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Analytical, numerical, and biomedical aspects of boundary value problems for third-order elliptic-type equations with singular coefficients

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Mathematical modeling of various real-world processes frequently leads to boundary value problems (BVPs) for third-order partial differential equations of mixed and composite types, which have no classical analogues in mathematical physics. Foundational studies by A.V. Bitsadze and M.S. Salakhitdinov first addressed well-posed boundary value problems for degenerate equations of third-order mixed and mixed-composite types. A key approach in their work involved representing the general solution of a composite-type equation as a sum of functions, which proved essential for operators constructed as products of permuted differential operators. Following these foundational contributions, the study of third-order partial differential equations involving Lavrentiev-Bitsadze, Gellerstedt, heat conduction, and string-type operators has been further advanced by both international and Uzbek mathematicians. Despite these developments, boundary value problems for third-order equations of parabolic-hyperbolic and elliptic-hyperbolic types with singular coefficients remain largely unexplored. In this article, we formulate and investigate boundary value problems for a third-order elliptic equation with a singular coefficient. The existence and uniqueness of classical solutions are rigorously proved. A new extremum principle for third-order equations is developed and applied to establish uniqueness. The existence of a solution is reduced to a singular integral equation of normal type, which is regularized using the classical Carleman-Vekua method, leading to an equivalent Fredholm equation of the second kind. The analytical framework is complemented by a numerical scheme that verifies the theoretical results and illustrates the qualitative behavior of solutions near the degenerate boundary. Furthermore, a numerical illustration is provided to demonstrate the stability and smoothness of the obtained solutions even in the presence of singular coefficients. Finally, the potential biomedical relevance of the model is discussed through its application to steady-state diffusion processes in tumor tissues.

Keywords: Analogue of the Dirichlet problem, representation of the general solution, third-order elliptic equation, extremum principle, regularization method, singular equation of normal type, finite-difference scheme, stability analysis, degenerate boundary, diffusion in tumor tissue

AMS Mathematics Subject Classification: 35J70, 35J25, 35J75, 65N12, 35B40, 92C50.

1 Introduction

The first fundamental studies for model equations of composite and mixed-composite types were conducted by A.V. Bitsadze and M.S. Salakhitdinov [1] in the early 1960s. Correct boundary value problems for third-order elliptic-hyperbolic and parabolic-hyperbolic type equations, where the principal part of the operator contains a derivative with respect to x or y, were first investigated in the works of A.V. Bitsadze, M.S. Salakhitdinov [1], T.D. Djuraev [2], and M.S. Salakhitdinov [3]. In these

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studies, the general solution of a mixed-composite type equation was represented as a sum of functions, a key idea for operators constructed as products of commutative differential operators. This direction was further developed in subsequent works devoted to various classes of third-order partial differential equations [4–6].

In works [7, 8] methods were proposed for solving third-order parabolic-hyperbolic and elliptic-hyperbolic equations by reducing them to inverse problems for second-order mixed-type equations with unknown right-hand sides. Numerical approaches to related boundary value problems were considered in [9]. It was also established that the coefficients of lower-order terms have a significant influence on the formulation and solvability of boundary value problems [2, 10]. Inverse problems for second-order mixed-type equations with unknown right-hand sides in various domains were examined in [11, 12]. Other types of problems for third-order mixed-composite equations were addressed in [13], while works [14, 15] considered boundary value problems for loaded parabolic-hyperbolic equations of third order in different domains.

Despite these developments, boundary value problems for third-order elliptic-type equations with singular coefficients have not been systematically studied so far.

In the present work, we study boundary value problems for a third-order elliptic-type equation with a singular coefficient. Theorems on the existence and uniqueness of classical solutions are proved based on the extremum principle and the theory of singular and Fredholm integral equations. An explicit Green's function is constructed using the method of double-layer potentials, and the integral representation of the solution is obtained.

In addition to the theoretical analysis, a numerical approach is developed to illustrate the behavior and stability of solutions near the singular boundary. The computational results confirm the analytical predictions and demonstrate the influence of the singular coefficient on the solution profile. Furthermore, the obtained results have potential applications in modeling steady-state diffusion and transport phenomena in heterogeneous biological media, including tumor tissues, where singular or degenerate coefficients naturally arise due to spatially variable diffusivity or irregular geometry.

Novelty and Contribution

In contrast to the well-known studies devoted to third-order mixed-type or elliptic equations with regular coefficients, this paper addresses boundary value problems for a third-order elliptic-type equation containing a singular coefficient. New formulations of Dirichlet- and Neumann-type problems are proposed for this class of equations. The corresponding Green's function is constructed for the first time, and the existence and uniqueness theorems for the classical solution are established using the extremum principle together with the theory of singular and Fredholm integral equations.

In addition, the paper complements the analytical framework with numerical simulations and an applied interpretation relevant to biomedical contexts, such as diffusion processes in tumor microenvironments. These results extend and generalize the classical works of Bitsadze and Salakhitdinov to a broader class of singular elliptic operators, providing both theoretical and applied contributions to the study of degenerate diffusion-type equations.

2 Problem statement

We consider the following third-order partial differential equation:

$$\frac{\partial}{\partial x}\left(y^m u_{xx} + u_{yy} + \frac{\beta_0}{y}u_y\right) = 0, \tag{1}$$

where $(x, y) \in \Omega$, $\Omega \subset \mathbb{R}^2$ is a bounded domain described below, and the parameters satisfy

$$m > 0, \quad -\frac{m}{2} < \beta_0 < 1. \tag{2}$$

Here and throughout the paper we use the notation $\beta = \frac{m+2\beta_0}{2(m+2)}$.

