges to a curve γ on F, then γ also has bounded turning variation.

This follows from Theorem 2 and the assumption that F does not have special supporting planes.

6 An analogue of a geodesic on a surface in R_3^1

The degeneracy of the metric in Galilean space R_3^1 prevents the definition of a geodesic via standard metric methods. The distance between two points lying on different special planes is equal to the interval between those planes [13].

An interesting phenomenon arises: all curves connecting two given points that do not lie on the same special plane have equal length. This effect can be interpreted within Newtonian mechanics as worldline length invariance connecting given events. In other words, in Newtonian mechanics, time is independent of the velocity of bodies. Hence, the natural question arises — by what criterion can a curve on the surface be distinguished as a substitute for the shortest path, i.e., a curve possessing properties similar to those of a Euclidean geodesic?

Let F be a convex surface in Galilean space. Consider a family $\{\gamma\}$ of curves lying on the surface and connecting two given points on the Galilean surface.

Definition 1. A curve γ from the family $\{\gamma\}$ that has the least turning variation is called the shortest curve between the given points on the surface.

This provides one possible definition of a geodesic in R_3^1.

Accordingly, a geodesic is defined as a continuous curve that is the shortest (in the sense of minimal turning variation) over each of its sufficiently small subarcs. A triangle on the surface is defined as a figure homeomorphic to a circle and bounded by three such shortest curves. A geodesic triangle is defined as a figure bounded by three geodesics.

7 Intrinsic curvature of a convex surface

The degeneracy of the metric in Galilean space necessitates a new approach to defining the intrinsic geometry of a surface. Intrinsic geometry includes those properties of a surface that depend on the properties of its metric. In Euclidean space, the intrinsic curvature of a convex surface is fully determined by the internal metric of the surface. A similar approach in Galilean space does not yield satisfactory results. Therefore, we attempt to study the intrinsic geometry of a convex surface using its extrinsic geometry in Galilean space. We define the intrinsic curvature of a set on a convex surface in Galilean space by analogy with the Euclidean case, initially for three types of "elementary" sets: open triangles, open geodesics, and points. An open triangle excludes its vertices and sides; its sides do not lie on special planes. An open geodesic is a geodesic excluding its endpoints.

For an open triangle T on F, the curvature is defined as

$$\omega(T) = \alpha - \beta + \gamma.$$

Here α, β, γ are the triangle's angles, and each side lies in a different half-space determined by a special plane through the vertex.

The curvature of an open geodesic is taken to be zero.

A point's curvature on a convex surface is defined as the total angle around the vertex of the tangent cone at that point.

We consider sets on a convex surface that do not share common points pairwise. Such sets are called "elementary". Based on the definition of elementary set curvature, we define a bounded set's curvature on a convex surface.

Definition 2. The intrinsic curvature of a bounded closed set on a convex surface is defined as the infimum of the curvatures of all elementary sets containing it.

We define the intrinsic curvature of Borel sets on a convex surface as the least upper bound (supremum) of the curvatures of all bounded closed subsets contained in it.

The definitions of intrinsic curvature of a set on a convex surface in Galilean space given above are analogous to those in Euclidean geometry. The main difference lies in how the curvature of the "elementary" (or "basic") sets is defined.

Let M be an "elementary" set on a convex surface F. Suppose it can be represented as a disjoint union of basic sets

$$M = \sum_{i=1}^{n} B_i.$$

Then, the intrinsic curvature of the set M is defined as the sum of the curvatures of its basic components:

$$\omega(M) = \sum_{i=1}^{n} \omega(B_i).$$

The intrinsic curvature of a Borel set on a convex surface is defined as the exact least upper bound of the curvatures of all bounded closed subsets contained in it.

The curvature value of a set on a convex surface does not depend on the particular way it is decomposed into basic sets.

This fact, along with the non-negativity and complete additivity of the intrinsic curvature of a convex surface for elementary sets, is proved in the same way as in Euclidean geometry. This is justified by the observation that the cylindrical mapping of a convex surface can be regarded as the projection of its spherical mapping onto a cylinder. The generating curve of the cylinder corresponds to a great circle on the unit sphere. As a result, the cylindrical mapping of a convex surface in Galilean space inherits all the essential properties of the spherical mapping. These properties ensure the correctness of the intrinsic curvature's stated properties.

Theorem 3. The intrinsic curvature of a Borel set on a convex surface is equal to its extrinsic curvature.

Proof. The concept of extrinsic curvature is defined in [8]. The authors show the cylindrical mapping is a projection of the Euclidean spherical mapping onto the sphere in the isotropic space R_3^1. The isotropic sphere is interpreted as the co-Euclidean plane S_2^1. To prove the theorem, it suffices to show that the curvature of basic sets equals the area of their cylindrical image. Indeed, the spherical image of an open triangle maps to a triangle on the co-Euclidean plane. The quantity defining the curvature of the open triangle on F equals the area of the triangle on S_3^1. The intrinsic curvature of an open geodesic equals the area of its cylindrical image, which is a curve on the plane.