Note that the condition $m > 0$ ensures the degeneracy of ellipticity near the boundary $y = 0$. The inequality $-\frac{m}{2} < \beta_0 < 1$ is independent and characterizes the strength of the singular term $\frac{\beta_0}{y}u_y$.

Remark 1. Equation (1) is written in a divergence form with respect to the variable x. Integrating with respect to x formally leads to a second-order equation with an unknown function of y in the right-hand side.

However, the third-order formulation is essential since it allows:

(i) a natural representation of the solution in the form $u(x, y) = v(x, y) + \omega(y)$;

(ii) a correct formulation of boundary conditions involving derivatives along the curve σ;

(iii) the application of extremum principles adapted to higher-order operators.

Thus, equation (1) serves as the primary model, while an equivalent reduced second-order equation is introduced in the subsequent analysis.

The curve σ is a smooth Jordan curve located in the upper half-plane $y \geq 0$, connecting the points $A(-1, 0)$ and $B(1, 0)$.

We assume that the curve σ satisfies the following conditions:

1) The curve σ intersects any line $x = const$ at only one point.

2) The functions $x(s)$, $y(s)$, which provide the parametric equation of the curve σ, have continuous derivatives $x'(s)$, $y'(s)$, which do not simultaneously vanish, and have second derivatives satisfying a Hölder condition of order κ, $0 < \kappa < 1$, in the interval $0 \leq s \leq l$.

3) The curve ends with arbitrarily small arcs BB' and AA' of the normal contour σ_0 :

$$x^2 + \frac{4}{(m+2)^2}y^{m+2} = 1, \quad (y \geq 0). \tag{3}$$

4) Each line $y = c,\ 0 \leq c < h$, intersects σ at two points, and the line $y = h$ has a single common point $N(0, h)$ (the point of tangency) with σ. We denote the parts of the arc σ AN and BN by σ_1 and σ_2 respectively.

Here Ω is the domain bounded by the curve σ and the segment $J = \{(x, 0) : -1 \leq x \leq 1\}$.

Problem AD (analogous to the Dirichlet problem). It is required to find a function $u(x, y)$, with the following properties:

1) $u(x, y) \in C\left(\bar{\Omega}\right) \cup C^1\left(\Omega \cup \sigma_1 \cup J\right)$, where $u_x(u_y)$ may approach infinity of an order less than $(1 - 2\beta)$ at points $A\left(-1, 0\right)$ and $B\left(1, 0\right)$;

2) *u(x,y)* is a twice continuously differentiable solution of equation (1) in region Ω;

3) $u(x, y)$ satisfies the conditions

$$u(x, y)|_{\sigma} = \varphi(s), \quad s \in [0, l]\,, u\left(x, +0\right) = \tau\left(x\right), \quad (x, 0) \in \bar{J}, \tag{4}$$

$$\left.\frac{\partial u(x, y)}{\partial n}\right|_{\sigma_1} = g(s), \quad s \in \left[\frac{l}{2}, l\right], \tag{5}$$

where n is the inward normal, s is curve arc length σ, measured from the point $B\left(1, 0\right)$, l is curve length σ, $\varphi\left(s\right),\ g\left(s\right),\ \tau\left(x\right)$ are given functions, wherein

$$\varphi(l) = \tau(-1), \quad \varphi(0) = \tau(1), \tag{6}$$

$$\varphi(\xi(s), \eta(s)) = \eta^2(s)\varphi_1(\xi(s), \eta(s)), \quad \varphi_1(s) \in C\left[0, l\right], \quad g(s) \in C^2\left(\frac{l}{2}, l\right), \tag{7}$$

$$\tau\left(x\right) \in C(\bar{J}) \cap C^2(J), \tag{8}$$

the function $g(s)$ may have singularities of order less than one at $s \to l$, but at $s \to \frac{l}{2}$ the function is bounded.

Problem AN (analogous to the Neumann problem). It is required to find a function $u(x,y)$, with the following properties:

1) $u(x,y) \in C\left(\bar{\Omega}\right) \cup C^1\left(\Omega \cup \sigma_1 \cup J\right)$, where $u_x(u_y)$ may approach infinity of an order less than $(1-2\beta)$ at points $A\left(-1,0\right)$ and $B\left(1,0\right)$;

2) *u(x,y)* is a twice continuously differentiable solution of equation (1) in region Ω;

3) $u(x,y)$ satisfies the conditions (4)–(6) and

$$u(x,y)|_{\sigma} = \varphi(s), \quad s \in [0,l]\,, \quad u_y(x,y)|_{y=+0} = \nu\left(x\right), \quad (x,0) \in J,$$

where $\nu(x)$ is a given function, where

$$\nu\left(x\right) \in C^2(J), \tag{9}$$

and function $\nu(x)$ may have singularities of order less than $1-2\beta$ at the endpoints of J.

Without loss of generality, we can assume

$$u(A) = u(B) = u'(A) = u'(B) = 0, \tag{10}$$

where derivatives are taken along the tangent to σ [16]. Due to these conditions, the function $\varphi(s)$ admits a suitable representation.

3 *Investigation of the problem AD*

Any regular solution of equation (1) in the region Ω can be represented in the form [16]:

$$u(x,y) = \upsilon(x,y) + \omega(y), \tag{11}$$

here $\upsilon\left(x,y\right)$ is a regular solution of the equation

$$L\upsilon \equiv y^m \upsilon_{xx} + \upsilon_{yy} + \frac{\beta_0}{y}\upsilon_y = 0, \quad (x,y) \in D, \tag{12}$$

and $\omega(y)$ is an arbitrary twice continuously differentiable function, and without loss of generality, we can assume that the function $\omega(y)$ satisfies the conditions

$$\omega(0) \ = \omega(h) = 0. \tag{13}$$

We will assume that everywhere along the arc σ_1 except at point $N(0,h)$ the following condition is satisfied (see condition (10))

$$\frac{dx}{dn} \neq 0. \tag{14}$$

Consequently, the problem AD reduces to the problem AD^* of determining in the region Ω a regular solution $\upsilon(x,y)$ of equation (12) that satisfies the conditions

$$\upsilon|_{\sigma} = \varphi(s) - \omega\left(y(s)\right), \quad s \in [0,l]\,, \upsilon|_{y=0} = \tau(x), \quad -1 \le x \le 1, \tag{15}$$

$$\left.\frac{\partial \upsilon(x,y)}{\partial n}\right|_{\sigma_1} = g(s) - \omega'(y(s))\frac{dy}{dn}, \quad s \in \left(\frac{l}{2}, l\right). \tag{16}$$

4 Uniqueness of solution to the problem AD

Theorem 1. If there exists a solution to problem AD, then it is unique if and only if conditions (13) and (14) are satisfied.

Proof. Let $\tau(x) \equiv \varphi(s) \equiv g(s) \equiv 0$, then by virtue of (15), in region Ω the regular solution $\upsilon(x, y)$ of equation (12) satisfies the conditions

$$\upsilon|_{\sigma} = -\omega(y(s)),\ s \in [0, l]\,, \upsilon|_{y=0} = 0,\ x \in [-1, 1],\ \left.\frac{\partial \upsilon(x, y)}{\partial n}\right|_{\sigma_1} = -\omega'(y)\frac{dy}{dn},\ s \in \left(\frac{l}{2}, l\right). \tag{17}$$

A regular solution $\upsilon(x, y)$ of equation (12) inside the domain Ω cannot attain a positive maximum and a negative minimum [16, 17]. By the extremum principle for elliptic equations [18], it follows that the solution $\upsilon(x, y)$ of equation (12) in $\bar{\Omega}$ reaches on $\overline{AB} \bigcup \bar{\sigma}$ its positive maximum and negative minimum. From (13) and the second condition of (17), it follows that $\upsilon(x, y)$ does not attain a positive maximum and negative minimum at point $N(0, h)$ and on the segment AB. Therefore, considering (14), we prove that the function $\upsilon(x, y)$ cannot attain a positive maximum and negative minimum on the open arcs σ_1 and σ_2. Suppose the opposite, let $\upsilon(x, y)$ attain its positive maximum (negative minimum) at some interior point s_0 of the arc σ_1. Then, since $\upsilon(x, y)$ on σ_1 takes on the values of the function $\omega(y(s))$, the necessary condition for an extremum gives at this point

$$\left.\frac{\partial \omega(y(s))}{\partial s}\right|_{\sigma_1} = \omega'(y(s))\frac{dy}{ds} = -\omega'(y(s))\frac{dx}{dn}.$$

From here, based on (14), we obtain $\omega'(y(s_0)) = 0$. Then, due to the last condition (17) with regard to (13), at the considered point we have $\frac{\partial \upsilon}{\partial n} = 0$. But this last equality contradicts the known property of harmonic functions, namely, that at a boundary point of a positive maximum (negative minimum), it is $\frac{\partial \upsilon}{\partial n} < 0$ $\left(\frac{\partial \upsilon}{\partial n} > 0\right)$ [19]. Therefore, $\upsilon(x, y)$ cannot attain a positive maximum and negative minimum on the arcs σ_1. By virtue of the first condition in (17), we conclude that the function $z\,(x, y)$ cannot attain an extremum other than zero on the arc σ_2.

Hence, based on the extremum principle [4] and equalities (13), we conclude that $\upsilon(x, y) \equiv \omega(y) \equiv 0$ in $\bar{\Omega}$. Therefore, from (11), we have $u(x, y) \equiv 0,\ (x, y) \in \bar{\Omega}$. This proves the uniqueness of the solution to problem AD. □

5 Existence of a solution to the problem AD

Let the curve σ satisfy conditions 1)–4).

Definition 1. The Green's function for the Dirichlet problem for equation (12) with conditions (15) is called the function, $G(\xi, \eta; x, y)$ which:

1) is a regular solution of equation (12) everywhere in the domain Ω, except for the point (x, y);
2) satisfies the boundary condition

$$G\,(\xi, \eta; x, y)|_{\sigma \cup \bar{J}} = 0, \qquad (x, y) \in \Omega; \tag{18}$$

3) can be represented as

$$G(\xi, \eta; x, y) = g(\xi, \eta; x, y) + \vartheta(\xi, \eta; x, y). \tag{19}$$

Here

$$g(\xi, \eta; x, y) = k\left(\frac{4}{m+2}\right)^{4\beta-2} \frac{\left(1-\rho^2\right)^{1-2\beta}}{\left(r_1{}^2\right)^{\beta}} F\left(1-\beta,\ 1-\beta,\ 2-2\beta;\ 1-\rho^2\right),$$

where $F(a,b,c;z)$ denotes the Gauss hypergeometric function, $g(\xi,\eta;x,y)$ is the fundamental solution of equation (12), and $\vartheta(\xi,\eta;x,y)$ is a regular solution of equation (12) everywhere inside the region Ω, satisfying the conditions

$$\vartheta\left(\xi(s),\eta(s);x,y\right)|_\sigma=-g\left(\xi(s),\eta(s),x,y\right)|_\sigma,\ (x,y)\in\Omega,\quad \vartheta\left(\xi,\eta;x,y\right)|_{\eta=0}=0, \tag{20}$$

$$\left.\begin{matrix} r^2 \\ {r_1}^2 \end{matrix}\right\}=(x-\xi)^2+\frac{4}{(m+2)^2}\left(y^{(m+2)/2}\mp\eta^{(m+2)/2}\right)^2,\quad \rho^2=\frac{r^2}{{r_1}^2},$$

$$k=\frac{1}{4\pi}\left(\frac{4}{m+2}\right)^{2-2\beta}\frac{\Gamma^2\left(1-\beta\right)}{\Gamma\left(2-2\beta\right)},\quad \beta=\frac{m+2\beta_0}{2(m+2)}.$$

According to a well-known formula [20], it follows that when $r\to 0$ and $\rho\to 0$ $(y>0)$ the function $g(\xi,\eta;x,y)$ has a logarithmic singularity [17] and satisfies condition $g(\xi,0;x,y)=0$ for all ξ.