The total angle around the vertex of a cone is taken to be equal to the area of its cylindrical image.

The theorem for any Borel set on a convex surface follows from the fact that the cylindrical mapping of a convex surface in Galilean space is a central projection of the Euclidean spherical mapping. □

Theorem 4. Intrinsic curvature is a non-negative and fully additive function on Borel sets of a convex surface.

Proof. The extrinsic curvature of convex surfaces in Galilean space is a non-negative and fully additive function on Borel sets of the surface. Therefore, intrinsic curvature, being equal to extrinsic curvature, also possesses these properties. □

8 *Gauss–Bonnet formula in Galilean space*

The results obtained in the previous sections allow us to approach a generalization of the Gauss–Bonnet formula for an arbitrary domain on a convex surface in Galilean space. However, a completely new difficulty arises here, related to the discontinuity of the angle between vectors when a vector traverses a closed region. In particular, when one of the vectors is parallel to a singular plane, the angle between vectors becomes unbounded. To eliminate this peculiarity, the domain must satisfy certain conditions.

Let Q be a convex domain on a convex surface F, that has no singular supporting planes, and is bounded by smooth curves

$$\alpha_1, \alpha_2, \ldots, \alpha_k, \quad \beta_1, \beta_2, \ldots, \beta_p.$$

Assume that the curves α_1 and β_p, as well as α_k and β_1, share common endpoints A and B, respectively. The points A and B lie on the singular planes that bound the domain. The directions of the curves α_1, β_p at point A, and α_k, β_1 at point B, are not parallel to the singular planes.

Let φ_i and ψ_j denote the angles between the curves (α_i, α_{i+1}) and (β_j, β_{j+1}), respectively. Let φ and ψ denote the angles at the points A and B, respectively.

A domain Q satisfying the above conditions is called admissible.

Then, the following theorem holds.

Theorem 5. Let $D \subset F$ be an admissible domain on a convex surface F in Galilean space. Then the generalized Gauss–Bonnet formula holds:

$$\iint\limits_D K\, d\sigma = \varphi + \psi - \sum_{i=1}^{k} \left[\varphi_i + \int\limits_{\alpha_i} k(\alpha_i)\, ds \right] - \sum_{j=1}^{n} \left[\psi_j + \int\limits_{\beta_j} k(\beta_j)\, ds \right],$$

where:

- K is the *Gaussian curvature* on the surface F,
- $d\sigma$ is the *surface area element*,
- $k(\alpha_i)$, $k(\beta_j)$ are the *geodesic curvatures* of the boundary curves,
- ds is the *arc length element*,
- φ_i, ψ_j are the *turning angles* between boundary curve segments,
- φ, ψ are the *interior angles* at the corner points A and B.

Proof. We begin by computing the intrinsic curvature of a convex geodesic polygon on a convex polyhedral surface. Let F_n be a sequence of convex polyhedral surfaces converging to a convex surface F that has no singular supporting planes.

Let Q_n be a geodesic polygon on F_n, bounded by geodesic arcs α_{in} and β_{jm}, such that Q_n consists of a collection of non-overlapping geodesic triangles. These triangles are chosen in such a way that none of their sides lie on singular planes. Furthermore, the vertices of the polyhedron F_n contained in Q_n are the vertices of these triangles.

By definition, the intrinsic curvature $\omega(Q_n)$ of the polygon Q_n is equal to the sum of the intrinsic curvatures of the sets contained within it:

$$\omega(Q_n) = \sum \omega(T_e) + \sum \omega(X_m) + \sum \omega(L_n),$$

where:

- T_e are the open triangles in the triangulation,
- X_m are the vertices of the triangles T_e contained in Q_n,
- L_n are the sides of the triangles (excluding endpoints).

The intrinsic curvature $\omega(L_n) = 0$ for all segments L_n, since geodesic arcs have zero intrinsic curvature except at their endpoints.

The vertices of triangles T_e in Q_n are of two types:

1. vertices located *inside* the polygon Q_n,
2. vertices lying *on the boundary* of the polygon.

The boundary vertices are further subdivided into:

- points lying on A_n or B_n,
- points lying on the geodesic arcs α_{in} or β_{jm}.

The sum of all angles around an interior vertex of Q_n is equal to the negative of the intrinsic curvature at that vertex. The angle at a boundary vertex equals the turning angle of the boundary at that point.

Thus, we obtain:

$$\sum \omega(T_e) = \varphi_n + \psi_n - \sum_{i=1}^{k} (\varphi_{in} + \Delta\alpha_{in}) - \sum_{j=1}^{p} (\psi_{jn} + \Delta\beta_{jn}) - \sum \omega(X_m),$$

where $\Delta\alpha_{in}$, $\Delta\beta_{jn}$ denote the total turning (geodesic curvature integrals) along the respective arcs α_{in}, β_{jn}.