Let us construct the Green's function using the double-layer potential

$$W(x,y)=\int\limits_0^l\mu(t)\ \eta^{\beta_0}(t)A_t\left[g\left(\xi(t),\eta(t);x,y\right)\right]dt, \tag{21}$$

where $\mu(t)\in C\left[0,l\right]$, and $A_t\left[g\left(\xi,\eta;x,y\right)\right]=\eta^m\frac{\partial g}{\partial\xi}\cdot\frac{d\eta}{dt}-\frac{\partial g}{\partial\eta}\cdot\frac{d\xi}{dt}$ is normal derivative [17].

Lemma 1. If the curve σ satisfies conditions 1)–2) and the density of $\mu(t)\in C\left[0,l\right]$, then the following formulas are valid

$$W_i(s)=-\frac{1}{2}\mu(s)+\int\limits_0^l\mu(t)K(s,t)dt, \tag{22}$$

$$W_e(s)=\frac{1}{2}\mu(s)+\int\limits_0^l\mu(t)K(s,t)dt. \tag{23}$$

Here $W_i(s)$ and $W_e(s)$ denote the interior and exterior boundary limits of the double-layer potential $W(x,y)$, respectively, and the kernel $K(s,t)$ is defined by

$$K(s,t)=\eta^{\beta_0}(t)A_t\big[g(\xi(t),\eta(t);x(s),\eta(s))\big],\quad (\xi(t),\eta(t))\in\sigma.$$

Proof. The proof of the lemma is carried out in the same way as in the works [17, 20]. □

Note that the potential $W(x,y)$ is a regular solution of equation (12) in each part of the upper half-plane that does not intersect either with curve σ, or with the x-axis. The double-layer potential $W(x,y)$ is defined for all points in the upper half-plane.

Let Ω_0 be a normal domain bounded by segment $[-1,1]$ of the x-axis and the normal curve (3), then the Green's function for the Dirichlet problem for equation (12) with conditions (15) is explicitly written as

$$G_0(\xi,\eta;x,y)=g(\xi,\eta;x,y)-R^{-2\beta}g(\xi,\eta;\bar{x},\bar{y}), \tag{24}$$

where $R^2=x^2+\frac{4}{(m+2)^2}y^{m+2},\ \bar{x}=\frac{x}{R^2},\ \bar{y}^{\frac{m+2}{2}}=y^{\frac{m+2}{2}}/R^2$.

The Green's function for an arbitrary domain, as in the case of the Gellerstedt equation [20], is constructed on the basis of the representation (24) using the double-layer potential (22) and satisfies the boundary condition (18). It can be represented as

$$G(\xi,\eta;x,y)=G_0(\xi,\eta;x,y)+H(\xi,\eta;x,y),$$

where

$$H(\xi,\eta;x,y)=\int\limits_0^l \lambda(t;\xi,\eta)\,G_0(\xi(t),\eta(t);x,y)dt.$$

Now we find the density $\lambda(t;x,y)$. According to representation (19), we seek the function $\vartheta\,(\xi,\eta;x,y)$ in the form of a double-layer potential (25):

$$\vartheta(\xi,\eta;x,y)=\int\limits_0^l \lambda(t;x,y)\eta^{\beta_0}(t)A_t\left[g\left(\xi(t),\eta(t);\xi,\eta\right)\right]dt. \tag{25}$$

Taking into account equalities (22), (23) and the boundary conditions (20), we obtain an integral equation for the density $\lambda\,(t,\,x,\,y)$:

$$\lambda(s;\,x,\,y)-2\int\limits_0^l K(s,t)\lambda\,(t;\,x,\,y)\,dt=2g\left(\xi(s),\,\eta(s);x,\,y\right). \tag{26}$$

The right-hand side of the equation is a continuous function of s(point $(x,\,y)\in\Omega$), kernel $K\,(s,t)$ has a weak singularity, and the number 2 is not its eigenvalue [17]. Therefore, equation (26) is solvable, and its continuous solution can be written as

$$\lambda\,(s;\,x,\,y)=2g\left(\xi(s),\eta(s);\,x,\,y\right)+4\int\limits_0^l R\,(s,t;2)g\left(\xi(t),\eta(t);\,x,\,y\right)dt, \tag{27}$$

where $R\,(s,t;2)$ the kernel resolvent $K\,(s,t)\,;\,(\xi(s),\,\eta(s))\in\sigma$.

Substituting (27) into (25), we find the function $\vartheta(\xi,\eta;x,y)$.

Theorem 2. If conditions (2), (3), (7)–(9) are satisfied, then in region Ω a solution to problem AD exists.

Proof. Using the Green's function in region Ω the solution to problem (15) for equation (12) can be represented as follows (see, (21)):

$$\upsilon(x,y)=\int\limits_{-1}^{1}\tau(\xi)\left[\eta^{\beta_0}(\xi)\frac{\partial G(\xi,\eta;x,y)}{\partial\eta}\right]_{\eta=0}d\xi+\int\limits_{\sigma}\left[\varphi(s)-\omega(\eta(s))\right]A_s\left[G(\xi(s),\,\eta(s);\,x,y)\right]ds, \tag{28}$$

where $\omega(\eta(s))$ is an unknown function to be determined.

I. Let the curve σ coincide with the normal contour σ_0, then we find the unknown function $\omega(y)$. Due to (25) and the form of the function $g(\xi,\eta;x,y)$ from (28), we find

$$\upsilon(x,y)=k(1-\beta_0)y^{1-\beta_0}\int\limits_{-1}^{1}\tau(t)\left\{\left[(x-t)^2+4y^{m+2}/(m+2)^2\,\right]^{\beta-1}-\right.$$

$$\left.-\left[(1-xt)^2+4t^2y^{m+2}/(m+2)^2\,\right]^{\beta-1}\right\}dt-k(1-\beta)(m+2)(1-R^2)y^{1-\beta_0}\times$$

$$\times\int\limits_{\sigma_0}\left[\varphi(s)-\omega(\eta(s))\right]\;\left(r_1^2\right)^{\beta-2}F(1-\beta,2-\beta,2-2\beta;1-\rho^2)ds. \tag{29}$$

Transitioning to polar coordinates $x = \cos\theta,\ y = ((m+2)\sin\theta/2\ )^{2/(m+2)}$ in formula (29), we have

$$\upsilon(R,\theta_0) = k\left(\frac{m+2}{2}R\sin\theta_0\right)^{1-2\beta}\int\limits_{-1}^{1}\tau(\xi)\left[\left(R^2-2R\xi\cos\theta_0+\xi^2\right)^{\beta-1}-\right.$$

$$\left.-\left(1-2R\xi\cos\theta_0 R^2\xi^2\right)^{\beta-1}\right]d\xi + k(1-\beta)(m+2)\left(\frac{m+2}{2}R\sin\theta_0\right)^{1-2\beta}\left(1-R^2\right)\int\limits_0^{\pi}\{\varphi(\theta)-$$

$$-\omega\left[\left(\frac{m+2}{2}R\sin\theta\right)^{1-2\beta}\right]\right\}\left(r_1^2\right)^{\beta-2}F\left(1-\beta,2-\beta,2-2\beta,1-\rho^2\right)\sin\theta\,d\theta. \tag{30}$$

To satisfy condition (16), we first compute the derivatives of the function $\upsilon(R,\theta_0)$ defined by formula (30) with respect to R and θ_0, and then, after integrating the obtained expressions by parts, we take the limit as $R\to 1$ $\left(\frac{\pi}{2}\le\theta\le\pi\right)$, obtaining

$$\gamma(\theta_0)\sin\theta_0 + \int\limits_0^{\pi}\gamma(\theta)\left[M_1(\theta_0,\theta)+M_2(\theta_0,\theta)\right]d\theta = f(\theta_0), \tag{31}$$

where

$$\gamma(\theta_0) = \omega'\left[\left((m+2)\sin\theta_0/2\ \right)^{1-2\beta}\right].$$

Due to conditions (7)–(9), we conclude that the kernel $M_1(\theta_0,\theta)$ at $\theta=\theta_0$ diverges to first-order infinity, i.e., it is singular, while $M_2(\theta_0,\theta)$ has a weak singularity and is a Fredholm kernel. $f(\theta_0)\in C^2\left(\frac{\pi}{2},\pi\right)$ and $f(\theta_0)$ are bounded at $\theta_0\to\ \frac{\pi}{2}$, but at $\theta_0\to\pi$ are unbounded.

In (31), we will split the integral from 0 to π into two parts: from 0 to $\frac{\pi}{2}$ and from $\frac{\pi}{2}$ to π. In the first part, we will substitute the variable according to the formula $\theta=\pi-\theta_1$, and then instead of θ_1 we will again write θ. Then, due to $\gamma(\theta)=\gamma(\pi-\theta)$ equation (32) can be written as

$$\gamma(\theta_0)\sin\theta_0 + \frac{\cos\theta_0}{2\pi}\int\limits_{\frac{\pi}{2}}^{\pi}\gamma(\theta)\left(ctg\frac{\theta-\theta_0}{2}-ctg\frac{\theta+\theta_0}{2}\right)d\theta+$$

$$+\int\limits_{\frac{\pi}{2}}^{\pi}M(\theta_0,\theta)\,\gamma(\theta)d\theta = f(\theta_0),\quad \theta_0\in\left(\frac{\pi}{2},\pi\right), \tag{32}$$

where the kernel $M(\theta_0,\theta)$ has a weak singularity [3].

Performing the substitution $t=e^{i\theta},\ t_0=e^{i\theta_0}$ in (32) taking into account

$$\frac{1}{2}\left(ctg\frac{\theta-\theta_0}{2}-ctg\frac{\theta+\theta_0}{2}\right)d\theta = \left(\frac{1}{t-t_0}-\frac{1}{1-t\,t_0}\right)dt + \frac{1+t_0}{1-t\,t_0}dt,$$

$$\sin\theta_0=\frac{t_0^2-1}{2\,i\,t_0},\ \cos\theta_0=\frac{t_0^2+1}{2t_0},\ \theta=\frac{1}{i}\ln t,\ \theta_0=\frac{1}{i}\ln t_0,\ d\theta=\frac{1}{i}\cdot\frac{dt}{t},$$

we obtain

$$a(t_0)\gamma_1(t_0) + \frac{b(t_0)}{\pi i}\int\limits_{c_0}\gamma_1(t)\left(\frac{1}{t-t_0}-\frac{1}{1-t\,t_0}\right)dt + \int\limits_{c_0}N(t_0,t)\,\gamma_1(t)dt = f_1(t_0), \tag{33}$$

where c_0 is the contour of integration representing a quarter of a circle θ_0,

$$\gamma_1(t_0) = \gamma\left(\frac{1}{i}\ \ln t_0\right),\quad a(t_0) = 1 - t_0^2,\quad b(t_0) = 1 + t_0^2,$$

$$f_1(t_0) = -2i\, t_0 f\left(\frac{1}{i}\ lnt_0\right),\quad f_1(e^{i\,\theta_0}) = -2i\, e^{i\,\theta_0} f(\theta_0),$$

$$N\left(t_0, t\right) = \frac{(1+t_0)\left(1+t_0^2\right)}{\pi i\,(1 - t\,t_0)} - \frac{2t_0}{t} M\left(\frac{1}{i}\ln t_0;\ \frac{1}{i}\ln t\right).$$