Hence, the total intrinsic curvature of the polygon Q_n is

$$\omega(Q_n) = \varphi_n + \psi_n - \sum_{i=1}^{k} (\varphi_{in} + \Delta\alpha_{in}) - \sum_{j=1}^{p} (\psi_{jn} + \Delta\beta_{jn}) - \sum \omega(X_m).$$

Finally, passing to the limit and applying arguments analogous to those used in Euclidean geometry, we obtain the required formula. □

In Galilean space, consider a closed surface F possessing two conical points A and B, each admitting a singular supporting plane. Assume that S_A and S_B are the tangent cones at points A and B, respectively. Let the total angles around the vertices of these cones be γ_A and γ_B.

Then the Gauss–Bonnet formula for the closed surface F takes the form

$$\int_{\Phi} K\, d\sigma = \gamma_A + \gamma_B,$$

where:

- K is the *Gaussian curvature*,
- $d\sigma$ is the *surface area element*,
- γ_A, γ_B are the *total cone angles* at the conical points A and B.

This formula reflects the concentration of curvature at the conical points on the surface and generalizes the classical result to surfaces with isolated singularities in Galilean geometry.

Conclusion

This work presents a systematic exposition of the intrinsic geometry of convex surfaces in Galilean space. It is shown that, despite the degeneracy of the metric, it is possible to construct a consistent theory that incorporates the notions of length, angle, geodesics, and curvature. One of the key results is the formulation and proof of an analogue of the Gauss–Bonnet theorem, valid for convex surfaces without special supporting planes. It is also demonstrated that the intrinsic curvature coincides with the extrinsic curvature defined via cylindrical mapping, highlighting the deep connection between the intrinsic and extrinsic properties of convex geometry in Galilean space. The results obtained may serve as a foundation for further investigations of geometric structures in non-smooth spaces and have potential applications in mechanics, optics, and relativity theory, where space-time models may admit degenerate metrics. These results can be applied in classical mechanics, where Galilean space models Newtonian spacetime. They may also be useful in optics and relativity theory for studying degenerate metrics.

Author Contributions

All authors contributed equally to this work.

Conflict of Interest

The authors declare no conflict of interest.

References

1 Alexandrov, A.D. (2005). *Vnutrenniaia geometriia vypuklykh poverkhnostei [Intrinsic geometry of convex surfaces]*. Classics of Soviet Mathematics [in Russian].

2 Roschel, O. (1984). *Die Geometrie des Galileischen Raumes [The Geometry of the Galilean Space].* Graz: Forschungszentrum Graz, Mathematisch-Statistische Sektion [in German].

3 Sultanov, B.M., Kurudirek, A., & Ismoilov, Sh.Sh. (2023). Development and isometry of surfaces in Galilean space G3. *Mathematics and Statistics, 11*(6), 965–972. https://doi.org/10.13189/ms.2023.110612

4 Amorim, R.G.G., Santos, A.F., Vaz, K., & Ulhoa-Zettl, S.C. (2025). On the Pauli–Schrödinger coupling to the Galilean gravity. *International Journal of Modern Physics A, 40*(26), Article 2550109. https://doi.org/10.1142/S0217751X2550109X

5 Thiandoum, A. (2025). Curves of constant breadth in Galilean 4-space. *Afrika Matematika, 36*, Article 70. https://doi.org/10.1007/s13370-025-01296-8

6 Milin-Sipus, Z., & Divjak, B. (2011). Translation surface in the Galilean space. *Glasnik Matematicki, 46*(66), 455–469. https://doi.org/10.3336/gm.46.2.14

7 Dede, M., Ekici, C., & Goemans, W. (2018). Surfaces of revolution with vanishing curvature in Galilean 3-Space. *Journal of Mathematical Physics, Analysis, Geometry, 14*(2), 141–152. https://doi.org/10.15407/mag14.02.141

8 Artykbaev, A. (1988). Total angle about the vertex of a cone in Galilean space. *Mathematical Notes, 43*(5), 379–382. https://doi.org/10.1007/BF01158845

9 Ashyralyev, A., & Artykbaev, A. (2025). A measure associated with a convex surface and its limit cone. *Lobachevskii Journal of Mathematics, 46*(5), 2312–2316. https://doi.org/10.1134/S1995080224607987

10 Sharipov, A., Topvoldiyev, F., & Usmonkhujaev, Z. (2024). On one property of a conditional full angle. *Mathematics and Statistics, 12*(5), 420–427. https://doi.org/10.13189/ms.2024.120503

11 Topvoldiyev, F., & Sharipov, A. (2024). On defects of polyhedra isometric on sections at vertices. *AIP Conference Proceedings, 3004*(1), Article 030011. https://doi.org/10.1063/5.0199937

12 Pogorelov, A.V. (1969). *Vneshniaia geometriia vypuklykh poverkhnostei [The Extrinsic Geometry of Convex Surfaces].* Moscow: Nauka, Main Editorial Office for Physical and Mathematical Literature [in Russian].

13 Sachs, H. (1990). *Isotrope Geometrie des Raumes [Isotropic geometry of space].* Wiesbaden: Vieweg+Teubner Verlag [in German]. https://doi.org/10.1007/978-3-322-83785-1

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