Transitioning to the question of the solvability of the singular integral equation (33), first of all, we note that it is an equation of the normal type [18], i.e. $a^2(t_0) + b^2(t_0) = 2(1 + t_0^4) \neq 0$. Its index is zero in the class h [5], which is bounded at $t_0 \to i$, but at $t_0 \to -1$ unbounded.

The singular integral equation (33) will be reduced by the well-known Carleman–Vekua regularization method [18], to an equivalent second-kind Fredholm equation, and returning to the old variables, we have

$$\chi(\theta_0) + \int\limits_{\pi/2}^{\pi} \nu_1(\theta_0, \theta)\chi(\theta)d\theta = f_2(\theta_0),\quad \theta_0 \in \left(\frac{\pi}{2}, \pi\right). \tag{34}$$

From the class h and the properties of the Cauchy-type integral [18], we conclude that the right-hand side of equation (34) is bounded at $\theta_0 \to \pi/2$ and can diverge to infinity of order not exceeding ε_2 at $\theta_0 \to \pi$, while the kernel $N_1(\theta_0, \theta)$ has a weak singularity [2].

According to the theory of Fredholm integral equations [18] and from the uniqueness of the solution to problem AD (see Theorem 1), we conclude that the integral equation (34) is uniquely solvable in class $C^2\left(\pi/2\,;\pi\right)$, moreover $\gamma(\theta_0)$ may have a singularity of order less than ε_2 at $\theta_0 \to \pi$, and at $\theta_0 \to \pi/2$ it is bounded, and its solution is given by the formula:

$$\gamma(\theta_0) = f_2(\theta_0) - \int\limits_{\pi/2}^{\pi} \nu_1^*(\theta_0, \theta)\, f_2(\theta)d\theta,\quad \theta_0 \in \left(\frac{\pi}{2}, \pi\right), \tag{35}$$

here $N_1^*(\theta_0, \theta)$ is the kernel resolvent $N_1(\theta_0, \theta)$.

From (35) we find the unknown function $\omega(y)$. Then, the solution to equation (12) satisfying conditions (15) is determined by formula (29). From this and from the general representation of (11), we find the solution to the problem AD. This proves the existence of a solution to the problem AD in the case when σ coincides with σ_0.

II. The proof for a general curve σ follows the same scheme as above, with appropriate modifications related to the geometry of the boundary, and therefore is omitted for brevity. □

Theorem 3. If conditions (2), (3), (7), (20) are satisfied, then in region Ω there exists a unique regular solution to the problem AN.

Theorem 3 is proven using the extremum principle and the method of integral equations using the property of the Green's function.

6 Numerical Illustration

To verify the analytical results established in the previous sections and to illustrate the qualitative behavior of solutions, we consider the corresponding reduced problem for the function $v(x, y)$ governed by equation (12), which is obtained from the original third-order equation (1) via the representation of the solution. Numerical simulations not only confirm the correctness of the proposed analytical framework but also provide valuable insight into the behavior of solutions, especially near the degenerate boundary where analytical evaluation is challenging [21–23].

$$y^m u_{xx} + u_{yy} + \frac{\beta_0}{y} u_y = 0, \quad (x, y) \in \Omega = [-1, 1] \times [0, 1], \tag{36}$$

subject to the boundary conditions

$$u(-1, y) = 0, \quad u(1, y) = 0, \quad u(x, 0) = 0, \quad u(x, 1) = \sin(\pi x).$$

The boundary function $\sin(\pi x)$ at $y = 1$ ensures smoothness and compatibility with the homogeneous conditions at $x = \pm 1$, producing a nontrivial, well-behaved solution suitable for numerical analysis.

Finite-Difference Scheme and Regularization

Regularization techniques for degenerate or singular diffusion models are widely used in numerical analysis [24, 25], while analytical estimates are studied in [26]. A discrete analogue of (36) was constructed using second-order central finite differences in both spatial directions:

$$y_j^m \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h_x^2} + \frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{h_y^2} + \frac{\beta_0}{y_j} \frac{u_{i,j+1} - u_{i,j-1}}{2h_y} = 0,$$

for $i = 1, \ldots, N_x - 1$, $j = 1, \ldots, N_y - 1$, with grid steps $h_x = h_y = 0.05$.

The term $\frac{\beta_0}{y} u_y$ contains a singularity at $y = 0$. To avoid numerical instability, a regularization was introduced by replacing y with $\max(y_j, \varepsilon)$, where $\varepsilon = 10^{-4}$. This modification preserves the accuracy of the approximation while preventing division by zero, and is fully consistent with the boundary condition $u(x, 0) = 0$.

The resulting sparse linear algebraic system was symmetric and diagonally dominant, allowing the use of the conjugate gradient method for efficient numerical solution.

Mesh Convergence and Stability Analysis

To examine the convergence of the finite-difference scheme, computations were performed on uniform grids with $h_x = h_y = \{0.1, 0.05, 0.025\}$. The discrete L_2-error was estimated by comparing solutions on successive meshes. The error decreased by approximately a factor of four when the grid step was halved, confirming the second-order accuracy of the scheme. No oscillations or instabilities were observed even near the degenerate line $y = 0$, demonstrating the numerical stability of the regularized formulation.

Effect of Parameters m and β_0

The parameters m and β_0 control the degree of elliptic degeneracy and the strength of the singular term, respectively. Table 1 shows representative values of the computed maximum and minimum of the solution for several parameter sets.

Table 1

Influence of parameters m and β_0 on the solution profile

(m, β_0)	$\max u(x,y)$	$\min u(x,y)$	Qualitative behavior
(1.0, 0.2)	0.42	-0.38	smooth, symmetric profile
(2.0, 0.5)	0.55	-0.53	sharper gradients near $y = 0$
(3.0, 0.8)	0.67	-0.61	strong localization near upper boundary

An increase in either m or β_0 enhances the anisotropy of diffusion and causes the solution to flatten near $y = 0$, illustrating the physical effect of singular diffusion suppression in heterogeneous media.

Numerical Results

Table 2 provides representative numerical values of the computed solution for $m = 2$, $\beta_0 = 0.5$.

Table 2

Selected numerical values of $u(x, y)$ for $m = 2$, $\beta_0 = 0.5$

$y\backslash x$	-1.0	-0.5	0.0	0.5	1.0
0.0	0.000	0.000	0.000	0.000	0.000
0.2	-0.182	-0.094	0.000	0.108	0.194
0.4	-0.361	-0.189	0.000	0.215	0.381
0.6	-0.533	-0.279	0.000	0.319	0.546
0.8	-0.712	-0.366	0.000	0.420	0.702
1.0	0.000	-1.000	0.000	1.000	0.000

Figures 1 and 2 show the resulting solution surfaces and contour maps, revealing the smooth and symmetric nature of $u(x, y)$.

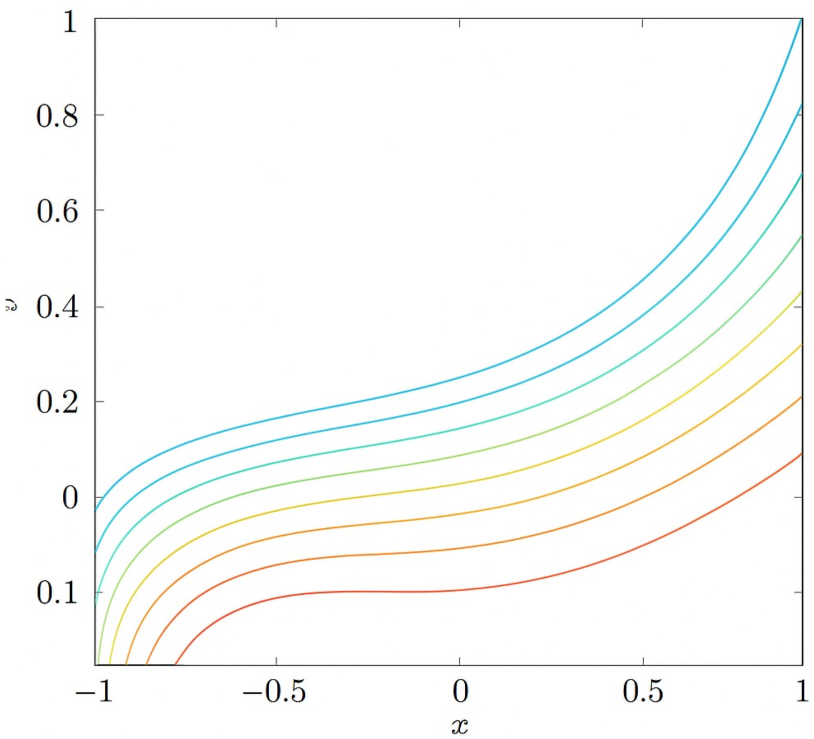

Figure 1. Contour plot of the numerical solution $u(x, y)$ for $m = 2$, $\beta_0 = 0.5$

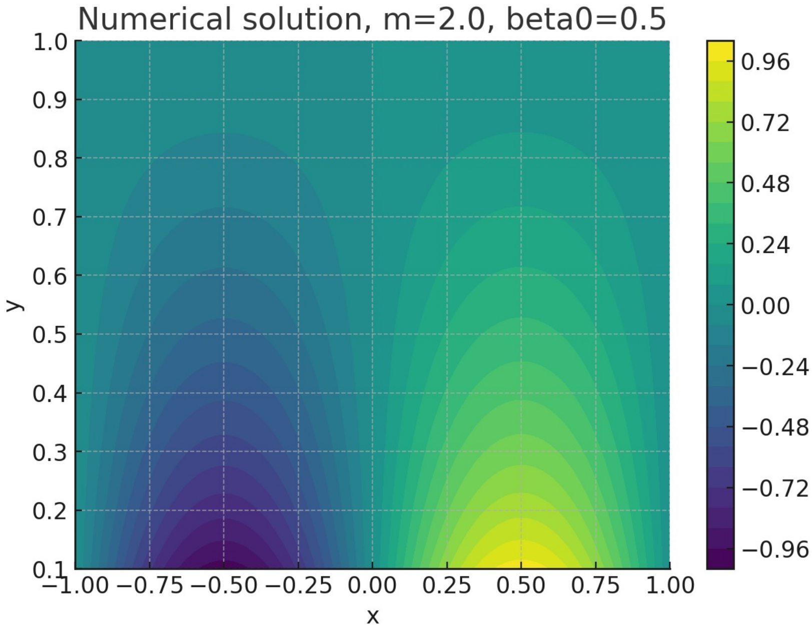

Figure 2. Color map of the numerical solution $u(x, y)$ for $m = 2$, $\beta_0 = 0.5$

Discussion and Interpretation

The numerical simulations confirm the theoretical predictions on the stability and smoothness of the classical solution, even in the presence of a singular coefficient. The results also demonstrate that the singularity primarily influences the diffusion rate in the y-direction, suppressing the amplitude of $u(x, y)$ near the degenerate line.

From a physical perspective, this effect is analogous to diffusion slow-down in heterogeneous biological media with variable permeability. Such models are relevant in biomedical contexts, including tumor growth and drug transport, where nonuniform tissue properties naturally give rise to singular or degenerate diffusion coefficients.

Conclusion of Numerical Analysis

The extended numerical analysis demonstrates that:
- the finite-difference scheme is second-order accurate and stable under mild regularization;
- the parameters m and β_0 significantly affect the localization and anisotropy of the solution;
- the computational results fully support the analytical existence and uniqueness theorems established earlier.

Hence, the combination of theoretical and numerical analyses provides a consistent and comprehensive understanding of third-order elliptic-type equations with singular coefficients, establishing a foundation for further research on their applications in complex physical and biomedical systems.

Applications in Oncology

Boundary value problems for degenerate elliptic-type equations with singular coefficients have recently gained considerable attention in the modeling of biological processes such as tumor growth, nutrient diffusion, and drug transport in cancerous tissues [27–29].

In this biomedical context, the governing equation

$$y^m u_{xx} + u_{yy} + \frac{\beta_0}{y} u_y = 0$$

can be interpreted as a steady-state diffusion model describing the spatial concentration $u(x, y)$ of oxygen, nutrients, or therapeutic agents within a heterogeneous tumor domain.

The coefficient y^m characterizes anisotropic or spatially varying diffusivity, which decreases toward necrotic regions ($y \to 0$), while the singular term $\frac{\beta_0}{y} u_y$ accounts for enhanced gradients and transport resistance near the tumor core or vascular interfaces. Such singular terms naturally arise in multilayer diffusion models, where permeability changes sharply across tumor boundaries, or in reduced radial formulations of spherical tumor growth, where the coordinate y represents the distance from the tumor center. The singular point $y = 0$ corresponds to a low-activity necrotic zone, where both diffusion and metabolic processes are significantly reduced.

Biophysical interpretation of parameters. The exponent m defines the degree of spatial inhomogeneity of diffusion, reflecting the structural heterogeneity of tumor tissues ($m \in [1, 3]$ for dense or fibrotic tumors). The parameter β_0 quantifies transport resistance or gradient intensity near the tumor center; larger β_0 values correspond to stronger attenuation of concentration fluxes in poorly vascularized regions.

Connection with numerical results. The numerical experiments presented in Section 6 demonstrated that for typical biological parameters ($m = 2$, $\beta_0 = 0.5$), the computed solutions preserve stability and smoothness even in the presence of singularities. This confirms that the proposed model adequately captures realistic diffusion behavior in heterogeneous media, where diffusion slows down but remains continuous as it approaches necrotic or impermeable regions.

Practical relevance. The analytical methods developed in this paper based on the Green's function construction and the degenerate double-layer potential enable accurate modeling of concentration fields in domains with irregular boundaries and spatially varying diffusivity. Such formulations are particularly relevant to:

- simulating oxygen and nutrient distribution in avascular tumor spheroids;
- analyzing the penetration of chemotherapeutic agents through layered tumor tissues with variable permeability;
- investigating steady-state profiles of diffusive signaling molecules influencing tumor-host interactions and microenvironmental feedbacks.

Thus, the combined analytical and numerical framework provides a rigorous mathematical foundation for describing stationary diffusion and transport processes in tumor tissues with spatial heterogeneity. These results contribute to the development of realistic and interpretable cancer models that can later be extended by incorporating reaction-diffusion or proliferation mechanisms to capture tumor evolution dynamics.

Conclusion

In this study, we examined boundary value problems for a class of degenerate elliptic-type equations with singular coefficients, where degeneracy occurs along a portion of the boundary according to a power-type law. A constructive analytical approach was proposed for obtaining the Green's function using the theory of degenerate double-layer potentials, guaranteeing the existence and uniqueness of classical solutions. The resulting integral representation explicitly reflects both the boundary geometry and the degeneracy structure of the problem.

Compared with standard elliptic models [30–32], the inclusion of a singular coefficient introduces essential mathematical challenges: the loss of uniform ellipticity near the degenerate boundary requires weighted functional formulations and special handling of singular kernels. The present approach extends previous studies on degenerate and related elliptic operators [33,34] by accommodating irregular and biologically motivated geometries, which enhances its practical applicability.

A distinctive aspect of this work lies in the introduction of a *degenerate double-layer potential*, whose analytical properties make it particularly suitable for modeling diffusion phenomena in heterogeneous biological media. Such structures are characteristic of tumor-host systems, where diffusivity can vanish or vary drastically near necrotic or vascular regions [35, 36]. Both analytical and numerical results confirm that the developed formulation preserves stability and smoothness even in the presence of singular coefficients, consistent with physical and biological expectations.

Beyond biomedical applications, the proposed framework can be extended to other applied fields such as porous media flow, heat transfer in nonuniform materials, and population dynamics, where similar singularities and degeneracies naturally occur.

Future research will focus on refining numerical solvers for the associated integral equations, incorporating nonlocal and time-fractional effects, and validating the model through comparison with experimental or clinical data. Overall, the developed theoretical and computational methodology provides a solid mathematical basis for analyzing degenerate diffusion processes in complex and heterogeneous environments.

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Author Contributions

D.B. Eshmamatova: Conceptualization, supervision, and analytical methodology.
N.K. Ochilova: Formal analysis, computations, and manuscript preparation.
All authors reviewed and approved the final manuscript.

Conflicts of interest

The authors declare no conflict of interest.

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